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On automorphic descent from GL_7 to G_2

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Abstract. In this paper, we study the functorial descent from self-contragredient cuspidal automorphic representations π of $GL_7(\mathbb{A})$ with $L^S(s, \pi, \wedge^3)$ having a pole at $s = 1$ to the split exceptional group $G_2(\mathbb{A})$, using Fourier coefficients associated to two nilpotent orbits of E_7. We show that one descent module is generic, and under suitable local conditions, it is cuspidal and π is a weak functorial lift of each of its irreducible summands. This establishes the first functorial descent involving the exotic exterior cube L-function. However, we show that the other descent module supports not only the nondegenerate Whittaker–Fourier integral on $G_2(\mathbb{A})$ but also every degenerate Whittaker–Fourier integral. Thus it is generic, but not cuspidal.

Keywords. Fourier coefficients of automorphic forms, functorial descent, exterior cube L-function, split exceptional group G_2

1. Introduction

In the theory of automorphic forms one of the major open problems is to construct functorial correspondences between automorphic forms on different groups. This has been accomplished in particular cases by various methods, including the converse theorem, the theta correspondence, the trace formula, and the theory of functorial descent.

The theory of functorial descent was pioneered by Ginzburg, Rallis, and Soudry. It serves as a complement to the constructions of functorial liftings, and can be used to characterize the image of a functorial lifting.

We briefly recall these notions. Let F be a number field, $\mathbb{A}$ its adele ring, and H a connected reductive F-group. Given an irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $H(\mathbb{A})$ we obtain a finite set S of places of F and a semisimple conjugacy class $\{t_{\pi_v}\}$ in ${}^L H$ for each $v \notin S$. We say that two automorphic representations π and π' are *nearly equivalent* if $\{t_{\pi_v}\} = \{t_{\pi'_v}\}$ for all v outside a finite set. Given an L-homomorphism $\varphi : {}^L H \to {}^L G$ we say that an irreducible automorphic representation Π of $G(\mathbb{A})$ is a *weak functorial lift*, relative to φ, of an irreducible automorphic representation π of $H(\mathbb{A})$ if

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$\{t_{\Pi_v}\} = \{\varphi(t_{\pi_v})\}$ for all v outside a finite set. Clearly, in this situation, every element of the near equivalence class of Π is also a weak functorial lift of every element of the near equivalence class of π. We also say that π is a *weak functorial descent* of Π. The Langlands functoriality conjecture then predicts that the set of weak functorial lifts is nonempty for all π and all φ. This has been proved in a number of cases, though the general case is still very much open.

Supposing that a lifting exists, one may ask what its image is. Here again, the general case is open but the problem has been solved in some cases. For example, Ginzburg, Rallis and Soudry showed, using descent together with the lifting results of Cogdell, Kim, Piatetski-Shapiro, and Shahidi, that an automorphic representation of $GL_{2n}(\mathbb{A})$ is a weak functorial lift from a generic cuspidal representation of $SO_{2n+1}(\mathbb{A})$ (for the inclusion $Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$) if and only if it is an isobaric sum $\tau_1 \boxplus \cdots \boxplus \tau_r$ of distinct cuspidal representations τ_i of $GL_{2n_i}(\mathbb{A})$ for $1 \le i \le r$, such that $L^S(s, \tau_i, \wedge^2)$ has a pole at $s = 1$ for each i. In particular, a cuspidal representation of $GL_{2n}(\mathbb{A})$ has a weak functorial descent to $SO_{2n+1}(\mathbb{A})$ if and only if its exterior square L-function has a pole. Notice that $Sp_{2n}(\mathbb{C})$ is embedded into $GL_{2n}(\mathbb{C})$ as the stabilizer of a point in general position in the exterior square representation. Ginzburg, Rallis and Soudry also obtained similar results for other classical groups, as well as metaplectic groups.

The connection between the exterior square L-function and the lifting is clear. It was an earlier result of Ginzburg, Rallis, and Soudry that $L^S(s, \tau, \wedge^2)$ has a pole at $s = 1$ whenever τ is a weak functorial lift relative to the above inclusion. Moreover, this result was predicted by the functoriality and generalized Ramanujan conjectures, before it was proved. If a cuspidal representation τ of $GL_{2n}(\mathbb{A})$ is the weak functorial lift of a cuspidal representation σ of $SO_{2n+1}(\mathbb{A})$ relative to the inclusion $Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$, then $L^S(s, \tau, \wedge^2) = L^S(s, \sigma, \wedge_0^2)\zeta^S(s)$, where $\wedge_0^2$ is the second fundamental representation of $Sp_{2n}(\mathbb{C})$, which satisfies $\wedge^2 = \wedge_0^2 \oplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation. Clearly $\zeta^S(s)$ has a pole at $s = 1$ for all finite sets S. Further, the functoriality conjecture predicts that $L^S(s, \sigma, \wedge_0^2)$ should be the standard L-function of the weak functorial lift of σ to $GL_{\dim \wedge_0^2}$, relative to $\wedge_0^2$. This lift may not be cuspidal, but the generalized Ramanujan conjecture predicts that σ will be tempered at all places, in which case its lift will be as well. This forces the cuspidal support of any weak functorial lift to be unitary, which is sufficient to ensure nonvanishing of its L-function on the line $\mathrm{Re}(s) = 1$.

In general, by the same reasoning, if r is a finite-dimensional representation of ${}^LG$ and the image of $\varphi : {}^LH \to {}^LG$ is contained in the stabilizer of some nonzero point in the space of r, and if π is an irreducible globally generic cuspidal representation of $H(\mathbb{A})$ then $L^S(s, \Pi, r)$ is expected to have a pole at $s = 1$ for any weak functorial lift Π of π to G relative to φ.

The descent results of Ginzburg, Rallis, and Soudry point to a converse result: if $L^S(s, \Pi, r)$ has a pole at $s = 1$, then Π should be a weak functorial lift relative to the inclusion of a reductive group which stabilizes a nonzero point in the space of r. (A more refined conjecture is given in [28].)

The descent method of Ginzburg, Rallis, and Soudry has been extended to GSpin groups (which are not classical, but have classical L-groups) in [20]. The paper [12] investigates the extension of the method of descent into exceptional groups. Ginzburg has also investigated descent from E_6 to F_4, together with the first named author, in an unpublished preprint. In this paper, we investigate an interesting case in the exceptional group GE_7.

The method may be described as follows. Suppose that there is a reductive group A such that

- G is a Levi subgroup of A,
- r appears in the restriction to ${}^L G$ of the adjoint representation of ${}^L A$,
- H is the stabilizer in A of some $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{a}$ of A.

Then the descent method proceeds by the following steps:

(1) Take an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$.

(2) Consider Eisenstein series on $A(\mathbb{A})$ induced from π. The L-function $L^S(s, \pi, r)$ appears in the constant term of these Eisenstein series. Consider the corresponding residual representation.

(3) Consider a Fourier coefficient attached to the $\mathfrak{sl}_2$-triple with stabilizer H. This Fourier coefficient will map automorphic forms on $A(\mathbb{A})$ to smooth automorphic functions of uniformly moderate growth on $H(\mathbb{A})$ (or in some cases the metaplectic double cover of $H(\mathbb{A})$). Applying this Fourier coefficient to our residual representation, we obtain a space of functions on $H(\mathbb{A})$ (or its double cover) which we call the *descent module*.

For example, in the classical work of Ginzburg, Rallis and Soudry, the group GL_{2n} appears as a Levi of SO_{4n}, and for suitable $\mathfrak{sl}_2$-triples in $\mathfrak{so}_{4n}$ the stabilizer in SO_{4n} is isomorphic to SO_{2n+1}. We remark that in some cases $L^S(s, \pi, r)$ will appear in the constant term *along with other L-functions*, and it will be necessary to add some assumption above and beyond $L^S(s, \pi, r)$ having a pole. For example, in the descent from GL_{2n} to $\widetilde{Sp}_{2n}$ one must assume that the exterior square L-function has a pole at 1, *and* that the standard L-function is nonvanishing at $1/2$.

As mentioned, in some cases the descent module consists of genuine functions on a metaplectic double cover. Since this does not apply to the case we consider in this paper, we will not go further into this. We remark that while the functions in the descent module are easily seen to be smooth, invariant by $H(F)$ on the left, of uniformly moderate growth, and finite under translations of a maximal compact subgroup of $H(\mathbb{A})$, it is not easy to see whether or not they are finite under the action of the center of the universal enveloping algebra. So, they are not necessarily automorphic forms.

In the classical work of Ginzburg, Rallis, and Soudry, it is possible to show that descent module is cuspidal (hence L^2, so that its closure is a Hilbert space direct sum of irreducibles), and that every summand is a weak descent of the original representation on $GL_{2n}(\mathbb{A})$. Moreover, it is orthogonal to the kernel of the nondegenerate Whittaker–Fourier integral on $H(\mathbb{A})$, which implies that it is multiplicity free and that every summand is globally generic. In some cases, it can even be shown that the descent module is irreducible. In [20], it is shown that the descent module is cuspidal, that every summand is a weak descent, and that the nondegenerate Whittaker–Fourier integral does not vanish on the descent module (so at least one summand is globally generic). The stronger result – that the descent module is orthogonal to the kernel of the nondegenerate Whittaker–Fourier integral – should follow from work in progress of Asgari, Cogdell, and Shahidi.

There are a number of cases where the conditions above are satisfied with A being one of the exceptional groups. In this paper we consider the case when $A = GE_7$, and $G = GL_7 \times GL_1$. The embedding of $GL_7 \times GL_1$ into GE_7 can be chosen so that r is the product of the $\wedge^3$ representation of GL_7 and the standard representation of GL_1. We show that it suffices to consider the case when the automorphic representation of GL_7 is self-contragredient and the character of GL_1 is trivial. The group $GL_7 \times GL_1$ acts on our space with a Zariski open orbit and the stabilizer of any point in this orbit is the product of the center of GE_7 and a subgroup of GL_7 of G_2 type. (See [11, pp. 356–357], and Lemma 6.2.1 below.) The stabilizer of any nonzero point which is not in the Zariski open orbit is not reductive. Thus we consider irreducible self-contragredient cuspidal automorphic representations π of $GL_7(\mathbb{A})$ such that the $\wedge^3$ L-function has a pole at $s = 1$, i.e., of G_2 type by Definition 4.2.10. The philosophy discussed above predicts that such cuspidal representations should be weak functorial lifts from G_2. We first construct square integrable residual representations of $GE_7(\mathbb{A})$. At this point, an interesting feature emerges: it turns out that there are *two* orbits of $\mathfrak{sl}_2$-triples in $\mathfrak{e}_7$ with stabilizers of G_2 type. Thus, we have two different Fourier coefficients which we can apply to obtain two descent modules on the exceptional group $G_2(\mathbb{A})$. In this paper we study both descent modules.

A similar situation was considered previously in [14], where the authors consider three different orbits of a group of type D_4, all of which have a stabilizer of type A_1. However, the two orbits considered in our paper are not related to one another by the automorphism group of $\mathfrak{e}_7$, whereas the three orbits considered in [14] are permuted by the automorphism group of $\mathfrak{d}_4$.

The functorial lifting corresponding to this case is known, at least for generic cuspidal representations. By [15] generic cuspidal representations of $G_2(\mathbb{A})$ can be lifted to $Sp_6(\mathbb{A})$ using the minimal representation of E_7. It can then be lifted to GL_7 using the work of Cogdell–Kim–Piatetski-Shapiro–Shahidi [6], Arthur [1], and Cai–Friedberg–Kaplan [4]. It is very natural to ask whether the descent from GL_7 to G_2 could be constructed by combining the descent from GL_7 to Sp_6 from [16] with the theta-type correspondence from Sp_6 to G_2 in [15]. To the best of our understanding, this should be possible, but would require proving the following conjecture.

Conjecture 1.0.1. *Let π be an irreducible self-contragredient cuspidal automorphic representation of $GL_7(\mathbb{A})$ such that $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$, and let σ denote the irreducible descent of π to $Sp_6(\mathbb{A})$. Then σ has trivial central character and satisfies the three equivalent conditions of* [13, Theorem 1.1].

An analogy with the earlier work of Ginzburg–Rallis–Soudry, as well as [20], would predict that the descent module should be cuspidal, support the nondegenerate Whittaker–Fourier integral, and be a direct sum of weak descents of our original cuspidal representation of GL_7. In this respect, the two descent modules behave totally differently.

In one case we prove that the descent module is generic, and under suitable local conditions, it is cuspidal and π is a weak functorial lift of each irreducible summand. One piece that is missing, in comparison to [16, 20], is a means of showing that when π is self-contragredient and $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$, the Satake parameters of the components of π at unramified places must contain conjugacy classes of $G_2(\mathbb{C})$. We show cuspidality under the assumption that at least one of them does, and weak functorial lifting under the assumption that all but finitely many of them do. In particular, we prove the following theorem (see Theorem 7.0.1).

Theorem 1.0.2. *Let F be a number field and let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A}_F)$. Suppose that the following conditions hold:*

(1) *The partial L-function $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$ for some finite set S.*
(2) *For almost all places v of F at which π_v is unramified, the Satake parameter of the local component π_v is conjugate in $GL_7(\mathbb{C})$ to an element of $r_7(G_2(\mathbb{C}))$, where r_7 is the standard representation of G_2.*

Then there exists a globally generic cuspidal automorphic representation σ of $G_2(\mathbb{A}_F)$ such that for almost all places v of F at which σ_v is unramified, the Satake parameter of π_v is conjugate in $GL_7(\mathbb{C})$ to the Satake parameter of σ_v.

We believe that it should be possible to replace the second condition with the weaker condition that π is self-contragredient or has trivial central character. That is, we have the following conjecture.

Conjecture 1.0.3. *Let π be an irreducible self-contragredient cuspidal automorphic representation of $GL_7(\mathbb{A})$ such that $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$. Then for almost all places v of F at which π_v is unramified, the Satake parameter of the local component π_v is conjugate in $GL_7(\mathbb{C})$ to an element of $r_7(G_2(\mathbb{C}))$, where r_7 is the standard representation of G_2.*

This conjecture turns out to be equivalent to Conjecture 1.0.1. More generally, if π satisfies conditions (1) and (2) of Theorem 1.0.2, then its descent to G_2 contains an irreducible generic cuspidal automorphic representation of $G_2(\mathbb{A})$, which we may theta-lift to $Sp_6(\mathbb{A})$ using the lifting from [15]. By a result of Savin, [35, Appendix A], the lifting is generic, and lifts weakly to π (which forces it to be cuspidal due to the Strong Multiplicity One Theorem for GL_7), and so, by Strong Multiplicity One Theorem for Sp_6, it contains the descent of π, which therefore satisfies the equivalent conditions of [13]. Conversely, if the descent of π to Sp_6 satisfies the equivalent conditions of [13], then it is the theta lift of a generic cuspidal representation of $G_2(\mathbb{A})$, and this lifting is functorial. It follows that π itself is a functorial lift from G_2 and condition (2) of Theorem 1.0.2 is satisfied.

The descent method is constructive and makes use of an Eisenstein series on the similitude exceptional group GE_7. We prove that this Eisenstein series has a pole whenever condition (1) of Theorem 1.0.2 is satisfied. In fact, we could replace condition (1) with the hypothesis that the Eisenstein series has a pole. Indeed, for any cuspidal automorphic representation of GL_7 such that the Eisenstein series has a pole, the descent method produces a space of functions on G_2 which is globally generic in the sense that the Whittaker integral does not vanish identically on it; see Remark 7.1.18. Under condition (2) we are able to prove that it is cuspidal and that all of its irreducible components lift weakly to π, but in each of these proofs, condition (2) can be replaced by a weaker hypothesis applied to the residue of the Eisenstein series; see Remarks 7.4.33 and 7.5.7.

The result above establishes the first functorial descent which involves the exotic exterior cube L-function. This is an important step towards fully understanding the Langlands functoriality from G_2 to GL_7 which is not an endoscopic type. As pointed out to us by Michael Harris, Theorem 1.0.2 has interesting applications already, for example, to [3, Conjecture 11.6] and the surjectivity of local Langlands correspondence [35].

The other descent module behaves totally differently. It supports not only the nondegenerate Whittaker–Fourier integral on $G_2(\mathbb{A})$, but also every *degenerate* Whittaker–Fourier integral. Thus it is generic, but *not* cuspidal. It has a nontrivial constant term for each proper parabolic of G_2, and its constant terms for the two maximal parabolics are generic representations of $GL_2(\mathbb{A})$. And this holds for *every* cuspidal representation of $GL_7(\mathbb{A})$ such that the $\wedge^3$ L-function has a pole! See Theorem 8.0.1.

This outcome is not entirely without precedent. Descent constructions in the exceptional group F_4 were previously studied in [12] from a different point of view. In [12], Ginzburg introduces a general family of lifting integrals which interpolates between theta-type liftings at one end of the spectrum and descent constructions at the other end. He also introduces a "dimension equation" which is said to hold in every known case where an integral of his type gives a functorial correspondence. He then uses the dimension equation to decide which automorphic representations to apply a Fourier coefficient to (instead of using a residual representation obtained from a pole of $L^S(s, \pi, r)$).

This approach makes sense from the perspective of the techniques which are used to prove genericity and cuspidality, namely identities of unipotent periods. The approach taken in [12] is to take the unipotent period obtained by composing the descent Fourier coefficient with either a Whittaker integral or a constant term on the stabilizer H, and relate this period to some combination of coefficients attached to $\mathfrak{sl}_2$-triples and constant terms.

One case of particular interest is when $A = F_4$, $G = GSp_6$, r is the spin representation of ${}^L G = GSpin_7(\mathbb{C})$, and $H = G_2$. In this case, it is shown in [12] that

(1) the nondegenerate Whittaker–Fourier integral of the descent module of any representation $\mathcal{E}$ can be expressed in terms of coefficients attached to the orbits F_4, $F_4(a_1)$, and $F_4(a_2)$, as well as the constant term along the C_3 parabolic, and
(2) the constant terms of the descent module can be expressed in terms of exactly the same four unipotent periods!

This is very similar to our result, which relates both the nondegenerate Whittaker–Fourier integral and all degenerate Whittaker–Fourier integrals of the descent to the *same* unipotent period on GE_7. This period is not one of the types considered by Ginzburg, but it is in a more general family, introduced by Gomez, Gourevitch and Sahi [17].

Another case which has been studied somewhat is when $A = E_8$, $G = GE_6 \times GL_1$, r is 27-dimensional, and $H = F_4$. This case is considered in work in progress of Ginzburg and the first named author. In that case, also, it appears that the descent module is generic, but not cuspidal.

Having established that the descent is not cuspidal, it is no longer clear that it has a decomposition into irreducibles, or even a spectral decomposition in terms of cuspidal data. Moreover, there would seem to be little reason to think that its irreducible subquotients – should they exist – will be weak descents of the original cuspidal representation of $GL_7(\mathbb{A})$. Indeed, if our representation of $GL_7(\mathbb{A})$ was a weak functorial lift of a cuspidal representation of $G_2(\mathbb{A})$ which is not CAP, then no weak descent of it has a constant term – and the descent module does. If one is still optimistic enough to believe that the descent module contains a generic weak descent of our cuspidal representation of $GL_7(\mathbb{A})$, then one is led to the questions of what *else* it contains, and whether this "extra" depends on the choice of the representation.

Another natural question is the following: *what* other *automorphic representations of $GL_7(\mathbb{A})$ should descend to $G_2(\mathbb{A})$? And can our construction generalize to construct their descents?* For example, there is a lifting, constructed in [15] and shown to be functorial in [13], attached to the embedding $SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$. If we compose this with an embedding $G_2(\mathbb{C}) \hookrightarrow GL_7(\mathbb{C})$ the result is conjugate to the map

$$g \mapsto \begin{pmatrix} g & & \\ & 1 & \\ & & {}^t g^{-1} \end{pmatrix}.$$

Thus, if an irreducible cuspidal automorphic representation π of $G_2(\mathbb{A})$ is the lift of a cuspidal representation τ of $PGL_3(\mathbb{A})$ then the lift of π to $GL_7(\mathbb{A})$ is the isobaric sum $\tau \boxplus \mathbf{1} \boxplus \widetilde{\tau}$, where $\mathbf{1}$ is the one-dimensional trivial representation of $GL_1(\mathbb{A})$. Thus, it is very natural to ask whether π can be recovered from $\tau \boxplus \mathbf{1} \boxplus \widetilde{\tau}$, by some generalization of our construction. (Note that this would then give an alternative construction of the lifting from [15].) We hope to return to this and related questions in the future.

The organization of the paper is as follows: We introduce some notation in Section 2, preliminaries and some general results in Section 3, the A_6 Levi and the residual representation of the similitude exceptional group $GE_7(\mathbb{A})$ in Section 4, and the nilpotent orbit A_6 of E_7 in Section 5. Then we introduce in Section 6 the two descent Fourier coefficients attached to the two nilpotent orbits, from which we obtain two descent modules. In Section 7, we show that one descent module is generic, and under suitable local conditions, it is cuspidal and having π as a weak functorial lift of each irreducible summand. In Section 8, we show that the other descent module supports not only the nondegenerate Whittaker–Fourier integral on $G_2(\mathbb{A})$ but also every degenerate Whittaker–Fourier integral. Thus it is generic, but not cuspidal.

We used three software packages for computations: LiE [7], GAP [34], and Sage [33]. LiE was used for computations involving the action of the Weyl group on the root and weight lattices and their duals. GAP, and the packages QuaGroup [9], SLA 0.14 [10], and UNIPOT 1.2 [18], were used for many computations involving nilpotent elements of the Lie algebra $\mathfrak{e}_7$ and their adjoint orbits. Sage was used for symbolic manipulation of multivariate polynomials – especially for performing computations using matrices over multivariate polynomial rings. These matrices were formed by loading integer matrices obtained from GAP into Sage and then forming linear combinations with coefficients in the polynomial ring. Our code is available at [19].

2. Notation

Let F be a number field, $\mathbb{A}$ its adele ring, and $\mathbb{A}_{\text{fin}}$ its ring of finite adeles. (Our results are restricted to number fields because we make use of [17]. We expect that both the results of [17] and our results should extend to function fields, except possibly for a few small primes. For a discussion of the relevant issues, see [17, Remark 5.1.4].)

We shall consider automorphic representations of the similitude exceptional group GE_7. This group can be realized as the maximal Levi subgroup of split E_8 whose derived group is of type E_7. For us, this will be the definition. The derived group is in fact the unique split connected simply connected quasi-simple group of type E_7. For the split group E_8, we label the simple roots as follows:

$$
\begin{array}{ccccccccccccc}
\alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \\
0 & -- & 0 & -- & 0 & -- & 0 & -- & 0 & -- & 0 & -- & 0 \\
 & & & & | & & & & & & & & \\
 & & & & 0 & & & & & & & & \\
 & & & & \alpha_2 & & & & & & & &
\end{array}
$$

We assume that GE_7 is equipped with a choice of split maximal torus T and Borel subgroup B. We write Φ for the set of roots of T in GE_7, Φ^+ for the set of positive roots determined by the choice of B, and Δ for the set of simple roots. If H is a T-stable subgroup of GE_7, we denote the set of roots of T in H by $\Phi(H, T)$. For $\alpha \in \Phi$ we denote the corresponding root subgroup by U_α and the corresponding coroot $\mathbb{G}_{\text{m}} \to T$ by $\alpha^\vee$. Let $\mathfrak{t}$ and $\mathfrak{u}_\alpha$ be the Lie algebras of T and U_α, respectively. We use exponential notation for rational characters and cocharacters: $t \mapsto t^\alpha$, $t \in T$, and $a \mapsto a^{\alpha^\vee}$, $a \in \mathbb{G}_{\text{m}}$. We sometimes also use the notation $h(t_1, \dots, t_8) = \prod_{i=1}^8 t_i^{\alpha_i^\vee}$. We also equip GE_7 with a realization in the sense of [32], i.e. a family $\{x_\alpha : \mathbb{G}_{\text{a}} \to U_\alpha\}$ of parametrizations of the root subgroups (subject to some compatibility relations). This determines a basis of the Lie algebra $\mathfrak{ge}_7$. Indeed, for each root α the differential Dx_α of x_α is an isomorphism $\mathbb{G}_{\text{a}} \to \mathfrak{u}_\alpha$ and we denote $Dx_\alpha(1)$ by X_α. The differential of $\alpha^\vee : \mathbb{G}_{\text{m}} \to T$ is an injective map $D\alpha^\vee : \mathbb{G}_{\text{a}} \to \mathfrak{t}$, and we denote $D\alpha^\vee(1)$ by H_α. Then $\{X_\alpha : \alpha \in \Phi(GE_7, T)\} \cup \{H_{\alpha_i} : 1 \le i \le 8\}$ is a basis for $\mathfrak{ge}_7$, and by taking a suitable realization, we can arrange for it to be a Chevalley basis. We choose the Chevalley basis so that the structure constants match those employed by GAP [34]. As mentioned previously, we used LiE and SageMath, in addition to GAP, for computations. Structure constants are not involved in the type of computations for which LiE was used, and SageMath was used to manipulate polynomial-linear combinations of integer matrices obtained from GAP, which ensures compatibility of structure constants between the computations done using GAP and Sage. We also fix a GE_7-invariant bilinear form κ on $\mathfrak{ge}_7$ such that $\kappa(X_\alpha, X_{-\alpha}) = 1$ for each root α.

We denote the Weyl group of GE_7 relative to T by W. We denote the simple reflection attached to the simple root α_i by $w[i]$, and the product $w[i_1]\dots w[i_l]$ by $w[i_1\dots i_l]$. There is a standard representative for $w[i]$, namely $\dot{w}[i] := x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1)$. This then gives rise to a standard representative $\dot{w}[i_1\dots i_l] := \dot{w}[i_1]\dots\dot{w}[i_l]$ for $w[i_1\dots i_l]$. But note that $\dot{w}[i_1\dots i_l]$ depends on the expression for $w[i_1\dots i_l]$ as a word in the simple reflections and not only on the Weyl group element.

Let $P = MU$ be the standard parabolic subgroup of GE_7 whose unipotent radical contains U_{α_i} if and only if $i = 2$, with Levi subgroup M and unipotent radical U. Then M is isomorphic to $GL_7 \times GL_1$ (see Lemma 4.1.1 for details). Let Q be the standard parabolic subgroup of GE_7 whose unipotent radical contains U_{α_i} if and only if $i = 4$ or 6. More generally, for $S \subset \{1, 2, 3, 4, 5, 6, 7\}$, let $P_S = M_S U_S$ denote the standard parabolic subgroup whose Levi subgroup M_S contains the root subgroups attached to the simple roots $\{\alpha_i : i \in S\}$ and whose unipotent radical U_S contains the root subgroups attached to the simple roots $\{\alpha_i : i \notin S\}$. Hence, $P = P_{\{1,3,4,5,6,7\}}$ and $Q = P_{\{1,2,3,5,7\}}$. We also fix once and for all a maximal compact subgroup K of $GE_7(\mathbb{A})$.

We shall also consider automorphic representations of the split exceptional group G_2. We denote the long simple root of G_2 by β and the short one by α. For $\gamma \in \{\beta, \alpha\}$ we let P_γ denote the maximal parabolic subgroup of G_2 whose Levi M_γ contains the root subgroup U_γ attached to γ. We let N_γ denote the unipotent radical of P_γ.

Let $\mathfrak{g}_2$ and $\mathfrak{gl}_7$ be the Lie algebras of G_2 and GL_7, respectively. Following [11] we embed $\mathfrak{g}_2$ into $\mathfrak{gl}_7$ by letting it act on a seven-dimensional vector space. We order the basis vectors as follows: $v_4, v_3, v_1, u, w_1, w_3, w_4$. Then it follows from the formulae in [11, p. 354] that the matrices of Y_1 and Y_2 (using notation [11, p. 340]) are

$$\begin{pmatrix} 0 &&&&&& \\ 1 & 0 &&&&& \\ & 0 & 0 &&&& \\ && 1 & 0 &&& \\ &&& 2 & 0 && \\ &&&& 0 & 0 & \\ &&&&& -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 &&&&&& \\ 0 & 0 &&&&& \\ & -1 & 0 &&&& \\ && 0 & 0 &&& \\ &&& 0 & 0 && \\ &&&& 1 & 0 & \\ &&&&& 0 & 0 \end{pmatrix},$$

respectively. The matrices attached to H_1 and H_2 are easily computed by looking at the images of H_1 and H_2 under the weights.

Weight	H_1	H_2
α	2	−1
β	−3	2
$\omega_1 = 2\alpha + \beta$	1	0
$\omega_1 - \alpha$	−1	1
$\omega_1 - \alpha - \beta$	2	−1
$\omega_1 - 2\alpha - \beta$	0	0
$\omega_1 - 3\alpha - \beta$	−2	1
$\omega_1 - 3\alpha - 2\beta$	1	−1
$\omega_1 - 4\alpha - 2\beta$	−1	0

The matrices are

$$\begin{pmatrix} 1 &&&&&& \\ & -1 &&&&& \\ && 2 &&&& \\ &&& 0 &&& \\ &&&& -2 && \\ &&&&& 1 & \\ &&&&&& -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 &&&&&& \\ & 1 &&&&& \\ && -1 &&&& \\ &&& 0 &&& \\ &&&& 1 && \\ &&&&& -1 & \\ &&&&&& 0 \end{pmatrix},$$

respectively. Finding the action of X_1 and X_2 takes a little work. In some cases, we use our knowledge about the set of weights. For example $X_1 w_3$ must be zero because w_3 is weight $\omega_1 - 3\alpha - 2\beta$ and $\omega_1 - 2\alpha - 2\beta$ is not a weight of this representation. For the others we use our knowledge of the action of Y_1, Y_2, H_1, H_2, and bracket relations. For example, since $X_1 v_4 = 0$, it follows that

$$X_1 v_3 = X_1 Y_1 v_4 = (H_1 + Y_1 X_1) v_4 = H_1 v_4 = v_4.$$

After similar computations we find that the matrices of X_1 and X_2 are

$$\begin{pmatrix} 0 & 1 &&&&& \\ & 0 & 0 &&&& \\ && 0 & 2 &&& \\ &&& 0 & 1 && \\ &&&& 0 & 0 & \\ &&&&& 0 & -1 \\ &&&&&& 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 &&&&& \\ & 0 & -1 &&&& \\ && 0 & 0 &&& \\ &&& 0 & 0 && \\ &&&& 0 & 1 & \\ &&&&& 0 & 0 \\ &&&&&& 0 \end{pmatrix},$$

respectively. Finally, for a matrix g we denote the transpose by ${}^t g$. When g is a square matrix, we also denote by ${}_t g$ the transpose about the second diagonal, which may be obtained by conjugating ${}^t g$ by the matrix $\begin{pmatrix} && 1 \\ & \iddots & \\ 1 && \end{pmatrix}$, i.e., with ones from lower left corner to upper right corner and zeros elsewhere.

