Rankeya Datta · Takumi Murayama

Tate algebras and
Frobenius non-splitting of excellent regular rings

Abstract. An excellent ring of prime characteristic for which the Frobenius map is pure is also Frobenius split in many commonly occurring situations in positive characteristic commutative algebra and algebraic geometry. However, using a fundamental construction from rigid geometry, we show that excellent F-pure rings of prime characteristic are not Frobenius split in general, even for Euclidean domains. Our construction uses the existence of a complete non-Archimedean field k of characteristic p with no nonzero continuous k-linear maps $k^{1/p} \to k$. An explicit example of such a field is given based on ideas of Gabber, and may be of independent interest. Our examples settle a long-standing open question in the theory of F-singularities whose origin can be traced back to when Hochster and Roberts introduced the notion of F-purity. The excellent Euclidean domains we construct also admit no nonzero R-linear maps $R^{1/p} \to R$. These are the first examples that illustrate that F-purity and Frobenius splitting define different classes of singularities for excellent domains, and are also the first examples of excellent domains with no nonzero p^{-1}-linear maps. The latter is particularly interesting from the perspective of the theory of test ideals.

Keywords. Tate algebra, Frobenius splitting, F-purity, p^{-1}-linear map, excellent ring, convergent power series

1. Introduction

Let R be a Noetherian ring of prime characteristic $p > 0$. A key theme in characteristic p commutative algebra and algebraic geometry is to use the *Frobenius map*

$$F_R : R \to F_{R*}R, \quad r \mapsto r^p,$$

to study the singularities of R (see the surveys [33,43,46]). This line of study was initiated by Kunz, who proved that R as above is regular if and only if F_R is faithfully flat [27,

Rankeya Datta: Department of Mathematics, University of Missouri,
Columbia, MO 65211, USA; rankeya.datta@missouri.edu

Takumi Murayama: Department of Mathematics, Purdue University,
West Lafayette, IN 47907-2067, USA; murayama@purdue.edu

Thm. 2.1]. Following Kunz's results and building on their work showing that rings of invariants of reductive groups are Cohen–Macaulay [22], Hochster and Roberts defined R to be *F-pure* if F_R is pure in the following sense: for every R-module M, the base change $M \to M \otimes_R F_{R*}R$ is injective [23, p. 121]. Since faithfully flat maps are pure [30, Thm. 7.5 (i)], it follows that regular rings of characteristic p are F-pure. In particular, F-pure rings form a natural class of singular characteristic p Noetherian rings.

In their study of the cohomology of Schubert varieties, Mehta and Ramanathan defined R to be *Frobenius split* if F_R splits as a map of R-modules, that is, if F_R admits an R-linear left inverse [31, Def. 2]. Since split maps are automatically pure, Frobenius split rings are F-pure. Whether the converse holds in nice situations was a mystery, prompting the following folklore question (see for instance [43, Rem. 1.12]):

Question 1.1. *Let R be an* excellent *Noetherian ring of prime characteristic. If R is F-pure, is it necessarily Frobenius split?*

For the definition of an excellent ring, we refer the reader to [16, Déf. 7.8.2].

Our main result shows that even excellent *regular* rings are not Frobenius split in general.

Theorem A. *For every prime $p > 0$, there exists a complete non-Archimedean field $(k, | \cdot |)$ of characteristic p such that the Tate algebra $T_n(k) := k\{X_1, \ldots, X_n\}$ is not Frobenius split for each $n > 0$. In fact, $T_n(k)$ admits no nonzero $T_n(k)$-linear maps $F_{T_n(k)*}T_n(k) \to T_n(k)$ for each $n > 0$.*

The origin of Question 1.1 can be traced to the aforementioned work of Hochster and Roberts on the purity of Frobenius, where the equivalence of F-purity and Frobenius splitting is observed when F_R is a finite map [23, Cor. 5.3]. Finiteness of Frobenius implies R is excellent by [28, Thm. 2.5]. Question 1.1 also has an affirmative answer when R is essentially of finite type over a complete local ring by [10, Thm. 3.1.1]. Thus, the notions of F-purity and Frobenius splitting coincide for all rings appearing in the study of algebraic varieties over positive characteristic fields.

The Tate algebra $T_n(k)$, introduced by Tate [47], is the analogue of the polynomial ring $k[X_1, \ldots, X_n]$ in the world of rigid analytic geometry and shares many of its good properties (see [5] and Theorem 2.7). For example, $T_n(k)$ is an n-dimensional Jacobson unique factorization domain that is regular (hence is F-pure), and is excellent by a theorem of Kiehl [25, Thm. 3.3]. Theorem A says that despite the similarities between $T_n(k)$ and $k[X_1, \ldots, X_n]$, rigid analytic geometry and classical algebraic geometry are fundamentally different from the perspective of Frobenius splittings. Theorem A should also be contrasted with [11, Thm. 3.2], where Smith and the first author show that if R is a Noetherian domain of characteristic p whose fraction field K satisfies $[K^{1/p} : K] < \infty$, then R is excellent precisely when R admits a nonzero R-linear map $F_{R*}R \to R$.

We establish Theorem A by first proving the following necessary and sufficient criterion for Frobenius splitting of the Tate algebra $T_n(k)$ in terms of a topological property of the non-Archimedean field k.

Theorem B. *Let $(k, | \cdot |)$ be a complete non-Archimedean field of characteristic $p > 0$. The following are equivalent:*

- *$T_n(k)$ is Frobenius split for each integer $n > 0$.*
- *$T_n(k)$ has a nonzero $T_n(k)$-linear map $F_{T_n(k)*}T_n(k) \to T_n(k)$ for each integer $n > 0$.*
- *There exists a nonzero continuous k-linear map $f : k^{1/p} \to k$.*

The equivalent statements above are a subset of those we prove in Theorem 3.1. Here, we view $k^{1/p}$ as a normed k-vector space with the canonical norm that extends the norm on k, and then equip $k^{1/p}$ and k with the corresponding metric topologies to be able to talk about continuous maps.

Using this topological characterization, Theorem A follows by explicitly constructing a non-Archimedean field $(k, | \cdot |)$ for which $k^{1/p}$ admits no nonzero continuous linear functionals (see Theorem 5.2). The existence of such fields is suggested by Gerritzen [14] and Kiehl [25] (see also [6, p. 63]), who take significant care to show that the Tate algebra is Japanese and excellent when $[k^{1/p} : k]$ is infinite. However, we were unable to locate an explicit example in the literature, and the example in Theorem 5.2 is due to Ofer Gabber. Similar constructions of valuative fields with infinite p-degree have been studied by Rzepka (formerly Blaszczok) and Kuhlmann [1], although the connection with the existence of continuous functionals was not made (see Remark 5.4).

We would like to isolate one consequence of Theorem 3.1 to emphasize the simplicity of the examples obtained in this paper.

Corollary C. *There exists an excellent Euclidean domain R of characteristic $p > 0$ such that R admits no nonzero R-linear maps $F_{R*}R \to R$. Moreover, one can choose R to be local and Henselian as well.*

Corollary C follows from Theorem 3.1 and the aforementioned example of a non-Archimedean field k with no nonzero continuous linear functionals $k^{1/p} \to k$ upon taking R to be the Tate algebra $T_1(k)$. For this, one uses the well-known fact that like $k[X]$, the Tate algebra $T_1(k)$ is a Euclidean domain (see Theorem 2.7 (vi)). In addition, we obtain a local and Henselian example by proving an analogue of Theorem 3.1 for *convergent power series rings* (see Definition 4.1) in Theorem 4.4. Corollary C should be contrasted with the well-known fact that a Noetherian complete local F-pure ring is always Frobenius split (see Remark 5.6). Thus, our local Henselian example shows that Question 1.1 fails even for the class of excellent Noetherian local rings that are closest in behavior to complete local rings.

Maps of the form

$$F_{R*}R \to R \qquad (1)$$

are used extensively in prime characteristic commutative algebra and algebraic geometry. One of the first such examples comes from looking at the stalks of the *Cartier operator,* which is a map of the form $F_{X*}\omega_X \to \omega_X$ on a smooth variety X over a perfect field of prime characteristic [7]. Blickle and Böckle later called maps of the form (1) p^{-1}-*linear maps* [3, p. 86]. The reason for this terminology is that for $r \in R$ and $x \in F_{R*}R$,

a p^{-1}-linear map ϕ satisfies

$$\phi(r^p x) = \phi(r \cdot x) = r\phi(x),$$

where the first equality follows from the R-module structure on $F_{R*}R$ induced by restriction of scalars via the Frobenius map F_R.

