J. Eur. Math. Soc. 25, 4221–4289 (2023) DOI 10.4171/JEMS/1264

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The cohomological Hall algebra of a surface and factorization cohomology

Received April 11, 2019

Abstract. For a smooth quasi-projective surface S over $\mathbb{C}$ we consider the Borel–Moore homology of the stack of coherent sheaves on S with compact support and make this space into an associative algebra by a version of the Hall multiplication. This multiplication involves data (virtual pullbacks) governing the derived moduli stack, i.e., the perfect obstruction theory naturally existing on the non-derived stack. By restricting to sheaves with support of given dimension, we obtain several types of Hecke operators. In particular, we study $R(S)$, the Hecke algebra of 0-dimensional sheaves. For the case $S = \mathbb{A}^2$, we show that $R(S)$ is an enveloping algebra and identify it, as a vector space, with the symmetric algebra of an explicit graded vector space. For a general S, we find the graded dimension of $R(S)$, using the techniques of factorization cohomology.

Keywords. Cohomological Hall algebra, virtual pullback

Contents

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Mathematics Subject Classification (2020): Primary 14F08; Secondary 18M75, 32G13

0. Introduction

0.1. Motivation

A large part of the classical theory of automorphic forms for GL_n over functional fields can be interpreted in terms of Hall algebras of abelian categories [32, 33]. Relevant here is $Coh(C)$, the category of coherent sheaves on a smooth projective curve $C/\mathbb{F}_q$. Taking the Hall algebra of $Bun(C)$, the subcategory of vector bundles, produces (unramified) automorphic forms, while $Coh_0(C)$, the category of torsion sheaves, gives rise to the Hecke algebra.

The classical Hall algebra of a category such as $Coh(C)$ consists of functions on ($\mathbb{F}_q$-) points of the moduli stack of objects and so admits various modifications (cf. [14, Ch. 8]). Most important is the *cohomological Hall algebra* (COHA) where we take the *cohomology of the stack* instead of the space of functions on the set of its points [37]. This allows us to work over more general fields such as $\mathbb{C}$.

Study of Hall algebras (classical or cohomological) of the categories $Coh(S)$ for varieties S of dimension $d > 1$ can therefore be considered as a higher-dimensional analog of the theory of automorphic forms. In this paper we consider the case of surfaces ($d = 2$) over $\mathbb{C}$ and study their COHA. In this case we have a whole new range of motivations coming from gauge theory, where cohomology of the moduli spaces of instantons is an object of longstanding interest [1, 8, 49].

0.2. *Description of the results*

The familiar 2-fold subdivision into *automorphic forms vs. Hecke operators* now becomes 3-fold: we have the categories $Coh_m(S)$, $m = 0, 1, 2$, of *purely m-dimensional sheaves* (see §4.1). Here, $Coh_2(S)$ consists of vector bundles, while $Coh_0(S)$ is the category of punctual sheaves. An important feature is that the COHA of $Coh_{m-1}(S)$ acts on that of $Coh_m(S)$ by *Hecke operators*.

We denote by $R(S)$ the COHA of the category $Coh_0(S)$. It is the most immediate analog of the unramified Hecke algebra of the classical theory and we relate it to objects studied before.

In the *flat case* $S = \mathbb{A}^2$, the algebra $R(\mathbb{A}^2)$ is identified with the direct sum, over $n \geq 0$, of the GL_n-equivariant Borel–Moore homology of the *commuting varieties* of $\mathfrak{gl}_n$.

Our first main result, Theorem 6.1.4, shows that $R(\mathbb{A}^2)$ is an enveloping algebra and is identified, as a graded vector space, with the symmetric algebra of an explicit graded vector space Θ. It is convenient to write $\Theta = H_\bullet^{\mathrm{BM}}(\mathbb{A}^2) \otimes \Theta'$, where the first factor is 1-dimensional, in homological degree 4.

For a general surface S, the algebra $R(S)$ is non-commutative. Our second main result, Theorem 7.1.6, provides a version of the Poincaré–Birkhoff–Witt theorem for $R(S)$. It exhibits a system of generators as well as determines the graded dimension of $R(S)$. More precisely, it establishes an isomorphism of graded vector spaces

$$\sigma : \mathrm{Sym}(H_\bullet^{\mathrm{BM}}(S) \otimes \Theta') \simeq R(S). \tag{0.2.1}$$

Like the classical PBW isomorphism for enveloping algebras, σ is given by the symmetrized product map on the space of generators.

0.3. *Role of factorization algebras*

Our proof of Theorem 6.1.4 is based on the techniques of factorization homology [9, 18, 21, 44]. More precisely, we consider the cochain lift $\mathcal{R}(S)$ of $R(S)$. This can be seen as a homotopy associative algebra whose cohomology is $R(S)$. For any open set $U \subset S$ we have a similarly defined algebra $\mathcal{R}(U)$. Further, one can consider U to be any open set in the complex analytic topology. In this case $\mathrm{Coh}_0(U)$ can be considered as an analytic stack and so its Borel–Moore homology and our entire construction of the COHA make sense.

We prove in Theorem 7.5.4 that the assignment $U \mapsto \mathcal{R}(U)$ is a *factorization coalgebra in the category of* E_1- (*i.e., homotopy associative dg*-) *algebras*. This is a reflection of a more fundamental fact: $U \mapsto \mathrm{Coh}_0(U)$ is a *factorization algebra in the category of analytic stacks* (see Proposition 7.4.3). These considerations allow us to lift σ to a morphism of factorization coalgebras in the category of dg-vector spaces and deduce the global isomorphism from the local one, i.e., from the case when S is an open ball which is equivalent to that of $S = \mathbb{A}^2$.

In fact, the identification (0.2.1) is suggestive of *non-abelian Poincaré duality* (NAPD) (compare [44, Thm. 5.5.6.6]), although it does not seem to be a formal consequence of it. NAPD can be extended to include, for instance, the classical Atiyah–Bott theorem on the cohomology of $\mathrm{Bun}_G(\Sigma)$, the moduli stack of holomorphic G-bundles on a compact Riemann surface Σ (see [18]). In the latter setting, we have $H^\bullet(BG) = \mathrm{Sym}(V)$ (with V being the space of characteristic classes for G-bundles), and

$$H^\bullet(\mathrm{Bun}_G(\Sigma)) \simeq \mathrm{Sym}(H_\bullet(\Sigma) \otimes V).$$

0.4. Derived nature of the COHA

As a vector space, our COHA is the Borel–Moore homology of the Artin stack $\mathrm{Coh}(S)$ (the moduli stack of objects of *Coh*(S)), i.e., it is the cohomology of the dualizing complex:

$$H_\bullet^{\mathrm{BM}}(\mathrm{Coh}(S)) = H^{-\bullet}(\mathrm{Coh}(S), \omega_{\mathrm{Coh}(S)}).$$

Since S is a surface, $\mathrm{Coh}(S)$ is singular due to obstructions encoded by Ext^2, so the dualizing complex is highly non-trivial. However, $\mathrm{Coh}(S)$ is in fact a truncation of a finer object, the *derived moduli stack* $R\mathrm{Coh}(S)$, smooth in the derived sense [62,63]. While the vector space underlying our COHA depends on $\mathrm{Coh}(S)$ alone, the multiplication makes appeal to the derived structure: we use the refined pullbacks corresponding to the perfect obstruction theories on $\mathrm{Coh}(S)$ and on the related stack of short exact sequences. So our construction has the appearance of applying some cohomology theory to the derived stack $R\mathrm{Coh}(S)$ itself and using its natural functorialities for morphisms of derived stacks. More recently, this approach has been implemented by M. Porta and F. Sala [55] at the K-theoretical level.

0.5. Relation to other work

The COHA of a surface that we consider here is a non-linear analog of the COHA associated to the preprojective algebra of the Jordan quiver considered in [58]; see, e.g., [59] for the case of arbitrary quivers. M. Kontsevich and Y. Soibelman [37] introduced cohomological Hall algebras for 3-dimensional Calabi–Yau categories, by taking cohomology of the moduli stack of objects with coefficients in the natural perverse sheaf of "vanishing cycles" with respect to the Chern–Simons functional. Although the details of the approach have been worked out only for quiver-type situations (see, e.g., [10] for a comparison with [58]), it seems applicable, in principle, to the category of compactly supported coherent sheaves on any 3-dimensional Calabi–Yau manifold M. In particular, our COHA for a surface S should be related to the Kontsevich–Soibelman COHA for M the total space of the anticanonical bundle on S.

Instead of Borel–Moore homology of the stack $\mathrm{Coh}(S)$, one can take its Chow groups or its algebraic K-theory, in particular one can study K-theoretic analogs of the Hecke operators. This approach was developed by A. Negut [48] who studied the K-theoretic effect of explicit Hecke correspondences on the moduli spaces, and very recently by Y. Zhao [65] who defined independently the K-theoretic Hall algebra of 0-dimensional sheaves by a method similar to ours. On the other hand, algebraic K-theory, being a more rigid object than homology, does not easily localize on the complex analytic topology and so determining the size of the resulting objects is more difficult.

In the particular case where S is the cotangent bundle of a smooth curve, other versions of the COHA (of 0-dimensional sheaves and of purely 1-dimensional sheaves) of S appeared recently in [47, 57].

After this paper first appeared on arXiv, there have been some important new developments. Thus, M. Porta and F. Sala [55] have defined a categorical and a K-theoretical version of the COHA for surfaces, using the derived enhancement of the stack of coherent sheaves. Further, A. Khan [36] introduced a motivic framework for Borel–Moore cohomology for Artin stacks which could potentially simplify the treatment of some questions considered in this paper.

0.6. Structure of the paper

In §1 we discuss the basic generalities on groupoids and stacks, including higher stacks understood as homotopy sheaves of simplicial sets. We pay special attention to Dold–Kan and Maurer–Cartan (Deligne) stacks associated to 3-term complexes and dg-Lie algebras. These constructions are used in §2 to describe stacks of extensions (needed for defining the Hall multiplication) and filtrations (needed to prove associativity).

In §3 we define and study the Borel–Moore homology of Artin stacks. This concept, which is a topological analog of A. Kresch's concept of Chow groups for Artin stacks [38], can be easily defined once we have a good formalism of constructible derived categories and their functorialities f^{-1}, Rf_*, Rf_c, $f^!$. While in the "classical" approach (sheaves first, complexes later) this may present complications (cf. [39, 50] for a discussion), the modern point of view of homotopy descent [19] allows a straightforward definition of the *enhanced* derived category of a stack as the ∞-categorical limit of the corresponding categories for schemes. The desired functorialities are also inherited from the case of schemes. We study virtual pullback in this context.

The COHA is defined in §4, first as a vector space, then as an associative algebra.

In §5 we consider subalgebras in the COHA corresponding to sheaves with various conditions on the dimension of support. These subalgebras play the role of Hecke algebras, since they act on other subspaces in COHA (corresponding to sheaves whose dimension of support is bigger) by natural "Hecke operators" (operators formally dual to those of the Hall multiplication).

In §6 we study the flat Hecke algebra $R(\mathbb{A}^2)$ by relating it to the earlier work on commuting varieties in $\mathfrak{gl}_n$. Here we prove Theorem 6.1.4.

Finally, in §7 we globalize the considerations of §6 by describing the global Hecke algebra $R(S)$ as the factorization (co)homology of an appropriate factorization (co)algebra. This leads to the proof of Theorem 7.1.6 .

The paper has two appendices. Appendix A, logically preceding the entire paper, provides a reminder on ∞-categories and dg-categories. Appendix B spells out the homotopy unique nature of Euler (top Chern) classes and orientation classes at the cochain level. It logically depends on §§1–3 (i.e., assumes the formalism of stacks presented in these sections) but precedes §7 for which it provides necessary material.

1. Generalities on stacks

1.1. Groupoids and simplicial sets

A *groupoid* is a category G in which all morphisms are invertible. We write $G = \{G_1 \rightrightarrows G_0\}$ where $G_0 = \mathrm{Ob}(G)$ is the class of objects and $G_1 = \mathrm{Mor}(G)$ is the class of morphisms. For an essentially small groupoid G let $\pi_0(G)$ be the set of isomorphisms classes of objects of G. For any object $x \in G_0$ let $\pi_1(G, x) = \mathrm{Aut}_G(x)$ be the automorphism group of x. All groupoids in what follows will be assumed essentially small.

The small groupoids form a 2-category $\mathfrak{Gpd}$. For any groupoids G_1, G_2 we have a groupoid whose objects are functors $G_1 \to G_2$ and morphisms are natural transformations of functors. We will refer to 1-morphisms of $\mathfrak{Gpd}$ as simply *morphisms of groupoids*. Considered with this notion of morphisms, the groupoids form a category which we denote Gpd. Let Eq $\subset$ Mor(Gpd) be the class of equivalences of groupoids.

Proposition 1.1.1. *Let $f : G \to G'$ be a morphism of groupoids. Suppose that f induces a bijection of sets $\pi_0(G) \to \pi_0(G')$ and, for any $x \in \mathrm{Ob}(G)$, an isomorphism of groups $\pi(G, x) \to \pi_1(G', f(x))$. Then f is an equivalence of groupoids.*

Proof. The conditions just mean that f is essentially surjective and fully faithful hence an equivalence. ∎

For a category C let $\Delta^\circ C$ be the category of simplicial objects in C. In particular, we will use the category $\Delta^\circ Set$ of simplicial sets and $\Delta^\circ Ab$ of simplicial abelian groups. For a simplicial set X let $|X|$ be its geometric realization. A morphism $f : X \to X'$ of simplicial sets is called a *weak equivalence* if it induces a homotopy equivalence $|X| \to |X'|$. In this case we write $X \sim X'$. Let W be the class of weak equivalences.

We also associate to any simplicial set X its *fundamental groupoid* ΠX. Objects of ΠX are vertices of X, i.e., elements $x \in X_0$, and for $x, y \in X_0$, the set $\mathrm{Hom}_{\Pi X}(x, y)$ consists of homotopy classes of paths in $|X|$ joining x and y. Let $\pi_0(X)$ be the set of connected components of $|X|$, and for each $i \geq 1$ and $x \in X_0$ let $\pi_i(X, x)$ be the topological homotopy group of $|X|$ at x.

Dually, the *nerve* NG of a groupoid G is a simplicial set with the set of m-simplices being

$$N_m G = G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1 \quad (m \text{ times}). \tag{1.1.2}$$

The topological homotopy groups of NG match those defined above algebraically for G:

$$\pi_0(NG) = \pi_0(G), \quad \pi_1(NG, x) = \pi_i(G, x), \quad \pi_i(NG, x) = 0, \quad i \geq 2.$$

A simplicial set is *of groupoid type* if it is weak equivalent to the nerve of some groupoid. We denote by $\Delta^\circ \mathit{Set}^{\leq 1} \subset \Delta^\circ \mathit{Set}$ the full subcategory of simplicial sets of groupoid type.

Proposition 1.1.3. (a) *A simplicial set X is of groupoid type if and only if $\pi_i(X, x) = 0$ for each $i \geq 2$ and $x \in X_0$. In that case $X \simeq N\Pi X$.*

(b) *The functors Π and N yield quasi-inverse equivalences of homotopy categories* $\Delta^\circ \mathit{Set}^{\leq 1}[\mathrm{W}^{-1}] \simeq \mathrm{Gpd}[\mathrm{Eq}^{-1}]$. ∎

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{C}(\mathcal{A})$ the category of cochain complexes $K = (K^n, d^n : K^{n-1} \to K^n)_{n \in \mathbb{Z}}$ over $\mathcal{A}$ bounded below, with morphisms being morphisms of complexes. For $n \in \mathbb{Z}$ we denote by $\mathrm{C}^{\leq n}(\mathcal{A})$ the category of complexes concentrated in degrees $\leq n$. For $K \in \mathrm{C}(\mathcal{A})$ we denote by

$$K^{\leq n} = \{\cdots \xrightarrow{d^{n-1}} K^{n-1} \xrightarrow{d^n} K^n \to 0 \to \cdots\} \in \mathrm{C}^{\leq n}(\mathcal{A}),$$

$$\tau_{\leq n} K = \{\cdots \xrightarrow{d^{n-1}} K^{n-1} \xrightarrow{d^n} \mathrm{Ker}(d^{n+1}) \to 0 \to \cdots\} \in \mathrm{C}^{\leq n}(\mathcal{A})$$

its *stupid* and *cohomological* truncation in degrees $\leq n$. Note that $\tau_{\leq n}$ sends quasi-isomorphisms of complexes to quasi-isomorphisms.

Examples 1.1.4 (Dold–Kan groupoids). Let *Ab* denote the category of abelian groups.

(a) Given a 3-term complex over *Ab*

$$K = \{K^{-1} \xrightarrow{d^0} K^0 \xrightarrow{d^1} K^1\},$$

we have the *action groupoid*

$$\varpi K = \mathrm{Ker}(d^1)//K^{-1} := \{K^{-1} \times \mathrm{Ker}(d^1) \rightrightarrows \mathrm{Ker}(d^1)\}$$

whose set of objects is $\mathrm{Ker}(d^1)$ and whose morphisms $s \to t$ are given by $\{h \in K^{-1} \,;\, s + d^0(h) = t\}$. Then we have

$$\pi_0(\varpi K) = H^0(K), \quad \pi_1(\varpi K, s) = H^{-1}(K), \quad \forall s \in \mathrm{Ob}\, \varpi K.$$

(b) The *Dold–Kan correspondence* $\mathrm{DK} : \mathrm{dg}^{\leq 0}Ab \to \Delta^\circ Ab$ associates to a $\mathbb{Z}_{\leq 0}$-graded complex K the simplicial abelian group $\mathrm{DK}(K)$ such that

- $\mathrm{DK}(K)_0 = K^0$,
- the set of edges joining $s, t \in K^0$ is $\{h \in K^{-1}\,;\, s + d^0(h) = t\}$,
- 2-simplices with given 1-faces are in bijection with certain elements of K^{-2}, and so on; see, e.g., [64, §8.4.1].

For each $i \geq 0$, we have an isomorphism $\pi_i(\mathrm{DK}(K)) \simeq H^{-i}(K)$ which is independent of the base point. In fact, the correspondence preserves the respective standard model structures. In particular, for a 3-term complex K as in (a), we have

$$\varpi K = \Pi \mathrm{DK}(\tau_{\leq 0} K). \tag{1.1.5}$$

Examples 1.1.6 (Maurer–Cartan groupoids). We will use a non-abelian generalization of Examples 1.1.4, due to Deligne; see [22, 23] and references therein, Hinich [27] and Getzler [20].

(a) Consider a (possibly infinite-dimensional) dg-Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ situated in degrees $[0, 2]$:

$$\mathfrak{g} = \{\mathfrak{g}^0 \xrightarrow{d^0} \mathfrak{g}^1 \xrightarrow{d^1} \mathfrak{g}^2\}.$$

Thus $\mathfrak{g}^0$ is an ordinary complex Lie algebra. We assume that it is nilpotent, so we have the nilpotent group $G^0 = \exp(\mathfrak{g}^0)$. By definition, G^0 consists of formal symbols e^y, $y \in \mathfrak{g}^0$ (so G^0 is identified with $\mathfrak{g}^0$ as a set), with the multiplication given by the Campbell–Hausdorff formula. The set of Maurer–Cartan elements of $\mathfrak{g}$ is

$$\mathbf{mc}(\mathfrak{g}) = \big\{x \in \mathfrak{g}^1\,;\, d^1 x + \tfrac{1}{2}[x, x] = 0\big\}.$$

The group G^0 acts on $\mathbf{mc}(\mathfrak{g})$ by the formula

$$e^y * x = e^{\mathrm{ad}}(y)(x) + \frac{1 - e^{\mathrm{ad}(y)}}{\mathrm{ad}(y)}(d^1(y)) \tag{1.1.7}$$

(see [23, p. 45]). We define the *Maurer–Cartan groupoid*[1] (or *Deligne groupoid*) of $\mathfrak{g}$ to be the action groupoid

$$\mathrm{MC}(\mathfrak{g}) = \mathbf{mc}(\mathfrak{g})//G^0 := \{G^0 \times \mathbf{mc}(\mathfrak{g}) \rightrightarrows \mathbf{mc}(\mathfrak{g})\}.$$

Note that if the dg-Lie algebra $\mathfrak{g}$ is abelian, i.e., if it reduces to a 3-term cochain complex, then $G^0 = \mathfrak{g}^0$ and it acts on $\mathbf{mc}(\mathfrak{g}) = \mathrm{Ker}(d^1)$ by translation, so we have $\mathrm{MC}(\mathfrak{g}) = \varpi(\mathfrak{g}[1])$ where ϖ is as in Example 1.1.4 (a).

[1] In this paper we use the terms "Maurer–Cartan groupoid" and "Maurer–Cartan stack" in order to avoid clashes with the algebro-geometric notion of Deligne–Mumford stacks.

(b) More generally, let $\mathfrak{g}$ be any nilpotent dg-Lie algebra over $\mathbb{C}$. The *Maurer–Cartan simplicial set* $\mathbf{mc}_\bullet(\mathfrak{g})$ is defined by

$$\mathbf{mc}_n(\mathfrak{g}) = \mathbf{mc}(\mathfrak{g} \otimes_{\mathbb{C}} \Omega^\bullet_{\mathrm{pol}}(\Delta^n)),$$

where $\Omega^\bullet_{\mathrm{pol}}(\Delta^n)$ is the commutative dg-algebra of polynomial differential forms on the standard n-simplex [20, 27]. Further, in [20] it is proved that if $\mathfrak{g}$ is concentrated in degrees $[0, 2]$ then $N_\bullet(\mathrm{MC}(\mathfrak{g}))$ is weak equivalent to $\mathbf{mc}_\bullet(\mathfrak{g})$.

Proposition 1.1.8. *A quasi-isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of nilpotent dg-Lie algebras induces a weak equivalence of simplicial sets $\mathbf{mc}_\bullet(\mathfrak{g}_1) \to \mathbf{mc}_\bullet(\mathfrak{g}_2)$. In particular:*

(a) *If $\mathfrak{g}_1$, $\mathfrak{g}_2$ are concentrated in degrees $[0, 2]$, then ϕ induces an equivalence of groupoids* $\mathrm{MC}(\mathfrak{g}_1) \to \mathrm{MC}(\mathfrak{g}_2)$.

(b) *A quasi-isomorphism $K_1 \to K_2$ of cochain complexes as in Example 1.1.4* (a) *induces an equivalence of groupoids $\varpi K_1 \to \varpi K_2$.* ■

Let now $p : \mathfrak{g} \to \mathfrak{h}$ be a surjective morphism of dg-Lie algebras, both situated in degrees $[0, 2]$. Let $\mathfrak{n} \subset \mathfrak{g}$ be the kernel of p and assume that there is an embedding $i : \mathfrak{h} \to \mathfrak{g}$ with $p \circ i = 1$ such that $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ is the semidirect product.

We have a functor of groupoids $p_* : \mathrm{MC}(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{h})$. Recall that for a functor $\phi : C \to D$ and an object $x \in \mathrm{Ob}(D)$, the *fiber category* ϕ/x consists of pairs (y, h) with $y \in \mathrm{Ob}(C)$ and $h : \phi(y) \to x$ a morphism in D, with the obvious notion of morphisms of such pairs. If C, D are groupoids, so is ϕ/x. We apply this when $C = \mathrm{MC}(\mathfrak{g})$, $D = \mathrm{MC}(\mathfrak{h})$ and $\phi = p_*$. We get the fiber category p_*/x. On the other hand, the object $x \in \mathrm{Ob}(\mathrm{MC}(\mathfrak{h}))$ being an element of $\mathbf{mc}(\mathfrak{h})$, it gives a new differential $d_x = d - \mathrm{ad}(x)$ on $\mathfrak{n}$, where we abbreviate $x = i(x)$. Let $\mathfrak{n}_x$ be the dg-Lie algebra with underlying Lie algebra $\mathfrak{n}$ and differential d_x.

Proposition 1.1.9. *The fiber category p_*/x is equivalent to the groupoid* $\mathrm{MC}(\mathfrak{n}_x)$. ■

1.2. Stacks and homotopy sheaves

Let $\mathscr{S}$ be a Grothendieck site. We recall that a *stack* (of essentially small groupoids) on $\mathscr{S}$ is a presheaf of groupoids $B : T \mapsto B(T)$, $T \in \mathrm{Ob}(\mathscr{S})$, satisfying the 2-categorical descent condition extending that for sheaves of sets (see [52] for background). The stacks on $\mathscr{S}$ form a 2-category $\mathfrak{St}_{\mathscr{S}}$. We will refer to 1-morphisms of $\mathfrak{St}_{\mathscr{S}}$ as *morphisms of stacks* and will denote by $St_{\mathscr{S}}$ the category of stacks on $\mathscr{S}$ with these morphisms. Let $\mathrm{Eq} \subset \mathrm{Mor}(St_{\mathscr{S}})$ be the class of equivalences of stacks.

Remark 1.2.1. For most purposes, the above 1-categorical point of view on stacks will be sufficient. However, in various constructions below such as gluing, the full 2-categorical structure on $\mathfrak{St}_{\mathscr{S}}$ becomes important. In particular, as with objects of any 2-category, to define a stack "uniquely" (e.g., naively, in a way "independent" of some choices) means, more formally, to define it *uniquely up to an equivalence which is defined uniquely up to a unique isomorphism*.

A stack of groupoids B gives rise to a sheaf of sets $\underline{\pi}_0(B)$ on $\mathscr{S}$, obtained by sheafifying the presheaf $T \mapsto \pi_0(B(T))$. Similarly, for any $T \in \mathrm{Ob}(\mathscr{S})$ and any object $x \in B(T)$ we have a sheaf of groups $\underline{\pi}_1(B, x)$ on T, i.e., on the site $\mathscr{S}/T$, obtained by sheafifying the presheaf $T' \mapsto \pi_1(B(T'), x|_{T'})$, where $x|_{T'}$ is the pullback by the morphism $T' \to T$.

Proposition 1.2.2. *Let $f : B \to B'$ be a morphism in $St_{\mathscr{S}}$ which induces an isomorphism of sheaves of sets $\underline{\pi}_0(B) \to \underline{\pi}_0(B')$ and an isomorphism of sheaves of groups $\underline{\pi}_1(B, x) \to \underline{\pi}_1(B', f(x))$ for any $T \in \mathrm{Ob}(\mathscr{S})$, $x \in \mathrm{Ob}(B(T))$. Then f is an equivalence of stacks.*

Proof. Follows from Proposition 1.1.1 by sheafification. ■

Let $\Delta^\circ Set_{\mathscr{S}}$ be the category of presheaves of simplicial sets on $\mathscr{S}$. Recall [63] that such a presheaf X is called a *homotopy sheaf* or an ∞-*stack* if it satisfies descent in the homotopy sense. We denote by $St^\infty_{\mathscr{S}}$ the category of homotopy sheaves of simplicial sets on $\mathscr{S}$ and by $W \subset \mathrm{Mor}(St^\infty_{\mathscr{S}})$ the class of weak equivalences (defined stalkwise). A homotopy sheaf X gives rise to a sheaf of sets $\underline{\pi}_0(X)$ on $\mathscr{S}$ and, for any $T \in \mathrm{Ob}(\mathscr{S})$ and any vertex $x \in X(T)_0$, a sheaf of groups $\underline{\pi}_i(X, x)$ on $\mathscr{S}/T$. We have:

Proposition 1.2.3. *Let $f : X \to X'$ be a morphism in $St^\infty_{\mathscr{S}}$. Suppose f induces an isomorphism of sheaves of sets $\underline{\pi}_0(X) \to \underline{\pi}_0(X')$ and, for each $T \in \mathrm{Ob}(\mathscr{S})$ and $x \in X(T)_0$, an isomorphism of sheaves of groups $\underline{\pi}_i(X, x) \to \underline{\pi}_i(X', f(x))$. Then f is a weak equivalence.*

Proof. If $\mathscr{S}$ is a point, this is the standard: a map of simplicial sets is a weak equivalence iff it induces an isomorphism of homotopy groups. The case of general $\mathscr{S}$ is obtained from this by sheafification. ■

Any homotopy sheaf X gives a stack of groupoids ΠX on $\mathscr{S}$, defined by taking $T \mapsto \Pi X(T)$. Any stack of groupoids B on $\mathscr{S}$ gives rise to a homotopy sheaf $N(B)$ taking T to the nerve of the groupoid $B(T)$. A homotopy sheaf X is called of *groupoid type* if it is weak equivalent to $N(B)$ for some stack B. We denote by $St^{\infty,\leq 1}_{\mathscr{S}} \subset St^\infty_{\mathscr{S}}$ the full category of homotopy sheaves of groupoid type.

Proposition 1.2.4. (a) *A homotopy sheaf X is of groupoid type if and only if $\underline{\pi}_i(X, x) = 0$ for each $T \in \mathrm{Ob}(\mathscr{S})$, $x \in X(T)_0$ and $i \geq 2$.*

(b) *The functors Π and N induce mutually quasi-inverse equivalences of homotopy categories $St^{\infty,\leq 1}_{\mathscr{S}}[\mathrm{W}^{-1}] \simeq St_{\mathscr{S}}[\mathrm{Eq}^{-1}]$.* ■

1.3. Artin and f-Artin stacks

In this paper all schemes, algebras, etc., will be considered over the base field $\mathbb{C}$ of complex numbers. Let $\widetilde{\mathscr{A}ff}$ be the category of affine schemes over $\mathbb{C}$ equipped with the étale topology. We refer to [40, 52] for general background on *Artin stacks*, i.e., stacks of groupoids on $\widetilde{\mathscr{A}ff}$ with a smooth atlas and a representable, quasi-compact, quasi-separated diagonal.

Examples 1.3.1. (a) Let $G = \{G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0\}$ be a groupoid in the category of schemes of finite type such that the source and target maps s, t are smooth morphisms. It gives rise to an Artin stack which we denote by $\|G\|$. By definition, $\|G\|$ is the stack associated with the prestack

$$T \mapsto \{\mathrm{Hom}(T, G_1) \rightrightarrows \mathrm{Hom}(T, G_0)\}.$$

(b) In particular, let G be an affine algebraic group acting on a scheme Z of finite type. Then we have the *action groupoid* $\{G \times Z \rightrightarrows Z\}$ in the category of schemes of finite type. The corresponding Artin stack is denoted $Z//G$ and is called the *quotient stack* of Z by G. Explicitly, for $T \in \widetilde{\mathscr{A}ff}$ the groupoid $(Z//G)(T)$ is identified with the category of pairs (P, u), where P is a G-torsor over T (locally trivial in étale topology) and $u : P \to Z$ is a G-equivariant map.

Definition 1.3.2. An Artin stack B is called

(a) *of finite type* if it is equivalent to the stack of the form $\|G\|$ for a groupoid G as in Example 1.3.1 (a);

(b) an *f-Artin stack* if it is locally of finite type;

(c) a *quotient* (resp. *locally quotient) stack* if it is equivalent (resp. locally equivalent) to the stack of the form $Z//G$ where Z, G are as in Example 1.3.1 (b).