3. Preliminaries and some general results

3.1. Fourier coefficients attached to nilpotent orbits

In this section, we recall Fourier coefficients of automorphic forms attached to nilpotent orbits, following the formulation in [17]. Let G be a reductive group defined over F, or a central extension of finite degree. Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Let $\mathfrak{g}$ be the Lie algebra of $G(F)$ and u be a nilpotent element in $\mathfrak{g}$. The element u defines a function on $\mathfrak{g}(\mathbb{A})$:

$$\psi_u : \mathfrak{g}(\mathbb{A}) \to \mathbb{C}^\times$$

by $\psi_u(x) = \psi(\kappa(u, x))$, where κ is a G-invariant symmetric bilinear form on $\mathfrak{g}(\mathbb{A})$ which is nondegenerate on every simple summand of $\mathfrak{g}$ (such as the Killing form, or a convenient scalar multiple).

Given any semisimple element $s \in \mathfrak{g}$, under the adjoint action, $\mathfrak{g}$ is decomposed into a direct sum of eigenspaces $\mathfrak{g}^s_i$ of h corresponding to eigenvalues i. For any $r \in \mathbb{Q}$, let $\mathfrak{g}^s_{\geq r} = \bigoplus_{r' \geq r} \mathfrak{g}^s_{r'}$. The element s is called *rational semisimple* if all its eigenvalues are in $\mathbb{Q}$. Given a nilpotent element u, a *Whittaker pair* is a pair (s, u) with $s \in \mathfrak{g}$ being a rational semisimple element, and $u \in \mathfrak{g}^s_{-2}$. The element s in a Whittaker pair (s, u) is called a *neutral element* for u if there is a nilpotent element $v \in \mathfrak{g}$ such that (v, s, u) is an $\mathfrak{sl}_2$-triple in this case we call (s, u) a *neutral pair*. For any $X \in \mathfrak{g}$, let $\mathfrak{g}_X$ be the centralizer of X in $\mathfrak{g}$.

Given any Whittaker pair (s, u), define an anti-symmetric form ω_u on $\mathfrak{g}$ by $\omega_u(X, Y) := \kappa(u, [X, Y])$. Let $\mathfrak{u}_s = \mathfrak{g}^s_{\geq 1}$ and let $\mathfrak{n}_{s,u} = \ker(\omega_u)$ be the radical of $\omega_u|_{\mathfrak{u}_s}$. Then $[\mathfrak{u}_s, \mathfrak{u}_s] \subset \mathfrak{g}^s_{\geq 2} \subset \mathfrak{n}_{s,u}$. By [17, Lemma 3.2.6], $\mathfrak{n}_{s,u} = \mathfrak{g}^s_{\geq 2} + \mathfrak{g}^s_1 \cap \mathfrak{g}_u$. Note that if the Whittaker pair (s, u) comes from an $\mathfrak{sl}_2$-triple (v, s, u), then $\mathfrak{n}_{s,u} = \mathfrak{g}^s_{\geq 2}$. Let $U_s = \exp(\mathfrak{u}_s)$ and $N_{s,u} = \exp(\mathfrak{n}_{s,u})$ be the corresponding unipotent subgroups of G. Abusing notation, we define a character of $N_{s,u}$ by $\psi_u(n) = \psi(\kappa(u, \log(n)))$. Let $N'_{s,u} = N_{s,u} \cap \ker(\psi_u)$. Then $U_s/N'_{s,u}$ is a Heisenberg group with center $N_s/N'_{s,u}$. It follows that for each Whittaker pair (s, u), ψ_u defines a character of $N_{s,u}(\mathbb{A})$ which is trivial on $N_{s,u}(F)$. Let $\mathfrak{m}_s = \mathfrak{g}^s_0$ and $M_s = \exp(\mathfrak{m}_s)$. Then $P_s = M_s U_s$ is a parabolic subgroup of G with Levi subgroup M_s and unipotent radical U_s.

Assume that π is an automorphic representation of $G(\mathbb{A})$. Define a *degenerate Whittaker–Fourier coefficient* of $\varphi \in \pi$ by

$$\mathcal{F}_{s,u}(\varphi)(g) = \int_{N_{s,u}(F)\backslash N_{s,u}(\mathbb{A})} \varphi(ng)\overline{\psi_u(n)}\, dn, \quad g \in G(\mathbb{A}). \tag{3.1.1}$$

Let $\mathcal{F}_{s,u}(\pi) = \{\mathcal{F}_{s,u}(\varphi) : \varphi \in \pi\}$. If s is a neutral element for u, then $\mathcal{F}_{s,u}(\varphi)$ is also called a *generalized Whittaker–Fourier coefficient* of φ. We are interested in the collection of neutral pairs (s, u) such that $\mathcal{F}_{s,u}(\varphi) \neq 0$. It is easy to see that this set is preserved by the natural action of $G(F)$ on $\mathfrak{g} \times \mathfrak{g}$. We shall refer to an orbit for the action of $G(F)$ on the nilpotent subvariety of $\mathfrak{g}$ as a *rational nilpotent orbit.* By a *stable nilpotent orbit* we shall mean the intersection of $\mathfrak{g}$ with a $G(\overline{F})$-orbit in $\mathfrak{g} \otimes_F \overline{F}$, where $\overline{F}$ is the algebraic closure of F. The (global) *wave-front set* $\mathfrak{n}(\pi)$ of π is defined to be the set of rational nilpotent orbits $\mathcal{O}$ such that $\mathcal{F}_{s,u}(\pi)$ is nonzero for some Whittaker pair (s, u) with $u \in \mathcal{O}$ and s being a neutral element for u. Note that if $\mathcal{F}_{s,u}(\pi)$ is nonzero for some Whittaker pair (s, u) with $f \in \mathcal{O}$ and s being a neutral element for u, then it is nonzero for any such Whittaker pair (s, u), since the non-vanishing property of such Fourier coefficients does not depend on the choice of representatives of $\mathcal{O}$. Let $\mathfrak{n}^m(\pi)$ be the set of maximal elements in $\mathfrak{n}(\pi)$ under the natural order of nilpotent orbits.

Assume that π is an admissible representation of $G(F_v)$, where v is a finite place of F. Then similarly we can define a twisted Jacquet module of π by $\mathcal{J}_{N_{s,u},\psi_u}(\pi)$ and consider the (local) wave-front set $\mathfrak{n}(\pi)$ and the subset $\mathfrak{n}^m(\pi)$.

The following theorem is one of the main results in [17].

Theorem 3.1.2 ([17, Theorem C]). *Let π be an automorphic representation of $G(\mathbb{A})$. Given two Whittaker pairs (s, u) and (s', u), with s being a neutral element for u, if $\mathcal{F}_{s',u}(\pi)$ is nonzero, then $\mathcal{F}_{s,u}(\pi)$ is also nonzero.*

In the following, we prove a slightly generalized version of Theorem 3.1.2 using similar arguments.

Assume that (s, u) and (s', u) are two Whittaker pairs with the same u such that $\mathfrak{g}_u \cap \mathfrak{g}^s_{\geq 1} \subset \mathfrak{g}^{s'}_{\geq 1}$. Let $z = s' - s \in \mathfrak{g}_u$. And for any rational number $0 \leq t \leq 1$, let $s_t = s + tz$, $\mathfrak{u}_t = \mathfrak{g}^{s_t}_{\geq 1}$, $\mathfrak{v}_t = \mathfrak{g}^{s_t}_{>1}$, and $\mathfrak{w}_t = \mathfrak{g}^{s_t}_1$. The number t is called *regular* if $\mathfrak{u}_t = \mathfrak{u}_{t+\epsilon}$ for any small enough $\epsilon \in \mathbb{Q}$; and t is called *critical* if it is not regular. For convenience, we say that 0 is critical and 1 is regular. Fix a Lagrangian $\mathfrak{m} \subset \mathfrak{g}^z_0 \cap \mathfrak{g}^s_1$ and let

$$\mathfrak{l}_t = \mathfrak{m} + (\mathfrak{w}_t \cap \mathfrak{g}^z_{<0}) + \mathfrak{v}_t + (\mathfrak{w}_t \cap \mathfrak{g}_u),$$
$$\mathfrak{r}_t = \mathfrak{m} + (\mathfrak{w}_t \cap \mathfrak{g}^z_{>0}) + \mathfrak{v}_t + (\mathfrak{w}_t \cap \mathfrak{g}_u).$$

Note that $\mathfrak{l}_t$ and $\mathfrak{r}_t$ defined here agree with those in [17] by applying [17, Lemma 3.2.6]. For $i, j \in \mathbb{Q}$, let

$$\mathfrak{g}_{i,j} = \{X \in \mathfrak{g} : [s, X] = iX, \ [z, X] = jX\}.$$

Then one can see that $\mathfrak{w}_t = \bigoplus_{i+tj=1} \mathfrak{g}_{i,j}$, $\mathfrak{v}_t = \bigoplus_{i+tj>1} \mathfrak{g}_{i,j}$, t is a critical number if and only if there exists (i, j) such that $i + tj = 1$ and $j \neq 0$, and t is a regular number if and only if $\mathfrak{w}_t = \mathfrak{g}_{1,0} = \mathfrak{g}^z_0 \cap \mathfrak{g}^s_1$. And we can rewrite $\mathfrak{l}_t$ and $\mathfrak{r}_t$ as follows:

$$\mathfrak{l}_t = \mathfrak{m} + \bigoplus_{i+tj=1,\, j<0} \mathfrak{g}_{i,j} + \mathfrak{v}_t + \Big(\bigoplus_{i+tj=1,\, j>0} \mathfrak{g}_{i,j}\Big) \cap \mathfrak{g}_u + \mathfrak{g}_{1,0} \cap \mathfrak{g}_u, \tag{3.1.3}$$

$$\mathfrak{r}_t = \mathfrak{m} + \bigoplus_{i+tj=1,\, j>0} \mathfrak{g}_{i,j} + \mathfrak{v}_t + \Big(\bigoplus_{i+tj=1,\, j<0} \mathfrak{g}_{i,j}\Big) \cap \mathfrak{g}_u + \mathfrak{g}_{1,0} \cap \mathfrak{g}_u. \tag{3.1.4}$$

We summarize the results in [17, Lemma 3.2.7] in the following lemma.

Lemma 3.1.5 ([17, Lemma 3.2.7]). *Assume that (s, u) and (s', u) are two Whittaker pairs with the same u such that $\mathfrak{g}_u \cap \mathfrak{g}^s_{\geq 1} \subset \mathfrak{g}^{s'}_{\geq 1}$. Then the following properties hold.*

(1) *For any $t \geq 0$, $\mathfrak{l}_t$ and $\mathfrak{r}_t$ are maximal isotropic subspaces of $\mathfrak{u}_t$ and $[\mathfrak{l}_t, \mathfrak{r}_t] \subset \mathfrak{l}_t \cap \mathfrak{r}_t$. And*

$$\mathfrak{u}_t/\ker(\omega_u|_{\mathfrak{u}_t}) = \mathfrak{w}_t/(\mathfrak{w}_t \cap \mathfrak{g}^z_0 + \mathfrak{w}_t \cap \mathfrak{g}_u)$$

defines a symplectic structure, with the image of $\mathfrak{l}_t$ and $\mathfrak{r}_t$ being two complementary Lagrangians.

(2) *Suppose that $0 \leq t < t'$, and that all the elements in the open interval (t, t') are regular. Then $\mathfrak{r}_t \subset \mathfrak{l}_{t'}$.*

In the following lemma, we analyze the precise structure of $\mathfrak{l}_{t'}/\mathfrak{r}_t$, in the situation of Lemma 3.1.5 (2).

Lemma 3.1.6. *Assume that (s, u) and (s', u) are two Whittaker pairs with the same u such that $\mathfrak{g}_u \cap \mathfrak{g}^s_{\geq 1} \subset \mathfrak{g}^{s'}_{\geq 1}$. Suppose that $0 \leq t < t'$, and that all the elements in the open interval (t, t') are regular. Then, $\mathfrak{l}_{t'}/\mathfrak{r}_t = (\bigoplus_{i+t'j=1,\, j>0} \mathfrak{g}_{i,j}) \cap \mathfrak{g}_u$, preserving ψ_u.*

Proof. By (3.1.3) and (3.1.4),

$$\mathfrak{l}_{t'} = \mathfrak{m} + \bigoplus_{i+t'j=1,\, j<0} \mathfrak{g}_{i,j} + \mathfrak{v}_{t'} + \Big(\bigoplus_{i+t'j=1,\, j>0} \mathfrak{g}_{i,j}\Big) \cap \mathfrak{g}_u + \mathfrak{g}_{1,0} \cap \mathfrak{g}_u. \tag{3.1.7}$$

$$\mathfrak{r}_t = \mathfrak{m} + \bigoplus_{i+tj=1,\, j>0} \mathfrak{g}_{i,j} + \mathfrak{v}_t + \Big(\bigoplus_{i+tj=1,\, j<0} \mathfrak{g}_{i,j}\Big) \cap \mathfrak{g}_u + \mathfrak{g}_{1,0} \cap \mathfrak{g}_u. \tag{3.1.8}$$

Since $0 \le t < t'$, and all the elements in the open interval (t, t') are regular, one can see that

$$\bigoplus_{i+tj=1,\, j>0} \mathfrak{g}_{i,j} + \mathfrak{v}_t = \bigoplus_{i+t'j=1,\, j<0} \mathfrak{g}_{i,j} + \mathfrak{v}_{t'}.$$

Therefore,

$$\mathfrak{l}_{t'} + \Big(\bigoplus_{i+tj=1,\, j<0} \mathfrak{g}_{i,j}\Big) \cap \mathfrak{g}_u = \mathfrak{r}_t + \Big(\bigoplus_{i+t'j=1,\, j>0} \mathfrak{g}_{i,j}\Big) \cap \mathfrak{g}_u.$$

Note that if $i + tj = 1$ and $j < 0$, then $i + j < 1$. Hence, $\bigoplus_{i+tj=1,\, j<0} \mathfrak{g}_{i,j} \subset \mathfrak{g}^{s'}_{<1}$. Since $\mathfrak{g}_u \cap \mathfrak{g}^{s}_{\ge 1} \subset \mathfrak{g}^{s'}_{\ge 1}$, $(\bigoplus_{i+tj=1,\, j<0} \mathfrak{g}_{i,j}) \cap \mathfrak{g}_u = \{0\}$. Therefore, $\mathfrak{l}_{t'}/\mathfrak{r}_t = (\bigoplus_{i+t'j=1,\, j>0} \mathfrak{g}_{i,j}) \cap \mathfrak{g}_u$, preserving ψ_u.

This completes the proof of the lemma. ■

For a Whittaker pair (s, u), let $\mathfrak{l}_s \subset \mathfrak{u}_s$ be any maximal isotropic subalgebra with respect to the form ω_u. And let $L_s = \exp(\mathfrak{l}_s)$. Then ψ_u can be extended trivially to a character of $L_s(k)\backslash L_s(\mathbb{A})$. Let π be an automorphic representation of $G(\mathbb{A})$. Define the following Fourier coefficient of $f \in \pi$:

$$\mathcal{F}^{L_s}_{s,u}(f)(g) = \int_{L_s(k)\backslash L_s(\mathbb{A})} f(ng)\overline{\psi_u(n)}\, dn, \quad g \in G(\mathbb{A}). \tag{3.1.9}$$

Let $\mathcal{F}^{L_s}_{s,u}(\pi) = \{\mathcal{F}^{L_s}_{s,u}(f) : f \in \pi\}$.

Next, we recall a lemma.

Lemma 3.1.10 ([17, Lemma 6.0.2]). *Let π be an automorphic representation of $G(\mathbb{A})$. Then $\mathcal{F}_{s,u}(\pi) \ne 0$ if and only if $\mathcal{F}^{L_s}_{s,u}(\pi) \ne 0$.*

The next theorem is the global analogue of [17, Corollary 3.0.3] with essentially the same proof. To be complete, we sketch it in the following.

Theorem 3.1.11. *Let π be an automorphic representation of $G(\mathbb{A})$. Assume that (s, u) and (s', u) are two Whittaker pairs with the same u such that $\mathfrak{g}_u \cap \mathfrak{g}^{s}_{\ge 1} \subset \mathfrak{g}^{s'}_{\ge 1}$. If $\mathcal{F}_{s',u}(\pi)$ is nonzero, then $\mathcal{F}_{s,u}(\pi)$ is also nonzero.*

Proof. Let (s, u) and (s', u) be two Whittaker pairs as in the statement. Then it is clear that $s' - s \in \mathfrak{g}_u$.

Let $t_0 = 0 < t_1 < \cdots < t_k$ be all the critical numbers. Let $t_{k+1} = 1$. Then, for $0 \le i \le k$, all the rational numbers in the open interval (t_i, t_{i+1}) are regular. Let $R_{t_i} = \exp(\mathfrak{r}_{t_i})$ and $L_{t_{i+1}} = \exp(\mathfrak{l}_{t_{i+1}})$. Assume that $\mathcal{F}^{R_{t_{i+1}}}_{s_{t_{i+1}},u}(\pi) \ne 0$; then we have $\mathcal{F}^{L_{t_{i+1}}}_{s_{t_{i+1}},u}(\pi) \ne 0$ by Lemma 3.1.10. By Lemma 3.1.5, $\mathfrak{r}_{t_i} \subset \mathfrak{l}_{t_{i+1}}$, and by Lemma 3.1.6,

$\mathfrak{l}_{t_{i+1}}/\mathfrak{r}_{t_i} = (\bigoplus_{\ell+t_{i+1}j=1,\, j>0} \mathfrak{g}_{\ell,j}) \cap \mathfrak{g}_u \subset \mathfrak{w}_{t_{i+1}} \cap \mathfrak{g}_u$, which is abelian and normalizes ψ_u. Then it is clear that $\mathcal{F}^{R_{t_i}}_{s_{t_i},u}(\pi) \neq 0$.

Note that $\mathcal{F}^{R_{t_{k+1}}}_{s_{t_{k+1}},u}(\pi) = \mathcal{F}_{s',u}(\pi) \neq 0$. Therefore, by the above discussion, $\mathcal{F}_{s,u}(\pi) = \mathcal{F}_{s_{t_0},u}(\pi) = \mathcal{F}^{R_{t_0}}_{s_{t_0},u}(\pi) \neq 0$. This completes the proof of the theorem. ■

3.2. A few general results

Before we turn to matters that are specific to the problem of descent from GL_7 to G_2 by way of GE_7, we would like to present some results in a general setting. These are related to the general problem of computing the twisted Jacquet module

$$\mathcal{J}_{U,\psi_U}(\mathrm{Ind}_Q^G \chi),$$

where G is a reductive p-adic group, Q is a parabolic subgroup of G, U is a subgroup of the unipotent radical of a second parabolic subgroup, P, of G, U is normalized by P, χ is a character of Q, and ψ_U is a character of U. In this direction, the most general result of which we are aware is [2, Theorem 5.2]. This result considers a set-up which is more general than the one we shall consider here, but it has the defect that one must check a certain finiteness condition which, for many applications, is unnecessary.

The group P acts on the space of characters of U by $p \cdot \psi_U(u) = \psi_U(p^{-1}up)$. In fact, this action may be realized as the rational representation of P dual to its action on $U/(U,U)$. Let R_{ψ_U} denote the stabilizer of ψ_U in P. Then for any admissible representation π of G, the twisted Jacquet module $\mathcal{J}_{U,\psi_U}(\pi)$ has the structure of an R_{ψ_U}-module.

We assume that G is equipped with a choice of minimal parabolic subgroup P_0 and that P and Q are both standard, i.e., both contain P_0. We also choose a maximal split torus T_0 contained in P_0. The space $\mathrm{Ind}_Q^G \chi$ has a filtration by P-modules I_w indexed by the elements of $Q\backslash G/P$. As representatives, we choose minimal-length elements of the relative Weyl group. The P-module I_w corresponding to w may be realized as $\text{c-ind}^P_{P\cap w^{-1}Qw} \chi\delta_Q^{1/2} \circ \mathrm{Ad}(\dot{w})$, where $\dot{w}$ is any representative for w in G and c-ind is the compact induction.

We say that $p \in P$ is *w-admissible* if $p \cdot \psi_U$ is trivial on $U \cap w^{-1}Qw$. (Clearly this is relative to ψ_U and Q.)

Lemma 3.2.1. *For each w, the set of w-admissible elements is a subvariety of P.*

Proof. Write $[U/(U,U)]^*$ for the rational representation of P that is dual to $U/(U,U)$. Then ψ_U corresponds to an element X of $[U/(U,U)]^*(F)$. Let V denote the image of $U \cap w^{-1}Qw$ in $U/(U,U)$. Then p is w-admissible if and only if $\langle \mathrm{Ad}(p).X, v\rangle = 0$ for all $v \in V$. Here $\langle\,,\rangle$ is the canonical pairing between $U/(U,U)$ and $[U/(U,U)]^*$. Taking a basis of V we obtain a finite number of polynomial conditions in p which define the w-admissible subvariety. ■

Now fix w and let X_w denote the open subset of w-inadmissible elements in P. Let I_w^o denote $\{f \in I_w : \mathrm{supp}(f) \subset X_w\}$. Then I_w^o is an R_{ψ_U}-submodule of I_w. Let $\overline{I}_w$

denote the quotient, so we have a short exact sequence of R_{ψ_U}-modules

$$0 \to I_w^o \to I_w \to \overline{I}_w \to 0.$$

Lemma 3.2.2.

$$\mathcal{J}_{U,\psi_U}(I_w^o) = 0, \quad \textit{hence} \quad \mathcal{J}_{U,\psi_U}(I_w) \cong \mathcal{J}_{U,\psi_U}(\overline{I}_w).$$

Proof. In general, for an admissible representation (π, V) of P the kernel of the map $V \to \mathcal{J}_{U,\psi_U}(V)$ is the subspace of elements v such that

$$\int_N \overline{\psi_U(n)}\pi(n).v\, dn = 0$$

for some compact subgroup N of U. In the case of an induced representation, this is equivalent to

$$\int_N f(pn)\overline{\psi_U(n)}\, dn = 0 \quad \forall p \in P.$$

For each fixed p,

$$\int_N f(pn)\overline{\psi_U(n)}\, dn = \int_N f(np)\overline{p \cdot \psi_U(n)}\, dn,$$

where $p \cdot \psi_U(u) = \psi_U(p^{-1}up)$. It is clear that if $p \cdot \psi_U$ is nontrivial on $U \cap w^{-1}Qw$, then this integral will be zero for all sufficiently large N, and if $f \in I_w^o$, then this holds for all p in the support of f. We need to show that N can be chosen independently of p. This follows because $p \cdot \psi_U$ depends continuously on p and the support of f is compact modulo $P \cap w^{-1}Qw$. ■

For each w in our set of representatives for $Q\backslash G/P$ let $P_w = P \cap w^{-1}Qw$. Note that the w-admissible subvariety of P is a union of P_w, R_{ψ_U}-double cosets.

Lemma 3.2.3. *Assume that w-admissible subvariety of P is a single P_w, R_{ψ_U}-double coset $P_w x R_{\psi_U}$. Then, as an R_{ψ_U}-module, $\overline{I}_w \cong \text{c-ind}_{R_{\psi_U} \cap x^{-1}w^{-1}Qwx}^{R_{\psi_U}} \chi\delta_Q^{1/2} \circ \text{Ad}(\dot{w}x)$.*

Proof. Recall that I_w^o is the subset of elements of I_w whose support is in the open set X_w of inadmissible elements. So, the canonical quotient map $I_w \to I_w/I_w^o = \overline{I}_w$ may be realized as restriction to the admissible subvariety. Write $\overline{I}_w^{(1)}$ for this realization of $\overline{I}_w$ as a subspace of $C^\infty(P_w x R_{\psi_U})$.

Clearly, each element $f \in \overline{I}_w^{(1)}$ is determined by the function $h_f(r) = f(xr) \in C^\infty(R_{\psi_U})$. Thus we obtain a second realization of $\overline{I}_w$ as a subspace of $C^\infty(R_{\psi_U})$ which we denote $\overline{I}_w^{(2)}$. We claim that $\overline{I}_w^{(2)}$ is precisely $\text{c-ind}_{R_{\psi_U} \cap x^{-1}w^{-1}Qwx}^{R_{\psi_U}} \chi\delta_Q^{1/2} \circ \text{Ad}(\dot{w}x)$.

It is clear that $h_f(pr) = \chi\delta_Q^{1/2}(\dot{w}xpx^{-1}\dot{w}^{-1})h_f(r)$ for each $p \in R_{\psi_U} \cap x^{-1}w^{-1}Qwx$, and $r \in R_{\psi_U}$. Moreover, since $R_{\psi_U} \cap x^{-1}w^{-1}Qwx\backslash R_{\psi_U}$ maps injectively into $P \cap w^{-1}Qw$, the support of h_f will be compact modulo $x^{-1}w^{-1}Qwx$. Thus $\overline{I}_w^{(2)}$ is contained in $\text{c-ind}_{R_{\psi_U}\cap x^{-1}w^{-1}Qwx}^{R_{\psi_U}} \chi\delta_Q^{1/2} \circ \mathrm{Ad}(\dot{w}x)$. What remains is to show that this map from $\overline{I}_w^{(2)}$ to $\text{c-ind}_{R_{\psi_U}\cap x^{-1}w^{-1}Qwx}^{R_{\psi_U}} \chi\delta_Q^{1/2} \circ \mathrm{Ad}(\dot{w}x)$ is surjective.

Given $h \in \text{c-ind}_{R_{\psi_U}\cap x^{-1}w^{-1}Qwx}^{R_{\psi_U}} \chi\delta_Q^{1/2} \circ \mathrm{Ad}(\dot{w}x)$, we can choose a compact open set Ω such that h is supported on $(R_{\psi_U} \cap x^{-1}w^{-1}Qwx)\Omega$, a compact open subgroup K_1 of R_{ψ_U} such that that h is right-K_1-invariant, and a compact open subgroup K_2 of P such that $K_2 \cap R_{\psi_U} = K_1$. Then we can define

$$f(g) = \begin{cases} \chi\delta_Q^{1/2}(\dot{w}q\dot{w}^{-1})h(r), & g = qxrk,\ q \in P_w,\ r \in R_{\psi_U},\ k \in K_2, \\ 0, & g \notin P_w x R_{\psi_U} K_2. \end{cases}$$

■

Using the form κ, the space $[U/(U,U)]^*$ may be identified with a subspace $[U/(U,U)]^-$ of the Lie algebra $\mathfrak{u}_P^-$ of the unipotent radical U_P^- of the parabolic that is opposed to P. It is important to keep in mind that this identification is an isomorphism of M_P-modules, where M_P is the Levi of P, but that it is not an isomorphism of P-modules. More precisely, the form κ gives us a linear isomorphism $\mathfrak{g}_{\text{der}} \to \mathfrak{g}_{\text{der}}^*$ that sends $X \in \mathfrak{g}_{\text{der}}$ to the linear form $Y \mapsto \kappa(X,Y)$. Here, $\mathfrak{g}_{\text{der}}$ is the derived subalgebra of $\mathfrak{g}$. We can decompose $\mathfrak{g}$ into irreducible M_P-submodules and those that are not contained in $\mathfrak{m}_P$ come in dual pairs. More precisely, each irreducible in $\mathfrak{u}_P$ is paired with an irreducible in $\mathfrak{u}_P^-$. The Lie algebra of U is a direct sum of irreducible components in $\mathfrak{u}_P$ so its dual is identified with a subspace of $\mathfrak{u}_P^-$. Then the dual of the quotient $U/(U,U)$ is a subspace of the dual of U. Since (U,U) is M_P-invariant, $[U/(U,U)]^*$ is again a direct sum of irreducible M_P-submodules of $\mathfrak{u}_P^-$. Notice that $X \in [U/(U,U)]^-$ implies $\mathrm{Ad}(m)X \in [U/(U,U)]^-$ for all m in M but not $\mathrm{Ad}(p)X \in [U/(U,U)]^-$ for p in P but not in M.

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{q}^- \oplus \mathfrak{u}_Q$ where $\mathfrak{q}^-$ is the Lie algebra of the parabolic $\mathfrak{q}^-$ opposed to Q and $\mathfrak{u}_Q$ is the Lie algebra of the unipotent radical of Q. Conjugating by w we have also $\mathfrak{g} = \mathrm{Ad}(w^{-1})\mathfrak{q}^- \oplus \mathrm{Ad}(w^{-1})\mathfrak{u}_Q$.

Lemma 3.2.4.

$$[U/(U,U)]^- = ([U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{q}^-) \oplus ([U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{u}_Q).$$

Proof. Let M_Q be the standard Levi factor of Q (containing T_0). Let Z_{M_Q} denote its center, and $A_{M_Q} = Z_{M_Q} \cap T_0$. Because the space $[U/(U,U)]^-$ is preserved by $w^{-1}A_{M_Q}w$, we can decompose $[U/(U,U)]^-$ into eigenspaces of $w^{-1}A_{M_Q}w$. If λ is one of the eigencharacters, then $\lambda \circ \mathrm{Ad}(w)$ is either trivial or a relative root for the torus A_{M_Q}. If it is trivial or negative then the λ-eigenspace lies in $\mathrm{Ad}(w^{-1})\mathfrak{q}^-$ and if it is positive then the λ-eigenspace lies in $\mathrm{Ad}(w^{-1})\mathfrak{u}_Q$.