The notion of a p^{-1}-linear map was used by Hara and Takagi [19], Schwede [40], and Blickle [2], among others, to give an alternative approach to the theory of *test ideals*. Test ideals were originally defined by Hochster and Huneke in their celebrated theory of tight closure [21], and later extended to the context of pairs by Hara and Yoshida [20] and Takagi [45]. Quite surprisingly, test ideals were shown to be related to the characteristic zero notion of multiplier ideals by work of Smith [42], Hara [18], Hara and Yoshida [20], and Takagi [45]. Since then, algebraists and geometers have used test ideals to establish many prime characteristic analogues of characteristic 0 results whose proofs require deep vanishing theorems that are known to fail in positive characteristic [17, 37]. See the surveys [4, 33, 41, 43, 46] for various applications. More recently, Schwede's insight [40] to use compatibly split subschemes defined via p^{-1}-linear maps as a foundation for the theory of test ideals, building on the work of Mehta and Ramanathan [31] and Lyubeznik and Smith [29], has further advanced our understanding of positive characteristic rings and varieties. In this context, Corollary C serves to caution us that although excellent rings have nonzero p^{-1}-linear maps in many geometric situations, they may fail to have such maps in general. In particular, the approach to test ideals using p^{-1}-linear maps will require a different formulation if one hopes to generalize the theory to arbitrary excellent rings and schemes over $\mathbf{F}_p$. Even an extension of the theory encompassing Tate algebras and their quotients may need a new approach.

While our emphasis so far has been on nonexistence results, Theorem 3.1 has positive consequences as well. For instance, we can show that $T_n(k)$ is Frobenius split for many commonly occurring non-Archimedean fields of positive characteristic.

Corollary D. *Let $(k, |\cdot|)$ be a complete non-Archimedean field of characteristic $p > 0$. For each $n > 0$, the Tate algebra $T_n(k)$ is Frobenius split in the following cases:*

(i) *$(k, |\cdot|)$ is spherically complete.*

(ii) *$k^{1/p}$ has a dense k-subspace V that has a countable k-basis, hence in particular if $[k^{1/p} : k] < \infty$.*

(iii) *$|k^\times|$ is not discrete, and the norm on $k^{1/p}$ is polar.*

A spherically complete non-Archimedean field (see Definition 2.13) is a generalization of a field equipped with a discrete valuation whose corresponding valuation ring is complete. The notion of polarity in (iii) is a more technical condition due to Schikhof generalizing (i) and (ii), which appears in Theorem 2.15 (iii). The proof of Corollary D requires some knowledge of non-Archimedean functional analysis that we will summarize in Section 2.2. The essential fact is that when k is a non-Archimedean field of the

above two types, then variants of the Hahn–Banach theorem for normed spaces over **R** and **C** also hold for normed spaces over k. In particular, this gives a wealth of nonzero continuous linear functionals of $k^{1/p}$, thereby allowing us to use Theorem 3.1.

Organization of the paper

In Section 2, we review all definitions and results from non-Archimedean analysis that are used in the rest of the paper. Our aim in writing this section has been to motivate some of the constructions of non-Archimedean geometry by relating them to more familiar constructions for polynomial rings. In Section 3, we first prove Theorem 3.1 and then use it to deduce Frobenius splitting of Tate algebras in some cases in Corollary D. In Section 4, we adapt the proof of Theorem 3.1 to the local setting of convergent power series rings (see Theorem 4.4), which later gives us a wealth of local examples for which Question 1.1 has a negative answer. Finally, Section 5 contains Gabber's example of a non-Archimedean field k such that $k^{1/p}$ admits no nonzero continuous linear functionals. Theorem A and Corollary C are then proved as straightforward consequences of Theorem 3.1 by working over the aforementioned example. Their analogues for the local ring of convergent power series are observed as well.

Notation

All rings will be commutative with identity. If R is a ring of prime characteristic $p > 0$, then the *Frobenius map* on R is the ring map

$$F_R \colon R \to F_{R*}R, \quad r \mapsto r^p.$$

The notation $F_{R*}R$ is used to emphasize the fact that the target of the Frobenius map has the (left) R-algebra structure given by $r \cdot x = r^p x$. A p^{-1}-*linear map on R is the datum of an R-linear map $F_{R*}R \to R$. Thus, a Frobenius splitting of R is a p^{-1}-linear map on R that sends 1 to 1.

    If R is a domain, we will sometimes identify $F_{R*}R$ with the ring $R^{1/p}$ of p-th roots of elements in R. Under this identification, the Frobenius map $F_R \colon R \to F_{R*}R$ corresponds to the inclusion $R \hookrightarrow R^{1/p}$.

2. A review of non-Archimedean analysis

We will use basic results from non-Archimedean functional analysis to produce our counterexamples and to prove that the Tate algebra is Frobenius split in some commonly occurring situations. Therefore, we collect all relevant definitions and results we will need in the present section to make the paper easier to navigate for readers unfamiliar with the non-Archimedean world. All results appearing in this section are well-known.

2.1. Non-Archimedean fields, normed spaces, and Tate algebras

Definition 2.1. A *real-valued field* $(k, | \cdot |)$ is a pair consisting of a field k and a non-Archimedean valuation $| \cdot |: k \to \mathbf{R}_{\geq 0}$ (written multiplicatively), which is a function that satisfies:

- $|x| = 0$ if and only if $x = 0$;
- $|x + y| \leq \max\{|x|, |y|\}$; and
- $|xy| = |x| \cdot |y|$.

The *valuation ring of k* is the subring $k^\circ := \{x \in k : |x| \leq 1\}$. This is a local ring with maximal ideal $k^{\circ\circ} := \{x \in k : |x| < 1\}$. The *value group* of k is the group $|k^\times|$.

Convention 2.2. By a *non-Achimedean field* we mean a real-valued field that is complete with respect to the metric $|x - y|$ induced by $| \cdot |$, and whose value group $|k^\times|$ is not the trivial group. This latter assumption implies that k° has Krull dimension 1 [30, Thm. 10.7].

Throughout this paper, we will only consider real-valued fields that are non-Archimedean, with the exception of the auxiliary field M in Theorem 5.2.

We will use the following well-known property of valuations.

Lemma 2.3 (see [5, Prop. 2.1/2]). *Let $(k, | \cdot |)$ be a non-Archimedean field. For $x, y \in k$, if $|x| \neq |y|$, then*

$$|x + y| = \max\{|x|, |y|\}.$$

We next define the analogue of a vector space over a non-Archimedean field.

Definition 2.4. Let $(k, | \cdot |)$ be a non-Archimedean field. A *normed space* $(E, \| \cdot \|)$ over k is a k-vector space E with a *norm* $\| \cdot \|: E \to \mathbf{R}_{\geq 0}$ that satisfies the following properties:

- $\|x\| = 0$ if and only if $x = 0$;
- $\|x + y\| \leq \max\{\|x\|, \|y\|\}$; and
- If $c \in k$ and $x \in E$, then $\|cx\| = |c| \cdot \|x\|$.

If E is complete in the metric induced by $\| \cdot \|$, then E is called a *Banach space* over k.

A *Banach k-algebra* $(A, \| \cdot \|)$ is a k-algebra A such that $(A, \| \cdot \|)$ is a k-Banach space and such that the norm $\| \cdot \|$ satisfies the following additional property:

- $\|xy\| \leq \|x\| \cdot \|y\|$.

The norm $\| \cdot \|$ is *multiplicative* if equality holds in the above inequality.

All Banach algebras considered in this paper will be multiplicative.

Remark 2.5. (i) A finite-dimensional vector space E over a non-Archimedean field $(k, | \cdot |)$ can always be given the structure of a k-Banach space in a canonical way. Indeed, if $\{x_1, \ldots, x_n\}$ is a k-basis of E, then for every element $x = \sum_i a_i x_i$ where $a_i \in k$, one can define

$$\|x\| := \max_{1 \leq i \leq n} |a_i|.$$

Now, because k is complete with respect to $|\cdot|$, every norm on E is equivalent to the one just defined, even though the above norm depends on the choice of a basis [5, App. A, Thm. 1]. It is fairly straightforward to verify that E is complete with respect to the above norm.

(ii) When ℓ is an algebraic extension of k, by expressing ℓ as a filtered colimit of finite subextensions, one can show that there exists a unique (not just equivalent) norm on ℓ that extends the norm on k [5, App. A, Thm. 3]. However, if $[\ell : k] = \infty$, then ℓ need not be complete with respect to the metric induced by this norm.