All the stacks we will use will be f-Artin. Let the 2-category $\mathfrak{S}t$ and the category St be the full 2-subcategory in $\mathfrak{S}t_{\widetilde{\mathscr{A}ff}}$ and the full subcategory in $St_{\widetilde{\mathscr{A}ff}}$ formed by f-Artin stacks.

Let $\mathscr{A}ff \subset \widetilde{\mathscr{A}ff}$ be the category of affine schemes *of finite type* with its étale topology. We note that f-Artin stacks are determined by their restrictions to $\mathscr{A}ff$, and so we can and will consider them as stacks of groupoids on $\mathscr{A}ff$.

Given an f-Artin stack B, let $\mathfrak{S}t_B$ be the 2-category of f-Artin stacks over B, i.e., of f-Artin stacks X together with a morphism of stacks $X \to B$. Objects of $\mathfrak{S}t_B$ can, equivalently, be seen as stacks of groupoids over the Grothendieck site $\mathscr{A}ff_B$ formed by affine schemes T of finite type together with a morphism of stacks $f : T \to B$. Thus, an f-Artin stack X over B can be seen as associating to each $T \in \mathscr{A}ff_B$ a groupoid $X(T)$.

2. Stacks of extensions and filtrations

2.1. Cone stacks

We refer to [50,52] for general background on quasi-coherent sheaves on Artin stacks. For an f-Artin stack B we denote by $QCoh(B)$ (resp. $Coh(B)$) the category of quasi-coherent (resp. coherent) sheaves of $\mathcal{O}_B$-modules. By a *vector bundle* we mean a locally free sheaf of finite rank.

Let B be an f-Artin stack and $R = \bigoplus_{i \in \mathbb{N}} R^i$ be a graded quasi-coherent sheaf of $\mathcal{O}_B$-algebras such that $R^0 = \mathcal{O}_B$, R^1 is coherent and R is generated by R^1 locally over B. The relative affine B-scheme $C = \mathrm{Spec}\ R$ is called a *cone* over B (see, e.g., [5, §1]).

If $\mathcal{E}$ is a coherent sheaf over B, we get the *associated cone* $C(\mathcal{E}) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_B}(\mathcal{E}))$ which is an affine group scheme over B. Its value (the set of points) on $(T \xrightarrow{f} B) \in \mathscr{A}ff_B$ is $\mathrm{Hom}_{\mathcal{O}_T}(f^*\mathcal{E}, \mathcal{O}_T)$. We call such a cone an *abelian cone*.

For instance, the *total space* of a vector bundle $\mathcal{E}$ over X is defined as

$$\mathrm{Tot}(\mathcal{E}) = C(\mathcal{E}^\vee) = \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_B}(\mathcal{E}^\vee)$$

where $\mathcal{E}^\vee$ is the dual sheaf of $\mathcal{O}_B$-modules. For any affine B-scheme $f : T \to B$ we have

$$\mathrm{Tot}(\mathcal{E})(T) = H^0(T, f^*\mathcal{E}). \tag{2.1.1}$$

Thus, a section $s \in H^0(B, \mathcal{E})$ is the same as a morphism $B \to \mathrm{Tot}(\mathcal{E})$ of schemes over B.

Any cone $C = \mathrm{Spec}\ R$ is canonically a closed subcone of the abelian cone $\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_B}(R^1))$, called the *abelian hull* of C.

Example 2.1.2. Let $d : \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles on B. We denote by $\underline{\mathrm{Ker}}(d) \subset \mathcal{E}$ the sheaf-theoretic kernel of d. On the other hand, let $\pi : \mathrm{Tot}(\mathcal{E}) \to B$ be the projection. The morphism d determines a section s of the vector bundle $\pi^*\mathcal{F}$ on $\mathrm{Tot}(\mathcal{E})$, and we define the abelian cone $\mathrm{Ker}(d) \subset \mathrm{Tot}(\mathcal{E})$ as the zero locus of this section. We note that $H^0(B, \underline{\mathrm{Ker}}(d)) \subset H^0(B, \mathcal{E})$ consists precisely of those sections s which, considered as morphisms $B \to \mathrm{Tot}(\mathcal{E})$, factor through the substack $\mathrm{Ker}(d)$.

A *morphism of abelian cones* over B is, by definition a morphism of group schemes over B. Given a morphism of abelian cones $E \to F$, we have an action of the affine group scheme E over B on F. Hence, we can form the quotient Artin stack $F/\!/E$. Stacks of this form are called *abelian cone stacks*.

2.2. Total spaces of perfect complexes

Let B be an f-Artin stack. We denote by $C_{\mathrm{qcoh}}(B)$ the category formed by complexes of $\mathcal{O}_B$-modules with quasi-coherent cohomology. Let qis be the class of quasi-isomorphisms in $C_{\mathrm{qcoh}}(B)$ and $D_{\mathrm{qcoh}}(B) = C_{\mathrm{qcoh}}(b)[\mathrm{qis}^{-1}]$ be the corresponding derived category. For any integers $p \le q$ let $C^{[p,q]}_{\mathrm{qcoh}}(B) \subset C_{\mathrm{qcoh}}(B)$ be the full subcategory formed by complexes situated in degrees from p to q.

Definition 2.2.1. Let $\mathcal{C} \in C_{\mathrm{qcoh}}(B)$ and $p \le q$ be integers.

(a) $\mathcal{C}$ is *strictly* $[p, q]$*-perfect* if $\mathcal{C}$ is quasi-isomorphic to a complex of vector bundles

$$\{\mathcal{C}^p \xrightarrow{d^{p+1}} \mathcal{C}^{p+1} \xrightarrow{d^{p+2}} \cdots \xrightarrow{d^q} \mathcal{C}^q\}$$

situated in degrees from p to q. This complex is called a *presentation* of $\mathcal{C}$.

(b) $\mathcal{C}$ is $[p, q]$*-perfect* if, locally on B, it is strictly $[p, q]$-perfect, and moreover the set of open substacks $U \subset B$ such that $\mathcal{C}|_U$ is strictly $[p, q]$-perfect is filtering with respect to the partial order by inclusion.

For a $[p,q]$-perfect complex $\mathcal{C}$ and an open $U \subset B$ as above we will refer to a quasi-isomorphism $\mathcal{C}|_U \to \mathcal{C}_U$, with $\mathcal{C}_U$ strictly $[p,q]$-perfect, as a *presentation* of $\mathcal{C}$ over U.

A $[p,q]$-perfect complex $\mathcal{C}$ has a *virtual rank* $\mathrm{vrk}(\mathcal{C})$ which is a $\mathbb{Z}$-valued locally constant function on B, i.e., a function constant on each connected component of B. It is defined in terms of a presentation of $\mathcal{C}$ as

$$\mathrm{vrk}(\mathcal{C}) = \sum_{i=p}^{q} (-1)^i \,\mathrm{rk}(\mathcal{C}^i).$$

We will be interested in making sense of total spaces of perfect complexes using (2.1.1) as a motivation (cf. [62, §3.3]).

Definition 2.2.2. (a) Let $\mathcal{C} \in C^{\leq 0}_{\mathrm{qcoh}}(B)$. We define the simplicial presheaf $\mathrm{Tot}^\infty(\mathcal{C})$ on $\mathscr{A}\!f\!f_B$ by

$$\mathrm{Tot}^\infty(\mathcal{C})(T) = \mathrm{DK}(H^0(T, f^*\mathcal{C})), \quad (T \xrightarrow{f} B) \in \mathscr{A}\!f\!f_B.$$

(b) Let $\mathcal{C} \in C^{[-1,0]}_{\mathrm{qcoh}}(B)$. We define the prestack of groupoids $\mathrm{Tot}(\mathcal{C})$ on $\mathscr{A}\!f\!f_B$ by

$$\mathrm{Tot}(\mathcal{C})(T) = \varpi(H^0(T, f^*\mathcal{C})), \quad (T \xrightarrow{f} B) \in \mathscr{A}\!f\!f_B.$$

We call $\mathrm{Tot}(\mathcal{C})$ the *total space* of $\mathcal{C}$.

Proposition 2.2.3. (a) *Let* $\mathcal{C} \in C^{\leq 0}_{\mathrm{qcoh}}(B)$. *The simplicial presheaf* $\mathrm{Tot}^\infty(\mathcal{C})$ *is a homotopy sheaf. For any* $x \in \mathrm{Tot}^\infty(\mathcal{C})(T)_0$ *we have* (*independently of the choice of base points*)

$$\underline{\pi}_i(\mathrm{Tot}^\infty(\mathcal{C})) = \underline{H}^{-i}(\mathcal{C}), \quad i \geq 0.$$

A morphism $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ *in* $C^{\leq 0}_{\mathrm{qcoh}}(B)$ *induces a morphism of homotopy sheaves* $\phi_\flat : \mathrm{Tot}^\infty(\mathcal{C}_1) \to \mathrm{Tot}^\infty(\mathcal{C}_2)$, *which is an equivalence if* ϕ *is a quasi-isomorphism.*

(b) *Let* $\mathcal{C} \in C^{[-1,0]}_{\mathrm{qcoh}}(B)$. *The prestack* $\mathrm{Tot}(\mathcal{C})$ *on* $\mathscr{A}\!f\!f_B$ *is a stack. The homotopy sheaf* $\mathrm{Tot}^\infty(\mathcal{C})$ *is of groupoid type and* $\Pi\mathrm{Tot}^\infty(\mathcal{C}) = \mathrm{Tot}(\mathcal{C})$. *In particular, the total space is functorial and takes quasi-isomorphisms* ϕ *to isomorphisms* $\phi_\flat$.

Proof. Part (a) follows from the fact that $\mathcal{C}$ is a sheaf and from the properties of the Dold–Kan correspondence. Part (b) follows by Proposition 1.2.4. ∎

Recall that a stack morphism f is called an *l.c.i.*, i.e., a *locally complete intersection morphism*, if it factorizes as $f = p \circ i$ where p is a smooth map and i is a regular immersion.

Proposition 2.2.4. (a) *Let* $\mathcal{C} \in C^{[-1,0]}_{\mathrm{qcoh}}(B)$ *be strictly* $[-1,0]$-*perfect. Then we have a canonical equivalence of stacks of groupoids* $u : \mathrm{Tot}(\mathcal{C}) \to \mathrm{Tot}(\mathcal{C}^0)/\!/\mathrm{Tot}(\mathcal{C}^{-1})$ *on* $\mathscr{A}\!f\!f_B$.

(b) *Let* $\mathcal{C} \in C^{[-1,0]}_{\mathrm{qcoh}}(B)$ *be* $[-1,0]$-*perfect. Then* $\mathrm{Tot}(\mathcal{C})$ *is an Artin stack over* B.

(c) *For any morphism* ϕ *of* $[-1,0]$-*perfect complexes, the induced morphism* $\phi_\flat$ *of stacks is an l.c.i.*

Proof. Part (a) is similar to the proof of [26, Lem. 0.1]. That is, look at any $(T \xrightarrow{f} B) \in \mathscr{A}ff_B$. By definition, the groupoid $\mathrm{Tot}(\mathcal{C})(T)$ is the category whose objects are elements x of $H^0(T, f^*\mathcal{C}^0)$ and a morphism $x \to x'$ is an element of $H^0(T, f^*\mathcal{C}^{-1})$ mapped by d^0 to $x' - x$. At the same time, the groupoid $(\mathrm{Tot}(\mathcal{C}^1)//\mathrm{Tot}(\mathcal{C}^0))(T)$ is the category of pairs consisting of an $f^*\mathcal{C}^{-1}$-torsor P over T and an $f^*\mathcal{C}^{-1}$-equivariant morphism $P \to \mathcal{C}^0$ of sheaves over T. We see that the former category is the full subcategory of the second consisting of data with the torsor P being the standard trivial one, $P = f^*\mathcal{C}^{-1}$. This defines a fully faithful functor u_T, and such functors for all T give the sought-for morphism of stacks u. Now, since T is affine, $H^1(T, f^*\mathcal{C}^{-1}) = 0$ and so any torsor P above is trivial. This means that the functor u is (locally) essentially surjective, hence an equivalence of stacks. This proves (a). Parts (b) and (c) follow from (a). ■

Example 2.2.5. Now, let $\mathcal{C}$ be a strictly $[-1, 1]$-perfect complex

$$\mathcal{C} = \{\mathcal{C}^{-1} \xrightarrow{d^0} \mathcal{C}^0 \xrightarrow{d^1} \mathcal{C}^1\}. \tag{2.2.6}$$

The stupid truncation $\mathcal{C}^{\leq 0} = \{\mathcal{C}^{-1} \to \mathcal{C}^0\}$ is strictly $[-1, 0]$-perfect. We denote by

$$\pi : \mathrm{Tot}(\mathcal{C}^0) \to B, \quad \overline{\pi} : \mathrm{Tot}(\mathcal{C}^{\leq 0}) = \mathrm{Tot}(\mathcal{C}^0)//\mathrm{Tot}(\mathcal{C}^{-1}) \to B$$

the projections. We recall from Example 2.1.2 (c) the abelian cone $\mathrm{Ker}(d^1) \subset \mathrm{Tot}(\mathcal{C}^0)$ given as the zero locus of the section s of $\pi^*\mathcal{C}^1$ induced by d^1.

Proposition 2.2.7. (a) *If $\mathcal{C}$ is strictly $[-1, 1]$-perfect, then we have a canonical equivalence of stacks $\mathrm{Ker}(d^1)//\mathcal{C}^{-1} \to \mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$, i.e., the section s descends to a section $\overline{s}$ of $\overline{\pi}^*\mathcal{C}^1$, and $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$ is the zero locus of $\overline{s}$.*

(b) *If $\mathcal{C}$ is $[-1, 1]$-perfect, then $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$ is an Artin stack over B.*

Proof. The proof of (a) is completely analogous to the proof of Proposition 2.2.4 (a), with $\mathcal{C}^0$ replaced by $\underline{\mathrm{Ker}}(d^1)$. Part (b) follows from (a). ■

We call $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$ the *truncated total space* of $\mathcal{C}$.

Proposition 2.2.8. *Let $\mathcal{C}$ be a $[-1, 1]$-perfect complex and $(T \xrightarrow{f} B) \in \mathscr{A}ff_B$.*

(a) *For all $s \in \mathrm{Ob}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})(T))$ we have*

$$\underline{\pi}_0(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})) \simeq \underline{H}^0(\mathcal{C}), \quad \underline{\pi}_1(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}), s) \simeq \underline{H}^{-1}(f^*\mathcal{C}).$$

(b) *The truncated total space of $[-1, 1]$-perfect complexes is functorial and takes quasi-isomorphisms ϕ to isomorphisms $\phi_\flat$.* ■

Proof. Part (a) is a consequence of Proposition 2.2.7. Part (b) follows from (c). More precisely, a morphism (resp. quasi-isomorphism) $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ of $[-1, 1]$-perfect complexes yields a morphism (resp. quasi-isomorphism) $\tau_{\leq 0}\mathcal{C}_1 \to \tau_{\leq 0}\mathcal{C}_2$ and the statement follows from Proposition 2.2.3 (b). ■

2.3. Stacks of extensions

We now consider the following general situation. Let B be an f-Artin stack and $p: Y \to B$ be a scheme of finite type over B. Let $\mathcal{E}$, $\mathcal{F}$ be coherent sheaves over Y which are flat over B. We can form the object $\mathcal{C} \in D^b_{\mathrm{qcoh}}(B)$ given by

$$\mathcal{C} = Rp_* \, R\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{E})[1].$$

Let SES be the stack over B classifying short exact sequences $0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0$ of coherent sheaves over Y. That is, for any B-scheme $T \in \mathcal{A}ff_B$ the objects of the groupoid $\mathrm{SES}(T)$ are short exact sequences

$$0 \to \mathcal{E}|_T \to \mathcal{G} \to \mathcal{F}|_T \to 0 \tag{2.3.1}$$

of coherent sheaves of $\mathcal{O}_{Y \times_B T}$-modules, and the morphisms are the isomorphisms of such sequences identical on the boundary terms. We then have

$$\pi_0(\mathrm{SES}(T)) = \mathrm{Ext}^1_{\mathcal{O}_{Y \times_B T}}(\mathcal{F}|_T, \mathcal{E}|_T), \quad \pi_1(\mathrm{SES}(T), \mathcal{G}) = \mathrm{Ext}^0_{\mathcal{O}_{Y \times_B T}}(\mathcal{F}|_T, \mathcal{E}|_T), \tag{2.3.2}$$

for any object $\mathcal{G}$ of $\mathrm{SES}(T)$. This implies identifications of sheaves of sets on $\mathcal{A}ff_B$, and of sheaves of groups on $\mathcal{A}ff_T$:

$$\underline{\pi}_0(\mathrm{SES}) = \underline{H}^0(\mathcal{C}), \quad \underline{\pi}_1(\mathrm{SES} \times_B T, \mathcal{G}) = \underline{H}^{-1}(\mathcal{C}|_T). \tag{2.3.3}$$

These identifications, together with those of Proposition 2.5.2 (b), suggest the following.

Proposition 2.3.4. *Assume that the complex $\mathcal{C}$ is $[-1,1]$-perfect. Then we have an equivalence* $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}) = \mathrm{SES}$ *of cone stacks over B.*

Proof. As aklready pointed out, the $\underline{\pi}_0$ and $\underline{\pi}_1$ of the two stacks $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$ and SES are isomorphic. So it remains to construct a morphism of stacks inducing these identifications. For this, we first make some general discussion.

We recall [7, 35, 61] that for any Artin stack Z the category $D^b_{\mathrm{qcoh}}(Z)$ has a *dg-thickening*, i.e., there is a pretriangulated dg-category $C_{\mathrm{qcoh}}(Z)$ with the same objects and spaces of morphisms being upgraded to complexes $\mathrm{RHom}_{C_{\mathrm{qcoh}}(Z)}(\mathcal{K}, \mathcal{L})$ of $\mathbb{C}$-vector spaces such that

$$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{K}, \mathcal{L}) = H^0 \, \mathrm{RHom}_{C_{\mathrm{qcoh}}(Z)}(\mathcal{K}, \mathcal{L}).$$

The complex RHom above can be explicitly found as

$$\mathrm{RHom}_{C_{\mathrm{qcoh}}(Z)}(\mathcal{K}, \mathcal{L}) = \mathrm{Hom}^\bullet_{\mathcal{O}_Z}(I(\mathcal{K}), I(\mathcal{L})), \tag{2.3.5}$$

where $I(\mathcal{K})$ is a fixed injective resolution of $\mathcal{K}$ for each $\mathcal{K}$.

We now specialize to the case

$$Z = Y \times_B T, \quad \mathcal{K} = \mathcal{F}|_T, \quad \mathcal{L} = \mathcal{E}|_T[1],$$

where $T \in \mathcal{A}ff_B$ is an affine B-scheme. The complex of $\mathbb{C}$-vector spaces

$$\tau_{\leq 0} \, \mathrm{RHom}_{C_{\mathrm{qcoh}}(Z)}(\mathcal{F}|_T, \mathcal{E}|_T[1])$$

has cohomology only in degrees 0 and -1, given by the Ext groups in (2.3.2). We consider the simplicial set

$$X(T) = \mathrm{DK}\big(\tau_{\leq 0}\,\mathrm{RHom}_{C_{\mathrm{qcoh}}(Z)}(\mathcal{F}|_T, \mathcal{E}|_T[1])\big),$$

which is of groupoid type by Proposition 1.1.3 (a). Its vertices are morphisms of complexes $I(\mathcal{F}|_T) \to I(\mathcal{E}|_T[1])$. The cone of such a morphism is a complex of sheaves which has only one cohomology sheaf, in degree -1, and this sheaf $\mathcal{G}$ fits into a short exact sequence as in (2.3.1). In this way we get a morphism of groupoids

$$h(T) : \Pi X(T) \to \mathrm{SES}(T).$$

At the same time, by (1.1.5), the groupoid $\Pi X(T)$ is equivalent to the groupoid $\Gamma H^0(T, \mathcal{C}|_T)$ in Example 1.1.4 (a), hence to $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})(T)$ by Proposition 2.2.8 (a). Combining these constructions for all $T \in \mathcal{A}ff_B$, we get a homotopy sheaf X of simplicial sets on $\mathcal{A}ff_B$ of groupoid type, together with an equivalence and a morphism of stacks

$$\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}) \simeq \Pi X \xrightarrow{h} \mathrm{SES}\,.$$

The morphism h induces the required identification on $\underline{\pi}_0$ and $\underline{\pi}_1$, so it is an equivalence of stacks. Proposition 2.3.4 is proved. ∎

2.4. *Maurer–Cartan stacks*

We now describe a non-abelian generalization of the construction of §2.2. Let B be an f-Artin stack and $(\mathcal{G}, d, [-,-])$ be an $\mathcal{O}_B$-dg-Lie algebra with quasi-coherent cohomology. In other words, $\mathcal{G}$ is a Lie algebra object in the symmetric monoidal category $(C_{\mathrm{qcoh}}(B), \otimes_B)$. We will assume that $\mathcal{G}$ is nilpotent. We define the *Maurer–Cartan ∞-stack* of $\mathcal{G}$ to be the simplicial presheaf $\mathbf{mc}_\bullet(\mathcal{G})$ on $\mathcal{A}ff_B$ defined by

$$\mathbf{mc}_\bullet(\mathcal{G})(T) = \mathbf{mc}_\bullet(H^0(T, f^*\mathcal{G})).$$

Here $(T \xrightarrow{f} B)$ is an object of $\mathcal{A}ff_B$, and we apply the functor $\mathbf{mc}_\bullet$ to the dg-Lie algebra $H^0(T, f^*\mathcal{G})$ over $\mathbb{C}$.

Proposition 2.4.1. (a) *The simplicial presheaf* $\mathbf{mc}_\bullet(\mathcal{G})$ *is a homotopy sheaf.*

(b) *A morphism* (*resp. quasi-isomorphism*) $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ *of nilpotent* $\mathcal{O}_B$*-dg-Lie algebras induces a morphism* (*resp. weak equivalence*) *of homotopy sheaves* $\phi_\flat : \mathbf{mc}_\bullet(\mathcal{G}_1) \to \mathbf{mc}_\bullet(\mathcal{G}_2)$.

Proof. Part (b) follows from Proposition 1.1.8 by sheafification. ∎

Assume that the dg-Lie algebra $\mathcal{G}$ is situated in degrees $[0, 2]$, i.e.,

$$\mathcal{G} = \{\mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \mathcal{G}^2\}. \tag{2.4.2}$$

Then we define the stack $\mathrm{MC}(\mathcal{G})$ of groupoids on $\mathscr{A}ff_B$ by

$$\mathrm{MC}(\mathcal{G})(T) = \mathrm{MC}(H^0(T, \mathcal{G}|_T)).$$

We call $\mathrm{MC}(\mathcal{G})$ the *Maurer–Cartan stack* associated to a 3-term $\mathcal{O}_B$-dg-Lie algebra $\mathcal{G}$.

Proposition 2.4.3. *If $\mathcal{G}$ is situated in degrees $[0,2]$, then the simplicial sheaf $\mathbf{mc}_\bullet(\mathcal{G})$ is of groupoid type and $\Pi\mathbf{mc}_\bullet(\mathcal{G}) = \mathrm{MC}(\mathcal{G})$.* ■

Let $\mathcal{G}$ be any $\mathcal{O}_B$-dg-Lie algebra with quasi-coherent cohomology. As for complexes, we call $\mathcal{G}$ *strictly* $[0,2]$*-perfect* if it is quasi-isomorphic, as an $\mathcal{O}_B$-dg-Lie algebra, to a 3-term dg-Lie algebra (2.4.2) with each $\mathcal{G}^i$ being a vector bundle on B. We say that $\mathcal{G}$ is $[0,2]$*-perfect* if, locally on B, it is strictly $[0,2]$-perfect, and moreover the set of open substacks $U \subset B$ such that $\mathcal{G}|_U$ is strictly $[0,2]$-perfect, is filtering with respect to the partial order by inclusion.

We now assume that $\mathcal{G}$ is a strictly $[0,2]$-perfect dg-Lie algebra as in (2.4.2). Then we have the closed substack $\mathbf{mc}(\mathcal{G}) \subset \mathrm{Tot}(\mathcal{G}^1)$ "given by the equation $d^1x + \frac{1}{2}[x,x] = 0$", with two equivalent definitions:

(mc1) For any affine B-scheme $T \xrightarrow{f} B$ we have a dg-Lie algebra $H^0(T, \mathcal{G}|_T)$, and we define

$$\mathbf{mc}(\mathcal{G})(T) = \mathbf{mc}(H^0(T, \mathcal{G}|_T)).$$

(mc2) The stack $\mathbf{mc}(\mathcal{G})$ is the zero locus of the section $s_{\mathcal{G}}$ of $\pi^*\mathcal{G}^2$ given by the *curvature*

$$\mathcal{G}^1 \to \mathcal{G}^2, \quad x \mapsto d^1x + \tfrac{1}{2}[x,x]. \tag{2.4.4}$$

Since the Lie algebra $\mathcal{G}^0$ is nilpotent, we have a sheaf of groups $G^0 = \exp(\mathcal{G}^0)$ on B by Malcev theory, which acts on the stack $\mathbf{mc}(\mathcal{G})$ as in (1.1.7), and we can consider the quotient stack $\mathbf{mc}(\mathcal{G})//G^0$. Consider also the quotient stack

$$\mathrm{Tot}(\mathcal{G}^{\le 1}) = \mathrm{Tot}(\mathcal{G}^1)//G^0$$

and denote its projection to B by $\overline{\pi}$.

Proposition 2.4.5. (a) *Let $\mathcal{G}$ be a strictly $[0,2]$-perfect dg-Lie algebra as in* (2.4.2).

(a1) *We have an equivalence of stacks $u : \mathrm{MC}(\mathcal{G}) \to \mathbf{mc}(\mathcal{G})//G^0$, so $\mathrm{MC}(\mathcal{G})$ is an Artin stack.*

(a2) *The section $s_{\mathcal{G}}$ of the bundle $\pi^*\mathcal{G}^2$ on $\mathrm{Tot}(\mathcal{G}^1)$ descends to a section $\overline{s}_{\mathcal{G}}$ of the bundle $\overline{\pi}^*\mathcal{G}^2$ on $\mathrm{Tot}(\mathcal{G}^{\le 1})$, and the substack $\mathrm{MC}(\mathcal{G}) \subset \mathrm{Tot}(\mathcal{G}^{\le 1})$ is the zero locus of $\overline{s}_{\mathcal{G}}$.*

(b) *If $\mathcal{G}$ is a $[0,2]$-perfect $\mathcal{O}_B$-dg-Lie algebra, then the simplicial sheaf $\mathbf{mc}_\bullet(\mathcal{G})$ is of groupoid type. The stack of groupoids $\mathrm{MC}(\mathcal{G}) := \Pi\mathbf{mc}_\bullet(\mathcal{G})$ is an Artin stack over B.*

Proof. Part (a1) is proved similarly to Proposition 2.2.4 (a), using the fact that, G^0 being a unipotent sheaf of groups, any f^*G^0-torsor over any $T \in \mathscr{A}ff_B$ is trivial. Part (a2) follows from (a) and from the equivalence of the two definitions (mc1) and (mc2) of the stack $\mathbf{mc}(\mathcal{G})$. Part (b) follows because being of groupoid type and being an Artin stack over B are properties local on B. ■

Example 2.4.6. If the dg-Lie algebra $\mathcal{G}$ is abelian, i.e., it reduces to a [0, 2]-perfect complex on B, then $\mathrm{MC}(\mathcal{G}) = \mathrm{Tot}(\tau_{\leq 0}(\mathcal{G}[1]))$.

Let us now globalize the considerations of Proposition 1.1.9 as follows. Let $p : \mathcal{G} = \mathcal{H} \ltimes \mathcal{N} \to \mathcal{H}$ be a split extension of strictly [0, 2]-perfect dg-Lie algebras on B. The B-scheme $\pi_{\mathcal{H}} : \mathbf{mc}(\mathcal{H}) \to B$ carries a strictly [0, 2]-perfect dg-Lie algebra $\tilde{\mathcal{N}}$ which is equal to $\pi_{\mathcal{H}}^* \mathcal{N}$ as a graded sheaf of $\mathcal{O}_{\mathbf{mc}(\mathcal{H})}$-Lie algebras, with the differential d_x at a point $x \in \mathbf{mc}(\mathcal{H})$ defined as above. The action of the sheaf of groups H^0 on $\mathbf{mc}(\mathcal{H})$ extends to a compatible action on $\tilde{\mathcal{N}}$, so that $\tilde{\mathcal{N}}$ descends to a strictly [0, 2]-perfect dg-Lie algebra on the stack $\mathrm{MC}(\mathcal{H})$. We denote this descended dg-Lie algebra by the same symbol $\tilde{\mathcal{N}}$. Note that $\mathrm{MC}(\tilde{\mathcal{N}})$ is a stack over $\mathrm{MC}(\mathcal{H})$, hence over B. Now, we have the following global analogue of Proposition 1.1.9.

Proposition 2.4.7. *The stacks* $\mathrm{MC}(\mathcal{G})$ *and* $\mathrm{MC}(\tilde{\mathcal{N}})$ *over* B *are isomorphic.*

Proof. For each affine B-scheme $T \in \mathcal{A}ff_B$, we have a split exact sequence of dg-Lie algebras

$$0 \to H^0(T, \mathcal{N}|_T) \to H^0(T, \mathcal{G}|_T) \xrightarrow{p} H^0(T, \mathcal{H}|_T) \to 0$$

which gives rise to a functor $p_* : \mathrm{MC}(H^0(T, \mathcal{G}|_T)) \to \mathrm{MC}(H^0(T, \mathcal{H}|_T))$ with the fiber category over an object x equivalent to $\mathrm{MC}(H^0(T, \mathcal{H}|_T)_x)$. This yields the following isomorphism of groupoids over $\mathrm{MC}(H^0(T, \mathcal{H}|_T))$:

$$\mathrm{MC}(H^0(T, \mathcal{G}|_T)) = \mathrm{MC}(H^0(T, \tilde{\mathcal{N}}|_T)).$$ ■

2.5. *Stacks of filtrations*

Let B be an f-Artin stack and $p : Y \to B$ be a scheme over B, locally of finite type. Let $\mathcal{E}_{01}, \mathcal{E}_{12}, \mathcal{E}_{23}$ be coherent sheaves over Y which are flat over B. We define FILT to be the stack over B classifying filtered coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ over Y, together with identifications $\mathcal{E}_{0j}/\mathcal{E}_{0i} \simeq \mathcal{E}_{ij}$ for $ij = 12, 23$. We have a sheaf of associative dg-algebras over B defined by

$$\mathcal{G} = \bigoplus_{ij<kl} Rp_* \underline{\mathrm{RHom}}(\mathcal{E}_{kl}, \mathcal{E}_{ij}), \quad 01 < 12 < 23. \tag{2.5.1}$$

We will consider $\mathcal{G}$ as a sheaf of dg-Lie algebras using the supercommutator. Then we have the following generalization of Proposition 2.3.4.

