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Between reduced powers and ultrapowers

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Abstract. One of our results is a transfer principle between ultrapowers and reduced powers associated with the Fréchet ideal. Although motivated by the Elliott classification programme, this result applies to any axiomatizable category. We also show that there exists a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} such that for every countable (or separable metric) structure B in a countable language the quotient map from the reduced power associated with the Fréchet ideal onto an ultrapower has a right inverse. While the transfer principle is proved without appealing to additional set-theoretic axioms, the conclusion of the latter theorem relies on the Continuum Hypothesis and it is independent of the standard axioms of set theory. We also prove that in the category of C^* -algebras, tensoring with the C^* -algebra of all continuous functions on the Cantor space preserves elementary equivalence. As a side note, neither the Jiang–Su algebra \mathcal{Z} nor any UHF algebra share this property.

Keywords. Ultrapowers, reduced powers, functorial classification, saturated models, P-points, Continuum Hypothesis

Ultrapowers (see §1.1 for the definitions) have been a part of the standard toolbox in logic, combinatorics, functional analysis, and algebra for decades. It should be remarked that in this paper, \mathcal{U} is a nonprincipal ultrafilter on \mathbb{N} . A typical application of ultrapowers in set theory is concerned with ultrapowers of transitive models of (a large enough fragment of) ZFC, and it is desirable that these ultrapowers be well-founded. This requires \mathcal{U} to be σ -closed, hence excluding nonprincipal ultrafilters on \mathbb{N} and other small sets. Ultrapowers are frequently used as a ‘magnifying glass’ focused on a countable (or separable metric) structure A . Taking the ultrapower of A results in a saturated (see §1.2) elementary extension $A^{\mathcal{U}}$ of A . Various asymptotic phenomena of sequences in A propagate to $A^{\mathcal{U}}$, where they are witnessed by its *elements*. This often makes the arguments more succinct and (to those familiar with the method) transparent. This method is a textbook example of Shelah’s Mountain Air Thesis [42, 6.4(b)].

Ultrapowers were introduced independently to operator algebras, shortly before Łoś’s introduction of ultrapowers to logic (see the first page of [43]). The method of classifying separable operator algebras by studying their position inside an ultrapower dates

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back to the seminal [30] and [11]. McDuff's results were adapted to C^* -algebras in [12], with a small twist. Instead of ultrapowers, Effros and Rosenberg used the asymptotic sequence algebra $\ell_\infty(A)/c_0(A)$ (known to logicians as the reduced power associated with the Fréchet ideal). This algebra is at some level easier to grasp: Unlike ultrapowers, it is canonical and its construction does not require any form of the Axiom of Choice. Although reduced powers (and reduced products) have been studied by logicians for decades, the absence of Łoś's Theorem renders them not as effective as the ultrapowers (see however footnote 2).

The fact that ultrapowers are often interchangeable with asymptotic sequence algebras within the theory of C^* -algebras is quite puzzling. One of the objectives of this paper is to establish a relation between ultrapowers and reduced powers associated with the Fréchet ideal of countable (and separable metric) structures that explains this phenomenon. Our main result transcends the category of C^* -algebras and is applicable to the category of structures in any countable language (discrete or metric). The direct motivation for this work stems from a concrete problem encountered in Elliott's classification programme for nuclear C^* -algebras.

In 1989, Elliott conjectured that a certain functor F (presently known as *Elliott's invariant*) classifies a large and important class of C^* -algebras (see [36], [44], and also [13] for a direct approach without using ultrapowers or reduced powers).¹ Ultrapowers are used in the classification programme of C^* -algebras in situations where the simplicity of a massive quotient algebra is desirable (as in the Kirchberg–Phillips classification of Kirchberg algebras [34]) and in the stably finite case, when one takes direct advantage of the tracial ultrapower whose fibres are ultrapowers of II_1 factors (for an excellent example of this technique see [39]). With the increased sophistication, one feature of the asymptotic sequence algebras not shared by the ultrapowers came to be seen as indispensable; we will return to this in a moment.

A distinguishing requirement of the Elliott classification is its *functoriality*: The functor F is required to have two properties, existence and uniqueness, that we now describe in a greater generality needed later on.

Definition 0.1. If $F: \mathbb{K} \rightarrow \mathbb{L}$ is a functor, a morphism $\alpha: F(A) \rightarrow F(B)$ is *realized* by a morphism $\Phi: A \rightarrow B$ if $F(\Phi) = \alpha$. More generally, given an arrow $\iota: B \rightarrow C$ into another object C , it is said that α is *realized* by a morphism $\Phi: A \rightarrow C$ if $F(\iota) \circ \alpha$ is realized by Φ .

The functor F satisfies *existence* if for all separable C^* -algebras A and B , every morphism $\alpha: F(A) \rightarrow F(B)$ is realized by a morphism $\Phi: A \rightarrow B$.

The statement of uniqueness involves a notion of equivalence between morphisms. Two $*$ -homomorphisms, Ψ_1 and Ψ_2 , from A into C are (*approximately*) *unitarily equivalent* if there is a net of unitaries u_λ such that $\text{Ad } u_\lambda \circ \Psi_2$ converges to Ψ_1 in the point-norm topology.

¹For a curious reader, this is the class of nuclear, separable, unital, simple, C^* -algebras that satisfy the Universal Coefficient Theorem (UCT) and tensorially absorb the Jiang–Su algebra \mathcal{Z} ; but this is beside the point.

The functor F satisfies *uniqueness* if all morphisms that realize the same α are approximately unitarily equivalent. Since approximately unitarily equivalent morphisms share their Elliott invariant, functorial classification requires F to be an isomorphism of categories, with the appropriately identified arrows in the domain.

An intermediate step in proving that a morphism $\alpha: F(A) \rightarrow F(B)$ is realized by some $\Phi: A \rightarrow B$ often involves a massive extension (an ultrapower or an asymptotic sequence algebra) C of B , with the diagonal embedding $\iota: B \rightarrow C$. In this step, one proves that α is realized (in the sense of the second part of Definition 0.1) by $\Phi: A \rightarrow C$. If C is the asymptotic sequence algebra $B^\infty := \ell^\infty(B)/c_0(B)$, finding such Φ is, together with uniqueness, all that it takes to prove the existence. We take a moment to briefly describe the reason for this. Suppose B is a metric structure. Every injection $f: \mathbb{N} \rightarrow \mathbb{N}$ defines an endomorphism Φ_f of B^∞ , by its action on the representing sequences (see §1.1):

$$\Phi_f((a_i)_{i \in \mathbb{N}}) = (a_{f(i)})_{i \in \mathbb{N}}. \quad (0.1)$$

The following is [23, Theorem 4.3] and (essentially) [34, Proposition 1.37] (it is proven using a variant of the approximate intertwining technique).²

Theorem. *If A and B are C^* -algebras and A is separable, then a $*$ -homomorphism $\Psi: A \rightarrow B^\infty$ is unitarily equivalent to a $*$ -homomorphism whose range is included in (the diagonal copy of) B if and only if it is approximately unitarily equivalent to $\Phi_f \circ \Psi$ for every increasing $f: \mathbb{N} \rightarrow \mathbb{N}$. ■*

As an immediate corollary, if F is a functor whose domain is the category of C^* -algebras that satisfies uniqueness and A and B are separable, then (using the terminology from the second sentence of Definition 0.1) a morphism $\alpha: F(A) \rightarrow F(B)$ is realized by a $*$ -homomorphism $\Phi: A \rightarrow B$ if and only if it is realized by a $*$ -homomorphism $\Phi: A \rightarrow B^\infty$.

If \mathcal{U} is an ultrafilter and f is a function from the index set of \mathcal{U} into itself, then f is either constant on a set of \mathcal{U} or it sends a set in \mathcal{U} to a set that does not belong to \mathcal{U} (e.g., [15, Lemma 9.4.5]). Because of this, a map of the form Φ_f as in (0.1) is an endomorphism of an ultrapower if and only if it is the identity map, hence there is no analog of the reindexing technique for ultrapowers in place of the asymptotic sequence algebras. Ironically, it is often easier to find Φ that realizes a morphism $\alpha: F(A) \rightarrow F(B)$ in an ultrapower of B .

To recapitulate: For C^* -algebras A and B , a morphism $\alpha: F(A) \rightarrow F(B)$ can be realized by a $*$ -homomorphism from A into $B^\mathcal{U}$, but this is not quite as helpful as realizing α by a $*$ -homomorphism from A into B^∞ . This ‘gap’ begs a question, asked by Chris Schafhauser and Aaron Tikuisis and answered by the following (for an arbitrary metric structure B , by B^∞ we denote the reduced power $\prod_{\text{Fin}} B$; in the case of C^* -algebras, this reduces to $\ell^\infty(B)/c_0(B)$ as used above).

²This result comes across as a special instance of a general model-theoretic result that can be applied to metric structures with an appropriately defined and well-behaved notion of an inner automorphism, waiting to be isolated.

Theorem A. *Suppose F is a functor whose domain is a category \mathbb{K} of structures in some countable language. For separable A and B in \mathbb{K} , a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} , and a morphism $\alpha: F(A) \rightarrow F(B)$ the following are equivalent:*

- (1) *The morphism α is realized by a morphism $\Phi: A \rightarrow B^{\mathcal{U}}$ for some (or any) nonprincipal ultrafilter \mathcal{U} on \mathbb{N} .*
- (2) *The morphism α is realized by a morphism $\Phi: A \rightarrow B^{\infty}$.*

Theorem A and our other results are stated and proved for structures in any countable language (discrete or metric). This theorem is really a corollary of a more fundamental result, Theorem 5.2. As the latter result’s statement is unsuitable for inclusion in a short introduction, instead we state its consequence that uses the Continuum Hypothesis.

For a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} , by $\pi_{\mathcal{U}}: B^{\infty} \rightarrow B^{\mathcal{U}}$ we denote the quotient map. A right inverse to $\pi_{\mathcal{U}}$ is a homomorphism $\Theta: B^{\mathcal{U}} \rightarrow B^{\infty}$ such that $\pi_{\mathcal{U}} \circ \Theta = \text{id}_{B^{\mathcal{U}}}$.

Theorem B. *The Continuum Hypothesis implies that there exists a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} such that for every separable metric structure B in a separable language the quotient map $\pi_{\mathcal{U}}: B^{\infty} \rightarrow B^{\mathcal{U}}$ has a right inverse.*

If the Continuum Hypothesis holds, then Theorem A is a consequence of Theorem B. Fortunately, the contentious issue of the true cardinality of the continuum (and a possibly even more contentious issue whether there is such a thing as a true cardinality of the continuum) can be ignored in the present context because Theorem 5.2 suffices for all practical purposes. Suffice it to say that a metamathematical detour involving the Continuum Hypothesis was instrumental in finding a ZFC-result to which the (very interesting, in the author’s opinion) question of the cardinality of the continuum is completely irrelevant.

In the case when B belongs to an abelian category, the conclusion of Theorem B asserts that the exact sequence (with $c_{\mathcal{U}}(B) = \ker(\pi_{\mathcal{U}})$)

$$0 \rightarrow c_{\mathcal{U}}(B) \rightarrow B^{\infty} \xrightarrow{\pi_{\mathcal{U}}} B^{\mathcal{U}} \rightarrow 0$$

splits. Unlike $\pi_{\mathcal{U}}$, its right inverse is not canonical and does not necessarily exist. A more precise version of Theorem B involves one of the most common types of special ultrafilters on \mathbb{N} (see Definition 1.1).

