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Niclas Technau · Nadav Yesha

On the correlations of n^α mod 1

Abstract. A well known result in the theory of uniform distribution modulo 1 (which goes back to Fejér and Csillag) states that the fractional parts $\{n^\alpha\}$ of the sequence $(n^\alpha)_{n\geq 1}$ are uniformly distributed in the unit interval whenever $\alpha > 0$ is not an integer. For sharpening this knowledge to local statistics, the k-level correlation functions of the sequence $(\{n^\alpha\})_{n\geq 1}$ are of fundamental importance. We prove that for each $k \geq 2$, the k-level correlation function R_k is Poissonian for almost every $\alpha > 4k^2 - 4k - 1$.

Keywords. Uniform distribution modulo 1, correlation functions, local statistics

1. Introduction

A real-valued sequence $(\vartheta_n)_{n\geq 1}$ is called *equidistributed* or *uniformly distributed modulo* 1 if each subinterval $[a, b] \subseteq [0, 1]$ gets its fair share of fractional parts $\{\vartheta_n\}$ in the sense that

$$\frac{1}{N} \# \{n \leq N : \{\vartheta_n\} \in [a, b]\} \xrightarrow[N\to\infty]{} b - a.$$

The notion of uniform distribution modulo 1 has been studied intensively since the beginning of the twentieth century, originating in Weyl's seminal paper *Über die Gleichverteilung von Zahlen mod. Eins* [25]. Notable instances of such sequences, as Weyl proved, are $\vartheta_n = \alpha n^d$ where $d \geq 1$ is an integer and α is irrational.

In this paper we study another natural family of sequences whose fractional parts are equidistributed, namely

$$\vartheta_n = n^\alpha, \tag{1.1}$$

where $\alpha > 0$ is *non-integer*. The equidistribution modulo 1 of these sequences (and more generally sequences of the form $\vartheta_n = \beta n^\alpha$ with $\beta \neq 0$ and non-integer $\alpha > 0$) is a corollary of Fejér's Theorem (see, e.g., [14, Cor. 2.1]) in the regime $0 < \alpha < 1$, which was extended to $\alpha > 1$ by Csillag [9].

Niclas Technau: School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel;
niclast@mail.tau.ac.il

Nadav Yesha: Department of Mathematics, University of Haifa, Haifa 3498838, Israel;
nyesha@univ.haifa.ac.il

In the last couple of decades the theory of equidistribution modulo 1 acquired a new facet which has developed into a highly active area of research: *local* (or *fine-scale*) statistics, which measure the behaviour of a sequence on the scale of the mean gap $1/N$. These statistics are able to distinguish between different equidistributed sequences, and are designed to quantify the *randomness* of a sequence; they are determined (see, e.g., [15, Appendix A]) by the the *k-level correlation functions*, which are therefore fundamental objects in this context. We first introduce the simplest correlation function, namely the pair correlation function.

1.1. The pair correlation function

The *pair correlation function* $R_2(x)$, defined as the limit distribution (if it exists)

$$\lim_{N \to \infty} \frac{1}{N} \#\left\{ 1 \le m \ne n \le N : \vartheta_n - \vartheta_m \in \frac{1}{N} I + \mathbb{Z} \right\} = \int_I R_2(x)\, dx \quad (I \subseteq \mathbb{R}), \quad (1.2)$$

measures the distribution of spacings between pairs of elements modulo 1 (not necessarily consecutive) on the scale of $1/N$. In particular, we say that the pair correlation function is *Poissonian* if $R_2 \equiv 1$, which is the pair correlation function of a sequence of independent random variables drawn uniformly in the unit interval (Poisson process). Since Poissonian pair correlation implies equidistribution modulo 1 (see [12]), studying the pair correlation function can also be viewed as a natural sharpening of the theory of uniform distribution modulo 1.

Being the most analytically accessible local statistic, the pair correlation of sequences modulo 1 has attracted considerable attention starting with the work of Rudnick and Sarnak [18], who showed that for any $d \ge 2$, the sequence $(\{\alpha n^d\})_{n \ge 1}$ has Poissonian pair correlation for almost all $\alpha \in \mathbb{R}$. Let us stress that often parametric families of sequences are investigated, as results for individual sequences are rarities. Indeed, even in the quadratic case of $(\{\alpha n^2\})_{n \ge 1}$, showing Poissonian pair correlation even for simple choices of α, say $\alpha = \sqrt{2}$, is an open problem.

Rudnick and Sarnak's result is an instance of a more general metric theory of the pair correlation of sequences of the form

$$\vartheta_n(\alpha) = \alpha a_n, \quad (1.3)$$

where $(a_n)_{n \ge 1}$ is a strictly increasing sequence of positive integers. The interest in a systematic metric theory of the pair correlation property have recently gained momentum, following the work of Aistleitner, Larcher and Lewko [5]. A crucial observation for this development was the central role of the additive energy $E(A_N)$ of the truncation $A_N := \{a_n : n \le N\}$, that is,

$$E(A_N) = \#\{(a, b, c, d) \in A_N^4 : a + b = c + d\}.$$

With the observation $N^2 \le E(A_N) \le N^3$ in mind, it was proved in [5] that if there is some $\epsilon > 0$ such that

$$E(A_N) = O(N^{3-\epsilon}),$$

then the fractional parts of the sequence (1.3) have metric Poissonian pair correlation, i.e., Poissonian pair correlation for almost all $\alpha \in \mathbb{R}$. The previous assumption for identifying metric Poissonian pair correlation was slackened considerably by Bloom and Walker [7], requiring only

$$E(A_N) = O(N^3 (\log N)^{-C})$$

with a universal constant $C > 0$. For further results on the additive energy $E(A_N)$ and applications, see [4, 6, 16, 24].

Until recently, there were only a few results about the pair correlation of sequences which are not dilated integer sequences as in (1.3). Metric Poissonian pair correlation was recently established by Rudnick and the first author [20] for dilations of *non-integer*, *lacunary* sequences (i.e., sequences satisfying $\liminf_{n\to\infty} a_{n+1}/a_n > 1$). Another family of non-integer lacunary sequences are the sequences $\vartheta_n(\alpha) = \alpha^n$ where $\alpha > 1$; these were studied by Aistleitner and Baker [1] who showed Poissonian pair correlation for almost all $\alpha > 1$. As for sequences of the form (1.1), El-Baz, Marklof and Vinogradov [10] showed that the pair correlation of the sequence $(\{\sqrt{n}\})_{n\geq1,\ \sqrt{n}\notin\mathbb{Z}}$ is Poissonian – this may come as a surprise in light of the non-Poissonian nearest neighbour spacing distribution established by Elkies and McMullen [11] (see §1.4 below). Lutsko, Sourmelidis and the first named author [17] recently showed that $(\{\beta n^\alpha\})_{n\geq1}$ has Poissonian pair correlation for any $\alpha \in (0, 14/41)$ and any $\beta > 0$. Metric Poissonian pair correlation was also recently established for sequences of the form (1.1) (see [3, 21]).

1.2. Higher order correlation functions

The definition of the pair correlation function naturally extends to higher correlation functions $R_k(\mathbf{x})$ ($k \geq 2$) which detect the distribution of scaled spacings between k-tuples of elements modulo 1. Rather than working with boxes in $\mathbb{R}^{k-1}$, it will be technically more convenient (and equivalent) to define $R_k(\mathbf{x})$ via functions in $C_c^\infty(\mathbb{R}^{k-1})$, the class of C^∞-functions from $\mathbb{R}^{k-1}$ to $\mathbb{R}$ with compact support. Let $\mathcal{X}_k = \mathcal{X}_k(N)$ denote the set of distinct integer k-tuples $(x_1, \ldots, x_k)$ satisfying $1 \leq x_i \leq N$, and for $\mathbf{x} \in \mathcal{X}_k$ denote

$$\Delta(\mathbf{x}, (\vartheta_n)) := (\vartheta_{x_1} - \vartheta_{x_2}, \ldots, \vartheta_{x_{k-1}} - \vartheta_{x_k}) \in \mathbb{R}^{k-1}.$$

Definition 1.1. Given a compactly supported function $f : \mathbb{R}^{k-1} \to \mathbb{R}$, we define the *k-level correlation sum* by

$$R_k(f, (\vartheta_n), N) := \frac{1}{N} \sum_{\mathbf{x} \in \mathcal{X}_k} \sum_{\mathbf{m} \in \mathbb{Z}^{k-1}} f\big(N(\Delta(\mathbf{x}, (\vartheta_n)) - \mathbf{m})\big). \tag{1.4}$$

The (limiting) *k-level correlation function* $R_k(\mathbf{x})$ is defined as the limit distribution (if it exists)

$$\lim_{N\to\infty} R_k(f, (\vartheta_n), N) = \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) R_k(\mathbf{x}) \, d\mathbf{x} \quad (f \in C_c^\infty(\mathbb{R}^{k-1})). \tag{1.5}$$

In particular, we say that the k-level correlation function is *Poissonian* if $R_k \equiv 1$, which is the k-level correlation function of independent uniform random variables.

In contrast to the well-developed metric theory of the pair correlation property for sequences of the shape (1.3), much less is known about the triple and higher order correlation functions whose analysis is much more involved. To the best of our knowledge, only for rapidly growing sequences are there fully satisfactory results – Rudnick and Zaharescu [22] proved metric Poissonian k-level correlation for any $k \geq 2$ when $(a_n)_{n \geq 1}$ is a lacunary sequence of integers; this was recently extended by Chaubey and the second author [8] to real-valued lacunary sequences. Furthermore, it is shown in [2] that for almost all $\alpha > 1$, the sequence $(\alpha^n)_{n \geq 1}$ has Poissonian k-level correlation for any $k \geq 2$.

The study of higher correlation functions of polynomially growing sequences, even in the presence of strong arithmetic structure, consists of only a handful of partial results. A notable example is due to Rudnick, Sarnak and Zaharescu [19, Thm. 1], who showed Poissonian k-level correlation for any $k \geq 2$ for $(\{\alpha n^2\})_{n \geq 1}$ along special subsequences of N when α is well approximable by rationals. Indeed, the polynomial growth of the sequence (1.1) is a main challenge in the present work.

1.3. Main results

We study the correlation functions of the sequences (1.1). To simplify the notation, we write $R_k(f, \alpha, N)$ instead of $R_k(f, (n^\alpha), N)$.

Theorem 1.2. *Let $k \geq 2$. The k-level correlation function of $(\{n^\alpha\})_{n \geq 1}$ is Poissonian for almost every $\alpha > 4k^2 - 4k - 1$. In particular, the pair correlation is Poissonian for almost every $\alpha > 7$.*

In order to prove Theorem 1.2, we will take an L^2 approach. The expected value of the k-level correlation sum (when averaging over α) is asymptotic to $\int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x}$ as will be shown in Proposition 5.1 (for $k = 2$) and Proposition 6.8 (for $k > 2$). For technical reasons that will become apparent below, it is convenient to multiply $\int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x}$ by the harmless combinatorial factor

$$C_k(N) := \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{k-1}{N}\right), \tag{1.6}$$

which is exactly the number of elements of $\mathcal{X}_k$ divided by N^k. The following *definition* for the variance is therefore natural.

Definition 1.3. *Let $\mathcal{I} \subseteq \mathbb{R}_{>0}$ be an interval. The *variance* of the k-level correlation sum $R_k(f, \alpha, N)$ with respect to $\mathcal{I}$ is defined as*

$$\mathrm{Var}(R_k(f, \cdot, N), \mathcal{I}) := \int_{\mathcal{I}} \left(R_k(f, \alpha, N) - C_k(N) \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x}\right)^2 d\alpha.$$

We will deduce Theorem 1.2 from the following variance bound.

Theorem 1.4. *Let $k \geq 2$, $A > 4k^2 - 4k - 1$ and $\mathcal{I} = [A, A+1]$. There exists $\rho = \rho(A) > 0$ such that*

$$\mathrm{Var}(R_k(f, \cdot, N), \mathcal{I}) = O(N^{-\rho}) \quad \text{as } N \to \infty. \tag{1.7}$$

The (largest admissible) $\rho(A)$ in (1.7) that our method delivers increases in $A > 0$; the condition $A > 4k^2 - 4k - 1$ is required to ensure that $\rho > 0$.

Remark 1.5. Fix any $\beta \neq 0$; the generalization of the above theorems to the sequences $(\{\beta n^\alpha\})_{n \geq 1}$ is straightforward.