Proposition 2.5.2. *Assume that* $\mathcal{G}$ *is a strictly* [0, 2]*-perfect dg-Lie algebra on* B. *Then we have an equivalence* $\mathrm{MC}(\mathcal{G}) = \mathrm{FILT}$ *of stacks over* B.

Proof. Let SES_{012} be the stack over B classifying short exact sequences

$$\mathcal{E}_{012} = \{0 \to \mathcal{E}_{01} \to \mathcal{E}_{02} \to \mathcal{E}_{12} \to 0\} \tag{2.5.3}$$

of coherent sheaves over Y. Then FILT is the stack over SES_{012} classifying short exact sequences

$$\mathcal{E}_{0123} = \{0 \to \mathcal{E}_{02} \to \mathcal{E}_{03} \to \mathcal{E}_{23} \to 0\}, \tag{2.5.4}$$

and $\mathcal{G} = \mathcal{H} \ltimes \mathcal{N}$ where

$$\mathcal{N} = Rp_*\underline{\mathrm{Hom}}(\mathcal{E}_{23}, \mathcal{E}_{01} \oplus \mathcal{E}_{12}), \quad \mathcal{H} = Rp_*\underline{\mathrm{Hom}}(\mathcal{E}_{12}, \mathcal{E}_{01}).$$

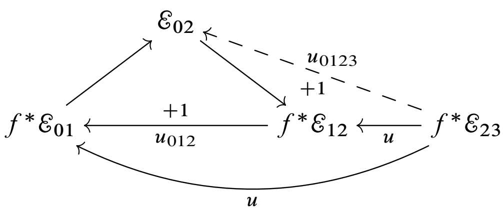
Since the dg-Lie algebra $\mathcal{H}$ is abelian, by Example 2.4.6 and Proposition 2.3.4 the stacks $\mathrm{MC}(\mathcal{H})$, SES_{012} are equivalent, and $\mathcal{N}$ gives an abelian strictly $[0, 2]$-perfect dg-Lie algebra $\tilde{\mathcal{N}}$ over SES_{012}. Further, by Proposition 2.4.7, we have $\mathrm{MC}(\mathcal{G}) = \mathrm{MC}(\tilde{\mathcal{N}})$ as stacks over SES_{012}. So it remains to prove that $\mathrm{MC}(\tilde{\mathcal{N}})$ is the stack over SES_{012} classifying short exact sequences (2.5.4).

Let $T \xrightarrow{f} B$ be an affine B-scheme. Suppose the object $\mathcal{E}_{012}$ of $\mathrm{SES}_{012}(T)$ is the cone of a morphism u_{012} in $\mathrm{RHom}^1_{Y \times_B T}(f^*\mathcal{E}_{12}, f^*\mathcal{E}_{01})$. Thus, given injective resolutions of $f^*\mathcal{E}_{ij}$ for each i, j, the complex $\mathcal{E}_{02}$ is quasi-isomorphic to the complex $C(u_{012}) = I_{12} \oplus I_{01}$ where the differential is the sum of the differentials of I_{12}, I_{01} and the composition with u_{012}, viewed as a morphism of complexes of sheaves $I_{12} \to I_{01}[1]$.

Next, we have $\tilde{\mathcal{N}} = \pi^*_{\mathcal{H}}\mathcal{N}$ as a graded sheaf, and the differential d_{012} of $\tilde{\mathcal{N}}$ at the point $\mathcal{E}_{012}$ is given by

$$d_{012}(u) = d(u) - \mathrm{ad}(u_{012})(u), \quad \forall u \in \mathrm{Hom}_{Y \times_B T}(f^*\mathcal{E}_{23}, f^*\mathcal{E}_{01} \oplus f^*\mathcal{E}_{12});$$

see Proposition 1.1.9 and the discussion before it. In our case $\mathrm{ad}(u_{012})(u)$ reduces to the composition $u_{012}u$. Thus, the condition for u to satisfy the equation $d_{012}(u) = 0$ is equivalent to saying that it lifts to a morphism of complexes $f^*\mathcal{E}_{23} \to C(u_{012})$, i.e., to a dotted arrow u_{0123} in the diagram



The cone of such an arrow defines $\mathcal{E}_{03}$ with a short exact sequence (2.5.4). We have thus constructed a morphism $\mathrm{MC}(\tilde{\mathcal{N}}) \to \mathrm{FILT}$ of stacks over SES_{012}, and it is easy to check that this morphism is an equivalence. ■

3. Borel–Moore homology of stacks and virtual pullbacks

3.1. BM homology and operations for schemes

We fix a field $\mathbf{k}$ of characteristic 0 which will serve as the field of coefficients for (co)homology. The cases $\mathbf{k} = \mathbb{Q}$ or $\mathbf{k} = \mathbb{Q}_l$ will be the most important. For basics on simplicial categories, ∞-categories and dg-categories, see §A and the references there. By dgVect $=$ dgVect$_{\mathbf{k}}$ we denote the dg-category of cochain complexes over $\mathbf{k}$. We recall the standard formalism of constructible derived categories of complexes of $\mathbf{k}$-vector spaces and their functorialities [34], together with its ∞-categorical enhancement.

Let Sch denote the category of schemes of finite type over $\mathbb{C}$. For a scheme $T \in$ Sch we denote by $\mathrm{C}(T)$ the category of constructible complexes of sheaves of $\mathbf{k}$-vector spaces on $T(\mathbb{C})$. Let $\mathrm{D}(T) = \mathrm{C}(T)[\mathrm{Qis}^{-1}]$ be the constructible derived category, i.e., the localization of $\mathrm{C}(T)$ by the class of quasi-isomorphisms. We denote by $\mathrm{D}(T)_{\mathrm{dg}}$ and $\mathrm{D}(T)_{\infty}$ the dg- and ∞-categorical enhancements of $\mathrm{D}(T)$ defined as in §A.2. If $\mathbf{k} = \mathbb{Q}_l$, we can use the étale l-adic version of the constructible derived category [50,51]. It admits similar enhancements.

These categories carry the Verdier duality functor which we denote by $\mathbb{D}$. For a morphism $f : S \to T$ in Sch we have the usual functorialities

$$\mathrm{D}(S) \underset{f^{-1}, f^!}{\overset{Rf_*, f_!}{\rightleftarrows}} \mathrm{D}(T)$$

with their standard adjunctions; see [34] for the case of classical topology or [50,51] for the case of étale topology. They extend to the above enhancements and we will be using these extensions.

We denote by $\omega_T = p^! \mathbf{k}$, $p : T \to \mathrm{pt}$, the dualizing complex of T. The *Borel–Moore homology* of T and its complex of Borel–Moore chains are defined by

$$H_\bullet^{\mathrm{BM}}(T) = H^{-\bullet}(T, \omega_T), \quad C_\bullet^{\mathrm{BM}}(T) = R\Gamma(T, \omega_T). \tag{3.1.1}$$

The Poincaré–Verdier duality implies that

$$H_\bullet^{\mathrm{BM}}(T) = H_c^\bullet(T)^*. \tag{3.1.2}$$

A morphism $f : S \to T$ in Sch is called *strongly orientable of relative dimension* $m \in \mathbb{Z}$ if there is an isomorphism $\underline{\mathbf{k}}_S \to f^! \underline{\mathbf{k}}_T[m]$ in $\mathrm{D}(S)$. A choice of such an isomorphism is called a *strong orientation* of f. For not necessarily connected S we can speak of relative dimension being a locally constant function on S, with the obvious modifications of the above.

Recall that $H_\bullet^{\mathrm{BM}}$ is covariantly functorial with respect to proper morphisms. By (3.1.1), an oriented morphism $f : S \to T$ of relative dimension m gives rise to a pullback map $f^* : H_\bullet^{\mathrm{BM}}(T) \to H_{\bullet+m}^{\mathrm{BM}}(S)$, and such maps are compatible with compositions of oriented morphisms.

Examples 3.1.3. (a) A smooth morphism f of dimension d is strongly oriented of relative dimension $2d$.

(b) An l.c.i. (locally complete intersection) morphism is a morphism $f : S \to T$ represented as a composition $f = p \circ i$ where p is smooth and i is a regular embedding. Thus an l.c.i. morphism f has a well-defined *dimension* d, which is a locally constant $\mathbb{Z}$-valued function on S. If the embedding i is strongly oriented, then f is also strongly oriented of relative dimension $2d$, hence gives rise to a pullback morphism f^*. Note that the map f^* still make sense for any l.c.i. morphism; see, e.g., [51, §2.17].

Example 3.1.4. Let $\mathcal{E}$ be a rank r vector bundle on T. We recall that the rth Chern class $c_r(\mathcal{E}) \in H^{2r}(T, \mathbf{k})$ is the obstruction to the existence of a continuous section of $\mathcal{E}$ which does not vanish anywhere. Let s be any section of $\mathcal{E}$. We denote the zero locus of s with its embedding into T by $T_s \xrightarrow{i_s} T$. In this situation we have the *refined rth Chern class*

$$c_r(\mathcal{E}, s) \in H^{2r}_{T_s}(T, \mathbf{k}) = H^{2r}(T_s, i_s^! \underline{\mathbf{k}}_T)$$

whose image in $H^{2r}(T, \mathbf{k})$ is $c_r(\mathcal{E})$, yieldding a *virtual pullback* map $s^! : H^{\mathrm{BM}}_\bullet(T) \to H^{\mathrm{BM}}_{\bullet-2r}(T_s)$. More precisely, following [17, §7.3], we introduce the *bivariant cohomology* of any morphism $f : S \to T$ to be

$$H^\bullet(S \xrightarrow{f} T) = H^\bullet(S, f^! \underline{\mathbf{k}}_T).$$

Recall that:

(a) We have $H^\bullet(S \xrightarrow{\mathrm{Id}} S) = H^\bullet(S, \mathbf{k})$, while $H^\bullet(S \to \mathrm{pt}) = H^{\mathrm{BM}}_{-\bullet}(S)$.

(b) For a composable pair of maps $S \xrightarrow{f} T \xrightarrow{g} U$ we have the product

$$H^\bullet(S \xrightarrow{f} T) \otimes H^\bullet(T \xrightarrow{g} U) \to H^\bullet(S \xrightarrow{gf} U).$$

So, taking $U = \mathrm{pt}$, each $h \in H^d(S \xrightarrow{f} T)$ gives rise to a map $u_h : H^{\mathrm{BM}}_\bullet(T) \to H^{\mathrm{BM}}_{\bullet-d}(S)$.

We deduce that $c_r(\mathcal{E}, s) \in H^{2r}(T_s \xrightarrow{i_s} T)$ defines a map $H^{\mathrm{BM}}_\bullet(T) \to H^{\mathrm{BM}}_{\bullet-2r}(T_s)$.

The construction of $c_r(\mathcal{E}, s)$ is as follows. We consider the embedding $T \xrightarrow{0} \mathrm{Tot}(\mathcal{E})$ as the zero section. It is strongly oriented of relative dimension $2r$ [17, Props. 4.1.3, 7.3.2], hence we get a class $\eta \in H^{2r}_T(\mathrm{Tot}(\mathcal{E}))$. Now T_s is the intersection of T with Γ_s, the graph of s inside $\mathrm{Tot}(\mathcal{E})$, and $c_r(\mathcal{E}, s)$ is the image of η under the restriction map

$$H^{2r}_T(\mathrm{Tot}(\mathcal{E}), \underline{\mathbf{k}}) \to H^{2r}_{T \cap \Gamma_s}(\Gamma_s, \underline{\mathbf{k}}) = H^{2r}_{T_s}(T, \underline{\mathbf{k}}).$$

See also [51, §2.17] for a different approach.

Proposition 3.1.5. *Let $\mathcal{E}$ be a vector bundle on T of rank r and let $p : \mathrm{Tot}(\mathcal{E}) \to B$ be the projection. The pullback $p^* : H^{\mathrm{BM}}_\bullet(T) \to H^{\mathrm{BM}}_{\bullet+r}(\mathrm{Tot}(\mathcal{E}))$ is an isomorphism.* ■

Remark 3.1.6. For $T \in \mathrm{Sch}$ let $A_m(T)$ be the Chow group of m-dimensional cycles in T. We have the canonical *class map* $\mathrm{cl} : A_m(T) \to H^{\mathrm{BM}}_{2m}(T)$. All the above constructions (proper pushforwards, l.c.i. pullbacks, virtual pullbacks) have natural analogs for the Chow groups [16], which are compatible, via cl, with the sheaf-theoretical constructions described above.

3.2. BM homology and operations for stacks

The formalism of constructible derived categories and their functorialities extends to f-Artin stacks. For the case $\mathbf{k} = \mathbb{Q}_l$ and étale topology this is done in [50, 51]. Another approach using ∞-categorical limits, which we outline below, is applicable for the complex analytic topology, any $\mathbf{k}$, as well as for the case of analytic stacks in §7.3. It is an adaptation of the approach used in [19, §3.1.1] for ind-coherent sheaves to the constructible case. All stacks in this sections will be f-Artin.

Let B be a stack. We denote by Sch_B the category formed by schemes T of finite type over $\mathbb{C}$ together with a morphism of stacks $T \to B$. We define

$$\mathrm{D}(B)_\infty = \varprojlim_{\{T\to B\}} \mathrm{D}(T)_\infty, \quad \mathrm{D}(B)_{\mathrm{dg}} = \varprojlim_{\{T\to B\}} \mathrm{D}(T)_{\mathrm{dg}}, \tag{3.2.1}$$

the ∞-categorical projective limit, resp. dg-categorical (homotopy) projective limit over the category Sch_B, with respect to the pullback functors. Note that $\mathrm{D}(B)_\infty$, resp. $\mathrm{D}(B)_{\mathrm{dg}}$ also carries the Verdier duality $\mathbb{D}$ induced by such dualities on the $\mathrm{D}(T)_\infty$, resp. $\mathrm{D}(T)_{\mathrm{dg}}$ above.

We compare this with the following. Let Z be a scheme of finite type over $\mathbb{C}$ with an action of an affine algebraic group G. Then we have action groupoid $\{G \times Z \rightrightarrows Z\}$ in the category of schemes, so its nerve $N_\bullet\{G \times Z \rightrightarrows Z\}$ is a simplicial scheme defined as in (1.1.2). The *Bernstein–Lunts equivariant derived constructible ∞-category* of Z is

$$\mathrm{D}(Z, G)_\infty = \varprojlim_{[n]\in\Delta^\circ} \mathrm{D}(N_n\{G \times Z \rightrightarrows Z\})_\infty.$$

It is an ∞-categorical version of the definition from [6]. Just as in [6], if $\mathcal{F}^\bullet$is a $G(\mathbb{C})$-equivariant constructible complex on $Z(\mathbb{C})$, then

$$\mathrm{Ext}^\bullet_{\mathrm{D}(Z,G)_\infty}(\underline{\mathbf{k}}_Z, \mathcal{F}^\bullet) = H^\bullet_{G(\mathbb{C})}(Z(\mathbb{C}), \mathcal{F}^\bullet)$$

is the topological equivariant (hyper)cohomology.

Proposition 3.2.2. *The ∞-category $\mathrm{D}(Z, G)_\infty$ is identified with $\mathrm{D}(Z//G)_\infty$.*

Proof. Each $N_n\{G \times Z \rightrightarrows Z\}$ is an affine scheme over Z, therefore over $Z//G$. In fact,

$$N_n\{G \times Z \rightrightarrows Z\} = Z \times_{Z//G} \cdots \times_{Z//G} Z \quad (n \text{ times}).$$

So $N_\bullet\{G \times Z \rightrightarrows Z\}$ is the nerve of the (smooth) morphism $Z \to Z//G$, which we can see as a 1-element covering of $Z//G$ in the smooth topology. Our statement therefore means that $\mathrm{D}(-)_\infty$ satisfies (∞-categorical) descent with respect to this covering. A more general statement is true: $\mathrm{D}(-)_\infty$ as a functor from stacks to ∞-categories satisfies descent (for any covering) in the smooth topology. This statement is a formal consequence of the corresponding, obvious, statement for schemes: $\mathrm{D}(-)_\infty$ as a functor from Sch to ∞-categories satisfies descent (for any covering) in the smooth topology. ∎

Given a morphism of stacks $f : B \to C$, the composition with f defines a functor $f_\square : \mathrm{Sch}_B \to \mathrm{Sch}_C$, hence a functor which we denote

$$f^{-1} : \mathrm{D}(C)_\infty = \varprojlim_{(U\to C)} \mathrm{D}(U) \to \varprojlim_{(T\to B\xrightarrow{f} C)} \mathrm{D}(T) = \mathrm{D}(B)_\infty.$$

The right adjoint functor to f^{-1} is denoted by $Rf_* : \mathrm{D}(C)_\infty \to \mathrm{D}(B)_\infty$.

We further define the functors

$$f^! = \mathbb{D} \circ f^{-1} \circ \mathbb{D} : \mathrm{D}(C)_\infty \to \mathrm{D}(B)_\infty, \quad Rf_! = \mathbb{D} \circ Rf_* \circ \mathbb{D} : \mathrm{D}(B)_\infty \to \mathrm{D}(C)_\infty.$$

In particular, we have the *dualizing complex* $\omega_B = \mathbb{D}(\underline{\mathbf{k}}_B) = p^!(\mathbf{k})$, where $p : B \to \mathrm{pt}$ (see [39]). Note that, for each affine algebraic group G over $\mathbb{C}$, we have $\omega_{BG} \simeq \underline{\mathbf{k}}_{BG}[-2\dim(G)]$, while for each smooth complex variety S, $\omega_S \simeq \underline{\mathbf{k}}_S[2\dim(S)]$.

We define the *Borel–Moore homology* and *cohomology with compact support* of an (f-Artin) stack B by

$$H_\bullet^{\mathrm{BM}}(B) = H^{-\bullet}(B, \omega_B), \quad H_c^\bullet(B, \underline{\mathbf{k}}_B) = H^\bullet(Rp_!\underline{\mathbf{k}}_B). \tag{3.2.3}$$

The Poincaré–Verdier duality extends from schemes of finite type to f-Artin stacks and implies that $H_\bullet^{\mathrm{BM}}(B) = H_c^\bullet(B, \underline{\mathbf{k}}_B)^*$. By gluing the corresponding properties of schemes, we find that $H_\bullet^{\mathrm{BM}}$ is covariantly functorial for proper morphisms and has pullbacks with respect to l.c.i. morphisms.

Remark 3.2.4. The BM homology for stacks is the topological analog of the Chow groups for stacks as defined by Kresch [38].

We also note the following (see [38, Thm. 2.1.12]).

Proposition 3.2.5. *Let $\mathcal{C}^\bullet = \{\mathcal{C}^{-1} \to \mathcal{C}^0\}$ be a two-term strictly perfect complex on B of virtual rank r, with the total space* $\mathrm{Tot}(\mathcal{C}^\bullet) = \mathcal{C}^0 /\!/ \mathcal{C}^{-1} \xrightarrow{\pi} B$. *Then π is a smooth morphism, hence it is strongly oriented of relative dimension $2r$, and $\pi^* : H_\bullet^{\mathrm{BM}}(B) \to H_\bullet^{\mathrm{BM}}(\mathrm{Tot}(\mathcal{C}))$ is an isomorphism if B admits a stratification by global quotients* [38, Def. 3.5.3], *in particular if B is locally quotient.* ■

3.3. *Virtual pullback for a perfect complex*

Let B be a stack and $\mathcal{E}$ be a vector bundle of rank r over B. Let $s \in H^0(B, \mathcal{E})$ be a section of $\mathcal{E}$ and

$$i : B_s = \{s = 0\} \hookrightarrow B$$

be the inclusion of the zero locus of s, which is a closed substack. The section s gives a regular embedding in the total space of $\mathcal{E}$, which we also denote by $s : B \to \mathrm{Tot}(\mathcal{E})$. The construction of Example 3.1.4 extends (by gluing) from schemes to stacks and gives the *refined pullback morphism*, or *refined Gysin morphism*,

$$s^! : H_\bullet^{\mathrm{BM}}(B) \to H_{\bullet-2r}^{\mathrm{BM}}(B_s), \tag{3.3.1}$$

making the following diagram commute:

$$\begin{array}{ccc} H_\bullet^{\mathrm{BM}}(B) & \xrightarrow{s^!} & H_{\bullet-2r}^{\mathrm{BM}}(B_s) \\ {\scriptstyle s_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ H_\bullet^{\mathrm{BM}}(\mathrm{Tot}(\mathcal{E})) & \underset{\pi^*}{\xleftarrow{\sim}} & H_{\bullet-2r}^{\mathrm{BM}}(B) \end{array}$$

Remark 3.3.2. The map $s^!$ is the BM-homology analog of the refined pullback on Chow groups for Artin stacks, which is a particular case of [45, Construction 3.6], or of [38, §3.1] which uses deformation to the normal cone.

Now, let $\mathcal{C}$ be a strictly $[-1,1]$-perfect complex on B and

$$\pi : \mathrm{Tot}(\mathcal{C}^{\leq 0}) \to B, \quad q : \mathrm{Tot}(\tau_{\leq 0}\mathcal{C}) \to B$$

be the obvious projections. The differential d^1 of $\mathcal{C}$ gives a section $s_{\mathcal{C}}$ of the vector bundle $\pi^*\mathcal{C}^1$ on $\mathrm{Tot}(\mathcal{C}^{\leq 0})$ whose zero locus is the cone stack $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$, yielding the diagram

$$\begin{array}{ccccc} & & \pi^*\mathcal{C}^1 & \xleftarrow{s_{\mathcal{C}}} & \mathrm{Tot}(\mathcal{C}^{\leq 0}) \\ & & {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle i} \\ B & \xleftarrow{\pi} & \mathrm{Tot}(\mathcal{C}^{\leq 0}) & \xhookleftarrow{i} & \mathrm{Tot}(\tau_{\leq 0}\mathcal{C}) \end{array}$$

such that $q = \pi \circ i$. By Proposition 3.2.5 (see also [38, Thm. 2.1.12]), the pullback along π defines a morphism

$$\pi^* : H_\bullet^{\mathrm{BM}}(B) \xrightarrow{\sim} H_{\bullet+2\,\mathrm{vrk}(\mathcal{C}^{\leq 0})}^{\mathrm{BM}}(\mathrm{Tot}(\mathcal{C}^{\leq 0})),$$

which is an isomorphism if B admits a stratification by global quotients. Further, we have the refined pullback map on Borel–Moore homology

$$s_{\mathcal{C}}^! : H_{\bullet+2\,\mathrm{vrk}(\mathcal{C}^{\leq 0})}^{\mathrm{BM}}(\mathrm{Tot}(\mathcal{C}^{\leq 0})) \to H_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}^{\mathrm{BM}}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})).$$

We define the *virtual pullback* associated with $\mathcal{C}$ to be the composite map

$$q_{\mathcal{C}}^! = s_{\mathcal{C}}^! \circ \pi^* : H_\bullet^{\mathrm{BM}}(B) \to H_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}^{\mathrm{BM}}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})).$$

By Proposition 2.2.8, the stack $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$ depends only on the isomorphism class of the complex $\mathcal{C}$ in $D^b_{\mathrm{coh}}(B)$ and not on the choice of the presentation (2.2.6).

Proposition 3.3.3. *Let $\mathcal{C}$ be a strictly $[-1,1]$-perfect complex on B. The virtual pullback $q_{\mathcal{C}}^!$ depends only on the isomorphism class of the strictly $[-1,1]$-perfect complex $\mathcal{C}$ in $D^b_{\mathrm{coh}}(B)$.*

Proof. Fix two presentations $\mathcal{C}_1, \mathcal{C}_2$ of the complex $\mathcal{C}$ as in (2.2.6), with

$$\mathcal{C}_k = \{\mathcal{C}_k^{-1} \xrightarrow{d_k^0} \mathcal{C}_k^0 \xrightarrow{d_k^1} \mathcal{C}_k^1\}, \quad k = 1, 2,$$

and fix a quasi-isomorphism $\phi : \mathcal{C}_1 \to \mathcal{C}_2$. By functoriality of the total space and the truncated total space, we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_1) & \overset{\phi_\flat}{=\!=\!=} & \mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_2) \\
{\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\
\mathrm{Tot}(\mathcal{C}_1^{\leq 0}) & \xrightarrow{\phi_\flat} & \mathrm{Tot}(\mathcal{C}_2^{\leq 0}) \\
{\scriptstyle \pi_1}\big\downarrow & \swarrow{\scriptstyle \pi_2} & \\
B & &
\end{array}$$

We claim that the following triangle commutes:

$$\begin{array}{ccc}
H_\bullet^{\mathrm{BM}}(B) & \xrightarrow{q^!_{\mathcal{C}_1}} & H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C}_1)}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_1)) \\
& {\scriptstyle q^!_{\mathcal{C}_2}}\searrow & \big\Vert{\scriptstyle (\phi_\flat)_*} \\
& & H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C}_2)}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_2)).
\end{array}$$

To prove this, we must prove that

$$s^!_{\mathcal{C}_1} \circ \pi_1^* = \phi_\flat^* \circ s^!_{\mathcal{C}_2} \circ \pi_2^*.$$

By Proposition 2.2.4, the map $\phi_\flat : \mathrm{Tot}(\mathcal{C}_1^{\leq 0}) \to \mathrm{Tot}(\mathcal{C}_2^{\leq 0})$ is an l.c.i. Hence there is a Gysin map $(\phi_\flat)^*$ and we have expressions through the local Chern classes associated to the sections s_{C_i} of $\pi_i^*\mathcal{C}_i^1$, $i = 1, 2$:

$$s^!_{\mathcal{C}_1} \circ \pi_1^* = c_{\mathrm{rk}(\mathcal{C}_1^1)}(\pi_1^*\mathcal{C}_1^1, s_{\mathcal{C}_1}) \circ \phi_\flat^* \circ \pi_2^*,$$
$$\phi_\flat^* \circ s^!_{\mathcal{C}_2} \circ \pi_2^* = \phi_\flat^* \circ c_{\mathrm{rk}(\mathcal{C}_2^1)}(\pi_2^*\mathcal{C}_2^1, s_{\mathcal{C}_2}) \circ \pi_2^*.$$

The proposition is a consequence of the following version of the excess intersection formula.

Lemma 3.3.4. *Let $f : B_1 \to B_2$ be a morphism of stacks which is an l.c.i. of relative dimension $r_2 - r_1$. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be vector bundles on B_1, B_2 of ranks r_1, r_2 and sections s_1, s_2 of $\mathcal{E}_1$, $\mathcal{E}_2$. Let $h : \mathcal{E}_1 \to f^*\mathcal{E}_2$ be a vector bundle homomorphism such that $h \circ s_1 = s_2 \circ f$, which yields a fiber diagram*

$$\begin{array}{ccccc}
(B_2)_{s_2} & \overset{i_2}{\hookrightarrow} & B_2 & \xrightarrow{s_2} & \mathrm{Tot}(\mathcal{E}_2) \\
{\scriptstyle g}\big\uparrow & & {\scriptstyle f}\big\uparrow & & {\scriptstyle h}\big\uparrow \\
(B_1)_{s_1} & \overset{i_1}{\hookrightarrow} & B_1 & \xrightarrow{s_1} & \mathrm{Tot}(\mathcal{E}_1)
\end{array}$$

where g is an isomorphism. Then we have a commutative square

$$\begin{array}{ccc} H_\bullet^{\mathrm{BM}}(B_1) & \xrightarrow{c_{r_1}(\mathcal{E}_1, s_1)} & H^{\mathrm{BM}}_{\bullet-2r_1}((B_1)_{s_1}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ H^{\mathrm{BM}}_{\bullet-2r_1+2r_2}(B_2) & \xrightarrow{c_{r_2}(\mathcal{E}_2, s_2)} & H^{\mathrm{BM}}_{\bullet-2r_1}((B_2)_{s_2}) \end{array}$$

■

Finally, let now B be an Artin stack and let $\mathcal{C}$ be any $[-1,1]$-perfect complex on B. Let $\mathfrak{U}$ be a filtering open cover of B consisting of open substacks U such that $\mathcal{C}|_U$ is strictly $[-1,1]$-perfect. We have

$$H_\bullet^{\mathrm{BM}}(B) = \varprojlim_{U\in\mathfrak{U}} H_\bullet^{\mathrm{BM}}(U), \quad H_\bullet^{\mathrm{BM}}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})) = \varprojlim_{U\in\mathfrak{U}} H_\bullet^{\mathrm{BM}}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}|_U)). \tag{3.3.5}$$

Definition 3.3.6. A *coherent perfect system* on a $[-1,1]$-perfect complex $\mathcal{C}$ on B is a collection of quasi-isomorphisms $\phi_U : \mathcal{C}|_U \to \mathcal{C}_U$ and $\phi_{V\subset U} : \mathcal{C}_U|_V \to \mathcal{C}_V$ for each $U, V \in \mathfrak{U}$ with $V \subset U$ such that $\mathcal{C}_U$ is a strictly $[-1,1]$-perfect complex on U with a presentation as in (2.2.6), and $\phi_V = \phi_{V\subset U} \circ \phi_U|_V$.

Given a coherent perfect system on $\mathcal{C}$, we define the virtual pullback

$$q^!_{\mathcal{C}} : H_\bullet^{\mathrm{BM}}(B) \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}))$$

as the map

$$q^!_{\mathcal{C}} = \varprojlim_{U\in\mathfrak{U}} ((\phi_U)^* \circ q^!_{\mathcal{C}_U}). \tag{3.3.7}$$

Remark 3.3.8. If $\mathcal{C}$ is a strictly $[-1,1]$-perfect complex on the stack B, then its total space has a *dg-stack* structure given by

$$\mathrm{Tot}(\mathcal{C}) = \big(\mathrm{Tot}(\mathcal{C}^{\leq 0}), (\mathrm{Sym}(\pi^*(\mathcal{C}^1)^\vee[1]), \partial_s)\big), \tag{3.3.9}$$

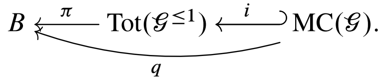
that is, the stack $\mathrm{Tot}(\mathcal{C}^{\leq 0})$ equipped with the sheaf of commutative dg-algebras which is the Koszul complex of the section s above. This dg-stack gives rise to a *derived stack* in the sense of [62]. The derived stack $\mathrm{Tot}(\mathcal{C})$ depends, up to a natural equivalence, only on the isomorphism class of the complex $\mathcal{C}$ in $D^b_{\mathrm{coh}}(B)$. We expect a direct conceptual interpretation of the virtual pullback $q^!_{\mathcal{C}}$ in terms of the derived stack $\mathrm{Tot}(\mathcal{C})$. However, this would require a well-behaved Borel–Moore homology theory for derived stacks and we do not know how to do it.