Take $X \in [U/(U,U)]^-$. Then using this eigenspace decomposition we can write $X = X_1 + X_2$ where $X_1 \in [U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{q}^-)$ and $X_2 \in ([U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{u}_Q)$. ■

Notice that p is w-admissible if and only if the projection of $\mathrm{Ad}(p)X$ onto $[U/(U,U)]^-$ is in $[U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{u}_Q$.

Now write U_P for the unipotent radical of the parabolic P. Inside $[U/(U,U)]^*$ we have the subspace of $[U/(U,U_P)]^*$ of linear forms which corresponds to the space of characters of U that are trivial on (U,U_P). This is an M_P-invariant subspace which we can identify with a subspace $[U/(U,U_P)]^-$ of $[U/(U,U)]^-$.

If $X \in [U/(U,U_P)]^-$ and $p = mu$ with $m \in M_P$ and $u \in U_P$ then the projection of $\mathrm{Ad}(p).X$ onto $[U/(U,U)]^-$ is $\mathrm{Ad}(m).X$. Put differently, if ψ_U is trivial on (U,U_P) then U_P fixes ψ_U, and hence $p \cdot \psi_U = m \cdot \psi_U$.

Assume now that ψ_U is trivial on (U,U_P). Then $p = mu$ is w-admissible if and only if $\mathrm{Ad}(m).X$ is in $[U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{u}_Q$, or, equivalently, if $\mathrm{Ad}(wm)X \in \mathfrak{u}_Q$. In particular, X must be conjugate to an element of the subspace $[U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{u}_Q$.

Corollary 3.2.5. *If ψ_U is trivial on (U,U_P) and the space $[U/(U,U)]^- \cap \mathrm{Ad}(w^{-1})\mathfrak{u}_Q$ does not contain any elements of the orbit of X, then the w-admissible subvariety of P is empty.*

Corollary 3.2.6. *Suppose that ψ_U is trivial on (U,U_P) and the w-admissible subvariety of P is nonzero. Then the nilpotent element X attached to ψ_U is conjugate to an element of $\mathfrak{u}_Q$.*

Corollary 3.2.7. *If ψ_U is trivial on (U,U_P) and the space $\mathfrak{u}_Q$ does not contain any elements of the orbit of X, then the w-admissible subvariety of P is empty for all w, and*

$$\mathcal{J}_{(U,\psi_U)}(\mathrm{Ind}_Q^G(\chi)) = 0.$$

Corollary 3.2.8. *Let $\mathcal{O}$ be the Richardson orbit of Q (the largest stable orbit that intersects $\mathfrak{u}_Q$). Let $\mathcal{O}'$ be a stable orbit that is greater than or not related to $\mathcal{O}$. Let (s,u) be any Whittaker pair with $u \in \mathcal{O}'$. Let $U = \exp(\mathfrak{g}^s_{\geq 2})$. Then*

$$\mathcal{J}_{(U,\psi_u)}(\mathrm{Ind}_Q^G(\chi)) = 0.$$

Proof. Let $P = \exp(\mathfrak{g}^s_{\geq 0})$; then $U_P = \exp(\mathfrak{g}^s_{\geq 1})$. The previous corollary applies to this situation, since $(U,U_P) = \exp(\mathfrak{g}^s_{\geq 3})$ and ψ_u is trivial on it. ∎

Corollary 3.2.9. *Let $\mathcal{O}$ be the Richardson orbit of Q. Let $\mathcal{O}'$ be a stable orbit that is greater than or not related to $\mathcal{O}$. Let (s,u) be any Whittaker pair with $u \in \mathcal{O}'$. Then $\mathcal{J}_{N_{s,u},\psi_u}(\mathrm{Ind}_Q^G\,\chi) = 0$.*

Proof. Define U as in the previous corollary. Then the conclusion follows from the definition of $\mathcal{J}_{N_{s,u},\psi_u}$, because $\mathcal{J}_{N_{s,u},\psi_u}(\pi)$ is a quotient of $\mathcal{J}_{(U,\psi_u)}(\pi)$ for any π. ∎

Remark 3.2.10. (1) Suppose that the weighted Dynkin diagram of $\mathcal{O}$ consists of 0's and 2's (namely $\mathcal{O}$ is even) and let Q be the parabolic whose Levi contains the simple roots labeled 0 and whose unipotent radical contains the simple roots labeled 2. Then $\mathcal{O}$ is the Richardson orbit of Q (see [8, Theorem 7.1.1, Theorem 7.1.6, Corollary 7.1.7]).

(2) Corollary 3.2.9 can also be deduced from the argument in [29, Section II.1.3].

4. The A_6 Levi of GE_7 and Eisenstein series

Recall that $P = MU$ is the standard parabolic subgroup of GE_7 whose unipotent radical contains U_{α_i} if and only if $i = 2$, with Levi subgroup M and unipotent radical U. In this section, we show that this Levi subgroup M which is of type A_6 is isomorphic to $GL_7 \times GL_1$. Then we introduce the Eisentein series associated to P whose residues at $s = 1$ generate a residual representation. This residual representation serves as automorphic kernel of our descent construction.

4.1. The A_6 Levi

Lemma 4.1.1. *The group M is isomorphic to $GL_7 \times GL_1$.*

Proof. Recall that the derived group of a Levi subgroup of a simply connected group is simply connected. In particular, the derived group M_{der} of M is simply connected, semisimple, of type A_6. This means that it is isomorphic to SL_7. To pin down a particular isomorphism we first require that $T \cap M_{\text{der}}$ is mapped to the standard torus of SL_7 (the diagonal elements), and $B \cap M_{\text{der}}$ is mapped to the standard Borel of SL_7 (the upper triangular elements). Any isomorphism satisfying these requirements induces a bijection on the set of simple roots which respects the structure of the root system. There are only two such bijections. For reasons which will become apparent, we choose to map α_7 to the first simple root of SL_7 and α_1 to the last. These conditions determine the isomorphism up to conjugation by an element of $T \cap M_{\text{der}}$. To make it unique, we can use the parametrizations x_α: there is a unique isomorphism $\iota_0 : M_{\text{der}} \to SL_7$ such that

$$x_{\alpha_1}(r) \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 && \\ &&&&& 1 & r \\ &&&&&& 1 \end{pmatrix}, \quad x_{\alpha_3}(r) \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 & r & \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix}, \quad \dots, \quad x_{\alpha_7}(r) \mapsto \begin{pmatrix} 1 & r &&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix}.$$

Now M is the product of its derived group and the maximal torus T. A general element of T is of the form $\prod_{i=1}^{8} t_i^{\alpha_i^\vee}$. Of course $\prod_{i \neq 2,8} t_i^{\alpha_i^\vee}$ lies in M which is mapped to (under ι_0)

$$\begin{pmatrix} t_7 &&&&&& \\ & t_7^{-1}t_6 &&&&& \\ && t_6^{-1}t_5 &&&& \\ &&& t_5^{-1}t_4 &&& \\ &&&& t_4^{-1}t_3 && \\ &&&&& t_3^{-1}t_1 & \\ &&&&&& t_1^{-1} \end{pmatrix}.$$

Since

$$(t_2^{\alpha_2^\vee} t_8^{\alpha_8^\vee})^{\alpha_j} = \begin{cases} t_2^{-1}, & j = 4, \\ t_8^{-1}, & j = 7, \\ 1, & \text{otherwise}, \end{cases}$$

we can extend ι_0 to a homomorphism $\iota_1 : M \to GL_7$ such that

$$\iota_1(t_2^{\alpha_2^\vee}) = \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& t_2 && \\ &&&&& t_2 & \\ &&&&&& t_2 \end{pmatrix}, \quad \iota_1(t_8^{\alpha_8^\vee}) = \begin{pmatrix} t_8^{-1} &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix}.$$

For any $m \in M$, assume that $m = m_0 t_2(m)^{\alpha_2^\vee} t_8(m)^{\alpha_8^\vee}$, where $m_0 \in M_{\text{der}}$. Define the map

$$\iota : M \to GL_7 \times GL_1, \quad m \mapsto (\iota_1(m), t_2(m)),$$

which is a group homomorphism. We claim that ι is an isomorphism between M and $GL_7 \times GL_1$. Indeed, assume that $\iota(m) = (I_7, 1)$. Then $t_2(m) = 1$. Consequently, $\det(\iota_1(m)) = t_8^{-1}(m)$, which is equal to $\det(I_7) = 1$. Hence, $\iota_1(m) = \iota_0(m_0) = I_7$. Since ι_0 is an isomorphism, we see that m_0 is the identity of M. Hence $m = m_0 t_2(m)^{\alpha_2^\vee} t_8(m)^{\alpha_8^\vee}$ is the identity of M. Therefore, ι is an isomorphism. This completes the proof of the lemma. ■

Remark 4.1.2. The inverse of ι can be described explicitly as follows: for $g \in GL_7$ write $g = g_1 \left(\begin{smallmatrix} a^{-1} & \\ & I_6 \end{smallmatrix} \right)$, with $g_1 \in SL_7$; then

$$\iota^{-1}(g, b) = \iota_0^{-1}(g_1) a^{\alpha_8^\vee} b^{\alpha_2^\vee - \alpha_5^\vee - 2\alpha_6^\vee - 3\alpha_7^\vee - 4\alpha_8^\vee}.$$

Remark 4.1.3. The center of GE_7 is the image of $2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee + 2\alpha_8^\vee$.

Remark 4.1.4. Recall that there is a notion of duality on split algebraic groups (by means of their root data) which underlies the definition of the L-group. By this duality, the isomorphism $\iota : M \to GL_7 \times GL_1$ induces a dual isomorphism $\iota^\vee : GL_7 \times GL_1 \to M$.

Remark 4.1.5. For $1 \le i \le 7$ let e_i denote the rational character of the standard maximal torus of GL_7 which maps a matrix to its ith diagonal entry. Treat e_i also as a rational character of $GL_7 \times GL_1$ which is trivial on the second factor and let e_8 denote projection onto the second factor, so that $e_1, \dots, e_8$ is a $\mathbb{Z}$-basis for the lattice of rational characters of the standard maximal torus of $GL_7 \times GL_1$. Let $e_1^*, \dots, e_8^*$ be the dual basis for the lattice of cocharacters. Then we see at once that

$$\alpha_7^\vee = e_1^* - e_2^*, \quad \alpha_6^\vee = e_2^* - e_3^*, \quad \alpha_5^\vee = e_3^* - e_4^*, \quad \alpha_4^\vee = e_4^* - e_5^*, \quad \alpha_3^\vee = e_5^* - e_6^*,$$
$$\alpha_1^\vee = e_6^* - e_7^*, \quad \alpha_2^\vee = e_5^* + e_6^* + e_7^* + e_8^*, \quad \alpha_8^\vee = -e_1^*.$$

4.2. Eisenstein series

Let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ and $\chi : \mathbb{A}^\times \to \mathbb{C}^\times$ a Hecke character. Having fixed above an isomorpism $\iota : M \to GL_7 \times GL_1$, we may regard $\pi \otimes \chi$ as an irreducible cuspidal automorphic representation of $M(\mathbb{A})$. Restriction maps the lattice $X(M)$ of rational characters of M isomorphically onto a subgroup of the lattice $X(T)$ of rational characters of T. This sublattice is generated by the second and eighth fundamental weights ϖ_2 and ϖ_8. We denote their preimages in $X(M)$ by $\widetilde{\varpi}_2$ and $\widetilde{\varpi}_8$. Then $\widetilde{\varpi}_8$ extends to a generator for the lattice of rational characters of GE_7 itself. Abusing notation, we still denote this extension by $\widetilde{\varpi}_8$. Let P be the standard parabolic whose Levi is M. We consider the family of induced representations $\mathrm{Ind}_{P(\mathbb{A})}^{GE_7(\mathbb{A})}(\pi \otimes \chi) \cdot |\widetilde{\varpi}_2|^s$, $s \in \mathbb{C}$ (normalized induction), and the corresponding space of Eisenstein series.

Lemma 4.2.1. *The ratio of products of partial L-functions appearing in the constant term of these Eisenstein series is*

$$\frac{L^S(s, \pi \otimes \chi, \wedge^3 \times \mathrm{St}) L^S(2s, \widetilde{\pi} \otimes \chi^2\omega_\pi, \mathrm{St} \times \mathrm{St})}{L^S(s+1, \pi \otimes \chi, \wedge^3 \times \mathrm{St}) L^S(2s+1, \widetilde{\pi} \otimes \chi^2\omega_\pi, \mathrm{St} \times \mathrm{St})} \tag{4.2.2}$$

Proof. This is standard from the Gindikin–Karpalevic formula and the L-group formalism. The Lie algebra of the unipotent radical of the parabolic $P^\vee$ is a direct sum of two irreducible $M^\vee$-submodules. The highest weights correspond to the coroots $\alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee + 3\alpha_4^\vee + 3\alpha_5^\vee + 2\alpha_6^\vee + \alpha_7^\vee$, and $2\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_5^\vee + 2\alpha_6^\vee + \alpha_7^\vee$. We must view the corresponding coroots as weights on the maximal torus of $GL_7(\mathbb{C}) \times GL_1(\mathbb{C})$. In terms of the basis $e_1^*, \dots, e_8^*$ these two cocharacters are $e_1^* + e_2^* + e_3^* + e_8^*$ and $e_1^* + e_2^* + e_3^* + e_4^* + e_5^* + e_6^* + 2e_8^*$, respectively. The highest weight of $\wedge^3$ is $e_1^* + e_2^* + e_3^*$, and projection to the GL_1 factor is e_8^* and determinant of the GL_7 factor. The weight $e_1^* + e_2^* + e_3^* + e_4^* + e_5^* + e_6^*$ is the highest weight of the $\wedge^6$ representation, which can also be regarded as the dual to the standard representation twisted by the determinant. ∎

Let $w_0 = w[243154234565423143542765423143542654376542]$, which is the longest Weyl word which is reduced by the Weyl group of GL_7 on both the left and the right. By [30, II.1.7] the constant term of the Eisenstein series applied to a section f of the induced space is given by $f + M(w_0).f$, where $M(w_0)$ is the standard intertwining operator as in [30, II.1.6]. By [30, IV.1.11], $M(w_0).f$ can have at most a simple pole at $s = 1$. By [26, (3.1) and (3.5, c)], it follows that (4.2.2) can have at most a simple pole at $s = 1$.

Since the standard L-functions of cuspidal representations of $GL(n)$ are nonzero on the half-plane $\mathrm{Re}(s) > 1$ (see [24, Theorem 5.3]) and are entire on the whole complex plane, $\frac{L^S(2s,\widetilde{\pi}\otimes\chi^2\omega_\pi,\mathrm{St}\times\mathrm{St})}{L^S(2s+1,\widetilde{\pi}\otimes\chi^2\omega_\pi,\mathrm{St}\times\mathrm{St})}$ has no pole and no zero at $s = 1$. So, $\frac{L^S(s,\pi\otimes\chi,\wedge^3\times\mathrm{St})}{L^S(s+1,\pi\otimes\chi,\wedge^3\times\mathrm{St})}$ has at most a simple pole at $s = 1$. Moreover, from [26, (3.5, b)] a pole of the intertwining operator in the half-plane $\mathrm{Re}(s) \geq 1$ must come from $\frac{L^S(s,\pi\otimes\chi,\wedge^3\times\mathrm{St})}{L^S(s+1,\pi\otimes\chi,\wedge^3\times\mathrm{St})}$.

Proposition 4.2.3. *If the Eisenstein series has a pole in the half-plane* $\mathrm{Re}(s) > 0$, *then the residual representation is square integrable.*

Proof. This is an easy application of the square integrability in [30, I.4.11]. ∎

According to [27, Lemma 7.5], the Eisenstein series can have a square integrable residue only if $\pi \otimes \chi \circ \mathrm{Ad}(\dot{w}_0) \cong \pi \otimes \chi$. We investigate what this condition says explicitly about π and χ.

Lemma 4.2.4. *There is a representative $\dot{w}_0$ for w_0 such that the automorphism of $GL_7 \times GL_1$ induced by* $\mathrm{Ad}(\dot{w}_0)$ *and our choice of isomorphism $M \to GL_7 \times GL_1$ is*

$$(g,a) \mapsto \left({}_t g^{-1} \frac{a^3}{\det g}, \frac{a^8}{(\det g)^3} \right),$$

where ${}_t g$ is defined at the end of Section 2.

Proof. For any choice of representative, $\mathrm{Ad}(\dot{w}_0)$ induces an automorphism of $GL_7 \times GL_1$ which preserves the chosen torus and Borel. When such an automorphism is restricted to SL_7 there are two possibilities: either it is given by conjugation by an element of the torus of GL_7 (in which case we can adjust the representative $\dot{w}_0$ to make it trivial), or else it is given by $g \mapsto {}_t g^{-1}$ composed with conjugation by an element of the torus of GL_7 (in which case we can adjust the representative $\dot{w}_0$ to make it $g \mapsto {}_t g^{-1}$).

By inspecting the action of w_0 on the simple coroots, one can see that $\mathrm{Ad}(\dot{w}_0)$ maps $h(t_1, \dots, t_8)$ to

$$h\left(\frac{t_7 t_8}{t_2}, \frac{t_8^3}{t_2}, \frac{t_6 t_8^3}{t_2^3}, \frac{t_5 t_8^5}{t_2^3}, \frac{t_4 t_8^4}{t_2^3}, \frac{t_3 t_8^3}{t_2^2}, \frac{t_1 t_8^2}{t_2}, t_8 \right).$$

If we push this through the isomorphism with $GL_7 \times GL_1$, it becomes

$$\left(\begin{pmatrix} \frac{t_7}{t_8} &&&&&& \\ & \frac{t_6}{t_7} &&&&& \\ && \frac{t_5}{t_6} &&&& \\ &&& \frac{t_4}{t_5} &&& \\ &&&& \frac{t_2 t_3}{t_4} && \\ &&&&& \frac{t_1 t_2}{t_3} & \\ &&&&&& \frac{t_2}{t_1} \end{pmatrix}, t_2 \right) \mapsto \left(\begin{pmatrix} \frac{t_1 t_8}{t_2} &&&&&& \\ & \frac{t_3 t_8}{t_1 t_2} &&&&& \\ && \frac{t_4 t_8}{t_3 t_2} &&&& \\ &&& \frac{t_5 t_8}{t_4} &&& \\ &&&& \frac{t_6 t_8}{t_5} && \\ &&&&& \frac{t_7 t_8}{t_6} & \\ &&&&&& \frac{t_8^2}{t_7} \end{pmatrix}, \frac{t_8^3}{t_2} \right).$$

We see that on the torus of SL_7 (obtained by setting $t_2 = t_8 = 1$) this agrees with $g \mapsto {}_t g^{-1}$. In general, it can be expressed as $(t, t_2) \mapsto ({}_t t^{-1} t_8, t_8^3/t_2)$, and t_8 can be expressed as $t_2^3/\det g$. ∎

Corollary 4.2.5. *If η is a character, write $\eta \cdot \pi$ for the twist of π by $\eta \circ$* det. *Then for any π, χ we have*

$$\pi \otimes \chi \circ \mathrm{Ad}(\dot{w}_0) \cong (\omega_\pi^{-1} \chi^{-3} \cdot \tilde{\pi}) \otimes (\omega_\pi^3 \chi^8).$$

Corollary 4.2.6. *If*

$$\pi \otimes \chi \circ \mathrm{Ad}(\dot{w}_0) \cong \pi \otimes \chi,$$

then there is a self-contragredient cuspidal representation π_0 with trivial central character, and a character η such that $\pi \cong \eta^{-1}\pi_0$ and $\chi = \eta^3$.

Proof. If $\chi = \omega_\pi^3 \chi^8$ then $\omega_\pi^3 = \chi^{-7}$, so $\chi = (\omega_\pi \chi^2)^{-3}$. Setting $\eta = \omega_\pi^{-1} \chi^{-2}$, we have $\chi = \eta^3$ and $\omega_\pi = \chi^{-2} \eta^{-1} = \eta^{-7}$. Then $\omega_\pi^{-1} \chi^{-3} \cdot \tilde{\pi} = \eta^{-2} \tilde{\pi}$. If this is isomorphic to π then $\pi_0 := \pi \otimes \eta$ is self-contragredient with trivial central character. ∎

Remark 4.2.7. $L^S(s, \eta^{-1}\pi_0 \otimes \eta^3, \wedge^3 \otimes \mathrm{St}) = L^S(s, \pi_0, \wedge^3)$.

Remark 4.2.8. If a representation π of GL_7 is self-contragredient, then $L^S(s, \pi, \mathrm{sym}^2)$ has a simple pole at $s = 1$. Indeed, each self-contragredient representation of GL_n is of either orthogonal type ($L^S(s, \pi_0, \mathrm{sym}^2)$ has a pole) or symplectic type ($L^S(s, \pi_0, \wedge^2)$ has a pole). When n is odd, π_0 must be of orthogonal type, because $L^S(s, \pi_0, \wedge^2)$ has no poles in the odd case (see [23, 25, 31]).

Corollary 4.2.6 implies that a cuspidal representation whose twisted $\wedge^3$ L-function has a pole is simply a twist of a representation whose untwisted $\wedge^3$ L-function has a pole. Since there is no essential loss of generality, we shall henceforth restrict our attention to untwisted $\wedge^3$ L-function, i.e., we shall assume that χ is trivial. In this case we get the following simplification of Corollary 4.2.6.

Lemma 4.2.9. *If $L^S(s, \pi, \wedge^3)$ has a pole, then $\pi = \eta \cdot \pi_0$ where η is cubic, π_0 is self-contragredient with trivial central character and $L^S(s, \pi_0, \mathrm{sym}^2)$ has a pole at $s = 1$.*

Definition 4.2.10. An irreducible cuspidal automorphic representation π of $GL_7(\mathbb{A})$ is said to be *of G_2 type* if it is self-contragredient and $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$.

Remark 4.2.11. By [26, Theorem 1], if $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$, then it is simple. By Lemma 4.2.9, if an irreducible cuspidal automorphic representation π of $GL_7(\mathbb{A})$ is of G_2 type, then the central character of π is trivial and $L^S(s, \pi, \mathrm{sym}^2)$ has a pole at $s = 1$.

Proposition 4.2.12. *If π is of G_2 type then the Eisenstein series has a simple pole at $s = 1$.*

Proof. We have already explained that the Eisenstein series has the same poles as $\frac{L^S(s,\pi,\wedge^3)}{L^S(s+1,\pi,\wedge^3)}$ in $\mathrm{Re}(s) \geq 1$.

The exterior cube L-function is holomorphic at 2 by [26, Lemma 5.1], so a pole at 1 will be inherited by the ratio and hence the Eisenstein series. ∎

Definition 4.2.13. When π is of G_2 type, we can see that the Eisenstein series above has a simple pole at $s = 1$. Denote the residual representation by $\mathcal{E}_\pi$.

Remark 4.2.14. (1) It is possible for the Eisenstein series to have a pole at 1 even if $L^S(s, \pi, \wedge^3)$ has no pole, namely, if $L(s, \pi, \wedge^3)$ vanishes at $s = 2$. One expects that this does not occur. For example, if Langlands functoriality holds, then $L^S(s, \pi, \wedge^3)$ is simply the standard L-function of the $\wedge^3$ lift of π. This lift does not need to be cuspidal, but if the Ramanujan conjecture also holds, then both π and its lift will be tempered at every place, so that the lift will be an isobaric sum of unitary cuspidal representations. In this case its standard L-function is holomorphic and nonvanishing in $\mathrm{Re}(s) > 1$.

(2) For similar reasons, one expects that $L^S(s, \pi, \wedge^3)$ will have no poles other than possibly at 0 and 1 with poles at 0 and 1 arising when the trivial character is an isobaric summand of the $\wedge^3$ lift.

(3) If π is of G_2 type, then $L^S(s, \pi, \wedge^3)$ must be nonvanishing at $s = 2$, since the intertwining operator can have at most a simple pole.

(4) If π is not of G_2 type but $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$, then we can still obtain a residual representation $\mathcal{E}_\pi$.

Lemma 4.2.15. *If an irreducible automorphic representation π of $GL_7(\mathbb{A})$ is the weak functorial lift of an irreducible automorphic representation σ of $G_2(\mathbb{A})$, then*

(1) *π is nearly equivalent to its contragredient $\widetilde{\pi}$,*

(2) $L^S(s, \pi, \wedge^3) = L^S(s, \pi, \mathrm{sym}^2)L^S(s, \pi)$.

Proof. The embedding of G_2 into GL_7 factors through an embedding of the special orthogonal group $SO_7 \hookrightarrow GL_7$, it follows that if π is a weak functorial lift associated with this embedding, then $\pi_v \cong \widetilde{\pi}_v$ at every unramified place v.

Write $\Gamma_{a,b}$ for the irreducible representation of $G_2(\mathbb{C})$ with highest weight $a\varpi_1^{G_2} + b\varpi_2^{G_2}$. (Here $\varpi_1^{G_2}, \varpi_2^{G_2}$ are the fundamental weights of $G_2(\mathbb{C})$.) The seven-dimensional "standard" representation of $G_2(\mathbb{C})$ is $\Gamma_{1,0}$. Then $\wedge^3\Gamma_{1,0} \cong \Gamma_{0,0} \oplus \Gamma_{1,0} \oplus \Gamma_{2,0}$, while $\mathrm{sym}^2\,\Gamma_{1,0} \cong \Gamma_{0,0} \oplus \Gamma_{2,0}$, so $\wedge^3\Gamma_{1,0} \cong \mathrm{sym}^2\,\Gamma_{1,0} \oplus \Gamma_{1,0}$. It follows that for π the weak functorial lift of σ we have

$$\begin{aligned} L^S(s, \pi, \wedge^3) &= L^S(s, \sigma, \wedge^3\Gamma_{1,0}) = L^S(s, \sigma, \mathrm{sym}^2\,\Gamma_{1,0})L^S(s, \sigma, \Gamma_{1,0}) \\ &= L^S(s, \pi, \mathrm{sym}^2)L^S(s, \pi). \end{aligned}$$ ∎

Lemma 4.2.16. *If an irreducible cuspidal representation π of $GL_7(\mathbb{A})$ is the weak functorial lift of an irreducible cuspidal representation σ of $G_2(\mathbb{A})$, then π is self-contragredient and $L^S(s, \pi, \wedge^3)$ has a simple pole at $s = 1$.*

Proof. From Lemma 4.2.15 (1), and strong multiplicity 1 for GL_7, it follows that $\pi = \widetilde{\pi}$.

From Lemma 4.2.15 (2), we have

$$L^S(s, \pi, \wedge^3) = L^S(s, \pi, \mathrm{sym}^2)L^S(s, \pi).$$

Now, $L^S(s, \pi)$ is holomorphic and nonvanishing in $\mathrm{Re}(s) \geq 1$, while $L^S(s, \pi, \mathrm{sym}^2)$ has a simple pole at $s = 1$, because π is self-contragredient. Note that self-contragredient representations of $GL_7(\mathbb{A})$ are automatically of orthogonal type. It follows that $L^S(s, \pi, \wedge^3)$ has a simple pole at $s = 1$. ∎

5. The nilpotent orbit A_6 of E_7

In this section we consider the rational orbit structure for the nilpotent orbit of E_7 whose Bala–Carter label is A_6 and whose weighted Dynkin diagram is

$$\begin{array}{ccccccccccc} 0 & - & 0 & - & 2 & - & 0 & - & 2 & - & 0 \\ & & & & | & & & & & & \\ & & & & 0 & & & & & & \end{array}$$

We will show that this nilpotent orbit consists of a single rational orbit, and the residual representation $\mathcal{E}_\pi$ has a nonzero generalized Whittaker–Fourier coefficient attached to it.

First, we introduce some notation related to nilpotent orbits. One of the most convenient ways to specify a nilpotent orbit $\mathcal{O}$ in a reductive Lie algebra is by a weighted Dynkin diagram. This method of specifying nilpotent elements relies on two facts:

(1) Orbits of nilpotent elements are in bijection with orbits of $\mathfrak{sl}_2$-triples [5, Theorem 5.5.11].

(2) Once a split maximal torus T and a base Δ of simple roots (relative to T) have been fixed, each $\mathfrak{sl}_2$-triple is conjugate to a triple (v, s, u) such that $s \in \mathfrak{t}$, and $\alpha(s) \geq 0$ for all $\alpha \in \Delta$. (Since each torus is contained in a maximal one, all maximal tori are conjugate, and every weight is in the Weyl orbit of a dominant one.)

Definition 5.0.1. The semisimple element $s = s_{\mathcal{O}}$ as above is called the *standard semisimple element* attached to the orbit $\mathcal{O}$ in question. Let $P_{\mathcal{O}} = M_{\mathcal{O}} U_{\mathcal{O}}$ be the parabolic subgroup $P_s = M_s U_s$ defined in Section 3.1, with Levi subgroup $M_{\mathcal{O}} = M_s$ and unipotent radical $U_{\mathcal{O}} = U_s$.

Each element s of $\mathfrak{t}$ determines a weighted Dynkin diagram

$$\begin{matrix} \alpha_1(s) & \alpha_3(s) & \alpha_4(s) & \alpha_5(s) & \alpha_6(s) & \alpha_7(s) \\ & & \alpha_2(s) & & & \end{matrix}$$

The weighted Dynkin diagram of a nilpotent orbit is then the weighted Dynkin diagram of its standard semisimple element.

The map from $\mathfrak{t}$ to weighted Dynkin diagrams is not injective, but each fiber has a unique element which is contained in the span of the coroots of G. For any nilpotent orbit, the standard semisimple element is contained in this subspace of $\mathfrak{t}$. In addition, if the weights of the Dynkin diagram are integral, then the diagram canonically determines a homomorphism from the root lattice into $\mathbb{Z}$, i.e., a coweight. Whenever convenient, we will use integrally weighted Dynkin diagrams to specify coweights, nilpotent orbits, and elements of $\mathfrak{t}$.