Theorem C. *Suppose that the Continuum Hypothesis holds. For a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} the following are equivalent:*

- (1) *For every separable metric structure B in a countable language the quotient map $\pi_{\mathcal{U}}: B^{\infty} \rightarrow B^{\mathcal{U}}$ has a right inverse.*
- (2) *\mathcal{U} is a P -point.*

The salient feature of both $\pi_{\mathcal{U}}$ and its right inverse, that each one of them is equal to the identity on the diagonal copies of B in B^{∞} and $B^{\mathcal{U}}$, is essential for the applications. It is preserved in the following poor man’s version of Theorem B that applies to all nonprincipal ultrafilters on \mathbb{N} .

Theorem D. *Suppose that the Continuum Hypothesis holds, \mathcal{U} is an ultrafilter on \mathbb{N} , and B is a separable metric structure. Then there exists a surjective homomorphism $\Phi_{\mathcal{U}}: B^{\infty} \rightarrow B^{\mathcal{U}}$ that is equal to the identity on the diagonal copy of B in B^{∞} and has a right inverse.*

Theorem 5.2 is a variant of Theorem D proved without any additional set-theoretic axioms, yet sufficiently strong to imply Theorem A. Since our main results depend on both the choice of an ultrafilter and the model of ZFC, it is unlikely that they, or even Theorem A, could have been discovered without a nontrivial use of logic.

The following (motivated by the observations given in §3) was pivotal in discovering our main results.

Theorem E. *There is a covariant functor K from the category of metric structures into itself such that for every B there is an embedding $\iota_{B,K}: B \rightarrow KB$ with the following property. If the Continuum Hypothesis holds and if B is a separable metric structure in a separable language, then there is an isomorphism $\Lambda: (KB)^{\mathcal{U}} \rightarrow B^{\infty}$ such that $\iota_{B,\infty} = \Lambda \circ \iota_{B,K}$.*

A proof of Theorem E is given at the end of §3.

In the case of C^* -algebras, the list of equivalences in Theorem C can be expanded by adding analogs to an influential result due to Sato and Kirchberg–Rørdam (see [28, Theorem 3.3], [38, Lemma 2.1]). If B is a C^* -subalgebra of a C^* -algebra C , then the relative commutant of B inside C is

$$C \cap B' = \{c \in C \mid bc = cb \text{ for all } b \in B\}.$$

Following Kirchberg, we say that an ideal J in a C^* -algebra is a σ -ideal if for every countable subset J_0 of J there exists a positive contraction $e \in J$ such that $ea = a$ for all $a \in J_0$ [27, 28].

Theorem F. *For a P -point \mathcal{U} on \mathbb{N} and a separable C^* -algebra B the following statements hold.³*

- (1) $\pi_{\mathcal{U}}[B^{\infty} \cap B'] = B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[B]'$.
- (2) Every separable C^* -subalgebra A of B^{∞} satisfies

$$\pi_{\mathcal{U}}[B^{\infty} \cap A'] = B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[A]'$$

- (3) The kernel of $\pi_{\mathcal{U}}, c_{\mathcal{U}}(B)$, is a σ -ideal in B^{∞} .

If B is a UHF algebra and the Continuum Hypothesis holds, and \mathcal{U} is any nonprincipal ultrafilter on \mathbb{N} , then (1) is equivalent to each of the following:

³Operator algebraists should take a note that in this theorem, and elsewhere in this note, $B^{\mathcal{U}}$ denotes the ultrapower of a metric structure, in particular the norm ultrapower of B if B is a C^* -algebra, and not the tracial ultrapower (except in Example 8.1, where the metric structure is the hyperfinite II_1 factor). This choice of notation was made for consistency, and it hopefully does not lead to confusion or alienation.

- (4) \mathcal{U} is a *P*-point.
 (5) The exact sequence $0 \rightarrow c_{\mathcal{U}}(B) \rightarrow B^{\infty} \rightarrow B^{\mathcal{U}} \rightarrow 0$ splits.

Can any of Theorems B–F be proven without appealing to an additional set-theoretic assumption? This depends on the theory of the structures in question (see Example 5.1). In [21, Corollary D] it was proven that in a forcing extension of the set-theoretic universe introduced in [14], for every separable C^* -algebra A and every ultrafilter \mathcal{U} on \mathbb{N} , $(A \otimes C(K))^{\mathcal{U}}$ is not isomorphic to A^{∞} or even to a C^* -subalgebra of B^{∞} for any separable C^* -algebra B . In [21, Theorem C] the analogous result, in the same forcing extension, was proven for models that satisfy the order property as witnessed by a quantifier-free formula (this includes C^* -algebras and II_1 factors). Thus in this forcing extension the conclusion of each of Theorems B, D, and E fails for models of many unstable theories. In this model the real line cannot be covered by fewer than 2^{\aleph_0} meager sets, and therefore in it *P*-points, and even selective ultrafilters, exist (this is well-known; see, e.g., [15, Proposition 8.5.7]). This implies that the conclusion of Theorem C and the equivalence of (4) and (5) in Theorem F also fail in this model.

One more thing. The question which C^* -algebras have the property that taking the minimal tensor product with them preserves elementary equivalence was raised in [17, Question 3.10.5]. In §6 we give a positive answer for $C(K)$ (K denotes the Cantor space) and a negative answer for \mathcal{Z} (the Jiang–Su algebra) and all UHF algebras.

Our results are proved by model-theoretic and set-theoretic analysis of the structure $(B^{\infty}, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ (see §1). In §2 we compute the theory of a structure of this sort (a ‘Feferman–Vaught-type’ result). In §3 we define a functor K such that B^{∞} is elementarily equivalent to KB for every metric structure B . In §4 we prove that $(B^{\infty}, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ is countably saturated when \mathcal{U} is a *P*-point. The proofs of Theorems A–D can be found in §5. In §6 we prove that in the category of C^* -algebras, tensoring with $C(K)$ preserves elementarity and tensoring with the Jiang–Su algebra or a UHF algebra does not. Theorem F is proved in §7, and we conclude with a few general remarks in §8. All proofs are given in the case of metric structures but they yield proofs in the case of classical, discrete, structures—simply ignore the epsilons and the deltas.

1. Preliminaries

We follow the standard model-theoretic terminology, as presented in [29] or [9]. For model theory of metric structures and C^* -algebras see [2] and [17], respectively, as well as [15, §16]. In this logic, the interpretations of formulas are \mathbb{R} -valued and the analog of the Lindenbaum boolean algebra of a theory is a real Banach algebra. All of our results are stated in model theory of metric structures. They specialize to classical model theory by considering discrete structures as structures in $\{0, 1\}$ -metric. The special discrete variants of our results, although novel, nontrivial, and with a potential for applications, will not be stated explicitly. Readers not interested in continuous logic can omit all arguments that involve approximating elements of a model up to some $\varepsilon > 0$ and still obtain complete proofs.

We denote by $\varphi^A(\bar{a})$ the interpretation of a formula $\varphi(\bar{x})$ at a tuple \bar{a} of the same sort as \bar{x} in the structure A .

1.1. *Reduced powers*

Given a metric language \mathcal{L} , an infinite indexed family of \mathcal{L} -structures C_j , for $j \in \mathbb{J}$,⁴ and an ideal \mathcal{J} on \mathbb{J} , the *reduced product* is the quotient of $\prod_j C_j$ corresponding to the pseudometric

$$d(a, b) = \inf_{X \in \mathcal{J}} \sup_{j \in \mathbb{J} \setminus X} d(a_j, b_j).$$

If \mathcal{L} is a discrete language, then the domain of the reduced product associated with \mathcal{J} is the set of equivalence classes on $\prod_{j \in \mathbb{J}} C_j$ where a and b are identified if and only if the set $\{j \in \mathbb{J} \mid a_j \neq b_j\}$ belongs to \mathcal{J} .

In the case when \mathcal{L} is a multisorted language, this is used to define every sort of the reduced product. All function symbols are interpreted in the natural way, pointwise, and the relation symbols R are interpreted by (\bar{a} denotes a tuple of the appropriate sort)

$$R^{\prod_{\mathcal{J}} C_j}(\bar{a}) := \inf_{X \in \mathcal{J}} \sup_{j \in \mathbb{J} \setminus X} R^{C_j}(\bar{a}_j)$$

in the metric case. In the classical, discrete case, we define the interpretation of R by setting $R^{\prod_{\mathcal{J}} C_j}(\bar{a})$ to hold if and only if $\{j \in \mathbb{J} \mid R(\bar{a}_j) \text{ fails}\} \in \mathcal{J}$.

The reduced product associated to \mathcal{J} is denoted $\prod_{\mathcal{J}} C_j$. In the case when all C_j are equal to B we write $B^{\mathcal{J}}$ for $\prod_{\mathcal{J}} B$. The two extremal cases are most important. If \mathcal{J} is the Fréchet ideal, denoted Fin , then $B^{\mathcal{J}}$ is denoted B^∞ , and if \mathcal{J} is a maximal ideal then $B^{\mathcal{J}}$ is denoted $B^{\mathcal{U}}$ (where \mathcal{U} is the complement of \mathcal{J}) and called *ultrapower*. If \mathcal{U} is disjoint from \mathcal{J} , then we have a natural quotient map $\pi_{\mathcal{U}}: B^{\mathcal{J}} \rightarrow B^{\mathcal{U}}$.

An element b of $B^{\mathcal{J}}$ is determined by a representing sequence $(b_i)_{i \in \mathbb{J}}$. We routinely commit the crime of identifying the elements of $B^{\mathcal{J}}$ with the corresponding representing sequences in $\prod_{i \in \mathbb{J}} B_i$ (the latter is considered as a sorted product, where sort S is interpreted as $\prod_{i \in \mathbb{J}} S(B_i)$, where $S(B_i)$ is the interpretation of sort S in B_i). This practice is similar to the analogous well-established practice in the case of L_p spaces: It is innocuous, as long as one knows what they are doing. For an ideal \mathcal{J} we define the diagonal embedding

$$\iota_{B, \mathcal{J}}: B \rightarrow B^{\mathcal{J}} \tag{1.1}$$

by sending b to the constant representing the sequence (b, b, \dots) and identify B with its diagonal image in $B^{\mathcal{J}}$. We write $\iota_{B, \infty}$ for $\iota_{B, \text{Fin}}$. We denote by \bar{a} a tuple (a_0, \dots, a_{n-1}) of an unspecified length (but ‘of the appropriate sort’, which depends on the context). The arity of \bar{a} will be routinely suppressed, and we will write $\bar{a} \in B$ for $\bar{a} \in B^n$ where n is the arity of \bar{a} . For the sake of brevity, variables are sometimes omitted and a formula

⁴The only index set used in our main results is $\mathbb{J} = \mathbb{N}$, but some of the intermediate (yet quotable) results hold in larger generality.