The principal example of a Banach k-algebra is the Tate algebra.

Definition 2.6 (see [5, Def. 2.2/2]). Let $(k, |\cdot|)$ be a non-Archimedean field. For every positive integer $n > 0$, the *Tate algebra* in n indeterminates over k is the k-subalgebra

$$T_n(k) := k\{X_1, \ldots, X_n\} := \left\{ \sum_{\nu \in \mathbf{Z}_{\geq 0}^n} a_\nu X^\nu : \begin{array}{l} a_\nu \in k \text{ and } |a_\nu| \to 0 \\ \text{as } \nu_1 + \cdots + \nu_n \to \infty \end{array} \right\}$$

of the formal power series $k[\![X_1, \ldots, X_n]\!]$ in n indeterminates over k. An element of $T_n(k)$ is called a *restricted power series*. For $n = 1$, we will denote the indeterminate in $T_1(k)$ by just X instead of X_1.

The Tate algebra becomes a k-Banach algebra when equipped with the *Gauss norm*, which is defined as follows. For every element $\sum_{\nu \in \mathbf{Z}_{\geq 0}^n} a_\nu X^\nu \in T_n(k)$, we set

$$\left\| \sum_{\nu \in \mathbf{Z}_{\geq 0}^n} a_\nu X^\nu \right\| := \max_{\nu \in \mathbf{Z}_{\geq 0}^n} |a_\nu|.$$

One can show that the Gauss norm is multiplicative [5, pp. 13–14].

The remarkable fact is that $T_n(k)$ shares many properties of the polynomial ring $k[X_1, \ldots, X_n]$. We collect these properties for readers who may be unfamiliar with $T_n(k)$.

Theorem 2.7. *Let $(k, |\cdot|)$ be a non-Archimedean field, and let n be a positive integer. Then the Tate algebra $T_n(k)$ satisfies the following properties:*

(i) *$T_n(k)$ is Noetherian.*

(ii) *$T_n(k)$ is a unique factorization domain.*

(iii) *$T_n(k)$ is Jacobson, that is, every radical ideal is the intersection of the maximal ideals containing it.*

(iv) *All maximal ideals of $T_n(k)$ are generated by n elements and have height n. In particular, $T_n(k)$ has Krull dimension n.*

(v) *$T_n(k)$ is regular.*

(vi) *$T_1(k)$ is a Euclidean domain with associated Euclidean function $T_1 \smallsetminus \{0\} \to \mathbf{Z}_{\geq 0}$ given by mapping a restricted power series $f = \sum_{i=0}^{\infty} a_i X^i$ to the largest index N such that $|a_N| = \|f\|$.*

(vii) *$T_n(k)$ is excellent.*

Indication of proof. (i)–(iv) are proved in [5, Props. 2.2/14–17]. (v) follows from (iv), since the latter implies that all localizations of $T_n(k)$ at maximal ideals are regular local. (vi) follows from [5, Cor. 2.2/10]. Finally, (vii) is the hardest property to show, and can be found in [25, Thm. 3.3] (see also [8, §1.1]). Key ingredients in the proofs of all these properties are the Weierstrass division and Weierstrass preparation theorems [5, Thm. 2.2/8 and Cor. 2.2/9]. ∎

For the proofs of the Weierstrass division and preparation theorems, one works with certain distinguished elements in $T_n(k)$ that we now introduce. These elements should be thought of as power series analogues of essentially monic polynomials (with respect to one of the variables) in polynomial rings.

Definition 2.8 (see [5, Def. 2.2/6]). A restricted power series $g = \sum_{\nu=0}^{\infty} g_\nu X_n^\nu \in T_n(k)$ with coefficients $g_\nu \in T_{n-1}(k)$ is called X_n-*distinguished of order* $s \in \mathbf{Z}_{\geq 0}$ if the following hold:

- g_s is a unit in $T_{n-1}(k)$; and
- $\|g_s\| = \|g\|$ and $\|g_s\| > \|g_\nu\|$ for every $\nu > s$.

Remark 2.9. (i) An element $f \in T_n(k)$ is a unit if and only if the constant term of f has absolute value strictly bigger than the absolute value of the coefficients of all other terms of f [5, Cor. 2.2/4]. In the language of distinguished elements, $f \in T_n(k)$ is a unit precisely when it is X_n-distinguished of order 0.

 (ii) An X_n-distinguished $g \in T_n(k)$ always has a term that just involves the variable X_n. Indeed, if $g = \sum_{\nu=0}^{\infty} g_\nu X_n^\nu \in T_n(k)$ is distinguished of order s, then g_s, being a unit, has a nonzero constant term. Then g has a term of the form $a_s X_n^s$, where $a_s \in k^\times$ is the constant term of g_s.

(iii) Every nonzero element of $T_1(k) = k\{X\}$ is X-distinguished of order equal to the value of the element under the associated Euclidean function that gives $T_1(k)$ the structure of a Euclidean domain (see Theorem 2.7 (vi)).

The next result can be thought of as the Tate algebra analogue of a technique used in the proof of Noether normalization for finite type algebras over a field that makes polynomials essentially monic in one of the variables upon applying a suitable ring automorphism. In fact, this result is used to prove the rigid-analytic analogue of Noether normalization for quotients of Tate algebras.

Lemma 2.10 (see [5, Lem. 2.2/7]). *Given finitely many nonzero elements* $g_1, \ldots, g_r \in T_n(k)$, *there is a k-algebra automorphism (automatically continuous)*

$$\sigma: T_n(k) \to T_n(k), \quad X_i \mapsto \begin{cases} X_i + X_n^{\alpha_i} & \text{for } i < n, \\ X_n & \text{for } i = n, \end{cases}$$

for some $\alpha_1, \ldots, \alpha_{n-1} \in \mathbf{Z}_{\geq 0}$ *such that the elements* $\sigma(g_1), \ldots, \sigma(g_r)$ *are X_n-distinguished.*

2.2. Continuous maps, spherically complete fields, and the Hahn–Banach extension property

We begin by giving alternative characterizations of continuous maps of normed spaces. Recall that by Convention 2.2, all non-Archimedean fields are complete and non-trivially valued. Given a normed space $(E, \|\cdot\|)$, we will say that a subset $S \subseteq E$ is *bounded* if and only if there exists some $a \in \mathbf{R}_{\geq 0}$ such that S is contained in the closed ball $B_a(0)$ of radius a centered at $0 \in E$.

Lemma 2.11. *Let $(k, |\cdot|)$ be a non-Archimedean field and $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be normed spaces. Then, for a k-linear map $f : E \to F$, the following are equivalent:*

(i) *f is continuous.*

(ii) *f maps null sequences to null sequences.*

(iii) *f maps null sequences to bounded sequences.*

(iv) *f maps bounded sets to bounded sets.*

(v) *There exist $a, b \in \mathbf{R}_{>0}$ such that $f(B_a(0)) \subseteq B_b(0)$.*

(vi) *There exists $B \in \mathbf{R}_{>0}$ such that for all $x \in E$, we have $\|f(x)\|_F \leq B \cdot \|x\|_E$.*

Proof. Since a null sequence converges to 0, (i)$\Rightarrow$(ii) follows by the continuity of f and the fact that f maps 0 to 0. (ii)$\Rightarrow$(iii) holds because null sequences are bounded.

For (iii)$\Rightarrow$(iv), assume for contradiction that there exists some $a \in \mathbf{R}_{>0}$ such that $f(B_a(0))$ is unbounded. Then there exists a sequence $(x_n)_n \subseteq B_a(0)$ and a sequence $(c_n)_n \subseteq k^\times$ such that

$$\sqrt{\|f(x_n)\|_F} \geq |c_n|$$

for all $n > 0$, and $|c_n| \to \infty$ as $n \to \infty$. Here we use the non-triviality of $|k^\times|$ to get the sequence $(c_n)_n$ with the said properties. Since $(x_n)_n$ is bounded, $(c_n^{-1} x_n)_n$ is a null sequence whose image is an unbounded sequence, because

$$\|f(c_n^{-1} x_n)\|_F = |c_n^{-1}| \cdot \|f(x_n)\|_F \geq \sqrt{\|f(x_n)\|_F}.$$

This contradicts (iii), completing the proof of (iii)$\Rightarrow$(iv).