To study the nilpotent orbit A_6, we consider the parabolic subgroup $Q = LV$ whose Levi L contains the root subgroups attached to $\alpha_1, \alpha_2, \alpha_3, \alpha_5$ and α_7 and whose unipotent radical V contains the root subgroups attached to the other simple roots. The derived group of L is isomorphic to $SL_3 \times SL_2 \times SL_2 \times SL_2$, and we can map L into $GL_3 \times GL_2 \times GL_2 \times GL_2$ so that the induced map on Lie algebras maps $\sum_{i=1}^{8} t_i H_{\alpha_i} + \sum_{i=1,2,3,5,7} x_i X_{\alpha_i} + y_i X_{-\alpha_i}$ to

$$\left(\begin{pmatrix} t_3 - t_4 & x_3 & \\ y_3 & t_1 - t_3 & x_1 \\ & y_1 & -t_1 \end{pmatrix}, \begin{pmatrix} t_2 - t_4 & x_2 \\ y_2 & -t_2 \end{pmatrix}, \begin{pmatrix} t_5 - t_6 & x_5 \\ y_5 & t_4 - t_5 \end{pmatrix}, \begin{pmatrix} t_7 - t_8 & x_7 \\ y_7 & t_6 - t_7 \end{pmatrix} \right).$$

The image is

$$\{(g_1, g_2, g_3, g_4) \in GL_3 \times GL_2 \times GL_2 \times GL_2 : \det g_1 = \det g_2\}. \tag{5.0.2}$$

Denote the isomorphism from L to (5.0.2) by ι_L. Denote the projection of $GL_3 \times GL_2 \times GL_2 \times GL_2$ onto the ith factor by p_i for $i = 1, 2, 3, 4$. We write D for the differential, i.e., the induced map on Lie algebras. Thus, for example $Dp_2 \circ D\iota_L$ maps $\mathfrak{l} \to \mathfrak{gl}_2$.

The space of characters of V is identified with the sum of the root spaces $\mathfrak{g}_{-\alpha}$ attached to roots α such that $\alpha = \sum_{i=1}^{7} c_i\alpha_i$ and $2c_4 + 2c_6 = 2$. Clearly, this is the direct sum of two subspaces

$$\mathfrak{v}_1^- := \bigoplus_{\alpha:c_4=1,\, c_6=0} \mathfrak{g}_{-\alpha} \quad \text{and} \quad \mathfrak{v}_2^- := \bigoplus_{\alpha:c_4=0,\, c_6=1} \mathfrak{g}_{-\alpha}.$$

Lemma 5.0.3. *Write GSO_4 for the usual split similitude orthogonal group in four variables. In other words, let*

$$J_4 = \begin{pmatrix} &&&1\\ &&1&\\ &1&&\\ 1&&& \end{pmatrix}, \qquad GSO_4 := \left\{g \in GL_4 : gJ_4{}^tg = \lambda(g)J_4,\ \lambda(g) \in GL_1\right\}.$$

There is a surjective homomorphism of algebraic groups $\mathrm{pr} : GL_2 \times GL_2 \to GSO_4$,

$$\mathrm{pr}\left(\begin{pmatrix} a_1 & b_1\\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2\\ c_2 & d_2 \end{pmatrix}\right) = \begin{pmatrix} a_1 & b_1 &&\\ c_1 & d_1 &&\\ && a_1 & -b_1\\ && -c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 && -b_2 &\\ & a_2 && b_2\\ -c_2 && d_2 &\\ & c_2 && d_2 \end{pmatrix},$$

which satisfies $\lambda(\mathrm{pr}(g_1, g_2)) = \det g_1 \det g_2$.

Proof. Write E_{ij} for the 2×2 matrix with a 1 at the i, j entry and zeros elsewhere. Then pr sends (g_1, g_2) to the matrix of the linear operator $X \mapsto g_1 X\, {}^tg_2$ relative to the ordered basis $(E_{1,1}, E_{2,1}, -E_{1,2}, E_{2,2})$ of $\mathrm{Mat}_{2\times 2}$. Notice that the coordinate vector for the matrix $\left(\begin{smallmatrix} a & b\\ c & d \end{smallmatrix}\right)$ relative to this ordered basis is ${}^t[a \quad c \quad -b \quad d]$. Thus the quadratic form determined by the matrix J_4 corresponds to twice the determinant form on $\mathrm{Mat}_{2\times 2}$, from which it easily follows that $GL_2 \times GL_2$ maps into GSO_4 (which can also be checked by hand on the matrices above). The formula for $\lambda \circ \mathrm{pr}$ also follows easily.

It remains to show that the map is surjective. It suffices to show that the image contains all four root subgroups and the full torus, and this is straightforward. ∎

Lemma 5.0.4. *There is an isomorphism of vector groups $\iota_{\mathfrak{v}_2^-} : \mathfrak{v}_2^- \to \mathrm{Mat}_{2\times 2}$ which is compatible with ι_L in the sense that*

$$\iota_{\mathfrak{v}_2^-}(\mathrm{Ad}(\iota_L^{-1}(g_1, g_2, g_3, g_4)).X) = g_3\iota_{\mathfrak{v}_2^-}(X)g_4^{-1}.$$

Proof. We consider the action of $SL_3 \times SL_2 \times SL_2 \times SL_2$ on $\mathfrak{v}_2^-$, and easily see that the copies of SL_2 attached to the roots α_5 and α_7 act nontrivially, while the copy of SL_2 attached to α_2 and the SL_3 factor act trivially. There is a unique four-dimensional representation of $SL_2 \times SL_2$ on which both factors act trivially. Hence, the given action on $\mathrm{Mat}_{2\times 2}$ is one realization of it, while inclusion into $SL_3 \times SL_2 \times SL_2 \times SL_2$ at the third and fourth positions composed with $\mathrm{Ad} \circ \iota_L^{-1}$ is another.

To construct a specific isomorphism we start by matching our preferred highest weight vectors and generating the correspondence on the complete bases of weight vectors.

Thus, we map $X_{-0000010}$ (a highest weight vector in $\mathfrak{v}_2^-$) to E_{12} (a highest weight vector in $\mathrm{Mat}_{2\times 2}$). Then, since the differential of ι_L maps $X_{-0000100}$ to $(E_{21}, 0)$, it follows that $\mathrm{ad}(X_{-0000100})X_{-0000010}$ must be mapped to $E_{21} \cdot E_{12} = E_{22}$. Of course $\mathrm{ad}(X_{-0000100})X_{-0000010}$ is a scalar multiple of $X_{-0000110}$. The scalar depends on the structure constants of our realization (or equivalently of the corresponding Chevalley basis). Using GAP, we find $[X_{-0000100}, X_{-0000010}] = X_{-0000110}$. Continuing in this fashion, we compute[1]

$$\iota_{\mathfrak{v}_2^-}(x_{0000010}X_{-0000010}+x_{0000011}X_{-0000011}+x_{0000110}X_{-0000110}+x_{0000111}X_{-0000111}) = \begin{pmatrix} x_{0000011} & x_{0000010} \\ x_{0000111} & x_{0000110} \end{pmatrix}.$$

What remains is to check that the action of $t_4^{\alpha_4^\vee} t_6^{\alpha_6^\vee} t_8^{\alpha_8^\vee}$ is the same on both sides. And this is easy, since

$$\begin{pmatrix} t_6^{-1} & \\ & t_4 \end{pmatrix} \begin{pmatrix} x_{0000011} & x_{0000010} \\ x_{0000111} & x_{0000110} \end{pmatrix} \begin{pmatrix} t_8^{-1} & \\ & t_6 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{t_8}{t_6} x_{0000011} & \frac{1}{t_6^2} x_{0000010} \\ t_4 t_8 x_{0000111} & \frac{t_4}{t_6} x_{0000110} \end{pmatrix}$$
$$= \begin{pmatrix} \left(t_4^{\alpha_4^\vee} t_6^{\alpha_6^\vee} t_8^{\alpha_8^\vee}\right)^{-0000011} x_{0000011} & \left(t_4^{\alpha_4^\vee} t_6^{\alpha_6^\vee} t_8^{\alpha_8^\vee}\right)^{-0000010} x_{0000010} \\ \left(t_4^{\alpha_4^\vee} t_6^{\alpha_6^\vee} t_8^{\alpha_8^\vee}\right)^{-0000111} x_{0000111} & \left(t_4^{\alpha_4^\vee} t_6^{\alpha_6^\vee} t_8^{\alpha_8^\vee}\right)^{-0000110} x_{0000110} \end{pmatrix}. \quad \blacksquare$$

Lemma 5.0.5. *There is an isomorphism of vector groups $\iota_{\mathfrak{v}_1^-} : \mathfrak{v}_1^- \to \mathrm{Mat}_{3\times 4}$ which is compatible with ι_L in the sense that*

$$\iota_{\mathfrak{v}_1^-}(\mathrm{Ad}(\iota_L^{-1}(g_1, g_2, g_3, g_4)).X) = g_1 \iota_{\mathfrak{v}_1^-}(X)\,\mathrm{pr}(g_2, g_3)^{-1}.$$

Proof. This is proved by the same method. We record only the essential information. The correspondence between roots α such that X_α lies in $\mathfrak{v}_1^-$ and entries in an element of $\mathrm{Mat}_{3\times 4}$ is succinctly expressed by the following matrix:

$$\begin{pmatrix} -0101100 & -0001100 & -0101000 & -0001000 \\ -0111100 & -0011100 & -0111000 & -0011000 \\ -1111100 & -1011100 & -1111000 & -1011000 \end{pmatrix}.$$

In the next matrix we record the image of $t_4^{\alpha_4^\vee} t_6^{\alpha_6^\vee} t_8^{\alpha_8^\vee}$ under these twelve roots:

$$\begin{pmatrix} t_6 & t_6/t_4 & 1/t_4 & 1/t_4^2 \\ t_6 t_4 & t_6 & 1 & 1/t_4 \\ t_6 t_4 & t_6 & 1 & 1/t_4 \end{pmatrix}.$$

Each entry matches exactly the effect of multiplying by $\mathrm{diag}(t_4^{-1}, 1, 1)$ on the left and $\mathrm{diag}(t_6 t_4, t_6, 1, t_4^{-1})$ on the right. Finally, one has to check that $\mathrm{diag}(t_6 t_4, t_6, 1, t_4^{-1})^{-1} = \mathrm{pr}\left(\left(\begin{smallmatrix} t_4^{-1} & \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_6^{-1} & \\ & t_4 \end{smallmatrix}\right)\right)$. $\blacksquare$

[1] We remark that the scalars are not important for the present argument – only the correspondence between roots and entries is really needed.

Next we compute the rational orbit structure for the action of $GL_3 \times GSO_4$ on $\mathrm{Mat}_{3\times 4}$ by $(g_1, g_2).Y = g_1 Y g_2^{-1}$. Write $\mathrm{Mat}_{3\times 3}^{\mathrm{sym}}$ for the space of 3×3 symmetric matrices. The group $GL_3 \times GL_1$ acts by $(g, a).Z = agZ\,{}^t g$. We have a map $\mathrm{Mat}_{3\times 4} \to \mathrm{Mat}_{3\times 3}^{\mathrm{sym}}$ given by $Y \mapsto Y J_4 {}^t Y$. Clearly

$$(g_1 Y g_2^{-1}) J_4 {}^t(g_1 Y g_2^{-1}) = \lambda(g_2^{-1}) g_1 Y J_4 {}^t Y.$$

Thus Y_1 and Y_2 lie in the same $GL_3 \times GSO_4$-orbit if and only if $Y_1{}^t Y_1$ lies in the same $GL_3 \times GL_1$-orbit as $Y_2{}^t Y_2$. It is clear that $\mathrm{Rank}\, Y$ and $\mathrm{Rank}\, Y\,{}^t Y$ are both invariants of a $GL_3 \times GSO_4$-orbit, and that the latter is bounded by the former. It is relatively easy to show that $\{Y \in \mathrm{Mat}_{3\times 4} : \mathrm{Rank}\, Y = i, \mathrm{Rank}\, Y\,{}^t Y = j\}$ is nonempty and a single $GL_3 \times GSO_4$-orbit for $(i, j) = (0, 0), (1, 0), (1, 1), (2, 0)$, and $(2, 1)$. Also, one can easily find a matrix Y of rank 2 such that $Y\,{}^t Y = \mathrm{diag}(a, b, 0)$ for any a, b.

Lemma 5.0.6. *Take F a field and $Y \in \mathrm{Mat}_{3\times 4}(F)$ of rank 3. Then there exists $g \in GL_3$ such that $(gY\,{}^t Y\,{}^t g)$ is of the form*

$$\begin{pmatrix} & & 1 \\ & a & \\ 1 & & \end{pmatrix}.$$

Proof. Write V for the span of the rows of Y. We choose a suitable basis for V such that the quadratic form attached to J_4, when written in terms of the new basis, has a matrix of the specified form.

We may write $\mathrm{Mat}_{1\times 4} = W_1 \oplus W_2$ where W_1, W_2 are two-dimensional isotropic subspaces. Since $\dim V > \dim W_1$, there exist nontrivial elements of v which project to 0 in W_1. That is, $V \cap W_1 \neq 0$. Likewise $V \cap W_2 \neq 0$. Select $v_1 \in V \cap W_1$ and $v_2 \in V \cap W_2$.

First suppose that v_1 is orthogonal to v_2. Then the span of v_1 and v_2 is a maximal isotropic subspace W_1'. Select v_3 in the orthogonal complement of W_1' and then replace v_1, v_2 by a new basis v_1', v_2' for W_1' such that $v_2' J_4 v_3 = 0$ and $v_1' J_4 v_3 = 1$. Then the basis v_1', v_2', v_3 fits the bill.

Now suppose that v_1 is not orthogonal to v_2, and let v_3 be any element of V which is linearly independent of v_1 and v_2. Then there exist a, b such that $v_3 - bv_1 - cv_2$ is orthogonal to both v_1 and v_2, and the basis v_1, v_3, v_2 fits the bill. ∎

Corollary 5.0.7.

$$\{Y \in \mathrm{Mat}_{3\times 4}(F) : \mathrm{Rank}\, Y\,{}^t Y = 3\}$$

is a Zariski open $GL_3(F) \times GSO_4(F)$-orbit over any field F.

Proof. The set is clearly Zariski open. We have shown that each orbit with $\mathrm{Rank}\, Y\,{}^t Y = 3$ contains an element with

$$Y\,{}^t Y = \begin{pmatrix} & & 1 \\ & a & \\ 1 & & \end{pmatrix}.$$

If the rank is 3 then a is nonzero and we can scale by a^{-1} in GL_1 and then act by $\mathrm{diag}(a,1,1)$ in GL_3 to get $\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, which completes the proof that our set is a single orbit. ■

Corollary 5.0.8. *The nilpotent orbit A_6 consists of a single rational orbit.*

Proof. We know that each rational orbit in A_6 has a representative that lies in $\mathfrak{v}_1^-(F) \oplus \mathfrak{v}_2^-(F)$, and that two elements of this space lie in the same $G(F)$-orbit if and only if they lie in the same $L(F)$-orbit. We can identify $\mathfrak{v}_1^-(F) \oplus \mathfrak{v}_2^-(F)$ with $\mathrm{Mat}_{3\times 4}(F) \oplus \mathrm{Mat}_{2\times 2}(F)$. It is clear that the action of $L(F)$ preserves the Zariski open subset $\{(Y,X) \in \mathrm{Mat}_{3\times 4}(F) \times \mathrm{Mat}_{2\times 2}(F) : \mathrm{Rank}\, Y\,{}^tY = 3, \mathrm{Rank}\, X = 2\}$. We show that this set is a single $L(F)$-orbit. Take two elements (Y_1, X_1) and (Y_2, X_2). Recall that L is identified with $\{(g_1, g_2, g_3, g_4) \in GL_3 \times GL_2 \times GL_2 \times GL_2 : \det g_1 = \det g_2\}$, and note that $(g_1, g_2, g_3, g_4) \mapsto (g_1, \mathrm{pr}(g_2, g_3))$ gives a surjective mapping onto $GL_3 \times GSO_4$. Thus, there exists (g_1, g_2, g_3) such that $\mathrm{Ad}(g_1, g_2, g_3, I_2).(Y_1, X_1) = (Y_2, X_2')$. Then $\mathrm{Ad}(I_3, I_2, I_2, X_2^{-1}(X_2')).(Y_2, X_2') = (Y_2, X_2)$. ■

It will be convenient to select a representative for our open orbit. A representative in $\mathrm{Mat}_{3\times 4} \times \mathrm{Mat}_{2\times 2}$ would be

$$\left(\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right).$$

A convenient representative in $\mathfrak{v}_1^- \oplus \mathfrak{v}_2^-$ would be $X_{-0101100} + X_{-0111000} + X_{-0011100} + X_{-1011000} + X_{-0000110} + X_{-0000011}$. This will correspond to the above pair of matrices up to some signs. In particular, it will be an element of the correct orbit. Let $w_0 = w[2431542346542237654]$. (This notation for an element of the Weyl group was introduced in Section 2.) Then there is a representative $\dot{w}_0$ for w_0 such that

$$\begin{aligned} \mathrm{Ad}(\dot{w}_0).(X_{-0101100} + X_{-0111000} + X_{-0011100} + X_{-1011000} + X_{-0000110} + X_{-0000011}) \\ = X_{-\alpha_4} + X_{-\alpha_7} + X_{-\alpha_1} + X_{-\alpha_5} + X_{-\alpha_6} + X_{-\alpha_3}. \end{aligned}$$

This nilpotent element corresponds to the regular orbit of the A_6 Levi. We remark that if a standard representative $\ddot{w}_0$ is used then

$$\begin{aligned} \mathrm{Ad}(\ddot{w}_0).(X_{-0101100} + X_{-0111000} + X_{-0011100} + X_{-1011000} + X_{-0000110} + X_{-0000011}) \\ = -X_{-\alpha_4} + X_{-\alpha_7} - X_{-\alpha_1} + X_{-\alpha_5} + X_{-\alpha_6} - X_{-\alpha_3}. \end{aligned}$$

For the sake of completeness, we record our findings regarding the rational orbit decomposition of $\mathrm{Mat}_{3\times 4}$.

Proposition 5.0.9. *The set*

$$\{Y \in \mathrm{Mat}_{3\times 4} : \mathrm{Rank}\, Y = i, \ \mathrm{Rank}\, Y J_4\, {}^tY = j\}$$

is nonempty if and only if either $0 \le j \le i \le 2$, *or* $i = 3$ *and* $2 \le j \le 3$. *It is a single* $GL_3 \times GSO_4$*-orbit unless* $i = j = 2$, *in which case it is a union of orbits which are in one-to-one correspondence with the action of* $GL_2 \times GL_1$ *on* $\mathrm{Mat}_{2\times 2}^{\mathrm{sym}}$.

Theorem 5.0.10. $\mathcal{E}_\pi$ *has a nonzero generalized Fourier coefficient attached to the rational nilpotent orbit labeled by* A_6.

Proof. Take $u = X_{-\alpha_4} + X_{-\alpha_7} + X_{-\alpha_1} + X_{-\alpha_5} + X_{-\alpha_6} + X_{-\alpha_3}$ and a rational semisimple s' element which acts by 2 on each simple root space. Then $\mathcal{F}_{s',u}$ maps an automorphic form to the GL_7 nondegenerate Whittaker–Fourier integral of its constant term along the A_6 parabolic. It is clear that the residual representation supports this coefficient. Therefore, by Theorem 3.1.2, it also supports $\mathcal{F}_{s,u}$, where s is a neutral element for u. ∎

Remark 5.0.11. We expect that in fact $\mathfrak{n}^m(\mathcal{E}_\pi) = \{A_6\}$. Indeed, we expect that if π is of G_2 type then at each unramified place v, π_v is attached to a semisimple conjugacy class of $GL_7(\mathbb{C})$ which intersects the subgroup $G_2(\mathbb{C})$. By Corollary 3.2.8 and Remark 3.2.10, it follows from the discussion in §7.3.2 below that if there is even one unramified finite place where this condition holds, then $\mathfrak{n}^m(\mathcal{E}_\pi) = \{A_6\}$.

6. Descent Fourier coefficients and descent modules

From the table [5, pp. 403–404], we learn that there are two conjugacy classes of $\mathfrak{sl}_2$-triples in GE_7 such that the stabilizer is of G_2 type. They are known as A_5'' and $A_2 + 3A_1$. For the sake of completeness, we consider Fourier coefficients and associated descent modules attached to each of them.

6.1. A_5''

The weighted Dynkin diagram of this orbit is $\begin{smallmatrix} 2&0&0&0&2&2 \\ &&0&&& \end{smallmatrix}$. Let s be the standard semisimple element attached to the orbit. Then the Levi subgroup whose Lie algebra is $\mathfrak{g}_0^s$ is the semidirect product of a derived group isomorphic to $Spin_8$ and a four-dimensional torus, while the space $\mathfrak{g}_{-2}$ is the direct sum of two nonisomorphic irreducible eight-dimensional representations of this Levi and one one-dimensional representation. On each eight-dimensional representation we have a $Spin_8$-invariant quadratic form, which is unique up to scalar (see [11, Exercise 20.38]). The Levi acts on $\mathfrak{g}_{-2}^{s_{A_5''}}$ with an open orbit. It is not hard to check that in this case the open orbit consists of triples such that each eight-dimensional component is anisotropic relative to the $Spin_8$-invariant form and the one-dimensional component is nonzero (see [22]). The stabilizer of any point in this open orbit is the product of the center of GE_7 and a group isomorphic to G_2. It is not hard to check that

$$f_0 := X_{-0000001} + X_{-1111000} + X_{-1011100} + X_{-0101110} + X_{-0011110}$$

is in this open orbit. The corresponding copy of $\mathfrak{g}_2$ is generated by

$$X_{\pm 0001000}, \quad X_{\pm 0100000} - X_{\pm 0010000} + X_{\pm 0000100},$$

and we embed G_2 into GE_7 so that $X_{\pm\alpha} = X_{\pm 0100000} - X_{\pm 0010000} + X_{\pm 0000100}$ and $X_{\pm\beta} = X_{\pm 0001000}$. Recall that $P_{A_5''} = M_{A_5''}U_{A_5''} = P_s = M_sU_s$ is the parabolic subgroup defined as in Section 3.1, where s is the standard semisimple element (see Definition 5.0.1) attached to A_5'', $M_{A_5''} = M_s$ is the Levi subgroup, and $U_{A_5''} = U_s$ is the unipotent radical. Then $U_{A_5''}$ contains U_{α_i} if and only if $i \neq 2, 3, 4, 5$. Let $\psi^{f_0}_{U_{A_5''}}$ be the character of $U_{A_5''}(F)\backslash U_{A_5''}(\mathbb{A})$ attached to f_0.

Definition 6.1.1. Let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type (as in Definition 4.2.10). Let $\mathcal{E}_\pi$ be the residual representation as in Definition 4.2.13. We define the corresponding *descent module* $\mathcal{D}_\pi = \mathcal{D}_\pi^{A_5''}$ to be

$$\mathcal{D}_\pi := \{\varphi^{(U_{A_5''},\psi^{f_0}_{U_{A_5''}})}|_{G_2(\mathbb{A})} : \varphi \in \mathcal{E}_\pi\},$$

where

$$\varphi^{(U_{A_5''},\psi^{f_0}_{U_{A_5''}})}(g) := \int_{U_{A_5''}(F)\backslash U_{A_5''}(\mathbb{A})} \varphi(ug)\overline{\psi^{f_0}_{U_{A_5''}}(u)}\, du, \quad g \in GE_7(\mathbb{A}).$$

6.2. $A_2 + 3A_1$

The weighted Dynkin diagram of this orbit is $\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & & 2 & & & \end{smallmatrix}$. Recall that M is the standard Levi subgroup isomorphic to $GL_7 \times GL_1$, P is the standard parabolic which contains it, and U is the unipotent radical of P. Then $P = MU = P_{A_2+3A_1} = M_{A_2+3A_1}U_{A_2+3A_1}$ as in Definition 5.0.1, $M = M_{A_2+3A_1}$, $U = U_{A_2+3A_1}$.

Let $e_0 = X_{-1122100} + X_{-1112110} + X_{-1111111} + X_{-0112210} + X_{-0112111}$ and

$$\psi_U^{e_0}(u) = \psi(u_{1122100} + u_{1112110} + u_{1111111} + u_{0112210} + u_{0112111})$$

be the corresponding character of $U(F)\backslash U(\mathbb{A})$. We write $u \in U$ as $\prod_\alpha x_\alpha(u_\alpha)$ with the roots taken in some fixed order. The coordinate u_α is independent of the choice of order provided the second coordinate of α is 1.

Lemma 6.2.1. *The stabilizer of $\psi_U^{e_0}$ in M is the product of the center and a group isomorphic to G_2.*

Proof. We can identify the space of characters of $U(F)\backslash U(\mathbb{A})$ with the space

$$\mathfrak{u}_2^{(-1)} = \bigoplus_{\langle\alpha,\varpi_2^\vee\rangle=-1} \mathfrak{u}_\alpha.$$

As representation of $\mathfrak{gl}_7$, this representation is isomorphic to the exterior cube representation of GL_7. It is well known (see [11, pp. 356–357]) that GL_7 acts on this representation with an open orbit, and that the stabilizer of any point in this open orbit is of G_2 type. Using SageMath, with adjoint matrices from GAP, we verified that $\psi_U^{e_0}$ is fixed by

$$x_{1000000}(a)x_{0001100}(-a^2)x_{0000100}(2a)x_{0001000}(a)x_{0000001}(-a),$$
$$x_{0010000}(b)x_{0000010}(b), \quad x_{-0010000}(b)x_{-0000010}(b),$$
$$x_{-1000000}(a)x_{-0001100}(a^2)x_{-0000100}(a)x_{-0001000}(2a)x_{-0000001}(-a).$$

These subgroups generate a split subgroup of GL_7 of G_2 type. The stabilizer also contains the center of GE_7. It remains to prove that the stabilizer is no larger. For this purpose it suffices to prove that our character corresponds to a point in the open orbit. In [11, p. 357] a specific point in the open orbit is written down; it is a sum of five weight vectors. We easily check that these five weights correspond to the five roots which appear in $\psi_U^{e_0}$. Over an algebraically closed field, the torus acts transitively on the set of linear combinations of these five weight vectors so that all five coefficients are nonzero. Therefore the point corresponding to $\psi_U^{e_0}$ is also in the open orbit. ∎

We remark that the embedding of G_2 into GL_7 obtained in this way agrees with the one from [11].

It is convenient to know that the roots in $\mathrm{supp}(\psi_U^{e_0})$ can be simultaneously conjugated to simple roots. Let $R_1 = \{1122100, 1112110, 1111111, 0112210, 0112111\}$, and $w_6 = w[423546542314376542]$. Then $w_6 \cdot R_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$.

Definition 6.2.2. Let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type (as in Definition 4.2.10). Let $\mathcal{E}_\pi$ be the residual representation as in Definition 4.2.13. We define the corresponding *descent module* $\mathcal{D}_\pi = \mathcal{D}_\pi^{A_2+3A_1}$ to be

$$\mathcal{D}_\pi := \{\varphi^{(U,\psi_U^{e_0})}|_{G_2(\mathbb{A})} : \varphi \in \mathcal{E}_\pi\},$$

where

$$\varphi^{(U,\psi_U^{e_0})}(g) := \int_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U^{e_0}}(u)\, du, \quad g \in GE_7(\mathbb{A}).$$

Remark 6.2.3. The embedding of G_2 which comes from the orbit $A_2 + 3A_1$ is closely related to the appearance of $\wedge^3$ in the constant term. Indeed, $L^S(s, \pi, r)$ appears in the constant term of an Eisenstein series of a group G if and only if r appears in the action of the relevant Levi of ${}^L G$ on the nilpotent radical of the Lie algebra of the corresponding parabolic. That is, r appears equipped with a realization as a space of nilpotent elements. In fact, the realization of $\wedge^3$ is precisely as the space $\mathfrak{g}_2^s$ where s is the standard semisimple element attached to $A_2 + 3A_1$. That is, the embedding of $G_2(\mathbb{C})$ into $GE_7(\mathbb{C})$ on the L-group side as the stabilizer of a point in the representation obtained from an L-function, and the embedding of G_2 into GE_7 as the stabilizer of a Fourier coefficient are essentially the same embedding. This phenomenon does not occur in the classical situation of [16], as it requires self-contragredientity of both the group denoted by H and the one denoted by A in our discussion of the general set-up in the introduction.

In the introduction we remarked on prior work of Ginzburg where $H = G_2$ and $A = F_4$, as well as prior work of Ginzburg–Hundley where $H = F_4$ and $A = E_8$, where the descent modules fail to be cuspidal. It is noteworthy that in both of those cases, H and A are self-contragredient and the embedding of H into A obtained from the L-function is the *only* embedding of H into A.

7. The A_5'' case

Recall from Definition 6.1.1 that in the A_5'' case the descent module $\mathcal{D}_\pi$ is defined by applying the Fourier coefficient $(U_{A_5''}, \psi^{f_0}_{U_{A_5''}})$ from Section 6.1 to the residual representation $\mathcal{E}_\pi$, where π is an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type. In this section, we prove the following theorem.