$\varphi(\bar{x})$ is written as φ . When dealing with tuples of representing sequences, in order to avoid confusion the entries of \bar{a} will be denoted $a(0), \dots, a(n - 1)$. If \bar{a} is an n -tuple of elements of $B^{\mathcal{J}}$, then $a_j(i)$, for $j \in \mathbb{J}$, is a representing sequence of the i -th entry of \bar{a} .

1.2. Types and saturation

We recall the definitions of a condition and a type. Fix a metric language \mathcal{L} . A *condition* is an expression $\varphi(\bar{x}) = r$, where $\varphi(\bar{x})$ is an \mathcal{L} -formula and r is a real number. It is *satisfied* by a tuple \bar{a} of an appropriate sort in an \mathcal{L} -structure A if $\varphi^A(\bar{a}) = r$. An *n -type* is a set of conditions in variables x_i , for $i < n$. A *type* is an n -type for some n . (Types in variables \bar{x} can be construed as functionals on the Lindenbaum algebra of all \mathcal{L} -formulas all of whose free variables are among \bar{x} , see [17, §4.7] or [15, §16.1]; although very intuitive, this formulation will not be used explicitly.) A type $\mathbf{t}(\bar{x})$ is *realized* by a tuple \bar{a} in an \mathcal{L} -structure A if each of its conditions is satisfied by \bar{a} . It is *satisfiable* in an \mathcal{L} -structure A if for every finite list of conditions $\varphi_i(\bar{x}) = r_i$, for $i < n$, in $\mathbf{t}(\bar{x})$ and every $\varepsilon > 0$ some \bar{a} in A satisfies $\max_{i < n} |\varphi_i^A(\bar{a}) - r_i| < \varepsilon$.

A type is *countable* if it is countable as a set of conditions. An \mathcal{L} -structure A is *countably saturated* if every countable, satisfiable type in the language obtained by expanding \mathcal{L} by adding constants for the elements of A is realized in the expansion of A to this language obtained by interpreting these constants in the natural way.⁵ Both ultrapowers and reduced products associated with the Fréchet ideal are countably saturated (see [9] for the discrete logic, in the metric case see [2] for ultrapowers and [20] for the reduced powers associated with the Fréchet ideal, or see [15, §16] for both).

1.3. P-points

The space of ultrafilters on \mathbb{N} is naturally identified with the Čech–Stone compactification $\beta\mathbb{N}$ of \mathbb{N} . Then the space of nonprincipal ultrafilters is identified with the remainder (corona) $\beta\mathbb{N} \setminus \mathbb{N}$.

Definition 1.1. An ultrafilter \mathcal{U} on \mathbb{N} is a *P-point* if for every sequence $X_n \in \mathcal{U}$, for $n \in \mathbb{N}$, there exists $X \in \mathcal{U}$ such that $X \setminus X_n$ is finite for all n .

Equivalently, \mathcal{U} is a P-point if the intersection of every countable family of open neighbourhoods of \mathcal{U} includes an open neighbourhood of \mathcal{U} . Not every ultrafilter on \mathbb{N} is a P-point: By compactness, any countable subset X of $\beta\mathbb{N} \setminus \mathbb{N}$ has a nonempty set of accumulation points, and none of its members is a P-point. By a classical result of W. Rudin, the Continuum Hypothesis implies that P-points exist [37]. It is relatively consistent with ZFC that there are no P-points in $\beta\mathbb{N} \setminus \mathbb{N}$ at all (see [41, §6.4] and [10]).

⁵In [40], such models are called countably compact. However, for countable (or separable) languages our definition of countable saturation coincides with Shelah’s.

2. Theories of reduced powers

In this section we compute the theory of the structure $(B^\infty, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ from the theory of B . Towards this goal, we prove a strengthening of Ghasemi’s Feferman–Vaught theorem [24]. It applies to structures of the form

$$(B^{\mathcal{J}}, B^{\mathcal{I}}, \pi_{\mathcal{I}})$$

where B is an \mathcal{L} -structure, \mathcal{J} and \mathcal{I} are ideals on the index set \mathbb{J} such that $\mathcal{J} \subseteq \mathcal{I}$, and $\pi_{\mathcal{I}}: B^{\mathcal{J}} \rightarrow B^{\mathcal{I}}$ is the quotient map. Let \mathcal{L}^2 denote the language of this structure. It has two ‘meta-sorts’ corresponding to $B^{\mathcal{J}}$ and $B^{\mathcal{I}}$, and each \mathcal{L} -sort corresponds to one sort in each of the two meta-sorts. It also has the function symbol $\pi_{\mathcal{I}}$ interpreted as the quotient map.

In order to state our main result, let $\mathcal{L}_{\text{BA}, \mathbb{U}}$ be the language of Boolean algebras with the Boolean operations $\wedge, \vee, \complement, 0$, and 1 , also equipped with constants Z_t^ζ for every \mathcal{L}^2 -formula ζ and every $t \in \mathbb{Q}$, and with an additional predicate \mathbb{U} . The following is modelled on the epymorphic notion from [24] and [15, Definition 16.3.2].

Definition 2.1. For $k \geq 2$ and language \mathcal{L} , an \mathcal{L}^2 -formula $\varphi(\bar{x})$ is *k-determined*⁶ if objects with the following properties exist:

- (1) A finite set $\mathbb{F}[\varphi, k]$ of \mathcal{L} -formulas whose free variables are included in the free variables of $\varphi(\bar{x})$,
- (2) $\mathcal{L}_{\text{BA}, \mathbb{U}}$ -sentences $\theta_l^{\varphi, k}$, for $0 \leq l \leq k$, such that the following conditions hold:
 - (a) All constants appearing in $\theta_l^{\varphi, k}$ are among Z_t^ζ , for $\zeta \in \mathbb{F}[\varphi, k]$, $t \in \mathbb{Q} \cap [0, 1]$.
 - (b) Each $\theta_l^{\varphi, k}$ is *increasing*, i.e., if $\bar{X} = (X_i)$ are its free variables and $A_i \leq A'_i$ for all i are elements of a Boolean algebra \mathbb{B} , then $(\theta_l^{\varphi, k})^{\mathbb{B}}(\bar{A})$ implies $(\theta_l^{\varphi, k})^{\mathbb{B}}(\bar{A}')$.

These objects are required to satisfy the following. Given \mathcal{L} -structures $(M_j)_{j \in \mathbb{J}}$ and ideals $\mathcal{J} \subseteq \mathcal{I}$ on \mathbb{J} , for $\zeta(\bar{x}) \in \mathbb{F}[\varphi, k]$, we write $\theta_l^{\varphi, k}[\bar{a}]$ for the value of $\theta_l^{\varphi, k}$ in the quotient Boolean algebra $\mathcal{P}(\mathbb{J})/\mathcal{I}$ with $([X]_{\mathcal{I}})$ denotes the equivalence class of X modulo \mathcal{I} ⁷

$$Z_{l/k}^\zeta := [\{j \mid (\zeta(\bar{a}_j))^{M_j} > l/k\}]_{\mathcal{I}}$$

and with $\mathbb{U}(Z_{l/k}^\zeta)$ true if and only if $Z_{l/k}^\zeta \notin \mathcal{I}$. Then the following holds (writing M for the structure $(\prod_j M_j/\mathcal{I}, \prod_j M_j/\mathcal{I}, \pi_{\mathcal{I}})$):

- (3) $\varphi^M(\bar{a}) > (l + 1)/k$ implies $\theta_l^{\varphi, k}[\bar{a}]$ and
- (4) $\theta_l^{\varphi, k}[\bar{a}]$ implies $\varphi^M(\bar{a}) > (l - 1)/k$.

⁶In [24] the definition of ‘k-determined’ is different, but in both cases this notion is used only within the proof of a theorem.

⁷Note that the value of $Z_{l/k}^\zeta$ depends on \bar{a} .

Definition 2.1 asserts that the value of $\varphi^M(\bar{a})$ is determined up to $2/k$ by a finite set of formulas $\theta_l^{\varphi,k}$ for $0 \leq l \leq k$, which are in turn determined by the evaluations of the formulas in the finite set $\mathbb{F}[\varphi, k]$ in every M_j .

Theorem 2.2. *For every metric language \mathcal{L} and every $k \geq 2$, every \mathcal{L}^2 -formula is k -determined.*

Proof. The proof is analogous to the proofs of [24, Theorem 3.1] and [15, Theorem 16.3.3], where it was proven that the formulas in a reduced product of metric structures are k -determined (with an appropriate definition). We will follow the template of the latter proof and indicate only the necessary changes. The proof proceeds by finding the required objects and demonstrating that they are as required in the case of an arbitrary $(M_j)_{j \in \mathbb{J}}$, ideals \mathcal{I} and \mathcal{J} on \mathbb{J} such that $\mathcal{I} \subseteq \mathcal{J}$, with

$$\tilde{M} := (\prod_j M_j / \mathcal{I}, \prod_j M_j / \mathcal{J}, \pi_{\mathcal{J}}),$$

and $\bar{a} \in \tilde{M}$ of the same sort as \bar{x} in the formula $\varphi(\bar{x})$ being considered.

By induction on complexity of the formula φ , it suffices to prove that the set of all k -determined formulas satisfies the following closure properties:

- (1) All atomic formulas are k -determined.
- (2) If φ is k -determined, so is $\frac{1}{2}\varphi$.
- (3) If φ and ψ are $2k$ -determined, then $\varphi \dot{-} \psi$ is k -determined.
- (4) If φ is k -determined, so are $\sup_x \varphi$ and $\inf_x \varphi$ for every variable x .

Only the treatment of case (1) is different from that in [15, Theorem 16.3.3], and it splits into two cases.

In the first case, the terms of the atomic formula φ are evaluated in the meta-sort corresponding to $\prod_j M_j / \mathcal{I}$. Then let $\mathbb{F}[\varphi, k] := \{\varphi\}$ and let $\theta_l^{\varphi,k}$ be the formula $Z_{l/k}^\varphi \neq 0$. This formula is clearly increasing. Since

$$\varphi(\bar{a})^M = \limsup_{j \rightarrow \mathcal{I}} \varphi^{M_j}(\bar{a}_j),$$

we see that $\varphi(\bar{a})^M > (l + 1)/k$ implies $\theta_l^{\varphi,k}[\bar{a}]$. Similarly, $\theta_l^{\varphi,k}[\bar{a}]$ implies $\varphi(\bar{a})^M > (l - 1)/k$, as required.

In the second case, the terms of the atomic formula φ are evaluated in the meta-sort corresponding to $\prod_j M_j / \mathcal{J}$. Then let $\mathbb{F}[\varphi, k] := \{\varphi\}$ and let $\theta_l^{\varphi,k}$ be the formula $\mathbb{U}(Z_{l/k}^\varphi)$. This formula is clearly increasing. Also,

$$\varphi(\bar{a})^M = \limsup_{j \rightarrow \mathcal{J}} \varphi^{M_j}(\bar{a}_j),$$

and since $\mathbb{U}(Z)$ if and only if $[Z]_{\mathcal{J}} \neq 0$, the proof proceeds as in the first case.

The treatment of (2)–(4) is identical to that in [15, Theorem 16.3.3], and therefore omitted. This completes the inductive proof. ■

In the proof of countable saturation (Theorem 4.2) we will need the following theorem that applies to an arbitrary language \mathcal{L} and is stated in the terminology introduced at the beginning of Section 2.

