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Sharp local well-posedness for quasilinear wave equations with spherical symmetry

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Abstract. In this paper, we prove a sharp local well-posedness result for spherically symmetric solutions to quasilinear wave equations with rough initial data, when the spatial dimension is three or higher. Our approach is based on Morawetz type local energy estimates with fractional regularity for linear wave equations with variable C^1 coefficients, which rely on the multiplier method, weighted Littlewood–Paley theory, duality and interpolation. Together with weighted linear and nonlinear estimates (including weighted trace estimates, Hardy's inequality, a fractional chain rule and a fractional Leibniz rule) which are adapted to the problem, the well-posedness result is proved by iteration. In addition, our argument yields almost global existence for $n = 3$ and global existence for $n \geq 4$ when the initial data are small and spherically symmetric with almost critical Sobolev regularity.

Keywords. Quasilinear wave equation, local energy estimates, KSS estimates, trace estimates, fractional chain rule, fractional Leibniz rule, unconditional uniqueness, almost global existence

Contents

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1. Introduction

Let $n \geq 3$. We are interested in the local well-posedness of spherically symmetric solutions of the Cauchy problem for a quasilinear wave equation with low regularity

$$\Box u + g(u)\Delta u = a(u)u_t^2 + b(u)|\nabla u|^2, \quad (t,x) \in (0,T) \times \mathbb{R}^n\,, \tag{1.1}$$

$$u(0,x) = u_0(x) \in H^s_{\rm rad}(\mathbb{R}^n), \quad \partial_t u(0,x) = u_1(x) \in H^{s-1}_{\rm rad}(\mathbb{R}^n), \tag{1.2}$$

where $\Box = -\partial_t^2 + \Delta$, g, a, b are smooth functions, $g(0) = 0$ and $\Box + g(u)\Delta$ satisfies the uniform hyperbolicity condition. Here, $H^s_{\rm rad}$ stands for the subspace of spherically symmetric functions in the usual Sobolev space H^s.

Equation (1.1) is scale-invariant in the sense that $u_\lambda(t,x) = u(t/\lambda, x/\lambda)$ solves (1.1) for every $\lambda > 0$ whenever $u(t,x)$ is a solution. This gives us the critical homogeneous Sobolev space $\dot{H}^{s_c}$ with

$$s_c = n/2,$$

which is known to be a lower bound of s such that the problem is well-posed in H^s. On the other hand, another characteristic feature of wave equations is the propagation of singularities along the light cone, which heuristically yields ill-posedness at the regularity level $s \leq s_l = (n+5)/4$.

For semilinear wave equations, that is, with $g \equiv 0$, the problem can be shown to be locally well-posed in H^s for $s > n/2 + 1 - 1/q$ with $q = \max(2, 4/(n-1))$ by $L_t^{q+} L_x^\infty$ Strichartz estimates. Moreover, it is known that the problem is locally well-posed if

$$s > \max(s_c, s_l),$$

and ill-posed in general if $s < s_c$ or $s \le s_l$; see Ponce and Sideris [40] ($n = 3$), Tataru [46] ($n \ge 5$) and Zhou [53] ($n = 2, 4$) for positive results, and Lindblad [28, 29] ($n = 3$), Fang–Wang [8] ($n \ge 6$) and Liu–Wang [32] ($n \le 5$) for negative results. The critical well-posedness remains open for higher dimensions ($n \ge 6$).

If the nonlinearity is of the first null-form, that is, $a(u)u_t^2 + b(u)|\nabla u|^2 = c(u)(u_t^2 - |\nabla u|^2)$ for some $c(u)$, improved local well-posedness results are also available, which state that $s > s_c$ is sufficient for local well-posedness; see, e.g., Klainerman–Machedon [22] ($n = 3$), Klainerman–Selberg [26] ($n \ge 2$), and also Liu–Wang [32].