Theorem 7.0.1. *Assume that π is an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type, and $\mathcal{D}_\pi$ is defined as in Definition 6.1.1. Then*

(1) *$\mathcal{D}_\pi$ is generic.*

(2) *Suppose that there exists a finite place v_0 such that π_{v_0} is a principal series representation of $GL_7(F_{v_0})$ which is attached to a semisimple conjugacy class of $GL_7(\mathbb{C})$, and intersects the subgroup $G_2(\mathbb{C})$. Then $\mathcal{D}_\pi$ is cuspidal.*

(3) *Suppose that for almost all finite places v, π_v is a principal series representation of $GL_7(F_v)$ which is attached to a semisimple conjugacy class of $GL_7(\mathbb{C})$, and intersects the subgroup $G_2(\mathbb{C})$. Then π is a weak functorial lift of each irreducible summand of $\mathcal{D}_\pi$.*

7.1. Genericity of the A_5'' descent module

The purpose of this section is to prove that the descent module $\mathcal{D}_\pi$ is generic. The proof can be explained using the language of “unipotent periods” introduced in [20]. Let $U^{G_2}_{\max}$ be the standard maximal unipotent subgroup of G_2. Let ψ^{G_2} be any character of $U^{G_2}_{\max}$. Then the composite $(U^{G_2}_{\max}, \psi^{G_2}) \circ (U_{A_5''}, \psi^{f_0}_{U_{A_5''}})$ makes sense as a unipotent period on $C^\infty(GE_7(F)\backslash GE_7(\mathbb{A}))$. Explicitly, it maps $\varphi \in C^\infty(GE_7(F)\backslash GE_7(\mathbb{A}))$ to

$$\int_{U^{G_2}_{\max}(F)\backslash U^{G_2}_{\max}(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi(u_1u_2g)\overline{\psi^{f_0}_{U_{A_5''}}}(u_1)\overline{\psi^{G_2}}(u_2)\, du_1\, du_2.$$

In our discussion of unipotent periods it is helpful to note that

$$S \longleftrightarrow \prod_{\alpha\in S} U_\alpha$$

is a bijection

$$\{S \subset \Phi : \alpha, \beta \in S, \alpha+\beta \in \Phi \cup \{0\} \Rightarrow \alpha+\beta \in S\}$$
$$\longleftrightarrow \{T\text{-stable unipotent subgroups of } GE_7\}.$$

Thus, it is often convenient to specify a unipotent subgroup V of GE_7 by identifying $\Phi(V, T)$. We adopt a convenient abuse of notation. Let V be a T-stable unipotent subgroup of GE_7 and let ψ_V be a character of it. We shall call $\{\alpha \in \Phi(V, T) : \psi_V|_{U_\alpha(\mathbb{A})} \not\equiv 1\}$ the "support" of ψ_V and denote it supp ψ_V. We denote by (V, ψ_V) or $\varphi^{(V,\psi_V)}$ the following attached unipotent period:

$$\int_{V(F)\backslash V(\mathbb{A})} \varphi(vg)\overline{\psi_V}(v)\, dv, \quad g \in GE_7(\mathbb{A}).$$

Given two unipotent periods (V, ψ_V) and (U, ψ_U), if $\varphi^{(V,\psi_V)}$ is left-invariant by $U(F)$, then we denote the composed period by $(U, \psi_U) \circ (V, \psi_V)$.

We recall the concept of equivalence of unipotent periods. Write $\mathcal{P}_1 \mid \mathcal{P}_2$ if $\mathcal{P}_2$ vanishes identically on any automorphic representation on which $\mathcal{P}_1$ vanishes identically. Two periods $\mathcal{P}_1$ and $\mathcal{P}_2$ are said to be equivalent (denoted $\mathcal{P}_1 \sim \mathcal{P}_2$) if $\mathcal{P}_1 \mid \mathcal{P}_2$ and $\mathcal{P}_2 \mid \mathcal{P}_1$.

In the study of Fourier coefficients of automorphic forms, in particular concerning the global nonvanishing property, a technical lemma from [16] has been very useful in the theory. We recall it as follows. Let G be any connected reductive group defined over F. Let C be an F-subgroup of a maximal unipotent subgroup of G, and let ψ_C be a nontrivial character of $[C] = C(F)\backslash C(\mathbb{A})$. Suppose X, Y are two unipotent F-subgroups satisfying the following conditions:

(1) X and Y normalize C;

(2) $X \cap C$ and $Y \cap C$ are normal in X and Y, respectively, and both $(X \cap C)\backslash X$ and $(Y \cap C)\backslash Y$ are abelian;

(3) $X(\mathbb{A})$ and $Y(\mathbb{A})$ preserve ψ_C;

(4) ψ_C is trivial on $(X \cap C)(\mathbb{A})$ and $(Y \cap C)(\mathbb{A})$;

(5) $[X, Y] \subset C$;

(6) there is a nondegenerate pairing $(X \cap C)(\mathbb{A})\backslash X(\mathbb{A}) \times (Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A}) \to \mathbb{C}^*$, given by $(x, y) \mapsto \psi_C([x, y])$, which is multiplicative in each coordinate, and identifies the set $(Y \cap C)(F)\backslash Y(F)$ with the dual of $X(F)(X \cap C)(\mathbb{A})\backslash X(\mathbb{A})$, and $(X \cap C)(F)\backslash X(F)$ with the dual of $Y(F)(Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A})$.

Let $B = CX$ and $D = CY$, and extend ψ_C trivially to characters of $[B] = B(F)\backslash B(\mathbb{A})$ and $[D] = D(F)\backslash D(\mathbb{A})$, which will be denoted by ψ_B and ψ_D respectively. When there is no confusion, we will denote both ψ_B and ψ_D by ψ_C.

Lemma 7.1.1 ([16, Lemma 7.1 and Corollary 7.1]). *Assume that (C, ψ_C, X, Y) satisfies all the above conditions. Let f be an automorphic function of uniformly moderate growth on $G(\mathbb{A})$. Then*

$$\int_{[B]} f(vg)\overline{\psi_B}(v)\, dv = \int_{(X\cap C)(\mathbb{A})\backslash X(\mathbb{A})} \int_{[D]} f(uxg)\overline{\psi_D}(u)\, du\, dx, \quad \forall g \in G(\mathbb{A}).$$

The right hand side of the above equality is convergent in the sense

$$\int_{(X\cap C)(\mathbb{A})\backslash X(\mathbb{A})} \left| \int_{[D]} f(uxg)\overline{\psi_D}(u)\, du \right| dx < \infty,$$

and this convergence is uniform as g varies in compact subsets of $G(\mathbb{A})$. Moreover, $(B, \psi_B) \sim (D, \psi_D)$.

We consider the unipotent period $(U_1, \psi^{\underline{a}}_{U_1})$ where U_1 is the T-stable unipotent group attached to the set of positive roots whose complement is $\{1011000, 0001110, 1010000, 0000110, 1000000, 0000010\}$. Also $\psi^{\underline{a}}_{U_1}(u) = \psi(u_{0000001} + u_{1111000} + u_{1011100} + u_{0101110} + u_{0011110} + a_1 u_{\alpha_2} + a_2 u_{\alpha_3} + a_3 u_{\alpha_5} + a_4 u_{\alpha_4})$. For $\underline{a} = (a_1, a_2, a_3, a_4) \in F^4$, we define a character $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ of $U^{G_2}_{\max}$ by $\psi^{\underline{a}}_{U^{G_2}_{\max}}(u) = \psi(a_4 u_\beta + (a_1 - a_2 + a_3) u_\alpha)$.

Lemma 7.1.2. *The period $(U_1, \psi^{\underline{a}}_{U_1})$ is equivalent to the composed period $(U^{G_2}_{\max}, \psi^{\underline{a}}_{U^{G_2}_{\max}})$ $\circ\, (U_{A''_5}, \psi^{f_0}_{U_{A''_5}})$.*

Proof. The proof consists of three applications of the "exchange lemma", Lemma 7.1.1. Each time, the group X is a product of two commuting root subgroups $U_{\gamma_1}, U_{\gamma_2}$ of GE_7, and there are three roots $\beta_1, \beta_2, \beta_3$ of GE_7 and a root δ of G_2 such that $\mathfrak{g}_2 \cap \bigoplus_{i=1}^{3} \mathfrak{u}_{\beta_i} = \mathfrak{u}_\delta$. For the group Y we may use any complement to U_δ in $U_{\beta_1} U_{\beta_2} U_{\beta_3}$. The roots which determine the groups X and Y in the successive applications of Lemma 7.1.1 are given in the table below.

X	Y	δ
1000000, 0000010	0111000, 0101100, 0011100	$2\alpha + \beta$
1010000, 0000110	0101000, 0011000, 0001100	$\alpha + \beta$
1011000, 0001110	0100000, 0010000, 0000100	α

Checking conditions (1) to (6) for Lemma 7.1.1 is similar to the proof of Lemma 8.1.3. ∎

Note that the character $\psi^{\underline{a}}_{U_1}$ is attached to

$$f_{\underline{a}} := f_0 + a_1 X_{-\alpha_2} + a_2 X_{-\alpha_3} + a_3 X_{-\alpha_5} + a_4 X_{-\alpha_4}.$$

Lemma 7.1.3. (1) *Let X be a nilpotent element of $\mathfrak{e}_7$. Then X is in the closure of A_6 if and only if* $\mathrm{ad}(X)^{14} = 0$. *In this case* $\mathrm{ad}(X)^{13}$ *is also* 0.

(2) *Let X be in the closure of A_6. Then X is in A_6 itself if and only if* $\mathrm{ad}(X)^{12} \neq 0$.

Proof. To any nilpotent element $X \in \mathfrak{e}_7$ we may associate the *rank sequence* $(\mathrm{rank}\, \mathrm{ad}(X)^k)_{k=0}^{\infty}$. (All but finitely many entries are zero.) It is clear that the rank sequence is an invariant of the stable orbit of X. In general the map from stable orbits to rank sequences is not injective, but one can check (using GAP, for example) that for $\mathfrak{e}_7$ it is. This lemma can then be proved by inspecting the rank sequences for all nilpotent orbits in $\mathfrak{e}_7$, obtained from GAP, while using the chart in [5, p. 442] to see which orbits are in the closure of A_6. ∎

Lemma 7.1.4. (1) *For $\underline{a}$ in general position, $f_{\underline{a}}$ is in the orbit $E_7(a_4)$.*

(2) *The orbit of $f_{\underline{a}}$ is in the closure of A_6 if and only if at least one of the following conditions holds:*

(a) $a_4 = 0$;

(b) $a_3 = 0$ *and* $a_1 = a_2$;

(c) $a_1 = a_3c_1(c_1 + 2)$ *and* $a_2 = a_3c_1(c_1 + 1)$ *for some* c_1.

(3) *If* $a_4 = 0$, *or* $a_3 = 0$ *and* $a_1 = a_2$, *then the orbit of* $f_{\underline{a}}$ *is strictly less than* A_6.

(4) *If* $a_1 = a_3c_1(c_1 + 2)$ *and* $a_2 = a_3c_1(c_1 + 1)$, *then* $f_{\underline{a}}$ *is in* A_6 *if and only if* a_4 *and* $a_1 - a_2 + a_3$ *are both nonzero, i.e., the character* $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ *is generic.*

Proof. Using GAP and SageMath, we compute that for $\underline{a}$ in general position, $\operatorname{Rank}\operatorname{ad}(f_{\underline{a}})^{14} = 1, \operatorname{Rank}\operatorname{ad}(f_{\underline{a}})^{13} = 2, \operatorname{Rank}\operatorname{ad}(f_{\underline{a}})^{12} = 4$. It follows that for $\underline{a}$ in general position, $f_{\underline{a}}$ is an element of the orbit $E_7(a_4)$. An element f of $\mathfrak{e}_7$ lies in the closure of A_6 if and only if $\operatorname{ad}(f)^{13} = 0$. It lies in A_6 itself if and only if $\operatorname{Rank}\operatorname{ad}(f)^{12} = 3$. Further, $\operatorname{Rank}\operatorname{ad}(f_{\underline{a}})^{14} = 0$ if and only if $a_4 = 0$ or

$$(a_1 - a_2)^2 + a_3(a_1 - 2a_2) = 0. \tag{7.1.5}$$

If $a_4 = 0$, then $\operatorname{Rank}\operatorname{ad}(f_{\underline{a}})^{11} = 0$, and $f_{\underline{a}}$ is in an orbit which is less than A_6. If $a_3 = 0$ and $a_2 = a_1$, the same is true.

If $a_3 \neq 0$ then we may let $b_1 = a_1 - a_2$, and (7.1.5) becomes $b_1 - a_2 = -b_1^2/a_3$. Then letting $c_1 = b_1/a_3$, this becomes $a_2 = c_1a_3 + c_1^2a_3$. Also $a_1 = c_1a_3 + a_2 = 2c_1a_3 + c_1^2a_3$. We may compute $\operatorname{ad}(f_{\underline{a}})$, with a_1, a_2 defined by these formulas, using SageMath. After dropping all rows and columns that consist entirely of zeros, we obtain a 9×9 matrix, all of whose entries are divisible by $462a_3^2a_4(c_1 + 1)^2$, which is easily seen to be rank 3 if this expression is nonzero. Further, when a_1, a_2 are defined by these formulas, we have $a_1 - a_2 + a_3 = (c_1 + 1)a_3$. From this we conclude that for any $\underline{a}$ such that $f_{\underline{a}} \in A_6$, the character $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ is generic. ∎

Remark 7.1.6. Note that the character $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ is trivial if and only if $a_4 = a_1 - a_2 + a_3 = 0$. We found that in this case $f_{\underline{a}}$ is always in the orbit A_5''.

Lemma 7.1.7. *Let* U_2 *be the* T*-stable unipotent subgroup such that*

$$\Phi(T, U_2) = \Phi^+ \smallsetminus \{0000100, 0000110, 0001100, 0010000, 0011000, 1010000\},$$

and let $\psi^{\underline{a}}_{U_2} : U_2(\mathbb{A}) \to \mathbb{C}^\times$ *be the character of* $U_2(F)\backslash U_2(\mathbb{A})$ *given by*

$$\psi^{\underline{a}}_{U_2}(u) = \psi(u_{0000111} + u_{0101100} + u_{0001110} + u_{0111000} + u_{1011000} + a_3u_{0000010} + a_4u_{0011100} + a_1u_{0100000} + a_2u_{1000000}).$$

Let $*$ *denote entrywise multiplication in* F^4:

$$(c_1, c_2, c_3, c_4) * (a_1, a_2, a_3, a_4) = (c_1a_1, c_2a_2, c_3a_3, c_4a_4).$$

Then there exists $\underline{c} = (c_1, c_2, c_3, c_4) \in \{\pm 1\}^4$ *such that* $(U_1, \psi^{\underline{a}}_{U_1}) \sim (U_2, \psi^{\underline{c}*\underline{a}}_{U_2})$ *for all* $\underline{a} \in F^4$.

Proof. Conjugate by a suitable representative of $w[5631]$. For any representative, $\dot{w}[5631]$, we have $\dot{w}[5631]x_\alpha(r)\dot{w}[5631]^{-1} = x_{w[5631]\alpha}(c_{\dot{w}[5631],\alpha}r)$, for some constant $c_{\dot{w}[5631],\alpha}$ which depends on α, the choice of representative $\dot{w}[5631]$, and the structure constants of the Chevalley basis. Moreover, there exist representatives such that $c_{\dot{w}[5631],\alpha} \in \{\pm 1\}$ for all α. Since the five roots from the original A_5'' character can be simultaneously conjugated to simple roots, it follows that we can adjust our representative by an element of the torus to make these five coefficients 1. ■

The character $\psi_{U_2}^{\underline{a}}$ is attached to $\mathrm{Ad}(\dot{w}[5631]) f_{\underline{c}*\underline{a}}$, which is, of course, in the same orbit as $f_{\underline{c}*\underline{a}}$. We have seen in Lemma 7.1.4 that if this orbit is greater than or equal to A_6, then $\psi_{U_{\max}^{G_2}}^{\underline{c}*\underline{a}}$ will be a generic character of $U_{\max}^{G_2}$. But the set of such characters is permuted transitively by the torus of G_2. Hence, all such characters are equivalent. That is, $(U_2, \psi_{U_2}^{\underline{a}}) \sim (U_2, \psi_{U_2}^{\underline{b}})$ whenever the nilpotent elements attached to $\psi_{U_2}^{\underline{a}}$ and $\psi_{U_2}^{\underline{b}}$ are both attached to orbits that are greater than or equal to A_6.

Lemma 7.1.8. *Let U_3 be the T-stable unipotent subgroup such that*

$$\Phi(T, U_2) = \Phi^+ \smallsetminus \{0000001, 0000100, 0001000, 0001100, 0010000, 0011000\},$$

and let $\psi_{U_3}^{\underline{a}} : U_3(\mathbb{A}) \to \mathbb{C}^\times$ be the character given by $\psi_{U_3}^{\underline{a}}(u) = \psi(u_{0001110} + u_{0101100} + u_{0000111} + u_{0111000} + u_{1010000} + a_3 u_{0000011} + a_4 u_{0011100} + a_1 u_{0101000} + a_2 u_{1000000})$. Then there exists $\underline{d} \in \{\pm 1\}^4$ such that $(U_2, \psi_{U_2}^{\underline{a}}) \sim (U_3, \psi_{U_3}^{\underline{d}*\underline{a}})$ for all $\underline{a} \in F^4$.

Proof. Exchange α_7 for 0000110 and α_4 for 1010000, applying Lemma 7.1.1, and then conjugate by a suitable representative for $w[47]$. ■

Lemma 7.1.9. *Let U_5 be the unipotent subgroup attached to $E_7(a_4)$. Thus U_{α_i} is in U_5 for $i = 1, 4$ and 7. Let U_4 be the subgroup of U_5 defined by the condition $u_{\alpha_4} = 0$. And $\psi_{U_4}^{\underline{a}}$ be the character of this group defined by the same formula as $\psi_{U_3}^{\underline{a}}$. Then $(U_3, \psi_{U_3}^{\underline{a}}) \sim (U_4, \psi_{U_4}^{\underline{a}})$.*

Proof. We exchange 0100000 for 0011000, 0000010 for 0001100, and then 0000110 for 0000001, applying Lemma 7.1.1. ■

Proposition 7.1.10. *For $\underline{a} \in F^4$ and $b \in F$, let $\psi_{U_5}^{\underline{a},b}$ be the character given by $\psi_{U_5}^{\underline{a},b}(u x_{\alpha_4}(r)) = \psi_{U_4}^{\underline{a}}(u)\psi(br)$ for $u \in U_4(\mathbb{A})$ and $r \in \mathbb{A}$. Then an automorphic representation supports the period $(U_4, \psi_{U_4}^{\underline{a}})$ if and only if it supports $(U_5, \psi_{U_5}^{\underline{a},b})$ for some b.*

Proof. Given an automorphic form φ we perform Fourier expansion of $\varphi^{(U_4, \psi_{U_4}^{\underline{a}})}$ along the one-dimensional unipotent group $U_{\alpha_4}(F)\backslash U_{\alpha_4}(\mathbb{A})$. ■

Let $M_{\{2,3,5,6\}}$ be the standard Levi subgroup of GE_7 which contains U_{α_i} if and only if $i = 2, 3, 5$, or 6. (Thus, $M_{\{2,3,5,6\}}$ is the standard Levi factor of a standard parabolic whose unipotent radical is the group U_5.)

Proposition 7.1.11. *Let* $y_{\underline{a}} = X_{-0001110} + X_{-0101100} + X_{-0000111} + X_{-0111000} + X_{-1010000} + a_3X_{-0000011} + a_4X_{-0011100} + a_1X_{-0101000} + a_2X_{-1000000}$, *which is the nilpotent element associated to* $\psi_{U_4}^{\underline{a}}$ *and* $\psi_{U_5}^{\underline{a},0}$. *Let* $e_0' = X_{-1010000} + X_{-0000011} + X_{-0111000} + X_{-0101100} + X_{-0011100} + X_{-0001110}$. *If* $y_{\underline{a}}$ *is in the orbit* A_6 *then there exists* m *in* $M_{\{2,3,5,6\}}$ *such that* $\mathrm{Ad}(m).y_{\underline{a}} = e_0'$. *In particular, if* $\psi_{U_5}^{e_0'}$ *is the character of* $U_5(\mathbb{A})$ *attached to* e_0', *then the periods* $(U_5, \psi_{U_5}^{\underline{a},0})$ *and* $(U_5, \psi_{U_5}^{e_0'})$ *are equivalent.*

Proof. Computations very similar to those in the proof of Lemma 7.1.4 show that $y_{\underline{a}}$ is in A_6 if and only if $a_4, a_3 \neq 0$, $a_1 = 2c_1a_3 + c_1^2a_3$, $a_2 = -(c_1^2a_3 + c_1a_3)$, with $c_1 \neq -1$. Let

$$u_1(b_1,b_2,b_3,b_4,b_5) = x_{0100000}(b_1)x_{0010000}(b_2)x_{0000100}(b_3)x_{0000010}(b_4)x_{0000110}(b_5),$$
$$l_1(b_1,b_2,b_3,b_4,b_5) = x_{-0100000}(b_1)x_{-0010000}(b_2)x_{-0000100}(b_3)x_{-0000010}(b_4) \times x_{-0000110}(b_5).$$

Then $u_1(a_3a_4c_1, -(a_3c_1^2 + a_3c_1), c_1a_3, -a_3a_4c_1^2, a_3^2a_4c_1^2)$ maps $y_{\underline{a}}$ to

$$X_{-0001110} + X_{-0101100} + X_{-0000111} + X_{-0111000} + X_{-1010000} + (a_3 + a_3c_1)X_{-0000011} + a_4X_{-0011100}.$$

Then acting on this by

$$l_1\left(\frac{1}{2a_3a_4(c_1+1)}, -\frac{1}{2a_3(c_1+1)}, -\frac{1}{a_3(c_1+1)}, -\frac{1}{2a_3a_4(c_1+1)}, -\frac{1}{4a_3^2a_4(c_1+1)^2}\right)$$

produces

$$X_{-0001110} + X_{-0101100} + X_{-0111000} + X_{-1010000} + (a_3 + a_3c_1)X_{-0000011} + a_4X_{-0011100}.$$

Then acting by a suitable torus element produces e_0'. ∎

Lemma 7.1.12. *Let* $w_3 = w[24315423465423765]$. *Then there is a representative* $\dot{w}_3$ *for* w_3 *in* $GE_7(F)$ *such that* $\dot{w}_3e_0' = X_{-\alpha_1} + X_{-\alpha_3} + X_{-\alpha_4} + X_{-\alpha_5} + X_{-\alpha_6} + X_{-\alpha_7}$.

Proof. One may check (using LiE for example) that w_3 maps the six roots which appear in the expression for e_0' to the six negative simple roots in the GL_7 subgroup. It follows that the identity holds up to nonzero scalars for any representative $\dot{w}[24315423465423765]$. We may then adjust by an element of $T(F)$ to make all the scalars 1. ∎

Remark 7.1.13. Let $s = {}^{2\,0\,\overset{2}{0}\,0\,0\,2}$ be the standard semisimple element attached to the orbit $E_7(a_4)$. Then $(U_5, \psi_{U_5}^{e_0'}) = \mathcal{F}_{s,e_0'}$.

Lemma 7.1.14. *Let* $s' = w_3^{-1}.{}^{2\,2\,\overset{2}{2}\,2\,2\,2}$. *Then* $\mathcal{F}_{s',e_0'}(\mathcal{E}_\pi) \neq 0$.

Proof. Let $e_2 = w_3 e_0' = X_{-\alpha_1} + X_{-\alpha_3} + X_{-\alpha_4} + X_{-\alpha_5} + X_{-\alpha_6} + X_{-\alpha_7}$ and $s'' = \begin{smallmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ & & 2 & & & \end{smallmatrix}$. Then as in the proof of Theorem 5.0.10, $\mathcal{F}_{s'',e_2}$ maps an automorphic form to the GL_7 nondegenerate Whittaker–Fourier integral of its constant term along the A_6 parabolic. Therefore, $\mathcal{F}_{s'',e_2}(\mathcal{E}_\pi) \neq 0$. Since for $\varphi \in \mathcal{E}_\pi$ we have $\mathcal{F}_{s',e_0'}(\varphi)(g) = \mathcal{F}_{s'',e_2}(\varphi)(w_3 g)$, it follows that $\mathcal{F}_{s',e_0'}(\mathcal{E}_\pi) \neq 0$. ∎

Lemma 7.1.15. *$\mathcal{F}_{s,e_0'} \mid \mathcal{F}_{s',e_0'}$. Hence, $\mathcal{F}_{s,e_0'}(\mathcal{E}_\pi) \neq 0$.*

Proof. By Theorem 3.1.11, we only need to check that

$$\mathfrak{g}_u \cap \mathfrak{g}^s_{\geq 1} \subset \mathfrak{g}^{s'}_{\geq 1}.$$

Here $u = e_0'$, $s = \begin{smallmatrix} 2 & 0 & 2 & 0 & 0 & 2 \\ & & 0 & & & \end{smallmatrix}$. In order to check this condition, it is convenient to embed u into a neutral pair. The element u is in the orbit A_6 and it is not hard to check that $w[4].u$ lies in the unipotent radical determined by $\begin{smallmatrix} 0 & 0 & 2 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$. It follows that $w[4].u$ forms a neutral pair with $\begin{smallmatrix} 0 & 0 & 2 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix}$, and thence u forms a neutral pair with

$$w[4].\begin{smallmatrix} 0 & 0 & 2 & 0 & 2 & 0 \\ & & 0 & & & \end{smallmatrix} = \begin{smallmatrix} 0 & 2 & -2 & 2 & 2 & 0 \\ & & 2 & & & \end{smallmatrix} =: s_0.$$

Now, we know that $\mathfrak{g}_u \subset \mathfrak{g}^{s_0}_{\leq 0}$. Hence $\mathfrak{g}_u \cap \mathfrak{g}^s_{\geq 1} \subset \mathfrak{g}^{s_0}_{\leq 0} \cap \mathfrak{g}^s_{\geq 1}$. It is not hard to check that $\mathfrak{g}^{s_0}_{\leq 0} \cap \mathfrak{g}^s_{\geq 1}$ is the sum of the root subgroups attached to the following roots:

$$\{1011000, 0001000, 0101000, 0011000, 0001100, 1000000, 0000001\}$$

and from there its not hard to check that $\mathfrak{g}^{s_0}_{\leq 0} \cap \mathfrak{g}^s_{\geq 1} \subset \mathfrak{g}^{s'}_{\geq 1}$.

In fact, it turns out that $s' = 7s - 6s_0$. It immediately follows that if s acts on X with a positive eigenvalue, and s_0 acts on X with a nonpositive eigenvalue, then s' acts on X with a positive eigenvalue, which is what we wanted. ∎

Corollary 7.1.16. *If $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ is generic, then $\mathcal{E}_\pi$ supports $(U^{G_2}_{\max}, \psi^{\underline{a}}_{U^{G_2}_{\max}}) \circ (U_{A_5''}, \psi^{f_0}_{U_{A_5''}})$.*

Proof. If $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ is generic, then – since $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ depends only on $a_1 - a_2 + a_3$ – we may assume that $a_1 = a_2 = 0$. In this case, by Lemma 7.1.4, the element $f_{\underline{a}}$ is in A_6. Hence, if $\underline{c}$ and $\underline{d}$ are as in Lemmas 7.1.7 and 7.1.8 respectively, then $y_{\underline{c}*\underline{d}*\underline{a}}$, which is conjugate to $f_{\underline{a}}$, is also in A_6. From Proposition 7.1.11, Lemmas 7.1.12 and 7.1.15, and Remark 7.1.13, it follows that $\mathcal{E}_\pi$ supports $(U_5, \psi^{\underline{a},0}_{U_5})$. Then, by Proposition 7.1.10 and Lemmas 7.1.9, 7.1.8, 7.1.7, and 7.1.2, it supports $(U^{G_2}_{\max}, \psi^{\underline{a}}_{U^{G_2}_{\max}}) \circ (U_{A_5''}, \psi^{f_0}_{U_{A_5''}})$ as well. ∎

Reformulating Corollary 7.1.16 gives the main theorem of this section.

Theorem 7.1.17. *$\mathcal{D}_\pi$ is generic.*

Remark 7.1.18. (1) It can be shown that for each $\underline{a} \in F^4$ there is a unique $b \in F$ such that the nilpotent element attached to the character $\psi^{\underline{c}*\underline{d}*\underline{a},b}_{U_5}$ is in the closure of A_6, and that this element is in A_6 if and only if $\psi^{\underline{a}}_{U^{G_2}_{\max}}$ is a generic character.

(2) If π is not of G_2 type but $L^S(s,\pi,\wedge^3)$ has a pole at $s=1$, then Theorem 7.1.17 is still valid for the residual representation $\mathcal{E}_\pi$ with exactly the same proof.

(3) It follows from the proof above that given any irreducible automorphic representation Π of $GE_7(\mathbb{A})$, if $\mathcal{F}_{s,e'_0}(\Pi)\neq 0$ then the $(U_{A''_5},\psi^{f_0}_{U_{A''_5}})$-Fourier coefficients of Π are generic. In particular, this applies to the residue of our Eisenstein series at 1, whenever it exists.

7.2. *Local descent*

Since the results of [17] hold in both the local and global settings, the same set of arguments given in the global setting above also provides a local analogue.

Theorem 7.2.1. *Let F_v be a nonarchimedean local field. Suppose that an irreducible admissible representation Π_v of $GE_7(F_v)$ supports the twisted Jacquet module attached to $(U_5,\psi^{\underline{a},0}_{U_5})$ with $y_{\underline{a}}$ (see Proposition 7.1.11) in the orbit A_6. Then the $(U_{A''_5},\psi^{f_0}_{U_{A''_5}})$-twisted Jacquet module of Π_v supports twisted Jacquet modules attached to $U^{G_2}_{\max}$ and all generic characters of $U^{G_2}_{\max}$. In particular, this holds when Π_v is the local component of any irreducible summand of $\mathcal{E}_\pi$ where π has the property that $L^S(s,\pi,\wedge^3)$ has a pole at $s=1$.*

7.3. *Unramified constituents of $\mathcal{E}_\pi$*

7.3.1. Unramified lifting. Let χ be an unramified character of $GL_7(F_v)$ where F_v is nonarchimedean. Recall that our isomorphism of the Levi M of GE_7 with $GL_7\times GL_1$ maps $h(t_1,\dots,t_8)$ to

$$\begin{pmatrix} t_8^{-1}t_7 &&&&&& \\ & t_7^{-1}t_6 &&&&& \\ && t_6^{-1}t_5 &&&& \\ &&& t_5^{-1}t_4 &&& \\ &&&& t_4^{-1}t_2t_3 && \\ &&&&& t_3^{-1}t_2t_1 & \\ &&&&&& t_1^{-1}t_2 \end{pmatrix}.$$

Thus, it identifies χ with a matrix $\tilde{t}=\mathrm{diag}(\tilde{t}_1,\dots,\tilde{t}_7)$ in $GL_7(\mathbb{C})$ such that

$$\chi(h(t_1,\dots,t_8))=\tilde{t}_1^{n_7-n_8}\tilde{t}_2^{n_6-n_7}\tilde{t}_3^{n_5-n_6}\tilde{t}_4^{n_4-n_5}\tilde{t}_5^{n_2+n_3-n_4}\tilde{t}_6^{n_1+n_2-n_3}\tilde{t}_7^{n_2-n_1},$$

where $n_i=\mathrm{ord}(t_i)$ for $i=1,\dots,8$.