Corollary 2.3. *For every finite set \mathbb{G} of \mathcal{L}^2 -formulas⁸ and every $\varepsilon > 0$ there are a finite set $\mathbb{F}[\mathbb{G}, \varepsilon]$ of \mathcal{L} -formulas and $\delta > 0$ with the following property. If $(M_j)_{j \in \mathbb{J}}$ are \mathcal{L} -structures and \mathcal{I} and \mathcal{J} are ideals on \mathbb{J} such that $\mathcal{I} \subseteq \mathcal{J}$, then with $M = (\prod_j M_j / \mathcal{I}, \prod_j M_j / \mathcal{J}, \pi_{\mathcal{J}})$ for all $\bar{a}_j \in M_j$ and $\bar{b}_j \in M_j$ of the appropriate sort,*

$$\max_{\zeta \in \mathbb{F}[\varphi, \varepsilon]} \limsup_{j \rightarrow \mathcal{I}} |\zeta^{M_j}(\bar{a}_j) - \zeta^{M_j}(\bar{b}_j)| < \delta$$

implies $\max_{\varphi \in \mathbb{G}} |\varphi^M(\bar{a}) - \varphi^M(\bar{b})| < \varepsilon$.

The displayed formula does not refer to \mathcal{J} , but since $\mathcal{J} \supseteq \mathcal{I}$ no information is lost and this is not a problem.

Proof of Corollary 2.3. Choose $k > 2/\varepsilon$ and $\delta = 1/k$. Then the conclusion follows from the fact that each $\varphi \in \mathbb{G}$ is k -determined and the finiteness of \mathbb{G} . ■

3. The functor K and elementary embeddings

For the definitions of conditions, types, and saturation see §1.2. The structure B^∞ is countably saturated ([20], see also [15, §16.5]). If B is separable and the Continuum Hypothesis holds, then B^∞ has both cardinality and density character equal to \aleph_1 , and it is therefore saturated. Since any two elementarily equivalent saturated structures are isomorphic (this is a classical result of Keisler; see, e.g., [2] or [15, §16] for the continuous variant), B^∞ is isomorphic to the ultrapower of any one of its separable elementary submodels C , via an isomorphism that extends the identity map on C . This observation begs the following question ($\iota_{B, \infty}: B \rightarrow B^\infty$ is the diagonal embedding):

Question 3.1. *Given a countable language \mathcal{L} , is there a functor K from the category of separable \mathcal{L} -structures into itself such that for every B there exist an embedding $\iota_{B, K}: B \rightarrow KB$ and an elementary embedding $\Psi: KB \rightarrow B^\infty$ such that $\iota_{B, \infty} = \Psi \circ \iota_{B, K}$?*

A positive answer is given in §3.1, giving a satisfactory explanation to the fact that B^∞ behaves as an ultrapower. It involves a canonical construction of a separable substructure KB of B^∞ for every separable B such that the ultrapower of KB is isomorphic to B^∞ via an isomorphism that commutes with the diagonal embeddings of B if the Continuum Hypothesis holds (Theorem E).⁹

⁸By adding dummy variables, we may assume that all formulas in \mathbb{G} have the same tuple of free variables \bar{x} .

⁹Caveat: the embedding Ψ is, unlike $\iota_{B, \infty}$ and $\iota_{B, K}$, not canonical.

3.1. The functor K

Given a metric structure A and a compact Hausdorff space X , let

$$C(X, A) = \{f: X \rightarrow A \mid f \text{ is continuous}\}.$$

This construction is functorial, because a morphism $f: A \rightarrow B$ defines a morphism from $C(X, A)$ to $C(X, B)$ that sends g to $f \circ g$. Let $\iota_{B,K}: B \rightarrow KB$ be the diagonal embedding that sends $b \in B$ to the corresponding constant function. We will identify B with its diagonal image in KB . In the case when X is the Cantor space, K , we denote this functor by K and in particular write KA for $C(K, A)$.

If X is a topological space and A is a metric structure, then a function $f: X \rightarrow A$ is *locally constant* if there exists a partition of X into clopen sets, $X = \bigsqcup_{i < m} U_i$, such that the restriction of f to U_i is constant for every i .

Example 3.2. (1) For every structure A define an inductive system of structures A_n , for $n \in \mathbb{N}$, by $A_0 = A$ and $A_{n+1} = A_n \oplus A_n$ for all n , with the connecting maps $a \mapsto (a, a)$. The inductive limit $\lim_n A_n$ is isomorphic to the substructure of KA consisting of locally constant functions.

- (2) If A is discrete then KA consists of locally constant functions as described in (1).
- (3) If A is a metric structure, then KA is the completion of the structure of all locally constant functions from K into A with respect to the uniform metric, $d_\infty(f, g) := \max_{x \in K} d(f(x), g(x))$.
- (4) If A is a C^* -algebra then $KA \cong A \otimes C(K)$, where the isomorphism respects the diagonal copies of A and $C(K)$. This is a well-known general fact about C^* -algebras. One way to see it is to note that the algebraic tensor product, $A \odot C(K)$, is isomorphic to the structure whose elements are locally constant functions as described in (1). Since $C(K)$ is abelian, the algebraic tensor product has the unique C^* -norm, with the completion isomorphic to $A \otimes C(K)$.

Lemma 3.3 below provides a template for the conclusion of Theorem D. The condition that both maps be 1-Lipschitz is important because in logic of metric structures every function symbol is equipped with a modulus of uniform continuity (1-Lipschitz maps are also known as contractions in the theory of operator algebras).

Lemma 3.3. *For any structure B there is a surjective homomorphism $\pi_0: KB \rightarrow B$ such that $\iota_{B,K}: B \rightarrow KB$ is the right inverse of π_0 , π_0 is equal to the identity on the diagonal copy of B in KB , and both π_0 and $\iota_{B,K}$ are 1-Lipschitz.*

Proof. Fix a point in K , denoted 0, and let π_0 be the evaluation map at 0. Then π_0 and $\iota_{B,K}$ clearly have the required properties. ■

The fact that the right inverse π_0 depends on the choice of a point in K will come back to haunt us. This lack of canonicity of π_0 is one of the reasons why the proof of Theorem B uses the Continuum Hypothesis.

Proposition 3.4. *If B is a metric structure and \mathcal{I} is an ideal on \mathbb{N} such that the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless then there is an embedding $\Psi: KB \rightarrow B^{\mathcal{I}}$ such that $\iota_{B,\mathcal{I}} = \Psi \circ \iota_{B,K}$.*

Proof. A subset of \mathbb{N} is called \mathcal{I} -positive if it does not belong to \mathcal{I} . Since $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is an atomless Boolean algebra, we can recursively find \mathcal{I} -positive sets X_s , for $s \in \{0, 1\}^{<\mathbb{N}}$, with the following properties for all s :

- (1) $X_{\emptyset} = \mathbb{N}$.
- (2) Each X_s is equal to the disjoint union $X_{s \smallfrown 0} \sqcup X_{s \smallfrown 1}$.

We continue the proof by elaborating Example 3.2 (1). For $n \geq 1$ let

- (3) $K_n B := \{a \in B^\infty \mid (\forall s \in \{0, 1\}^n)(\exists a(s) \in A)(\forall j \in X_s) a_j = a(s)\}$.

Then $K_n B \cong B^{2^n}$ and $K_n B \subseteq K_{n+1} B$ for all n .

A well-known back-and-forth argument shows that all countable atomless Boolean algebras are isomorphic (e.g., [8, Proposition 1.4.5]). Since the Boolean algebra $\text{Clop}(K)$ of clopen subsets of K is countable and atomless, it is isomorphic to the subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ generated by X_s , for $s \in \{0, 1\}^{<\mathbb{N}}$. By fixing an isomorphism, we can identify $K_n B$ as defined in (3) with a substructure of B^∞ . These identifications are isometric and compatible with one another. By Example 3.2 (3), $\bigcup_n K_n B$ is dense in KB , and we have an isometric isomorphism between KB and the closure of $\bigcup_n K_n B$ inside $B^{\mathcal{I}}$. This isometric isomorphism is the required embedding Ψ . ■

Proposition 3.5. *Suppose that \mathcal{L} , B , \mathcal{I} , and Ψ are as in Proposition 3.4.*

- (1) *The embedding $\Psi: KB \rightarrow B^{\mathcal{I}}$ is elementary and $\iota_{B,\mathcal{I}} = \Psi \circ \iota_{B,K}$.*
- (2) *If \mathcal{U} is an ultrafilter disjoint from \mathcal{I} , $\pi_{\mathcal{U}}: B^{\mathcal{I}} \rightarrow B^{\mathcal{U}}$ is the quotient map, and $\pi_0: KA \rightarrow A$ is the evaluation map as in Lemma 3.3, then the embedding Ψ can be chosen so that the embedding*

$$(\Psi, \iota_{B,\mathcal{I}}): (KB, B, \pi_0) \rightarrow (B^{\mathcal{I}}, B^{\mathcal{U}}, \pi_{\mathcal{U}})$$

is elementary.

Proof. (1) We will identify KB with its image under Ψ . By the Tarski–Vaught test [17, Theorem 2.6.1] it suffices to prove that if $\varphi(\vec{x}, y)$ is a formula and \vec{a} in KB is a tuple of the appropriate sorts, then¹⁰

$$\inf_{y \in S(KB)} \varphi^{B^\infty}(\vec{a}, y) = \inf_{y \in S(B^\infty)} \varphi^{B^\infty}(\vec{a}, y).$$

This is equivalent to asserting that for every $\varepsilon > 0$ and every $d \in B^\infty$ there exists $c \in KB$ such that $\varphi^{B^\infty}(\vec{a}, c) < \varphi^{KB}(\vec{a}, d) + \varepsilon$.

¹⁰All variables are sorted, thus in \inf_y the variable y ranges over the appropriate sort S . In the case of C^* -algebras, each sort is an n -ball of the C^* -algebra for some fixed n , hence $y \in S(KB)$ is equivalent to ‘ $y \in KB$ and $\|y\| \leq n$.’

Since the locally constant functions are dense in KB (Example 3.2 (3)), by possibly changing ε we can assume that every entry of \bar{a} is locally constant (i.e., a step function from K into B). Let X_s , for $s \in \{0, 1\}^{<\mathbb{N}}$, be the subsets of \mathbb{N} constructed in the proof of Proposition 3.4 used to define Ψ . Fix an m large enough so that every entry of \bar{a} is constant on X_s for every $s \in \{0, 1\}^m$.

By Corollary 2.3 there are $n \geq 1, \delta > 0$, and \mathcal{L} -formulas $\zeta_i(\bar{x}, y)$, for $i < n$, with the same free variables as φ , such that for all \bar{b}, \bar{b}', c , and c' in B^∞ of the appropriate sorts, $\limsup_j \max_{i < n} |\zeta_i^{\bar{B}}(\bar{b}_j, c_j) - \zeta_i^{\bar{B}}(\bar{b}'_j, c'_j)| \leq \delta$ implies $|\varphi^{B^\infty}(\bar{b}, c) - \varphi^{B^\infty}(\bar{b}', c')| < \varepsilon$.