3.4. Virtual pullback for Maurer–Cartan stacks

Let B be an Artin stack of finite type and $\mathcal{G}$ be a strictly $[0,2]$-perfect dg-Lie algebra over B as in (2.4.2). We now define a virtual pullback

$$q^!_{\mathcal{G}} : H_\bullet^{\mathrm{BM}}(B) \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{G})}(\mathrm{MC}(\mathcal{G}))$$

using the diagram



In order to define the map $q_{\mathscr{G}}^! = s_{\mathscr{G}}^! \circ \pi^*$ as in §3.3, we must check that the pullback morphism

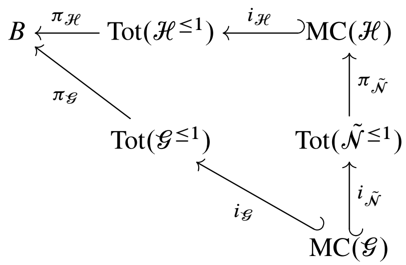
$$\pi^* : H_\bullet^{\mathrm{BM}}(B) \to H_{\bullet+2\,\mathrm{vrk}(\mathscr{G}^{\le 1})}^{\mathrm{BM}}(\mathrm{Tot}(\mathscr{G}^{\le 1}))$$

and the refined pullback

$$s_{\mathscr{G}}^! : H_{\bullet+2\,\mathrm{vrk}(\mathscr{G}^{\le 1})}^{\mathrm{BM}}(\mathrm{Tot}(\mathscr{G}^{\le 1})) \to H_{\bullet+2\,\mathrm{vrk}(\mathscr{G})}^{\mathrm{BM}}(\mathrm{MC}(\mathscr{G}))$$

are well-defined. The refined pullback is defined as in the previous sections, using the fact that $\mathrm{MC}(\mathscr{G})$ is the zero locus of the section s of the bundle $\pi^*\mathscr{G}^2$ on $\mathrm{Tot}(\mathscr{G}^{\le 1})$ associated with the curvature (2.4.4). The pullback map π^* is well-defined, because π is a vector bundle stack, hence is smooth although non-representable.

Next, we study the behavior of the virtual pullback under extensions of dg-Lie algebras. Let $\mathscr{G} = \mathscr{H} \ltimes \mathscr{N}$ and $\tilde{\mathscr{N}} = \pi_{\mathscr{H}}^* \mathscr{N}$ be as in §2.4. Note that Proposition 2.4.7 allows us to write the commutative diagram



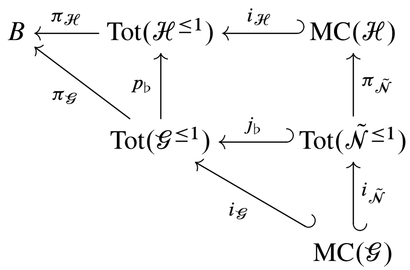
The virtual pullback maps $q_{\mathscr{G}}^!$, $q_{\tilde{\mathscr{N}}}^!$ and $q_{\mathscr{H}}^!$ are defined as above.

Proposition 3.4.1. *We have the equality* $q_{\mathscr{G}}^! = q_{\tilde{\mathscr{N}}}^! \circ q_{\mathscr{H}}^!$.

Proof. Let $s_{\mathscr{G}}, s_{\tilde{\mathscr{N}}}, s_{\mathscr{H}}$ be the sections of the bundles $\pi_{\mathscr{G}}^*\mathscr{G}^2$, $\pi_{\tilde{\mathscr{N}}}^*\tilde{\mathscr{N}}^2$, $\pi_{\mathscr{H}}^*\mathscr{H}^2$ associated with the curvature maps of $\mathscr{G}$, $\tilde{\mathscr{N}}$, $\mathscr{H}$ respectively. We must prove that

$$s_{\mathscr{G}}^! \circ \pi_{\mathscr{G}}^* = s_{\tilde{\mathscr{N}}}^! \circ \pi_{\tilde{\mathscr{N}}}^* \circ s_{\mathscr{H}}^! \circ \pi_{\mathscr{H}}^*.$$

First, observe that we have the diagram whose square is a fiber square



and the maps $p_\flat$, $j_\flat$ are given by the functoriality of the total space of a $[-1,0]$-complex. Note further that we have vector bundle homomorphisms

$$\pi_{\mathcal{G}}^* \mathcal{G}^2 \to (p_\flat)^* \pi_{\mathcal{H}}^*(\mathcal{H}^2), \quad \pi_{\tilde{\mathcal{N}}}^* \tilde{\mathcal{N}}^2 \to (j_\flat)^* \pi_{\mathcal{G}}^* \mathcal{G}^2.$$

These vector bundle homomorphisms being compatible with the sections $s_{\mathcal{G}}$, $s_{\tilde{\mathcal{N}}}$ and $s_{\mathcal{H}}$, the claim follows from the functoriality of the refined pullback with respect to pullback by smooth maps. ■

4. The COHA of a surface

4.1. The COHA as a vector space

Let S be a smooth connected quasi-projective surface over $\mathbb{C}$. Let $\mathrm{Coh}(S)$ be the stack of coherent sheaves on S with proper support. It is not smooth because the deformation theory can be obstructed due to Ext^2.

Proposition 4.1.1. $\mathrm{Coh}(S)$ *is a locally quotient f-Artin stack.*

Proof. This is standard [40, Thm. 4.6.2.1]. Here are the details for future use in Prop. 4.3.2. Let $\overline{S}$ be a smooth projective variety containing S an an open set. Then $\mathrm{Coh}(S)$ is an open substack in $\mathrm{Coh}(\overline{S})$. So it is enough to assume that S is projective which we will. Let $\mathcal{O}(1)$ be the ample line bundle on S induced by a projective embedding. The stack $\mathrm{Coh}(S)$ splits into disjoint union

$$\mathrm{Coh}(S) = \bigsqcup_{h \in \mathbf{k}[[t]]} \mathrm{Coh}^{(h)}(S),$$

where $\mathrm{Coh}^{(h)}(S)$ consists of sheaves $\mathcal{F}$ with Hilbert polynomial h, i.e., such that

$$\dim H^0(S, \mathcal{F}(n)) = h(n), \quad n \gg 0.$$

For any $N \in \mathbb{N}$, let $\mathrm{Coh}^{(h,N)}(S) \subset \mathrm{Coh}^{(h)}(S)$ be the open substack formed by $\mathcal{F}$ such that for each $n \geq N$ two conditions hold:

(a) $H^i(S, \mathcal{F}(n)) = 0$, $i > 0$,

(b) the canonical map $H^0(S, \mathcal{F}(n)) \otimes \mathcal{O}(-n) \to \mathcal{F}$ is surjective.

Now, for any coherent sheaf $\mathcal{E}$ on a scheme B, let $Quot_{\mathcal{E}}$ be the scheme such that, for any B-scheme $T \to B$, the set of T-points $Quot_{\mathcal{E}}(T)$ is the set of surjective sheaf homomorphisms $\mathcal{E}|_T \to \mathcal{F}$ where $\mathcal{F}$ is flat over T, modulo the equivalence relation

$$(q : \mathcal{E}|_T \to \mathcal{F}) \sim (q' : \mathcal{E}|_T \to \mathcal{F}') \iff \mathrm{Ker}(q) = \mathrm{Ker}(q').$$

Let $Quot^{(h,N)}(S)$ be the open subscheme of $Quot_{\mathcal{O}(-N)^{\oplus h(N)}}$ formed by the equivalence classes of surjections $\phi : \mathcal{O}(-N)^{\oplus h(N)} \to \mathcal{F}$ with $\mathcal{F} \in \mathrm{Coh}^{(h,N)}(S)$ such that $\phi(N)$ induces an isomorphism $H^0(S, \mathcal{O})^{\oplus h(N)} \to H^0(S, \mathcal{F}(N))$. Then the stack $\mathrm{Coh}^{(h,N)}(S)$ is isomorphic to the quotient stack of $Quot^{(h,N)}(S)$ by the obvious action of the group $GL_{h(N)}$. It is a stack of finite type and, as $N \to \infty$, the substacks $\mathrm{Coh}^{(h,N)}(S)$ form an open exhaustion of $\mathrm{Coh}^{(h)}(S)$. ■

4.2. *The induction diagram*

Let SES be the Artin stack classifying short exact sequences

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0 \tag{4.2.1}$$

of coherent sheaves with proper support over S. Morphisms in SES are isomorphisms of such sequences. We then have the *induction diagram*

$$\mathrm{Coh}(S) \times \mathrm{Coh}(S) \xleftarrow{q} \mathrm{SES} \xrightarrow{p} \mathrm{Coh}(S), \tag{4.2.2}$$

where the map p projects a sequence (4.2.1) to $\mathcal{G}$, while q projects it to $(\mathcal{E}, \mathcal{F})$.

Proposition 4.2.3. *The morphism p is schematic* (*representable*) *and proper.*

Proof. For any coherent sheaf $\mathcal{G}$ on S with proper support, the Grothendieck Quot scheme $Quot_{\mathcal{G}}$ is proper. ∎

4.3. *The derived induction diagram*

We have the projections

$$\mathrm{Coh}(S) \times \mathrm{Coh}(S) \xleftarrow{p_{12}} \mathrm{Coh}(S) \times \mathrm{Coh}(S) \times S \xrightarrow{p_{13}, p_{23}} \mathrm{Coh}(S) \times S.$$

Consider the tautological coherent sheaf $\mathcal{U}$ over $\mathrm{Coh}(S) \times S$ and the complex of coherent sheaves over $\mathrm{Coh}(S) \times \mathrm{Coh}(S)$ given by

$$\mathcal{C} = R(p_{12})_* \underline{\mathrm{RHom}}(p_{23}^* \mathcal{U}, p_{13}^* \mathcal{U})[1]. \tag{4.3.1}$$

Its fiber at a point $(\mathcal{E}, \mathcal{F})$ is the complex of vector spaces $\mathrm{RHom}_S(\mathcal{F}, \mathcal{E})[1]$. Given a substack $X \subset Coh(S)$, let $\mathcal{U}_X = \mathcal{U}|_{X \times S}$ and $\mathcal{C}_X = \mathcal{C}|_{X \times X}$ be the restrictions of $\mathcal{U}$ and $\mathcal{C}$.

Proposition 4.3.2. (a) *The complex $\mathcal{C}$ is $[-1, 1]$-perfect and admits a perfect coherent system.*

(b) *The complex $\mathcal{C}_X$ is strictly $[-1, 1]$-perfect if $X = \mathrm{Coh}_0(S)$.*

Proof. As in the proof of Proposition 4.1.1, the statements reduce to the case when S is projective, which we assume. We also keep the notation from that proof. Fix two polynomials $h, h' \in \mathbf{k}[t]$ and let $\mathcal{E} \in \mathrm{Coh}^{(h)}(S)$ and $\mathcal{F} \in \mathrm{Coh}^{(h')}(S)$ be two fixed coherent sheaves on S with Hilbert polynomials h, h'. Since S is smooth of dimension 2, we can fix a locally free resolution $\mathcal{P}^\bullet = \{\mathcal{P}^{-2} \to \mathcal{P}^{-1} \to \mathcal{P}^0\}$ of $\mathcal{F}$. If we know that the $\mathcal{P}^i$ are "sufficiently negative" with respect to $\mathcal{E}$, i.e., for each $i \in [-2, 0]$ and $j > 0$ the space $\mathrm{Ext}^j_S(\mathcal{P}^i, \mathcal{E}) = H^j(S, (\mathcal{P}^i)^\vee \otimes \mathcal{E})$ vanishes, then the complex of vector spaces $\mathrm{RHom}_S(\mathcal{F}, \mathcal{E})[1]$ is represented by the complex

$$\mathrm{Hom}_S(\mathcal{P}^0, \mathcal{E}) \to \mathrm{Hom}_S(\mathcal{P}^{-1}, \mathcal{E}) \to \mathrm{Hom}_S(\mathcal{P}^{-2}, \mathcal{E}) \tag{4.3.3}$$

situated in degrees $[-1, 1]$. In order to achieve this, we define, in a standard way,

$$\mathcal{P}^0 = H^0(S, \mathcal{F}(N_0)) \otimes \mathcal{O}(-N_0) \xrightarrow{\mathrm{ev}_0} \mathcal{F}, \quad N_0 \ll 0,$$

with ev_0 being the evaluation map. Then we set $\mathcal{K}_0 = \mathrm{Ker}(\mathrm{ev}_0) \overset{\varepsilon_1}{\hookrightarrow} \mathcal{P}^0$ and

$$\mathcal{P}^{-1} = H^0(S, \mathcal{K}_0(N_1)) \otimes \mathcal{K}(-N_1) \xrightarrow{\mathrm{ev}_1} \mathcal{K}_0, \quad N_1 \ll N_0,$$

and $\mathcal{P}^{-2} = \mathrm{Ker}(\mathrm{ev}_1) \overset{\varepsilon_2}{\hookrightarrow} \mathcal{P}^{-1}$. Then by Hilbert's syzygy theorem, $\mathcal{P}^{-2}$ is locally free, and

$$\{\mathcal{P}^{-2} \xrightarrow{d^{-2}=\varepsilon_2} \mathcal{P}^{-1} \xrightarrow{d^{-1}=\varepsilon_1 \circ \mathrm{ev}_1} \mathcal{P}^0\} \xrightarrow{\mathrm{ev}_0} \mathcal{F}$$

is a locally free resolution of $\mathcal{F}$. Further, if $N_1 \ll N_0 \ll 0$ are sufficiently negative with respect to $\mathcal{E}$ and $\mathcal{F}$, then the dimensions (denoted by r_{-1}, r_0, r_1) of the terms of the complex (4.3.3) are determined by h, h' and N_0, N_1. For fixed $N_1 \ll N_0 \ll 0$ the locus of $(\mathcal{E}, \mathcal{F})$ for which it is true forms an open substack $U_{N_1,N_0,h,h'}$ in $\mathrm{Coh}^{(h)}(S) \times \mathrm{Coh}^{(h')}(S)$. On $U_{N_1,N_0,h,h'}$, the complex $\mathcal{C}$ is then represented by a complex of vector bundles whose ranks are r_{-1}, r_0, r_1, so it is strictly perfect. Further, as $N_1, N_0 \to -\infty$, the substacks $U_{N_1,N_0,h,h'}$ form an open exhaustion of $\mathrm{Coh}^{(h)}(S) \times \mathrm{Coh}^{(h')}(S)$. This proves (a).

To see (b), we notice that for 0-dimensional $\mathcal{E}$ and $\mathcal{F}$ with given h and h', i.e., with given dimensions of $H^0(S, \mathcal{E})$ and $H^0(S, \mathcal{F})$, one can choose N_0, N_1 in a universal way. ∎

Let now $X \subset \mathrm{Coh}(S)$ be a substack whose points are closed under extensions in $\mathrm{Coh}(S)$. Let $\mathrm{SES}_X \subset \mathrm{SES}$ be the substack which classifies all short exact sequences of coherent sheaves over S which belong to X. We abbreviate $\mathcal{U} = \mathcal{U}_X$, $\mathcal{C} = \mathcal{C}_X$ and $\mathrm{SES} = \mathrm{SES}_X$. Assume further that the complex $\mathcal{C}$ over $X \times X$ is strictly $[-1, 1]$-perfect. Fix a presentation of $\mathcal{C}$ as in Example 2.2.5.

Proposition 4.3.4. *The stack* $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C})$ *is isomorphic to* SES.

Proof. Apply Proposition 2.3.4 with $Y = X \times X \times S$ and $\mathcal{F} = p_{23}^*\mathcal{U}$, $\mathcal{E} = p_{13}^*\mathcal{U}$. ∎

Thus, for all X as above we have the following diagram of f-Artin stacks:

$$X \times X \xleftarrow{\pi} \mathrm{Tot}(\mathcal{C}^{\leq 0}) \overset{i}{\hookleftarrow} \mathrm{SES} \xrightarrow{p} X \tag{4.3.5}$$

with $q = \pi \circ i$, which can be viewed as a refinement of the induction diagram (4.2.2). We call this diagram the *derived induction diagram*.

4.4. The COHA as an algebra

We apply the analysis of §3.3 to all diagrams (5.2.1) as X runs over the set of open substacks of finite type of $\mathrm{Coh}(S)$ such that the complex $\mathcal{C}$ in (4.3.1) is strictly $[-1, 1]$-perfect over $X \times X$. Note that the stack $\mathrm{Coh}(S)$ is covered by all such X's by the proof of Proposition 4.3.2. Since the map p is representable and proper, the pushforward p_* in Borel–Moore homology is well-defined. Hence, we have the maps

$$H_\bullet^{\mathrm{BM}}(X \times X) \xrightarrow{q_{\mathcal{C}}^!} H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{SES}) \xrightarrow{p_*} H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(X),$$

which, by (3.3.5), give rise to the maps

$$H_\bullet^{\mathrm{BM}}(\mathrm{Coh}(S) \times \mathrm{Coh}(S)) \xrightarrow{q_{\mathcal{C}}^!} H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{SES}) \xrightarrow{p_*} H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{Coh}(S)).$$

Composing the maps $q_{\mathcal{C}}^!$, p_* and the exterior product

$$H_\bullet^{\mathrm{BM}}(X) \otimes H_\bullet^{\mathrm{BM}}(X) \to H_\bullet^{\mathrm{BM}}(X \times X),$$

we get the map

$$m : H_\bullet^{\mathrm{BM}}(X) \otimes H_\bullet^{\mathrm{BM}}(X) \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(X), \tag{4.4.1}$$

and, by (3.3.5), the map

$$m : H_\bullet^{\mathrm{BM}}(\mathrm{Coh}(S)) \otimes H_\bullet^{\mathrm{BM}}(\mathrm{Coh}(S)) \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{Coh}(S)).$$

The first main result of this paper is the following theorem. It is proved in the next section.

Theorem 4.4.2. *The map m equips $H_\bullet^{\mathrm{BM}}(X)$ and $H_\bullet^{\mathrm{BM}}(\mathrm{Coh}(S))$ with an associative* **k**-*algebra structure.* ∎

4.5. Proof of associativity

We must prove the associativity of the map m. It is enough to do it for $H_\bullet^{\mathrm{BM}}(X)$. To do that, we consider the Artin stack FILT classifying flags of coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ over S such that the sheaves $\mathcal{E}_{01}$, $\mathcal{E}_{12}$, $\mathcal{E}_{23}$ defined by $\mathcal{E}_{ij} = \mathcal{E}_{0j}/\mathcal{E}_{0i}$ belong to the substack $X \subset \mathrm{Coh}(S)$. For any $i < j$ we introduce a copy X_{ij} of the stack X parametrizing sheaves $\mathcal{E}_{ij}$. For any $i < j < k$ we introduce a copy SES_{ijk} of the stack SES parametrizing short exact sequences

$$0 \to \mathcal{E}_{ij} \to \mathcal{E}_{ik} \to \mathcal{E}_{jk} \to 0.$$

Then we have the fiber diagrams of stacks

$$\begin{array}{ccccc} \mathrm{FILT} & \xrightarrow{x} & \mathrm{SES}_{023} & \xrightarrow{p} & X_{03} \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle q} & & \\ \mathrm{SES}_{012} \times X_{23} & \xrightarrow{p \times 1} & X_{02} \times X_{23} & & \\ \downarrow{\scriptstyle q \times 1} & & & & \\ X_{01} \times X_{12} \times X_{23} & & & & \end{array} \tag{4.5.1}$$

and

$$\begin{array}{ccccc} \mathrm{FILT} & \xrightarrow{v} & \mathrm{SES}_{013} & \xrightarrow{p} & X_{03} \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle q} & & \\ X_{01} \times \mathrm{SES}_{123} & \xrightarrow{1\times p} & X_{01} \times X_{13} & & \\ {\scriptstyle 1\times q}\downarrow & & & & \\ X_{01} \times X_{12} \times X_{23} & & & & \end{array} \tag{4.5.2}$$

given by

$$\begin{aligned} x(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) &= (\mathcal{E}_{02} \subset \mathcal{E}_{03}), \quad & y(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) &= (\mathcal{E}_{01} \subset \mathcal{E}_{02}, \mathcal{E}_{23}), \\ v(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) &= (\mathcal{E}_{01} \subset \mathcal{E}_{03}), \quad & w(\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}) &= (\mathcal{E}_{01}, \mathcal{E}_{12} \subset \mathcal{E}_{13}). \end{aligned}$$

We must prove that

$$p_* \circ q^!_{\mathcal{C}} \circ (p_* \times 1) \circ (q^!_{\mathcal{C}} \times 1) = p_* \circ q^!_{\mathcal{C}} \circ (1 \times p_*) \circ (1 \times q^!_{\mathcal{C}}).$$

Note that the morphisms x, z are both proper and representable and that we have the following equalities of stack homomorphisms:

$$(q \times 1) \circ y = (1 \times q) \circ w, \quad p \circ v = p \circ x.$$

We claim that there are virtual pullback homomorphisms $y^!_{\mathcal{C}}$ and $w^!_{\mathcal{C}}$ such that

$$\begin{aligned} x_* \circ y^!_{\mathcal{C}} &= q^!_{\mathcal{C}} \circ (p_* \times 1), \\ v_* \circ w^!_{\mathcal{C}} &= q^!_{\mathcal{C}} \circ (1 \times p_*), \\ y^!_{\mathcal{C}} \circ (q^!_{\mathcal{C}} \times 1) &= w^!_{\mathcal{C}} \times (1 \times q^!_{\mathcal{C}}). \end{aligned} \tag{4.5.3}$$

The complex $\mathcal{C}_{023} = (p \times 1)^* \mathcal{C}$ on $\mathrm{SES}_{012} \times X_{23}$ and the complex $\mathcal{C}_{013} = (1 \times p)^* \mathcal{C}$ on $X_{01} \times \mathrm{SES}_{123}$ are both strictly $[-1, 1]$-perfect. Since the squares in the diagrams (4.5.1), (4.5.2) are Cartesian, by Proposition 2.3.4 we have stack isomorphisms

$$\begin{aligned} \mathrm{Tot}(\tau_{\leq 0} \mathcal{C}_{023}) &= \mathrm{SES}_{012} \times_{X_{02}} \mathrm{SES}_{023} = \mathrm{FILT}, \\ \mathrm{Tot}(\tau_{\leq 0} \mathcal{C}_{013}) &= \mathrm{SES}_{123} \times_{X_{13}} \mathrm{SES}_{013} = \mathrm{FILT}. \end{aligned}$$

Therefore, we have virtual pullback maps

$$\begin{aligned} y^!_{\mathcal{C}} = y^!_{\mathcal{C}_{023}} &: H^{\mathrm{BM}}_{\bullet}(\mathrm{SES}_{012} \times X_{23}) \to H^{\mathrm{BM}}_{\bullet + 2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{FILT}), \\ w^!_{\mathcal{C}} = w^!_{\mathcal{C}_{013}} &: H^{\mathrm{BM}}_{\bullet}(X_{01} \times \mathrm{SES}_{123}) \to H^{\mathrm{BM}}_{\bullet + 2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{FILT}) \end{aligned}$$

associated with the complexes $\mathcal{C}_{023}$ and $\mathcal{C}_{013}$. Then the first two equations in (4.5.3) follow from the following base change property of virtual pullbacks.

Lemma 4.5.4. *Let B, B' be Artin stacks of finite type, $\mathcal{C}$ be a strictly* $[-1,1]$*-perfect complex on B, and $f : B' \to B$ be a representable and proper morphism of stacks. Then the complex $\mathcal{C}' := f^*\mathcal{C}$ on B' is strictly* $[-1,1]$*-perfect and gives rise to the Cartesian square*

$$\begin{array}{ccc} \mathrm{Tot}(\tau_{\le 0}\mathcal{C}') & \xrightarrow{\;g\;} & \mathrm{Tot}(\tau_{\le 0}\mathcal{C}) \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ B' & \xrightarrow{\;f\;} & B \end{array}$$

Further, we have the following equality of maps:

$$g_* \circ q'^{!}_{\mathcal{C}'} = q^{!}_{\mathcal{C}} \circ f_* : H^{\mathrm{BM}}_{\bullet}(B') \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(\mathrm{Tot}(\tau_{\le 0}\mathcal{C})). \qquad \blacksquare$$

Now, we concentrate on the third equation in (4.5.3). To do this, we first apply Proposition 2.5.2 to the stack homomorphism

$$p : Y = X_{01} \times X_{12} \times X_{23} \times S \to B = X_{01} \times X_{12} \times X_{23}$$

and to the coherent sheaves $\mathcal{E}_{ij} = p_{ij}^*\mathcal{U}$ with $ij = 01, 12, 23$ given by the pullback of the tautological sheaf $\mathcal{U}$ by the obvious projections $Y \to X \times S$. The sheaf $\mathcal{G}$ of associative dg-algebras in (2.5.1) is a strictly $[0,2]$-perfect dg-Lie algebra on B. So, Proposition 2.5.2 yields an equivalence of stacks over B

$$\mathrm{MC}(\mathcal{G}) \simeq \mathrm{FILT}.$$

More precisely, we realize $\mathcal{G}$ as a semidirect product in two ways, $\mathcal{G} = \mathcal{H} \ltimes \mathcal{N} = \mathcal{H}' \ltimes \mathcal{N}'$, where

$$\begin{aligned} \mathcal{N} &= Rp_*\underline{\mathrm{Hom}}(\mathcal{E}_{23}, \mathcal{E}_{01} \oplus \mathcal{E}_{12}), & \mathcal{H} &= Rp_*\underline{\mathrm{Hom}}(\mathcal{E}_{12}, \mathcal{E}_{01}), \\ \mathcal{N}' &= Rp_*\underline{\mathrm{Hom}}(\mathcal{E}_{12} \oplus \mathcal{E}_{23}, \mathcal{E}_{01}), & \mathcal{H}' &= Rp_*\underline{\mathrm{Hom}}(\mathcal{E}_{23}, \mathcal{E}_{12}). \end{aligned}$$

Then the proof of Proposition 2.5.2 yields the following isomorphisms of stacks:

$$\begin{aligned} \mathrm{MC}(\mathcal{H}) &= \mathrm{SES}_{012} \times X_{23}, \\ \mathrm{MC}(\mathcal{H}') &= X_{01} \times \mathrm{SES}_{123}, \\ \mathrm{MC}(\mathcal{G}) &= \mathrm{MC}(\tilde{\mathcal{N}}) = \mathrm{SES}_{012} \times_{X_{02}} \mathrm{SES}_{023} = \mathrm{FILT}, \\ \mathrm{MC}(\mathcal{G}) &= \mathrm{MC}(\tilde{\mathcal{N}}') = \mathrm{SES}_{123} \times_{X_{13}} \mathrm{SES}_{013} = \mathrm{FILT}. \end{aligned}$$

In particular, we can identify the diagram

$$\begin{array}{ccccc} & & \pi^*\mathcal{C}^1_{023} & \xleftarrow{\;s\;} & \mathrm{Tot}(\mathcal{C}^{\le 0}_{023}) \\ & & \big\uparrow & & \big\uparrow \\ \mathrm{SES}_{012} \times X_{23} & \xleftarrow{\;\pi\;} & \mathrm{Tot}(\mathcal{C}^{\le 0}_{023}) & \xleftarrow{\;i\;} & \mathrm{FILT} \end{array} \qquad y : \mathrm{FILT} \to \mathrm{SES}_{012} \times X_{23}$$

with the diagram

$$\begin{array}{ccccc} & & \pi^*_{\tilde{\mathcal{N}}}\tilde{\mathcal{N}}^2 & \xleftarrow{s_{\tilde{\mathcal{N}}}} & \mathrm{Tot}(\tilde{\mathcal{N}}^1)//\tilde{N}^0 \\ & & \uparrow & & \uparrow \\ \mathrm{MC}(\mathcal{H}) & \xleftarrow{\pi_{\tilde{\mathcal{N}}}} & \mathrm{Tot}(\tilde{\mathcal{N}}^1)//\tilde{N}^0 & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \mathrm{MC}(\mathcal{G}) \end{array} \qquad q_{\tilde{\mathcal{N}}} : \mathrm{MC}(\mathcal{G}) \to \mathrm{MC}(\mathcal{H})$$

We deduce that $y^!_{\mathcal{C}} = q^!_{\tilde{\mathcal{N}}}$. Similarly, we get

$$q^!_{\mathcal{C}} \times 1 = q^!_{\mathcal{H}}, \quad w^!_{\mathcal{C}} = q^!_{\tilde{\mathcal{N}}'}, \quad 1 \times q^!_{\mathcal{C}} = q^!_{\mathcal{H}'}.$$

So the third equation in (4.5.3) follows from Proposition 3.4.1. This finishes the proof of Theorem 4.4.2.

4.6. *Chow groups and K-theory versions of COHA*

Given an f-Artin stack B, we denote by $A_\bullet(B)$ its rational Kresch–Chow groups, as in [38]. We denote by $K(B)$ the Grothendieck group of the category of coherent sheaves on B. The construction in §3.3 makes sense as well for $A_\bullet$ and K-theory, yielding virtual pullback morphisms

$$q^!_{\mathcal{C}} : A_\bullet(\mathrm{Coh}(S) \times \mathrm{Coh}(S)) \to A_{\bullet+\mathrm{vrk}(\mathcal{C})}(\mathrm{SES}),$$
$$q^!_{\mathcal{C}} : K(\mathrm{Coh}(S) \times \mathrm{Coh}(S)) \to K(\mathrm{SES}),$$

associated with the complex $\mathcal{C}$ in (4.3.1). Composing them with the pushforward $p_* : A_\bullet(\mathrm{SES}) \to A_\bullet(\mathrm{Coh}(S))$ and $p_* : K(\mathrm{SES}) \to K(\mathrm{Coh}(S))$ by the map p in (4.2.2), we get an associative ring structure on $A_\bullet(\mathrm{Coh}(S))$ and on $K(\mathrm{Coh}(S))$.

A definition of the K-theoretic COHA of finite length coherent sheaves over S was independently proposed along these lines in the recent paper of Zhao [65].

5. Hecke operators

5.1. *Hecke patterns and Hecke diagrams*

We continue to assume that S is a smooth quasi-projective surface over $\mathbb{C}$. Recall that $\mathrm{Coh}(S)$ is the stack of coherent sheaves on S with proper support.

Definition 5.1.1. A *Hecke pattern* for S is a pair (X, Y) of substacks in $\mathrm{Coh}(S)$ with the following properties:

(H1) X is open and Y is closed.

(H2) For any short exact sequence

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0 \qquad (5.1.2)$$

with $\mathcal{G} \in X$ and $\mathcal{F} \in Y$, we have $\mathcal{E} \in X$.

(H3) Y is closed under extensions, i.e., if in (5.1.2) we have $\mathcal{E}, \mathcal{F} \in Y$, then $\mathcal{G} \in Y$.