If $\tilde{t}\in G_2(\mathbb{C})$ then $\tilde{t}_3=\tilde{t}_1/\tilde{t}_2$, $\tilde{t}_4=1$, $\tilde{t}_5=\tilde{t}_2/\tilde{t}_1$, $\tilde{t}_6=\tilde{t}_2^{-1}$, and $\tilde{t}_7=\tilde{t}_1^{-1}$, hence

$$\chi(h(t_1,\dots,t_8))=\tilde{t}_1^{n_1-2n_2-n_3+n_4+n_5-n_6+n_7-n_8}\tilde{t}_2^{-n_1+2n_3-n_4-n_5+2n_6-n_7}.$$

We can rephrase this as follows. Let $\lambda_1 = \varpi_1 - 2\varpi_2 - \varpi_3 + \varpi_4 + \varpi_5 - \varpi_6 + \varpi_7 - \varpi_8$, and $\lambda_2 = -\varpi_1 + 2\varpi_3 - \varpi_4 - \varpi_5 + 2\varpi_6 - \varpi_7$, and let χ_i be the unramified character of $GL_1(F_v)$ attached to $\tilde{t}_i$ for $i = 1, 2$. Then

$$\chi(t) = \chi_1(t^{\lambda_1})\chi_2(t^{\lambda_2}) \quad \text{for } t = h(t_1, \dots, t_8) \in M. \tag{7.3.1}$$

This element $\tilde{t} \in G_2(\mathbb{C}) \subset GL_7(\mathbb{C})$ also determines a character μ of the standard torus of G_2. If α is the short simple root of G_2 and β is the long simple root, then $\alpha^\vee$ is the long simple coroot and is identified with the long simple root of the dual group, while $\beta^\vee$ is identified with the short simple root of the dual. Then

$$\mu(t_1^{\alpha^\vee} t_2^{\beta^\vee}) = \left(\frac{\tilde{t}_1}{\tilde{t}_2}\right)^{n_2}\left(\frac{\tilde{t}_2^2}{\tilde{t}_1}\right)^{n_1} = \tilde{t}_1^{-n_1+n_2}\tilde{t}_2^{2n_1-n_2},$$

where $n_i = \operatorname{ord}(t_i)$ for $i = 1, 2$.

7.3.2. Degeneration. Recall that P is the standard parabolic subgroup of GE_7 whose unipotent radical contains U_{α_i} if and only if $i = 2$, and Q is the standard parabolic subgroup of GE_7 whose unipotent radical contains U_{α_i} if and only if $i = 4$ or 6.

Suppose now that π_v is a principal series representation of $GL_7(F_v)$ which is attached to a character of the form (7.3.1). We consider the representation $\operatorname{Ind}_{P(F_v)}^{GE_7(F_v)} \pi_v \cdot |\tilde{\varpi}_2|$. If π_v is the local component of a cuspidal representation π of G_2 type, then the residual representation $\mathcal{E}_\pi$ is a quotient of $\operatorname{Ind}_{P(\mathbb{A})}^{GE_7(\mathbb{A})} \pi \cdot |\tilde{\varpi}_2|$. It may be reducible, but it is in the discrete spectrum, and if Π is any irreducible summand, then Π_v is a quotient of $\operatorname{Ind}_{P(F_v)}^{GE_7(F_v)} \pi_v \cdot |\tilde{\varpi}_2|$. Moreover, if Π_v is unramified, then it is the unique unramified constituent of $\operatorname{Ind}_{P(F_v)}^{GE_7(F_v)} \pi_v \cdot |\tilde{\varpi}_2|$.

Lemma 7.3.2. *Let w_6 be $w[423546542314376542]$ as in Section 6.2, so w_6 maps the five roots in the character $\psi_U^{e_0}$ to $\{\alpha_i : i = 1, 2, 3, 5, 7\}$. Let w_0 denote the longest element of the Weyl group of GE_7 which is reduced by P on the left and right. Then $w_6 w_0$ maps λ_1 to $\varpi_4 - \varpi_6 - \varpi_8$, λ_2 to $-\varpi_4 + 2\varpi_6 - \varpi_8$, and ϖ_2 to $\rho_Q - \rho_B + (3/2)\varpi_8$.*

Proof. This can be checked using a computer software package such as LiE. ■

Since $w_6 w_0 \lambda_1$ pairs trivially with all coroots in the Levi of Q, it induces a rational character ν_1 of this Levi. Similarly, $w_6 w_0 \lambda_2$ induces a rational character ν_2.

Corollary 7.3.3. *The unramified constituent of $\operatorname{Ind}_{P(F_v)}^{GE_7(F_v)} \pi_v |\tilde{\varpi}_2|$ is equal to that of*

$$\operatorname{Ind}_{Q(F_v)}^{GE_7(F_v)} (\chi_1 \circ \nu_1)(\chi_2 \circ \nu_2)\tilde{\varpi}_8^{3/2}. \tag{7.3.4}$$

Proposition 7.3.5. *Let (s, u) be a Whittaker pair such that u is contained in an orbit which is greater than or not related to A_6. Let $U = \exp(\mathfrak{g}^s_{\geq 2})$. Then both $\mathcal{J}_{(U,\psi_u)}$ and $\mathcal{J}_{N_{s,u},\psi_u}$ kill the representation (7.3.4).*

Proof. This follows from Corollary 3.2.8 (cf. Remark 3.2.10). ■

7.4. Cuspidality of the A_5'' descent module

The purpose of this section is to show that $\mathcal{D}_\pi$ is cuspidal, provided that there exists a finite place v_0 such that π_{v_0} is a principal series representation of $GL_7(F_{v_0})$ which is attached to a character of the form (7.3.1). There are two maximal parabolic subgroups of G_2. Recall that β denotes the long simple root of G_2 and α denotes the short one, and for $\gamma \in \{\beta, \alpha\}$, P_γ denotes the maximal parabolic subgroup of G_2 whose Levi M_γ contains the root subgroup U_γ attached to γ. Finally, N_γ denotes the unipotent radical of P_γ.

7.4.1. Constant term along N_α.

Lemma 7.4.1. *Let $h_{P_\alpha} = 2\alpha^\vee + 4\beta^\vee$. This is the standard semisimple element of G_2 which is attached to the parabolic P_α.*

(1) *The embedding of G_2 into GE_7 identifies h_{P_α} with $2\alpha_2^\vee + 2\alpha_3^\vee + 4\alpha_4^\vee + 2\alpha_5^\vee$.*

(2) *The weight attached to this semisimple element is* $\begin{smallmatrix} -2 & 0 & 2 & 0 & -2 & 0 \\ & & 0 & & & \end{smallmatrix}$.

(3) *The Weyl element $w_{P_\alpha} = w[134567245631]$ maps this weight to the dominant weight* $\begin{smallmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix}$.

(4) *Let*

$$f_1 = X_{-0100000} + X_{-0011000} + X_{-0001100} + X_{-0000110} + X_{-0000011}.$$

Then there exists a representative $\dot{w}_{P_\alpha}$ for w_{P_α} which maps f_0 to f_1.

Proof. The embedding fixed in Section 6.1 maps $\alpha^\vee$ to $\alpha_2^\vee + \alpha_4^\vee + \alpha_5^\vee$ and $\beta^\vee$ to $\alpha_4^\vee$ so (1) is clear. Parts (2) and (3) can be checked using LiE. As for part (4), let

$$S_{f_1} = \{-0100000, -0011000, -0001100, -0000110, -0000011\}.$$

Then we can first check using LiE that w_{P_α} maps the five roots γ such that X_γ appears in f_0 to the five roots of S_{f_1}. This ensures that any representative $\dot{w}_{P_\alpha}$ maps f_0 to $\sum_{\gamma \in S_{f_1}} c_\gamma X_\gamma$ for some quintuple $(c_\gamma)_{\gamma \in S_{f_1}}$ of elements of $F^\times$. Since $\dot{w}_{P_\alpha}$ is unique up to an arbitrary element of the torus $T(F)$, it suffices to show that for any such quintuple $(c_\gamma)_{\gamma \in S_{f_1}}$, there is an element $t \in T(F)$ which acts on X_γ by c_γ for each $\gamma \in S_{f_1}$. Since E_8 is of adjoint type and GE_7 contains the full torus of E_8, the elements of $F^\times$ by which t acts on $X_{\alpha_1}, \dots, X_{\alpha_7}$ can be chosen arbitrarily. Since S_{f_1} is a subset of a basis of the root lattice, it follows that the scalars by which t acts on $\{X_\gamma : \gamma \in S_{f_1}\}$ can be chosen arbitrarily as well. ■

Lemma 7.4.2. *Let U_1 be the unipotent subgroup of GE_7 such that $\Phi(U_1, T) = \Phi^+(GE_7, T) \smallsetminus \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, 1010000, 0000110\}$. Let $\psi_{U_1}^{f_0}$ be the character of U_1 determined by f_0, and let* tri *denote the trivial character of N_α. Then the composed period $(N_\alpha, \mathrm{tri}) \circ (U_{A_5''}, \psi_{U_{A_5''}}^{f_0})$ is equivalent to $(U_1, \psi_{U_1}^{f_0})$.*

Proof. This follows from the exchange lemma (Lemma 7.1.1). (Cf. Lemma 7.1.2.) ■

Now let $U_2 = w_{P_\alpha} U_1 w_{P_\alpha}^{-1}$. Then it follows from Lemma 7.4.1 (4) that $(U_1, \psi_{U_1}^{f_0})$ is equivalent to $(U_2, \psi_{U_2}^{f_1})$.

Lemma 7.4.3. *Let S_3 be the set which consists of all positive roots of E_7 except*

0000001, 0000100, 0010000, 1000000, 1010000, 1011000, 1011100, 1011110, 1011111,

and in addition contains −1000000, −1010000. *This set is closed under addition, and hence determines a unipotent subgroup U_3. The nilpotent element f_1 determines a character of $U_3(\mathbb{A})$ which we denote by $\psi_{U_3}^{f_1}$. Then $(U_2, \psi_{U_2}^{f_1})$ is equivalent to $(U_3, \psi_{U_3}^{f_1})$.*

Proof. We apply the exchange lemma (Lemma 7.1.1) six times, exchanging −1111100 for 1122100, −1111000 for 1112100, −1011111 for 1111111, −1011110 for 1111110, −1011100 for 1111100, and −1011000 for 1111000. ∎

Lemma 7.4.4. *For $a, b \in F$, let $f_2(a,b) = f_1 + aX_{-1011110} + bX_{-1011111}$. Let U_4 be the product of U_3 and the two-dimensional unipotent group corresponding to* 1011110 *and* 1011111. *Then*

$$\left(U_3, \psi_{U_3}^{f_1}\right) = \sum_{a,b\in F} \left(U_4, \psi_{U_4}^{f_2(a,b)}\right).$$

Proof. This follows from taking the Fourier expansion on the two-dimensional unipotent group corresponding to 1011110 and 1011111. ∎

Lemma 7.4.5. *The element $f_2(a,b)$ lies in the orbit $D_6(a_1)$ unless $a = b = 0$.*

Proof. This was checked using GAP and SageMath. An element X of $\mathfrak{e}_7$ is in $D_6(a_1)$ if and only if $\operatorname{Rank} \operatorname{ad}(X)^k$ is given as in the table for the listed values of k.

k	10	11	12	13	14
$\operatorname{Rank} \operatorname{ad}(X)^k$	11	6	3	2	1

GAP was used to obtain adjoint matrices for a Chevalley basis of $\mathfrak{e}_7$. These were then loaded into SageMath, in order to work in the polynomial ring $\mathbb{Z}[a,b]$. The matrices $\operatorname{ad}(f(a,b))^k$ were then computed, starting with $k = 1$ and continuing until the zero matrix was obtained. Next, we deleted any rows and columns consisting entirely of zeros to obtain a sequence of smaller matrices, which we refer to as the nonzero parts of the matrices $\operatorname{ad}(f(a,b))^k$. Clearly, each matrix has the same rank as its nonzero part. Next, we computed the ranks of the matrices $\operatorname{ad}(f(a,b))^k$, deducing that as an element of $\mathfrak{e}_7(\mathbb{Z}[a,b])$, $f_2(a,b)$ lies in the orbit $D_6(a_1)$. This implies that for any specific scalars a and b, $f_2(a,b)$ lies in the Zariski closure of $D_6(a_1)$.

Now, each stable orbit which is less than $D_6(a_1)$ is contained in the closure of either $E_7(a_5)$ or D_5. (See, for example, the diagram in [5, p. 442].)

If X lies in the closure of $E_7(a_5)$ then $\operatorname{ad}(X)^{14} = 0$. The nonzero part of $\operatorname{ad}(f(a,b))^{14}$ is

$$\begin{pmatrix} -1716a^2 & -1716ab \\ -1716ab & -1716b^2 \end{pmatrix}.$$

We deduce that if $f(a,b)$ is in the closure of $E_7(a_5)$ then $a = b = 0$.

If X lies in the closure of D_5, then rank $\mathrm{ad}(X)^{11} \le 4$. A suitable permutation of the rows and columns of the nonzero part of $\mathrm{ad}(f(a,b))^{11}$ puts it into the form

$$\begin{pmatrix} 0 & 0 & 0 & A \\ 0 & 0 & B & 0 \\ 0 & -{}^tB & 0 & 0 \\ -{}^tA & 0 & 0 & 0 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 264a & -330a & -66b & -528a^2 & -330a^2 & -528ab & 330ab \\ 66a & 330b & -264b & 0 & -528ab & -330ab & -528b^2 & 330b^2 \end{pmatrix},$$

$$B = \begin{pmatrix} -594a & -528a^2 & -528ab & 1188a^2 & 1188ab \\ -594b & -528ab & -528b^2 & 1188ab & 1188b^2 \end{pmatrix}.$$

It is fairly easy to see that if $(a,b) \neq (0,0)$, then A and $-{}^tA$ are of rank 2, B and $-{}^tB$ are of rank 1, and $\mathrm{ad}(f(a,b))^{11}$ is of rank 6. ∎

Lemma 7.4.6. *Let $U_4' = w[31]U_4w[13]$, which is the unipotent radical of a parabolic subgroup and contains the root subgroup U_{α_i} attached to the simple root α_i if $i = 2, 3,$ or 6. Let $\dot{w}[31]$ be a representative for $w[31]$ and $f_2'(a,b) = \mathrm{Ad}(\dot{w}[31]) f_2(a,b)$. Then for any smooth automorphic function φ,*

$$\varphi^{(U_4, \psi_{U_4}^{f_2(a,b)})}(g) = \varphi^{(U_4', \psi_{U_4'}^{f_2'(a,b)})}(\dot{w}[31]g).$$

In particular, the periods $(U_4, \psi_{U_4}^{f_2(a,b)})$ and $(U_4', \psi_{U_4'}^{f_2'(a,b)})$ are equivalent.

Proposition 7.4.7. *Let $\mathcal{E} = \bigotimes_v \mathcal{E}_v$ be an irreducible automorphic representation of $GE_7(\mathbb{A})$ and assume that there is a finite place v_0 such that $\mathcal{E}_{v_0}$ is induced from a character of the group Q from Section 7.3.2. Then $\mathcal{E}$ does not support the coefficient $(U_4', \psi_{U_4'}^{f_2'(a,b)})$ for $(a,b) \neq (0,0)$.*

Proof. This follows from Corollary 3.2.7, since the Richardson orbit of Q is A_6 (cf. Remark 3.2.10) and $f_2'(a,b)$ is in $D_6(a_1)$ by Lemma 7.4.5. ∎

Proposition 7.4.8. *Let S_5 be the set which consists of all positive roots of E_7 except*

$$0000001, 0000100, 0010000, 1000000, 1010000.$$

Then for any smooth automorphic function φ,

$$\varphi^{(U_4, \psi_{U_4}^{f_1})}(g) = \int_{\mathbb{A}} \int_{\mathbb{A}} \varphi^{(U_5, \psi_{U_5}^{f_1})}(x_{-1000000}(r_1) x_{-1010000}(r_2) g)\, dr_1\, dr_2.$$

In particular, $(U_4, \psi_{U_4}^{f_2(0,0)})$ is equivalent to $(U_5, \psi_{U_5}^{f_2(0,0)})$.

Proof. This is another application of the exchange lemma (Lemma 7.1.1). ∎

Lemma 7.4.9. *Let U_6 be the product of U_5 and the two-dimensional unipotent group $U_{\alpha_1}U_{\alpha_1+\alpha_3}$. For $a,b \in F$, let $f_3(a,b) = f_1 + aX_{-\alpha_1} + bX_{-1010000}$. Then $(U_5, \psi_{U_5}^{f_2(0,0)}) = \sum_{a,b\in F}(U_6, \psi_{U_6}^{f_3(a,b)})$.*

Proof. This is again just a Fourier expansion. ■

Lemma 7.4.10. *The residual representation $\mathcal{E}_\pi$ does not support the period $(U_6, \psi_{U_6}^{f_3(0,0)})$.*

Proof. This holds because U_6 contains the full unipotent radical of the standard maximal parabolic subgroup of E_7 whose Levi is of type D_6, and the character $\psi_{U_6}^{f_3(0,0)}$ is trivial on this subgroup. Thus $(U_6, \psi_{U_6}^{f_3(0,0)})$ factors through the constant term attached to this maximal parabolic. But that parabolic is not associate to the one used in constructing our Eisenstein series, so neither the Eisenstein series nor its residue will support this constant term. ■

Lemma 7.4.11. *If $(a,b) \neq (0,0)$ then $f_3(a,b)$ lies in the orbit D_6.*

Proof. We use the same method which we used above to find the orbit of $f_2(a,b)$. ■

Proposition 7.4.12. *Let $\mathcal{E} = \bigotimes_v \mathcal{E}_v$ be an irreducible automorphic representation of $GE_7(\mathbb{A})$ and assume that there is a finite place v_0 such that $\mathcal{E}_{v_0}$ is induced from a character of the group Q from Section 7.3.2. Then $\mathcal{E}$ does not support the coefficient $(U_6, \psi_{U_6}^{f_3(a,b)})$ for $(a,b) \neq (0,0)$.*

Proof. This follows from Corollary 3.2.7 and Lemma 7.4.11, because the Richardson orbit of Q is A_6 (cf. Remark 3.2.10). ■

Hence, we have the following theorem.

Theorem 7.4.13. *Let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type, such that π_{v_0} is induced from a character of the form (7.3.1) at some finite place v_0. Then the constant term of $\mathcal{E}_\pi^{(U_{A_5''}, \psi_{U_{A_5''}}^{f_0})}$ along N_α is zero.*

7.4.2. Constant term along N_β. Let $h_{P_\beta} = 4\alpha^\vee + 6\beta^\vee$. This is the standard semisimple element of G_2 which is attached to the parabolic P_β. The embedding of G_2 into GE_7 identifies h_{P_β} with $4\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee$. The weight attached to this semisimple element is $\begin{smallmatrix}-4 & 2 & 0 & 2 & -4 & 0\\ & & 2 & & & \end{smallmatrix}$. The Weyl element $w_{P_\beta} = w[3,4,1,3,2,4,5,6,7,4,3,2,4,5,6,4,3,1]$ maps this to the dominant weight $\begin{smallmatrix}0 & 2 & 0 & 0 & 0 & 0\\ & & 0 & & & \end{smallmatrix}$.

Lemma 7.4.14. *Let U_1 be the unipotent subgroup of GE_7 such that $\Phi(U_1,T) = \Phi^+(GE_7,T) \smallsetminus \{0001000, 1011000, 0001110, 1010000, 0000110, 1000000, 0000010\}$. Let $\psi_{U_1}^{f_0}$ be the character of U_1 determined by f_0, and let tri denote the trivial character of $N_\beta(\mathbb{A})$. Then the composed period $(N_\beta, \mathrm{tri}) \circ (U_{A_5''}, \psi_{U_{A_5''}}^{f_0})$ is equivalent to $(U_1, \psi_{U_1}^{f_0})$.*

Proof. This follows from the exchange lemma (Lemma 7.1.1). (Cf. Lemma 7.1.2.) ∎

Now let $U_2 = w_{P_\beta} U_1 w_{P_\beta}^{-1}$ and

$$f_1 = X_{-0100000} + X_{-0001000} + X_{-0000100} + X_{-0000010} + X_{-0000001}.$$

Then there exists a representative $\dot{w}_{P_\beta}$ for w_{P_β} which maps f_0 to f_1, so $(U_1, \psi_{U_1}^{f_0})$ is equivalent to $(U_2, \psi_{U_2}^{f_1})$.

Lemma 7.4.15. *Let S_3 be the set which consists of all positive roots of E_7 except*

$$0010000, 0011000, 0011100, 0111000, 1000000, 1010000, 1011000,$$

$$1011100, 1011110, 1111000, 1111100, 1122100, 1122110, 1122210,$$

and in addition contains

$$-1111000, -1011100, -1011000, -0011000, -1122100, -1010000, -0010000.$$

This set is closed under addition, and hence determines a unipotent subgroup U_3. The nilpotent element f_1 determines a character of $U_3(\mathbb{A})$ which we denote $\psi_{U_3}^{f_1}$. Then $(U_2, \psi_{U_2}^{f_1})$ is equivalent to $(U_3, \psi_{U_3}^{f_1})$.

Proof. We apply the exchange lemma (Lemma 7.1.1) five times, exchanging -1122210 for 1123210, -1122111 for 1122211, -0011100 for 0011110, -0111000 for 0111100, -1122110 for 1122111. ∎

Lemma 7.4.16. *For $a \in F$, let $f_2(a) = f_1 + aX_{-1122210}$. Let U_4 be the product of U_3 and the one-dimensional unipotent group corresponding to* 1122210. *Then*

$$(U_3, \psi_{U_3}^{f_1}) = \sum_{a \in F} (U_4, \psi_{U_4}^{f_2(a)}).$$

Proof. This follows from taking the Fourier expansion on the one-dimensional unipotent group corresponding to 1122210. ∎

Lemma 7.4.17. *The element $f_2(a)$ lies in the orbit $D_6(a_1)$ unless $a = 0$.*

Proof. The method is similar to that of Lemma 7.4.5. ∎

Proposition 7.4.18. *Let $\mathcal{E} = \bigotimes_v \mathcal{E}_v$ be an irreducible automorphic representation of $GE_7(\mathbb{A})$ and assume that there is a finite place v_0 such that $\mathcal{E}_{v_0}$ is induced from a character of the group Q from Section 7.3.2. Then $\mathcal{E}$ does not support the coefficient $(U_4, \psi_{U_4}^{f_2(a)})$ for $a \neq 0$.*

Proof. Recall that for $S \subset \{1, 2, 3, 4, 5, 6, 7\}$, P_S denotes the standard parabolic subgroup whose Levi contains the root subgroups attached to the simple roots $\{\alpha_i : i \in S\}$ and whose unipotent radical contains the root subgroups attached to the simple roots $\{\alpha_i : i \notin S\}$. Let $w = w[425423413]$. Let $U_4' = wU_4w^{-1}$, which is contained in the unipotent radical of

$P_{\{2,3,5,6\}}$. Let $\dot{w}$ be a representative for w and $f_2'(a) = \mathrm{Ad}(\dot{w}) f_2(a)$. Then for any smooth automorphic function φ,

$$\varphi^{(U_4, \psi_{U_4}^{f_2(a)})}(g) = \varphi^{(U_4', \psi_{U_4'}^{f_2'(a)})}(\dot{w} g).$$

In particular, the periods $(U_4, \psi_{U_4}^{f_2(a)})$ and $(U_4', \psi_{U_4'}^{f_2'(a)})$ are equivalent.

Hence, it suffices to show that $\mathcal{E}$ does not support the coefficient $(U_4', \psi_{U_4'}^{f_2'(a)})$ for $a \neq 0$. This follows from Corollary 3.2.7 and Lemma 7.4.17, because the Richardson orbit of Q is A_6 (see Remark 3.2.10). ∎

Proposition 7.4.19. *Let S_5 be the set which consists of all positive roots of E_7 except*

$$0010000, 0011000, 0111000, 1000000, 1010000, 1011000, 1111000, 1122100,$$

and in addition contains $-1010000, -0010000$. *Then* $(U_4, \psi_{U_4}^{f_2(0)})$ *is equivalent to* $(U_5, \psi_{U_5}^{f_2(0)})$.

Proof. This is another application of the exchange lemma (Lemma 7.1.1) five times: exchanging -1011100 for 1011110, -1111000 for 1111100, -1122100 for 1122110, -0011000 for 0011100, -1011000 for 1011100. ∎

Lemma 7.4.20. *Let U_6 be the product of U_5 and the one-dimensional unipotent group $U_{1122100}$. For $a \in F$, let $f_3(a) = f_1 + aX_{-1122100}$. Then*

$$(U_5, \psi_{U_5}^{f_2(0)}) = \sum_{a \in F} (U_6, \psi_{U_6}^{f_3(a)}).$$

Proof. This is again just a Fourier expansion. ∎

Lemma 7.4.21. *If $a \neq 0$ then $f_3(a)$ lies in the orbit D_6.*

Proof. The method is similar to that of Lemma 7.4.5. ∎

Proposition 7.4.22. *Let $\mathcal{E} = \bigotimes_v \mathcal{E}_v$ be an irreducible automorphic representation of $GE_7(\mathbb{A})$ and assume that there is a finite place v_0 such that $\mathcal{E}_{v_0}$ is induced from a character of the group Q from Section 7.3.2. Then $\mathcal{E}$ does not support the coefficient $(U_6, \psi_{U_6}^{f_3(a)})$ for $a \neq 0$.*

Proof. Let $U_6' = w[3,4,1,3] U_6 w[3,4,1,3]$. Let $\dot{w}[3,4,1,3]$ be a representative for $w[3,4,1,3]$ and $f_3'(a) = \mathrm{Ad}(\dot{w}[3,4,1,3]) f_3(a)$. Then for any φ,

$$\varphi^{(U_6, \psi_{U_6}^{f_3(a)})}(g) = \varphi^{(U_6', \psi_{U_6'}^{f_3'(a)})}(\dot{w}[3,4,1,3] g).$$

In particular, the periods $(U_6, \psi_{U_6}^{f_3(a)})$ and $(U_6', \psi_{U_6'}^{f_3'(a)})$ are equivalent.

Hence, it suffices to show that $\mathcal{E}$ does not support the coefficient $(U_6', \psi_{U_6'}^{f_3'(a)})$ for $a \neq 0$.

Now, write s_{D_6} for the standard semisimple element attached to the orbit D_6. Let V_{D_6} be the unipotent group whose Lie algebra is $\mathfrak{g}_{\geq 2}^{s_{D_6}}$. Then $U_6' = V_{D_6} U_{0000100} U_{0001100}$, and $\psi_{U_6'}^{f_3'(a)}$ is trivial on $U_{0000100}U_{0001100}$. So $\varphi^{(U_6', \psi_{U_6'}^{f_3'(a)})}$ may be written as a double integral with the inner integral being $\varphi^{(V_{D_6}, \psi_{V_{D_6}}^{f_3'(a)})}$. So, it suffices to show that the coefficient $(V_{D_6}, \psi_{V_{D_6}}^{f_3'(a)})$ vanishes on $\mathcal{E}$. This follows from Corollary 3.2.7 and Lemma 7.4.21, because the Richardson orbit of Q is A_6 (see Remark 3.2.10), and D_6 is greater than A_6. The role of "P" in Corollary 3.2.7 is played by $P_{\{4\}}$. ∎

Lemma 7.4.23. *Let U_7 be the product of U_6 and the two-dimensional unipotent group $U_{0111000}U_{1111000}$. For $a, b \in F$, let $f_4(a,b) = f_1 + aX_{-0111000} + bX_{-1111000}$. Then*

$$(U_6, \psi_{U_6}^{f_3(0)}) = \sum_{a,b \in F} (U_7, \psi_{U_7}^{f_4(a,b)}).$$

Proof. This is again just a Fourier expansion. ∎

Lemma 7.4.24. *If $(a,b) \neq (0,0)$ then $f_4(a,b)$ lies in the orbit $D_6(a_1)$.*

Proof. The method is similar to that of Lemma 7.4.5. ∎

Proposition 7.4.25. *Let $\mathcal{E} = \bigotimes_v \mathcal{E}_v$ be an irreducible automorphic representation of $GE_7(\mathbb{A})$ and assume that there is a finite place v_0 such that $\mathcal{E}_{v_0}$ is induced from a character of the group Q from Section 7.3.2. Then $\mathcal{E}$ does not support the coefficient $(U_7, \psi_{U_7}^{f_4(a,b)})$ for $(a,b) \neq (0,0)$.*

Proof. Let $U_7' = w[13]U_7 w[31]$ and $f_4'(a,b) = \mathrm{Ad}(\dot{w}[13]).f_4(a,b)$. Then $U_7' = U_{\{3,4\}}$. We apply Corollary 3.2.7 with $P = P_{\{3,4\}}$. Since the Richardson orbit of Q is A_6 (see Remark 3.2.10), it follows from Lemma 7.4.24 that $\mathcal{E}_{v_0}$ does not support the coefficient $(U_7', \psi_{U_7'}^{f_4'(a,b)})$, which is clearly equivalent to $(U_7, \psi_{U_7}^{f_4(a,b)})$. ∎

Proposition 7.4.26. *Let S_8 be the set which consists of all positive roots of E_7 except* 0010000, 1010000. *Then $(U_7, \psi_{U_7}^{f_4(0,0)})$ is equivalent to $(U_8, \psi_{U_8}^{f_4(0,0)})$.*

Proof. This is another application of the exchange lemma (Lemma 7.1.1) twice: exchanging -1010000 for 1011000, -0010000 for 0011000. ∎

Lemma 7.4.27. *Let U_9 be the product of U_8 and the two-dimensional unipotent group $U_{0010000}U_{1010000}$. For $a, b \in F$, let $f_5(a,b) = f_1 + aX_{-0010000} + bX_{-1010000}$. Then*

$$(U_8, \psi_{U_8}^{f_4(0,0)}) = \sum_{a,b \in F} (U_9, \psi_{U_9}^{f_5(a,b)}).$$

Proof. This is again just a Fourier expansion. ∎

Lemma 7.4.28. *If $(a,b) \neq (0,0)$ then $f_5(a,b)$ lies in the orbit D_6.*

Proof. The method is similar to that of Lemma 7.4.5. ∎

Proposition 7.4.29. *Let $\mathcal{E} = \bigotimes_v \mathcal{E}_v$ be an irreducible automorphic representation of $GE_7(\mathbb{A})$ and assume that there is a finite place v_0 such that $\mathcal{E}_{v_0}$ is induced from a character of the group Q from Section 7.3.2. Then $\mathcal{E}$ does not support the coefficient $(U_9, \psi_{U_9}^{f_5(a,b)})$ for $(a,b) \neq (0,0)$.*

Proof. Note that U_9 is the full unipotent radical of the parabolic $P_{\{1\}}$. We apply Corollary 3.2.7 with $P = P_{\{1\}}$. The result follows from Lemma 7.4.28, because the Richardson orbit of Q is A_6 (see Remark 3.2.10). ∎

Lemma 7.4.30. *The residual representation $\mathcal{E}_\pi$ does not support the period $(U_9, \psi_{U_9}^{f_5(0,0)})$.*

Proof. This holds because U_9 contains the full unipotent radical of the standard maximal parabolic subgroup $P_{\{1,2,4,5,6,7\}}$, and the character $\psi_{U_9}^{f_5(0,0)}$ is trivial on this subgroup. Thus $(U_9, \psi_{U_9}^{f_5(0,0)})$ factors through the constant term attached to this maximal parabolic. But that parabolic is not associate to the one used in constructing our Eisenstein series, so neither the Eisenstein series nor its residue will support this constant term. ∎

Hence, we have the following theorem.