1.4. Application: nearest neighbour spacing distribution

We may use Theorem 1.2 to obtain information about various local statistics which are determined by the k-level correlation functions $R_k(\mathbf{x})$. A natural statistic to consider is the *nearest neighbour spacing distribution* (also called the gap distribution), which is the limiting distribution $P(s)$ (if it exists) of the gaps between consecutive elements (modulo 1) of the sequence scaled to have a unit mean. More precisely, if we let

$$\vartheta_{(1)}^N \leq \vartheta_{(2)}^N \leq \cdots \leq \vartheta_{(N)}^N \leq \vartheta_{(N+1)}^N$$

denote the first $N + 1$ *ordered* elements of $\{\vartheta_n\}$, then $P(s)$ is defined as the limit distribution (if it exists)

$$\lim_{N \to \infty} g(x, (\vartheta_n), N) = \int_0^x P(s)\, ds \tag{1.8}$$

where

$$g(x, (\vartheta_n), N) := \frac{1}{N} \#\{n \leq N : N(\vartheta_{(n+1)}^N - \vartheta_{(n)}^N) \leq x\}.$$

A strong indication for randomness of a sequence $(\{\vartheta_n\})_{n \geq 1}$ is a Poissonian nearest neighbour distribution, that is, $P(s) = e^{-s}$, which is the nearest neighbour distribution of independent uniform random variables.

There are only a few examples in which one can determine the gap distribution (1.8). For dilations of lacunary sequences, metric Poissonian gap distribution follows from the aforementioned metric Poissonian k-level correlations established in [8,22], and similarly the gap distribution of $(\{\alpha^n\})_{n \geq 1}$ is Poissonian for almost all $\alpha > 1$. Another (deterministic) example is the work of Elkies and McMullen [11] on the fractional parts of the sequence $(\sqrt{n})_{n \geq 1}$. The gap distribution turns out to be non-standard (in particular not Poissonian) in this case, and is intimately related to the Haar measure on the space of translates of unimodular lattices in the plane. For the fractional parts of $(n^\alpha)_{n \geq 1}$ with $\alpha \in (0, 1) \setminus \{1/2\}$, Elkies and McMullen [11, Sec. 1] conjectured that the gap distribution is Poissonian. In fact, numerical experiments suggest that this may hold for most, and perhaps all, non-integer $\alpha \in \mathbb{R}_{>0} \setminus \{1/2\}$. In this regard, while Theorem 1.2 does not allow us to capture the gap distribution of $(\{n^\alpha\})_{n \geq 1}$ completely, it ensures that for almost all large values of α, the distribution functions $g(x, (n^\alpha), N)$ can be approximated by truncations of the Taylor series of $1 - e^{-x}$ as $N \to \infty$, so that the distribution of the gaps becomes *approximately* Poissonian.

Corollary 1.6. *Let* $K \geq 1$. *For almost all* $\alpha > 16K^2 + 8K - 1$, *we have the inequalities*

$$\sum_{1 \leq k \leq 2K} (-1)^{k+1} \frac{x^k}{k!} \leq \liminf_{N \to \infty} g(x, (n^\alpha), N)$$

$$\leq \limsup_{N \to \infty} g(x, (n^\alpha), N) \leq \sum_{1 \leq k \leq 2K-1} (-1)^{k+1} \frac{x^k}{k!}$$

holding for all $x \geq 0$.

Remark 1.7. The above approximation is meaningful (and rather efficient) when x is small relative to K. In particular, if x lies in a fixed interval and K is large, then the left- and right-hand sides are equal up to an error decaying superexponentially in K.

2. Outline of the argument

The analysis of each correlation sum R_k follows a general pattern. First, let us remark that we seek to show three intermediate objectives:

(1) Show that the expectation of $R_k(f, \cdot, N)$ is asymptotic to $\int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x}$.

(2) The variance of $R_k(f, \cdot, N)$ is $O(N^{-\rho})$ for some $\rho > 0$.

(3) Using the previous steps, deduce that $R_k(f, \cdot, N)$ converges almost surely to $\int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x}$.

In other words, we set out to show that R_k concentrates around its mean value, which we demonstrate to be the desired limit, by taking an L^2 approach.

As usual, the crux of the matter is to establish the variance bound. The third step is fairly routine requiring only minor adaptations from the standard arguments. For the sake of completeness, we decided to detail them.

Now let us explain how we bound the variance of the pair correlation sum

$$R_2(f, \alpha, N) = \frac{1}{N} \sum_{1 \leq x_1 \neq x_2 \leq N} \sum_{m \in \mathbb{Z}} f(N(x_1^\alpha - x_2^\alpha - m)), \tag{2.1}$$

for which the technical aspects of the analysis, which get more intricate as k increases, are still relatively simple. By using Poisson summation and a common truncation argument, the variance can be bounded by a sum of oscillatory integrals:

$$\mathrm{Var}(R_2(f, \cdot, N), \mathscr{J}) \ll \frac{1}{N^4} \sum_{n, m} \sum_{x_j, y_j} \left| \int_{\mathscr{J}} e(n(x_1^\alpha - x_2^\alpha) - m(y_1^\alpha - y_2^\alpha)) \, d\alpha \right| + N^{-t}, \tag{2.2}$$

where the constant $t > 0$ can be chosen arbitrarily large, and the summation constraints are given by

$$x_j, y_j \in [1, N] \quad (j = 1, 2), \quad x_1 > x_2, \, y_1 > y_2,$$
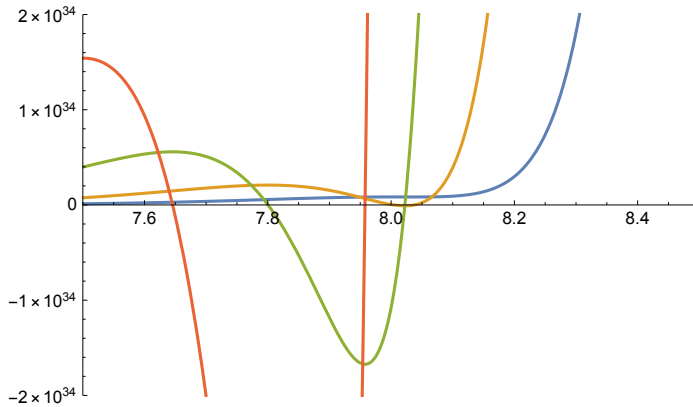$$n, m \in [-N^{1+\epsilon}, N^{1+\epsilon}], \quad n \neq 0, \, m \neq 0. \tag{2.3}$$

Fig. 2.1. Plot of the first four derivatives of the phase function $\phi(\alpha) = n(x_1^\alpha - x_2^\alpha) - m(y_1^\alpha - y_2^\alpha)$, where the blue curve is ϕ', the orange curve is ϕ'', the green curve is $\phi^{(3)}$ and the red curve is $\phi^{(4)}$. Here we use the following specifications for the plot: $n = 5135, m = 10000$, and $x_1 = 10000$, $x_2 = 1000$, $y_1 = 9500$, $y_2 = 7890$ in the range $\alpha \in [7.5, 8.5]$.

To establish the desired variance bound, we need to demonstrate that the right-hand side of (2.2) is, up to a constant, smaller than some fixed negative power of N.

In order to bound the above sum, we will establish a bound for each individual term with good dependence on the n, m, x_j, y_j parameters. To this end, we use an estimate derived from a suitable modification of van der Corput's lemma for oscillatory integrals. This provides us with a sharp bound for the individual terms and, furthermore, with the necessary uniformity in the parameters. For that estimate to be applicable, we need to ensure that at any point in $\mathcal{J}$ at least one of the first four[1] derivatives of the phase function is large. To demonstrate this largeness property is the crux of the matter. To verify it, we use a "repulsion principle" that quantifies how the smallness of the first three derivatives repels the fourth derivative from being small as well (see Figure 2.1 illustrating the first four derivatives of a phase function that we encounter).[2]

3. Preliminaries

Before proceeding, we will introduce some notation.

3.1. Notation

- The Bachmann–Landau big O notation is used in the usual sense, i.e., $f = O(g)$ as $x \to \infty$ means that there exists a constant $c > 0$ such that $|f(x)/g(x)| \le c$ for all x

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[1] For the k-level correlation sum we shall consider $2k$ derivatives.

[2] In the case of the k-level correlation sum we show that at least one of the first $2k$ derivatives is large.

sufficiently large. In order to simplify the notation, usually we will not keep track of the dependence of the implied constant c on other parameters. In particular, the dependence on a (fixed) test function f will not be explicitly mentioned.

- We will also use the Vinogradov symbols $\ll$ (and $\gg$) in their usual meaning in analytic number theory, that is, the statement $f \ll g$ denotes that $f = O(g)$.
- We will use the standard notation $e(z) = e^{2\pi i z}$.
- We denote by $[k] := \{1, \ldots, k\}$ the set of the first k natural numbers.
- We denote the shifted unit interval with left end point $A > 0$ by

$$\mathcal{J} = \mathcal{J}(A) := [A, A + 1]. \tag{3.1}$$

3.2. Tools from harmonic analysis

The bulk of our work is concerned with estimating one-dimensional oscillatory integrals

$$I(\phi, \mathcal{J}) := \int_{\mathcal{J}} e(\phi(\alpha)) \, d\alpha$$

where $\phi : \mathcal{J} \to \mathbb{R}$ is a C^∞-function (a so called *phase function*). The phase functions that we encounter are of the shape

$$\phi(\alpha) = \phi(\mathbf{u}, \mathbf{x}, \alpha) = \sum_{i \leq d} u_i x_i^\alpha, \quad \mathbf{u} = (u_1, \ldots, u_d), \ \mathbf{x} = (x_1, \ldots, x_d). \tag{3.2}$$

We wish to establish a bound with good dependence on the parameters $\mathbf{u}, \mathbf{x}$ – most importantly on the maximum norm $\|\mathbf{x}\|_\infty = \max_{i \leq d} |x_i|$. To this end, the following well-known lemma is useful.

Lemma 3.1 (Van der Corput's lemma). *Let $\phi : \mathcal{J} \to \mathbb{R}$ be a C^∞-function. Fix $d \geq 1$, and suppose that $|\phi^{(d)}(\alpha)| \geq \lambda > 0$ for all $\alpha \in \mathcal{J}$. If $d = 1$, suppose in addition that ϕ' is monotone on $\mathcal{J}$. Then there exists a constant $C_d > 0$ depending only on d such that*

$$|I(\phi, \mathcal{J})| \leq C_d \lambda^{-1/d}.$$

Proof. This classical bound follows by partial integration for $d = 1$, and then by induction on d; see Stein [23, Ch. VIII, Prop. 2]. ∎

Remark 3.2. A drawback of van der Corput's lemma is that the more complicated the phase function ϕ is (bearing the shape (3.2) of ϕ in mind), the more difficult it is to get acceptable lower bounds on the size of the minimum of the derivative $\phi^{(d)}$ for a given d. To remedy this, we use the following variant of van der Corput's lemma. The key feature is that for a non-trivial estimation of $I(\phi, \mathcal{J})$, we only require that at every point $\alpha \in \mathcal{J}$ at least one of the first d derivatives of ϕ is large, rather than requiring that one specific derivative is large throughout $\mathcal{J}$ (note that for ϕ as in (3.2), all the derivatives are on the same scale up to logarithmic factors). This amounts to estimating the function

$$M_d \phi(\alpha) := \max_{1 \leq i \leq d} |\phi^{(i)}(\alpha)|. \tag{3.3}$$

For phrasing this variant of van der Corput's lemma, there is a small price to pay: we need to control the number of zeros of $\phi^{(d)}$ on $\mathcal{J}$.

Lemma 3.3. *Let $\phi : \mathcal{J} \to \mathbb{R}$ be a C^∞-function, and let $d \geq 1$. Suppose that $\phi^{(d)}$ has at most k zeros, and that*

$$M_d\phi(\alpha) \geq \lambda > 0 \tag{3.4}$$

for all $\alpha \in \mathcal{J}$. If $d = 1$, suppose in addition that ϕ' is monotone on $\mathcal{J}$. Then there exists a constant $C_{d,k} > 0$ depending only on d and k such that

$$|I(\phi, \mathcal{J})| \leq C_{d,k}\lambda^{-1/d}.$$

Proof. Since $\phi^{(d)}$ has at most k zeros, Rolle's theorem implies that the number of zeros of any lower derivative $\phi^{(i)}$, $1 \leq i \leq d - 1$, is at most $k + d - i \leq k + d - 1$. Hence, by splitting the integral $I(\phi, \mathcal{J})$ into $O_{d,k}(1)$ integrals, we can assume without loss of generality that for any $1 \leq i \leq d - 1$ the function $\phi^{(i)}$ is monotone.