To a Hecke pattern (X, Y) we associate a version of the induction diagram (4.2.2) which we call the *Hecke diagram*

$$X \times Y \xleftarrow{q} \mathrm{SES}_{XXY} \xrightarrow{p} X. \tag{5.1.3}$$

Here SES_{XXY} is the moduli stack of short exact sequences (5.1.2) with $\mathcal{E}, \mathcal{G} \in X$ and $\mathcal{F} \in Y$, and the projections $q : \mathrm{SES}_{XXY} \to X \times Y$, $p : \mathrm{SES}_{XXY} \to Y$ associate to a sequence (5.1.2) the pair of sheaves $(\mathcal{E}, \mathcal{F})$ and the sheaf $\mathcal{G}$ respectively. We note the following analog of Propositions 4.2.3 and 4.3.4.

Proposition 5.1.4. (a) *The morphism p is schematic and proper.*

(b) *The morphism q identifies* SES_{XXY} *with an open substack in* $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_{XY})$*, where $\mathcal{C}_{XY}$ is the* $[0, 2]$*-perfect complex on $X \times Y$ defined as in* (4.3.1).

Proof. The fiber of p over $\mathcal{G}$ consists of subsheaves $\mathcal{E} \subset \mathcal{G}$ such that $\mathcal{E} \in X$ and $\mathcal{G}/\mathcal{E} \in Y$. Because of the property (H2) we can say that it consists of $\mathcal{E} \subset \mathcal{G}$ such that $\mathcal{G}/\mathcal{E} \in Y$. Since Y is closed in $\mathrm{Coh}(S)$, our fiber is a closed part of the Quot scheme of $\mathcal{G}$, hence proper. Part (a) is proved.

To prove (b), note that, similarly to Proposition 4.3.4, the full $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_{XY})$ is the stack $\mathrm{SES}_{X?Y}$ formed by short exact sequences (5.1.2) with $\mathcal{E} \in X$, $\mathcal{F} \in Y$ but $\mathcal{G}$ being an arbitrary coherent sheaf. Now, SES_{XXY} in the intersection of $\mathrm{SES}_{X?Y}$ with the preimage of $X \subset \mathrm{Coh}(S)$ under the projection to the middle term. Since X is open in $\mathrm{Coh}(S)$, we see that SES_{XXY} is open in $\mathrm{Tot}(\tau_{\leq 0}\mathcal{C}_{XY})$. ∎

5.2. *The derived Hecke action*

Let (X, Y) be a Hecke pattern for S. Denote $\mathcal{H}_X = H^{\mathrm{BM}}_\bullet(X)$ and $\mathcal{H}_Y = H^{\mathrm{BM}}_\bullet(Y)$. From the property (H3) we see, as in Theorem 4.4.2, that the derived induction diagram (5.2.1) for Y makes $\mathcal{H}_Y$ into an associative algebra. Further, similarly to (5.2.1), we have the diagram of f-Artin stacks which we call the *derived Hecke diagram*:

$$X \times Y \xleftarrow{\pi} \mathrm{Tot}(\mathcal{C}^{\leq 0}_{XY}) \xhookleftarrow{i} \mathrm{SES}_{XXY} \xrightarrow{p} X. \tag{5.2.1}$$

Here i identifies SES_{XXY} with an open subset of the zero locus of a section of the vector bundle $\pi^*\mathcal{C}^1_{XY}$ and so gives rise to the virtual pullback $i^!$. So as in §4.4, we define the map

$$\nu : \mathcal{H}_X \otimes \mathcal{H}_Y = H^{\mathrm{BM}}_\bullet(X) \otimes H^{\mathrm{BM}}_\bullet(Y) \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}\,\mathcal{C}_{XY}}(X) = \mathcal{H}_X.$$

Theorem 5.2.2. *The map ν makes $\mathcal{H}_X$ into a right module over the algebra $\mathcal{H}_Y$.*

Proof. Completely similar to that of Theorem 4.4.2. It is based on considering FILT_{XYY}, the stack of flags of coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ with $\mathcal{E}_{01} \in X$ and $\mathcal{E}_{02}/\mathcal{E}_{01}$, $\mathcal{E}_{03}/\mathcal{E}_{02} \in Y$. ∎

5.3. Examples of Hecke patterns

It is a general phenomenon that sheaves with support of lower dimension act, by Hecke operators, on sheaves with support of higher dimension. We consider several refinements of the condition on the dimension of support.

Definition 5.3.1. Let $0 \leq m \leq 2$.

(a) A coherent sheaf $\mathcal{F}$ on S with proper support is called *m-dimensional* if $\dim \operatorname{Supp}(\mathcal{F}) \leq m$. We denote by $\mathrm{Coh}_{\leq m} = \mathrm{Coh}_{\leq m}(S) \subset \mathrm{Coh}$ the substack formed by m-dimensional sheaves.

(b) We say that $\mathcal{F}$ is *purely m-dimensional* if any non-zero $\mathcal{O}_S$-submodule $\mathcal{F}' \subset \mathcal{F}$ is m-dimensional.

(c) We further say that $\mathcal{F}$ is *homologically m-dimensional* if it is m-dimensional and for any $\mathbb{C}$-point $x \in S$ we have $\mathrm{Ext}^j_{\mathcal{O}_S}(\mathcal{O}_x, \mathcal{F}) = 0$ for $0 \leq j < m$. We denote by $\mathrm{Coh}_m = \mathrm{Coh}_m(S) \subset \mathrm{Coh}$ the substack formed by homologically m-dimensional sheaves.

Proposition 5.3.2. (a) *For $m = 0$, the conditions "0-dimensional", "purely 0-dimensional" and "homologically 0-dimensional" are the same.*

(b) *For $m = 1$, the conditions "purely 1-dimensional" and "homologically 1-dimensional" are the same.*

(c) *For $m = 2$, the condition "purely 2-dimensional" is the same as "torsion free" while "homologically 2-dimensional" is the same as "vector bundle".*

Proof. Parts (a) and (b) are obvious, as is the first statement in (c). Let us show the second statement. Notice that condition of being homologically 2-dimensional, i.e., $\mathrm{Ext}^j(\mathcal{O}_x, \mathcal{F}) = 0$ for $j < 2$ and all x, is nothing but the maximal Cohen–Macaulay condition. Since S is assumed to be smooth, any maximal Cohen–Macaulay sheaf is locally free. ∎

We denote by $\mathrm{Coh}_m(S)$ the moduli stack of homologically 2-dimensional sheaves with proper support, and by $\mathrm{Coh}_{\mathrm{tf}}(S)$ denote the moduli stack of torsion free (i.e., purely 2-dimensional) sheaves.

Proposition 5.3.3. *The pairs of substacks* $(\mathrm{Coh}_1(S), \mathrm{Coh}_0(S))$, $(\mathrm{Coh}_2(S), \mathrm{Coh}_1(S))$, $(\mathrm{Coh}_{\mathrm{tf}}(S), \mathrm{Coh}_0(S))$ *and* $(\mathrm{Coh}_{\mathrm{tf}}(S), \mathrm{Coh}_1(S))$ *are Hecke patterns.*

To prove the proposition, we note that $\mathrm{Coh}_1(S)$ and $\mathrm{Coh}_0(S)$ are both open and closed in $\mathrm{Coh}(S)$. Further, $\mathrm{Coh}_2(S)$, the stack of vector bundles, is open, as is $\mathrm{Coh}_{\mathrm{tf}}(S)$. Further, all these stacks are closed under extensions. So it remains to prove the following.

Lemma 5.3.4. *Suppose we have a short exact sequence as in* (5.1.2).

(a) *If $\mathcal{G} \in \mathrm{Coh}_m(S)$ and $\mathcal{F} \in \mathrm{Coh}_{m-1}(S)$, then $\mathcal{E} \in \mathrm{Coh}_m(S)$.*

(b) *If $\mathcal{G} \in \mathrm{Coh}_{\mathrm{tf}}(S)$, then $\mathcal{E} \in \mathrm{Coh}_{\mathrm{tf}}(S)$.*

Proof. (a) Since $\mathcal{E} \subset \mathcal{G}$, it is clear that $\dim \operatorname{Supp}(\mathcal{E}) \leq m$. The vanishing of $\mathrm{Ext}^j(\mathcal{O}_x, \mathcal{E}')$ for $j < m$ follows at once from the long exact sequence of $\mathrm{Ext}^\bullet(\mathcal{O}_x, -)$ induced by the short exact sequence above. Part (b) is obvious: any subsheaf of a torsion free sheaf is torsion free. ∎

This ends the proof of Proposition 5.3.3.

Remark 5.3.5. The non-trivial part of the proposition says that homologically (or, what is the same, purely) 1-dimensional sheaves govern Hecke modifications of vector bundles on a surface.

5.4. Stable sheaves and Hilbert schemes

Let S be a smooth connected projective surface and $m = 0, 1$. We can apply the construction in §4.4 to the substack of m-dimensional sheaves $X = \mathrm{Coh}_{\leq m}(S)$ of $\mathrm{Coh}(S)$. We have the derived induction diagram (5.2.1), hence formula (4.4.1) yields an associative multiplication on $H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_{\leq m}(S))$.

Now, let $P(\mathcal{E}) : m \mapsto \chi(\mathcal{E}(m))$ be the Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on S, and $p(\mathcal{E}) = P(\mathcal{E})/(\text{leading coefficient})$ be the reduced Hilbert polynomial. The sheaf $\mathcal{E}$ is *stable* if it is pure and $p(\mathcal{F}) < p(\mathcal{E})$ for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$. Let $M_S(r, d, n)$ be the moduli space of rank r semistable sheaves with first Chern number d and second Chern number n. See [30] for a general background on these moduli spaces.

Theorem 5.4.1. (a) *The direct image by the closed embeddings* $\mathrm{Coh}_0(S) \subset \mathrm{Coh}_{\leq 1}(S) \subset \mathrm{Coh}(S)$ *gives algebra homomorphisms* $H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(S)) \to H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_{\leq 1}(S)) \to H_\bullet^{\mathrm{BM}}(\mathrm{Coh}(S))$.

(b) *The algebra* $H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_{\leq 1}(S))^{\mathrm{op}}$ *acts on* $\bigoplus_{d,n} H_\bullet^{\mathrm{BM}}(M_S(1, d, n))$.

(c) *The algebra* $H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(S))^{\mathrm{op}}$ *acts on* $\bigoplus_n H_\bullet^{\mathrm{BM}}(M_S(1, d, n))$ *for each* d.

Proof. Part (a) follows from base change. Parts (b), (c) are proved as in §5.2. Let us give more details on (b) part (c) is proved in a similar way.

First, let us consider the following more general setting: Let $X = \mathrm{Coh}(S)$ and $Y \subset \mathrm{Coh}(S)$ be the substack consisting of torsion free sheaves. Note that the substack $Y \subset X$ is both open and stable by subobjects. We claim that the algebra $H_\bullet^{\mathrm{BM}}(X)^{\mathrm{op}}$ acts on $H_\bullet^{\mathrm{BM}}(Y)$. To prove this, we consider the restrictions of $\mathrm{Tot}(\mathcal{C}^{\leq 0})$ and SES to the stack $Y \times X$ given by

$$\mathrm{Tot}(\mathcal{C}^{\leq 0})|_{Y \times X} = \pi^{-1}(Y \times X), \quad \mathrm{SES}|_{Y \times X} = q^{-1}(Y \times X).$$

Then the derived induction diagram (5.2.1) gives rise to the commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Coh}(S) \times \mathrm{Coh}(S) & \xleftarrow{\pi} & \mathrm{Tot}(\mathcal{C}^{\leq 0}) & \xhookleftarrow{i} & \mathrm{SES} & = & \mathrm{SES} & \xrightarrow{p} & \mathrm{Coh}(S) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y \times X & \xleftarrow{\bar{\pi}} & \mathrm{Tot}(\mathcal{C}^{\leq 0})|_{Y \times X} & \xhookleftarrow{\bar{\imath}} & \mathrm{SES}|_{Y \times X} & \xhookleftarrow{j} & \overline{\mathrm{SES}} & \xrightarrow{\bar{p}} & Y
\end{array}$$

where $\overline{\mathrm{SES}} = p^{-1}(Y)$ and j is the obvious open immersion of stacks $j : \overline{\mathrm{SES}} \subset \mathrm{SES}|_{Y \times X}$. Let $\bar{s}_{\mathcal{C}}$ be the restriction of the section $s_{\mathcal{C}}$ of $\pi^*\mathcal{C}^1$ to $Y \times X$. We define a map

$$\bar{m} : H_\bullet^{\mathrm{BM}}(Y) \otimes H_\bullet^{\mathrm{BM}}(X) \to H^{\mathrm{BM}}_{\bullet+2\,\mathrm{vrk}(\mathcal{C})}(Y) \tag{5.4.2}$$

as the composition of the exterior product and the composite map $\bar{p}_* \circ \bar{j}^* \circ \bar{s}^!_{\mathcal{C}} \circ \bar{\pi}^*$. We claim that the map $\bar{m}$ above defines an action of the algebra $H_\bullet^{\mathrm{BM}}(X)^{\mathrm{op}}$ on $H_\bullet^{\mathrm{BM}}(Y)$. Then the diagrams (4.5.1), (4.5.2) yield the following fiber diagrams of stacks:

$$\begin{array}{ccccc}
\overline{\mathrm{FILT}} & \xrightarrow{x} & \overline{\mathrm{SES}} & \xrightarrow{p} & Y \\
\Big\downarrow{\scriptstyle y} & & \Big\downarrow{\scriptstyle q} & & \\
\overline{\mathrm{SES}} \times X & \xrightarrow{p \times 1} & Y \times X & & \\
\Big\downarrow{\scriptstyle q \times 1} & & & & \\
Y \times X \times X & & & &
\end{array} \tag{5.4.3}$$

and

$$\begin{array}{ccccc}
\overline{\mathrm{FILT}} & \xrightarrow{v} & \overline{\mathrm{SES}} & \xrightarrow{p} & Y \\
\Big\downarrow{\scriptstyle w} & & \Big\downarrow{\scriptstyle q} & & \\
Y \times \mathrm{SES} & \xrightarrow{1 \times p} & Y \times X & & \\
\Big\downarrow{\scriptstyle 1 \times q} & & & & \\
Y \times X \times X & & & &
\end{array} \tag{5.4.4}$$

where $\overline{\mathrm{FILT}} \subset \mathrm{FILT}$ is the open substack classifying flags of coherent sheaves $\mathcal{E}_{01} \subset \mathcal{E}_{02} \subset \mathcal{E}_{03}$ over S such that $\mathcal{E}_{01}, \mathcal{E}_{02}, \mathcal{E}_{03} \in Y$. Then the claim is proved as in §4.5, replacing the diagrams (4.5.1), (4.5.2) by (5.4.3), (5.4.4).

Now, a rank 1 coherent sheaf is stable if and only if it is torsion free. Thus, setting $X = \mathrm{Coh}_{\leq 1}(S)$ and $Y \subset \mathrm{Coh}(S)$ to be the substack consisting of rank 1 torsion free sheaves, the argument above proves part (b). ∎

Remark 5.4.5. (a) The moduli space $M_S(1, \mathcal{O}_S, n)$ of rank 1 sheaves with trivial determinant and second Chern number n is canonically isomorphic to the Hilbert scheme $\mathrm{Hilb}^n(S)$. If S is a $K3$ surface, then $\mathrm{Hilb}^n(S)$ is further isomorphic to $M_S(1, 0, n)$.

(b) The rings $A_\bullet(\mathrm{Coh}_{\leq 1}(S))^{\mathrm{op}}$ and $K(\mathrm{Coh}_{\leq 1}(S))^{\mathrm{op}}$ act on

$$\bigoplus_{d,n} A_\bullet(M_S(1, d, n)), \quad \bigoplus_{d,n} K(M_S(1, d, n))$$

respectively, as in Theorem 5.4.1. The proofs are analogous to the proof in Borel–Moore homology.

6. The flat COHA

6.1. $R(\mathbb{A}^2)$ and commuting varieties

In this section we assume $S = \mathbb{A}^2$ and denote by

$$R(\mathbb{A}^2) = H^{\mathrm{BM}}_\bullet(\mathrm{Coh}_0(\mathbb{A}^2))$$

the COHA of 0-dimensional coherent sheaves on $\mathbb{A}^2$. We note that

$$\mathrm{Coh}_0(\mathbb{A}^2) = \bigsqcup_{n\geq 0} \mathrm{Coh}_0^{(n)}(\mathbb{A}^2),$$

where $\mathrm{Coh}_0^n(\mathbb{A}^2)$ is the stack of 0-dimensional sheaves $\mathcal{F}$ such that the *length* of $\mathcal{F}$, i.e., $\dim H^0(\mathcal{F})$, is equal to n. We further recall that

$$\mathrm{Coh}_0^{(n)}(\mathbb{A}^2) \simeq C_n // GL_n,$$

where C_n is the $n \times n$ *commuting variety*

$$C_n = \{(A, B) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})\,;\, [A, B] = 0\},$$

acted upon by GL_n (simultaneous conjugation). Indeed, a 0-dimensional coherent sheaf $\mathcal{F}$ on $\mathbb{A}^2$ of length n is the same as a $\mathbb{C}[x, y]$-module $H^0(\mathcal{F})$ which has dimension n over $\mathbb{C}$, i.e., can be represented by the space $\mathbb{C}^n$ with two commuting operators A, B, the actions of x and y. We recall

Proposition 6.1.1. *C_n is an irreducible variety of dimension $n^2 + n$. Therefore $\mathrm{Coh}_0^{(n)}(\mathbb{A}^2)$ is an irreducible stack of dimension n.* ∎

Accordingly, we have a direct sum decomposition

$$R(\mathbb{A}^2) = \bigoplus_{n\geq 0} R^n(\mathbb{A}^2), \quad R^n(\mathbb{A}^2) = H^{\mathrm{BM}}_\bullet(\mathrm{Coh}_0^{(n)}(\mathbb{A}^2)) = H^{\mathrm{BM}}_\bullet(C_n // GL_n),$$

where on the right we have the equivariant Borel–Moore homology of the topological space C_n. The algebra $R(\mathbb{A}^2)$ has a $\mathbb{Z}^2$-grading (compatible with multiplication), consisting of (in this order)

(a) the *length degree*, by the decomposition into the $\mathcal{H}^{(n)}_{\{x\}}$,

(b) the *homological degree*, where we put H^{BM}_i in degree i.

Define the $\mathbb{Z}^2$-graded vector space

$$\Theta = q^{-1} t \cdot \mathbf{k}[q, t], \quad \deg(q) = (0, -2), \quad \deg(t) = (1, 0). \tag{6.1.2}$$

The following is well-known (see, e.g., [59, §5.3] and the references there) and goes back to the Feit–Fine formula for the number of points in the commuting varieties over finite fields [15, (2)] and the purity of the Borel–Moore homology of the commuting stack $C_n // GL_n$ proved in [11].

Proposition 6.1.3. *As a $\mathbb{Z}^2$-graded vector space, $R(\mathbb{A}^2) \simeq \mathrm{Sym}(\Theta)$.* ■

The goal of this section is to prove the following.

Theorem 6.1.4. (a) *Θ has a natural structure of a graded Lie algebra.*

(b) *$R(\mathbb{A}^2)$ is isomorphic to $U(\Theta)$ as a graded algebra.*

(c) *The symmetrized product map yields a graded vector space isomorphism*

$$\mathrm{Sym}(\Theta) \simeq R(\mathbb{A}^2).$$

Before doing this, let us observe the following.

Proposition 6.1.5. *The algebra $R(\mathbb{A}^2)$ is the same as the COHA considered in* [58, §4.4] *in the case of the Jordan quiver.*

Proof. To prove this, we abbreviate $X_n = C_n//GL_n$, $S = \mathbb{A}^2$, and note that the tautological sheaf $\mathcal{U}$ over $X_n \times S$ is identified with the GL_n-equivariant sheaf over $C_n \times S$ given by $\mathcal{U} = \mathbb{C}^n \otimes \mathcal{O}_{C_n}$, with the $\mathcal{O}_{C_n}$-linear action of $\mathcal{O}_S = \mathbb{C}[x, y]$ such that x, y act as $A \otimes 1$, $B \otimes 1$ respectively on the fiber $\mathcal{U}|_{(A,B)}$. Let $\mathfrak{p}$ be the Lie algebra consisting of (n, m)-upper-triangular matrices in $\mathfrak{gl}_{n+m}$, and let $\mathfrak{u}$, $\mathfrak{l}$ be its nilpotent radical and its standard Levi subalgebra. Let P, U and L be the corresponding linear groups. Write $X_{n,m} = X_n \times X_m$ and $C_{n,m} = C_n \times C_m$. We identify $C_{n,m}$ with the commuting variety of the Lie algebra $\mathfrak{l}$ and $X_{n,m}$ with the moduli stack $C_{n,m}//L$. We have $\mathfrak{u} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m)$, and the perfect $[-1, 1]$-complex $\mathcal{E}$ over $X_{n,m}$ in (4.3.1) is identified with the L-equivariant Koszul complex of vector bundles over $C_{n,m}$ given by

$$\mathfrak{u} \otimes \mathcal{O}_{C_{n,m}} \xrightarrow{d^0} \mathfrak{u}^2 \otimes \mathcal{O}_{C_{n,m}} \xrightarrow{d^1} \mathfrak{u} \otimes \mathcal{O}_{C_{n,m}},$$

where the differentials over the $\mathbb{C}$-point (A, B) in $C_{n,m}$ are given by

$$d^0(u) = ([A, u], [B, u]), \quad d^1(v, w) = [A, w] - [B, v] = [A \oplus v, B \oplus w],$$

and the direct sum holds for the canonical isomorphism $\mathfrak{l} \times \mathfrak{u} \to \mathfrak{p}$. The total space $\mathrm{Tot}(\mathcal{E})$ of this complex, defined in (3.3.9), is a smooth derived stack over $X_{n,m}$ such that:

(a) The underlying Artin stack is the vector bundle stack $\mathcal{E}^0//\mathcal{E}^{-1}$ over $X_{n,m}$ such that

$$\mathcal{E}^{-1} = (C_{n,m} \times \mathfrak{u})//L, \quad \mathcal{E}^0 = (C_{n,m} \times \mathfrak{u}^2)//L.$$

It is isomorphic to the following quotient relative to the diagonal P-action:

$$\mathrm{Tot}(\mathcal{E}^{\leq 0}) = (C_{n,m} \times \mathfrak{u}^2)//P.$$

(b) The structural sheaf of derived algebras is the free P-equivariant graded-commutative $\mathcal{O}_{C_{n,m} \times \mathfrak{u}^2}$-algebra generated by the elements of $\mathfrak{u}^\vee$ in degree -1. The differential is given by the transpose of the Lie bracket $\mathfrak{u} \times \mathfrak{u} \to \mathfrak{u}$.

Therefore, the derived induction diagram (5.2.1) is

$$C_{n,m}/\!/L \overset{\pi}{\to} (C_{n,m}\times \mathfrak{u}^2)/\!/P \overset{i}{\hookleftarrow} \widetilde{C}_{n,m}/\!/P \overset{p}{\to} C_{n+m}/\!/GL_{n+m}, \tag{6.1.6}$$

where $\widetilde{C}_{n,m}$ is the commuting variety of the Lie algebra $\mathfrak{p}$. We can now compare the product

$$m : H_\bullet^{\mathrm{BM}}(X_n)\otimes H_\bullet^{\mathrm{BM}}(X_m)\to H_\bullet^{\mathrm{BM}}(X_{n+m})$$

in (4.4.1) with the multiplication in [58, §4.4]. We have the fiber diagram of stacks

$$\begin{array}{ccccc}
(C_{n,m}\times\mathfrak{u})/\!/P & \xleftarrow{\;f\;} & (C_{n,m}\times\mathfrak{u}^3)/\!/P & \xleftarrow{\;s\;} & (C_{n,m}\times\mathfrak{u}^2)/\!/P \\
{\scriptstyle 1\times 0}\uparrow & & {\scriptstyle 1\times 0}\uparrow & & \uparrow{\scriptstyle i} \\
C_{n,m}/\!/P & \xleftarrow{\;\pi\;} & (C_{n,m}\times\mathfrak{u}^2)/\!/P & \overset{i}{\hookleftarrow} & \widetilde{C}_{n,m}/\!/P
\end{array}$$

where 1 is the identity, 0 is the zero section, f is the projection to the third component of $\mathfrak{u}^3$ (which is a local complete intersection morphism) and $s = 1\times d^1$. Hence, the composite map $g = f\circ s$ is the Lie bracket $(A,B;v,w)\mapsto [A\oplus v, B\oplus w]$ and the composition rule of refined pullback morphisms implies that

$$g^!(x) = s^! f^!(x) = s^!\pi^*(x)$$

in $H_\bullet^{\mathrm{BM}}(\widetilde{C}_{n,m}/\!/P)$ for any class $x\in H_\bullet^{\mathrm{BM}}(X_n\times X_m)$. We deduce that the multiplication map m is the same as the multiplication considered in [58, §4.4]. ∎

6.2. *$R(\mathbb{A}^2)$ as a Hopf algebra*

As a first step in the proof of Theorem 6.1.4, we introduce on $R(\mathbb{A}^2)$ a compatible comultiplication.

Let $U\subset\mathbb{C}^2$ be any open set in the complex analytic topology. We denote by $Coh_0(U)$ the category of 0-dimensional coherent analytic sheaves on U. The corresponding moduli stack $\mathrm{Coh}_0(U)$ can be understood as a complex analytic stack in the sense of [54], i.e., a stack of groupoids on the site of Stein complex analytic spaces. It can also be understood in a more elementary way, as follows.

Let $C_n(U)\subset C_n$ be the open subset (in the complex analytic topology) formed by pairs (A,B) of commuting matrices for which the joint spectrum (the support of the corresponding coherent sheaf on $\mathbb{C}^2$) is contained in U. It is, therefore, a complex analytic space. Then we can define

$$\mathrm{Coh}_0^{(n)}(U) = C_m(U)/\!/GL_n(\mathbb{C}),$$

as the quotient analytic stack, and put

$$\mathrm{Coh}_0(U) = \bigsqcup_{n\geq 0}\mathrm{Coh}_0^{(n)}(U).$$

Using this interpretation, we define directly

$$R(U) = H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(U)) = \bigoplus_{n\geq 0} H_\bullet^{\mathrm{BM}}(C_n(U)//GL_n(\mathbb{C})) = \bigoplus_{n\geq 0} R^n(U).$$

The same considerations as in §4 make $R(U)$ into a graded associative algebra.

If $U' \subset U \subset \mathbb{C}^2$ are two open sets, then $C_n(U') \hookrightarrow C_n(U)$ is an open embedding, and we have maps of $\mathbb{Z}$-graded, resp. $\mathbb{Z}^2$-graded vector spaces

$$\rho^n_{U,U'} : H_\bullet^{\mathrm{BM}}(C_n(U)//GL_n(\mathbb{C})) \to H_\bullet^{\mathrm{BM}}(C_n(U')//GL_n(\mathbb{C})),$$
$$\rho_{U,U'} = \bigoplus_{n\geq 0} \rho^n_{U,U'} : R(U) \to R(U').$$

Proposition 6.2.1. (a) *$\rho_{U,U'}$ is an algebra homomorphism.*

(b) *If the embedding $U' \hookrightarrow U$ is a homotopy equivalence, then $\rho_{U,U'}$ is an isomorphism.*

(c) *If U is a disjoint union of open subsets $U_1, \dots, U_m$, then*

$$R(U) \simeq R(U_1) \otimes \cdots \otimes R(U_n).$$

Proof. Part (a) is clear from definitions. To show (b), we note that $C_n(U)$ and $C_n(U')$ are naturally stratified (by singularities), and, under our assumption, the embedding $C_n(U') \hookrightarrow C_n(U)$ is a homotopy equivalence relative to the stratifications, i.e., it induces homotopy equivalences on all the strata. By dévissage (a spectral sequence argument) this implies that the map

$$H_\bullet^{\mathrm{BM},GL_n(\mathbb{C})}(C_n(U)) = H^{-\bullet}_{GL_n(\mathbb{C})}(C_n(U), \omega_{C_n(U)}) \\ \to H^{-\bullet}_{GL_n(\mathbb{C})}(C_n(U'), \omega_{C_n(U')}) = H_\bullet^{\mathrm{BM},GL_n(\mathbb{C})}(C_n(U'))$$

is an isomorphism.

We abbreviate $GL_{n_1,\dots,n_m} = GL_{n_1} \times \cdots \times GL_{n_m}$. Then part (c) follows from the $GL_n(\mathbb{C})$-equivariant identifications

$$C_n(U) = \bigsqcup_{n_1+\cdots+n_m=n} \big(GL_n(\mathbb{C}) \times_{GL_{n_1,\dots,n_m}(\mathbb{C})} C_{n_1}(U_1) \times \cdots \times C_{n_m}(U_m)\big),$$

which reflect the fact that a length n sheaf $\mathcal{F}$ on U consists of sheaves $\mathcal{F}_i$ on U_i of lengths n_i summing to n. ■

Corollary 6.2.2. *If an open $U \subset \mathbb{C}^2$ is homeomorphic to a 4-ball, then $\rho_{\mathbb{C}^2,U} : R(\mathbb{C}^2) \to R(U)$ is an isomorphism.* ■

Let us now choose, once and for all, two disjoint round balls $U_1, U_2 \subset \mathbb{C}^2$. Define a morphism of $\mathbb{Z}^2$-graded vector spaces $\Delta : R(\mathbb{C}^2) \to R(\mathbb{C}^2) \otimes R(\mathbb{C}^2)$ as the composition

$$R(\mathbb{C}^2) \xrightarrow{\rho_{\mathbb{C}^2, U_1 \sqcup U_2}} R(U_1 \sqcup U_2) \simeq R(U_1) \otimes R(U_2) \xrightarrow{\rho^{-1}_{\mathbb{C}^2,U_1} \otimes \rho^{-1}_{\mathbb{C}^2,U_2}} R(\mathbb{C}^2) \otimes R(\mathbb{C}^2).$$

Proposition 6.2.3. (a) *Δ does not depend on the choice of the disjoint balls U_1, U_2.*

(b) *Δ makes $R(\mathbb{C}^2)$ into a cocommutative Hopf algebra.*

Proof. Any two admissible choices of U_1, U_2 are connected by a path of admissible choices, and Δ does not change along this path. This proves (a). To prove (b), note that all the maps in the above chain are compatible with the Hall multiplication, so Δ is a homomorphism of algebras. Its cocommutativity follows from (a) by interchanging U_1 and U_2, i.e., by connecting (U_1, U_2) and (U_2, U_1) by a path of admissible choices. Coassociativity is proved similarly by considering triples of disjoint balls. This proves that $R(\mathbb{C}^2)$ is a cocommutative bialgebra.