Furthermore, it is well-known that we can extend the admissible pairs for Strichartz estimates when the initial data are spherically symmetric or have a certain angular regularity; see Klainerman–Machedon [21], Sterbenz [44], Machihara–Nakamura–Nakanishi–Ozawa [33] and Fang–Wang [9]. With the help of this observation, we can improve the radial results to $s > 3/2$ for $n = 2$ and $s \ge 2$ for $n = 3$. We see that there is still a $1/2$ gap of regularity, between the positive results and the scaling regularity. When $n = 3$ in the case of radial small data, by exploiting local energy estimates and weighted fractional chain rule, the regularity assumption is improved to the almost critical assumption $s > 3/2$ in Hidano–Jiang–Lee–Wang [12], with previous results of Hidano–Yokoyama [16] for $s = 2$. In view of [12], it seems that the critical radial regularity for $n = 2$ is $s = 3/2$ against the scaling critical regularity $s = 1$. Concerning radial solutions to general semilinear hyperbolic systems in 3D under the null condition, global existence for small scaling invariant $\dot{W}^{2,1}(\mathbb{R}^3)$ data is known from Yin–Zhou [52].

The quasilinear problem (1.1) is much more delicate. By the classical energy argument, the problem is locally well-posed as long as $s > n/2 + 1$ [17]. Similar to the semilinear problem, the approach using $L^{q+} L^\infty$ Strichartz estimates has been intensively investigated. To make the argument work, we need to obtain Strichartz estimates for wave operators with variable coefficients. It is known that we have the full Strichartz estimates provided that the perturbation is $C^{1,1}$; see Smith [41], Tataru [48], as well as Kapitanskii [19] and Mockenhaupt–Seeger–Sogge [37] for previous results with a smooth perturbation. However, in view of application to the quasilinear problem (1.1), $u \in H^s$ with $s < n/2 + 1$ will only imply $g(u) \in C^{0,s-n/2}$, by Sobolev embedding, which means that it is desirable to obtain Strichartz estimates for wave operators with rough coefficients.

The first breakthrough was achieved through the independent works of Bahouri-Chemin [3] (Hadamard parametrix) and Tataru [47] (FBI transform), where weaker Strichartz estimates for a metric with limited regularity were obtained, giving the local well-posedness for

$$s > \frac{n+2}{2} - \begin{cases} 1/4, & n \ge 3, \\ 1/8, & n = 2. \end{cases}$$

This approach was developed further, to arrive at

$$s > \frac{n+2}{2} - \begin{cases} 1/3, & n \geq 3, \\ 1/6, & n = 2; \end{cases}$$

see Bahouri–Chemin [2] and Tataru [48]. To get further improvements, it is desirable to exploit the additional geometric information on the metric $g(u)$ and the solution u. In the work of Klainerman–Rodnianski [23], the nonlinear structure of solutions was exploited to obtain the improved result $s > 3 - \sqrt{3}/2$ when $n = 3$. Finally, the well-posedness for

$$s > \frac{n+2}{2} - \begin{cases} 1/2, & 3 \leq n \leq 5, \\ 1/4, & n = 2, \end{cases}$$

was proven by Klainerman–Rodnianski [24] for the Einstein-vacuum equations ($n = 3$), and by Smith–Tataru [42] for general quasilinear wave equations in all space dimensions, by constructing a parametrix using wave packets. Later, Q. Wang [51] gave an alternative proof for Smith–Tataru's result when $n = 3$, by the commuting vector fields approach.

When $n = 3, 2$, we know from Lindblad [30] and Liu–Wang [32] that the well-posedness result in [42] is sharp in general. However, concerning the Einstein-vacuum equations, the so-called bounded L^2 curvature conjecture (well-posedness in H^2) was verified in Klainerman–Rodnianski–Szeftel [25]. In contrast, Ettinger–Lindblad [7] proved an ill-posedness result in H^2 for Einstein-vacuum equations in the harmonic gauge.

In summary, the quasilinear problem is locally well-posed in H^s for

$$s > \begin{cases} (n+5)/4, & n \leq 3, \\ (n+1)/2, & n = 4, 5, \\ n/2 + 2/3, & n \geq 6, \end{cases}$$

in general.