We will now prove by induction on d that

$$|I(\phi, \mathcal{J})| \ll_d \lambda^{-1/d} \tag{3.5}$$

where the implied constant in (3.5) depends only on d. The case $d = 1$ follows directly from Lemma 3.1. Assume now this claim holds for $d - 1$ where $d \geq 2$. Let (a, b) be the (possibly empty) interval of $\alpha \in \mathcal{J}$ satisfying

$$M_{d-1}\phi(\alpha) = \max_{1 \leq i \leq d-1} |\phi^{(i)}(\alpha)| < \lambda.$$

We have

$$|I(\phi, \mathcal{J})| \leq \left|\int_A^a e(\phi(\alpha))\,d\alpha\right| + \left|\int_a^b e(\phi(\alpha))\,d\alpha\right| + \left|\int_b^{A+1} e(\phi(\alpha))\,d\alpha\right|. \tag{3.6}$$

By the assumption (3.4), the lower bound $|\phi^{(d)}(\alpha)| \geq \lambda$ holds for all $\alpha \in (a, b)$. Therefore Lemma 3.1 implies that

$$\left|\int_a^b e(\phi(\alpha))\,d\alpha\right| \ll_d \lambda^{-1/d}. \tag{3.7}$$

Outside (a, b), we have $M_{d-1}\phi(\alpha) \geq \lambda$. Thus, by the induction hypothesis,

$$\left|\int_A^a e(\phi(\alpha))\,d\alpha\right| + \left|\int_b^{A+1} e(\phi(\alpha))\,d\alpha\right| \ll_d \lambda^{-1/(d-1)}. \tag{3.8}$$

Note that we may suppose that $\lambda \geq 1$, since $|I(\phi, \mathcal{J})| \leq 1$, and for $\lambda < 1$ the desired bound plainly follows from $1 < \lambda^{-1/d}$. Now inserting the bounds (3.7) and (3.8) into (3.6) (the former dominates the latter due to our assumption $\lambda \geq 1$) gives the claimed bound (3.5). ∎

The following lemma indicates how oscillatory integrals arise in our analysis. For its proof, and later reference, we recall that for any smooth compactly supported function $g : \mathbb{R} \to \mathbb{R}$, the Fourier transform $\widehat{g}$ of g decays rapidly in the sense that for any arbitrarily large $t > 0$ we have

$$\widehat{g}(\xi) = O(\xi^{-t}) \quad \text{as } |\xi| \to \infty. \tag{3.9}$$

Lemma 3.4. *Let $f \in C_c^\infty(\mathbb{R})$, and let $\epsilon > 0$. Then for all $t > 0$ we have*

$$R_2(f, \alpha, N) = \frac{1}{N^2} \sum_{|n| \le N^{1+\epsilon}} \widehat{f}\left(\frac{n}{N}\right) \sum_{1 \le x_1 \ne x_2 \le N} e(n(x_1^\alpha - x_2^\alpha)) + O(N^{-t}) \tag{3.10}$$

as $N \to \infty$.

Proof. The Poisson summation formula applied to (2.1) yields the identity

$$R_2(f, \alpha, N) = \frac{1}{N^2} \sum_{n \in \mathbb{Z}} \widehat{f}\left(\frac{n}{N}\right) \sum_{1 \le x_1 \ne x_2 \le N} e(n(x_1^\alpha - x_2^\alpha)), \tag{3.11}$$

and we want to truncate the right-hand side.

Due to (3.9) and the trivial bound

$$\left| \sum_{1 \le x_1 \ne x_2 \le N} e(n(x_1^\alpha - x_2^\alpha)) \right| \le N^2,$$

we have

$$\sum_{|n| > N^{1+\epsilon}} \widehat{f}\left(\frac{n}{N}\right) \sum_{1 \le x_1 \ne x_2 \le N} e(n(x_1^\alpha - x_2^\alpha)) \ll N^{2+s} \sum_{n > N^{1+\epsilon}} n^{-s}$$

$$\ll N^{2+s-(1+\epsilon)(s-1)}. \tag{3.12}$$

Taking s so large that

$$s - (1 + \epsilon)(s - 1) = 1 + \epsilon - \epsilon s < -t,$$

the right-hand side of (3.12) is $< N^{2-t}$. This implies (3.10), concluding the proof. ∎

4. Repulsion principles

In order to make Lemma 3.3 usable for computing the pair and higher order correlations, we need to control the M-function (3.3) of functions as in (3.2). In the present section, we show that irrespective of the choice of α some derivative of such a function is large.

Recall that if

$$V_d = \begin{bmatrix} L_1 & L_2 & \cdots & L_d \\ L_1^2 & L_2^2 & \cdots & L_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ L_1^d & L_2^d & \cdots & L_d^d \end{bmatrix} \tag{4.1}$$

is the Vandermonde matrix corresponding to distinct nonzero numbers $L_1, \ldots, L_d$, then the inverse Vandermonde matrix is given by $V_d^{-1} = [a_{ij}]$, where

$$a_{ij} = \frac{(-1)^{j-1} \sum_{1 \le m_1 < \cdots < m_{d-j} \le d, \, m_1, \ldots, m_{d-j} \ne i} L_{m_1} \cdots L_{m_{d-j}}}{L_i \prod_{1 \le m \le d, \, m \ne i} (L_m - L_i)} \tag{4.2}$$

(see, e.g., [13, Ex. 40]).

We require the following lemma.

Lemma 4.1. *Let $d \ge 2$ be an integer, let*

$$2 \le x_1 < \cdots < x_d \le N$$

be real numbers, and denote $L_i = \log x_i$ $(1 \le i \le d)$. Let V_d be the Vandermonde matrix (4.1) corresponding to the numbers $L_1, \ldots, L_d$. Let $w \in \mathbb{R}^d$, and denote $y = V_d w$. Then for all $\epsilon > 0$, there exists a constant $C_{d,\epsilon} > 0$ depending only on d and ϵ such that

$$\|y\|_\infty \ge C_{d,\epsilon} \|w\|_\infty x_d^{1-d} N^{-\epsilon} \prod_{m=1}^{d-1} h_m,$$

where $h_m = x_{m+1} - x_m$ for $1 \le m \le d-1$.

Proof. Let $V_d^{-1} = [a_{ij}]$, where a_{ij} is given by (4.2). For every $1 \le i, j \le d$ we have

$$a_{ij} \ll_{d,\epsilon} N^\epsilon \frac{1}{\prod_{1 \le m \le d, \, m \ne i} |L_m - L_i|}, \tag{4.3}$$

where the implied constant in (4.3) depends only on d and ϵ.

For all $t > -1$, we have $\log(1 + t) \le t$. So, for $m = 1, \ldots, i-1$,

$$|L_m - L_i| = -\log \frac{x_m}{x_i} = -\log\left(1 + \frac{x_m - x_i}{x_i}\right) \ge \frac{x_i - x_m}{x_i} \ge \frac{h_m}{x_d},$$

and for $m = i+1, \ldots, d$,

$$|L_m - L_i| = -\log \frac{x_i}{x_m} = -\log\left(1 + \frac{x_i - x_m}{x_m}\right) \ge \frac{x_m - x_i}{x_m} \ge \frac{h_{m-1}}{x_d}.$$

Hence,

$$a_{ij} \ll_{d,\epsilon} \frac{x_d^{d-1} N^\epsilon}{\prod_{m=1}^{d-1} h_m}.$$

Thus, we have found a uniform bound for the elements of V_d^{-1}, and since all matrix norms are equivalent, we conclude that

$$\|w\|_\infty = \|V_d^{-1} y\|_\infty \le \|V_d^{-1}\|_\infty \|y\|_\infty \ll_{d,\epsilon} \frac{x_d^{d-1} N^\epsilon}{\prod_{m=1}^{d-1} h_m} \|y\|_\infty. \qquad \blacksquare$$

We can now bound the M-function (3.3) from below for functions of the form (3.2).

Lemma 4.2. *Let $d \geq 2$ be an integer, and let $u_1, \ldots, u_d$ be non-zero real numbers. Given real numbers $2 \leq x_1 < x_2 < \cdots < x_d \leq N$, we define*

$$\phi(\alpha) := \sum_{r \leq d} u_r x_r^\alpha \quad (\alpha \in \mathcal{J} = [A, A + 1]).$$

Furthermore, let $\epsilon > 0$ and define

$$\lambda = N^{-\epsilon} |u_d| x_d^{A+1-d} \prod_{m=1}^{d-1} h_m,$$

where $h_m = x_{m+1} - x_m$ $(1 \leq m \leq d - 1)$. Then there exists a constant $C_{d,\epsilon} > 0$, depending only on d and ϵ, such that

$$M_d \phi(\alpha) \geq C_{d,\epsilon} \lambda > 0 \quad \text{for all } \alpha \in \mathcal{J}. \tag{4.4}$$

Proof. Denote $w = (u_1 x_1^\alpha, \ldots, u_d x_d^\alpha)$, and let $L_i = \log x_i$ $(1 \leq i \leq d)$. Then

$$(\phi^{(1)}(\alpha), \ldots, \phi^{(d)}(\alpha))^T = V_d w^T,$$

where V_d is the Vandermonde matrix (4.1) corresponding to the numbers $L_1, \ldots, L_d$.
  By Lemma 4.1, we infer that

$$M_d \phi(\alpha) \geq C_{d,\epsilon} \|w\|_\infty x_d^{1-d} N^{-\epsilon} \prod_{m=1}^{d-1} h_m \geq C_{d,\epsilon} N^{-\epsilon} |u_d| x_d^{A+1-d} \prod_{m=1}^{d-1} h_m,$$

where $C_{d,\epsilon} > 0$ is a constant, depending only on d and ϵ. This is exactly (4.4). ∎

We require the following simple bound on the number of zeros of functions ϕ as in (3.2).

Lemma 4.3. *Let $d \geq 1$ be an integer, let $u_1, \ldots, u_d$ be non-zero real numbers, and let $x_1, \ldots, x_d$ be distinct (strictly) positive numbers. Then the function*

$$\phi(\alpha) = \sum_{r \leq d} u_r x_r^\alpha \quad (\alpha \in \mathbb{R})$$

has at most $d - 1$ zeros.

Proof. The proof is by induction on d. For $d = 1$ the statement is clear. Assume that it is true for $d - 1$ $(d \geq 2)$, and let

$$\phi(\alpha) = \sum_{r \leq d} u_r x_r^\alpha.$$

The zeros of ϕ are exactly the zeros of the function

$$\tilde{\phi}(\alpha) = \sum_{r \leq d-1} \tilde{u}_r \tilde{x}_r^\alpha + 1$$

where $\tilde{u}_r = \frac{u_r}{u_d}$ and $\tilde{x}_r = \frac{x_r}{x_d}$ $(1 \le r \le d-1)$, since $\phi(\alpha) = u_d x_d^\alpha \tilde{\phi}(\alpha)$. Moreover,

$$\tilde{\phi}'(\alpha) = \sum_{r \le d-1} v_r \tilde{x}_r^\alpha, \quad \text{where} \quad v_r = \tilde{u}_r \log \tilde{x}_r \ (1 \le i \le d-1).$$

Clearly, the numbers $v_1, \ldots, v_{d-1}$ are non-zero and $\tilde{x}_1, \ldots, \tilde{x}_{d-1}$ are distinct. Therefore, by the induction hypothesis, $\tilde{\phi}'$ has at most $d-2$ zeros. Hence, by Rolle's theorem, $\tilde{\phi}$ has at most $d-1$ zeros, completing the proof. ∎

We are ready to prove the main lemma of this section, obtaining an upper bound for integrals with phase functions of the form (3.2).

Lemma 4.4. *Let $d \ge 2$ be an integer, let $u_1, \ldots, u_d$ be non-zero real numbers, and let*

$$2 \le x_1 < x_2 < \cdots < x_d \le N$$

be real numbers. Denote

$$\phi(\alpha) := \sum_{r \le d} u_r x_r^\alpha \quad (\alpha \in \mathcal{J} = [A, A+1]).$$

Then for all $\epsilon > 0$, there exists a constant $C_{d,\epsilon} > 0$, depending only on d and on ϵ, such that

$$|I(\phi, \mathcal{J})| \le C_{d,\epsilon} \lambda^{-1/d}, \tag{4.5}$$

where

$$\lambda = N^{-\epsilon} |u_d| x_d^{A+1-d} \prod_{m=1}^{d-1} h_m, \tag{4.6}$$

and $h_m = x_{m+1} - x_m$ $(1 \le m \le d-1)$.

Remark 4.5. For $d = 1$, we clearly have the (sharper) bound

$$|I(\phi, \mathcal{J})| \le C \frac{1}{|u_1| x_1^A \log x_1},$$

where $C > 0$ is an absolute constant, which follows directly from Lemma 3.1.