It remains to prove that $R(\mathbb{C}^2)$ has an antipode. This is a standard argument using co-nilpotency (see, e.g., [42, §1.2]). That is, define

$$\overline{\Delta} : R(\mathbb{C}^2) \to R(\mathbb{C}^2) \otimes R(\mathbb{C}^2), \quad \overline{\Delta}(x) = \Delta(x) - (x \otimes 1 + 1 \otimes x),$$

and let $\overline{\Delta}^n : R(\mathbb{C}^2) \to R(\mathbb{C}^2)^{\otimes n}$ be the n-fold iteration of $\overline{\Delta}$. Then $R(\mathbb{C}^2)$ is *conilpotent*, that is, for any $x \in R(\mathbb{C}^2)$ there is n such that $\overline{\Delta}^m(x) = 0$ for $m \geq n$. Therefore the antipode $\alpha : R(\mathbb{C}^2) \to R(\mathbb{C}^2)$ is given by the following geometric series, terminating upon evaluation on any $x \in R(\mathbb{C}^2)$:

$$\alpha = \sum_{n=1}^{\infty} (-1)^n m_n \circ \overline{\Delta}^n,$$

where $m_n : R(\mathbb{C}^2)^{\otimes n} \to R(\mathbb{C}^2)$ is the n-fold multiplication. ■

Let $R(\mathbb{C}^2)_{\mathrm{prim}} = \{a \in R(\mathbb{C}^2)\,;\, \Delta(a) = a \otimes 1 + 1 \otimes a\}$ be the Lie algebra of primitive elements with the bracket $[a, b] = ab - ba$.

Corollary 6.2.4. (a) $R(\mathbb{C}^2)$ *is isomorphic, as a Hopf algebra, to the universal enveloping algebra of* $\mathrm{R}(\mathbb{C}^2)_{\mathrm{prim}}$.

(b) $R(\mathbb{C}^2)_{\mathrm{prim}} \simeq \Theta$ *as a* $\mathbb{Z}^2$*-graded vector space.*

Proof. Part (a) follows from the Milnor–Moore theorem. Part (b) follows from the Poincaré–Birkhoff–Witt theorem and from Proposition 6.1.3. ■

6.3. *Explicit primitive elements in* $R(\mathbb{A}^2)$

For any open $U \subset \mathbb{C}^2$ let $\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(U) \subset \mathrm{Coh}^{(n)}_0(U)$ be the closed analytic substack formed by 1-*point* coherent sheaves, i.e., sheaves whose support consists of precisely one point. In other words,

$$\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(\mathbb{C}^2) = C_{n,1\,\mathrm{pt}}(U)//GL_n(\mathbb{C}),$$

where $C_{n,1\,\mathrm{pt}}(U) \subset C_n(U)$ is the closed analytic subspace formed by pairs (A, B) of commuting matrices whose joint spectrum reduces to one point in $\mathbb{C}^2$ (but can be degenerate). Still more explicitly,

$$C_{n,1\,\mathrm{pt}}(U) = U \times NC_n,$$

where NC_n is the n by n *nilpotent commuting variety*

$$NC_n = \{(A, B) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})\,;\, [A, B] = A^n = B^n = 0\}.$$

In particular, we have the closed subvariety

$$C_{n,1\,\mathrm{pt}} = C_{n,1\,\mathrm{pt}}(\mathbb{C}^2) = \mathbb{C}^2 \times NC_n \subset C_n, \tag{6.3.1}$$

invariant under $GL_n(\mathbb{C})$. We recall

Proposition 6.3.2 ([3]). *NC_n is an irreducible algebraic variety of dimension $n^2 - 1$.* ■

The proposition implies that $C_{n,1\,\mathrm{pt}}$ is an irreducible variety of dimension $n^2 + 1$. So $\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(\mathbb{C}^2)$, its image in $\mathrm{Coh}^{(n)}_0(\mathbb{C}^2)$, is an irreducible stack of dimension 1, and it has the equivariant fundamental class

$$\theta_n = [C_{n,1\,\mathrm{pt}}] \in H_2^{\mathrm{BM}}(C_n//GL_n).$$

Further, let $\mathcal{E}_n$ be the trivial vector bundle of rank n on the GL_n-variety C_n, equipped with the vectorial representation of GL_n. We call $\mathcal{E}_n$ the *tautological sheaf*. Being an equivariant vector bundle, it has the equivariant Chern characters

$$\mathrm{ch}_i(\mathcal{E}_n) \in H^{2i}(C_n//GL_n), \quad i \geq 0,$$

and, for $i \geq 0$, $n \geq 1$, we define

$$\theta_{n,i} = \mathrm{ch}_i(\mathcal{E}_n) \cap \theta_n \in H^{\mathrm{BM}}_{2-2i}(C_n//GL_n) = R^{n,2-2i}(\mathbb{C}^2). \tag{6.3.3}$$

Comparing the $\mathbb{Z}^2$-grading of Θ, we see that the map

$$\alpha : \Theta \to R(\mathbb{C}^2), \quad t^n q^{i-1} \mapsto \theta_{n,i}, \tag{6.3.4}$$

is a morphism of $\mathbb{Z}^2$-graded vector spaces.

Proposition 6.3.5. (a) *α is injective, i.e., each $\theta_{n,i}$ is non-zero.*

(b) *$\theta_{n,i}$ is primitive.*

Proof. The claim (a) follows from [11, Thm. C] and the explicit computations in [11, §5] in the case of the Jordan quiver. More precisely, let Q_g be the quiver with one vertex and g loops. For each integer $n \geq 0$, let $\mathcal{M}(Q_g)_n$ be the coarse moduli space of semisimple n-dimensional representations of $\mathbb{C}Q_g$, i.e., the categorical quotient of $(\mathfrak{gl}_n)^g$ by the adjoint action of GL_n. We will abbreviate $\mathcal{M}(Q_g) = \bigsqcup_{n\geq 0} \mathcal{M}(Q_g)_n$. The direct sum of representations yields a finite morphism $\mathcal{M}(Q_g) \times \mathcal{M}(Q_g) \to \mathcal{M}(Q_g)$, hence a symmetric monoidal structure on the category $\mathrm{Perv}(\mathcal{M}(Q_g))$ of perverse sheaves on $\mathcal{M}(Q_g)$, which allows one to consider the nth symmetric power $\mathrm{Sym}^n(\mathcal{E})$ for any object $\mathcal{E}$ in $\mathrm{Perv}(\mathcal{M}(Q_g))$. Let $\mathrm{Sym}(\mathcal{E}) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathcal{E})$. Set $g = 3$ and fix an embedding $Q_2 \subset Q_3$. By [11], we have

$$\bigoplus_{n\geq 0} H_c^\bullet(C_n//GL_n) = H_c^\bullet\big(\mathcal{M}(Q_3), \mathrm{Sym}(\mathcal{BPS} \otimes H_c^\bullet(B\mathbb{C}^\times))\big)$$

$$\bigoplus_{n\geq 0} H_c^\bullet(C_{n,1\,\mathrm{pt}}//GL_n) = H_c^\bullet\big(\mathcal{M}(Q_3)_{1\,\mathrm{pt}}, \mathrm{Sym}(\mathcal{BPS} \otimes H_c^\bullet(B\mathbb{C}^\times))\big) \tag{6.3.6}$$

where $\mathcal{BPS} = \bigoplus_{n>0} \mathcal{BPS}_n$ and $\mathcal{BPS}_n$ is, up to some shift, the constant sheaf supported on the small diagonal $\mathbb{C}^3 \subset \mathcal{M}(Q_3)_n$. For each n, the closed subset $\mathcal{M}(Q_3)_{n,1\,\mathrm{pt}} \subset \mathcal{M}(Q_3)_n$ is the coarse moduli space of semisimple representations of $\mathbb{C}Q_3$ for which the underlying $\mathbb{C}Q_2$-module has a punctual support in $\mathbb{C}^2$. We have

$$\mathbb{C}^3 \subset \mathcal{M}(Q_3)_{n,1\,\mathrm{pt}} \subset \mathcal{M}(Q_3)_n.$$

Taking the direct summand in (6.3.6)

$$\mathcal{BPS}_n \otimes H_c^\bullet(B\mathbb{C}^\times) \subset \mathrm{Sym}\left(\mathcal{BPS} \otimes H_c^\bullet(B\mathbb{C}^\times)\right),$$

we get the commutative diagram

$$\begin{array}{ccc} H_c^\bullet\left(\mathcal{M}(Q_3)_n, \mathcal{BPS}_n \otimes H_c^\bullet(B\mathbb{C}^\times)\right)^* & \hookrightarrow & H_\bullet^{\mathrm{BM}}(C_n//GL_n) \\ f\uparrow & & h\uparrow \\ H_c^\bullet\left(\mathcal{M}(Q_3)_{n,1\,\mathrm{pt}}, \mathcal{BPS}_n \otimes H_c^\bullet(B\mathbb{C}^\times)\right)^* & \hookrightarrow & H_\bullet^{\mathrm{BM}}(C_{n,1\,\mathrm{pt}}//GL_n) \end{array}$$

The map f is invertible, and h is the pushforward by the closed embedding $C_{n,1\,\mathrm{pt}} \subset C_n$. We deduce that the class $\mathrm{ch}_i(\mathcal{E}_n) \cap [C_{n,1\,\mathrm{pt}}]$ is non-zero in $H_{2-2i}^{\mathrm{BM}}(C_{n,1\,\mathrm{pt}}//GL_n)$ and that its image by h is non-zero. This image is equal to the class $\theta_{n,i}$.

To prove (b), given an open $U \subset \mathbb{C}^2$, we define, in the same way as before, elements

$$\theta_{n,i}(U) \in R^{n,2-2i}(U) = H_{2-2i}^{\mathrm{BM}}(C_n(U)//GL_n(\mathbb{C})).$$

For $U' \subset U$ we have

$$\rho_{U,U'}(\theta_{i,n}(U)) = \theta_{n,i}(U').$$

For $U = U_1 \sqcup U_2$ being a disjoint union of two opens, a length n 0-dimensional sheaf $\mathcal{F}$ on U consists of two sheaves $\mathcal{F}_i$ on U_i of lengths n_i, $i = 1, 2$, such that $n_1 + n_2 = n$. This can be expressed by saying that

$$C_n(U_1 \sqcup U_2) = \bigsqcup_{n_1+n_2=n} \left(GL_n(\mathbb{C}) \times_{GL_{n_1,n_2}(\mathbb{C})} \left(C_{n_1}(U_1) \times C_{n_2}(U_2)\right)\right), \qquad (6.3.7)$$

from which we deduce the identification

$$R^n(U) = \bigoplus_{n_1+n_2=n} R^{n_1}(U_1) \otimes R^{n_2}(U_2). \qquad (6.3.8)$$

Let $\mathcal{E}_{n,U}$ be the tautological sheaf of $C_n(U)$, and similarly for U_1, U_2. With respect to the identification (6.3.7), we have

$$\mathcal{E}_{n,U} = \bigsqcup_{n_1+n_2=n} (\mathcal{E}_{n_1,U_1} \boxtimes \mathcal{O} \oplus \mathcal{O} \boxtimes \mathcal{E}_{n_2,U_2}).$$

Thus, the additivity of the Chern character gives

$$\mathrm{ch}_i(\mathcal{E}_{n,U}) = \sum_{n_1+n_2=n} \left(\mathrm{ch}_i(\mathcal{E}_{n_1,U_1}) \otimes 1 + 1 \otimes \mathrm{ch}_i(\mathcal{E}_{n_2,U_2})\right), \quad \forall i \geq 0. \qquad (6.3.9)$$

Since, under the identification (6.3.8), we have

$$\theta_n(U) = \theta_n(U_1) \otimes 1 + 1 \otimes \theta_n(U_2),$$

we deduce that also

$$\theta_{n,i}(U) = \theta_{n,i}(U_1) \otimes 1 + 1 \otimes \theta_{n,i}(U_2), \quad \forall i \geq 0.$$

Our statement follows from this and from the definition of Δ via ρ. ■

Corollary 6.3.10. *The space $R(\mathbb{C}^2)_{\mathrm{prim}}$ of primitive elements of $R(\mathbb{C}^2)$ coincides with the image $\alpha(\Theta)$. It is closed under the commutator $[a,b] = ab - ba$.* ■

Theorem 6.1.4 is proved. The symmetrized product map $\sigma : \mathrm{Sym}(\Theta) \to R(\mathbb{A}^2)$ is considered in detail in (7.1.5) below.

7. The COHA of a surface S and factorization homology

7.1. Statement of results

Let S be an arbitrary smooth quasi-projective surface and $R(S) = H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(S))$ be the corresponding cohomological Hall algebra. It is $\mathbb{Z}^2$-graded by (length, homological degree). We introduce a global analog of the space Θ generating the flat COHA $R(\mathbb{A}^2)$ from §6.3. Let

$$S \xleftarrow{p_n} \mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(S) \xrightarrow{i_n} \mathrm{Coh}^{(n)}_0(S) \tag{7.1.1}$$

be the stack of 1-pointed, length n sheaves on S with its canonical closed embedding i_n into $\mathrm{Coh}^{(n)}_0(S)$ and projection p_n to S (so $p_n(\mathcal{F})$ is the unique support point of $\mathcal{F}$). Proposition 6.3.2 implies that p_n is a morphism with all fibers irreducible of relative dimension -1 and therefore $\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(S)$ is irreducible of dimension $+1$. Moreover, we have a natural fundamental class in $H_2^{\mathrm{BM}}(\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(S))$ constructed as follows.

We consider the open subscheme $\mathrm{FCoh}^{(n)}_0(S) := \mathit{Quot}^{(n,0)}(S)$ of the quot-scheme formed by the equivalence classes of surjections $\phi : \mathcal{O}^n \to \mathcal{F}$ with $\mathcal{F} \in \mathrm{Coh}^{(n)}_0(S)$ such that ϕ induces an isomorphism $\mathbb{C}^n \to H^0(S, \mathcal{F})$. Let $\mathrm{FCoh}^{(n)}_{1\,\mathrm{pt}}(S) \subset \mathrm{FCoh}^{(n)}_0(S)$ be the closed subscheme formed by the equivalence classes of ϕ such that $\mathcal{F}$ is a 1-pointed sheaf. Then the stack $\mathrm{Coh}^{(n)}_0(S)$ is isomorphic to the quotient stack $\mathrm{FCoh}^{(n)}_0(S)//GL_n$, and $\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(S)$ is isomorphic to the quotient stack $\mathrm{FCoh}^{(n)}_{1\,\mathrm{pt}}(S)//GL_n$. Further, we have the projection $\mathrm{FCoh}^{(n)}_{1\,\mathrm{pt}}(S) \to S$ with fibers being identified with the variety NC_n of pairs of nilpotent commuting matrices (see §6.3). Since this variety is irreducible of dimension $n^2 - 1$, the scheme $\mathrm{FCoh}^{(n)}_{1\,\mathrm{pt}}(S)$ is an irreducible variety of dimension $n^2 + 1$ and has the fundamental class in $H^{\mathrm{BM}}_{2(n^2+1)}(\mathrm{FCoh}^{(n)}_{1\,\mathrm{pt}}(S))$. So the quotient stack by GL_n has the fundamental class in $H_2^{\mathrm{BM}}(\mathrm{Coh}^{(n)}_{1\,\mathrm{pt}}(S))$.

Therefore we have the pullback map $p_n^\dagger$ given by the composition

$$H_\bullet^{\mathrm{BM}}(S) = H^{4-\bullet}(S) \xrightarrow{p_n^*} H^{4-\bullet}(\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(S)) \to H_{\bullet-2}^{\mathrm{BM}}(\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(S)), \tag{7.1.2}$$

where the last arrow is the cap-product with the fundamental class of $\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(S)$.

Define the subspace

$$\Theta_n(S) = i_{n*} p_n^\dagger H_\bullet^{\mathrm{BM}}(S) \subset H_{\bullet-2}^{\mathrm{BM}}(\mathrm{Coh}_0^{(n)}(S)) = R^n(S).$$

Let $\mathcal{E}_n$ denote also the tautological sheaf on $\mathrm{Coh}_0^{(n)}(S)$ and further put, for $i \geq 0$,

$$\Theta_{n,i}(S) = \Theta_n(S) \cap \mathrm{ch}_i(\mathcal{E}_n) \subset R^n(S).$$

Proposition 7.1.3. *The canonical map $H_\bullet^{\mathrm{BM}}(S) \to \Theta_{n,i}(S)$ is an isomorphism.*

Proof. As before, we use the subscheme $\mathrm{FCoh}_0^{(n)}(S)$ whose quotient stack by GL_n is $\mathrm{Coh}_0^{(n)}(S)$. Let $T \subset GL_n$ be a maximal torus. Then the fixed points locus $\mathrm{FCoh}_0^{(n)}(S)^T$ is isomorphic to $\mathrm{FCoh}_0^{(1)}(S)^n = S^n$. Thus, we have a commutative diagram

$$\begin{array}{ccccc}
H_\bullet^{\mathrm{BM}}(S) & \xrightarrow{p_n^*} & H_\bullet^{\mathrm{BM},GL_n}(\mathrm{FCoh}_{1\,\mathrm{pt}}^{(n)}(S))_{\mathrm{loc}} & \xrightarrow{i_{n*}} & H_\bullet^{\mathrm{BM},GL_n}(\mathrm{FCoh}_0^{(n)}(S))_{\mathrm{loc}} \\
& \searrow^{a} & \uparrow b & & \uparrow c \\
& & H_\bullet^{\mathrm{BM}}(S) \otimes H_{GL_n,\mathrm{loc}}^\bullet & \xhookrightarrow{\Delta} & (H_\bullet^{\mathrm{BM}}(S^n) \otimes H_T^*)_{\mathrm{loc}}^{\mathfrak{S}_n}
\end{array}$$

where $H_G^\bullet = H^\bullet(BG)$ and the subscript “loc” means the tensor product by the fraction field $H_{GL_n,\mathrm{loc}}^\bullet$ of $H_{GL_n}^\bullet$ over $H_{GL_n}^\bullet$. The maps b, c are the pushforward by the closed embeddings $S \subset \mathrm{FCoh}_{1\,\mathrm{pt}}^{(n)}(S)$ and $S^n \subset \mathrm{FCoh}_0^{(n)}(S)$, which are invertible by the localization theorem in equivariant cohomology. The map Δ is the diagonal embedding. It is injective. The map a is equal to $\mathrm{Id} \otimes 1$, up to the cap-product by an invertible element in $H^\bullet(S) \otimes H_{GL_n,\mathrm{loc}}^\bullet$. It is injective. We deduce that the map

$$i_{n*} p_n^* : H_\bullet^{\mathrm{BM}}(S) \to H_\bullet^{\mathrm{BM},GL_n}(\mathrm{FCoh}_{1\,\mathrm{pt}}^{(n)}(S))$$

is injective as well. ∎

We define

$$\Theta(S) = \bigoplus_{n,i} \Theta_{n,i}(S) \subset R(S).$$

Thus, for $S = \mathbb{A}^2$ we see that $\Theta(\mathbb{A}^2)$ is identified with the graded space Θ from (6.1.2), embedded into R by the map α as in (6.3.4). We recall that $H_\bullet^{\mathrm{BM}}(\mathbb{A}^2)$ is 1-dimensional, concentrated in homological degree 4. Thus shifting the grading by putting

$$\Theta' = \Theta[0,-4] = qt \cdot \mathbf{k}[q,t], \tag{7.1.4}$$

we have, by Proposition 7.1.3, an identification of $\mathbb{Z}^2$-graded vector spaces

$$\Theta(S) \simeq H^{\mathrm{BM}}_{\bullet}(S) \otimes \Theta' \simeq H^{\mathrm{BM}}_{\bullet}(S//\mathbb{C}^{\times}) \otimes t\mathbf{k}[t].$$

We now consider the symmetrized product map $\sigma = \sigma : \mathrm{Sym}(\Theta(S)) \to R(S)$ defined as

$$\sigma = \sum_{n\geq 0} \sigma_n, \quad \sigma_n : \mathrm{Sym}^n(\Theta(S)) \to R(S), \quad \sigma_n(v_1 \bullet \cdots \bullet v_n) = \frac{1}{n!} \sum_{s \in S_n} v_{s(1)} * \cdots * v_{s(n)}. \tag{7.1.5}$$

Here $\bullet$ is the product in the symmetric algebra and $*$ is the Hall multiplication. The second main result of this paper is a version of the Poincaré–Birkhoff–Witt theorem for $R(S)$ which allows us to compute its graded dimension. It is proved in the next sections.

Theorem 7.1.6. *The map $\sigma : \mathrm{Sym}(\Theta(S)) \to R(S)$ is an isomorphism of $\mathbb{Z}^2$-graded vector spaces.* ∎

7.2. *Reminder on factorization algebras*

We follow the approach of [9,21]. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal model category. In particular, it has a class W of weak equivalences. We will consider three examples:

(a) $\mathcal{C} = \mathrm{Top}$ is the category of topological spaces (homotopy equivalent to a CW-complex), $\otimes$ is Cartesian product, and weak equivalences have the usual topological meaning.

(b) $\mathcal{C}$ is the category of Artin stacks, $\otimes$ is the Cartesian product of stacks and weak equivalences are equivalences of stacks.

(c) $\mathcal{C} = \mathrm{dgVect}$ is the category of cochain complexes, $\otimes$ is the usual tensor product and weak equivalences are quasi-isomorphisms.

Let M be a C^{∞} manifold of dimension n.

Definition 7.2.1. A *prefactorization algebra* on M valued in $\mathcal{C}$ is a rule $\mathcal{A}$ which associates

(a) to any open set $U \subset M$ an object $\mathcal{A}(U) \in \mathcal{C}$, so that $\mathcal{A}(\emptyset) = \mathbf{1}$;

(b) to any system $U_1, \ldots, U_p$ of disjoint open sets contained in an open set U_0 a morphism $\mu^{U_0}_{U_1,\ldots,U_p} : \mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_p) \to \mathcal{A}(U_0)$, such that the morphisms $\mu^{U_0}_{U_1,\ldots,U_p}$ satisfy associativity.

A *morphism of prefactorization algebras* $\sigma : \mathcal{A} \to \mathcal{A}'$ is a datum of morphisms $\sigma_U : \mathcal{A}(U) \to \mathcal{A}'(U)$ compatible with the structures. It is a *weak equivalence* if each σ_U is a weak equivalence.

A prefactorization algebra is, in particular, a *precosheaf* via the maps $\mu^{U_0}_{U_1}$, i.e., it is a covariant functor from the category of open subsets in M to $\mathcal{C}$.

Definition 7.2.2. An open covering of M is called a *Weiss covering* if any finite subset of M is contained in an open set of the covering.

Example 7.2.3. (a) Let $D \subset \mathbb{R}^n$ be the standard unit disk $\|x\| < 1$. A *disk* in M is an open subset which is homeomorphic to D. The open covering $\mathfrak{D}(M)$ of M generated by the disks of M is a Weiss covering. By definition, every open subset from $\mathfrak{D}(M)$ consists of a finite disjoint union of disks.

(b) A prefactorization algebra is called *locally constant* if for any inclusion of disks $U_0 \subset U_1$ the map $\mu_{U_1}^{U_0}$ is a weak equivalence.

Definition 7.2.4. (a) A prefactorization algebra $\mathcal{A}$ is called a (*homotopy*) *factorization algebra* if:

(a1) For any Weiss covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of any open set $U \subset M$ the natural morphism

$$\underrightarrow{\operatorname{holim}}\ \mathcal{N}_\bullet(\mathfrak{U}, \mathcal{A}) \to \mathcal{A}(U),$$

$$\mathcal{N}_\bullet(\mathfrak{U}, \mathcal{A}) := \Big\{ \cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} \coprod_{i,j,k \in I} \mathcal{A}(U_{ijk}) \overset{\rightarrow}{\rightrightarrows} \coprod_{i,j \in I} \mathcal{A}(U_{ij}) \rightrightarrows \coprod_{i \in I} \mathcal{A}(U_i) \Big\},$$

with $U_{ij} = U_i \cap U_j$, etc., is a weak equivalence (*codescent*).

(a2) $\mu_{U_1,\ldots,U_p}^{U_0}$ is a weak equivalence for any system $U_0, \ldots, U_p$ of open sets with $U_0 = U_1 \sqcup \cdots \sqcup U_p$ (*multiplicativity*).

(b) The *factorization homology* of M with coefficients in a factorization algebra $\mathcal{A}$ is the space of global cosections of $\mathcal{A}$, viewed as an object of $\mathcal{C}$, which we denote

$$\int_M \mathcal{A} = \mathcal{A}(M) \in \mathcal{C}.$$

Remark 7.2.5. (a) A multiplicative prefactorization algebra $\mathcal{A}$ is a factorization algebra if and only if for the particular Weiss covering $\mathfrak{D}(U)$ of any open subset $U \subset M$, the object $\mathcal{A}(U)$ is the homotopy colimit of the diagram

$$\coprod_{U_1, U_2 \in \mathfrak{D}(U)} \mathcal{A}(U_1 \cap U_2) \rightrightarrows \coprod_{U_1 \in \mathfrak{D}(U)} \mathcal{A}(U_1).$$

In particular, we have

$$\int_M \mathcal{A} = \underrightarrow{\operatorname{holim}}_{U \in \mathfrak{D}(M)} \mathcal{A}(U).$$

See [9, §A.4.3] for details.

(b) Any locally constant prefactorization algebra has a unique extension as a locally constant factorization algebra taking the same value on any disk, but possibly different values on other open sets [21, Rem. 24].

Sometimes it is convenient to use the dual language. By a *(pre)factorization coalgebra* $\mathcal{B}$ in $\mathcal{C}$ we will mean a (pre)factorization algebra in $\mathcal{C}^{\mathrm{op}}$. Thus, we have maps

$$\nu_{U_0}^{U_1,\ldots,U_p} : \mathcal{B}(U_0) \to \mathcal{B}(U_1) \otimes \cdots \otimes \mathcal{B}(U_p)$$

yielding a presheaf on M. For a factorization coalgebra $\mathcal{B}$ we have the *factorization cohomology* which we denote as

$$\oint_M \mathcal{B} = \mathcal{B}(M) = \underset{U\in\mathfrak{D}(M)}{\underleftarrow{\operatorname{holim}}} \mathcal{B}(U).$$

Let us record the following two statements for later use.

Proposition 7.2.6. *If $\mathcal{F}$ is a locally constant sheaf of* $\mathbf{k}$*-dg-vector spaces, then* $\operatorname{Sym}(\mathcal{F}) : U \mapsto \operatorname{Sym}_{\mathbf{k}}(\mathcal{F}(U))$ *is a locally constant factorization coalgebra.*

Note that $\operatorname{Sym}(\mathcal{F})$, as we define it, is not the same as the symmetric algebra of $\mathcal{F}$ in the symmetric monoidal category of sheaves of (dg-)vector spaces; in fact, it is not a sheaf in the usual sense.

Proof of Proposition 7.2.6. This is an analog of [9, Thm. 5.2.1] which deals with sheaves corresponding to C^∞ sections of vector bundles, and their symmetric products in the sense of bornological vector spaces. In our case the proof is similar but easier due to the absence of analytic difficulties. That is, call a covering $\mathfrak{U}$ an n-Weiss covering if each subset $I \subset M$ of cardinality $\leq n$ is contained in one of the opens of $\mathfrak{U}$. Then it suffices to show that $\operatorname{Sym}^n(\mathcal{F}) : U \mapsto \operatorname{Sym}^n_{\mathbf{k}}(\mathcal{F}(U))$ satisfies descent for n-Weiss coverings. This follows, as in the proof of [9, Thm. 5.2.1], from the fact that $\mathcal{F}^{\boxtimes n}$ is a sheaf of M^n. ∎

Proposition 7.2.7. *Let $\sigma : \mathcal{B} \to \mathcal{B}'$ be a morphism of factorization coalgebras. Suppose that for any disk $U \subset M$ the morphism $\sigma_U : \mathcal{B}(U) \to \mathcal{B}(U')$ is a weak equivalence. Then σ is a weak equivalence of factorization coalgebras; in particular, σ induces a weak equivalence $\sigma_M : \oint_M \mathcal{B} \to \oint_M \mathcal{B}'$.*

Proof. For any open U we realize σ_U by descent from the Weiss cover $\mathfrak{D}(U)$. ∎

7.3. Analytic stacks

For the analytic version of the theory of algebraic stacks we follow [54] (where, in fact, the case of higher and derived stacks is also considered).

An *analytic stack* is a stack of groupoids on the category of (possibly singular) Stein analytic spaces over $\mathbb{C}$, equipped with the Grothendieck topology consisting of open covers in the usual sense. The analytic stacks form a 2-category $\mathfrak{Stan}$ as well as a model category *Stan* where weak equivalences are equivalences of stacks.

For every scheme Y of finite type over $\mathbb{C}$ we have the analytic space Y^{an}, the analytification of Y. Further, or any Artin stack X over $\mathbb{C}$ we have the analytic stack X^{an}, the analytification X^{an} defined as the Kan extension of X from the category of affine schemes of finite type to the category of Stein analytic spaces; see [54, §4] or [29, §3]. Note that we have a canonical map

$$\eta_X : R\Gamma(X, \omega_X) \to R\Gamma(X^{\mathrm{an}}, \omega_{X^{\mathrm{an}}}). \tag{7.3.1}$$

If $X = Y$ is a scheme of finite type considered as an Artin stack, then $X^{\mathrm{an}} = Y^{\mathrm{an}}$ is the corresponding analytic space considered as an analytic stack.

We will need only analytic stacks of a special form, namely the *quotient analytic stacks* $Z//K$, where Z is an analytic space and K is a complex Lie group. For such stacks various concepts such as Borel–Moore homology, etc., can be defined directly in terms of equivariant homology of the topological spaces of $\mathbb{C}$-points, that is,

$$H^\bullet(Z//K, \omega_{Z//K}) = H^{\mathrm{BM},K}_{-\bullet}(Z(\mathbb{C}), \mathbb{C}) \tag{7.3.2}$$

is the equivariant Borel–Moore homology in the topological sense.

If Y is a scheme of finite type over $\mathbb{C}$ and G is an algebraic group over $\mathbb{C}$ acting on Y, then G^{an} is a complex Lie group and we have the quotient analytic stack $Y^{\mathrm{an}}//G^{\mathrm{an}}$. Note that in this case we have

$$(Y//G)^{\mathrm{an}} \simeq Y^{\mathrm{an}}//G^{\mathrm{an}}, \tag{7.3.3}$$

and the map $\eta_{Y//G}$ is a quasi-isomorphism, so that

$$H^\bullet(R\Gamma((Y//G)^{\mathrm{an}}, \omega_{(Y//G)^{\mathrm{an}}})) \simeq H^\bullet(R\Gamma(Y//G, \omega_{Y//G})) \simeq H^{\mathrm{BM},G(\mathbb{C})}_{-\bullet}(Y(\mathbb{C}), \mathbb{C}) \tag{7.3.4}$$

is the equivariant Borel–Moore homology in the topological sense, as above.