Comparing with the semilinear problem, we expect naturally that there should be improved well-posedness when the problem and the initial data are spherically symmetric. Actually, in [14], together with Hidano and Yokoyama, we proved that the 3-dimensional problem

$$\Box u + g(u)\Delta u = a u_t^2 + b|\nabla u|^2$$

is well-posed for small radial data in $H^2_{\text{rad}} \times H^1_{\text{rad}}$, with almost global existence of solutions, up to time $\exp(c/\|(\nabla u_0, u_1)\|_{H^1})$ (see also Zhou–Lei [54] for previous work on global existence with $a = b = 0$, for compactly supported $H^2_{\text{rad}} \times H^1_{\text{rad}}$ data). On the other hand, when $n = 2$, the improved local well-posedmess result for H^s_{rad} with $s > 3/2$ was suggested in Fang–Wang [10]. In addition, as already mentioned, when $n = 2, 3$, the long time well-posedness with small radial data in H^s_{rad} with $s > 3/2$ is known from Hidano–Jiang–Lee–Wang [12].

1.1. Main results

Let us turn to the first main result of this paper, concerning the physically important case, $n = 3$. It is known that the problem is well-posed in H^2_{rad} (at least for small data) and generally ill-posed in H^s_{rad} with $s < s_c = 3/2$. Heuristically, comparing with the semilinear results, we may expect well-posedness in H^s_{rad} for certain $s < 2$. However, H^2_{rad} is the lowest possible regularity we can obtain by the approach of ($L^2_t L^\infty_x$) Strichartz estimates, even in the radial case. To break the Sobolev regularity barrier $s = 2$, we need to circumvent the Strichartz estimates approach.

In the following result, we prove well-posedness in H^s_{rad} for any subcritical regularity, $s > 3/2$, which shows that there are no other obstacles to well-posedness in the radial case except scaling. As far as we know, this might be the first well-posedness result for three-dimensional quasilinear wave equations, which breaks the Sobolev regularity barrier, $s = 2$. More precisely, we prove the following, with a certain low frequency condition on u_1.

Theorem 1.1. *Let $n = 3$, $s \in (3/2, 2]$ and $s_0 \in [2 - s, s - 1]$. Consider (1.1)–(1.2) with $u_0 \in H^s_{\mathrm{rad}}$ and $u_1 \in \dot H^{s-1}_{\mathrm{rad}} \cap \dot H^{s_0-1}_{\mathrm{rad}}$. There exists $T_0 > 0$ such that the problem is (unconditionally) locally well-posed in the function space*

$$u \in L^\infty_t H^s_{\mathrm{rad}} \cap C^{0,1}_t H^{s-1}_{\mathrm{rad}} \cap C_t \dot H^{s_0}([0, T_0] \times \mathbb{R}^3), \quad \partial_t u \in C_t \dot H^{s_0-1}. \tag{1.3}$$

More precisely:

(1) (Existence) *There exists a universal constant $C > 0$ such that there exists a (weak) solution u satisfying (1.3) and*

$$\|\partial u\|_{L^\infty \dot H^\theta} + T^{-\mu/2}\|r^{-(1-\mu)/2} D^\theta \partial u\|_{L^2([0,T]\times\mathbb{R}^3)} \le C\|(\nabla u_0, u_1)\|_{\dot H^\theta}$$

for all $\theta \in \{s_0 - 1\} \cup [0, s - 1]$ and $T \in (0, T_0]$. Here $\mu = s - 3/2$, $D = \sqrt{-\Delta}$.

(2) (Uniqueness) *The solution is unconditionally unique in (1.3).*

(3) (Persistence of regularity) *Let T_* be the maximal time of existence (lifespan) for the solution in (1.3). If $(u_0, u_1) \in H^{s_1} \times H^{s_1-1}$ for some $s_1 \ge 3$, then the solution u is in $L^\infty H^{s_1} \times C^{0,1}_t H^{s_1-1}$ in $[0, T] \times \mathbb{R}^3$ for any $T < T_*$.*

(4) (Continuous dependence) *We also have continuous dependence on the data when $s_0 < s - 1$, in the following sense: for any $T \in (0, T_*)$, $s_1 \in (s_c, s)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\|(\nabla(u_0 - v_0), u_1 - v_1)\|_{\dot H^{s-1} \cap \dot H^{\max(s_1-2, s_0-1)}} \le \delta$, the corresponding solution $v \in L^\infty H^{s_1} \times C^{0,1}_t H^{s_1-1}$ is well-defined in $[0, T] \times \mathbb{R}^3$ and*

$$\|\partial(u - v)\|_{L^\infty(\dot H^{s_1-1} \cap \dot H^{\max(s_1-2, s_0-1)})} \le \varepsilon.$$