For $j \in \mathbb{N}$ define an element of \mathbb{R}^n by

$$\bar{r}(j) = ((\zeta_0^B(\bar{a}_j, \bar{d}_j), \dots, \zeta_{n-1}^B(\bar{a}_j, \bar{d}_j))).$$

Since every formula has bounded codomain, the set of all $\bar{r}(j)$ is included in a sufficiently large open neighbourhood of 0 in \mathbb{R}^n with a compact closure. We can therefore find a partition $\bigsqcup_{i < m'} Y_i$ refining $\bigsqcup_{s \in \{0, 1\}^m} X_s$ such that for every $i < m'$ and all j and j' in Y_i we have $\max_{i < m} |\bar{r}(j)_i - \bar{r}(j')_i| < \delta$. For every $j < m'$ there exists a unique $s(j) \in \{0, 1\}^m$ such that $Y_j \subseteq X_{s(j)}$. Also, $X_s = \bigcup_{s(j)=s} Y_j$ for all $s \in \{0, 1\}^m$.

Let l be such that $2^l > m'$. The sets X_t , for $t \in \{0, 1\}^{m+l}$, form a partition of \mathbb{N} into infinite sets that refines the partition X_s , for $s \in \{0, 1\}^m$. For every $s \in \{0, 1\}^m$, the set $\{j < m' \mid Y_j \subseteq X_s\}$ has fewer elements than the set $\{t \in \{0, 1\}^{m+l} \mid X_t \subseteq X_s\}$. Therefore there is a surjection $\{0, 1\}^{m+l} \rightarrow m', t \mapsto j(t)$, such that $s(j(t))$ is an initial segment of t for every t .

We can therefore choose a permutation χ of \mathbb{N} with the following properties:

- (a) $\chi[X_s] = X_s$ for all $s \in \{0, 1\}^m$.
- (b) $\chi[Y_j] = \bigsqcup \{X_t \mid t \in \{0, 1\}^{m+l}, j(t) = j\}$.

Every permutation of \mathbb{N} naturally acts on $B^\mathbb{N}$, by sending (b_j) to $(b_{\chi(j)})$. Since the Fréchet ideal is invariant under permutations, χ defines an automorphism, denoted Φ_χ , of B^∞ . Clearly such ‘permutation automorphism’ fixes the diagonal copy of B pointwise. Since $\Phi_\chi[X_s] = X_s$ for all $s \in \{0, 1\}^m$, we have $\Phi_\chi(\bar{a}) = \bar{a}$.

For each $j < m'$ choose $k(j) \in Y_j$. Define $c \in B^\infty$ by its representing sequence,

- (c) $c_i := d_{k(j(t))}$ if $t \in \{0, 1\}^{m+l}$ is such that $i \in X_t$.

Then $d(d_{\chi(i)}, c_i) < \delta$ for all $i \in \mathbb{N}$, hence $d(\Phi_\chi(d), c) \leq \delta$. Also, c is constant on each X_t , for $t \in \{0, 1\}^{m+l}$, and it therefore belongs to $\Psi[KB]$. By the choice of the sets Y_j we have

$$\max_{k < m} |\zeta_k^B(\bar{a}, c) - \zeta_k^B(\bar{a}, d)| \leq \delta,$$

and the choice of δ implies $\varphi^{B^\infty}(\bar{a}, c) < \varphi^{B^\infty}(\bar{a}, d) + \varepsilon$ as required.

(2) Unlike the proof of (1), in this proof we revisit the construction of Ψ from Proposition 3.4 and choose the sets X_s for $s \in \{0, 1\}^{<\mathbb{N}}$ with additional care. Since \mathcal{U} is an ultrafilter, for every partition of a \mathcal{U} -positive set into two pieces exactly one of the pieces is \mathcal{U} -positive. Since $\mathbb{N} = X_\emptyset$ is \mathcal{U} -positive, in the construction of Ψ we can choose the sets X_s so that $X_s \in \mathcal{U}$ implies $X_{s \smallfrown 0} \in \mathcal{U}$ for all $s \in \{0, 1\}^{<\mathbb{N}}$, hence $X_s \in \mathcal{U}$ if and only

if $s(i) = 0$ for all $i \in \text{dom}(s)$. Therefore $\Psi(\pi_0(f)) = \pi_{\mathcal{U}}(\Psi(f))$ for all $f \in KB$, and $(\Psi, \iota_{B, \mathcal{J}})$ is an embedding of (KB, B) into $(B^{\mathcal{J}}, B^{\mathcal{U}})$

The proof that this embedding is elementary is virtually identical to the proof of (1). The only extra care needs to be taken in the choice of the permutation χ . Since \mathcal{U} is an ultrafilter and the sets Y_i , for $i < m'$, form a partition of \mathbb{N} , there is a unique $i < m'$ such that $Y_i \in \mathcal{U}$. The permutation χ needs to satisfy $\chi[Y_i] \in \mathcal{U}$. In other words, with l and $j(t) < m'$ as in the proof of (1), χ is chosen so that $\chi[Y_i] \supseteq X_t$ for the unique $t \in \{0, 1\}^{m+l}$ such that $X_t \in \mathcal{U}$ and $j(t) = i$. Therefore Φ_χ defines an automorphism of $(B^\infty, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ and the proof proceeds as in (1). The other details are omitted. ■

The following application will be reformulated in §6.

Corollary 3.6. *Suppose that B and C are structures in the same language.*

- (1) *If B and C are elementarily equivalent, then so are KB and KC .*
- (2) *If $\Psi: B \rightarrow C$ is elementary, then $K\Psi: KB \rightarrow KC$ is elementary.*

Proof. By Proposition 3.5, KB is isomorphic to an elementary submodel of B^∞ and KC is isomorphic to an elementary submodel of C^∞ . By [24, Proposition 3.6] (or the results of §2), the operation of taking reduced product over Fin preserves elementary equivalence, and therefore B^∞ and C^∞ are elementarily equivalent and so are KB and KC by transitivity. This proves (1). To prove (2), note that if $\Psi: B \rightarrow C$ is an elementary embedding then $\Psi^\infty: B^\infty \rightarrow C^\infty$ is an elementary embedding by Theorem 2.2. Therefore $K\Psi$ is an elementary embedding of KB into KC . ■

Proof of Theorem E. Suppose the Continuum Hypothesis holds and fix a separable metric structure B in a separable language. We need to find an isomorphism between $(KB)^\mathcal{U}$ and B^∞ that commutes with the diagonal embeddings of B . By Proposition 3.5, there is an elementary embedding $\Psi: KB \rightarrow B^\infty$ such that $\iota_{B, \infty} = \Psi \circ \iota_{B, K}$. Since $(KB)^\mathcal{U}$ and B^∞ are both countably saturated ([15, Proposition 16.4.2] and [15, Theorem 16.5.1], respectively) and of cardinality $\aleph_1 = 2^{\aleph_0}$, they are both saturated. Since B is separable, Ψ can be extended to an isomorphism $\Lambda: (KB)^\mathcal{U} \rightarrow B^\infty$ [15, Theorem 16.7.5]. ■

4. Countable saturation of composite quotients

Theorem 4.2 is the main technical result of this section (see §1.2 for the explanation of the terminology). Its proof uses Proposition 4.1 which must have been known to Tarski. Since I could not find a reference to this result, a sketch of its proof is included.

Proposition 4.1. *The theory T_U of atomless Boolean algebras with an additional unary predicate U for an ultrafilter admits elimination of quantifiers.*

Proof. The easiest way to prove this may be to show that if A and B are countable models of T_U and $\Phi_0: A_0 \rightarrow B_0$ is an isomorphism between finitely generated submodels of A

and B , then Φ_0 extends to an isomorphism between A and B . Fix such A and B and an isomorphism $\Phi_0: A_0 \rightarrow B_0$.

To find Φ , note that a finitely generated submodel of a model of T_U is a finite Boolean algebra with a distinguished ultrafilter. This ultrafilter is uniquely determined by the atom that belongs to it. Write A and B as increasing unions of finite Boolean algebras, $A = \bigcup_j A_j$ and $B = \bigcup_j B_j$ such that $|A_j| = |B_j|$ for all j . By recursion find isomorphisms $\Phi_j: A_j \rightarrow B_j$ such that Φ_{j+1} extends Φ_j for all j . This sequence uniquely determines an isomorphism $\Phi: A \rightarrow B$ that extends Φ_0 .

This shows that in a countable model of T_U the type of a finite subset is uniquely determined by its quantifier-free type. Since the language is countable this holds in an arbitrary model of T_U and completes the proof. ■

Theorem 4.2. *For every structure B , if \mathcal{U} is a P -point ultrafilter on \mathbb{N} then the structure $(B^\infty, B^\mathcal{U}, \pi_\mathcal{U})$ is countably saturated.*

Proof. Fix B , \mathcal{U} , and a countable satisfiable type $\mathbf{t}(\bar{x})$ over

$$\tilde{B} := (B^\infty, B^\mathcal{U}, \pi_\mathcal{U}).$$

Enumerate $\mathbf{t}(\bar{x})$ as a sequence of conditions $\varphi_i(\bar{x}) = s_i$, for $i \in \mathbb{N}$. By composing φ_i with a piecewise linear function we may assume that its range is included in $[0, 1]$ for all i . By Corollary 2.3, for each $k \geq 1$ there exist a finite set of formulas $\mathbb{F}(k)$ and $d(k) \geq 1$ such that for all \bar{a} and \bar{b} in \tilde{B} of the appropriate sort,

$$\max_{\zeta \in \mathbb{F}(k)} \limsup_j |\zeta^B(\bar{a}_j) - \zeta^B(\bar{b}_j)| < 1/d(k)$$

implies $\max_{j \leq k} |\varphi_j^{\tilde{B}}(\bar{a}) - \varphi_j^{\tilde{B}}(\bar{b})| < 1/k$.

We may assume that the codomain of every $\zeta \in \mathbb{F}$ is $[0, 1]$ and that $\mathbb{F}(k) \subseteq \mathbb{F}(k + 1)$ for all k . Let $\mathbb{F} := \bigcup_k \mathbb{F}(k)$. For $\zeta \in \mathbb{F}$ and \bar{a} in \tilde{B} of the appropriate sort let

$$Z_t^\zeta(\bar{a}) := \{n \mid \zeta^{\tilde{B}}(\bar{a}_n) > t\}.$$

For $k \in \mathbb{N}$ and $S \subseteq d(k) \times \mathbb{F}(k)$, let (we declare hitherto undefined sets to be empty):

$$\Phi_{S,k}(\bar{a}) := \bigcap_{(j,\zeta) \in S} (Z_{j/d(k)}^\zeta(\bar{a}) \setminus Z_{(j+1)/d(k)}^\zeta(\bar{a})).$$

Also let

$$\begin{aligned} \Upsilon_k(\bar{a}) &:= \{S \subseteq d(k) \times \mathbb{F}(k) \mid \Phi_{S,k}(\bar{a}) \text{ is finite}\}, \\ \Upsilon_{k,\mathcal{U}}(\bar{a}) &:= \{S \subseteq d(k) \times \mathbb{F}(k) \mid \Phi_{S,k}(\bar{a}) \notin \mathcal{U}\}. \end{aligned}$$

From the choice of $\mathbb{F}(k)$ and $d(k)$, we have the following.