The proof of (iv)$\Rightarrow$(v) is clear. For (v)$\Rightarrow$(vi), by non-triviality of $|k^\times|$, we may choose a nonzero $c \in k$ such that $0 < |c| < 1$, $|c| \leq a$, and consequently $f(B_{|c|}(0)) \subseteq B_b(0)$. Let $x \in E$ and choose $m \in \mathbf{Z}$ such that

$$|c|^{m+2} < \|x\|_E \leq |c|^{m+1}. \tag{2}$$

We have $\|c^{-m} x\|_E \leq |c|$, and so

$$\|f(x)\|_F = |c|^m \|f(c^{-m} x)\|_F \leq |c|^m b \leq |c|^{-2} b \|x\|_E,$$

where the middle inequality follows because $f(B_{|c|}(0)) \subseteq B_b(0)$, and the final inequality follows from (2). Then taking $B := |c|^{-2} b$, we get (v)$\Rightarrow$(vi).

Finally, (vi)$\Rightarrow$(i) follows by using the ϵ-δ definition of continuity. This finishes the proof of the lemma. ∎

Thus, by Lemma 2.11, for a continuous linear map $f: (E, \|\cdot\|_E) \to (F, \|\cdot\|_F)$ of normed spaces,

$$\sup_{x \neq 0} \frac{\|f(x)\|_F}{\|x\|_E}$$

is finite. In other words, all continuous maps of normed spaces are *bounded continuous*.

We next introduce the Hahn–Banach extension property over non-Archimedean fields. The corresponding extension property over **R** and **C** is one of the most important results in functional analysis.

Definition 2.12 (see [34, p. 170]). A normed space $(E, \|\cdot\|)$ over a non-Archimedean field $(k, |\cdot|)$ satisfies the $(1 + \epsilon)$-*Hahn–Banach extension property* if for every subspace D of E, for every $\epsilon > 0$, and for every linear functional $f: D \to k$ such that $|f(x)| \leq \|x\|$ for all $x \in D$, there exists a linear functional $\widetilde{f}: E \to k$ extending f such that for all $x \in E$, we have

$$|\widetilde{f}(x)| \leq (1 + \epsilon)\|x\|.$$

We say E satisfies the *Hahn–Banach extension property* if ϵ can be chosen to be 0 for the extension $\widetilde{f}$, that is, if there exists an extension $\widetilde{f}$ such that $|\widetilde{f}(x)| \leq \|x\|$ for all $x \in E$.

The terminology "$(1 + \epsilon)$-Hahn–Banach extension property" is used in passing in [34], but it is convenient. Note that both f and its extension $\widetilde{f}$ are continuous by Lemma 2.11. In particular, the $(1 + \epsilon)$-Hahn–Banach extension property guarantees that E has many continuous linear functionals. We are particularly interested in conditions on the non-Archimedean field k which guarantee that every normed space over k satisfies the $(1 + \epsilon)$-Hahn–Banach extension property or the usual Hahn–Banach extension property. This leads to the notion of a spherically complete field, which should be viewed as a generalization of a (complete) non-Archimedean field with a discrete value group.

Definition 2.13. A non-Archimedean field $(k, |\cdot|)$ is *spherically complete* if, for every decreasing sequence $D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$ of closed disks, the intersection $\bigcap_n D_n$ is non-empty.

Remark 2.14.   (i) The defining property of spherical completeness implies completeness, even though our convention is that non-Archimedean fields are complete (see [38, p. 24]).

(ii) If $|k^\times| \cong \mathbf{Z}$, then k is spherically complete. In other words, the fraction field of a complete discrete valuation ring is spherically complete in the topology induced by the valuation (see [38, Cor. 2.4]).

(iii) There exist non-Archimedean fields that are not spherically complete. For example, $\mathbf{C}_p$, the completion of the algebraic closure of $\mathbf{Q}_p$, is not spherically complete [39, Cor. 20.6]. However, every non-Archimedean field admits an embedding into a spherically complete field [26, Satz 24] (see also [38, Thm. 4.49]).

(iv) A non-Archimedean field $(k, |\cdot|)$ is spherically complete if and only if k admits no proper immediate extensions [24, Thm. 4] (see also [38, Thm. 4.47]). Recall that we

say that an extension of real-valued fields $(k, |\cdot|_k) \hookrightarrow (\ell, |\cdot|_\ell)$ (i.e. an extension of fields such that $|\cdot|_\ell$ restricted to k equals $|\cdot|_k$) is an *immediate extension* if $|k^\times|_k = |\ell^\times|_\ell$ and the induced map on residue fields $k^\circ/k^{\circ\circ} \hookrightarrow \ell^\circ/\ell^{\circ\circ}$ is an isomorphism.

Every normed space over a spherically complete field satisfies the Hahn–Banach extension property. This, and some related results, are summarized below.

Theorem 2.15. *Let $(E, \|\cdot\|)$ be a normed space over a non-Archimedean field $(k, |\cdot|)$.*

(i) *If k is spherically complete (in particular, if k is discretely valued), then E satisfies the Hahn–Banach extension property.*

(ii) *If E has a dense subspace V which has a countable basis over k, then E satisfies the $(1 + \epsilon)$-Hahn–Banach extension property.*

(iii) *Assume that $|k^\times|$ is not discrete, and suppose the norm $\|\cdot\|$ on E is polar in the sense that for every*

$$x \notin E^\circ := \{x \in E : \|x\| \leq 1\},$$

there exists a linear functional $f : E \to k$ such that $|f(E^\circ)| \leq 1$ and $|f(x)| > 1$. For every finite-dimensional subspace D of E, if $f : D \to k$ is a linear map that satisfies $|f(x)| \leq \|x\|$ for all $x \in D$, then for all $\epsilon > 0$, there exists an extension $\widetilde{f} : E \to k$ of f such that $|\widetilde{f}(x)| \leq (1 + \epsilon)\|x\|$ for all $x \in E$.

Proof. (i) follows from [34, Thm. 4.1.1], (ii) from [34, Thm. 4.2.4], and (iii) from [34, Thm. 4.4.5]. In (i), the discretely valued case follows from the spherically complete case by Remark 2.14 (ii). ∎

Remark 2.16. In [34], norms are assumed to be *solid* in the sense of [34, Def. 3.1.1] (see [34, p. 93]). However, the solidity assumption is not used in the results [34, Thms. 4.1.1 and 4.2.4] cited for the proofs of (i) and (ii) above. Moreover, solidity is automatic if $|k^\times|$ is dense in $\mathbf{R}_{\geq 0}$, or equivalently, when $|k^\times|$ is not discrete. For (iii), the notion of polarity was defined by Schikhof as a common generalization of the situations in (i) and (ii) (see [34, Def. 4.4.1, Lem. 4.4.4, and Thm. 4.4.3]).

Theorem 2.15 raises the natural question of whether one can construct Banach spaces over a non-Archimedean field whose continuous dual space is trivial. The next result shows that such spaces exist for every non-Archimedean field that is not spherically complete because any non-Archimedean field embeds into one that is spherically complete by Remark 2.14 (iii).

Proposition 2.17. *Consider an extension of non-Archimedean fields $(k, |\cdot|_k) \hookrightarrow (\ell, |\cdot|_\ell)$ such that ℓ is spherically complete but k is not. Then there are no nonzero continuous k-linear maps $\ell \to k$.*

Proof. The proposition follows by setting E to be ℓ in the statement of [38, Cor. 4.3] or in the proof of [36, Thm. 2]. ∎

Remark 2.18. For the curious reader who prefers a more explicit example of a Banach space over a non-spherically-complete non-Archimedean field $(k, |\cdot|)$ that admits no nonzero continuous functionals, note that if ℓ^∞ is the k-Banach space of sequences of elements of k with bounded norms and c_0 is the closed subspace of null sequences, then k is spherically complete if and only if ℓ^∞/c_0 admits a nonzero continuous linear functional [34, Cor. 4.1.13].

3. Frobenius splitting of some Tate algebras

We begin by giving a topological characterization of Frobenius splitting of Tate algebras over non-Archimedean fields of prime characteristic, which is a stronger version of Theorem B.

Theorem 3.1. *Let $(k, |\cdot|)$ be a non-Archimedean field of characteristic $p > 0$. The following are equivalent:*

  (i) *$T_n(k)$ is Frobenius split for each integer $n > 0$.*

 (ii) *$T_n(k)$ has a nonzero p^{-1}-linear map for each integer $n > 0$.*

(iii) *There exists an integer $n > 0$ for which $T_n(k)$ has a nonzero p^{-1}-linear map.*

(iv) *$T_1(k)$ has a nonzero p^{-1}-linear map.*

 (v) *$T_1(k)$ is Frobenius split.*

(vi) *There exists a nonzero continuous k-linear map $f \colon F_{k*}k \to k$.*

Proof. (i)$\Rightarrow$(ii) is clear since Frobenius splittings are nonzero p^{-1}-linear maps. Similarly, so is (ii)$\Rightarrow$(iii).