Remark 1.2. The regularity assumption on the lifespan obtained in Theorem 1.1 is sharp in general. More precisely, we do not have well-posedness for data in some critical space, B, and possibly nonsubcritical space $\dot H^s$ with $s \le s_c = 3/2$. Actually, let $g = 0$, $a = 1$, $b = 0$, and ϕ, ψ be given nonnegative nontrivial, spherically symmetric C^∞_0 functions. Then it is well-known (see, e.g., John [18]) that for classical solutions, for any $\varepsilon > 0$, the lifespan T_* is finite for data $u_0 = \varepsilon\phi, u_1 = \varepsilon\psi$. By persistence of regularity, we know that T_* is the same as the lifespan for weak solutions. If the problem is still well-posed, then by continuous dependence at the trivial solution, there exists $\delta > 0$ such that $T_* \geq \delta$ for any data with critical norm $\|(u_0, u_1)\|_B \leq \delta$ and $\varepsilon_s = \|(\nabla_x u_0, u_1)\|_{\dot{H}^{s-1}} \leq \delta$. Let $\varepsilon \ll 1$ be such that the critical norm of the data is $\leq \delta$ and we obtain a solution u with $T_* < \infty$. For such fixed $\varepsilon > 0$, by rescaling we know that, for any $0 < \lambda \leq 1$, $u_\lambda(t, x) = u(t/\lambda, x/\lambda)$ solves the equation with rescaled data, and

$$\|(u_\lambda(0), \partial_t u_\lambda(0))\|_B = \|(u_0, u_1)\|_B \leq \delta, \quad \varepsilon_{s,\lambda} = \lambda^{s_c - s}\varepsilon_s \leq \varepsilon_s \leq \varepsilon_0, \quad T_{*,\lambda} = \lambda T_*.$$

This gives $0 < \delta \leq T_{*,\lambda} = \lambda T_* < \infty$ for any $0 < \lambda \leq 1$, which is clearly a contradiction. On the other hand, we have an auxiliary low frequency regularity assumption on the initial velocity $u_1 \in \dot{H}^{s_0-1}$, due to the second order of the equation and the limited regularity level $s < 2$. This assumption plays a key role in our analysis, to close the iteration, and it will be interesting to determine whether it is essential for the well-posedness result or not. Notice, however, that we do not need to assume that u_1 is in $\dot{H}^{s_0-1}$ when it is compactly supported.

Next, we present our high-dimensional well-posedness result.

Theorem 1.3. *Let $n \geq 4$ and $s = n/2 + \mu$ with*

$$\mu \in \begin{cases} (0, 1/2], & n \text{ odd}, \\ (0, 1), & n \text{ even}. \end{cases} \tag{1.4}$$

The problem (1.1)–(1.2) *is* (*unconditionally*) *locally well-posed in the function space*

$$u \in L_t^\infty H_{\text{rad}}^s \cap C_t^{0,1} H_{\text{rad}}^{s-1} \cap CH^1 \cap C^1 L^2. \tag{1.5}$$

More precisely:

(1) (Existence) *There exists a constant $C > 0$ such that for any data $(u_0, u_1) \in H_{\text{rad}}^s \times H_{\text{rad}}^{s-1}$, there exist $T > 0$ and a* (*weak*) *solution u in* (1.5) *in $[0, T] \times \mathbb{R}^n$ satisfying*

$$\|\partial u\|_{L^\infty \dot{H}^\theta} + T^{-\mu/2}\|r^{-(1-\mu)/2} D^\theta \partial u\|_{L^2([0,T]\times\mathbb{R}^n)} \leq C\|(\nabla u_0, u_1)\|_{\dot{H}^\theta}$$

for all $\theta \in [0, s-1]$.

(2) (Uniqueness) *The solution is unconditionally unique in* (1.5).