Proof of Lemma 4.4. For any $k \ge 0$, we have

$$\phi^{(k)}(\alpha) = \sum_{r \le d} v_r x_r^\alpha \quad \text{where} \quad v_r = u_r (\log x_r)^k.$$

Hence, by Lemma 4.3, for any k the function $\phi^{(k)}$ has at most $d-1$ zeros, and in particular this is true for $\phi^{(d)}$.

By Lemma 4.2, we have

$$M_d \phi(x) \gg_{d,\epsilon} \lambda > 0 \quad \text{for all } \alpha \in \mathcal{J}, \tag{4.7}$$

where λ is as in (4.6), and the implied constant in (4.7) depends only on d and ϵ. Hence, the bound (4.5) follows from Lemma 3.3. ∎

5. The pair correlation

The goal of this section is to prove the variance bound (1.7) for the pair correlation sum, i.e., for $k = 2$. This will outline the strategy for bounding the variance in the more technically involved case $k > 2$, which will be treated in the next section.

5.1. Computing the expectation

First, we show that the expectation of $R_2(f, \cdot, N)$ is asymptotic to the average of f.

Proposition 5.1. *Let* $f \in C_c^\infty(\mathbb{R})$ *and let* $\mathcal{J}$ *be as in* (3.1). *Then for all* $\epsilon > 0$,

$$\int_{\mathcal{J}} R_2(f, \alpha, N) \, d\alpha = \left(1 - \frac{1}{N}\right) \int_{-\infty}^{\infty} f(x) \, dx + O(N^{-\min(2,A)+\epsilon}) \tag{5.1}$$

as $N \to \infty$.

For the proof of Proposition 5.1 and for later reference, we require the subsequent lemma.

Lemma 5.2. *If* $n \neq 0$ *is a real number, then for all* $\epsilon > 0$,

$$\frac{1}{N^2} \sum_{1 \leq x_1 \neq x_2 \leq N} \left| \int_{\mathcal{J}} e(n(x_1^\alpha - x_2^\alpha)) \, d\alpha \right| = O_\epsilon \left(\frac{N^{-\min(2,A)+\epsilon}}{|n|} \right) \tag{5.2}$$

as $N \to \infty$, *where the implied constant depends only on* ϵ.

Proof. By relabelling if needed, we can assume that the summation in (5.2) is over $x_1 > x_2$. Consider the phase function

$$\phi(\alpha) = \phi(n, x_1, x_2, \alpha) := n(x_1^\alpha - x_2^\alpha).$$

Note that the first derivative

$$\phi'(\alpha) = n(x_1^\alpha \log x_1 - x_2^\alpha \log x_2)$$

is monotone and non-zero on $\mathcal{J}$. Hence, Lemma 3.1 yields

$$I(\phi, \mathcal{J}) \ll \frac{1}{\min_{\alpha \in \mathcal{J}} |\phi'(\alpha)|} = \frac{1}{|n|} \frac{1}{x_1^A \log x_1 - x_2^A \log x_2}, \tag{5.3}$$

where the implied constant in (5.3) is absolute.

Let $h := x_1 - x_2$. By the bound $\log(1 + t) \leq t$, we have

$$x_1^A \log x_1 - x_2^A \log x_2 \geq x_1^A (\log x_1 - \log x_2) = -x_1^A \log\left(1 - \frac{h}{x_1}\right) \geq x_1^{A-1} h.$$

Therefore,

$$\sum_{1 \leq x_2 < x_1 \leq N} \frac{1}{x_1^A \log x_1 - x_2^A \log x_2} \leq \sum_{1 < x_1 \leq N} \frac{1}{x_1^{A-1}} \sum_{1 \leq h < x_1} \frac{1}{h} = O_\epsilon(N^{-\min(0,A-2)+\epsilon}), \tag{5.4}$$

where the implied constant depends only on ϵ. Thus, (5.2) follows from (5.3) and (5.4). ∎

Proof of Proposition 5.1. Recall that by Lemma 3.4, for all $t > 0$ we have

$$R_2(f, \alpha, N) = \frac{1}{N^2} \sum_{|n| \leq N^{1+\epsilon}} \hat{f}\left(\frac{n}{N}\right) \sum_{1 \leq x_1 \neq x_2 \leq N} e(n(x_1^\alpha - x_2^\alpha)) + O(N^{-t})$$

$$= \left(1 - \frac{1}{N}\right) \int_{-\infty}^{\infty} f(x)\,dx + \frac{1}{N^2} \sum_{1 \leq |n| \leq N^{1+\epsilon}} \hat{f}\left(\frac{n}{N}\right) \sum_{1 \leq x_1 \neq x_2 \leq N} e(n(x_1^\alpha - x_2^\alpha))$$

$$+ O(N^{-t}).$$

Integrating over α, we have a bound ready for the summation over x_1, x_2 thanks to (5.2). Using $\hat{f} \ll 1$, this yields

$$\int_{\mathcal{J}} R_2(f, \alpha, N)\,d\alpha - \left(1 - \frac{1}{N}\right) \int_{-\infty}^{\infty} f(x)\,dx \ll N^{-\min(2, A) + \epsilon/2} \sum_{1 \leq |n| \leq N^{1+\epsilon}} \frac{1}{|n|} + N^{-t}.$$

Choosing t large enough will give our claim. ∎

5.2. *Proof of Theorem* 1.4 *for* $k = 2$

We can now proceed with the proof of the bound (1.7) for the pair correlation sum R_2, obtaining Theorem 1.4 in the particular case $k = 2$.

Proof of Theorem 1.4 *for* $k = 2$. Let $\epsilon > 0$. We denote by $\mathcal{S} = \mathcal{S}(N, \epsilon)$ the set of tuples $\mathbf{z} = (\mathbf{n}, \mathbf{x})$ with $\mathbf{n} = (n, -n, m, -m) \in \mathbb{Z}_{\neq 0}^4$ satisfying $\|\mathbf{n}\|_\infty \leq N^{1+\epsilon}$, and $\mathbf{x} = (x_1, x_2, y_1, y_2) \in \mathbb{Z}_{>0}^4$ satisfying $\|\mathbf{x}\|_\infty \leq N$ and

$$x_1 > x_2, \quad y_1 > y_2, \quad x_1 = \|\mathbf{x}\|_\infty. \tag{5.5}$$

From (3.10), the fact that $\hat{f} \ll 1$, and relabelling, we deduce that for all $t > 0$,

$$\mathrm{Var}(R_2(f, \cdot, N), \mathcal{J}) = \int_{\mathcal{J}} \left(R_2(f, \alpha, N) - \left(1 - \frac{1}{N}\right) \int_{-\infty}^{\infty} f(x)\,dx\right)^2 d\alpha$$

$$\ll \frac{1}{N^4} \sum_{\mathbf{z} \in \mathcal{S}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| + N^{-t} \tag{5.6}$$

with the phase function

$$\phi(\mathbf{z}, \alpha) = n(x_1^\alpha - x_2^\alpha) - m(y_1^\alpha - y_2^\alpha).$$

To proceed further, we split the parameter set $\mathcal{S}$ into three different regimes depending on several degeneracy conditions. Let

$$\mathcal{S}^1 := \{\mathbf{z} \in \mathcal{S} : n = m,\ \#\{x_1, x_2, y_1, y_2\} < 4\},$$
$$\mathcal{S}^2 := \{\mathbf{z} \in \mathcal{S} \setminus \mathcal{S}^1 : x_1 \neq y_1\}, \quad \mathcal{S}^3 := \{\mathbf{z} \in \mathcal{S} \setminus \mathcal{S}^1 : x_1 = y_1\},$$

so that

$$S = \bigsqcup_{i \le 3} S^i.$$

Further, we associate to each S^i, $i \le 3$, the term

$$T_i := \sum_{\mathbf{z} \in S^i} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|.$$

Inserting the definition of T_i into (5.6) yields

$$\text{Var}(R_2(f, \cdot, N), \mathcal{J}) \ll N^{-4} \sum_{r \le 3} T_r + N^{-t}, \tag{5.7}$$

and to verify (1.7) (when $k = 2$) it is enough to establish that for each r we have $T_r \ll N^{4-\rho}$ for some $\rho > 0$. We estimate the terms T_r in order of their index r.

Bounding T_1. Let

$$S^{1,1} := \{\mathbf{z} \in S^1 : \#\{x_1, x_2, y_1, y_2\} = 2\},$$
$$S^{1,2} := \{\mathbf{z} \in S^1 : \#\{x_1, x_2, y_1, y_2\} = 3\},$$

so that

$$T_1 = \sum_{\mathbf{z} \in S^{1,1}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| + \sum_{\mathbf{z} \in S^{1,2}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|.$$

For $\mathbf{z} \in S^{1,1}$, we have $x_1 = y_1$ and $x_2 = y_2$, and the phase function vanishes. Hence,

$$\sum_{\mathbf{z} \in S^{1,1}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| = \sum_{1 \le |n| \le N^{1+\epsilon}} \sum_{1 \le x_2 < x_1 \le N} 1 \ll N^{3+\epsilon}.$$

For $\mathbf{z} \in S^{1,2}$, we can assume without loss of generality that $x_1 = y_1$ and $x_2 \ne y_2$. The phase function then simplifies to $\alpha \mapsto n(y_2^\alpha - x_2^\alpha)$. Therefore,

$$\sum_{\mathbf{z} \in S^{1,2}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| \ll \sum_{1 \le |n| \le N^{1+\epsilon}} \sum_{1 \le x_1 \le N} \sum_{1 \le x_2 \ne y_2 \le N} |I(\alpha \mapsto n(y_2^\alpha - x_2^\alpha), \mathcal{J})|$$

$$= N \sum_{1 \le |n| \le N^{1+\epsilon}} \sum_{1 \le x_2 \ne y_2 \le N} |I(\alpha \mapsto n(y_2^\alpha - x_2^\alpha), \mathcal{J})|.$$

We apply Lemma 5.2 to deduce that

$$\sum_{1 \le |n| \le N^{1+\epsilon}} \sum_{1 \le x_2 \ne y_2 \le N} |I(\alpha \mapsto n(y_2^\alpha - x_2^\alpha), \mathcal{J})| \ll N^{2-\min(2,A)+\epsilon}.$$

Hence,

$$T_1 \ll N^{3+\epsilon} + N^{3-\min(2,A)+\epsilon} \ll N^{3+\epsilon}. \tag{5.8}$$

Bounding T_2. For $2 \le d \le 4$, let

$$S^{2,d} := \{\mathbf{z} \in S^2 : \#(\{x_1, x_2, y_1, y_2\} \setminus \{1\}) = d\},$$

so that

$$T_2 = \sum_{2 \le d \le 4} \sum_{\mathbf{z} \in \mathcal{S}^{2,d}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|. \tag{5.9}$$

For $\mathbf{z} \in \mathcal{S}^{2,d}$, the phase function ϕ consists of d non-constant terms with non-vanishing coefficients. Since $x_1 \ne y_1$, the leading coefficient of x_1^α is n. Invoking Lemma 4.4 (recall that $x_1 = \|\mathbf{x}\|_\infty$), we obtain

$$\sum_{\mathbf{z} \in \mathcal{S}^{2,d}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|$$

$$\ll N^{\epsilon/d} \sum_{1 \le |n|, |m| \le N^{1+\epsilon}} |n|^{-1/d} \sum_{x_1 \le N} x_1^{1-A/d-1/d} \sum_{h_1, \ldots, h_{d-1} \le x_1} h_1^{-1/d} \cdots h_{d-1}^{-1/d}$$

$$\ll N^{1+\epsilon+\epsilon/d} \sum_{1 \le |n| \le N^{1+\epsilon}} |n|^{-1/d} \sum_{x_1 \le N} x_1^{1-A/d-1/d} \sum_{h_1, \ldots, h_{d-1} \le x_1} h_1^{-1/d} \cdots h_{d-1}^{-1/d}.$$

The innermost summation over the $h_i \le x_1$ variables equals

$$\left(\sum_{h \le x_1} h^{-1/d} \right)^{d-1} \ll x_1^{d-2+1/d}.$$

Since the sum over n is $\ll N^{(1-1/d)(1+\epsilon)}$, we deduce that

$$\sum_{\mathbf{z} \in \mathcal{S}^{2,d}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| \ll N^{2-1/d+2\epsilon} \sum_{x_1 \le N} x_1^{d-1-A/d}$$

$$\ll N^{2-1/d-\min(0, A/d-d)+3\epsilon}. \tag{5.10}$$

Substituting (5.10) back into (5.9), we obtain

$$T_2 \ll \sum_{2 \le d \le 4} N^{2-1/d-\min(0, A/d-d)+3\epsilon} \ll N^{4-\rho} \tag{5.11}$$

for some $\rho > 0$ as long as $-2 - 1/d - A/d + d < 0$, or equivalently $A > d^2 - 2d - 1$, for all $2 \le d \le 4$, which is equivalent to $A > 7$.