For (iii)$\Rightarrow$(iv), we may assume $n > 1$. Let $\Phi \colon F_{T_n*}T_n \to T_n$ be a nonzero T_n-linear map, and choose $a \in F_{T_n*}T_n$ such that $\Phi(a) \neq 0$. By replacing Φ with $\Phi \circ F_{T_n*}(- \cdot a)$, we may assume that Φ is such that $\Phi(1) \neq 0$. We now modify Φ to create a T_n-linear map $\widetilde{\Phi} \colon F_{T_n*}T_n \to T_n$ such that 1 maps to an element not contained in the ideal $(X_1, \ldots, X_{n-1})$. By Lemma 2.10, there exists an automorphism $\sigma \colon T_n \to T_n$ such that $g = \sigma(\Phi(1))$ is X_n-distinguished in the sense of Definition 2.8. In particular, $g \notin (X_1, \ldots, X_{n-1})$ by Remark 2.9 (ii). Now consider the composition

$$\widetilde{\Phi} \colon F_{T_n*}T_n \xrightarrow{F_{T_n*}(\sigma^{-1})} F_{T_n*}T_n \xrightarrow{\Phi} T_n \xrightarrow{\sigma} T_n.$$

We claim that $\widetilde{\Phi}$ is T_n-linear, or said differently, that $\widetilde{\Phi}$ defines a p^{-1}-linear map on T_n. We will exploit the fact that σ is a ring automorphism. Let $h \in T_n$ and $f \in F_{T_n*}T_n$. We then have

$$\widetilde{\Phi}(h \cdot f) = \widetilde{\Phi}(h^p f) = (\sigma \circ \Phi)(\sigma^{-1}(h^p f)) = (\sigma \circ \Phi)(\sigma^{-1}(h)^p \sigma^{-1}(f)))$$
$$= \sigma(\sigma^{-1}(h) \cdot \Phi(\sigma^{-1}(f))) = h \cdot \sigma(\Phi(\sigma^{-1}(f))) = h \cdot \widetilde{\Phi}(f)$$

as desired. Since σ^{-1} maps 1 to 1, we see that $\widetilde{\Phi}$ maps 1 to the X_n-distinguished element g. Finally, consider the composition

$$F_{k\{X_n\}*}k\{X_n\} \hookrightarrow F_{T_n*}T_n \xrightarrow{\widetilde{\Phi}} T_n \xrightarrow{\pi} k\{X_n\},$$

where π is the quotient map sending $X_1, \ldots, X_{n-1}$ to 0. The composition defines a p^{-1}-linear map on $k\{X_n\}$, and sends 1 to a nonzero element in $k\{X_n\}$ because $g \notin (X_1, \ldots, X_{n-1}) = \ker(\pi)$.

For (iv)$\Rightarrow$(v), we will use the fact that T_1 is a Euclidean domain (Theorem 2.7 (vi)), hence in particular a principal ideal domain (PID). Let $K := \mathrm{Frac}(T_1)$ and let

$$\Phi \colon F_{T_1*}T_1 \to T_1$$

be a nonzero T_1-linear map. Since T_1 is a PID, there exists a nonzero $a \in T_1$ such that $\mathrm{im}(\Phi) = aT_1$. Then by construction, the composition

$$F_{T_1*}T_1 \xrightarrow{\Phi} T_1 \hookrightarrow K \xrightarrow{-\cdot a^{-1}} K$$

is a T_1-linear map whose image is T_1. Therefore, restricting the codomain of this composition to T_1 gives us a surjective T_1-linear map $\widetilde{\Phi} \colon F_{T_1*}T_1 \twoheadrightarrow T_1$. Let $x \in F_{T_1*}T_1$ be such that $\widetilde{\Phi}(x) = 1$. Then the composition

$$F_{T_1*}T_1 \xrightarrow{F_{T_1*}(-\cdot x)} F_{T_1*}T_1 \xrightarrow{\widetilde{\Phi}} T_1$$

maps $1 \in F_{T_1*}T_1$ to $1 \in T_1$, and hence T_1 is Frobenius split.

We now show (vi)$\Rightarrow$(i). Let $c \in F_{k*}k$ be such that $f(c) = b \neq 0$. Then the composition

$$\phi \colon F_{k*}k \xrightarrow{F_{k*}(-\cdot c)} F_{k*}k \xrightarrow{f} k \xrightarrow{-\cdot b^{-1}} k$$

is a continuous k-linear splitting of the Frobenius map $F_k \colon k \to F_{k*}k$. Now on $F_{T_n*}T_n$, we consider the map

$$\Phi \colon F_{T_n*}T_n \to T_n, \quad \sum_{v \in \mathbf{Z}_{\geq 0}^n} a_v X^v \mapsto \sum_{v \in p \cdot \mathbf{Z}_{\geq 0}^n} \phi(a_v) X^{v/p},$$

of T_n-modules, where $p \cdot \mathbf{Z}_{\geq 0}^n$ denotes the submonoid of $\mathbf{Z}_{\geq 0}^n$ where each coordinate is divisible by p, and for $v \in p \cdot \mathbf{Z}_{\geq 0}^n$, the multi-index v/p is obtained by dividing every coordinate by p. The map indeed defines a map to T_n since if $|a_v| \to 0$, then $|\phi(a_v)| \to 0$ by continuity of ϕ (see Lemma 2.11). Since Φ maps $1 \in F_{T_n*}T_n$ to $1 \in T_n$, we get a Frobenius splitting of T_n.

It remains to show (v)$\Rightarrow$(vi). Let $\Phi \colon F_{k\{X\}*}k\{X\} \to k\{X\}$ be a Frobenius splitting of $T_1 = k\{X\}$, and consider the composition

$$f \colon F_{k*}k \hookrightarrow F_{k\{X\}*}k\{X\} \xrightarrow{\Phi} k\{X\} \twoheadrightarrow \frac{k\{X\}}{(X)} \xrightarrow{\sim} k.$$

Note that f is a nonzero k-linear map since f maps $1 \in F_{k*}k$ to $1 \in k$. Assume for contradiction that f is not continuous, and choose by Lemma 2.11 a sequence $(a_i)_{i \in \mathbf{Z}_{\geq 0}} \subseteq F_{k*}k$ such that $|a_i| \to 0$ as $i \to \infty$ and such that denoting

$$\Phi(a_i) = \sum_{j=0}^{\infty} b_{i,j} X^j,$$

we have

$$|b_{i,0}| \geq i!$$

for all i.[1] Note that $f(a_i) = b_{i,0}$. Using this sequence $(a_i)_i$, we construct a restricted power series in $k\{X\}$ whose image under Φ does not land in $k\{X\}$. Let $m_0 := 0$, and for all $i \geq 1$, inductively choose $m_i \gg m_{i-1}$ such that

$$\max_{0 \leq r \leq i-1} |b_{m_r,i-r}| < |b_{m_i,0}|.$$

Note that such m_i exist because $|b_{i,0}| \to \infty$ as $i \to \infty$. We then have

$$\left| \sum_{r=0}^{i-1} b_{m_r,i-r} \right| \leq \max_{0 \leq r \leq i-1} |b_{m_r,i-r}| < |b_{m_i,0}| \tag{3}$$

by the non-Archimedean triangle inequality. Now consider the restricted power series

$$\sum_{r=0}^{\infty} a_{m_r} X^{rp} \in k\{X\}.$$

Applying the Frobenius splitting Φ to this power series we see that for all $i \in \mathbf{Z}_{\geq 0}$,

$$\Phi\left(\sum_{r=0}^{\infty} a_{m_r} X^{rp} \right) = \Phi\left(\sum_{r=0}^{i} a_{m_r} X^{rp} \right) + \Phi\left(\sum_{r=i+1}^{\infty} a_{m_r} X^{rp} \right)$$

$$= \sum_{r=0}^{i} \left(X^r \sum_{j=0}^{\infty} b_{m_r,j} X^j \right) + X^{i+1} \Phi\left(\sum_{r=i+1}^{\infty} a_{m_r} X^{(r-i-1)p} \right). \tag{4}$$

Since the last term in (4) is divisible by X^{i+1}, it does not contribute to the coefficient of X^i in $\Phi(\sum_{r=0}^{\infty} a_{m_r} X^{rp})$. For every $i \in \mathbf{Z}_{\geq 0}$, we then have

$$\left| \text{coefficient of } X^i \text{ in } \Phi\left(\sum_{r=0}^{\infty} a_{m_r} X^{rp} \right) \right|$$

$$= \left| \sum_{r=0}^{i} b_{m_r,i-r} \right| = \left| b_{m_i,0} + \sum_{r=0}^{i-1} b_{m_r,i-r} \right| = |b_{m_i,0}| \geq m_i!,$$

---

[1] We assume more than we need here so that our argument easily adapts to the setting of Theorem 4.4.

where the penultimate equality follows by Lemma 2.3 and by (3). Therefore $\Phi(\sum_{r=0}^{\infty} a_{m_r} X^{rp})$ cannot be a restricted power series, contradicting the assumption that Φ maps into T_1. Thus, f must be continuous. ∎

This topological characterization shows that Tate algebras are Frobenius split in many commonly occurring cases.