Theorem 7.4.31. *Let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type, such that π_{v_0} is induced from a character of the form (7.3.1) at some finite place v_0. Then the constant term of $\mathcal{E}_\pi^{(U_{A_5''}, \psi_{U_{A_5''}}^{f_0})}$ along N_β is zero.*

Therefore, Theorems 7.4.13 and 7.4.31 together imply the following theorem on the cuspidality of our descent module $\mathcal{E}_\pi^{(U_{A_5''}, \psi_{U_{A_5''}}^{f_0})}$.

Theorem 7.4.32. *Let π be an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type, such that π_{v_0} is induced from a character of the form (7.3.1) at some finite place v_0. Then $\mathcal{E}_\pi^{(U_{A_5''}, \psi_{U_{A_5''}}^{f_0})}$ is a cuspidal automorphic representation of $G_2(\mathbb{A})$.*

Remark 7.4.33. It follows from the proof above that given any irreducible automorphic representation Π of $GE_7(\mathbb{A})$, if $D_6(a_1), D_6 \notin \mathfrak{n}(\Pi)$ and the constant terms of Π along $P_{\{2,3,4,5,6,7\}}$ and $P_{\{1,2,4,5,6,7\}}$ are identically zero, then the $(U_{A_5''}, \psi_{U_{A_5''}}^{f_0})$-Fourier coefficients of Π are cuspidal.

7.5. Unramified local descent

The purpose of this section is to show that π is a weak functorial lift of each irreducible summand of $\mathcal{D}_\pi$, provided that for almost all finite places v, π_v is a principal series representation of $GL_7(F_v)$ which is attached to a character of the form (7.3.1).

Recall that $P_{A_5''} = M_{A_5''}U_{A_5''} = P_s = M_sU_s$ is the parabolic subgroup defined as in Section 3.1, where $s = s_{A_5''}$ is the standard semisimple element (see Definition 5.0.1) attached to A_5'', $M_{A_5''} = M_s$ is the Levi subgroup, and $U_{A_5''} = U_s$ is the unipotent radical.

We consider the twisted Jacquet module

$$\mathcal{J}_{U_{A_5''},\psi^{f_0}_{U_{A_5''}}}\left(\mathrm{Ind}_{Q(F_v)}^{GE_7(F_v)}(\chi_1\circ\nu_1)(\chi_2\circ\nu_2)\widetilde{\varpi}_8^{3/2}\right).$$

For χ_i and ν_i, see §7.3. To that end we study the space $Q(F_v)\backslash GE_7(F_v)/G_2(F_v)U_{A_5''}(F_v)$ of double cosets, where G_2 is embedded into $M_{A_5''}$ as the stabilizer of f_0.

For $\gamma \in Q(F_v)\backslash GE_7(F_v)/G_2(F_v)U_{A_5''}(F_v)$ we say that γ is *admissible* if we have $\psi^{f_0}_{U_{A_5''}}|_{U_{A_5''}\cap(\gamma^{-1}Q\gamma)} \equiv 1$. Each double coset contains elements of the form $w\mu$ with w in the Weyl group of minimal length in its $(Q, P_{A_5''})$-double coset, and $\mu \in M_{A_5''}(F_v)$. Indeed, μ may be taken modulo $G_2(F_v)$ on the right and $M_{A_5''}\cap w^{-1}Qw$ – which is a standard parabolic subgroup of $M_{A_5''}$– on the left. Then

$$\psi^{f_0}_{U_{A_5''}}|_{U_{A_5''}\cap(\gamma^{-1}Q\gamma)} \equiv 1 \iff \mu\cdot\psi^{f_0}_{U_{A_5''}}\Big|_{U_{A_5''}\cap(w^{-1}Qw)} \equiv 1.$$

Note also that $\mu\cdot\psi^{f_0}_{U_{A_5''}} = \psi^{\mathrm{Ad}(\mu).f_0}_{U_{A_5''}}$. Clearly $\mathrm{Ad}(\mu).f_0$ is in the open orbit for the action of $M_{A_5''}$ on $\mathfrak{g}_{-2}^{s_{A_5''}}$.

Lemma 7.5.1. *Let*

$$\Phi_{A_5''}(2) = \{\alpha\in\Phi : \langle\alpha, s_{A_5''}\rangle = 2\}.$$

Then $\sum_{\alpha\in\Phi_{A_5''}(2)} a_\alpha X_{-\alpha}$ *is in* A_5'' *if and only if*

$$(a_{0011110}a_{0101110} - a_{0001110}a_{0111110} - a_{0000110}a_{0112110} - a_{0000010}a_{0112210})^2$$
$$\times(a_{1011100}a_{1111000} - a_{1011000}a_{1111100} + a_{1010000}a_{1112100} + a_{1000000}a_{1122100})^2 a^2_{0000001} \neq 0.$$

Proof. Direct computation using SageMath, with adjoint matrices obtained using GAP. ■

Proposition 7.5.2. *The set of reduced representatives* w *for* $Q\backslash GE_7/P_{A_5''}$ *such that* $\psi^{f}_{U_{A_5''}}|_{U_{A_5''}\cap w^{-1}Qw} \equiv 1$ *for some* f *in the open* $M_{A_5''}$*-orbit of* $\mathfrak{g}_{-2}^{s_{A_5''}}$ *has only one element, namely,*

$$w_0 := w[4231435423165423143542654317654231435426543176].$$

Proof. If $\psi^{f}_{U_{A_5''}}|_{U_{A_5''}\cap w^{-1}Qw} \equiv 1$ with $f = \sum_{\alpha\in\Phi_{A_5''}(2)} a_\alpha X_{-\alpha}$, then

$$\{\alpha\in\Phi_{A_5''}(2) : w\alpha < 0\}$$

contains $\{\alpha\in\Phi_{A_5''}(2) : a_\alpha \neq 0\}$. If f is in the open orbit, then it follows from Lemma 7.5.1 that $\{\alpha\in\Phi_{A_5''}(2) : a_\alpha \neq 0\}$ contains

(1) the root 0000001,

(2) two roots of the form $1 * * * *00$ that add up to 2122100,

(3) two roots of the form $0 * * * *10$ that add up to 0112220.

One can check using LiE that $Q\backslash GE_7/P_{A_5''}$ has 786 elements. Of these, only 342 map 0000001 to a negative root. Of these 342 only 120 map two roots of the form $1 * * * *00$ that add up to 2122100 to negative roots, and of these 120 only one maps two roots of the form $0 * * * *10$ that add up to 0112220 to negative roots. Thus there is only one element of $Q\backslash GE_7/P_{A_5''}$ such that the set

$$\{f \in \mathfrak{g}_{-2}^{s_{A_5''}} : \psi^f_{U_{A_5''}}|_{U_{A_5''}\cap w^{-1}Qw} \equiv 1\}$$

contains elements of the orbit A_5''. This element is w_0. ∎

Lemma 7.5.3. *The orbit A_5'' is a single rational orbit.*

Proof. The space $\mathfrak{g}_{-2}^{s_{A_5''}}$ decomposes as a direct sum of three irreducible $M_{A_5''}$-modules: $\langle X_{0000001}\rangle$ and

$$\mathfrak{v}_{010} := \langle X_{0000010}, X_{0000110}, X_{0001110}, X_{0101110}, X_{0011110}, X_{0111110}, X_{0112110}, X_{0112210}\rangle,$$

$$\mathfrak{v}_{100} := \langle X_{1000000}, X_{1010000}, X_{1011000}, X_{1111000}, X_{1011100}, X_{1111100}, X_{1112100}, X_{1122100}\rangle.$$

We identify an element of $\mathfrak{g}_{-2}^{s_{A_5''}}$ with a triple $(\underline{x}, \underline{y}, z)$ where $\underline{x}$ and $\underline{y}$ are column vectors of size 8 and z is a scalar. The action of $M_{A_5''}$ on $\mathfrak{g}_{-2}^{s_{A_5''}}$ then induces a rational homomorphism $M_{A_5''} \to GL_8 \times GL_8 \times GL_1$. From Lemma 7.5.1, the triple $(\underline{x}, \underline{y}, z)$ corresponds to an element of A_5'' if $q_1(\underline{x})q_2(\underline{y})z \neq 0$, where q_1 and q_2 are two quadratic forms. The derived group of $M_{A_5''}$ is isomorphic to $Spin_8$, and its image in $GL_8 \times GL_8 \times GL_1$ preserves the forms q_1 and q_2. That is, the image the derived group is contained in $SO_8(q_1) \times SO_8(q_2) \times \{1\}$. By [21, Propositions 1 and 4], we can map any triple which corresponds to an element of A_5'' to one of the form

$$\left(\begin{bmatrix}0\\0\\0\\a\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\b\\1\\0\\0\\0\end{bmatrix}, z\right)$$

using an element of the derived group of $M_{A_5''}$. It then suffices to show that the torus of GE_7 contains an element t which acts by a^{-1} on $X_{-1111000}$, by b^{-1} on $X_{-0101110}$, by z^{-1} on $X_{-0000001}$ and by 1 on $X_{-1011100}$ and $X_{-0011110}$. Since the images of t under the seven simple roots of E_7 can be chosen arbitrarily, this is easy. ∎

Proposition 7.5.4. *Let $P_{1,w_0} := M_{A_5''} \cap w_0^{-1} Q w_0$. Then P_{1,w_0} acts transitively on*

$$\{f \in \mathfrak{g}_{-2}^{s_{A_5''}}(F) \cap A_5'' : \psi_{U_{A_5''}}^f|_{U_{A_5''} \cap w_0^{-1} Q w_0} \equiv 1\}.$$

In the language of §3.2, the w_0-admissible subvariety of $P_{A_5''}$ is equal to $P_{1,w_0} \cdot G_2 U_{A_5''}$.

Proof. Write $f \in \mathfrak{g}_{-2}^{s_{A_5''}}$ as $\sum_\alpha a_\alpha X_{-\alpha}$, and identify it with a triple $(\underline{x}, \underline{y}, z)$ as above, given by

$$^t\underline{x} = [a_{1000000} \quad a_{1010000} \quad a_{1011000} \quad a_{1111000} \quad a_{1011100} \quad a_{1111100} \quad a_{1112100} \quad a_{1122100}],$$
$$^t\underline{y} = [a_{0000010} \quad a_{0000110} \quad a_{0001110} \quad a_{0101110} \quad a_{0011110} \quad a_{0111110} \quad a_{0112110} \quad a_{0112210}].$$

The group P_{1,w_0} is the standard parabolic subgroup of $M_{A_5''}$ whose Levi contains $U_{\pm\alpha_2}$ and $U_{\pm\alpha_3}$, and whose unipotent radical contains U_{α_4} and U_{α_5}. This parabolic preserves a flag in each of the spaces $\mathfrak{v}_{100}$ and $\mathfrak{v}_{010}$ which is compatible with the order placed on the roots above. Specifically, write

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \underline{y}_3 \\ y_4 \\ y_5 \end{bmatrix},$$

where $\underline{x}_i$ is a column vector of size 2 for each i, $\underline{y}_3$ is a column vector of size 4, and y_i is a scalar for $i = 1, 2, 4, 5$. Then the standard Levi subgroup of P_{1,w_0} respects this decomposition. The condition $\psi_{U_{A_5''}}^f|_{U_{A_5''} \cap w_0^{-1} Q w_0} \equiv 1$ is equivalent to $a_{0112110} = a_{0112210} = a_{1112100} = a_{1122100} = 0$, i.e., to $\underline{x}_4 = 0$, $y_4 = y_5 = 0$.

The triple $(\underline{x}, \underline{y}, z)$ corresponds to an element of A_5'' if $z \neq 0$ and $\underline{x}$ and $\underline{y}$ are each anisotropic relative to a certain quadratic form (cf. Lemma 7.5.1). When $\underline{x}_4$, y_4 and y_5 are trivial, this forces $\underline{y}_3$ and $\left(\begin{smallmatrix} \underline{x}_2 \\ \underline{x}_3 \end{smallmatrix}\right)$ to be anisotropic.

The derived group of the Levi of P_{1,w_0} is isomorphic to $SL_2 \times SL_2$; its action on the $\underline{y}_3$ component of $\mathfrak{v}_{010}$ can be identified with the action of $SL_2 \times SL_2$ on 2×2 matrices by $(g_1, g_2) \cdot Y = g_1 Y g_2^{-1}$. Anisotropic elements correspond to matrices Y with $\det Y \neq 0$. Clearly, each such matrix is in the same orbit as a $\operatorname{diag}(a, 1)$ for some a. It follows that each f is in the same orbit as one with $a_{0011110} = a_{0101110} = 0$, $a_{0111110} = 1$. The condition $f \in A_5''$ forces $a_{0001110} \neq 0$. Once $\underline{y}_3$ is of this form, the subgroup of $SL_2 \times SL_2$ which preserves it is isomorphic to SL_2. The four-dimensional space corresponding to $\underline{x}_2$ and $\underline{x}_3$ can then be identified with 2×2 matrices with this SL_2 acting by $g \cdot X = gX$ (matrix multiplication). Once again, $\det X \neq 0$ for $\left(\begin{smallmatrix} \underline{x}_2 \\ \underline{x}_3 \end{smallmatrix}\right)$ anisotropic. Hence we can choose a suitable element of SL_2 so that $gX = \operatorname{diag}(b, 1)$. Hence we can arrange $a_{1111100} = 1$, $a_{1011100} = a_{1111000} = 0$. The condition $f \in A_5''$ then forces $a_{1011000} \neq 0$. Now, acting by a suitable element of the torus, we can arrange $a_{1011000} = a_{0001110} = a_{0000001} = 1$ without changing the existing conditions $a_{0111110} = a_{1111100} = 1$. Finally, we can act by an element $x_{0011000}(a) x_{0101100}(b) x_{0001100}(c) x_{0111000}(d)$ to make $\underline{x}_1$, y_1 and y_2 trivial. ∎

Proposition 7.5.5. *The twisted Jacquet module*

$$\mathcal{J}_{U_{A_5''},\psi^{f_0}_{U_{A_5''}}}\left(\mathrm{Ind}_Q^{GE_7}\left((\chi_1\circ\nu_1)(\chi_2\circ\nu_2)\widetilde{\varpi}_8^{3/2}\right)\right)$$

is isomorphic as a representation of G_2 *to* $\mathrm{Ind}_{B_{G_2}}^{G_2}\mu$, *where* μ *is given in Section 7.3.1 and* B_{G_2} *is the Borel subgroup of* G_2 *obtained by intersecting* G_2 *with our standard Borel of* GE_7.

Proof. It now follows from the results of §3.2 that

$$\mathcal{J}_{U_{A_5''},\psi^{f_0}_{U_{A_5''}}}\left(\mathrm{Ind}_Q^{GE_7}\left((\chi_1\circ\nu_1)(\chi_2\circ\nu_2)\widetilde{\varpi}_8^{3/2}\right)\right)=\mathcal{J}_{U_{A_5''},\psi^{f_0}_{U_{A_5''}}}(\overline{I}_{w_0}),$$

where

$$\overline{I}_{w_0}\cong \text{c-ind}_{G_2U_{A_5''}\cap w_0^{-1}Qw_0}^{G_2U_{A_5''}}\left((\chi_1\circ\nu_1)(\chi_2\circ\nu_2)\widetilde{\varpi}_8^{3/2}\right)\delta_Q^{1/2}\circ \mathrm{Ad}(w_0).$$

The group $G_2\cap w_0^{-1}Qw_0$ is the standard Borel subgroup of G_2, while $U_{A_5''}\cap w_0^{-1}Qw_0$ is the product of the root subgroups attached to the following five roots:

$$\{0112110, 0112210, 0112211, 1112100, 1122100\}.$$

Let J denote the sum of these five roots.

We compute

$$\begin{aligned}
J&=-2\varpi_1+2\varpi_4+\varpi_5-2\varpi_6-\varpi_7-\varpi_8,\\
\nu_1\circ\mathrm{Ad}(w_0)&=-\varpi_1+\varpi_4-\varpi_5-\varpi_6+\varpi_7+\varpi_8,\\
\nu_2\circ\mathrm{Ad}(w_0)&=-\varpi_4+2\varpi_5-2\varpi_7+\varpi_8,\\
\widetilde{\varpi}_8\circ\mathrm{Ad}(w_0)&=\widetilde{\varpi}_8,\\
\delta_Q^{1/2}&=3\varpi_4+2\varpi_6-13\varpi_8,\\
\delta_Q^{1/2}\circ\mathrm{Ad}(w_0)&=-8\varpi_1+3\varpi_4+2\varpi_5-8\varpi_6-2\varpi_7+13\varpi_8.
\end{aligned}$$

Each of these induces a rational character of the standard torus T_{G_2} of the embedded G_2. If the fundamental weights are denoted $\varpi_1^{G_2}$ and $\varpi_2^{G_2}$, then

$$\begin{aligned}
J&=\varpi_1^{G_2}+2\varpi_2^{G_2},\\
\nu_1\circ\mathrm{Ad}(w_0)|_{T_{G_2}}&=-\varpi_1^{G_2}+\varpi_2^{G_2},\\
\nu_2\circ\mathrm{Ad}(w_0)|_{T_{G_2}}&=2\varpi_1^{G_2}-\varpi_2^{G_2},\\
\widetilde{\varpi}_8\circ\mathrm{Ad}(w_0)|_{T_{G_2}}&=0,\\
\delta_Q^{1/2}\circ\mathrm{Ad}(w_0)|_{T_{G_2}}&=2\varpi_1^{G_2}+3\varpi_2^{G_2}.
\end{aligned}$$

Thus $(\chi_1\circ\nu_1)(\chi_2\circ\nu_2)\widetilde{\varpi}_8^{3/2}\circ\mathrm{Ad}(w_0)|_{T_{G_2}}$ is precisely the character μ given in Section 7.3.1, and an element h of $\overline{I}_{w_0}$ satisfies $h(utg)=\mu(t)\delta_Q^{1/2}(w_0tw_0^{-1})h(g)$ for u in the standard maximal unipotent of G_2 and $t\in T_{G_2}$.

Now, for $h \in \overline{I}_{w_0}$ let

$$W.h(g) := \int_{(U_{A_5''} \cap w_0^{-1} Q w_0) \backslash U_{A_5''}} h(ug) \overline{\psi_{U_{A_5''}}^{f_0}(u)}\, du.$$

(This is convergent, since the support of h is compact modulo $(U_{A_5''} G_2 \cap w_0^{-1} Q w_0)$.) Then the kernel of W is the kernel of the canonical map $\overline{I}_{w_0} \to \mathcal{J}_{U_{A_5''}, \psi_{U_{A_5''}}^{f_0}}(\overline{I}_{w_0})$. That is, the image of W is a concrete realization of $J_{U_{A_5''}, \psi_{U_{A_5''}}^{f_0}}(\overline{I}_{w_0})$. (The proof is the same as in [20, Section 10].) Further, direct computation shows that

$$\begin{aligned} &W.h(u_1 u_2 t g) \\ &= \psi_{U_{A_5''}}^{f_0}(u_1) \mu(t) \delta_Q^{1/2} \circ \mathrm{Ad}(w_0)(t) |t|^{-J}\, W.h(g), \quad u_1 \in U_{A_5''}, u_2 \in U_{\max}^{G_2}, t \in T_{G_2}, g \in G_2. \end{aligned}$$

But

$$(\delta_Q^{1/2} \circ \mathrm{Ad}(w_0) - J)|_{T_{G_2}} = \varpi_1^{G_2} + \varpi_2^{G_2} = \delta_{B_{G_2}}^{1/2}.$$

Hence restriction from $G_2 U_{A_5''}$ to G_2 is a linear isomorphism from the image of W onto $\mathrm{Ind}_{B_{G_2}}^{G_2}(\mu)$. ∎

Hence, we have proved the following theorem.

Theorem 7.5.6. *Assume that for almost all finite places v, π_v is a principal series representation of $GL_7(F_v)$ which is attached to a character of the form (7.3.1). Then every irreducible summand of $\mathcal{D}_\pi$ weakly functorially lifts to π.*

Remark 7.5.7. It follows from the proof above that given any irreducible automorphic representation Π of $GE_7(\mathbb{A})$ and any finite local place v, if Π_v has the form as in (7.3.4), then the $(U_{A_5''}, \psi_{U_{A_5''}}^{f_0})$-twisted Jacquet module of Π_v has the form $\mathrm{Ind}_{B_{G_2}}^{G_2} \mu$, where μ is given in Section 7.3.1.

8. The $A_2 + 3A_1$ case

Recall from Definition 6.2.2 that in the $A_2 + 3A_1$ case the descent module $\mathcal{D}_\pi$ is defined by applying the Fourier coefficient $(U, \psi_U^{e_0})$ from Section 6.2 to the residual representation $\mathcal{E}_\pi$, where π is an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type. In this section, we prove the following theorem.

Theorem 8.0.1. *Assume that π is an irreducible cuspidal automorphic representation of $GL_7(\mathbb{A})$ which is of G_2 type, and $\mathcal{D}_\pi$ is defined as in Definition 6.2.2. Then*

(1) *$\mathcal{D}_\pi$ is generic.*

(2) *$\mathcal{D}_\pi$ is* not *cuspidal. Actually, $\mathcal{D}_\pi$ supports all degenerate Whittaker–Fourier coefficients of G_2.*

We also study the unramified local descent as in Section 7.5, which is motivated by the question of whether irreducible subquotients of $\mathcal{D}_\pi$ would lift functorially back to π, and provides evidence that they might well not.

8.1. *Nonvanishing Fourier coefficients of the descent module*

The main goal of this subsection is to prove (in the following theorem) that the descent module supports the Whittaker–Fourier integral along the maximal unipotent of G_2 against *any* character of this group. In particular, it is globally generic, but not cuspidal, and its constant term along the Borel is nontrivial.

Theorem 8.1.1. *Recall that $U_{\max}^{G_2}$ is the standard maximal unipotent subgroup of G_2, and let ψ^{G_2} be any character of $U_{\max}^{G_2}(F)\backslash U_{\max}^{G_2}(\mathbb{A})$. Write $(U_{\max}^{G_2}, \psi^{G_2})$ for the corresponding (possibly) degenerate Whittaker–Fourier integral. That is, for any $f \in C^\infty(G_2(F)\backslash G_2(\mathbb{A}))$,*

$$f^{(U_{\max}^{G_2}, \psi^{G_2})}(g) := \int_{U_{\max}^{G_2}(F)\backslash U_{\max}^{G_2}(\mathbb{A})} f(ug)\overline{\psi^{G_2}}(u)\, du.$$

Then $(U_{\max}^{G_2}, \psi^{G_2})$ does not vanish identically on the descent module $\mathcal{D}_\pi$. That is, there is some $D \in \mathcal{D}_\pi$ such that $D^{(U_{\max}^{G_2}, \psi^{G_2})} \neq 0$.

Define $V_1 := UU_{\max}^{G_2}$ and define $\psi_{V_1} : V_1(F)\backslash V_1(\mathbb{A}) \to \mathbb{C}^\times$ by $\psi_{V_1}(u_1u_2) = \psi_U^{e_0}(u_1)\psi^{G_2}(u_2)$ for $u_1 \in U$ and $u_2 \in U_{\max}^{G_2}$ (this is a well-defined character of $V_1(F)\backslash V_1(\mathbb{A})$). Then the composed period $(U_{\max}^{G_2}, \psi^{G_2}) \circ (U, \psi_U^{e_0})$ is (V_1, ψ_{V_1}). Theorem 8.1.1 is therefore an immediate consequence of the following theorem.

Theorem 8.1.2. *The period (V_1, ψ_{V_1}) does not vanish identically on $\mathcal{E}_\pi$.*

Lemma 8.1.3. *Let*

$$S_2^0 = \left\{\begin{matrix}0100000, 0101000, 0111000, 0101100, 1111000, 0111100, 0101110, 1111100,\\ 0112100, 0111110, 0101111, 1112100, 1111110, 0112110, 0111111\end{matrix}\right\}.$$

Let $S_2 = \Phi^+ \smallsetminus S_2^0$ and $S_2' = S_2^0 \cup \{1223210, 1223211\}$. Let V_2 and V_2' be the T-stable unipotent subgroups of GE_7 corresponding to S_2 and S_2'.

Let ψ_{V_2} denote a character of V_2 such that supp ψ_{V_2} *is contained in*

$$\{1000000, 0010000, 0001000, 0000100, 0000010, 0000001,\\ 1111111, 1122100, 1112110, 0112210, 0112111\},$$

and $\psi_{V_2}|_{V_1(\mathbb{A})\cap V_2(\mathbb{A})} = \psi_{V_1}|_{V_1(\mathbb{A})\cap V_2(\mathbb{A})}$. Then for any automorphic function $f : GE_7(F)\backslash GE_7(\mathbb{A}) \to \mathbb{C}$ of uniformly moderate growth, and any $g \in GE_7(\mathbb{A})$,

$$f^{(V_1,\psi_{V_1})}(g) = \int_{(V_2\cap V_2'(\mathbb{A}))\backslash V_2'(\mathbb{A})} f^{(V_2,\psi_{V_2})}(v_2'g)\, dv_2'.$$

Moreover, $(V_1, \psi_{V_1}) \sim (V_2, \psi_{V_2})$.

Proof. The proof is by nine successive applications of Lemma 7.1.1. The applications come in three basic types. In the first type there are two roots $\beta_1 \in \Phi(M, T)$, $\gamma_1 \in \Phi(U, T)$ such that $X = U_{\gamma_1}$ and $Y = U_{\beta_1}$. In these cases $\mathfrak{g}_2 \cap \mathfrak{u}_{\gamma_1} = \{0\}$, and the roots β_1, γ_1 are given in the table below. Recall that $\mathfrak{g}_2$ is the Lie algebra of G_2.

In the second type, there are two roots $\beta_1, \beta_2 \in \Phi(M, T)$ and $\delta \in \Phi_{G_2}^{\mathrm{lg},+}$ (positive long roots of G_2) such that $\mathfrak{g}_2 \cap \mathfrak{u}_{\beta_1} \oplus \mathfrak{u}_{\beta_2} = \mathfrak{u}_\delta$. In these cases, there is a root $\gamma \in \Phi(U, T)$ such that $X = U_\gamma$ which has a pairing with $U_{\beta_1} U_{\beta_2}$ as in Lemma 7.1.1, and U_δ is the right kernel of this pairing. We may take Y to be any complement of U_δ in $U_{\beta_1} U_{\beta_2}$ so that the group D in Lemma 7.1.1 contains the whole group $U_{\beta_1} U_{\beta_2}$. For these cases, the roots β_1, β_2 and γ are given in the table below.

The third type is similar to the second, except that δ is a short root of G_2. In this case (see proof of Lemma 6.2.1), there are four roots $\beta_1, \beta_2, \beta_3, \beta_4 \in \Phi(M, T)$ such that $\mathfrak{g}_2 \cap \bigoplus_{i=1}^4 \mathfrak{u}_{\beta_i} = \mathfrak{u}_\delta$. Moreover, there is a unique pair of them such that the sum is another root $\beta_5 \in \Phi(M, T)$. The product $\prod_{i=1}^5 U_{\beta_i}$ is a T-stable subgroup. In fact, it is the smallest T-stable subgroup of GE_7 which contains U_δ. We denote it V_δ. It is two-step nilpotent with center U_{β_5}. In these cases the group X is a product $\prod_{i=1}^3 U_{\gamma_i}$ which has a pairing with V_δ as in Lemma 7.1.1, and $U_\delta U_{\beta_5}$, is the right kernel of this pairing. For Y, we may select any subgroup of V_δ which contains U_{β_5} such that the image in the abelian quotient V_δ / U_{β_5} is complementary to the image of U_δ. In the table below we give $\gamma_1, \gamma_2, \gamma_3$ and $\beta_1, \dots, \beta_5$ with β_5 in parentheses.

X	Y	δ
0100000	1011111	
0101000	0011111, 1011110	$3\alpha + 2\beta$
0111000	0001111, 1011100	$3\alpha + \beta$
0101100	0011110	
1111000, 0111100, 0101110	0000111, 0001110, 0011100, 1011000, (1011111)	$2\alpha + \beta$
1111100, 0112100, 0101111	0000011, 0011000, 0000110, 1010000, (0011110)	$\alpha + \beta$
0111110	0001100	
1112100	0000010, 0010000	β
1111110, 0112110, 0111111	0000001, 0001000, 0000100, 1000000, (0001100)	α

At the first stage, the group B is just V_1. At each later stage it is the group D obtained from the previous stage. At each stage the group C may be thought of as the subgroup of B obtained by deleting the roots listed under "X" in the table. More precisely, the Lie algebra $\mathfrak{c}$ of C is the largest subalgebra of the Lie algebra $\mathfrak{b}$ of B whose projection onto $\mathfrak{u}_{\gamma_i}$ is trivial for each i. The group D is the product of C and the root subgroups attached to the roots listed under "Y" in the table.