Bounding T_3. For $1 \le d \le 3$, let

$$\mathcal{S}^{3,d} := \{\mathbf{z} \in \mathcal{S}^3 : \#(\{x_1, x_2, y_2\} \setminus \{1\}) = d\},$$

so that

$$T_3 = \sum_{d \le 3} \sum_{\mathbf{z} \in \mathcal{S}^{3,d}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|.$$

For $\mathbf{z} \in \mathcal{S}^{3,d}$, the phase function ϕ consists of d non-constant terms with non-vanishing coefficients. Now we have $x_1 = y_1$, and therefore the leading coefficient of x_1^α is

$$l := n - m \ne 0.$$

Hence, Lemma 4.4 yields

$$\sum_{\mathbf{z} \in \mathcal{S}^{3,d}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|$$

$$\ll N^{\epsilon/d} \sum_{\substack{1 \le |m| \le N^{1+\epsilon} \\ 1 \le |l| \le 2N^{1+\epsilon}}} |l|^{-1/d} \sum_{x_1 \le N} x_1^{1-A/d-1/d} \sum_{h_1,\ldots,h_{d-1} \le x_1} h_1^{-1/d} \cdots h_{d-1}^{-1/d}$$

$$\ll N^{2-1/d-\min(0,A/d-d)+3\epsilon}.$$

Therefore,

$$T_3 \ll \sum_{d \le 3} N^{2-1/d-\min(0,A/d-d)+3\epsilon} \ll N^{4-\rho} \tag{5.12}$$

for some $\rho > 0$ as long as $A > d^2 - 2d - 1$ for all $1 \le d \le 3$, which is equivalent to $A > 2$.

To summarize, if $A > 7$, then inserting into (5.7) the estimates of the T_i from (5.8), (5.11), and (5.12), we find that

$$\mathrm{Var}(R_2(f, \cdot, N), \mathcal{J}) \ll N^{-\rho} \quad \text{for some } \rho > 0. \qquad \blacksquare$$

6. Higher order correlations

6.1. Expectation and variance in terms of oscillatory integrals

For $k \ge 2$ and $\epsilon > 0$, let $\mathcal{N}_{k-1}^{\epsilon} = \mathcal{N}_{k-1}^{\epsilon}(N)$ denote the set of integer $(k-1)$-tuples $(n_1, \ldots, n_{k-1})$ satisfying $|n_i| \le N^{1+\epsilon}$.

Recall that we denoted by $\mathcal{X}_k$ the set of k-tuples $(x_1, \ldots, x_k)$ of *distinct* integers satisfying $1 \le x_i \le N$. For $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{X}_k$, denote

$$\Delta(\mathbf{x}, \alpha) := (x_1^{\alpha} - x_2^{\alpha}, x_2^{\alpha} - x_3^{\alpha}, \ldots, x_{k-1}^{\alpha} - x_k^{\alpha}),$$

so that

$$R_k(f, \alpha, N) = \frac{1}{N} \sum_{\mathbf{m} \in \mathbb{Z}^{k-1}} \sum_{\mathbf{x} \in \mathcal{X}_k} f\left(N(\Delta(\mathbf{x}, \alpha) - \mathbf{m})\right).$$

The following lemma generalizes Lemma 3.4.

Lemma 6.1 (Truncated Poisson summation). *Let $k \ge 2$, $f \in C_c^{\infty}(\mathbb{R}^{k-1})$, and $\epsilon > 0$. Then for all $t > 0$ we have*

$$R_k(f, \alpha, N) = \frac{1}{N^k} \sum_{\mathbf{n} \in \mathcal{N}_{k-1}^{\epsilon}} \sum_{\mathbf{x} \in \mathcal{X}_k} \hat{f}\left(\frac{\mathbf{n}}{N}\right) e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle) + O(N^{-t}) \tag{6.1}$$

as $N \to \infty$.

Proof. By Poisson summation,

$$R_k(f, \alpha, N) = \frac{1}{N^k} \sum_{\mathbf{n} \in \mathbb{Z}^{k-1}} \sum_{\mathbf{x} \in \mathcal{X}_k} \hat{f}\left(\frac{\mathbf{n}}{N}\right) e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle).$$

Bounding the summation over $\mathbf{x} \in \mathcal{X}_k$ trivially yields

$$\sum_{\substack{\mathbf{n} \in \mathbb{Z}^{k-1} \\ \|\mathbf{n}\|_\infty > N^{1+\epsilon}}} \sum_{\mathbf{x} \in \mathcal{X}_k} \hat{f}\left(\frac{\mathbf{n}}{N}\right) e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle) \le N^k \sum_{\substack{\mathbf{n} \in \mathbb{Z}^{k-1} \\ \|\mathbf{n}\|_\infty > N^{1+\epsilon}}} \left| \hat{f}\left(\frac{\mathbf{n}}{N}\right) \right|.$$

By the rapid decay of $\hat{f}$, for $\mathbf{u} = (u_1, \dots, u_{k-1}) \in \mathbb{R}^{k-1}$ we have

$$\hat{f}(\mathbf{u}) = O\left(\frac{1}{(1 + |u_1|)^{s_1} \cdots (1 + |u_{k-1}|)^{s_{k-1}}} \right) \tag{6.2}$$

for any $s_1, \dots, s_{k-1} > 0$. In particular, for any $s > 0$ we have

$$\sum_{\substack{\mathbf{n} \in \mathbb{Z}^{k-1} \\ \|\mathbf{n}\|_\infty > N^{1+\epsilon}}} \left| \hat{f}\left(\frac{\mathbf{n}}{N}\right) \right| \ll N^{s+2(k-2)} \sum_{\substack{\mathbf{n} \in \mathbb{Z}^{k-1} \\ n_1 > N^{1+\epsilon}}} n_1^{-s} (1 + |n_2|)^{-2} \cdots (1 + |n_{k-1}|)^{-2}$$

$$\ll N^{2k-4+s} \sum_{n > N^{1+\epsilon}} n^{-s} \ll N^{2k-4+s-(1+\epsilon)(s-1)}. \tag{6.3}$$

Taking s so large that

$$2k - 4 + s - (1 + \epsilon)(s - 1) = 2k - 3 + \epsilon - \epsilon s < -t,$$

we find that the right-hand side of (6.3) is $< N^{-t}$, which gives our claim. ∎

Given $\mathbf{n} \in \mathbb{Z}^{k-1}$, we define the vector $\mathbf{u}(\mathbf{n}) = (u_1(\mathbf{n}), \dots, u_k(\mathbf{n})) \in \mathbb{Z}^k$ by

$$u_i(\mathbf{n}) := \begin{cases} n_1 & \text{if } i = 1, \\ n_i - n_{i-1} & \text{if } 2 \le i \le k-1, \\ -n_{k-1} & \text{if } i = k. \end{cases}$$

Note that the linear map $\mathbf{n} \mapsto \mathbf{u}(\mathbf{n})$ is injective. Moreover,

$$\|\mathbf{u}(\mathbf{n})\|_\infty \le 2\|\mathbf{n}\|_\infty \tag{6.4}$$

and

$$\sum_{i=1}^k u_i(\mathbf{n}) = 0. \tag{6.5}$$

Let

$$\mathcal{U}_k^\epsilon = \mathcal{U}_k^\epsilon(N) = \{\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{Z}^k : 1 \le \|\mathbf{u}\|_\infty \le 2N^{1+\epsilon}, u_1 + \cdots + u_k = 0\},$$

and note that the relations (6.4), (6.5) imply that $\mathbf{u}(\mathbf{n}) \in \mathcal{U}_k^\epsilon$ whenever $\mathbf{0}_{k-1} \ne \mathbf{n} \in \mathcal{N}_{k-1}^\epsilon$.

6.2. Degenerate regimes

Let $K > 0$ (below we will take either $K = k$ or $K = 2k$), let $\mathbf{u} = (u_1, \ldots, u_K) \in \mathbb{Z}^K$, $\mathbf{x} = (x_1, \ldots, x_K) \in \mathbb{Z}_{>0}^K$, and let $\phi(\alpha)$ be a function of the form

$$\phi(\alpha) = \phi(\mathbf{z}, \alpha) = \sum_{i \leq K} u_i x_i^\alpha, \quad \mathbf{z} = (\mathbf{u}, \mathbf{x}). \tag{6.6}$$

To utilize the repulsion principle, we require the derivative of the phase function ϕ to genuinely depend on all the x_i variables. While this is true throughout most of the regime, there are certain constellations of parameters where this basic property fails. To illustrate this phenomenon, let us consider the oscillatory integrals that we have already encountered when analysing the variance of the pair correlation sum,

$$\int_{\mathscr{J}} e(n(x_1^\alpha - x_2^\alpha) - m(y_1^\alpha - y_2^\alpha)) \, d\alpha, \quad \begin{array}{l} 1 \leq x_2 < x_1 \leq N, \\ 1 \leq y_2 < y_1 \leq N, \end{array} \quad 1 \leq |n|, |m| \leq N^{1+\epsilon}. \tag{6.7}$$

Already here (different kinds of) degeneracy issues arose, yet the combinatorics was still simple. Note that this integral can degenerate in essentially three different ways:

(1) Some of the variables x_i, y_i can be equal to 1, e.g., $x_2 = 1$ (in fact there can be at most two such variables).

(2) Some of the variables x_i, y_i could be identical, e.g., we may have $x_1 = y_1$; in fact, there can be at most two pairs of identical variables in (6.7).

(3) The variables n, m can be chosen in such a manner that the coefficients of some terms vanish. For instance, we may have $n = m$ and $x_1 = y_1$. Moreover, a particular scenario is that the variables are arranged in such a way that the phase function ϕ vanishes identically[3] when

$$x_1 = y_1, \quad x_2 = y_2, \quad n = m.$$

Fortunately, this is the only configuration for this scenario to happen, and there are only $O(N^{3+\epsilon})$ such parameters. Since the variance estimate is equipped with a normalization factor of N^{-4}, the contribution from this regime is negligible.

Each of these possible degeneracies will also occur when dealing with the expectation and the variance of higher correlation sums, and will need to be accounted for.

Given ϕ of the form (6.6), we define a measurement of how many variables x_i genuinely occur in the derivative $\phi'(\alpha)$. We can clearly write

$$\phi'(\alpha) = \sum_{i \leq d} w_i z_i^\alpha \tag{6.8}$$

---

[3]It turns out that ϕ can vanish identically only in the kind of integrals like (6.7) appearing in the variance bounds, but not in the kind of integrals involved in the expectation.

where $0 \le d \le K$, $\{z_1, \ldots, z_d\} \subseteq \{x_1, \ldots, x_K\}$, $z_1, \ldots, z_d \ge 2$ are distinct, and $w_1, \ldots, w_d \ne 0$. Moreover, by the independence of the functions $\alpha \mapsto z_i^\alpha$, the representation (6.8) is unique, and we say that ϕ is $(K-d)$-*degenerate*. Instead of 0-degenerate (corresponding to $d = K$) we say that ϕ is *non-degenerate*.

Let $\mathcal{E}_{k,d}^\epsilon = \mathcal{E}_{k,d}^\epsilon(N)$ (resp. $\mathcal{V}_{k,d}^\epsilon = \mathcal{V}_{k,d}^\epsilon(N)$) denote the set of all $\mathbf{z} = (\mathbf{u}, \mathbf{x}) \in \mathcal{U}_k^\epsilon \times \mathcal{X}_k$ (resp. $\mathbf{z} = (\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}) \in (\mathcal{U}_k^\epsilon)^2 \times \mathcal{X}_k^2$) such that $\phi(\mathbf{z}, \alpha)$ is $(k-d)$-degenerate (resp. $(2k-d)$-degenerate). Our main goal will be to bound the sums

$$S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J}) := \sum_{\mathbf{z} \in \mathcal{E}_{k,d}^\epsilon} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|, \quad S(\mathcal{V}_{k,d}^\epsilon, \mathcal{J}) := \sum_{\mathbf{z} \in \mathcal{V}_{k,d}^\epsilon} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|.$$

As we shall show, these quantities control the expectation and variance of R_k.