(3) (Persistence of regularity) *Let T_* be the lifespan. If $(u_0, u_1) \in H^{s_1} \times H^{s_1-1}$ for some $s_1 \geq [(n+4)/2]$, then the solution u is in $L^\infty H^{s_1} \times C_t^{0,1} H^{s_1-1}$ in $[0, T] \times \mathbb{R}^n$ for any $T < T_*$.*

(4) (Continuous dependence) *We also have continuous dependence on the data, in the H^{s_1} topology, for $s_1 \in (s_c, s)$.*

For the small data problem, we can give the following long time existence result.

Theorem 1.4 (Long time existence for small data). *Let $n \geq 3$ and $s > s_c = n/2$. Consider (1.1)–(1.2) with $(u_0, u_1) \in H^s_{\text{rad}} \times H^{s-1}_{\text{rad}}$. When $n = 3$, assuming further that $u_1 \in \dot{H}^{s-2}$, there exist $c > 0$ and $\delta > 0$ such that for any data with $\varepsilon_1 + \varepsilon_s < \delta$, the problem admits an almost global $L^\infty([0, T]; H^{\min(s,2)}(\mathbb{R}^3))$ solution, where*

$$T = \exp(c/(\varepsilon_1 + \varepsilon_s)), \tag{1.6}$$

$$\varepsilon_1 := \|(\nabla_x u_0, u_1)\|_{L^2}, \quad \varepsilon_s := \|(\nabla_x u_0, u_1)\|_{\dot{H}^{s-1}}, \quad \varepsilon_c = \|(\nabla u_0, u_1)\|_{\dot{B}^{s_c-1}_{2,1}}. \tag{1.7}$$

When $n \geq 4$, for any $s > s_c$ there exists $\varepsilon > 0$ such that the problem admits global solutions whenever $\varepsilon_s + \varepsilon_1 \leq \varepsilon$.

Remark 1.5. The lower bound (1.6) of the lifespan obtained in Theorem 1.4 for $n = 3$ is sharp in general, in terms of the order. Actually, as in Remark 1.2, for the sample case $a = 1$, $g = b = 0$, it is well-known that there exist data $(\varepsilon\phi, \varepsilon\psi)$ such that, for any $\varepsilon \in (0, 1]$, the lifespan of the classical solutions has upper bound $T_* \leq \exp(C/\varepsilon)$ for some $C > 0$. By the way, it is clear from the proofs of Theorems 1.1–1.3 that, when $|g| \ll 1$, we can obtain the following lower bound of the lifespan:

$$T_* \geq c(g, a, b, \varepsilon_c)\varepsilon_s^{-1/(s-s_c)}.$$

Moreover, when $\varepsilon_c \ll 1$,

$$T_* \geq c(g, a, b)\varepsilon_s^{-1/(s-s_c)} \exp(c(g, a, b)/\varepsilon_c).$$

On the other hand, under certain nonlinear conditions, such as null conditions, or many cases of weak null conditions, the problem admits global solutions with small data. See, e.g., [1, 31, 54] for global results with $a = b = 0$. In such situations, we naturally expect that global radial results with $s > s_c$ (or even in certain critical spaces like $\dot{B}^{s_c}_{2,1}$) still hold, which is an interesting further problem.

Remark 1.6. Although we state our results only for scalar quasilinear wave equations, it is clear from the proofs that they also apply to general multi-speed systems of quasilinear wave equations which permit spherically symmetric solutions. In particular, the system with multi-speeds ($c_j > 0$)

$$\partial_t^2 u^j - c_j^2(1 + g_j(u))\Delta u^j = Q^{j\alpha\beta}_{kl}(u)\partial_\alpha u^k \partial_\beta u^l, \quad 1 \leq j \leq N,$$

is locally well-posed in $H^s_{\text{rad}}(\mathbb{R}^n) \times (H^{s-1}_{\text{rad}}(\mathbb{R}^n) \cap \dot{H}^{s-2}(\mathbb{R}^n))$ for $n \geq 3$ and $s > n/2$ as long as it admits spherically symmetric solutions. A similar statement holds for

$$\partial_t^2 u^j - c_j^2(1 + g_j(u, \partial u))\Delta u^j = Q^{j\alpha\beta}_{kl}(u)\partial_\alpha u^k \partial_\beta u^l, \quad 1 \leq j \leq N,$$

in $H^s_{\text{rad}}(\mathbb{R}^n) \times (H^{s-1}_{\text{rad}}(\mathbb{R}^n) \cap \dot{H}^{s-3}(\mathbb{R}^n))$ when $n \geq 3$ and $s > (n+2)/2$.