Corollary D. *Let $(k, |\cdot|)$ be a complete non-Archimedean field of characteristic $p > 0$. For each $n > 0$, the Tate algebra $T_n(k)$ is Frobenius split in the following cases:*

  (i) *$(k, |\cdot|)$ is spherically complete.*

 (ii) *$k^{1/p}$ has a dense k-subspace V that has a countable k-basis, hence in particular if $[k^{1/p} : k] < \infty$.*

(iii) *$|k^{\times}|$ is not discrete, and the norm on $k^{1/p}$ is polar.*

We note that (i) and (ii) arose out of conversations with Eric Canton and Matthew Stevenson.

Proof of Corollary D. In each of the above cases, Theorem 2.15 implies that the identity map $\mathrm{id}_k : k \to k$ can be extended to a continuous k-linear map $k^{1/p} \to k$. We are then done by Theorem 3.1. ∎

4. A local construction

We now show how the topological characterization of Frobenius splittings of Tate algebras can be extended to a similar local construction involving convergent power series rings. This will in turn yield local examples of excellent regular rings that are not Frobenius split in Section 5.

Definition 4.1 (see [32, pp. 190–191]). Let $(k, |\cdot|)$ be a non-Archimedean field. For every positive integer $n > 0$, the *convergent power series ring* in n indeterminates over k is the k-subalgebra

$$
\begin{aligned}
K_n(k) &:= k\langle X_1, \ldots, X_n \rangle \\
&:= \left\{ \sum_{\nu \in \mathbf{Z}_{\geq 0}^n} a_\nu X^\nu : \begin{array}{l} a_\nu \in k \text{ and there exist } r_1, \ldots, r_n \in \mathbf{R}_{>0} \text{ and } M \in \mathbf{R}_{>0} \\ \text{such that } |a_\nu| \, r_1^{\nu_1} \ldots r_n^{\nu_n} \leq M \text{ for all } \nu \in \mathbf{Z}_{\geq 0}^n \end{array} \right\}
\end{aligned}
$$

of the formal power series $k[[X_1, \ldots, X_n]]$ in n indeterminates over k.[2] For $n = 1$, we will denote the indeterminate in $K_1(k)$ by just X instead of X_1.

---

[2]Following [15], the letter K is used instead of C because the German word for "convergent" is "konvergent."

Remark 4.2. It is clear that $T_n(k)$ is a subring of $K_n(k)$ for each $n > 0$. For $n = 1$, an element of $K_1(k)$ is a power series $\sum_i a_i X^i$ for which there exist real numbers $r, M > 0$ such that $|a_i| \le M r^{-i}$ for all $i \in \mathbf{Z}_{\ge 0}$. We can always assume $0 < r < 1$, and so $\sum_i a_i X^i$ is convergent precisely when the norms of its coefficients can be bounded by some exponential function.

The main properties of $K_n(k)$ are summarized below.

Theorem 4.3. *Let $(k, |\cdot|)$ be a non-Archimedean field, and let n be a positive integer. Then the convergent power series ring $K_n(k)$ satisfies the following properties:*

(i) *$K_n(k)$ is a Noetherian local ring of Krull dimension n whose maximal ideal is $(X_1, \ldots, X_n)$.*

(ii) *$K_n(k)$ is regular.*

(iii) *$K_n(k)$ is Henselian.*

(iv) *$K_n(k)$ is excellent.*

Indication of proof. (i)–(iii) are proved in [32, Thm. 45.5]. For (iv), in the proof that $T_n(k)$ is excellent, Kiehl observes that one can adapt the proof for $T_n(k)$ to show that $K_n(k)$ is also excellent [25, p. 89] (see also [8, Thm. 1.1.3]). ∎

With these preliminaries, one can now adapt the proof of Theorem 3.1 to obtain an analogous set of results for the local ring $K_n(k)$.

Theorem 4.4. *Let $(k, |\cdot|)$ be a non-Archimedean field of characteristic $p > 0$. The following are equivalent:*

(i) *$K_n(k)$ is Frobenius split for each integer $n > 0$.*

(ii) *$K_n(k)$ has a nonzero p^{-1}-linear map for each integer $n > 0$.*

(iii) *There exists an integer $n > 0$ for which $K_n(k)$ has a nonzero p^{-1}-linear map.*

(iv) *$K_1(k)$ has a nonzero p^{-1}-linear map.*

(v) *$K_1(k)$ is Frobenius split.*

(vi) *There exists a nonzero continuous k-linear map $f \colon F_{k*}k \to k$.*

Proof. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are clear.

For the proof of (iii)$\Rightarrow$(iv), we adapt the proof in Theorem 3.1 as follows. Instead of Lemma 2.10, one uses [15, Kap. I, Folgerung zu Satz 4.3] to find an automorphism $\sigma \colon K_n \to K_n$ such that $\sigma(\Phi(1))$ is X_n-distinguished in the sense of [15, Kap. I, §4.1].[3] The definition of X_n-distinguished still implies that $g \notin (X_1, \ldots, X_{n-1})$, and the rest of the proof is the same as in Theorem 3.1.

---

[3]While the definition of K_n in [15, Kap. I, §3.1] differs from that in Definition 4.1, they are equivalent by Abel's lemma on convergence of power series [15, Kap. I, Satz 1.2].

The proof of (iv)⇒(v) follows by adapting the corresponding proof in Theorem 3.1 using the observation that K_1, being a regular local ring of dimension 1, is a PID (Theorem 4.3).

For (vi)⇒(i), we may assume without loss of generality that f maps $1 \in F_{k*}k$ to $1 \in k$ as in Theorem 3.1. By Lemma 2.11 (vi), choose a positive real number B such that for all $x \in F_{k*}k$, we have $|f(x)| \leq B \cdot |x|$. Now on $F_{K_n*}K_n$, we consider the map

$$\Phi: F_{K_n*}K_n \to K_n, \qquad \sum_{\nu \in \mathbf{Z}_{\geq 0}^n} a_\nu X^\nu \mapsto \sum_{\nu \in p \cdot \mathbf{Z}_{\geq 0}^n} f(a_\nu) X^{\nu/p},$$

of K_n-modules, which we claim indeed maps to K_n. This is because if $r_1, \ldots, r_n$ and M are positive real numbers such that for every $\nu \in \mathbf{Z}_{\geq 0}^n$ we have $|a_\nu| r_1^{\nu_1} \cdots r_n^{\nu_n} \leq M$, then for every $\nu \in p \cdot \mathbf{Z}_{\geq 0}^n$, we have

$$|f(a_\nu)|(r_1^p)^{\nu_1/p}(r_2^p)^{\nu_2/p} \cdots (r_n^p)^{\nu_n/p} \leq B \cdot |a_\nu| r_1^{\nu_1} r_2^{\nu_2} \cdots r_n^{\nu_n} \leq B \cdot M.$$

Said differently, the defining condition of a convergent power series can be checked for $\sum_{\nu \in p \cdot \mathbf{Z}_{\geq 0}^n} f(a_\nu) X^{\nu/p}$ upon replacing $r_1, \ldots, r_n$ by $r_1^p, \ldots, r_n^p$ and M by $B \cdot M$. Since Φ maps $1 \in F_{K_n*}K_n$ to $1 \in K_n$, we get a Frobenius splitting of K_n.