Lemma 6.2. *Let $k \ge 2$, $f \in C_c^\infty(\mathbb{R}^{k-1})$, $\epsilon > 0$, $C_k(N)$ be as in (1.6), and*

$$E_k(N, \mathcal{J}, \epsilon) := \frac{1}{N^k} \sum_{0 < d \le k} S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J}).$$

Then for all $t > 0$, as $N \to \infty$, we have

$$\int_{\mathcal{J}} R_k(f, \alpha, N) \, d\alpha - C_k(N) \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x} \ll E_k(N, \mathcal{J}, \epsilon) + N^{-t}. \tag{6.9}$$

Proof. By Lemma 6.1,

$$\int_{\mathcal{J}} R_k(f, \alpha, N) \, d\alpha = \frac{1}{N^k} \sum_{\substack{\mathbf{x} \in \mathcal{X}_k \\ \mathbf{n} \in \mathcal{N}_{k-1}^\epsilon}} \hat{f}\left(\frac{\mathbf{n}}{N}\right) \int_{\mathcal{J}} e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle) \, d\alpha + O(N^{-t}).$$

We observe that for $\mathbf{n} = \mathbf{0}_{k-1}$, the number of corresponding $\mathbf{x} \in \mathcal{X}_k$ to choose from is

$$\#\mathcal{X}_k = N \cdot (N-1) \cdots (N-k+1).$$

Hence

$$\int_{\mathcal{J}} R_k(f, \alpha, N) \, d\alpha - C_k(N) \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x}$$

$$= \frac{1}{N^k} \sum_{\substack{\mathbf{x} \in \mathcal{X}_k \\ \mathbf{0}_{k-1} \ne \mathbf{n} \in \mathcal{N}_{k-1}^\epsilon}} \hat{f}\left(\frac{\mathbf{n}}{N}\right) \int_{\mathcal{J}} e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle) \, d\alpha + O(N^{-t}). \tag{6.10}$$

Clearly,

$$\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle = \sum_{1 \le i \le k-1} n_i (x_i^\alpha - x_{i+1}^\alpha) = n_1 x_1^\alpha - n_{k-1} x_k^\alpha + \sum_{2 \le i \le k-1} (n_i - n_{i-1}) x_i^\alpha$$

$$= \phi(\mathbf{u}(\mathbf{n}), \mathbf{x}, \alpha).$$

Thus,

$$\int_{\mathcal{J}} e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n}\rangle)\, d\alpha = I(\phi(\mathbf{u}(\mathbf{n}), \mathbf{x}, \cdot), \mathcal{J}).$$

Summing over the different $(k - d)$-degenerate regimes (and noting that k-degeneracy, corresponding to $d = 0$, cannot occur for $\mathbf{n} \neq \mathbf{0}_{k-1}$) implies

$$\sum_{\substack{\mathbf{x} \in \mathcal{X}_k \\ \mathbf{0}_{k-1} \neq \mathbf{n} \in \mathcal{N}_{k-1}^{\epsilon}}} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) \int_{\mathcal{J}} e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n}\rangle)\, d\alpha \ll \sum_{0 < d \leq k} \sum_{\substack{\mathbf{x} \in \mathcal{X}_k \\ \mathbf{0}_{k-1} \neq \mathbf{n} \in \mathcal{N}_{k-1}^{\epsilon} \\ \mathbf{z} = (\mathbf{u}(\mathbf{n}), \mathbf{x}) \in \mathcal{E}_{k,d}^{\epsilon}}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|.$$

Finally, by the injectivity of the map $\mathbf{n} \mapsto \mathbf{u}(\mathbf{n})$ we have

$$\sum_{\substack{\mathbf{x} \in \mathcal{X}_k \\ \mathbf{0}_{k-1} \neq \mathbf{n} \in \mathcal{N}_{k-1}^{\epsilon} \\ \mathbf{z} = (\mathbf{u}(\mathbf{n}), \mathbf{x}) \in \mathcal{E}_{k,d}^{\epsilon}}} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| \ll S(\mathcal{E}_{k,d}^{\epsilon}, \mathcal{J}),$$

which implies (6.9). ∎

The derivation of a bound for the variance of R_k in terms of oscillatory integrals is similar:

Lemma 6.3. *Let $k \geq 2$, $f \in C_c^{\infty}(\mathbb{R}^{k-1})$, and $\epsilon > 0$. Then, for all $t > 0$, we have*

$$\mathrm{Var}(R_k(f, \cdot, N), \mathcal{J}) \ll V_k(N, \mathcal{J}, \epsilon) + N^{-t} \tag{6.11}$$

as $N \to \infty$, where

$$V_k(N, \mathcal{J}, \epsilon) := \frac{1}{N^{2k}} \sum_{0 \leq d \leq 2k} S(\mathcal{V}_{k,d}^{\epsilon}, \mathcal{J}).$$

Proof. By Lemma 6.1, for all $s > 0$ we have

$$\mathrm{Var}(R_k(f, \cdot, N), \mathcal{J}) = \int_{\mathcal{J}} \left(N^{-k} \sum_{\substack{\mathbf{x} \in \mathcal{X}_k \\ \mathbf{0}_{k-1} \neq \mathbf{n} \in \mathcal{N}_{k-1}^{\epsilon}}} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n}\rangle) + O(N^{-s}) \right)^2 d\alpha.$$

Expanding the square and taking s sufficiently large, we find that for all $t > 0$,

$$\mathrm{Var}(R_k(f, \cdot, N), \mathcal{J}) = N^{-2k} \sum_{\substack{\mathbf{x}, \mathbf{y} \in \mathcal{X}_k \\ \mathbf{0}_{k-1} \neq \mathbf{n}, \mathbf{m} \in \mathcal{N}_{k-1}^{\epsilon}}} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) \widehat{f}\left(\frac{\mathbf{m}}{N}\right)$$

$$\times \int_{\mathcal{J}} e(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n}\rangle + \langle \Delta(\mathbf{y}, \alpha), \mathbf{m}\rangle)\, d\alpha + O(N^{-t}).$$

Repeating the arguments of Lemma 6.2 yields the claim. ∎

6.3. Proof of Theorem 1.4: the general case

As outlined above, we wish to relate the sums $S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J})$ and $S(\mathcal{V}_{k,d}^\epsilon, \mathcal{J})$ to quantities that only involve those $\mathbf{z}$ for which $\phi(\mathbf{z}, \alpha)$ is non-degenerate.

Fix $d, j \geq 1$, and for each $1 \leq i \leq d$, let $L_i : \mathbb{Z}^j \to \mathbb{Z}$ denote a non-zero linear map. Let $\mathbf{L} = (L_1, \ldots, L_d)$, and consider the sums

$$Q(d, j, \mathbf{L}, N, \mathcal{J}, \epsilon) := \sum_{\substack{\mathbf{m} \in \mathbb{Z}^j \\ 1 \leq \|\mathbf{m}\|_\infty \leq 2N^{1+\epsilon} \\ L_i(\mathbf{m}) \neq 0 \, (1 \leq i \leq d)}} \sum_{\substack{\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{Z}^d \\ 2 \leq t_i \leq N \text{ distinct}}} |I(\phi(\mathbf{L}(\mathbf{m}), \mathbf{t}, \cdot), \mathcal{J})|. \quad (6.12)$$

Here ϕ is as in (6.6) with $K = d$.

Surely bounding $S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J})$, $S(\mathcal{V}_{k,d}^\epsilon, \mathcal{J})$ in terms of Q requires weight factors accounting for the number of variables that $|I(\phi(\mathbf{z}, \cdot), \mathcal{J})|$ does not effectively depend upon – see the proofs of Propositions 6.5 and 6.6 below. But first, we deduce an appropriate bound for Q.

Lemma 6.4. *Fix $d, j \geq 1$. For $1 \leq i \leq d$, let $L_i : \mathbb{Z}^j \to \mathbb{Z}$ denote a non-zero linear map. Further, let $\mathbf{L} = (L_1, \ldots, L_d)$. For every $\epsilon > 0$, we have the bound*

$$Q(d, j, \mathbf{L}, N, \mathcal{J}, \epsilon) = O(N^{j-1/d-\min(A/d-d,0)+\epsilon}) \quad as \ N \to \infty. \quad (6.13)$$

Proof. Let $\mathbf{m}, \mathbf{t}$ be arbitrary elements in the summation of Q, and assume without loss of generality that $t_1 < \cdots < t_d$. By Lemma 4.4,

$$|I(\phi(\mathbf{L}(\mathbf{m}), \mathbf{t}, \cdot), \mathcal{J})| \ll_{d,\epsilon} |L_d(\mathbf{m})|^{-1/d} t_d^{-A/d+1-1/d} (h_1 \cdots h_{d-1})^{-1/d} N^{\epsilon/d},$$

where $h_i = t_{i+1} - t_i$ for $i = 1, \ldots, d-1$.

Since L_d is not the zero map, we can express one of the variables comprising $\mathbf{m}$ in terms of the other $j - 1$ variables and $l = L_d(\mathbf{m})$. Thus, the bound

$$Q(d, j, \mathbf{L}, N, \mathcal{J}, \epsilon)$$
$$\ll N^{(j-1)(1+\epsilon)+\epsilon/d} \sum_{1 \leq |l| \ll_{\mathbf{L}} N^{1+\epsilon}} |l|^{-1/d} \sum_{t \leq N} t^{-A/d+1-1/d} \sum_{\substack{h_i \leq t \\ i \leq d-1}} (h_1 \cdots h_{d-1})^{-1/d}$$
$$\ll N^{j-1/d-\min(A/d-d,0)+(j+1)\epsilon}$$

produces the required estimate. ∎

Proposition 6.5. *Let $k \geq 2$ and $\epsilon > 0$. If $0 < d \leq k$, then*

$$S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J}) = O(N^{k-1-1/d-\min(A/d-d,0)+\epsilon}) \quad as \ N \to \infty. \quad (6.14)$$

Proof. For (possibly empty) index sets $\mathcal{J}_1, \mathcal{J}_2 \subseteq [k]$, denote by $\mathcal{E}_{k,d}^\epsilon(\mathcal{J}_1, \mathcal{J}_2)$ the set of $\mathbf{z} = (\mathbf{u}, \mathbf{x}) \in \mathcal{E}_{k,d}^\epsilon$ such that

$$\{i \in [k] : x_i = 1\} = \mathcal{J}_1, \quad \{i \in [k] \setminus \mathcal{J}_1 : u_i = 0\} = \mathcal{J}_2.$$

Assume that the set $\mathcal{E}_{k,d}^{\epsilon}(\mathcal{J}_1, \mathcal{J}_2)$ is nonempty. Then $d = k - \#(\mathcal{J}_1 \cup \mathcal{J}_2)$, and since x_i are distinct we have $\#\mathcal{J}_1 \leq 1$.

Consider the sum

$$S(\mathcal{E}_{k,d}^{\epsilon}(\mathcal{J}_1, \mathcal{J}_2), \mathcal{J}) := \sum_{\mathbf{z} \in \mathcal{E}_{k,d}^{\epsilon}(\mathcal{J}_1, \mathcal{J}_2)} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})| \qquad (6.15)$$

and recall that the summation in (6.15) is over $\mathbf{z} = (\mathbf{u}, \mathbf{x})$ belonging to a subset of $\mathcal{U}_k^{\epsilon} \times \mathcal{X}_k$. We first determine the constraints on $\mathbf{u}$. By the conditions $u_i = 0$ ($i \in \mathcal{J}_2$) and $u_1 + \cdots + u_k = 0$, the summation is restricted to $\mathbf{u}$ whose entries linearly depend on $k - 1 - \#\mathcal{J}_2$ of the variables $u_1, \ldots, u_k$, i.e., there exists a set

$$\{j_1, \ldots, j_{k-1-\#\mathcal{J}_2}\} \subseteq [k]$$

such that each u_i is a linear combination of $u_{j_1}, \ldots, u_{j_{k-1-\#\mathcal{J}_2}}$ (since $\mathbf{u} \neq \mathbf{0}_k$, we have $\#\mathcal{J}_2 < k - 1$). For each $\mathbf{z} \in \mathcal{E}_{k,d}^{\epsilon}(\mathcal{J}_1, \mathcal{J}_2)$, we can then write

$$\phi(\mathbf{z}, \alpha) = \sum_{i \in [k] \setminus \mathcal{J}_2} L_i(u_{j_1}, \ldots, u_{j_{k-1-\#\mathcal{J}_2}}) x_i^{\alpha} \qquad (6.16)$$

where L_i are non-zero linear combinations of the variables $u_{j_1}, \ldots, u_{j_{k-1-\#\mathcal{J}_2}}$ (determined only by $\mathcal{J}_2$).