Claim 4.3. *For $\bar{a} \in B$, the sets $\Upsilon_k(\bar{a})$ and $\Upsilon_{k,\mathcal{U}}(\bar{a})$ determine the value of $\varphi_j^{\tilde{B}}(\bar{a})$ up to $1/k$.*

Proof. If the codomain of ζ is $[0, 1]$, then the sets $Z_{j/d(k)}^\zeta \setminus Z_{(j+1)/d(k)}^\zeta$, for $j \leq d(k)$, form a partition of \mathbb{N} . Therefore $\Upsilon_k(\bar{a})$ and $\Upsilon_{k,\mathcal{U}}(\bar{a})$, for $k \in \mathbb{N}$, together determine the isomorphism type of the finite Boolean algebra generated by $Z_{j/d(k)}^\zeta$, for $\zeta \in \mathbb{F}(k)$ and $0 \leq j \leq d(k)$ with a distinguished ultrafilter \mathcal{U} . By Proposition 4.1, the theory of atomless Boolean algebras with a distinguished ultrafilter admits elimination of quantifiers. Therefore the type of the tuple $Z_{j/d(k)}^\zeta$, for $j \leq d(k)$, in the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ with the distinguished ultrafilter \mathcal{U} is determined by its quantifier-free type. By the choice of $\mathbb{F}(k)$, this determines the value of $\varphi_j^{\bar{B}}(\bar{a})$ for all j . ■

Since the type $\mathbf{t}(\bar{x})$ is satisfiable, for every k there exists $\bar{b}(k)$ of the appropriate sort in B such that $\max_{i < k} |\varphi_i^{\bar{B}}(\bar{b}(k)) - s_i| < 1/k$. Let \mathcal{V} be a nonprincipal ultrafilter on \mathbb{N} (we could use \mathcal{U} , but we want to emphasize that the role of \mathcal{V} is different from that of \mathcal{U}). Since for a fixed k there are only finitely many possibilities for the sets $\Upsilon_k(\bar{a})$ and $\Upsilon_{k,\mathcal{U}}(\bar{a})$, there exist Υ_k^* and $\Upsilon_{k,\mathcal{U}}^*$ such that

$$\{n \in \mathbb{N} \mid \Upsilon_k(\bar{b}(n)) = \Upsilon_k^* \text{ and } \Upsilon_{k,\mathcal{U}}(\bar{b}(n)) = \Upsilon_{k,\mathcal{U}}^*\} \in \mathcal{V}.$$

Claim 4.4. *If $\Upsilon_k(\bar{b}) = \Upsilon_k^*$ and $\Upsilon_{k,\mathcal{U}}(\bar{b}) = \Upsilon_{k,\mathcal{U}}^*$ for all k then $\varphi_i^{\bar{B}}(\bar{b}) = s_i$ for all i and \bar{b} satisfies the type $\mathbf{t}(\bar{x})$.*

Proof. Fix j . By Claim 4.3, we have $\varphi_j^{\bar{B}}(\bar{b}) = \lim_{k \rightarrow \mathcal{V}} \varphi_j^{\bar{B}}(\bar{b}(k))$, and the latter limit is equal to s_j . ■

Since \mathcal{V} is an ultrafilter, we have

- (1) $\Upsilon_k^* = \Upsilon_{k'}^* \cap (d(k) \times \mathbb{F}(k))$ for all $k < k'$, and
- (2) $\Upsilon_{k,\mathcal{U}}^* = \Upsilon_{k',\mathcal{U}}^* \cap (d(k) \times \mathbb{F}(k))$ for all $k < k'$.

By refining the sequence $\{\bar{b}(k)\}$, we may assume that $\Upsilon_k(\bar{b}(k)) = \Upsilon_k^*$ and $\Upsilon_{k,\mathcal{U}}(\bar{b}(k)) = \Upsilon_{k,\mathcal{U}}^*$ for all k . We do not need \mathcal{V} anymore. (Clearly we did not really need an ultrafilter, but as we have already assumed that a nonprincipal ultrafilter on \mathbb{N} existed, this was an easy route towards a proof of the simultaneous variant of the Bolzano–Weierstrass theorem for countably many bounded sequences.)

The set $\mathbb{F} := \bigcup_k \mathbb{F}(k)$ is countable, and

$$\mathbb{S} := \{\Phi_{S,k}(\bar{b}(j)) \mid j \in \mathbb{N}, k \in \mathbb{N}, S \notin \Upsilon_{k,\mathcal{U}}(\bar{b}(j))\}$$

is a countable subset of \mathcal{U} . Since \mathcal{U} is a P-point, we can fix $X \in \mathcal{U}$ such that $X \setminus Y$ is finite for all $Y \in \mathbb{S}$. We will choose a finite $Y_k \subseteq \mathbb{N}$, for $2 \leq k$, such that for all k the following conditions hold:

- (3) $j \in \bigcup_{i \leq j} Y_i$ and $Y_i \cap Y_j = \emptyset$ if $i \neq j$.
- (4) $|\Phi_{S,k}(\bar{b}(k)) \cap Y_k| \geq k$ for all $S \subseteq d(k) \times \mathbb{F}(k)$ such that $S \notin \Upsilon_k^*$.
- (5) $\Phi_{S,k}(\bar{b}(k+1)) \subseteq \bigcup_{j \leq k} Y_j$ for all $S \subseteq d(k) \times \mathbb{F}(k)$ such that $S \in \Upsilon_k^*$.
- (6) $\Phi_{S,k}(\bar{b}(k+1)) \cap X \subseteq \bigcup_{j \leq k} Y_j$ for all $S \subseteq d(k) \times \mathbb{F}(k)$ such that $S \in \Upsilon_{k,\mathcal{U}}^*$.

We proceed to describe the recursive construction of the sequence (Y_k) . It is analogous to the corresponding construction in the proof that the reduced product of metric structures associated to the Fréchet filter is countably saturated [15, Theorem 16.4]. For $S \notin \Upsilon_2^*$ the set $\Phi_{S,2}(\bar{b}(2))$ is infinite and we have $|\Phi_{S,2}(\bar{b}(2)) \cap m(S)| \geq 2$ for a large enough $m(S)$. Also, if $S \in \Upsilon_{2,\mathcal{U}}^*$ then $\Phi_{2,\mathcal{U}}(\bar{b}(2)) \notin \mathbb{S}$, hence $\Phi_{S,\mathcal{U}}(\bar{b}(2)) \cap X$ is finite. We can therefore choose $m > 2$ large enough so that the set (identifying m with $\{0, \dots, m - 1\}$)

$$Y_2 := m \cup \bigcup \{ \Phi_{S,3}(\bar{b}(n(3))) \mid S \subseteq d(3) \times \mathbb{F}(3), S \in \Upsilon_3^* \} \\ \cup \bigcup \{ \Phi_{S,3}(\bar{b}(n(3))) \cap X \mid S \subseteq d(3) \times \mathbb{F}(3), S \in \Upsilon_{3,\mathcal{U}}^* \}$$

satisfies (3)–(6) with $k = 2$.

Suppose that $k \geq 2$ and the sets Y_2, \dots, Y_k have been chosen to satisfy (3)–(6). The set $\Phi_{S,k+1}(\bar{b}(k + 1))$ is infinite for every $S \notin \Upsilon_{k+1}^*$, and any large enough m satisfies $|m \cap (\Phi_{S,k+1}(\bar{b}(k + 1)) \setminus \bigcup_{j \leq k} Y_j)| \geq k + 1$ for all $S \notin \Upsilon_{k+1}^*$. Also, the set $\Phi_{S,k+2}(\bar{b}(k + 2))$ is finite for every $S \in \Upsilon_{k+1}^*$, and the set $\Phi_{S,k+2}(\bar{b}(k + 2)) \cap X$ is finite for every $S \in \Upsilon_{k+1,\mathcal{U}}^*$. Let $m \geq k + 1$ be sufficiently large. Then the set

$$Y_{k+1} := \left(m \setminus \bigcup_{j \leq k} Y_j \right) \\ \cup \bigcup \{ \Phi_{S,k+2}(\bar{b}(k + 2)) \mid S \subseteq d(k + 2) \times \mathbb{F}(k + 2), S \in \Upsilon_{k+2}^* \} \\ \cup \bigcup \{ \Phi_{S,k+2}(\bar{b}(k + 2)) \cap X \mid S \subseteq d(k + 2) \times \mathbb{F}(k + 2), S \in \Upsilon_{k+2,\mathcal{U}}^* \}$$

is finite and it satisfies (3)–(6) with $k + 1$ replacing k .

This describes the recursive construction. The sets Y_k , for $k \in \mathbb{N}$, are disjoint and by (5) their union includes \mathbb{N} , hence they form a partition of \mathbb{N} into finite sets. Define $\bar{a} \in M$ by its representing sequence

$$\bar{a}_j := \bar{b}(n(k))_j \quad \text{if } j \in Y_k.$$

Then $\Phi_{S,k}(\bar{a}) \cap Y_k = \Phi_{S,k}(\bar{b}(n(k)))$ for all k . To prove that \bar{a} realizes $\mathbf{t}(\bar{x})$, it suffices to prove $\Upsilon_k(\bar{a}) = \Upsilon_k^*$ and $\Upsilon_{k,\mathcal{U}}(\bar{a}) = \Upsilon_{k,\mathcal{U}}^*$ for all k . Fix k and $S \subseteq d(k)^2 \times \bar{m}(k)$.

If $S \notin \Upsilon_k^*$, then (4) implies $|\Phi_{S,k}(\bar{a})| \geq j$ for all $j \geq k$. Therefore $\Phi_{S,k}(\bar{a})$ is infinite, hence S is an infinite set in $\Upsilon_k(\bar{a})$.

Now suppose $S \in \Upsilon_k^*$. Then $S \in \Upsilon_j^*$ for all $j \geq k$. Fix any $l \geq k$. Then (5) and (1) together imply $\Phi_{S,l}(\bar{a}) \cap Y_j = \emptyset$ for all $j \geq l + 1$. Hence $\Phi_{S,l}(\bar{a})$ is finite, and since $l \geq k$ was arbitrary, $S \notin \Upsilon_k(\bar{a})$.

If $S \in \Upsilon_{k,\mathcal{U}}^*$, then $S \in \Upsilon_{j,\mathcal{U}}^*$ for all $j \geq k$. Then $\Phi_{S,k}(\bar{a}) \cap X \cap Y_j = \emptyset$ for all $j \geq k$, and since $X \in \mathcal{U}$ we have $\Psi_{S,k}(\bar{a}) \notin \mathcal{U}$.

If $S \notin \Upsilon_{k,\mathcal{U}}^*$, then $S \notin \Upsilon_{j,\mathcal{U}}^*$ for all $j \geq k$ and $Y_j \cap X \subseteq \Phi_{S,k}(\bar{a})$ for all $j \geq k$. Since $X \in \mathcal{U}$ and the set $X \setminus \bigcup_{j \geq k} Y_j$ is finite, $X \setminus \Phi_{S,k}(\bar{a})$ is also finite and $\Phi_{S,k}(\bar{a}) \in \mathcal{U}$.