Checking conditions (1) to (6) for Lemma 7.1.1 is fairly routine. The order in which the nine applications of Lemma 7.1.1 are carried out is important. It is useful to consider the bigrading in which the root subgroup U_γ, where $\gamma = \sum_{i=1}^7 c_i \alpha_i$, gets grading $(c_2, \sum_{i=1}^7 c_i - c_2)$. Notice that as the table is read top-to-bottom, the second component of this grading is nondecreasing in the column labeled "X" and nonincreasing in the column labeled "Y". This determines a partial ordering on the nine rows. It is fairly easy to check most of the conditions of Lemma 7.1.1 provided this partial ordering is respected, but (3) and (6) require some care, particularly for applications of the third type. We discuss the first application of the third type in some detail and leave all the remaining details to the reader.

For the first application of the third type, $X = U_{1111000}U_{0111100}U_{0101110} \cong \mathfrak{u}_{1111000} \oplus \mathfrak{u}_{0111100} \oplus \mathfrak{u}_{0101110}$, while $V_\delta = U_{0000111}U_{0001110}U_{0011100}U_{1011000}U_{1011111}$. The center of V_δ is $U_{\beta_5} = U_{1011111}$. The quotient $V_\delta(\mathbb{A})/U_{1011111}(\mathbb{A})$ may be identified with $\mathfrak{u}_{0000111} \oplus \mathfrak{u}_{0001110} \oplus \mathfrak{u}_{0011100} \oplus \mathfrak{u}_{1011000}$. The character of $C(\mathbb{A})$ which we consider is given by

$$\psi_C^{e_0}(\exp c) = \psi(\kappa(e_0, c)) \quad (c \in \mathfrak{c}(\mathbb{A})).$$

In order to check conditions (3) and (6) we must consider the pairing

$$\Upsilon(x, y) := \psi_C^{e_0}([x, y]),$$

where

$$[x, y] = xyx^{-1}y^{-1}, \quad x \in X(\mathbb{A}),\ y \in V_\delta(\mathbb{A}).$$

(It is trivial on $X(\mathbb{A}) \times U_{\beta_5}(\mathbb{A})$ and hence may be regarded as a pairing on the set $X(\mathbb{A}) \times V_\delta/U_{\beta_5}(\mathbb{A})$.) The pairing Υ satisfies

$$\Upsilon(\exp a, \exp b) = \psi(\kappa(e_0, [a, b])) = \psi(\omega_{e_0}(a, b)), \tag{8.1.4}$$

where

$$[a, b] = ab - ba, \quad a \in \mathfrak{u}_{1111000} \oplus \mathfrak{u}_{0111100} \oplus \mathfrak{u}_{0101110},$$
$$b \in \mathfrak{u}_{0000111} \oplus \mathfrak{u}_{0001110} \oplus \mathfrak{u}_{0011100} \oplus \mathfrak{u}_{1011000}.$$

To check condition (3), we have to check that $X(\mathbb{A})$ and $Y(\mathbb{A})$ preserve ψ_C. This amounts to checking that Υ is trivial on $X(\mathbb{A}) \times U_\delta/U_{\beta_5}(\mathbb{A})$ and on $Y(\mathbb{A}) \times U_\delta/U_{\beta_5}(\mathbb{A})$. The former is obvious, since $\mathfrak{u}_\delta = \mathfrak{g}_2 \cap \bigoplus_{i=1}^4 \mathfrak{u}_{\beta_i}$. The latter is also obvious, since $Y \subset V_\delta$ and V_δ/U_{β_5} is abelian. To check condition (6), we have to check that Υ is nondegenerate on $X(\mathbb{A}) \times Y(\mathbb{A})/U_{\beta_5}(\mathbb{A})$ for any Y such that Y/U_{β_5} is complementary to U_δ/U_{β_5}. In other words, we have to show that

$$\{y \in V_\delta(\mathbb{A}) : \Upsilon(x, y) = 1\ \forall x \in X(\mathbb{A})\} = U_\delta(\mathbb{A}).$$

By (8.1.4), this reduces to showing that

$$\{b \in \mathfrak{u}_{0000111} \oplus \mathfrak{u}_{0001110} \oplus \mathfrak{u}_{0011100} \oplus \mathfrak{u}_{1011000} : \\ \kappa(e_0, [a, b]) = 0\ \forall a \in \mathfrak{u}_{1111000} \oplus \mathfrak{u}_{0111100} \oplus \mathfrak{u}_{0101110}\} = \mathfrak{u}_\delta.$$

Now $\kappa(e_0, [a, b]) = -\kappa([b, e_0], a)$, which is certainly trivial if $b \in \mathfrak{u}_\delta$, since $[b, e_0] = 0$ for all $b \in \mathfrak{g}_2$. On the other hand, if $b \notin \mathfrak{u}_\delta = \mathfrak{g}_2 \cap \bigoplus_{i=1}^4 \mathfrak{u}_{\beta_i}$, then $[b, e_0]$ is nonzero, hence $\kappa([b, e_0], a) \neq 0$ for some $a \in \mathfrak{e}_7$ because κ is nondegenerate, and hence $\kappa([b, e_0], a) \neq 0$ for some $a \in \mathfrak{u}_{1111000} \oplus \mathfrak{u}_{0111100} \oplus \mathfrak{u}_{0101110}$ because κ respects the bigrading. ■

Remark 8.1.5. As noted, for applications of Lemma 7.1.1 of the second and third types, the group Y is not uniquely determined, but can be taken to be any complement to a given subgroup. This is the reason that ψ_{V_2} may be chosen with some degree of freedom. In addition we have a degree of freedom in the choice of ψ^{G_2}.

In order to proceed further, it will be convenient to write ψ_{V_2} and ψ^{G_2} explicitly in coordinates. There exist $a_1, a_3, a_4, a_5, a_6, a_7 \in F$ such that

$$\psi_{V_2}(v) = \psi(v_{1122100} + v_{1112110} + v_{1111111} + v_{0112210} + v_{0112111} \\ + a_1 v_{\alpha_1} + a_3 v_{\alpha_3} + a_4 v_{\alpha_4} + a_5 v_{\alpha_5} + a_6 v_{\alpha_6} + a_7 v_{\alpha_7})$$

for all $v \in V_2$. Then $\psi^{G_2}(u) = \psi((a_1+a_4+a_5+a_6)u_\alpha + (a_3+a_6)u_\beta)$ for all $u \in U_{\max}^{G_2}$. Rewrite ψ_{V_2} as $\psi_{V_2}^{\underline{a}}$ with $\underline{a} = \{a_1, a_3, a_4, a_5, a_6, a_7\}$.

Lemma 8.1.6. *Let*

$$S_3 = \Phi^+ \cup \{-\alpha_4\} \setminus \{0000001, 0001000, 0001100, 0001111, 0011000, 0101000, 0112100, \\ 0112111, 1011000, 1112100, 1112111, 1123211, 0100000, 0010000, 0000100, 0000010\},$$

and let V_3 be the corresponding T-stable unipotent subgroup. Let $\psi_{V_3}^{\underline{a}'} : V_3(\mathbb{A}) \to \mathbb{C}^\times$ be given by

$$\psi(v_{0000111} + v_{0111100} + v_{0101110} + v_{1010000} + v_{0011110} \\ + a_1' v_{0101100} + a_3' v_{0000011} + a_4' v_{0011100} + a_5' v_{\alpha_1} + a_6' v_{0111000} + a_7' v_{0001110}).$$

Let $w_4 = w[745632451342]$. Then there is a representative $\dot{w}_4$ for w_4 such that for each $\underline{a}$ there exists $\underline{a}'$ with a_i' being a nonzero scalar multiple of a_i and $f^{(V_3, \psi_{V_3}^{\underline{a}'})}(g) = f^{(V_2, \psi_{V_2}^{\underline{a}})}(\dot{w}_4 g)$ for all $f \in C^\infty(GE_7(F)\backslash GE_7(\mathbb{A}))$ and $g \in GE_7(\mathbb{A})$, whence $(V_2, \psi_{V_2}^{\underline{a}}) \sim (V_3, \psi_{V_3}^{\underline{a}'})$.

Proof. Let

$$\begin{aligned} R_1 &= \{1122100, 1112110, 1111111, 0112210, 0112111\}, \\ R_2 &= \{\alpha_i : 1 \le i \le 7,\ i \ne 2\}, \\ R_1' &= \{0000111, 0111100, 0101110, 1010000, 0011110\}, \\ R_2' &= \{0101100, 0000011, 0011100, 1000000, 0111000, 0001110\}. \end{aligned}$$

Then $w_4 R_1 = R_1'$ and $w_4 R_2 = R_2'$.

For any representative $\dot{w}_4$ for w_4, we have $V_3 = \dot{w}_4 V_2 \dot{w}_4^{-1}$, and

$$\psi_{V_2}^{\underline{a}}(\dot{w}_4^{-1} v \dot{w}_4) = \psi\Big(\sum_{\alpha \in R_1} c_{\dot{w}_4,\alpha} v_{w_4\alpha} + \sum_{\substack{i=1 \\ i \ne 2}}^{7} a_i c_{\dot{w}_4,\alpha_i} v_{w_4\alpha_i}\Big)$$

for some nonzero constants $c_{\dot{w}_4,\alpha}$ depending on the choice of the representative $\dot{w}_4$. The point is to show that $\dot{w}_4$ may be chosen so that $c_{\dot{w}_4,\alpha} = 1$ for all $\alpha \in R_1$. Now, $\dot{w}_4$ is unique up to an element of the maximal torus T of GE_7, so it suffices to check that the mapping $T \to GL_1^5$ induced by the five elements of R_1 is surjective. This follows from the fact that these five elements can be simultaneously conjugated to simple roots, as seen in Section 6.2. ■

Remark 8.1.7. Recall that the descent Fourier coefficient is attached to the standard semisimple element $\begin{smallmatrix}0&0&0&0&0&0\\&&2&&&\end{smallmatrix}$. The regular nilpotent orbit of $\mathfrak{g}_2$ is attached to a standard semisimple element of $\mathfrak{g}_2$, which may then be mapped to a semisimple element of $\mathfrak{ge}_7$, namely $\begin{smallmatrix}2&2&2&2&2&2\\&&-12&&&\end{smallmatrix}$. The sum is $\begin{smallmatrix}2&2&2&2&2&2\\&&-10&&&\end{smallmatrix}$. If we regard it as a coweight, it is not dominant. The dominant element of its Weyl orbit is $\begin{smallmatrix}2&0&2&0&0&2\\&&0&&&\end{smallmatrix}$, which is the standard semisimple element attached to a nilpotent orbit of E_7 whose Bala–Carter label is $E_7(a_4)$. The element w_4 maps $\begin{smallmatrix}2&2&2&2&2&2\\&&-10&&&\end{smallmatrix}$ to $\begin{smallmatrix}2&0&2&0&0&2\\&&0&&&\end{smallmatrix}$. This was the original motivation for considering w_4, V_3, and $\psi_{V_3}^{a'}$.

Lemma 8.1.8. *Let*

$$S_4 = \Phi^+ \cup \{-\alpha_4\} \setminus \{0000001, 0001000, 0001100, 0001111, 0011000, 0101000, \\ 0112100, 1011000, 0100000, 0010000, 0000100, 0000010\},$$

and let V_4 be the corresponding unipotent subgroup. Let $\psi_{V_4}^{a'}$ be the character such that $\psi_{V_4}^{a'}|_{V_3(\mathbb{A})} = \psi_{V_3}^{a'}$ and $\psi_{V_4}^{a'}|_{U_\gamma} \equiv 1$ for $\gamma \in \Phi(V_4,T) \setminus \Phi(V_3,T)$. Then $(V_3, \psi_{V_3}^{a'}) \,|\, (V_4, \psi_{V_4}^{a'})$.

Proof. One may write $(V_4, \psi_{V_4}^{a'})$ as a double integral with $(V_3, \psi_{V_3}^{a'})$ as inner integral. ■

Lemma 8.1.9. *Let $S_5 = S_4 \cup \{0001111, 0000001\} \setminus \{-\alpha_4, 0000110\}$. Let V_5 be the corresponding T-stable unipotent group. Let $\psi_{V_5}^{a'} : V_5(\mathbb{A}) \to \mathbb{C}^\times$ be the character such that $\psi_{V_5}^{a'}|_{V_4 \cap V_5(\mathbb{A})} = \psi_{V_4}^{a'}|_{V_4 \cap V_5(\mathbb{A})}$ and $\psi_{V_5}^{a'}|_{U_{\alpha_7} U_{0001111}(\mathbb{A})} \equiv 1$. Then*

$$f^{(V_4, \psi_{V_4}^{a'})}(g) = \int_{\mathbb{A}} \int_{\mathbb{A}} f^{(V_5, \psi_{V_5}^{a'})}\big(x_{-\alpha_4}(r_1) x_{0000110}(r_2) g\big)\, dr_1\, dr_2.$$

Moreover, $(V_5, \psi_{V_5}^{a'}) \sim (V_4, \psi_{V_4}^{a'})$.

Proof. This is another application of Lemma 7.1.1. ■

The key feature of V_5 is that it is contained in the unipotent subgroup attached to the weighted Dynkin diagram $\begin{smallmatrix}2&0&2&0&0&2\\&&0&&&\end{smallmatrix}$ for the orbit $E_7(a_4)$. Further supp $\psi_{V_5}^{a'}$ is contained in

$$\{\alpha_7, 0001110, 0011100, 0101100, 0111000, 0000011, \\ 0011110, 1010000, 0101110, 0111100, 0000111\},$$

which is contained in the two-graded piece for this weighting.

Let V_6 be the full unipotent group for ${}^{2\,0\,\overset{2}{0}\,0\,0\,2}$ (that is, all root subgroups with weights bigger than or equal to 2) and $\psi_{V_6}^{\underline{a}'}$ be the character of it with $\psi_{V_6}^{\underline{a}'}\big|_{V_5} = \psi_{V_5}^{\underline{a}'}$ and $\operatorname{supp}\psi_{V_6}^{\underline{a}'} = \operatorname{supp}\psi_{V_5}^{\underline{a}'}$. Then for any automorphic function f of uniformly moderate growth, $f^{(V_6,\psi_{V_6}^{\underline{a}'})}$ can be written as a double integral with inner integral $f^{(V_5,\psi_{V_5}^{\underline{a}'})}$. Hence $(V_5,\psi_{V_5}^{\underline{a}'}) \mid (V_6,\psi_{V_6}^{\underline{a}'})$. Notice that $(V_6,\psi_{V_6}^{\underline{a}'})$ is a unipotent period of the type considered in Section 3.1.

Lemma 8.1.10. *Let*

$$\begin{aligned} X_{\underline{a}'} = {} & X_{-1010000} + X_{-0000111} + X_{-0011110} + X_{-0101110} + X_{-0111100} \\ & + a_7' X_{-0001110} + a_4' X_{-0011100} + a_1' X_{-0101100} + a_6' X_{-0111000} \\ & + a_3' X_{-0000011} + a_5' X_{-1000000}, \end{aligned}$$

and

$$e_0' = X_{-1010000} + X_{-0000011} + X_{-0111000} + X_{-0101100} + X_{-0011100} + X_{-0001110}.$$

Then

(1) *$X_{\underline{a}'}$ is an element of the closure of the orbit A_6 if and only if*

$$-1716(a_1'a_3'a_5' + a_3'a_4'a_5' - 2a_1'a_3'a_7' - a_3'a_5'a_7' - a_5'a_6'a_7')^2 a_4'^2 a_5'^2 a_6'^2 = 0. \tag{8.1.11}$$

(2) *When $a_5' = 0$, the element $X_{\underline{a}'}$ lies in A_6 if and only if $a_1'a_3'a_4'a_6'a_7' \neq 0$.*

(3) *If $X_{\underline{a}'}$ is in A_6 then it is conjugate to e_0'.*

Proof. We may regard $X_{\underline{a}'}$ first as an element of the Lie algebra $\mathfrak{e}_7$ over a polynomial ring in six indeterminates and compute its rank sequence as such. This can be done, for example, by obtaining 133×133 matrices for $\operatorname{ad}(X_\gamma)$ for the relevant roots γ from GAP and then loading them into SageMath. This tells us what orbit $X_{\underline{a}'}$ lies in for $\underline{a}'$ in general position, and allows us to obtain polynomial conditions for $X_{\underline{a}'}$ to lie in a smaller orbit.

It turns out that for $\underline{a}'$ in general position, $X_{\underline{a}'}$ lies in the orbit $E_7(a_4)$. The largest value of k such that $X_{\underline{a}'}^k \neq 0$ is 14, and $X_{\underline{a}'}^{14}$ is rank 1, with only one nonzero entry. This nonzero entry is the left hand side of (8.1.11). As mentioned in Lemma 7.1.3, $X_{\underline{a}'}$ is in A_6 if and only if its 14th power is 0.

From the diagram in [5, p. 442], we see that there are three stable orbits which are less than $E_7(a_4)$ but not less than A_6. Their Bala–Carter labels are $D_5 + A_1$, $D_6(a_1)$, and D_5. For X in any of these orbits we have $\operatorname{rank}\operatorname{ad}(X)^{14} = 1$. This proves the first part.

It is then clear that $a_5' = 0$ implies $X_{\underline{a}'}$ is in the closure of A_6. Referring again to the diagram in [5, p. 442], we see that $\mathcal{O} < A_6 \Leftrightarrow \mathcal{O} \le E_7(a_5)$. By inspecting the rank sequences of these two orbits, we can see that if $X \in A_6$, then $\operatorname{rank}\operatorname{ad}(X)^{12} = 3$, while if $X \in E_7(a_5)$, then $\operatorname{rank}\operatorname{ad}(X)^{12} = 0$. When $a_5' = 0$, if we calculate the matrix $\operatorname{ad}(X_{\underline{a}'})^{12}$ (as an element of $\mathfrak{e}_7$ over a polynomial ring) and then discard all rows and columns which consist entirely of zeros, we obtain the following 3×3 matrix:

$$\begin{pmatrix} 0 & 0 & -462a_1'^3a_2'^2a_4'a_6'a_7'^2 \\ 0 & 924a_1'^2a_2'^2a_4'^2a_6'^2a_7'^2 & 0 \\ -462a_1'^3a_2'^2a_4'a_6'a_7'^2 & 0 & 0 \end{pmatrix}.$$

This completes the proof of the second part.

To prove the third part we consider

$$X'_{\underline{a}'} = X_{-1010000} + a_3'X_{-0000011} + a_6'X_{-0111000} + a_1'X_{-0101100} + a_4'X_{-0011100} + a_7'X_{-0001110},$$

and

$$u(b_1,\dots,b_5) := x_{-\alpha_2}(b_1)x_{-\alpha_3}(b_2)x_{-\alpha_5}(b_3)x_{-\alpha_5-\alpha_6}(b_4)x_{-\alpha_6}(b_5).$$

Using SageMath, one can check that for each $a_1', a_3', a_4', a_6', a_7'$ (all nonzero) there exist unique $b_1,\dots,b_5$ such that

$$\mathrm{Ad}(u(b_1,\dots,b_5)).X_{\underline{a}'} = X'_{\underline{a}'}.$$

These six roots which appear in $X'_{\underline{a}'}$ may be simultaneously conjugated to simple roots (cf. Lemma 7.1.12). Hence we can conjugate $X'_{\underline{a}'}$ to e_0' using a suitable element of the torus. ■

Corollary 8.1.12. *Let $\psi'_{V_6} : V_6 \to \mathbb{C}^\times$ be given by*

$$\psi'_{V_6}(v) = \psi(v_{0001110} + v_{0011100} + v_{0101100} + v_{0111000} + v_{0000011} + v_{1010000}).$$

Then for each $\underline{a}' = (a_1', a_3', a_4', 0, a_6', a_7')$ with $a_i' \neq 0$ for $i = 1,3,4,6,7$, there exists $\nu_{\underline{a}'} \in GE_6(F)$ such that $\nu_{\underline{a}'}V_6\nu_{\underline{a}'}^{-1} = V_6$ and $\psi'_{V_6}(\nu_{\underline{a}'}v\nu_{\underline{a}'}^{-1}) = \psi_{V_6}(v)$ for all $v \in V_6(\mathbb{A})$. Hence $f^{(V_6,\psi_{V_6})}(g) = f^{(V_6,\psi'_{V_6})}(\nu_{\underline{a}'}g)$ for all smooth automorphic functions $f : GE_7(F)\backslash GE_7(\mathbb{A}) \to \mathbb{C}$ and all $g \in GE_7(\mathbb{A})$, and in particular $(V_6, \psi_{V_6}^{\underline{a}'}) \sim (V_6, \psi'_{V_6})$.

This completes the proof of Theorem 8.1.1, since (V_6, ψ'_{V_6}) has appeared previously as $(U_5, \psi_{U_5}^{e_0'})$, and it was already shown in Lemma 7.1.15 that $\mathcal{E}_\pi$ supports this period.

8.1.1. Remarks. The proof of Theorem 8.1.1 can be summarized as follows. For $\underline{c} = (c_1, c_2)$, let $\psi_{\underline{c}}^{G_2}(u) = \psi(c_1u_\alpha + c_2u_\beta)$ for $u \in U_{\max}^{G_2}$. Then $(U_{\max}^{G_2}, \psi_{\underline{c}}^{G_2}) \circ (U, \psi_U^{e_0})$ divides $(V_6, \psi_{V_6}^{\underline{a}'})$ whenever $\underline{c}$ is the image of $\underline{a}'$ under a certain linear map. In this situation, every representation which supports $(V_6, \psi_{V_6}^{\underline{a}'})$ must also support $(U_{\max}^{G_2}, \psi_{\underline{c}}^{G_2}) \circ (U, \psi_U^{e_0})$. For any $\underline{c}$, we can choose $\underline{a}'$ which maps to $\underline{c}$ and corresponds to an element of the orbit A_6. The residual representation $\mathcal{E}_\pi$ supports the Fourier coefficient $(V_6, \psi_{V_6}^{\underline{a}'})$ whenever $\underline{a}'$ corresponds to an element of A_6. Therefore it supports $(U_{\max}^{G_2}, \psi_{\underline{c}}^{G_2}) \circ (U, \psi_U^{e_0})$ for all $\underline{c}$.

In particular, the conclusion applies not only to $\mathcal{E}_\pi$, but to *any* automorphic representation Π which supports the Fourier coefficient $(V_6, \psi_{V_6}^{\underline{a}'})$ whenever $\underline{a}'$ corresponds to an element of A_6. Moreover, it is reasonable to ask whether A_6 can be replaced by a smaller orbit. In this connection we note that taking $a'_3 = a'_5 = 1$ and the rest zero, or $a'_5 = a'_6 = 1$ and the rest zero, gives an element $X_{\underline{a}'}$ in the orbit $2A_2 + A_1$, which lies immediately above the orbit $A_2 + 3A_1$ attached to $\psi_U^{e_0}$.

If π is not of G_2 type but $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$, then Theorem 8.1.1 is still valid for the residual representation $\mathcal{E}_\pi$ with exactly the same proof.

8.2. Local descent

Since the results of [17] hold in both the local and global settings, the same set of arguments given in the global setting above also provides a local analogue.

Theorem 8.2.1. *Let F_v be a nonarchimedean local field. Suppose that an irreducible admissible representation Π_v of $GE_7(F_v)$ supports the twisted Jacquet module attached to $(V_6, \psi_{V_6}^{\underline{a}'})$ with $\underline{a}'$ now in F_v^6 corresponding to an element of A_6. Then the $(U, \psi_U^{e_0})$-twisted Jacquet module of Π_v supports (twisted and untwisted) Jacquet modules attached to $U_{\max}^{G_2}$ and all characters of $U_{\max}^{G_2}$. In particular, this holds when Π_v is the local component of any irreducible subquotient Π of $\mathcal{E}_\pi$ where π has the property that $L^S(s, \pi, \wedge^3)$ has a pole at $s = 1$.*

8.3. Unramified local descent

One may now consider the twisted Jacquet module

$$\mathcal{J}_{U,\psi_U^{e_0}}\big(\mathrm{Ind}_{Q(F_v)}^{GE_7(F_v)}(\chi_1 \circ \nu_1)(\chi_2 \circ \nu_2)\widetilde{\varpi}_8^{3/2}\big).$$

If π is an irreducible cuspidal automorphic representation of GL_7 with π_v being induced from a character of the form (7.3.1) and σ is an irreducible quotient of $\mathcal{E}_\pi$, then σ_v will be a quotient of this twisted Jacquet module.

The study of such a twisted Jacquet module is closely connected with the structure of the double coset space $Q(F_v)\backslash GE_7(F_v)/G_2(F_v)U(F_v)$. Notice that this space is infinite, since

$$\dim GE_7 = 134, \quad \dim Q + \dim G_2 + \dim U = 133.$$

This stands in contrast to the situation encountered in [16, 20], where [2, Theorem 5.2] could be applied.

Moreover, suppose we say that a double coset is *admissible* if its elements γ satisfy $\psi_U^{e_0}|_{U\cap(\gamma^{-1}Q\gamma)} \equiv 1$. Then we have

Lemma 8.3.1. *The set of admissible double cosets in $Q(F_v)\backslash GE_7(F_v)/G_2(F_v)U(F_v)$ is infinite.*

Proof. We can sort the elements of $Q(F_v)\backslash GE_7(F_v)/G_2(F_v)U(F_v)$ according to the elements of $Q(F_v)\backslash GE_7(F_v)/P(F_v)$. Of course this latter double coset space is finite and represented by elements of the Weyl group. We use elements w of the Weyl group that are of minimal length in their double coset. For each such w,

$$\delta \mapsto Q(F_v)w\delta G_2(F_v)U(F_v)$$

induces a bijection between the set of $Q(F_v)$, $G_2(F_v)U(F_v)$-double cosets in $Q(F_v)wP(F_v)$ and $(M(F_v)\cap w^{-1}Q(F_v)w)\backslash M(F_v)/G_2(F_v)$. Moreover, for $\delta\in M(F_v)$,

$$\psi_U^{e_0}|_{U\cap\delta^{-1}w^{-1}Qw\delta}\equiv 1 \iff [\delta\cdot\psi_U^{e_0}]|_{U\cap w^{-1}Qw}\equiv 1.$$

We consider the longest element w_0 of $Q(F_v)\backslash GE_7(F_v)/P(F_v)$, and show that

$$\{\delta\in(M(F_v)\cap w_0^{-1}Q(F_v)w_0)\backslash M(F_v)/G_2(F_v) : \delta\cdot\psi_U^{e_0}|_{U\cap w_0^{-1}Qw_0}\equiv 1\}$$

is infinite.

To do this we first compute $M\cap w_0^{-1}Qw_0$ and find that it is the product of the GL_1 factor of M and the parabolic of type $(2,2,3)$ in the Levi factor. Note that the dimension of this parabolic is 33.

If we let $GL_7(F_v)$ act on $\psi_U^{e_0}$, then the stabilizer is $G_2(F_v)$, and so the orbit is a variety of dimension 35. Recall that $\psi_U^{e_0}$ is identified with a nilpotent element X of $\mathfrak{ge}_7$, lying in $\mathfrak{g}_{-2}^s$ for the semisimple element ${}^{0\,0\,\overset{0}{2}\,0\,0\,0}$, and our variety is then identified with the GL_7-orbit of X in $\mathfrak{g}_{-2}^s$.

Finally, we compute that $\{\alpha\in\Phi(U,T) : w_0\alpha>0\}=\{1123321\}$. Because w_0 is of shortest length in Qw_0P, this implies that $U\cap w_0^{-1}Qw_0=U_{1123321}$. This means that the condition $\delta\cdot\psi_U^{e_0}|_{U\cap w_0^{-1}Qw_0}\equiv 1$ amounts to a single polynomial equation on the entries of δ, so we get a 34-dimensional subvariety. Clearly, our 33-dimensional parabolic cannot act transitively on this subvariety. ■

Lemma 8.3.2. *At least eight different $Q(F_v)$, $P(F_v)$-double cosets contain admissible $Q(F_v)$, $G_2(F_v)U(F_v)$-double cosets.*

Proof. Indeed, there are eight distinct Kostant representatives w for elements of $Q\backslash GE_7/P$ such that $\psi_U^{e_0}|_{U\cap w^{-1}Qw}\equiv 1$. That is, there are eight Kostant representatives such that

$$\{\delta\in(M(F_v)\cap w^{-1}Q(F_v)w)\backslash M(F_v)/G_2(F_v) : \delta\cdot\psi_U^{e_0}|_{U\cap w^{-1}Qw}\equiv 1\}$$

contains the identity. (And possibly other representatives such that it is nonempty but does not contain the identity.) ■

Remark 8.3.3. We expect that if π is of G_2 type then the local components of π at unramified places will be induced from characters of the form (7.3.1), with χ_1,χ_2 being unitary characters. However, we would expect that in general χ_1,χ_2 would not satisfy any special condition that would permit (7.3.4) to be reducible. The representation (7.3.4) has a P-module filtration parametrized by the elements of $Q\backslash GE_7/P$, and Lemma 8.3.2 suggests that at least eight of the P-modules in this filtration will have nontrivial twisted Jacquet modules. Thus the local unramified descent appears to be highly reducible.

This is consistent with our global results. We would expect an irreducible cuspidal automorphic representation π of G_2 type to be a weak functorial lift attached to the embedding $G_2(\mathbb{C})\hookrightarrow GL_7(\mathbb{C})$ of some generic cuspidal automorphic representation of $G_2(\mathbb{A})$. In the classical cases considered in [16, 20], the descent recovers the original cuspidal representation that was lifted (up to near equivalence). In our case, our global results let us know that the descent module also contains noncuspidal functions. In general, we would not expect any noncuspidal automorphic forms to lift weakly to π. Hence our noncuspidality result predicts that the descent module will not consist solely of automorphic forms which lift weakly to π.

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