The implication (v)⇒(vi) follows from the proof of the same implication in Theorem 3.1 by replacing T_1 by K_1 and the phrase "restricted power series" by "convergent power series" everywhere. This is because in Theorem 3.1, assuming that $F_{k*}k$ admits no nonzero continuous functionals $F_{k*}k \to k$, we construct a restricted (hence convergent) power series whose image under the Frobenius splitting is a power series whose coefficients have norms growing factorially. Therefore, this image cannot be a convergent power series by Remark 4.2. ∎

Remark 4.5. Let $\mathfrak{m}$ be the maximal ideal of $T_n(k)$ generated by the indeterminates. Then one can show the following are equivalent:

(i) $(T_n(k))_{\mathfrak{m}}$ is Frobenius split (resp. has a nonzero p^{-1}-linear map) for each integer $n > 0$.

(ii) $(T_n(k))_{\mathfrak{m}}^h$ is Frobenius split (resp. has a nonzero p^{-1}-linear map) for each integer $n > 0$.

(iii) There exists a nonzero continuous k-linear map $f: F_{k*}k \to k$.

(iii)⇒(i) follows from Theorem 3.1 and localization, and (i)⇒(ii) follows by base extension since the relative Frobenius of the map from a local ring to its Henselization is an isomorphism. The latter assertion follows by [44, Tag 097N and Tag 0F6W] because the Henselization of a local ring $(R, \mathfrak{m})$ is a filtered colimit of étale R-algebras. The implication (ii)⇒(iii) follows from the proof of (v)⇒(vi) in Theorem 3.1. This is because we have k-algebra inclusions

$$T_n \subseteq (T_n)_{\mathfrak{m}} \subseteq (T_n)_{\mathfrak{m}}^h \subseteq K_n$$

by the universal property of localization applied to $T_n \hookrightarrow K_n$ and the Henselian property of K_n (see Theorem 4.3 (iii)), and also because of the fact that for a Frobenius splitting $\Phi : F_{(T_n)_{\mathfrak{m}}^h *}(T_n)_{\mathfrak{m}}^h \to (T_n)_{\mathfrak{m}}^h$, if the composition

$$F_{k*}k \hookrightarrow F_{(T_n)_{\mathfrak{m}}^h *}(T_n)_{\mathfrak{m}}^h \xrightarrow{\Phi} (T_n)_{\mathfrak{m}}^h \twoheadrightarrow \frac{(T_n)_{\mathfrak{m}}^h}{\mathfrak{m}(T_n)_{\mathfrak{m}}^h} \xrightarrow{\sim} k$$

is not continuous, then we can construct a restricted power series (i.e. an element of T_n) whose image under Φ does not even land in K_n.

We also obtain an analogue of Corollary D for K_n.

Corollary 4.6. *Let $(k, |\cdot|)$ be a non-Archimedean field of characteristic $p > 0$. For every $n > 0$, the convergent power series ring $K_n(k)$ is Frobenius split in the following cases:*

(i) $(k, |\cdot|)$ *is spherically complete.*

(ii) $k^{1/p}$ *has a dense k-subspace V which has a countable k-basis, hence in particular if $[k^{1/p} : k] < \infty$.*

(iii) $|k^\times|$ *is not discrete, and the norm on $k^{1/p}$ is polar.*

Proof. Lifting $\mathrm{id}_k : k \to k$ to a continuous k-linear map $k^{1/p} \to k$ using Theorem 2.15, the corollary follows by Theorem 4.4. ∎

5. Gabber's example of a non-Archimedean field with no nonzero continuous p^{-1}-linear maps

Following ideas of Ofer Gabber, we now construct a non-Archimedean field $(k, |\cdot|)$ of prime characteristic $p > 0$ such that $F_{k*}k$ has no nonzero continuous k-linear functionals. The possibility of the existence of such examples is suggested by Gerritzen [14] and Kiehl [25] (see also [6, p. 63]), and while similar constructions of valuative fields with infinite p-degree had been studied by Rzepka and Kuhlmann [1], the connection with the existence of continuous functionals was not made (see Remark 5.4).

Recall from the "Notation" subsection in Section 1 that we can identify the Frobenius map $k \to F_{k*}k$ for k as above with the inclusion $k \hookrightarrow k^{1/p}$ of k into the field $k^{1/p}$ of p-th roots of elements in k. The field $k^{1/p}$ has a unique norm extending the one on k (see Remark 2.5 (ii)), and we will continue denoting this norm on $k^{1/p}$ by $|\cdot|$. In this new notation, we want to show that there exists a non-Archimedean field k with no nonzero continuous functionals $k^{1/p} \to k$.

The idea behind the example is to create a non-Archimedean field $(k, |\cdot|)$ which is not spherically complete, but such that k admits an extension $k \hookrightarrow \ell \hookrightarrow k^{1/p}$ where ℓ is spherically complete under the restriction of the norm on $k^{1/p}$ to ℓ. One then gets the desired result by applying Proposition 2.17.

We first review some standard constructions of fields associated to an *additive* value group Γ.

Definition 5.1 (see [35, §3, 12, §§2.8–2.9]). Let K be a field, and let $\Gamma \subseteq \mathbf{R}$ be an additive subgroup. The *field of generalized rational functions* is the field of fractions $K(t^\Gamma)$ of the ring of generalized polynomials

$$K[t^\Gamma] := \left\{\sum_{\gamma \in \Gamma} a_\gamma t^\gamma : \text{the set } \{\gamma \in \Gamma : a_\gamma \neq 0\} \text{ is finite}\right\}.$$

The field $K(t^\Gamma)$ embeds in the *Hahn series field*

$$K((t^\Gamma)) := \left\{\sum_{\gamma \in \Gamma} a_\gamma t^\gamma : \text{the set } \{\gamma \in \Gamma : a_\gamma \neq 0\} \text{ is well-ordered}\right\}.$$

These fields are compatibly valued with the analogue of the Gauss norm:[4]

$$\left\|\sum_{\gamma \in \Gamma} a_\gamma t^\gamma\right\| := \max_{a_\gamma \neq 0} e^{-\gamma}.$$

The value groups of $K(t^\Gamma)$ and $K((t^\Gamma))$ are clearly $e^{-\Gamma}$, and their residue fields are both K.

The Hahn series field $K((t^\Gamma))$ is spherically complete (hence also complete) with respect to the analogue of the Gauss norm [35, Thm. 1]. It is also called a Mal'cev–Neumann field (see [35, p. 88]).

Theorem 5.2. *Let K be a field of characteristic $p > 0$, and let $\Gamma \subseteq \mathbf{R}$ be an additive subgroup such that $\Gamma / p\Gamma$ is infinite. Consider the compositum*

$$M := K((t^{p\Gamma})) \, . \, K(t^\Gamma) \subseteq K((t^\Gamma))$$

of fields over $K(t^{p\Gamma})$ with norm induced by that on $K((t^\Gamma))$. Then the following properties hold:

  (i) *There is a bounded sequence $(r_i)_{i \in \mathbf{Z}_{\geq 0}}$ of pairwise distinct coset representatives of $p\Gamma$ in Γ.*

 (ii) *M is the algebraic extension of $K((t^{p\Gamma}))$ consisting of Hahn series in $K((t^\Gamma))$ whose exponents lie in finitely many cosets of $p\Gamma$ in Γ.*

(iii) *The completion $\widehat{M}$ of M is not spherically complete.*

(iv) *There are no nonzero continuous $\widehat{M}$-linear functionals $\widehat{M}^{1/p} \to \widehat{M}$.*

We first explicitly describe an additive subgroup Γ satisfying the hypotheses above.

Example 5.3. A simple example of a subgroup $\Gamma \subseteq \mathbf{R}$ satisfying the hypothesis of Theorem 5.2 is

$$\Gamma = \sum_{i \in \mathbf{Z}_{\geq 0}} \mathbf{Z} \cdot \{r_i\} \subseteq \mathbf{R}$$

generated by a decreasing sequence of positive real numbers r_i such that $r_i \to 0$ as $i \to \infty$ and $\{r_i\}_{i \in \mathbf{Z}_{\geq 0}}$ is linearly independent over $\mathbf{Q}$. Sequences of this form appear in [24, §5],

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[4]Note that even though Γ is written additively, the norm on $K((t^\Gamma))$ is a multiplicative valuation.

where Kaplansky constructs examples of non-Archimedean fields with nonunique spherical completions.

Proof of Theorem 5.2. We first show (i). Since $\Gamma/p\Gamma$ is infinite, the group Γ is not discrete (that is, isomorphic to $\mathbf{Z}$), and hence $p\Gamma$ is also not discrete. Consequently, both groups are dense in $\mathbf{R}$. Choose any sequence $(r_i)_{i \in \mathbf{Z}_{>0}}$ of representatives of pairwise distinct cosets of $p\Gamma$ in Γ. By the density of $p\Gamma$ in $\mathbf{R}$, for each r_i there exists $f_i \in p\Gamma$ such that $|r_i - f_i| < 1$. Then, replacing $(r_i)_i$ by $(r_i - f_i)_i$ gives a sequence of representatives of pairwise distinct cosets $p\Gamma$ in Γ such that $\{r_i - f_i\}_i \subseteq (-1, 1)$, proving (i).