This implies that \bar{a} satisfies the type \mathbf{t} in B . ■

If \mathcal{U} is not a P-point and B contains an infinite linear order definable by a quantifier-free formula, then $(B^\infty, B^\mathcal{U}, \pi_\mathcal{U})$ is not countably saturated. This simple observation is a basis for the proofs of Theorem C (1) \Rightarrow (2) and Theorem F (1) \Rightarrow (4).

5. Proofs of Theorems C, B, 5.2, D, and A (in this order)

The proofs referred to in the title amount to little more than putting together the results of the previous sections.

Proof of Theorem C. We first prove that if B is a separable metric structure in a separable language and \mathcal{U} is a P-point then the quotient map $\pi_{\mathcal{U}}: B^{\infty} \rightarrow B^{\mathcal{U}}$ has a right inverse.

By Proposition 3.5 (2), the structure $\tilde{B}_0 := (KB, B, \pi_0)$ is isomorphic to an elementary submodel of $\tilde{B} := (B^{\infty}, B^{\mathcal{U}}, \pi_{\mathcal{U}})$. By Theorem 4.2 and the Continuum Hypothesis, \tilde{B} is saturated. Since \tilde{B}_0 is separable, there is an isomorphism Φ between the ultrapower¹¹ $(\tilde{B}_0)^{\mathcal{U}}$ and \tilde{B} that extends the identity map on \tilde{B}_0 . The isomorphism Φ sends the ultrapower of the right inverse, Θ , of π_0 (Lemma 3.3) to a map $\Theta_{\mathcal{U}}: B^{\mathcal{U}} \rightarrow B^{\infty}$. Since all the properties required from Θ —being a homomorphism, right inverse to $\pi_{\mathcal{U}}$, and extending id_B —are elementary, Łoś’s Theorem implies that Θ is as required.

For the converse, suppose that \mathcal{U} is not a P-point, and fix $X_n \in \mathcal{U}$ for $n \in \mathbb{N}$ such that for every $X \in \mathcal{U}$ the set $X \setminus X_n$ is infinite for some $n \in \mathbb{N}$. Let B be (\mathbb{N}, \min) where \min is the obvious binary function. Define $b \in B^{\infty}$ by its representing sequence, $b_j = n$ if $j \in X_{n+1} \setminus X_n$ (taking $X_{-1} = \mathbb{N}$).

Assume for the sake of obtaining a contradiction that $\Theta: B^{\mathcal{U}} \rightarrow B^{\infty}$ is a right inverse to $\pi_{\mathcal{U}}$ and let $c = \Theta(b)$. The set $X = \{j \mid c_j = b_j\}$ belongs to \mathcal{U} because $\pi_{\mathcal{U}}(c) = b$. Fix the least n such that $X \setminus X_n$ is infinite. Identifying $n + 1 \in B$ with its diagonal image, we have $\min(b, n + 1) = n + 1$, but $\min(\Theta(b), n + 1) = \min(c, n + 1) \neq n + 1$; contradiction. ■

Proof of Theorem B. Since the Continuum Hypothesis implies that a P-point exists ([37]) this is an immediate consequence of Theorem C. ■

Even if \mathcal{U} is not a P-point and the Continuum Hypothesis fails, $\pi_{\mathcal{U}}$ may have a right inverse and the exact sequence $0 \rightarrow c_{\mathcal{U}}(B) \rightarrow B^{\infty} \rightarrow B^{\mathcal{U}} \rightarrow 0$ may split for some choices of B . If the structure B is finite then $B \cong B^{\mathcal{U}}$, and if the language of B has no function symbols then any selector for $\pi_{\mathcal{U}}$ splits the exact sequence as in Theorem B. The following example is more interesting.

Example 5.1. Let \mathcal{U} be any nonprincipal ultrafilter on \mathcal{U} . Let B be $\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$, considered as a vector space over $\mathbb{Z}/2\mathbb{Z}$. It is a structure in the language equipped with the symbol for addition and symbols for multiplication by scalars in $\mathbb{Z}/2\mathbb{Z}$. Model-theorists will notice that the theory of B is stable, equal to the theory of KB , and that it admits the elimination of quantifiers. Even without using this observation, one sees that both B^{∞} and $B^{\mathcal{U}}$ are $\mathbb{Z}/2\mathbb{Z}$ -vector spaces of dimension 2^{\aleph_0} , and transfinite recursion shows that the exact sequence $0 \rightarrow c_{\mathcal{U}}(B) \rightarrow B^{\infty} \rightarrow B^{\mathcal{U}} \rightarrow 0$ splits.

¹¹The fact that we use the same ultrafilter \mathcal{U} is unimportant.

The instances of Theorem A when \mathbb{K} is the category of C^* -algebras and F is any of the standard K -theoretic functors (in addition to K_0 and K_1 , this includes the algebraic K -theory, K -theory with coefficients, KK , and KL ; see, e.g., [3, 36]) follow from Theorem D (proved below) by the standard metamathematical absoluteness arguments (similar to, e.g., [1, Appendix 2]). The following variant of Theorem D is strong enough to imply the conclusion of Theorem A for any functor F as well as Theorem D itself.

Theorem 5.2. *Suppose B is a separable metric structure in a separable language \mathcal{L} . Then there exists a family \mathcal{F} of quadruples (C, D, π, Θ) with the following properties:*

- (1) C is a separable substructure of B^∞ .
- (2) D is a separable substructure of B^u .
- (3) $\pi: C \rightarrow D$ is a surjective homomorphism.
- (4) $\Theta: D \rightarrow C$ is a homomorphism such that $\pi \circ \Theta = \text{id}_D$.
- (5) (\mathcal{F} is σ -closed) *In the ordering on \mathcal{F} defined by $(C_1, D_1, \pi_1, \Theta_1) \leq (C_2, D_2, \pi_2, \Theta_2)$ if $C_1 \subseteq C_2$, $D_1 \subseteq D_2$, π_2 extends π_1 , and Θ_2 extends Θ_1 , for every countable increasing sequence $(C_n, D_n, \pi_n, \Theta_n)$ with $n \in \mathbb{N}$ in \mathcal{F} we have*

$$\left(\bigcup_n C_n, \bigcup_n D_n, \bigcup_n \pi_n, \bigcup_n \Theta_n \right) \in \mathcal{F}.^{12}$$

- (6) (\mathcal{F} is dense) *For every $(C, D, \pi, \Theta) \in \mathcal{F}$, every $c \in B^\infty$, and every $d \in B^u$ some $(C_1, D_1, \pi_1, \Theta_1) \in \mathcal{F}$ such that $c \in C_1$ and $d \in D_1$ extends (C, D, π, Θ) .*

Also, for any separable substructures C_0 of B^∞ and D_0 of B^u , there exists a quadruple (C, D, π, Θ) such that $C_0 \subseteq C$ and $D \subseteq D_0$.

A family \mathcal{F} with the properties (1)–(5) is called a σ -complete back-and-forth system in [15, Definition 8.2.8].

Proof of Theorem 5.2. Fix a separable metric structure B in a separable language \mathcal{L} . The structure (KB, B) is by Proposition 3.5 elementarily equivalent to (B^∞, B^u) . The latter structure is countably saturated since both of its components B^∞ and B^u are countably saturated and there is no relation between them. Therefore for any $\mathcal{V} \in \beta\mathbb{N} \setminus \mathbb{N}$ the structures $\mathfrak{B}_1 := (KB, B)^\mathcal{V}$ and $\mathfrak{B}_2 := (B^\infty, B^u)$ are countably saturated and elementarily equivalent, and there exists a σ -complete back-and-forth system \mathcal{E} of partial isomorphisms between separable substructures of \mathfrak{B}_1 and \mathfrak{B}_2 (see, e.g., [15, Proposition 16.6.4]). By Lemma 3.3, there is a surjective homomorphism $\pi_0: KB \rightarrow B$ such that $\iota_{B,K}: B \rightarrow KB$ is the right inverse of π_0 , π_0 is equal to the identity on the diagonal copy of B in KB , and both π_0 and $\iota_{B,K}$ are 1-Lipschitz. Consider the expansion of (KB, B) to $(KB, B, \pi_0, \iota_{B,K})$ (this is a metric structure because both maps are 1-Lipschitz). Its ultrapower is an expansion of \mathfrak{B}_1 , $(KB, B, \pi_0, \iota_{B,K})^\mathcal{V}$. The properties required in (3) and (4) of π and Θ are first-order and satisfied by π_0 and $\iota_{B,K}$. By Łoś’s Theorem, they are

¹²Functions are identified with their graphs, giving a meaning to the formulas $\bigcup_n \pi_n$ and $\bigcup_n \Theta_n$.

shared by $\pi_0^{\mathcal{U}}$ and $\iota_{B,K}^{\mathcal{U}}$. The partial isomorphisms in \mathcal{E} transfer these maps to maps with the required properties.

The last sentence is an easy consequence of (5) and (6). ■

Proof of Theorem D. This is an immediate consequence of Theorem 5.2 and a standard back-and-forth construction of length \aleph_1 using the Continuum Hypothesis (use, e.g., [15, Proposition 16.6.1]). ■

Proof of Theorem A. Fix separable metric structures A and B and a morphism $\alpha: F(A) \rightarrow F(B)$. If $\Psi: A \rightarrow B^\infty$ realizes $\iota_{B,\infty} \circ \alpha$, then $\pi_{\mathcal{U}} \circ \Psi$ realizes $\iota_{B,\mathcal{U}} \circ \alpha$. Now assume $\Psi: A \rightarrow B^{\mathcal{U}}$ realizes $\iota_{B,\mathcal{U}} \circ \alpha$. Since Ψ is continuous, $\Psi[A]$ is separable and by Theorem 5.2, there is a quadruple (C, D, π, Θ) such that C is a separable substructure of B^∞ that includes B , D is a separable substructure of $B^{\mathcal{U}}$ that includes $B \cup \Psi[A]$, $\pi: C \rightarrow D$ is a surjective homomorphism, $\Theta: D \rightarrow C$ is a homomorphism such that $\pi \circ \Theta = \text{id}_D$, and π and Θ commute with the diagonal embeddings of B . Then $\Theta \circ \Psi$ realizes $\iota_{B,\infty} \circ \alpha$. ■

6. Preservation of elementarity by tensor products of C^* -algebras

Some familiarity with the basic theory of C^* -algebras is required in this section (obviously). The question which operations on structures preserve elementarity was raised in [22], where preservation results for generalized products were proven. Reduced products of metric structures preserve elementarity by [24] and §2. Tensor products of modules do not preserve elementarity ([31]). The question whether tensoring with any infinite-dimensional C^* -algebra preserves elementary equivalence was asked in [17, Question 3.10.5].¹³ In addition to the intrinsic interest in preservation results, this was motivated by the general problem of the extent of definability of K -groups in C^* -algebras (see [17, §3.11 and §3.12] for some specific results). In the unital case the first step towards constructing K_0 and K_1 is tensoring with the algebra $\mathcal{K}(H)$ of compact operators, and it is not known whether tensoring with $\mathcal{K}(H)$ preserves elementarity. Tensoring with $C([0, 1])$ does not necessarily preserve elementary equivalence. This was proved in [17, Proposition 3.10.3] as an application of [33, Theorem 3.1] and the ‘range of invariant’ results from Elliott’s classification programme. Since $C(X) \otimes A$ is isomorphic to $C(X, A)$ and (K denotes the Cantor space) $K\Psi$ is $\Psi \otimes 1_{C(K)}$, Corollary 3.6 has the following consequence.