7.4. *The stack* Coh$_0$ *and factorization algebras*

Let Σ be a complex analytic surface. We view it as a C^∞ manifold of dimension 4 and consider open subsets $U \subset \Sigma$ in the complex analytic topology. For any such non-empty U we have the category $\mathit{Coh}_0(U)$ of 0-dimensional coherent sheaves on U (with finite support). We set $\mathit{Coh}_0(\emptyset) = \{\bullet\}$. We also have the analytic moduli stack $\mathrm{Coh}_0(U) = \bigsqcup_{n\geq 0} \mathrm{Coh}_0^{(n)}(U)$ parametrizing objects of $\mathit{Coh}_0(U)$, with its components given by the length, as in the algebraic case. Each component is explicitly realized as a quotient analytic stack

$$\mathrm{Coh}_0^{(n)}(U) = \mathrm{FCoh}_0^{(n)}(U)//GL_n(\mathbb{C}),$$

where $\mathrm{FCoh}_0^{(n)}(U)$ is the analytic space parametrizing pairs $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is a 0-dimensional coherent sheaf on U and $\phi : \mathbb{C}^n \to H^0(U, \mathcal{F})$ is an isomorphism. To see that $\mathrm{FCoh}_0^{(n)}(U)$ is well-defined as an analytic space, we note that the datum of ϕ is equivalent to the datum of the corresponding surjection $\underline{\phi} : \mathcal{O}_U^{\oplus n} \to \mathcal{F}$. Thus $\mathrm{FCoh}_0^{(n)}(U)$ is a locally closed analytic subspace in $\mathcal{Q}uot^{(n)}(\mathcal{O}_U^{\oplus n})$, the analytic analog of the Grothendieck Quot scheme parametrizing all length n quotients of $\mathcal{O}_U^{\oplus n}$. This makes the following fact clear.

Proposition 7.4.1. *Let S be a smooth quasi-projective algebraic surface over $\mathbb{C}$. Then we have an equivalence of analytic stacks*

$$\mathrm{Coh}_0(S^{\mathrm{an}}) \simeq \mathrm{Coh}_0(S)^{\mathrm{an}}.$$

In particular, $\mathrm{Coh}_0(\mathbb{C}^2)$ *is identified with the analytification of the Artin stack* $\mathrm{Coh}_0(\mathbb{A}^2)$.

■

If $U_1, \dots, U_n$ are disjoint open sets contained in the open subset $U_0 \subset \Sigma$, then we have an open embedding of analytic stacks

$$\alpha^{U_0}_{U_1,\dots,U_n} : \mathrm{Coh}_0(U_1) \times \cdots \times \mathrm{Coh}_0(U_n) \to \mathrm{Coh}_0(U_0). \tag{7.4.2}$$

Proposition 7.4.3. Coh_0 *is a factorization algebra on* Σ *with values in the category Stan.*

Proof. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a Weiss open cover of U. Let us understand more explicitly the analytic stack $\underrightarrow{\mathrm{holim}}\ \mathcal{N}_\bullet(\mathfrak{U}, \mathrm{Coh}_0)$, a homotopy colimit in the model category *Stan*, or equivalently the 2-categorical colimit of $\mathcal{N}_\bullet(\mathfrak{U}, \mathrm{Coh}_0)$ in the 2-category $\mathfrak{Stan}$. It is parametrized by pairs $(i \in I, \mathcal{F} \in Coh_0(U_i))$, the leftmost term in the diagram $\mathcal{N}_\bullet(\mathfrak{U}, \mathrm{Coh}_0)$, subject to coherent systems of identifications given by the rest of the diagram. These identifications say that two pairs $(i \in I, \mathcal{F} \in Coh_0(U_i))$ and $(j \in J, \mathcal{F} \in Coh_0(U_j))$ are identified whenever in the second pair $\mathcal{F}$ is *the same sheaf but living on* U_j. This happens whenever $\mathcal{F}$ lives in fact on $U_{ij} = U_i \cap U_j$. Further terms in the diagram $\mathcal{N}_\bullet(\mathfrak{U}, \mathrm{Coh}_0)$ impose coherence conditions on such identifications. This means that this homotopy colimit parametrizes 0-dimensional coherent sheaves which live *on some* U_i. But $\mathfrak{U}$ is a Weiss cover and every $\mathcal{F} \in Coh_0(U)$ has finite support, which therefore, must lie in some U_i. Thus, our homotopy colimit is identified with $\mathrm{Coh}_0(U)$. ∎

7.5. *Chain-level COHA as a factorization coalgebra*

For each open set $U \subset \Sigma$ as above we consider the complex of Borel–Moore chains of $\mathrm{Coh}_0(U)$

$$\mathcal{R}(U) = C^{\mathrm{BM}}_\bullet(\mathrm{Coh}_0(U)) := R\Gamma(\mathrm{Coh}_0(U), \omega_{\mathrm{Coh}_0(U)}).$$

Proposition 7.5.1. *The assignment* $\mathcal{R} : U \mapsto \mathcal{R}(U)$ *is a locally constant factorization coalgebra on* S *in the category* $\mathrm{C}(\mathrm{Vect}_{\mathbf{k}})$ (*complexes of* $\mathbf{k}$*-vector spaces*).

Proof. The fact that $\mathcal{R}$ is a factorization algebra follows from Proposition 7.4.3. The fact that $\mathcal{R}$ is locally constant is proved in the same way as Proposition 6.2.1 (b). ∎

Next, we upgrade this statement to take into account the Hall multiplication. The relevant concept here is that of a *homotopy associative* (E_1-)algebra which we now recall. We will use the language of operads; see, e.g., [9] for a brief background and additional references.

Definition 7.5.2. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category.

(a) An *operad* $\mathcal{P}$ in $\mathcal{C}$ is a system of

- (O1) objects $\mathcal{P}(r) \in \mathcal{C}$ with actions of S_r, given for $r \geq 0$,
- (O2) the unit morphism $\mathbf{1} \to \mathcal{P}(1)$,
- (O3) the *operadic compositions* for any $k, r_1, \dots, r_k$,

$$\mathcal{P}(k) \otimes \mathcal{P}(r_1) \otimes \cdots \otimes \mathcal{P}(r_k) \to \mathcal{P}(r_1 + \cdots + r_k).$$

These data must satisfy the axioms of equivariance, associativity and unit.

(b) An *algebra over an operad* $\mathcal{P}$ is an object $A \in \mathcal{C}$ together with S_r-equivariant morphisms $\mathcal{P}(r) \otimes A^{\otimes r} \to A$, $r \geq 0$, satisfying the axioms of unit and associativity.

We will use the case when $\mathcal{C} = \Delta^\circ Set$, $\mathcal{C} = \text{Top}$ or $\mathcal{C} = \text{C}(\text{Vect}_{\mathbf{k}})$. We will refer to these cases as *simplicial, topological* and *dg-operads*. Any topological operad $\mathcal{P}$ gives a simplicial operad $\text{Sing}(\mathcal{P})$ by passing to the singular simplicial sets of the $\mathcal{P}(r)$'s. It further gives a dg-operad $C_\bullet(\mathcal{P})$ formed by the singular chain complexes of the $\mathcal{P}(r)$ (considered, as usual, as cochain complexes with reverse indexation).

A *weak equivalence* of simplicial operads is a morphism $\mathcal{P} \to \mathcal{Q}$ of such operads such that for each r the morphism of simplicial sets $\mathcal{P}(r) \to \mathcal{Q}(r)$ is a weak equivalence, i.e., it induces a homotopy equivalence on the realizations.

Recall (A.1.2) that the category $\text{C}(\text{Vect}_{\mathbf{k}})$ is enriched in the category $\Delta^\circ Set$ of simplicial sets. Thus, for any simplicial operad $\mathcal{P}$ we can speak about *$\mathcal{P}$-algebras* in dgVect. Such an algebra is a cochain complex A together with morphisms of simplicial sets

$$\mathcal{P}(r) \to \text{Map}(A^{\otimes r}, A)$$

compatible with the S_r-actions and operadic compositions. It sends the image of $\mathbf{1} = \text{pt}$ to the identity map. Dually, a *$\mathcal{P}$-coalgebra* in dgVect is a complex B with morphisms of simplicial sets

$$\mathcal{P}(r) \to \text{Map}(B, B^{\otimes r})$$

satisfying similar compatibilities. If $\mathcal{P}$ is a topological operad, its (co)algebras in $\text{C}(\text{Vect}_{\mathbf{k}})$ are understood as (co)algebras over the simplicial operad $\text{Sing}(\mathcal{P})$.

Let $m \geq 1$. Let D_m be the topological *operad of little m-disks*. The space $D_m(r)$ parametrizes families $(B_1, \dots, B_r)$ of round m-dimensional open balls disjointly embedded into the standard unit ball $B = \{|x| < 1\}$ of $\mathbb{R}^m$; see, e.g., [9] for more details including the definition of the operadic compositions.

Definition 7.5.3. By an *E_m-operad* we mean a topological operad weakly equivalent to D_m. An *E_m-(co)algebra* in dgVect is a (co)algebra over an E_m-operad.

We can now formulate our upgrade of the chain level COHA.

Theorem 7.5.4. *$\mathcal{R}$ is a locally constant factorization coalgebra on Σ in the category of E_1-algebras.*

An E_1-algebra can be seen as a weakly (homotopy) associative dg-algebra; see discussion below.

7.6. Proof of Theorem 7.5.4

We first note that $D_1(r)$ is the union of $r!$ contractible components which are permuted by S_r. This means that algebras over D_1 (and so over any E_1-operad) can be described using the concept of a non-symmetric (non-Σ) operad [46, Def. 9]. A *non-symmetric operad* in a monoidal (not necessarily symmetric) category $\mathcal{C}$ is a datum $\mathcal{Q}$

of objects $\mathcal{Q}(r)$, $r \geq 0$ (no symmetric group action is required) together with a unit morphism $\mathbf{1} \to \mathcal{Q}(1)$ and the compositions as in (O3) satisfying the axioms of associativity and unit. Similarly, an algebra over a non-symmetric operad $\mathcal{Q}$ is an object $A \in \mathcal{C}$ together with morphisms $\mathcal{Q}(r) \otimes A^{\otimes r} \to A$ satisfying the axioms of unit and associativity.

Let $ND_1(r) \subset D_1(r)$ be the connected component formed by families $(B_1, \dots, B_r)$ of disjoint 1-disks (i.e., open intervals) in $B = (-1, 1)$ such that the centers of the B_i are positioned in the increasing order. Then $ND_1 = (ND_1(r))$ is a non-symmetric operad in Top with each $ND_1(r)$ contractible. Let us define an NE_1-*operad* to be any non-symmetric operad Q in Top with each $\mathcal{Q}(r)$ contractible. Given an NE_1-operad $\mathcal{Q}$, we can "symmetrize" it, forming an E_1-operad $S\mathcal{Q}$ with $S\mathcal{Q}(r) = \mathcal{Q}(r) \times S_r$ and the S_r acting via the second factor. This establishes an equivalence between the categories of NE_1-operads and E_1-operads, with the categories of algebras over the corresponding operads being identified as well.

Let us now consider dg-versions of the topological operads above and use slightly different notation for these versions. Let an $\mathbf{ne}_1$*-operad* be a non-symmetric dg-operad $\mathcal{K}$ such that each cochain complex $\mathcal{Q}(r)$ is situated in degrees ≤ 0 and quasi-isomorphic to $\mathbf{k}$. Because of Dold–Kan equivalence between $\mathrm{C}^{\leq 0}(\mathrm{Vect}_{\mathbf{k}})$ and $\Delta^\circ\mathrm{Vect}_{\mathbf{k}}$ (see Example 1.1.4 (b)), equipping a complex with a structure of an algebra over an NE_1-operad is the same as equipping it with the structure of an algebra over an $\mathbf{ne}_1$-operad.

An example of an $\mathbf{ne}_1$-operad is given by the *non-symmetric associative operad* $\mathcal{A}ss$ with $\mathcal{A}ss(r) = \mathbf{k}$ for all r and all the compositions being the identities. Dg-algebras over $\mathcal{A}ss$ are the same as associative dg-algebras.

So for the proof of Theorem 7.5.4 we exhibit an $\mathbf{ne}_1$-operad $\mathcal{K}$ and equip each $\mathcal{R}(U)$ with the structure of a $\mathcal{K}$-algebra in a way compatible with factorization coalgebra structure. The argument is an upgrade of the proof of Theorem 4.4.2 (associativity of COHA), so parts of the treatment will be brief.

For $r \geq 1$ let $\mathrm{FILT}^{(r)} = \mathrm{FILT}^{(r)}(U)$ be the stack $\mathrm{FILT}^{(r)}$ parametrizing flags of objects of $Coh_0(U)$,

$$E_1 \subset E_2 \subset\subset E_r.$$

For $r = 0$ we put $\mathrm{FILT}^{(0)} = \mathrm{pt}$. The stack $\mathrm{FILT}^{(r)}$ comes with the projections

$$\begin{array}{ccc} \mathrm{FILT}^{(r)} & \xrightarrow{\rho_r} & \mathrm{Coh}_0(U) \\ {\scriptstyle q_r}\big\downarrow & & \\ \mathrm{Coh}_0(U)^r & & \end{array}$$

$$\rho_r(E_1 \subset E_2 \subset \cdots \subset E_r) = E_r,$$
$$q_r(E_1 \subset E_2 \subset \cdots \subset E_r) = (E_1, E_2/E_1, \dots, E_r/E_{r-1}).$$

For $r = 0$ we have $\mathrm{Coh}(U)^0 = \mathrm{pt}$ and we define $q_0 : \mathrm{pt} \to \mathrm{pt}$ to be the identity map and $\rho_0 : \mathrm{pt} \to \mathrm{Coh}(U)$ to send pt to the zero sheaf.

Let $\mathcal{E}_i$, $i = 1, \dots, r$, be the ith tautological sheaf on $\mathrm{Coh}_0(U)^r \times U$ and $p_r : \mathrm{Coh}_0(U) \times U \to \mathrm{Coh}_0(U)$ be the projection.

Similarly to §2.5, we form the sheaf of associative dg-algebras (and, passing to the super-commutator, of dg-Lie algebras) on $\mathrm{Coh}_0(U)^r$,

$$\mathcal{G}_r = \mathcal{G}_r(U) = \bigoplus_{1 \le i < j \le r} Rp_{r*} \underline{\mathrm{RHom}}(\mathcal{E}_j, \mathcal{E}_i),$$

and find that $\mathrm{FILT}^{(r)} = \mathrm{MC}(\mathcal{G}_r)$ so that q_r is identified with the projection of the Maurer–Cartan stack. Therefore we have the diagram

$$\mathrm{Coh}_0(U)^r \xleftarrow{\pi_r} \mathrm{Tot}(\mathcal{G}_r^{\le 1}) \xleftarrow{i_r} \mathrm{FILT}^{(r)} \xrightarrow{\rho_r} \mathrm{Coh}_0(U), \tag{7.6.1}$$

in which the map i_r realizes $\mathrm{FILT}^{(r)}$ as the zero locus of the section s_r of $\pi_r^* \mathcal{G}_r^2$ given by the curvature map. This gives a virtual pullback $i_r^!$ on Borel–Moore homology. We get, so far *at the level of BM-homology*, the map

$$m_r = \rho_{r*} \circ i_r^! \circ \pi_r^* : R(U)^{\otimes r} \to R(U), \quad R(U) = H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(U)). \tag{7.6.2}$$

As in §4.5, we see that m_r is the r-fold product in the (associative) COHA $R(U)$.

Next, we notice that the family $(\mathcal{G}_r)_{r \ge 0}$ of dg-Lie algebras carries a kind of operadic structure. For $r_1, \dots, r_n \ge 0$ consider the summation map

$$\sigma_{r_1,\dots,r_n} : \mathrm{Coh}_0(U)^{r_1+\dots+r_n} \to \mathrm{Coh}(U)^n,$$
$$(\mathcal{F}_1, \dots, \mathcal{F}_{r_1+\dots+r_n}) \mapsto \Big(\bigoplus_{i=1}^{r_1} \mathcal{F}_i, \bigoplus_{i=r_1+1}^{r_1+r_2} \mathcal{F}_i, \dots, \bigoplus_{i=r_1+\dots+r_{n-1}+1}^{r_1+\dots+r_n} \mathcal{F}_i \Big).$$

Proposition 7.6.3. *We have a semidirect product decomposition, more precisely, an isomorphism*

$$\lambda_{r_1,\dots,r_n} : (\sigma_{r_1,\dots,r_n}^* \mathcal{G}_n) \rtimes (\mathcal{G}_{r_1} \boxplus \dots \boxplus \mathcal{G}_{r_n}) \xrightarrow{\simeq} \mathcal{G}_{r_1+\dots+r_n}$$

of sheaves of dg-Lie algebras on $\mathrm{Coh}(U)^{r_1+\dots+r_n}$.

The proposition means that we have a split exact sequence

$$0 \to \sigma_{r_1,\dots,r_n}^* \mathcal{G}_n \xrightarrow{a} \mathcal{G}_{r_1+\dots+r_n} \xrightarrow{b} \mathcal{G}_{r_1} \boxplus \dots \boxplus \mathcal{G}_{r_n} \to 0 \tag{7.6.4}$$

in which $\sigma_{r_1,\dots,r_n}^* \mathcal{G}_n$ is a dg-Lie ideal with quotient $\mathcal{G}_{r_1} \boxplus \dots \boxplus \mathcal{G}_{r_n}$.

Proof of Proposition 7.6.3. By construction, $\mathcal{G}_{r_1,\dots,r_n}$ consists of upper-triangular square matrices of size $r_1 + \dots + r_n$. We can decompose such a matrix into blocks of sizes $r_i \times r_j$, $1 \le i \le j \le n$. Of these, the diagonal blocks (of sizes $r_i \times r_i$) are upper-triangular, since the total matrix must be upper-triangular. These correspond to the $\mathcal{G}_{r_i}$. So the block-diagonal part is $\mathcal{G}_{r_1} \boxplus \dots \boxplus \mathcal{G}_{r_n}$. Similarly, the over-diagonal blocks are seen as representing $\sigma_{r_1,\dots,r_n}^* \mathcal{G}_n$. ∎

Proposition 7.6.5. *The isomorphisms $\lambda_{r_1,\dots,r_n}$ satisfy operadic associativity. That is, suppose that each r_i is decomposed as $r_i = r_{i,1} + \cdots + r_{i,m_i}$. Then the isomorphisms*

$$\lambda_{r_{i,1},\dots,r_{i,m_i}} : (\sigma^*_{r_{i,1},\dots,r_{i,m_i}} \mathcal{G}_{n_i}) \rtimes (\mathcal{G}_{r_{i,1}} \boxplus \cdots \boxplus \mathcal{G}_{r_{i,m_i}}) \xrightarrow{\simeq} \mathcal{G}_{r_{i,1}+\cdots+r_{i,m_i}} = \mathcal{G}_{r_i}, \quad i = 1, \dots, n,$$

together with $\lambda_{r_i,\dots,r_n}$, compose to $\lambda_{r_{1,1},\dots,r_{1,m_1},r_{2,1},\dots,r_{2,m_2},\dots,r_{n,1},\dots,r_{n,m_n}}$.

Proof. Straightforward verification in terms of matrices whose blocks are decomposed into further blocks. ∎

Next, we study the compatibilty of the curvature sections s_r on the $\pi_r^* \mathcal{G}_r^2$ with the semidirect product decompositions $\lambda_{r_1,\dots,r_n}$. Let $r = r_1 + \cdots + r_n$. On $\mathrm{Tot}(\mathcal{G}_r^{\leq 1})$ the sequence (7.6.4) gives a short exact sequence of vector bundles

$$0 \to \pi_r^* \sigma^*_{r_1,\dots,r_n} \mathcal{G}_n^2 \xrightarrow{\alpha} \pi_r^* \mathcal{G}_r^2 \xrightarrow{\beta} \pi_r^* (\mathcal{G}_{r_1}^2 \boxplus \cdots \boxplus \mathcal{G}_{r_n}^2) \to 0 \tag{7.6.6}$$

pulled back from $\mathrm{Coh}_0(U)^r$. We apply to this situation the analysis of §B.4, taking $X = \mathrm{Tot}(\mathcal{G}_r^{\leq 1})$ and viewing (7.6.6) as an instance of the sequence (B.4.1). The curvature section $s = s_r$ of the middle bundle gives rise to the section $s'' = s''_r = \beta(s)$ of $\pi_r^* (\mathcal{G}_{r_1}^2 \boxplus \cdots \boxplus \mathcal{G}_{r_n}^2)$ with zero locus $X_{s''} = \mathrm{Tot}(\mathcal{G}_r^{\leq 1})_{s''_r}$ and the section $s' = s'_r$ of $\pi_r^* \sigma^*_{r_1,\dots,r_n} \mathcal{G}_n^2$ over $X_{s''}$. To describe them we consider, for each $i = 1, \dots, n$, the stack $\mathrm{FILT}^{(r_i)}$ parametrizing flags $E_{i,1} \subset E_{i,2} \subset \cdots \subset E_{i,r_i} = E_i$ of objects of $\mathrm{Coh}_0(U)$. Let

$$\phi : \prod_{i=1}^n \mathrm{FILT}^{(r_i)} \to \mathrm{Coh}_0(U)^r = \prod_{i=1}^r \mathrm{Coh}_0(U)$$

be the projection which sends a tuple of flags as above to the tuple $(E_1, \dots, E_n)$ of their maximal elements. We also denote by

$$\pi_{r_1,\dots,r_n} : \mathrm{Tot}(\phi^* \mathcal{G}_n^{\leq 1}) \to \prod_{i=1}^n \mathrm{FILT}^{(r_i)}$$

the projection.

Proposition 7.6.7. (a) *s''_r is equal to* (*the pullback of*) *the tuple* $(s_{r_1}, \dots, s_{r_n})$ *considered as a section of the external direct sum.*

(b) *$X_{s''}$ is identified with* $\mathrm{Tot}(\phi^* \mathcal{G}_n^{\leq 1})$.

(c) *Under the identification of* (b), *the restriction of $\pi_r^* \sigma^*_{r_1,\dots,r_n} \mathcal{G}_n^2$ to $X_{s''}$ is identified with $\pi^*_{r_1,\dots,r_n} \phi^* \mathcal{G}_n^2$.*

(d) *Under the identification of* (c), *the section s'_r is identified with the pullback of s_n.*

Proof. (a) As in the proof of Proposition 7.6.3, let us view sections of $\mathcal{G}_r$ as upper-triangular $r \times r$ matrices subdivided into blocks of sizes $r_i \times r_j$. The projection b (whose pullback is β) associates to such a matrix x its block-diagonal part which we denote x_Δ. Thus $\beta(s_r)$ associates to x the block-diagonal part of the curvature, i.e., $(dx + (1/2)[x,x])_\Delta$. Since the block-diagonal subspace is a dg-Lie subalgebra, this equals $d(x_\Delta) + (1/2)[x_\Delta, x_\Delta]$ which corresponds to the pullback of $(s_{r_1}, \dots, s_{r_n})$.

(b) Let us represent a point of $\mathrm{Coh}_0(U)^r$, $r = r_1 + \cdots + r_n$, as a sequence of sheaves

$$\mathcal{F}_{1,1}, \dots, \mathcal{F}_{1,r_1}, \mathcal{F}_{2,1}, \dots, \mathcal{F}_{2,r_2}, \dots, \mathcal{F}_{n,1}, \dots, \mathcal{F}_{n,r_n}.$$

In terms of matrices $x \in \mathcal{G}_r^1$, vanishing of the block-diagonal part of the curvature of x means that the Ext-data for the $\mathcal{F}_{i,j}$ provided by x, integrate to n filtrations

$$E_{i,1} \subset E_{i,2} \subset \cdots \subset E_{i,r_i} = E_i, \quad E_{i,j}/E_{i,j-1} \simeq \mathcal{F}_{i,j}, \quad i = 1, \dots, n, \ j = 1, \dots, r_i,$$

i.e., we have a point of $\prod_{i=1}^n \mathrm{FILT}^{(r_i)}$. Further, the summation map $\sigma_{r_1,\dots,r_n}$ on the sequence of the $\mathcal{F}_{i,j}$ corresponds to the projection ϕ. The over-diagonal blocks of x assemble into a section of $\phi^*\mathcal{G}_n^{\le 1}$, whence the statement.

Part (c) is clear from the above. To see (d), notice that the pullback of s_n represents the over-diagonal blocks of the curvature of x. ■

We now apply the formalism of homotopy canonical Euler classes from Appendix B. Let $\mathcal{K}_r(U) = K_{\pi_r^*\mathcal{G}_r^2}$ be the parameter complex for the homotopy canonical orientation class of the bundle $\pi_r^*\mathcal{G}_r^2$ on $\mathrm{Tot}(\mathcal{G}_r^{\le 1})$ (see §B.3). Here $\mathcal{G}_r = \mathcal{G}_r(U)$, as above. By construction, each $\mathcal{K}_r(U)$ maps quasi-isomorphically to $\mathbf{k}$. The semidirect product decomposition of Proposition 7.6.3 and the pairings (B.4.8) of the K-complexes give morphisms of complexes

$$\mathcal{K}_n(U) \otimes \mathcal{K}_{r_1}(U) \otimes \cdots \otimes \mathcal{K}_{r_n}(U) \to \mathcal{K}_{r_1+\cdots+r_n}(U).$$

The operadic associativity of the isomorphisms $\lambda_{r_1,\dots,r_n}$ and the associativity of the pairings (B.4.8) imply that $\mathcal{K}(U) = (\mathcal{K}_r(U))_{r\ge 0}$ is a (non-symmetric) dg-operad. Since each $\mathcal{K}_r(U)$ is quasi-isomorphic to $\mathbf{k}$, we see that $\mathcal{K}(U)$ is an $\mathbf{ne}_1$-operad. Further, the correspondence $U \mapsto \mathcal{K}(U)$ forms a presheaf (in fact, a sheaf up to homotopy, by the above) of dg-operads on the analytic surface Σ. Let $\mathcal{K} = \mathcal{K}(\Sigma)$ be the operad of global sections.

Finally, let us upgrade the r-fold multiplication map (7.6.2) to the cochain level by analyzing the ambiguity. This map involves the virtual pullback $i_r^!$ which is defined in terms of the refined Chern class $c_d(\pi_r^*\mathcal{G}_r^2)$, $d = \mathrm{rk}(\mathcal{G}_r^2)$. Using the homotopy canonical cochain lifting $\widetilde{c}_d(\pi_r^*\mathcal{G}_r^2)$, $d = \mathrm{rk}(\mathcal{G}_r^2)$ (see (B.3.4)), we define a cochain level multiplication

$$\widetilde{m}_{r,U} : \mathcal{K}_r(U) \otimes \mathcal{R}(U)^{\otimes r} = \mathcal{K}_r(U) \otimes C_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(U))^{\otimes r} \to C_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(U)) = \mathcal{R}(U). \tag{7.6.8}$$

The multiplicativity of the $\widetilde{c}_d$ in short exact sequences (B.4.10) implies that the $\mu_{r,U}$ make $\mathcal{R}(U)$ into an algebra over the $\mathbf{ne}_1$-operad $\mathcal{K}(U)$ and therefore over $\mathcal{K} = \mathcal{K}(\Sigma)$. Further, these $\mathcal{K}$-algebra structures are clearly compatible with the factorization coalgebra structure on the presheaf $\mathcal{R} = (\mathcal{R}(U))$. This finishes the proof of Theorem 7.5.4. ■

7.7. *Proof of Theorem* 7.1.6

As before, let $\Sigma = S^{\mathrm{an}}$. For any open subset $U \subset \Sigma$ (in the complex analytic topology) we have the $\mathbb{Z}^2$-graded space $\Theta(U)$ and the symmetrized product map $\sigma_U : \mathrm{Sym}(\Theta(U)) \to R(U)$. Because of Proposition 7.4.1 and the identification (7.3.4), the map σ of Theorem 7.1.6 is identified with the global map σ_Σ, corresponding to $U = \Sigma$. Now, if U is a disk, then σ_U is an isomorphism by Theorem 6.1.4. We will deduce the global statement (for $U = \Sigma$) from these local ones.

For this, we upgrade the correspondence $U \mapsto \Theta(U)$ to a complex of sheaves $\mathcal{V}$ on Σ so that $\Theta(U) = \mathbb{H}^{-\bullet}(U, \mathcal{V})$ is the hypercohomology of U with coefficients in $\mathcal{V}$. That is, we define

$$\mathcal{V} = \omega_\Sigma \otimes_{\mathbf{k}} \Theta',$$

the tensor product of the dualizing complex ω_Σ and the graded vector space $\Theta' = \Theta[0, -4]$ (see (7.1.4)). Recall that Θ and therefore Θ' is spanned by the basis vectors $t^n q^{i-1}$, and such a vector is identified with $\theta_{n,i} = \mathrm{ch}_i(\mathcal{E}_n) \cap \theta_n \in R^{n,2-2i}(\mathbb{C}^2)$ (see (6.3.3)). Here, as we recall, $\mathcal{E}_n$ is the tautological rank n bundle on $\mathrm{Coh}_0^{(n)}(\mathbb{C}^2)$ whose fiber at a point represented by a coherent sheaf $\mathcal{F}$ is $H^0(\mathcal{F})$ and θ_n is the fundamental class of $\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(\mathbb{C}^2)$.

As before, we denote by the same symbol $\mathcal{E}_n$ the analogous tautological bundle on $\mathrm{Coh}_0^{(n)}(\Sigma)$ and, if necessary, its restriction to $\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(\Sigma)$.

Extending the construction of (7.1.2), we choose a cocycle representing the fundamental class of $\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(\Sigma)$ in H_2^{BM} and define the morphism

$$p_n^\dagger : p_n^* \omega_\Sigma \to p_n^! \omega_\Sigma[2] = \omega_{\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(\Sigma)}[2]$$

as the cup-product with this cocycle. Here $p_n : \mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(S) \to S$ is the projection defined in (7.1.1). Further, for each i we fix a cocycle representative $\widetilde{\mathrm{ch}}_i(\mathcal{E}_n)$ of $\mathrm{ch}_i(\mathcal{E}_n) \in H^{2i}(\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(\Sigma), \mathbf{k})$.

The sheaf $\mathcal{V}$ and the factorization coalgebra $\mathcal{R}$ are both presheaves with values in the category of cochain complexes. We define a morphism of presheaves $\tilde{\alpha} : \mathcal{V} \to \mathcal{R}$ as the composition of two morphisms: first, the morphism

$$\begin{aligned} R\Gamma(U, \omega_\Sigma) \otimes t^n q^{i-1} &\to R\Gamma(\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(U), \omega_{\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(U)}), \\ \gamma \otimes t^n q^{i-1} &\mapsto \widetilde{\mathrm{ch}}_i(\mathcal{E}_n) \cap p_n^\dagger(p_n^*(\gamma)), \end{aligned}$$

(here $p_n^*(\gamma)$ is an element of $R\Gamma(\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(U)), p_n^*\omega_U))$ and, second, the direct image morphism

$$\varepsilon_* : R\Gamma(\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(U), \omega_{\mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(U)}) \to R\Gamma(\mathrm{Coh}_0^{(n)}(U), \omega_{\mathrm{Coh}_0^{(n)}(U)}) = \mathcal{R}(U)^{(n)},$$

where $\varepsilon : \mathrm{Coh}_{1\,\mathrm{pt}}^{(n)}(U) \to \mathrm{Coh}_0^{(n)}(U)$ is the (closed) embedding.