There are d non-constant terms in the sum (6.16) corresponding to the indices $i \in [k] \setminus (\mathcal{J}_1 \cup \mathcal{J}_2)$ (note that if $\mathcal{J}_1$ is non-empty, then one of the terms in the sum (6.16) is constant). Moreover, for each $i \in \mathcal{J}_2$, the value of x_i does not affect $\phi(\mathbf{z}, \alpha)$, and x_i ranges between 2 and N, so that the function (6.16) appears $O(N^{\#\mathcal{J}_2})$ times in (6.15) upon summing over $\mathbf{z}$. Hence, letting $\mathbf{L} = (L_i)_{i \in [k] \setminus (\mathcal{J}_1 \cup \mathcal{J}_2)}$, we have

$$S(\mathcal{E}_{k,d}^{\epsilon}(\mathcal{J}_1, \mathcal{J}_2), \mathcal{J}) \ll N^{\#\mathcal{J}_2} Q(d, k - 1 - \#\mathcal{J}_2, \mathbf{L}, N, \mathcal{J}, \epsilon).$$

By Lemma 6.4, we have

$$N^{\#\mathcal{J}_2} Q(d, k - 1 - \#\mathcal{J}_2, \mathbf{L}, N, \mathcal{J}, \epsilon) \ll N^{k-1-1/d-\min(A/d-d,0)+\epsilon}.$$

Summing over all configurations $\mathcal{E}_{k,d}^{\epsilon}(\mathcal{J}_1, \mathcal{J}_2)$ completes the proof. ∎

The argument for majorizing $S(\mathcal{V}_{k,d}^{\epsilon}, \mathcal{J})$ by a weighted sum $Q(d, j, \mathbf{L}, N, \mathcal{J}, \epsilon)$ for some $d, j, \mathbf{L}$ is similar but the combinatorics is somewhat more technical.

Proposition 6.6. *Let $k \geq 2$ and $\epsilon > 0$. If $d = 0$ then*

$$S(\mathcal{V}_{k,0}^{\epsilon}, \mathcal{J}) = O(N^{2k-1+\epsilon}) \quad \text{as } N \to \infty, \qquad (6.17)$$

and if $0 < d \leq 2k$ then

$$S(\mathcal{V}_{k,d}^{\epsilon}, \mathcal{J}) = O(N^{2k-2-1/d-\min(A/d-d,0)+\epsilon}) \quad \text{as } N \to \infty. \qquad (6.18)$$

Proof. Let $0 \leq d \leq 2k$, and let $\mathcal{I}_1, \mathcal{I}'_1, \mathcal{I}_2, \mathcal{I}'_2, \mathcal{I}_3, \mathcal{I}'_3, \mathcal{I}_4, \mathcal{I}'_4 \subseteq [k]$ be (possibly empty) sets of indices. Fixing $\tau := (\mathcal{I}_1, \mathcal{I}'_1, \mathcal{I}_2, \mathcal{I}'_2, \mathcal{I}_3, \mathcal{I}'_3, \mathcal{I}_4, \mathcal{I}'_4)$, we denote by $\mathcal{V}^\epsilon_{k,d}(\tau)$ the set of vectors $\mathbf{z} = (\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}) \in \mathcal{V}^\epsilon_{k,d}$ for which

$$\{i \in [k] : x_i = 1\} = \mathcal{I}_1, \quad \{j \in [k] : y_j = 1\} = \mathcal{I}'_1,$$
$$\{i \in [k] \setminus \mathcal{I}_1 : \exists_{j(i) \in [k]} \, x_i = y_{j(i)}\} = \mathcal{I}_2,$$
$$\{j \in [k] \setminus \mathcal{I}'_1 : \exists_{i(j) \in [k]} \, x_{i(j)} = y_j\} = \mathcal{I}'_2,$$
$$\{i \in \mathcal{I}_2 : u_i + v_{j(i)} = 0, \text{ where } j(i) \text{ is such that } x_i = y_{j(i)}\} = \mathcal{I}_3,$$
$$\{j \in \mathcal{I}'_2 : u_{i(j)} + v_j = 0, \text{ where } i(j) \text{ is such that } x_{i(j)} = y_j\} = \mathcal{I}'_3,$$

and

$$\{i \in [k] \setminus (\mathcal{I}_1 \cup \mathcal{I}_2) : u_i = 0\} = \mathcal{I}_4$$
$$\{j \in [k] \setminus (\mathcal{I}'_1 \cup \mathcal{I}'_2) : v_j = 0\} = \mathcal{I}'_4.$$

Assume that the set $\mathcal{V}^\epsilon_{k,d}(\tau)$ is non-empty. Then $\#\mathcal{I}_2 = \#\mathcal{I}'_2$, $\#\mathcal{I}_3 = \#\mathcal{I}'_3$ and

$$d = 2k - (\#\mathcal{I}_1 + \#\mathcal{I}'_1 + \#\mathcal{I}_2 + \#\mathcal{I}_3 + \#\mathcal{I}_4 + \#\mathcal{I}'_4). \tag{6.19}$$

Consider the sum

$$S(\mathcal{V}^\epsilon_{k,d}(\tau), \mathcal{J}) := \sum_{\mathbf{z} \in \mathcal{V}^\epsilon_{k,d}(\tau)} |I(\phi(\mathbf{z}, \cdot), \mathcal{J})|. \tag{6.20}$$

We first consider the constraints on $\mathbf{u}, \mathbf{v}$ when summing in (6.20) over $\mathbf{z} = (\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}) \in \mathcal{V}^\epsilon_{k,d}(\tau)$:

(i) The conditions $u_i = 0$ ($i \in \mathcal{I}_4$) and $u_1 + \cdots + u_k = 0$ determine $\#\mathcal{I}_4 + 1$ of the variables u_i in terms of the other $k - 1 - \#\mathcal{I}_4$ variables u_i. Note that $\mathbf{u} \neq \mathbf{0}_k$, so that $\#\mathcal{I}_4 < k - 1$.

(ii) The conditions $v_j = 0$ ($j \in \mathcal{I}'_4$), $u_{i(j)} + v_j = 0$, $j \in \mathcal{I}'_3$, determine $\#\mathcal{I}'_4 + \#\mathcal{I}'_3$ of the variables v_j in terms of the variables u_i.

(iii) The condition $v_1 + \cdots + v_k = 0$ trivializes if $\mathcal{I}_3 \cup \mathcal{I}_4 = \mathcal{I}'_3 \cup \mathcal{I}'_4 = [k]$. Otherwise, if $\mathcal{I}'_3 \cup \mathcal{I}'_4 \neq [k]$, it determines another variable v_j ($j \notin \mathcal{I}'_3 \cup \mathcal{I}'_4$) in terms of the rest of the variables. If $\mathcal{I}'_3 \cup \mathcal{I}'_4 = [k]$ and $\mathcal{I}_3 \cup \mathcal{I}_4 \neq [k]$, it determines another variable u_i in terms of the rest of the variables.

To conclude, we have found that there exist sets

$$\{i_1, \ldots, i_l\} \subseteq [k], \quad \{j_1, \ldots, j_m\} \subseteq [k]$$

such that the variables $u_1, \ldots, u_k, v_1, \ldots, v_k$ linearly depend on $u_{i_1}, \ldots u_{i_l}, v_{j_1}, \ldots v_{j_m}$, and

$$l + m = (k - 1 - \#\mathcal{I}_4) + (k - \#\mathcal{I}'_3 - \#\mathcal{I}'_4) - 1 = 2k - 2 - \#\mathcal{I}'_3 - \#\mathcal{I}_4 - \#\mathcal{I}'_4 \tag{6.21}$$

unless $\mathcal{J}_3 \cup \mathcal{J}_4 = \mathcal{J}_3' \cup \mathcal{J}_4' = [k]$, in which case we have

$$l + m = (k - 1 - \#\mathcal{J}_4) + (k - \#\mathcal{J}_3' - \#\mathcal{J}_4') = 2k - 1 - \#\mathcal{J}_3' - \#\mathcal{J}_4 - \#\mathcal{J}_4'. \quad (6.22)$$

For each $\mathbf{z} \in \mathcal{V}_{k,d}^\epsilon(\tau)$, we can then write

$$\phi(\mathbf{z}, \alpha) = \sum_{i \in [k] \setminus (\mathcal{J}_3 \cup \mathcal{J}_4)} L_i(u_{i_1}, \dots u_{i_l}, v_{j_1}, \dots v_{j_m}) x_i^\alpha$$

$$+ \sum_{j \in [k] \setminus (\mathcal{J}_2' \cup \mathcal{J}_4')} L_j(u_{i_1}, \dots u_{i_l}, v_{j_1}, \dots v_{j_m}) y_i^\alpha, \quad (6.23)$$

where L_i are non-zero linear combinations of the variables $u_{i_1}, \dots u_{i_l}, v_{j_1}, \dots v_{j_m}$ (determined by τ).

The total number of non-constant terms in (6.23) is d (see (6.19); note that if at least one of the sets $\mathcal{J}_1$ or $\mathcal{J}_1'$ is non-empty, then one or two terms in the sum (6.16) are constant). For each $i \in \mathcal{J}_3 \cup \mathcal{J}_4$, the value of x_i does not affect $\phi(\mathbf{z}, \alpha)$, and x_i ranges between 2 and N. Likewise, for each $j \in \mathcal{J}_4'$, the value of y_j does not affect $\phi(\mathbf{z}, \alpha)$, and y_j ranges between 2 and N. Hence, the function (6.23) appears $O(N^{\#\mathcal{J}_3 + \#\mathcal{J}_4 + \#\mathcal{J}_4'})$ times in (6.20) upon summing over $\mathbf{z}$.

If $d = 0$, then the phase function is constant (in fact, it must vanish), and therefore

$$S(\mathcal{V}_{k,0}^\epsilon(\tau), \mathcal{J}) \ll N^{\#\mathcal{J}_3 + \#\mathcal{J}_4 + \#\mathcal{J}_4'} N^{(1+\epsilon)(l+m)} \ll N^{(2k-1)(1+\epsilon)}$$

in either of the cases (6.21), (6.22). If $d > 0$, we can choose

$$\mathbf{L} = (L_i, L_j)_{i \in [k] \setminus (\mathcal{J}_1 \cup \mathcal{J}_3 \cup \mathcal{J}_4), \, j \in [k] \setminus (\mathcal{J}_1' \cup \mathcal{J}_2' \cup \mathcal{J}_4')}$$

and get

$$S(\mathcal{V}_{k,d}^\epsilon(\tau), \mathcal{J}) \ll N^{\#\mathcal{J}_3 + \#\mathcal{J}_4 + \#\mathcal{J}_4'} Q(d, l + m, \mathbf{L}, N, \mathcal{J}, \epsilon).$$

Clearly, either $\mathcal{J}_3 \cup \mathcal{J}_4 \neq [k]$ or $\mathcal{J}_3' \cup \mathcal{J}_4' \neq [k]$, so that by (6.21) we have

$$Q(d, l + m, \mathbf{L}, N, \mathcal{J}, \epsilon) = Q(d, 2k - 2 - \#\mathcal{J}_3 - \#\mathcal{J}_4 - \#\mathcal{J}_4', \mathbf{L}, N, \mathcal{J}, \epsilon).$$

By Lemma 6.4, we establish the bound

$$N^{\#\mathcal{J}_3 + \#\mathcal{J}_4 + \#\mathcal{J}_4'} Q(d, 2k - 2 - \#\mathcal{J}_3 - \#\mathcal{J}_4 - \#\mathcal{J}_4', \mathbf{L}, N, \mathcal{J}, \epsilon)$$
$$\ll N^{2k-2-1/d-\min(A/d-d,0)+\epsilon}.$$

As there are $\ll_k 1$ sets $\mathcal{V}_{k,d}^\epsilon(\tau)$, summing over τ concludes the proof. ∎

Corollary 6.7. *For each $A > k^2 - k - 1$ there exists $\rho = \rho(A) > 0$ such that for any $0 < d \leq k$,*

$$S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J}) = O(N^{k-\rho}) \quad \text{as } N \to \infty. \quad (6.24)$$

Further, for each $A > 4k^2 - 4k - 1$ there exists $\rho = \rho(A) > 0$ such that for any $0 \leq d \leq 2k$,

$$S(\mathcal{V}_{k,d}^\epsilon, \mathcal{J}) = O(N^{2k-\rho}) \quad \text{as } N \to \infty. \quad (6.25)$$

Proof. Recall that by Proposition 6.5 we have

$$S(\mathcal{E}_{k,d}^\epsilon, \mathcal{J}) = O(N^{k-1-1/d-\min(A/d-d,0)+\epsilon}).$$

Hence, we obtain (6.24) when $-1 - 1/d - A/d + d < 0$, or equivalently $A > d(d-1) - 1$ for all $0 < d \le k$. Since $d \mapsto d(d-1) - 1$ is increasing on the interval $[1, k]$, it attains its maximum value at $d = k$, which is equal to $k^2 - k - 1$. Hence, (6.24) holds whenever $A > k^2 - k - 1$.