- Corollary 6.1.** (1) *If B and C are elementarily equivalent C^* -algebras, then so are $B \otimes C(K)$ and $C \otimes C(K)$.*
 (2) *If $\Psi: B \rightarrow C$ is an elementary embedding, then $\Psi \circ \text{id}_{C(K)}$ is an elementary embedding of $B \otimes C(K)$ into $C \otimes C(K)$.* ■

¹³This question was asked for the minimal (spatial) tensor product, but since in every example considered in [17] and in this section at least one of the factors is nuclear, in all examples under consideration $A \otimes B$ is necessarily the minimal tensor product.

The following result, all but proven in [17, §3.5], transpired during a conversation with Chris Schafhauser at the Fields Institute in 2019, and it is included with his kind permission (\mathcal{Z} denotes the Jiang–Su algebra [26]).

Proposition 6.2. *There are elementarily equivalent C^* -algebras A and B such that $A \otimes D$ and $B \otimes D$ are not elementarily equivalent if D is \mathcal{Z} or any UHF algebra D .*

Proof. The reader is assumed to be familiar with [17]. Let A be the unital, monotracial C^* -algebra constructed in [35, Theorem 1.4] such that $A^{\mathcal{U}}$ does not have a unique trace and let $B = A^{\mathcal{U}}$. Then A and B are elementarily equivalent, $A \otimes \mathcal{Z}$ is monotracial, and $B \otimes \mathcal{Z}$ is not. As the Cuntz–Pedersen nullset in $A \otimes \mathcal{Z}$ is definable (this was essentially proved in [32], see [17, Theorem 3.5.5 (3)] for a reformulation of Robert’s result), for every $\varepsilon > 0$ there exists $m(\varepsilon)$ such that every positive contraction a in $A \otimes \mathcal{Z}$ can be ε -approximated by a sum of $m(\varepsilon)$ commutators of elements of norm ≤ 1 . For a fixed $\varepsilon > 0$ this property can be expressed as a statement in the theory of $A \otimes \mathcal{Z}$. Since $B \otimes \mathcal{Z}$ does not have a unique trace, for a small enough ε the corresponding assertion fails in $B \otimes \mathcal{Z}$ and therefore $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$ are not elementarily equivalent.

If D is a UHF algebra then $A \otimes D$ is monotracial and since D absorbs \mathcal{Z} tensorially, so does $A \otimes D$. Therefore the Cuntz–Pedersen nullset of $A \otimes D$ is definable, and the proof proceeds as in the case of \mathcal{Z} . ■

A C^* -algebra A is \mathcal{Z} -absorbing if $A \otimes \mathcal{Z} \cong A$. This is a remarkably strong and important regularity property of nuclear C^* -algebras (see [44]). Also, \mathcal{Z} itself is \mathcal{Z} -absorbing ([26]), and being \mathcal{Z} -absorbing is (among separable C^* -algebras) axiomatizable [17, Theorem 2.5.2 (21)]. I am inclined to conjecture that tensoring with a fixed UHF algebra (or any nuclear C^* -algebra) preserves elementary equivalence among \mathcal{Z} -absorbing C^* -algebras, and that tensoring with the Cuntz algebra \mathcal{O}_2 preserves elementary equivalence of arbitrary C^* -algebras.

7. Applications to C^* -algebras

Proof of Theorem F. Fix a P-point \mathcal{U} and a separable C^* -algebra B .

(1) It suffices to prove $\pi_{\mathcal{U}}[B^{\infty} \cap B'] \supseteq B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[B]'$, since the other inclusion is automatic. Fix $c \in B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[B]'$. In the structure $(B^{\infty}, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ consider the 1-type of an element of B^{∞} consisting of the conditions $\pi_{\mathcal{U}}(x) = c$ and $\|[x, b_n]\| = 0$, for a fixed dense subset $\{b_n \mid n \in \mathbb{N}\}$ of the unit ball of B . In order to see that this type is satisfiable, fix $m \geq 1$. Then the set $X = \{j \mid \max_{n \leq m} \|[c_j, b_n]\| \leq 1/m\}$ belongs to \mathcal{U} . Define $d \in B^{\infty}$ by its representing sequence, $d_j = c_j$ if $j \in X$ and $d_j = 0$ if $j \notin X$. Then $\pi_{\mathcal{U}}(d) = c$ and $\max_{n \leq m} \|[d, b_n]\| \leq 2/m$. Since m was arbitrary, the type is satisfiable. By the countable saturation of $(B^{\infty}, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ (Theorem 4.2), it is realized by some $c' \in B^{\infty}$. Clearly $c' \in B^{\infty} \cap B'$ and $\pi_{\mathcal{U}}(c') = c$. Since c was an arbitrary element of $B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[B]'$, this completes the proof.

The proof of (2), that $\pi_{\mathcal{U}}[B^{\infty} \cap A'] = B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[A]'$ for every separable C^* -subalgebra A of B^{∞} , is analogous to that of (1) and therefore omitted.

(3) Recall that an ideal I in a C^* -algebra is a σ -ideal if for every countable $X \subseteq J$ there exists a positive contraction $a \in J$ such that $ab = ba = b$ for all $b \in X$. Fix a countable subset $\{a_n \mid n \in \mathbb{N}\}$ of $c_{\mathcal{U}}(B)$. Consider the type of an element of B^∞ with conditions $x \leq 0$, $\|x\| = 1$, $\|xa_n - a_n\| = 0$, and $\pi_{\mathcal{U}}(x) = 0$. Since $c_{\mathcal{U}}(B)$ has an approximate unit (every C^* -algebra does; see, e.g., [15, §1.8]), this type is satisfiable. By countable saturation of $(B^\infty, B^{\mathcal{U}}, \pi_{\mathcal{U}})$ it is realized by some c in B^∞ . Then c is as required.

For the remainder of the proof, we do not assume that \mathcal{U} is a P-point.

(1) implies (4): We need to prove that if \mathcal{U} is not a P-point and B is a UHF algebra then $\pi_{\mathcal{U}}[B^\infty \cap B'] \not\supseteq B^{\mathcal{U}} \cap \pi_{\mathcal{U}}[B]'$. This is similar to the proof of the analogous part of Theorem C. Suppose that \mathcal{U} is not a P-point and fix $X_n \in \mathcal{U}$ for $n \in \mathbb{N}$ such that for every $X \in \mathcal{U}$ the set $X \setminus X_n$ is infinite for some $n \in \mathbb{N}$. In B , identified with $\bigotimes_{\mathbb{N}} M_2(\mathbb{C})$, we can choose unitaries u_j and v_j for all j such that $\lim_{j \rightarrow \infty} \|[a, u_j]\| = 0$ for all $a \in B$ but $\|[u_j, v_n]\| = 2$ if $n \geq j$ (the construction is similar to that in the proof of [18, Lemma 3.2], where the analogous statement for the tracial norm was proven). Define $c \in B^{\mathcal{U}}$ by its representing sequence $c_j = u_n$ if $j \in X_n \setminus X_{n+1}$ (with $X_{-1} = \mathbb{N}$). Then $c \in B^{\mathcal{U}} \cap B'$. If $b \in B^\infty$ is such that $\pi_{\mathcal{U}}(b) = c$, then $X = \{n \mid \|b_n - c_n\| < 1/4\}$ belongs to \mathcal{U} . Let n be large enough to have $X \setminus X_n$ infinite. Then $\|[v_n, b]\| \geq \|[v_n, c]\| - \|v_n\|/2 > 0$, and therefore $b \notin B^\infty \cap B'$. Since b was an arbitrary $\pi_{\mathcal{U}}$ -preimage of c , this completes the proof.

(5) clearly implies (1), and we have already proved that (4) implies (5). This completes the proof. ■

8. Concluding remarks

Our main goal was to prove Theorems A–F and 5.2. Theorems 2.2 and 4.2 were therefore stated and proved in what appear to be special cases of more general results on multi-sorted structures consisting of various reduced powers of a single structure and quotient maps between them. This line of research may merit further attention.

Continuous fields of C^* -algebras are well-studied objects (see [4, §IV.1.6]), and a theory of continuous fields of metric structures will inevitably be developed. Can the conclusion of Corollary 6.1 be extended to the assertion that taking continuous fields of metric structures over the Cantor space preserves elementary equivalence? Or, what sort of a Feferman–Vaught–Ghasemi-style theorem holds for continuous fields of metric structures? (See, e.g., [5, Theorem 2.1] for a discrete version of the desired results.) It should be noted that [17, Proposition 3.10.3] implies that $A \prec B$ (this stands for ‘ A is an elementary submodel of B ’) does not imply $C([0, 1], A) \prec C([0, 1], B)$, and therefore even taking trivial continuous fields of metric structures does not necessarily preserve elementary equivalence.

Another class of massive (albeit typically not countably saturated—see [15, Exercise 16.8.36], but note the typo in its statement: ‘degree-1’ should be ‘quantifier-free’, since these C^* -algebras are countably degree-1 saturated by [15, Theorem 15.1.5]) quotient C^* -algebras derived from continuous fields deserves attention of model-theorists,

and I will use this opportunity to mention them. If B is a separable C^* -algebra, consider the algebra of all bounded continuous functions from $[0, \infty)$ into B , $C_b([0, \infty), B)$. The quotient of this algebra over the ideal $C_0([0, \infty), B)$ provides the setting for the E-theory [3, §25].

Tracial von Neumann algebras give an example of an axiomatizable category not closed under the operation of taking the reduced power. This is because the asymptotic sequence algebra B^∞ is infinite-dimensional and countably saturated, and therefore not a von Neumann algebra. Therefore the following quotient merits our attention.

Example 8.1. Let R denote the hyperfinite II_1 factor and let τ denote its unique tracial state. Then (R, τ) is naturally construed as a metric structure with respect to the ℓ_2 -norm, $\|a\|_2 := \sqrt{\tau(a^*a)}$ (see [19, §3.2]). Since the category of II_1 factors is axiomatizable, the ultrapower of R in this language (also known as the *tracial ultrapower*), $R^{\mathcal{U}}$, is a II_1 factor.¹⁴ The reduced power R^∞ is however not a II_1 factor, or even a von Neumann algebra. One reason for this is that in a von Neumann algebra every directed family of positive contractions has a supremum, and this is not the case with the reduced powers associated with the Fréchet ideal. This reduced power belongs to the class of C^* -algebras extensively studied in recent years (see [7], and [25] for their model-theoretic analysis).

The structure R^∞ was recently used in [6]. By the following, R^∞ is for all practical purposes isomorphic to an ultrapower of KR (where K is the functor defined in §3.1 and KR is embedded into R^∞ by using Proposition 3.4).

Proposition 8.2. *Consider R , KR , and R^∞ as metric structures with respect to the ℓ_2 -norm. Then R^∞ is a countably saturated elementary extension of KR .*

Proof. The first part follows from the first part of Proposition 3.5 and the second part is a consequence of [20, Theorem 2.1] (see also [15, Theorem 16.5.1]). ■

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¹⁴This fact had been known long before general ultrapowers of metric structures were defined; see the introduction to [19].

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