Since $\mathcal{V}$ is a sheaf with values in the category of cochain complexes, its symmetric algebra $\mathrm{Sym}(\mathcal{V})$ is a factorization coalgebra with values in this category, by Proposition 7.2.6. Since $\mathcal{R}$ is a factorization algebra in the category of E_1-algebras, we can define the symmetrized product $\widetilde{\sigma} : \mathrm{Sym}(\mathcal{V}) \to \mathcal{R}$ by setting

$$\widetilde{\sigma} = \sum_{n \geq 0} \widetilde{\sigma}_n,$$

where

$$\widetilde{\sigma}_n : \mathrm{Sym}^n(\mathcal{V}) \to \mathcal{R}, \quad \widetilde{\sigma}_n(v_1 \bullet \cdots \bullet v_n) = \frac{1}{n!} \sum_{s \in S_n} \mu_n(\widetilde{\alpha}(v_{s(1)}) \otimes \cdots \otimes \widetilde{\alpha}(v_{s(n)})), \tag{7.7.1}$$

lifting the map σ from (7.1.5). In other words, $\widetilde{\sigma}_n$ is the symmetrization of the map

$$\mu_n \circ (\widetilde{\alpha} \otimes \cdots \otimes \widetilde{\alpha}) : \mathcal{V}^{\otimes n} \to \mathcal{R}.$$

The map $\widetilde{\sigma}$ is a morphism of factorization coalgebras *in the category of* $\mathbb{Z}^2$*-graded cochain complexes*. Note that we do not claim (and it is not true) that $\widetilde{\sigma}$ is a morphism of factorization coalgebras in the category of E_1-algebras.

By the above, $\widetilde{\sigma}_U$ is a weak equivalence (of $\mathbb{Z}^2$-graded cochain complexes) for any U which is, topologically, a disk. Therefore $\widetilde{\sigma}$ is a weak equivalence (of factorization coalgebras in the category of $\mathbb{Z}^2$-graded cochain complexes) by Proposition 7.2.7. Taking $U = \Sigma$ we obtain Theorem 7.1.6.

7.8. E_4-structure on the flat COHA

By [21, 44], locally constant factorization (co)algebras on $\mathbb{R}^m$ with values in $\mathrm{C}(\mathrm{Vect}_{\mathbf{k}})$ can be identified with E_m-(co) algebras in $\mathrm{C}(\mathrm{Vect}_{\mathbf{k}})$, the identification associating to a (co)algebra $\mathcal{B}$ the object $\mathcal{B}(B)$ where $B \subset \mathbb{R}^m$ is the standard unit m-ball. Note that $\mathcal{B}(B)$ is weak equivalent to $\mathcal{B}(\mathbb{R}^d)$.

Let us specialize this to the case when $\mathcal{B} = \mathcal{R}$ and $m = 4$, since $\mathbb{C}^2 \simeq \mathbb{R}^4$. In this case we form the cochain complex $\mathcal{R}(B) \simeq \mathcal{R}(\mathbb{C}^2)$ whose cohomology is the flat Hecke algebra $R(B) \simeq R(\mathbb{C}^2)$ studied in §6. The general results above, applied to the category $\mathcal{C}$ of E_1-algebras, imply

Corollary 7.8.1. *$\mathcal{R}(\mathbb{C}^2)$ is an E_1-algebra in the category of E_4-coalgebras.* ∎

Remarks 7.8.2. (a) The E_4-coalgebra structure on $\mathcal{R}(\mathbb{C}^2)$ is a cochain level refinement of the comultiplication Δ on $R(\mathbb{C}^2)$ (see §6.2). While Δ is cocommutative, because it is independent of the choice of two distinct disks $U_1, U_2 \subset \mathbb{C}^2$, at the cochain level we do not seem to have cocommutativity since the space of choices of such pairs of disks is not contractible (it is precisely the space of binary operations in the operad D_4).

(b) By forming the Koszul dual to the E_1-algebra structure on $\mathcal{R}(\mathbb{C}^2)$, we obtain an E_1-coalgebra in the category of E_4-coalgebras, i.e., an E_5-coalgebra. Alternatively, forming the Koszul dual to the E_4-algebra structure, we obtain an E_5-algebra. This suggests that some 5-dimensional field theory may be relevant to this picture.

Appendix A. Basics on ∞-categories and dg-categories

A.1. ∞-categories

Let $\mathbf{k}$ be a field of characteristic 0. By $\mathrm{Vect} = \mathrm{Vect}_{\mathbf{k}}$ we denote the category of $\mathbf{k}$-vector spaces and by $\mathrm{C}(\mathrm{Vect}) = \mathrm{C}(\mathrm{Vect}_{\mathbf{k}})$ the category of complexes of $\mathbf{k}$-vector spaces bounded below, with morphisms being morphisms of complexes. By $\Delta^\circ \mathit{Set}$ we denote the category of simplicial sets. For a simplicial set Y we denote by $|Y|$ the geometric realization of Y. We say that Y is *contractible* if $|Y|$ is a contractible topological space. For a topological space T we denote by $\mathrm{Sing}(T)$ the singular simplicial set of T.

An ∞-*category* $\mathfrak{C}$ is a simplicial set $(\mathfrak{C}_n)_{n\geq 0}$ satisfying the partial Kan condition, with elements of $\mathfrak{C}_0$ called objects and elements of $\mathfrak{C}_1$ called morphisms.

Every ∞-category $\mathfrak{C}$ gives rise to an ordinary category $\mathrm{h}\mathfrak{C}$ known as the *homotopy category of* $\mathfrak{C}$. It has the same objects as $\mathfrak{C}$ and its morphisms are certain equivalence classes of morphisms in $\mathfrak{C}$. Further, $\mathfrak{C}$ contains the maximal Kan simplicial subset $\mathfrak{C}^{\mathrm{Kan}}$ with $\mathfrak{C}^{\mathrm{Kan}}_0 = \mathfrak{C}_0$, having the meaning of the subgroupoid of (weakly) invertible morphisms. We refer to [43] for more details.

A *simplicial category* is a category $\mathcal{C}$ enriched in $\Delta^\circ \mathit{Set}$, so that for any two objects $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ we are given a simplicial set $\mathrm{Map}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ with standard properties. A simplicial category $\mathcal{C}$ gives an ∞-category $\mathfrak{N}(\mathcal{C})$ with the same objects, as explained in [43].

A *dg-category* is a category $\mathcal{C}$ enriched in $\mathrm{C}(\mathrm{Vect})$, so that for any two objects $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ we are given a cochain complex $\mathrm{Hom}^\bullet_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ with standard properties. Any dg-category $\mathcal{C}$ gives rise to a $\mathbf{k}$-linear category $H^0\mathcal{C}$ with the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{H^0\mathcal{C}}(\mathcal{F}, \mathcal{G}) = H^0 \,\mathrm{Hom}^\bullet_{\mathcal{C}}(\mathcal{F}, \mathcal{G}). \tag{A.1.1}$$

Further, $\mathcal{C}$ can be made into a simplicial category (with the same objects) by

$$\mathrm{Map}(\mathcal{F}, \mathcal{G}) = \mathrm{DK}(\tau_{\leq 0} \mathrm{Hom}^\bullet_{\mathcal{C}}(\mathcal{F}, \mathcal{G})), \tag{A.1.2}$$

where DK is the Dold–Kan simplicial set associated to a $\mathbb{Z}_{\leq 0}$-graded complex (see [64, §8.4.1] and a discussion in Example 1.1.4). So it gives rise to an ∞-category denoted $N^{\mathrm{dg}}(\mathcal{C})$ (see [44]).

A.2. Enhanced derived categories

Let $\mathcal{A}$ be a $\mathbf{k}$-linear abelian category. We denote by $\mathrm{C}(\mathcal{A})$ the category of complexes over $\mathcal{A}$ with morphisms being morphisms of complexes. By $\mathrm{C}(\mathcal{A})_{\mathrm{dg}}$ we denote the dg-category with the same objects as $\mathrm{C}(\mathcal{A})$. For any two objects $\mathcal{F}, \mathcal{G}$ of $\mathrm{C}(\mathcal{A})_{\mathrm{dg}}$, the complex $\mathrm{Hom}_{\mathrm{C}(\mathcal{A})_{\mathrm{dg}}}(\mathcal{F}, \mathcal{G})$ is the graded $\mathbf{k}$-vector space $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ with the differential given by the commutation with $d_{\mathcal{F}}$ and $d_{\mathcal{G}}$. Thus $\mathrm{C}(\mathcal{A}) = H^0\mathrm{C}(\mathcal{A})_{\mathrm{dg}}$. By $\mathrm{D}(\mathcal{A}) = \mathrm{C}(\mathcal{A})[\mathrm{Qis}^{-1}]$ we denote the derived category of $\mathcal{A}$, i.e., the localization of $\mathrm{C}(\mathcal{A})$ by the class Qis of quasi-isomorphisms. There are three closely related *enhancements* of $\mathrm{D}(\mathcal{A})$ with the same objects:

(a) The *derived dg-category* $\mathrm{D}(\mathcal{A})_{\mathrm{dg}}$ with the property that $\mathrm{D}(\mathcal{A}) = H^0\,\mathrm{D}(\mathcal{A})_{\mathrm{dg}}$. If $\mathcal{A}$ has canonical injective resolutions $A \mapsto I(A)$, then we define (see [7])

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})_{\mathrm{dg}}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}^{\bullet}_{\mathrm{C}(\mathcal{A})_{\mathrm{dg}}}(I(\mathcal{F}), I(\mathcal{G})).$$

The complex on the RHS is also denoted $\mathrm{RHom}^{\bullet}(\mathcal{F}, \mathcal{G})$.

(b) The *simplicial derived category* $\mathrm{D}(\mathcal{A})_{\Delta}$ with the property that $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(\mathcal{F}, \mathcal{G}) = \pi_0 \mathrm{Map}_{\mathrm{D}(\mathcal{A})_{\Delta}}(\mathcal{F}, \mathcal{G})$. There are two homotopy equivalent ways of constructing $\mathrm{Map}_{\mathrm{D}(\mathcal{A})_{\Delta}}(\mathcal{F}, \mathcal{G})$:

(b1) Given the data in (a), we can define, as in (A.1.2),

$$\mathrm{Map}_{\mathrm{D}(\mathcal{A})_{\Delta}}(\mathcal{F}, \mathcal{G}) = \mathrm{DK}(\tau_{\leq 0}\, \mathrm{RHom}^{\bullet}(\mathcal{F}, \mathcal{G})).$$

(b2) The Dwyer–Kan simplicial localization procedure [12, 13] produces simplicial sets $\mathrm{Map}(\mathcal{F}, \mathcal{G})$, starting from the category $\mathrm{C}(\mathcal{A})$ and the class Qis of morphisms. We can take $\mathrm{Map}_{\mathrm{D}(\mathcal{A})_{\Delta}}(\mathcal{F}, \mathcal{G})$ to be the simplicial sets $\mathrm{Map}(\mathcal{F}, \mathcal{G})$. Further, we can use them to get an intrinsic definition of the $\mathrm{RHom}^{\bullet}(\mathcal{F}, \mathcal{G})$ in (a) by taking the normalized chain complexes and stabilizing with respect to the shift. This allows one to define $\mathrm{D}(\mathcal{A})_{\mathrm{dg}}$ even without the use of canonical injective resolutions.

(c) The *derived* ∞*-category* $\mathrm{D}(\mathcal{A})_{\infty}$ with the property that $\mathrm{h}\,\mathrm{D}(\mathcal{A})_{\infty} = \mathrm{D}(\mathcal{A})$. As in (b2), it can be defined intrinsically, as the ∞-categorical localization of $\mathrm{C}(\mathcal{A})$ by Qis [44].

Appendix B. Homotopy canonical Euler classes

The concept of *coherent homotopy uniqueness* of objects, morphisms, cohomology classes, etc., is implicit in the formalism of ∞-categories, as well as in homotopical algebra in general. In this appendix we spell out some instances of this concept in the dg-context.

B.1. Cocycles defined up to a contractible choice

Let V be a cochain complex over $\mathbf{k}$, and let $a \in H^d(V)$. Viewing a as a morphism $a : \mathbf{k} \to V[d]$ in $\mathrm{D}(\mathrm{Vect}_{\mathbf{k}})$, we can represent a (non-uniquely) by a diagram of morphisms of complexes $\mathbf{k} \xleftarrow{q} K \xrightarrow{\alpha} V[d]$, where q is a quasi-isomorphism. Such a diagram is just a right fraction representing the morphism a in $\mathrm{D}(\mathrm{Vect}_{\mathbf{k}}) = \mathrm{C}(\mathrm{Vect}_{\mathbf{k}})[\mathrm{Qis}^{-1}]$ as $a = \alpha q^{-1}$. We will refer to any such diagram as a d*-cocycle defined up to a contractible choice* and say that it *represents a up to a contractible choice*.

Examples B.1.1. (a) Suppose that $a \neq 0$ and $H^j(V) = 0$ for $j < d$. Let $Z^d(V) \subset V^d$ be the space of d-cocycles and $\gamma : Z^d(V) \to H^d(V)$ be the projection. Let

$$V_a^{\leq d} = \{\cdots \to V^{d-2} \to V^{d-1} \to \gamma^{-1}(\mathbf{k}a)\} \subset V^{\leq d}.$$

Then the projection to $\mathbf{k}a \simeq \mathbf{k}$ gives a quasi-isomorphism $V_a^{\leq d}[d] \xrightarrow{\sim} \mathbf{k}$ and the diagram

$$\mathbf{k} \xleftarrow{\sim} V_a^{\leq d}[d] \hookrightarrow V[d]$$

represents a up to a contractible choice.

(b) In particular, let $\mathcal{C}$ be a dg-category and $x, y \in \mathrm{Ob}(\mathcal{C})$ be such that $\mathrm{Hom}^\bullet_{\mathcal{C}}(x, y)$ has $H^j = 0$ for $j < 0$. Then any non-zero morphism $f : x \to y$ in $H^0\mathcal{C}$ is represented, up to a contractible choice, by the diagram

$$\mathbf{k} \xleftarrow{\sim} \mathrm{Hom}^{\leq 0}_{\mathcal{C}}(x, y)_f \hookrightarrow \mathrm{Hom}^\bullet_{\mathcal{C}}(x, y).$$

B.2. *Homotopy canonical orientation classes*

Let $X \xhookrightarrow{i} Y$ be a regular embedding of stacks of codimension d. We then have the canonical *orientation isomorphism* $\eta_{X/Y} : \underline{\mathbf{k}}_X \xrightarrow{\sim} i^!\underline{\mathbf{k}}_Y[2d]$ in the derived category $\mathrm{D}(X)$. If $X \xhookrightarrow{i} Y \xhookrightarrow{j} Z$ are two composable regular embeddings, with i of codimension d and j of codimension e, then ji is a regular embedding of codimension $d + e$ and $\eta_{X/Z} : \underline{\mathbf{k}}_X \to (ji)^!\underline{\mathbf{k}}_Z[2(d + e)]$ is equal to the composition

$$\underline{\mathbf{k}}_X \xrightarrow{\eta_{X/Y}} i^!\underline{\mathbf{k}}_Y[2d] \xrightarrow{i^!\eta_{Y/Z}[2d]} i^!j^!\underline{\mathbf{k}}_Z[2d + 2e]. \tag{B.2.1}$$

Passing to the dg-enhancements, we notice that $\eta_{X/Y}$ connects two objects which are quasi-isomorphic to single sheaves in degree 0 and so negative Ext's between these objects vanish. We are therefore in the situation of Example B.1.1 (b) and so the diagram

$$\mathbf{k} \xleftarrow{\sim} K_{X/Y} := \mathrm{Hom}^{\leq 0}_{\mathrm{D}(X)_{\mathrm{dg}}}(\underline{\mathbf{k}}_X, i^!\underline{\mathbf{k}}_Y[2d])_{\eta_{X/Y}} \hookrightarrow \mathrm{Hom}^\bullet_{\mathrm{D}(X)_{\mathrm{dg}}}(\underline{\mathbf{k}}_X, i^!\underline{\mathbf{k}}_Y[2d]) \tag{B.2.2}$$

represents $\eta_{X/Y}$ up to a contractible choice. We can write it as a canonical closed morphism in $\mathrm{D}(X)_{\mathrm{dg}}$ of degree 0,

$$\tilde{\eta}_{X/Y} : K_{X/Y} \otimes \underline{\mathbf{k}}_X \to i^!\underline{\mathbf{k}}_Y[2d]. \tag{B.2.3}$$

If $X \xhookrightarrow{i} Y \xhookrightarrow{j} Z$ are two composable regular embeddings as before, then the composition of Hom-complexes in the dg-category $\mathrm{D}(X)_{\mathrm{dg}}$ induces a composition

$$m_{X,Y,Z} : K_{Y/Z} \otimes K_{X/Y} \to K_{X/Z}, \tag{B.2.4}$$

and such compositions are associative for any composable triple of regular embeddings. The composition $m_{X,Y,Z}$ fits into the commutative square

$$\begin{array}{ccc} K_{Y/Z} \otimes K_{X/Y} \otimes \underline{\mathbf{k}}_X & \xrightarrow{K_{Y/Z} \otimes \tilde{\eta}_{X/Y}} & K_{Y/Z} \otimes i^!\underline{\mathbf{k}}_Y[2d] \\ {\scriptstyle m_{X,Y,Z} \otimes \underline{\mathbf{k}}_X} \downarrow & & \downarrow {\scriptstyle i^!\tilde{\eta}_{Y/Z}} \\ K_{X/Z} \otimes \underline{\mathbf{k}}_X & \xrightarrow{\tilde{\eta}_{X/Z}} & i^!j^!\underline{\mathbf{k}}_Z[2d + 2e] \end{array} \tag{B.2.5}$$

which underlies the identification of $\eta_{X/Z}$ with the composition (B.2.1).

B.3. Homotopy canonical Euler classes

Let $\mathcal{E}$ be a rank d vector bundle over a stack X. Let $s \in H^0(X, \mathcal{E})$ be a section. We consider it as a morphism $s : X \to \mathrm{Tot}(\mathcal{E})$. Let $i_s : X_s \to X$ be the embedding of the zero locus of s. We then have a Cartesian square of closed embeddings

$$\begin{array}{ccc} X_s & \xrightarrow{i_s} & X \\ {\scriptstyle i_s}\downarrow & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{s} & \mathrm{Tot}(\mathcal{E}) \end{array} \tag{B.3.1}$$

The zero section embedding $0 : X \hookrightarrow \mathrm{Tot}(\mathcal{E})$ is regular of codimension d, so we have the orientation isomorphism in $\mathrm{D}(X)$

$$\eta_{\mathcal{E}} := \eta_{X/\mathrm{Tot}(\mathcal{E})} : \underline{\mathbf{k}}_X \to 0^! \underline{\mathbf{k}}_{\mathrm{Tot}(\mathcal{E})}[2d]. \tag{B.3.2}$$

Applying i_s^{-1} to $\eta_{\mathcal{E}}$, we get a morphism in $\mathrm{D}(X_s)$

$$\underline{\mathbf{k}}_{X_s} = i_s^{-1}\underline{\mathbf{k}}_X \xrightarrow{i_s^{-1}\eta_{\mathcal{E}}} i_s^{-1}0^!\underline{\mathbf{k}}_{\mathrm{Tot}(\mathcal{E})}[2d] \xrightarrow{\text{B.C.}} i_s^! s^{-1}\underline{\mathbf{k}}_{\mathrm{Tot}(\mathcal{E})}[2d] = i_s^!\underline{\mathbf{k}}_X[2d], \tag{B.3.3}$$

where "B.C." means the base change morphism for the square (B.3.1) [34, Prop. III.1.9 (iii)]. The morphism (B.3.3) can be viewed as an element $c_d(E, s)$ in $H^{2d}_{X_s}(X, \mathbf{k})$ which is known as the *refined Euler* (*top Chern*) *class of* $(\mathcal{E}, s)$. Its image in $H^{2d}(X, \mathbf{k})$ is the usual Euler (top Chern) class $c_d(\mathcal{E})$.

Passing to dg-enhancements, we denote $K_{\mathcal{E}} := K^{\bullet}_{X/\mathrm{Tot}(\mathcal{E})}$. We can think of objects of the dg-categories $\mathrm{D}(Y)_{\mathrm{dg}}$ associated to various stacks Y as (systems of, see (3.2.1)) complexes consisting of flabby sheaves. Now, for a flabby sheaf the !-inverse image under a closed embedding is given by the sheaf of sections with support. With this understanding, the base change morphism in a Cartesian square of closed embeddings of topological spaces is a canonical morphism of sheaves. Therefore our conventions imply that the base change morphism in (B.3.3) is defined canonically (no choice needed). So lifting $\eta_{\mathcal{E}}$ to $\widetilde{\eta}_{\mathcal{E}} := \widetilde{\eta}_{X/\mathrm{Tot}(\mathcal{E})}$ as defined in §B.2, we upgrade the composite morphism (B.3.3) to a closed degree 0 morphism in $\mathrm{D}(X)_{\mathrm{dg}}$

$$\widetilde{c}_d(\mathcal{E}, s) : K_{\mathcal{E}} \otimes \underline{\mathbf{k}}_{X_s} \to i_s^!\underline{\mathbf{k}}_{\mathrm{Tot}(\mathcal{E})}[2d], \tag{B.3.4}$$

representing $c_d(\mathcal{E}, s)$ up to a contractible choice.

B.4. Multiplicativity of homotopy canonical Euler classes

Let

$$0 \to \mathcal{E}' \xrightarrow{a} \mathcal{E} \xrightarrow{b} \mathcal{E}'' \to 0 \tag{B.4.1}$$

be a short exact sequence of vector bundles on a stack X, of ranks d', d, d'' respectively, so $d = d' + d''$. We explain how to upgrade the multiplicativity relation $c_d(\mathcal{E}) = c_{d'}(\mathcal{E}')c_{d''}(\mathcal{E}'')$ in $H^{\bullet}(\mathrm{X}, \mathbf{k})$ to the level of homotopy canonical refined classes.

Let $s \in H^0(X, \mathcal{E})$ be a section. Then $s'' := b(s)$ is a section of $\mathcal{E}''$. Its zero locus $i_{s''} : X_{s''} \hookrightarrow X$ can be described, informally, as the locus of points $x \to X$ such that $s(x) \in \mathcal{E}'$. In particular, the bundle $i_{s''}^* \mathcal{E}'$ on $X_{s''}$ carries a section s' given by the restriction of s. The zero locus $(X_{s''})_{s'}$ of this latter section is nothing but X_s, so we have a commutative triangle of closed embeddings

$$\begin{array}{ccc} X_s = (X_{s''})_{s'} & \xrightarrow{i_{s'}} & X_{s''} \\ & \underset{i_s}{\searrow} & \downarrow{\scriptstyle i_{s''}} \\ & & X \end{array} \tag{B.4.2}$$

The multiplicativity of refined Euler classes at the cohomology level can be expressed as the commutativity of the triangle in $\mathrm{D}(X_s)$

$$\begin{array}{ccc} \underline{\mathbf{k}}_{X_s} & \xrightarrow{c_{r'}(i_{s''}^*\mathcal{E}', s')} & i_{s'}^! \underline{\mathbf{k}}_{X_{s''}}[2d'] \\ & \underset{c_d(\mathcal{E}, s)}{\searrow} & \downarrow{\scriptstyle i_{s'}^! c_{r''}(\mathcal{E}'', s'')[2d']} \\ & & i_{s'}^! i_{s''}^! \underline{\mathbf{k}}_X[2d' + 2d''] \end{array} \tag{B.4.3}$$

To prove this commutativity and to lift it to the homotopy canonical level, we denote by

$$\begin{array}{ccccc} \mathrm{Tot}(\mathcal{E}') & \xrightarrow{a} & \mathrm{Tot}(\mathcal{E}) & \xrightarrow{b} & \mathrm{Tot}(\mathcal{E}'') \\ & \underset{\pi'}{\searrow} & \downarrow{\scriptstyle \pi} & \underset{\pi''}{\swarrow} & \\ & & X & & \end{array} \tag{B.4.4}$$

the diagram of the total spaces induced by (B.4.1). We note that

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{E}') & \xrightarrow{a} & \mathrm{Tot}(\mathcal{E}) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{0''_{\mathcal{E}}} & \mathrm{Tot}(\mathcal{E}'') \end{array} \tag{B.4.5}$$

is a Cartesian square. Therefore the same base change argument as used in (B.3.3) gives a morphism of complexes

$$\mathrm{Hom}_{\mathrm{D}(X)_{\mathrm{dg}}}(\underline{\mathbf{k}}_X, 0^!_{\mathcal{E}''}\underline{\mathbf{k}}_{Tot(\mathcal{E}'')}[2d'']) \to \mathrm{Hom}^\bullet_{\mathrm{D}(\mathrm{Tot}(\mathcal{E}')_{\mathrm{dg}}}(\underline{\mathbf{k}}_{\mathrm{Tot}(\mathcal{E}')}, a^! \underline{\mathbf{k}}_{\mathrm{Tot}(\mathcal{E})}[2d''].$$

This morphism induces a morphism

$$K_{\mathcal{E}''} = K_{X/\mathrm{Tot}(\mathcal{E}'')} \to K_{\mathrm{Tot}(\mathcal{E}')/\mathrm{Tot}(\mathcal{E})}. \tag{B.4.6}$$

Also,

$$X \overset{0_{\mathcal{E}'}}{\to} \mathrm{Tot}(\mathcal{E}') \overset{a}{\to} \mathrm{Tot}(\mathcal{E}) \tag{B.4.7}$$

is a composable pair of regular embeddings with composition $0_{\mathcal{E}}$. Therefore composing the pairing (B.2.4) of this composable pair with the morphism (B.4.6), we get a pairing

$$m_{\mathcal{E}',\mathcal{E},\mathcal{E}''} : K_{\mathcal{E}''} \otimes K_{\mathcal{E}'} \to K_{\mathcal{E}}. \tag{B.4.8}$$

These pairings are associative for any admissible (locally split) filtration $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}$ of vector bundles.

Further, (B.4.2) and (B.4.7) combine into a diagram

$$\begin{array}{ccccc} X & \xrightarrow{0_{\mathcal{E}'}} & \mathrm{Tot}(\mathcal{E}') & \xrightarrow{a} & \mathrm{Tot}(\mathcal{E}) \\ {\scriptstyle i_s}\uparrow & & \uparrow{\scriptstyle s'} & & \uparrow{\scriptstyle s} \\ X_s & \xrightarrow{i_{s'}} & X_{s''} & \xrightarrow{i_{s''}} & X \end{array} \tag{B.4.9}$$

consisting of two Cartesian squares, whose concatenation (i.e., the outer perimeter diagram with horizontal edges composed) is the Cartesian square (B.3.1). We now notice that:

- The right square recovers $\widetilde{c}_{d''}(\mathcal{E}'', s'')$ by pullback, as in (B.3.3), from $\widetilde{\eta}_{\mathrm{Tot}(\mathcal{E}')/\mathrm{Tot}(\mathcal{E})}$. This follows from the square (B.4.5) which shows that $\widetilde{\eta}_{\mathrm{Tot}(\mathcal{E}')/\mathrm{Tot}(\mathcal{E})}$ is the image of $\widetilde{\eta}_{\mathcal{E}}$ under (B.4.6).
- The left square recovers $\widetilde{c}_{d'}(i_{s''}^{*}\mathcal{E}', s')$ by pullback from $\widetilde{\eta}_{\mathcal{E}'} = \widetilde{\eta}_{X/\mathrm{Tot}(\mathcal{E}')}$. This is because we can subdivide the square into two Cartesian squares

$$\begin{array}{ccc} X & \xrightarrow{0_{\mathcal{E}'}} & \mathrm{Tot}(\mathcal{E}') \\ {\scriptstyle i_{s''}}\uparrow & & \uparrow \\ X_{s''} & \xrightarrow{0_{i_{s''}^{*}\mathcal{E}'}} & \mathrm{Tot}(i_{s''}^{*}\mathcal{E}') \\ {\scriptstyle i_{s'}}\uparrow & & \uparrow{\scriptstyle s'} \\ X_s & \xrightarrow{i_{s'}} & X_{s''} \end{array}$$

  which show that $\widetilde{\eta}_{i_{s''}\mathcal{E}'}$ is the pullback of $\widetilde{\eta}_{\mathcal{E}'}$
- The composite (outer) square (B.3.1) recovers $\widetilde{c}_d(\mathcal{E}, s)$ by pullback from $\widetilde{\eta}_{\mathcal{E}}$ by definition.

So applying (B.2.5), we obtain a commutative square

$$\begin{array}{ccc} K_{\mathcal{E}''} \otimes K_{\mathcal{E}'} \otimes \underline{\mathbf{k}}_{X_s} & \xrightarrow{K_{\mathcal{E}''}\otimes \widetilde{c}_{d'}(i_{s''}^{*}\mathcal{E}', s')} & i_{s'}^{!}\underline{\mathbf{k}}_{X_{s''}}[2d'] \\ {\scriptstyle m_{\mathcal{E}',\mathcal{E},\mathcal{E}''}\otimes \underline{\mathbf{k}}_{X_s}}\downarrow & & \downarrow{\scriptstyle i_{s'}^{!}\widetilde{c}_{d''}(\mathcal{E}'',s'')[2d']} \\ K_{\mathcal{E}} \otimes \underline{\mathbf{k}}_{X_s} & \xrightarrow{\widetilde{c}_d(\mathcal{E},s)} & i_{s'}^{!} i_{s''}^{!}\underline{\mathbf{k}}_X[2d' + 2d''] \end{array} \tag{B.4.10}$$

which is a lift of (B.4.3) to the homotopy canonical level.

Acknowledgments. We are grateful to B. Feigin, G. Ginot, B. Hennion, A. Negut, M. Porta, N. Rozenblyum and Y. Soibelman for useful discussions. We would like to thank J. Schürrmann for pointing out an inaccuracy in an earlier version, and B. Davison for pointing out to us that the statement about commutativity of $R(\mathbb{A}^2)$ in an earlier version is not true. We are also grateful to the referee for numerous detailed remarks and suggestions that helped us improve the paper.

The results of this work have been presented at several conferences: CMND international conference on geometric representation theory and symplectic varieties, 06/2018, University of Notre Dame; Vertex algebras, factorization algebras and applications, 07/2018, Kavli IPMU; Vertex algebras and gauge theory, 12/2018, Simons Center; TCRT6, 01/2019, Academia Sinica). We are grateful to the organizers of these events for the invitations to speak.

Funding. The research of M.K. was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan, by the IAS School of Mathematics and, during the revision of the paper, by the JSPS KAKENHI grant 20H01794. The research of E.V. was supported by the grant ANR-18-CE40-0024 (ANR), the UMR7586 (CNRS) and the Institut Universitaire de France (IUF).

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