Now recall that Proposition 6.6 yields $S(\mathcal{V}_{k,0}^\epsilon, \mathcal{J}) = O(N^{2k-1+\epsilon})$ for $d = 0$ and

$$S(\mathcal{V}_{k,d}^\epsilon, \mathcal{J}) = O(N^{2k-2-1/d-\min(A/d-d,0)+\epsilon})$$

for $0 < d \le 2k$. Thus, (6.25) holds when $-2 - 1/d - A/d + d < 0$, i.e., $A > d(d-2) - 1$, for all $0 < d \le 2k$. Since $d(d-2) - 1$ is increasing as a function of $d \in [1, 2k]$, the maximum is attained at $d = 2k$ and is equal to $4k^2 - 4k - 1$. Therefore, (6.25) holds whenever $A > 4k^2 - 4k - 1$. ∎

Substituting the bound (6.24) in (6.9), we can now find a regime in which the expectation of $R_k(f, \cdot, N)$ is asymptotic to the average of f. We formulate the next proposition for $k > 2$, since for $k = 2$ Proposition 5.1 yielded a stronger result (holding for $A > 0$).

Proposition 6.8. *Let $k > 2$, $A > k^2 - k - 1$ and let $\mathcal{J}$ be given by* (3.1). *Then there exists $\rho = \rho(A) > 0$ such that*

$$\int_{\mathcal{J}} R_k(f, \alpha, N) \, d\alpha = \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x} + O(N^{-\rho}) \quad \text{as } N \to \infty.$$

Finally, Theorem 1.4 also follows from Corollary 6.7:

Proof of Theorem 1.4. The bound (1.7), $k \ge 2$, follows by substituting (6.25) in (6.11). ∎

7. Proofs of Theorem 1.2 and Corollary 1.6

With the variance bound from Theorem 1.4 at hand, we can deduce Theorem 1.2 by rather soft arguments from a general principle. Although the argument is fairly standard, we have not found it stated explicitly in the literature in a form that readily applies to our case, so we decided to give the details in full. The following proposition allows one to deduce Theorem 1.2 from the variance bound, recorded in Theorem 1.4, at once.

Proposition 7.1. *Let $k \ge 2$, let $\mathcal{J} \subset \mathbb{R}$ be a bounded interval, and let $c_k(N)$ be a sequence satisfying $c_k(N) \to 1$ as $N \to \infty$. Suppose we are given a real-valued sequence $(\vartheta_n(\alpha))_{n \ge 1}$ for each $\alpha \in \mathcal{J}$ so that $\mathcal{J} \ni \alpha \mapsto \vartheta_n(\alpha)$ is a continuous map for each fixed $n \ge 1$. Assume that there exists $\rho > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^{k-1})$,*

$$\int_{\mathcal{J}} \left(R_k(f, (\vartheta_n(\alpha)), N) - c_k(N) \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x} \right)^2 d\alpha = O(N^{-\rho}) \tag{7.1}$$

as $N \to \infty$, then the sequence $(\vartheta_n(\alpha))_{n \geq 1}$ has Poissonian k-level correlation for almost every $\alpha \in \mathcal{J}$.

First we record a useful lemma that allows us to pass from the convergence of a subsequence to the convergence of the entire sequence (extending [20, Lemma 3.1] which was established for $k = 2$).

Lemma 7.2. *Let $(\vartheta_n)_{n \geq 1}$ be a real-valued sequence. If there is an increasing sequence $(N_m)_{m \geq 1}$ of positive integers such that*

$$\lim_{m \to \infty} \frac{N_{m+1}}{N_m} = 1 \tag{7.2}$$

and

$$\lim_{m \to \infty} R_k(f, (\vartheta_n), N_m) = \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x} \tag{7.3}$$

for all $f \in C_c^\infty(\mathbb{R}^{k-1})$, then

$$\lim_{N \to \infty} R_k(f, (\vartheta_n), N) = \int_{\mathbb{R}^{k-1}} f(\mathbf{x}) \, d\mathbf{x} \tag{7.4}$$

for all $f \in C_c^\infty(\mathbb{R}^{k-1})$. Moreover, (7.4) holds for all indicator functions $f = 1_\Pi$ of boxes $\Pi \subseteq \mathbb{R}^{k-1}$.

Proof. First we argue that if the assumption (7.3) is true for all $f \in C_c^\infty(\mathbb{R}^{k-1})$ then it also holds for all indicator functions 1_Π of boxes $\Pi = [a_1, b_1] \times \cdots \times [a_{k-1}, b_{k-1}]$. Fix $\delta > 0$ and choose $f_-, f_+ \in C_c^\infty(\mathbb{R}^{k-1})$ such that $f_- \leq 1_\Pi \leq f_+$ and

$$\int_{\mathbb{R}^{k-1}} (f_+(\mathbf{x}) - f_-(\mathbf{x})) \, d\mathbf{x} < \delta.$$

Then by the definition of the correlation sum (1.4) we have

$$R_k(f_-, (\vartheta_n), N) \leq R_k(1_\Pi, (\vartheta_n), N) \leq R_k(f_+, (\vartheta_n), N).$$

Thus

$$\limsup_{m \to \infty} R_k(1_\Pi, (\vartheta_n), N_m) \leq \limsup_{m \to \infty} R_k(f_+, (\vartheta_n), N_m) = \int_{\mathbb{R}^{k-1}} f_+(\mathbf{x}) \, d\mathbf{x},$$

$$\liminf_{m \to \infty} R_k(1_\Pi, (\vartheta_n), N_m) \geq \liminf_{m \to \infty} R_k(f_-, (\vartheta_n), N_m) = \int_{\mathbb{R}^{k-1}} f_-(\mathbf{x}) \, d\mathbf{x}.$$

Therefore,

$$0 \leq \limsup_{m \to \infty} R_k(1_\Pi, (\vartheta_n), N_m) - \liminf_{m \to \infty} R_k(1_\Pi, (\vartheta_n), N_m)$$

$$\leq \int_{\mathbb{R}^{k-1}} (f_+(\mathbf{x}) - f_-(\mathbf{x})) \, d\mathbf{x} < \delta.$$

Since $\delta > 0$ was arbitrary, we conclude that

$$\lim_{m \to \infty} R_k(1_\Pi, (\vartheta_n), N_m) = \int_{\mathbb{R}^{k-1}} 1_\Pi(\mathbf{x}) \, d\mathbf{x}, \tag{7.5}$$

which proves (7.3) for $f = 1_\Pi$.

Given a positive integer N, we can find $m \geq 1$ such that $N_m \leq N < N_{m+1}$. Moreover,

$$R_k(1_\Pi, (\vartheta_n), N) = \frac{1}{N} \# \left\{ \mathbf{x} \in \mathcal{X}_k(N) : \vartheta_{x_i} - \vartheta_{x_{i+1}} \in \left(\frac{a_i}{N}, \frac{b_i}{N} \right) + \mathbb{Z}, \, i = 1, \ldots, k-1 \right\}.$$

The limit (7.2) implies that when N is sufficiently large we have $N_m/N = N_{m+1}/N = 1 + o(1)$. Given $\delta > 0$, we therefore let

$$\Pi' = [a_1 - \delta, b_1 + \delta] \times \cdots \times [a_{k-1} - \delta, b_{k-1} + \delta]$$

and see that for sufficiently large N

$$R_k(1_\Pi, (\vartheta_n), N)$$

$$= \frac{\frac{N_{m+1}}{N}}{N_{m+1}} \# \left\{ \mathbf{x} \in \mathcal{X}_k(N) : \vartheta_{x_i} - \vartheta_{x_{i+1}} \in \left(\frac{a_i \cdot \frac{N_{m+1}}{N}}{N_{m+1}}, \frac{b_i \cdot \frac{N_{m+1}}{N}}{N_{m+1}} \right) + \mathbb{Z}, \, i = 1, \ldots, k-1 \right\}$$

$$\leq \frac{1+\delta}{N_{m+1}} \# \left\{ \mathbf{x} \in \mathcal{X}_k(N_{m+1}) : \vartheta_{x_i} - \vartheta_{x_{i+1}} \in \left(\frac{a_i - \delta}{N_{m+1}}, \frac{b_i + \delta}{N_{m+1}} \right) + \mathbb{Z}, \, i = 1, \ldots, k-1 \right\},$$

where the right-hand side is $(1 + \delta) R_k(1_{\Pi'}, (\vartheta_n), N_{m+1})$. Thus, we conclude that

$$\limsup_{N \to \infty} R_k(1_\Pi, (\vartheta_n), N) \leq (1 + \delta) \limsup_{m \to \infty} R_k(1_{\Pi'}, (\vartheta_n), N_{m+1})$$

$$= (1 + \delta) \int_{\mathbb{R}^{k-1}} 1_{\Pi'}(\mathbf{x}) \, d\mathbf{x}.$$

Recalling the definition of Π', we clearly have

$$\int_{\mathbb{R}^{k-1}} 1_{\Pi'}(\mathbf{x}) \, d\mathbf{x} = \int_{\mathbb{R}^{k-1}} 1_\Pi(\mathbf{x}) \, d\mathbf{x} + O(\delta).$$

Since δ was arbitrary, we infer that

$$\limsup_{N \to \infty} R_k(1_\Pi, (\vartheta_n), N) \leq \int_{\mathbb{R}^{k-1}} 1_\Pi(\mathbf{x}) \, d\mathbf{x}$$

and a similar argument shows that

$$\liminf_{N \to \infty} R_k(1_\Pi, (\vartheta_n), N) \geq \int_{\mathbb{R}^{k-1}} 1_\Pi(\mathbf{x}) \, d\mathbf{x}.$$

This establishes (7.4) for all functions $f = 1_\Pi$. Since every $f \in C_c^\infty(\mathbb{R}^{k-1})$ can be approximated from below and from above by a linear combination of indicator functions of boxes, (7.4) holds for smooth compactly supported functions as well (by the same argument we detailed above to prove (7.5)). ∎

Proof of Proposition 7.1. For each $m \geq 1$, let $N_m = \lfloor m^{2/\rho} \rfloor$. For each fixed $f \in C_c^\infty(\mathbb{R}^{k-1})$ define

$$X_m(\alpha) = \left| R_k(f, (\vartheta_n(\alpha)), N_m) - c_k(N_m) \int_{\mathbb{R}^{k-1}} f(x)\, \mathrm{d}x \right|^2.$$

By (7.1), the L^1-norms of $X_m \geq 0$ on $\mathcal{J}$ are summable. Changing the order of summation and integration yields

$$\int_{\mathcal{J}} \sum_{m \geq 1} X_m(\alpha)\mathrm{d}\alpha < \infty,$$

and therefore for almost all $\alpha \in \mathcal{J}$ we have

$$\sum_{m \geq 1} X_m(\alpha) < \infty.$$

In particular, $X_m(\alpha) \to 0$ for almost all $\alpha \in \mathcal{J}$, and hence (7.3) is satisfied for almost all $\alpha \in \mathcal{J}$ for our fixed f; by a standard diagonal argument (approximating from above and below by functions f_i belonging to a countable dense set in $C_c^\infty(\mathbb{R}^{k-1})$), we conclude that for almost all $\alpha \in \mathcal{J}$, (7.3) holds for all $f \in C_c^\infty(\mathbb{R}^{k-1})$. Hence by Lemma 7.2, the limit (7.4) holds for almost every $\alpha \in \mathcal{J}$, completing the proof. ∎

To prove Corollary 1.6, we require a well-known relation between the gap distribution and the correlation functions. Let

$$\Delta_{k-1} = \left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{R}_{>0}^{k-1} : \sum_{1 \leq i \leq k-1} x_i < 1 \right\}$$

denote the standard open $k-1$-simplex. For $x > 0$, let $1_{x\Delta_{k-1}}$ be the indicator function of the dilation $x\Delta_{k-1}$.

Lemma 7.3. *Let $(\vartheta_n)_{n \geq 1}$ be a real-valued sequence, and let $K \geq 1$. For all $x > 0$, we have*

$$\sum_{2 \leq k \leq 2K+1} (-1)^k R_k(1_{x\Delta_{k-1}}, (\vartheta_n), N) \leq g(x, (\vartheta_n), N)$$

$$\leq \sum_{2 \leq k \leq 2K} (-1)^k R_k(1_{x\Delta_{k-1}}, (\vartheta_n), N).$$

Proof. The claim follows from [15, Lemma 11 and (A.2)]. ∎

Proof of Corollary 1.6. By Theorem 1.2, for almost all

$$\alpha > 4(2K+1)^2 - 4(2K+1) - 1 = 16K^2 + 8K - 1,$$

the k-level correlation functions R_k are Poissonian for all $2 \leq k \leq 2K+1$, so that as $N \to \infty$, $R_k(1_{x\Delta_{k-1}}, (\vartheta_n), N)$ converges to the volume of $x\Delta_{k-1}$ which is equal to $x^{k-1}/(k-1)!$. The claimed inequalities now follow from Lemma 7.3. ∎

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