We next show (ii). Note that M can be identified as a subfield of $K((t^\Gamma))$ by adjoining to $K((t^{p\Gamma}))$ the p-th roots t^γ of elements of the form $t^{p\gamma} \in K((t^{p\Gamma}))$, for $\gamma \in \Gamma$. Moreover, since every element of M lies in a subfield $K((t^{p\Gamma}))(t^{\gamma_1}, \ldots, t^{\gamma_n})$ for some finite set of elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, it follows that M consists of those Hahn series of $K((t^\Gamma))$ whose exponents lie in finitely many cosets of $p\Gamma$ in Γ.

We now show (iii). We first claim that the extensions

$$M \subseteq \hat{M} \subseteq K((t^\Gamma)) \tag{5}$$

are immediate extensions of real-valued fields. For this, it suffices to show that $M \subseteq K((t^\Gamma))$ is immediate. The field M has the same value group as $K((t^\Gamma))$ (the norm of the p-th root of $t^{p\gamma}$ equals $e^{-\gamma}$, for every $\gamma \in \Gamma$), and both M and $K((t^\Gamma))$ have residue field K, showing that $M \subseteq K((t^\Gamma))$ is immediate. Since spherically complete fields do not admit proper immediate extensions by Remark 2.14 (iv), to show that $\hat{M}$ is not spherically complete it therefore suffices to show that $\hat{M}$ is a proper subfield of $K((t^\Gamma))$ in the sequence (5) of immediate extensions. By (i), there is a bounded sequence $(r_i)_i$ of representatives of pairwise distinct cosets of $p\Gamma$ in Γ. By passing to a subsequence we may assume $(r_i)_i$ is strictly increasing or strictly decreasing. After possibly replacing every r_i by $-r_i$, we assume that (r_i) is strictly decreasing and bounded below by some $r \in \mathbf{R}$. The set $\{-r_i\}_{i \in \mathbf{Z}_{\geq 0}}$ is then well-ordered and bounded above by $-r$. We claim the Hahn series

$$f = \sum_{i \in \mathbf{Z}_{\geq 0}} t^{-r_i} \in K((t^\Gamma))$$

does not lie in $\hat{M}$. For this, it suffices to show that elements in M are bounded away from $\sum_i t^{-r_i}$. Let $g \in M$ be arbitrary. Since the r_i do not lie in $p\Gamma$, the series $g \in M$ can only contain finitely many of the $-r_i$ as exponents by (ii). Let i_g be such that $-r_{i_g}$ does not appear as an exponent in g. Then

$$\|f - g\| = \left\| t^{-r_{i_g}} + \sum_{\gamma \neq -r_{i_g}} b_\gamma t^\gamma \right\| \geq e^{r_{i_g}} > e^r$$

for some $b_\gamma \in K$, where the middle inequality follows by definition of the Gauss norm $\|\cdot\|$ on $K((t^\Gamma))$, and the last inequality follows from the fact that (r_i) is strictly decreasing and bounded below by r. Therefore, the ball of radius e^r centered at f contains no element

of M. Hence, M is not dense in $K((t^\Gamma))$, and consequently $\hat{M}$ is a proper subfield of $K((t^\Gamma))$ because M is dense in $\hat{M}$.

Finally, we show (iv). Since $\hat{M}$ is not spherically complete by (iii), it follows that there are no nonzero continuous $\hat{M}$-linear maps $K((t^\Gamma)) \to \hat{M}$ by Proposition 2.17. Since we have the inclusions

$$\hat{M} \subseteq K((t^\Gamma)) \subseteq \hat{M}^{1/p},$$

if $f : \hat{M}^{1/p} \to \hat{M}$ is a continuous $\hat{M}$-linear map such that $f(x) \neq 0$ then for every nonzero $a \in K((t^\Gamma))$ the composition

$$K((t^\Gamma)) \hookrightarrow \hat{M}^{1/p} \xrightarrow{\ -\cdot a^{-1}x\ } \hat{M}^{1/p} \xrightarrow{\ f\ } \hat{M}$$

is a continuous $\hat{M}$-linear map such that $a \mapsto f(x) \neq 0$. But this contradicts $K((t^\Gamma))$ having no nonzero continuous $\hat{M}$-linear functionals. ∎

Remark 5.4. In [1, Thm. 1.6], Rzepka and Kuhlmann give a more general construction of fields similar to those in Theorem 5.2, although in our special setting the arguments are simpler. For suitable spherically complete non-Archimedean fields $(K, |\cdot|)$ of characteristic $p > 0$, Rzepka and Kuhlmann construct a non-Archimedean field L fitting into a sequence

$$K \subseteq L \subseteq K^{1/p}$$

of immediate extensions, such that the completion $\hat{L}$ (denoted by L^c in [1, pp. 211–213]) of L is strictly contained in $K^{1/p}$. The argument in (iii) shows that $\hat{L}$ is not spherically complete, and the argument in (iv) shows that there are no nonzero continuous $\hat{L}$-linear functionals $\hat{L}^{1/p} \to \hat{L}$. We can therefore replace the field $\hat{M}$ constructed in Theorem 5.2 with Rzepka and Kuhlmann's field $\hat{L}$ in the proofs of Theorem A and Corollary C below.

The proofs of Theorem A and Corollary C are now a simple matter of interpreting Theorem 3.1 in light of the construction in Theorem 5.2.

Theorem A. *For every prime $p > 0$, there exists a complete non-Archimedean field $(k, |\cdot|)$ of characteristic p such that the Tate algebra $T_n(k) := k\{X_1, \ldots, X_n\}$ is not Frobenius split for each $n > 0$. In fact, $T_n(k)$ admits no nonzero $T_n(k)$-linear maps $F_{T_n(k)*} T_n(k) \to T_n(k)$ for each $n > 0$.*

Proof. Take k to be the non-Archimedean field $\hat{M}$ constructed in Theorem 5.2. We can then use Theorem 3.1 to conclude that the Tate algebras $T_n(k)$ cannot admit any nonzero p^{-1}-linear maps for each $n > 0$. ∎

Remark 5.5. One obtains an analogue of Theorem A for the regular local convergent power series rings $K_n(k)$ by using Theorem 4.4.

Corollary C. *There exists an excellent Euclidean domain R of characteristic $p > 0$ such that R admits no nonzero R-linear maps $F_{R*} R \to R$. Moreover, one can choose R to be local and Henselian as well.*

Proof. Take k to be the field $\hat{M}$ constructed in Theorem 5.2 and R to be the Tate algebra $T_1(k)$. Then R is a Euclidean domain by Theorem 2.7 (vi), and admits no nonzero R-linear maps $F_{R*}R \to R$ by Theorem A. To get a local and Henselian Euclidean domain R, one can choose $R = K_1(k)$ and then apply Theorem 4.4. ∎

Remark 5.6. A well-known argument using Matlis duality shows that a Noetherian complete local ring that is F-pure is always Frobenius split [13, Lem. 1.2]. Thus, Corollary C provides a stark contrast with the complete case, since it shows that Question 1.1 has a negative answer even for excellent local rings that behave the most like complete local rings, namely those that are Henselian.

Remark 5.7. Let k be the non-Archimedean field $\hat{M}$ from Theorem 5.2. Since there are no nonzero continuous k-linear maps $F_{k*}k \to k$, the valuation ring k° has no nonzero k°-linear maps $F_{k^\circ*}k^\circ \to k^\circ$. Indeed, any nonzero k°-linear map $f\colon F_{k^\circ*}k^\circ \to k^\circ$ extends to nonzero k-linear map $\tilde{f}\colon F_{k*}k \to k$ at the level of fraction fields. Since $\tilde{f}(B_1(0)) \subseteq B_1(0)$ by virtue of $\tilde{f}$ being an extension of f, it follows that $\tilde{f}$ is continuous by Lemma 2.11. But this is impossible by Theorem 5.2. This answers a question raised by the first author in [9, p. 25] about the existence of non-Frobenius split complete rank 1 valuation rings k° of a field k for which the extension $k \hookrightarrow k^{1/p}$ is not immediate (the group Γ in this case is not p-divisible).

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