In addition, the quasilinear part could be replaced by the D'Alembertian $\Box_{g(u)}$ with respect to the metric $ds^2 = -dt^2 + g(u)dx^2$, or by $\Box_{\mathfrak{g}} + g(u)\Delta$ when $\mathfrak{g}$ is a small, long range perturbation of the Minkowski metric:

$$
\begin{aligned}
&\mathfrak{g} = -K_0(t,r)^2 dt^2 + 2K_{01}(t,r)dtdr + K_1(t,r)^2 dr^2 + r^2 d\omega^2, \\
&|(K_0 - 1, K_{01}, K_1 - 1)| \ll 1, \\
&\sum_{j\geq 0} \|r^{|\gamma|-\mu}\langle r\rangle^{\mu}\partial^{\gamma}(K_0 - 1, K_{01}, K_1 - 1)\|_{L^{\infty}_{t,x}(1+|x|\sim 2^j)} \ll 1,
\end{aligned}
$$

for $1 \leq |\gamma| \leq [n/2] + 1$ (and $K_{01} = 0$ when $n = 3$).

1.2. Idea of proof

Let us discuss the idea of the proof. Basically, we rely on Morawetz type local energy estimates, instead of Strichartz estimates. In many works on dispersive and wave equations, Morawetz type local energy estimates have proven to be more fundamental and robust than Strichartz estimates, for many nonlinear problems.

To make this approach work for quasilinear wave equations, similar to the approach using Strichartz estimates, we prove a version of Morawetz type local energy estimates (Theorem 3.1) for linear wave equations with variable C^1 coefficients. It is this version of local energy estimates which makes it possible to relax the regularity requirement for quasilinear wave equations. The proof is based on the classical multiplier approach with well-chosen multipliers, which yields such estimates for small perturbations of the Minkowski metric. Furthermore, the property of finite speed of propagation is exploited to handle the general case of large perturbations.

With the help of the well-adapted Morawetz type local energy estimates (weighted space-time L^2 estimates), we are naturally led to develop the corresponding linear and nonlinear estimates involving weight functions. Among others, we prove weighted Sobolev type estimates (including weighted trace estimates, Proposition 2.2, and weighted Hardy inequality, Lemma 2.7), a weighted fractional chain rule (Theorem 2.3 and Proposition 2.8), as well as a weighted Leibniz rule (Theorem 2.4).

The Morawetz type local energy estimates of Theorem 3.1 are at the regularity level of $\dot{H}^1$. To make the approach work, we need to develop the corresponding version of local energy estimates at the regularity level $\dot{H}^s$ with $s > n/2$. With the help of interpolation, Littlewood–Paley theory involving weight functions, together with the weighted Sobolev type estimates from Proposition 2.2, Lemma 2.7 and Lemma 2.9, we prove a series of local energy estimates with positive fractional derivatives (Propositions 3.5, 3.6 and 3.9).

Equipped with all these linear and nonlinear estimates, we then use the standard iteration argument to establish local existence and uniqueness, as well as long time existence. In particular, for $n = 3$, to prove convergence of approximate solutions, we develop local energy estimates with negative regularity, and we need to make a certain low frequency requirement on the initial velocity, due to the second order of the equation and the limited regularity level $s < 2$.

Recall that, in the approach using Strichartz estimates, the proof of local existence immediately implies persistence of regularity and continuous dependence on the data, through Gronwall's inequality. By contrast, in our approach, the proofs of persistence of regularity and continuous dependence on the data are not so direct. For example, concerning persistence of regularity, we prove first the result from regularity index s to any $s \in (s_c, [(n+2)/2])$, and then to $s = [(n+4)/2]$, which is sufficient to conclude persistence of higher regularity.

This paper is organized as follows. In the next section, we recall and prove various basic linear and nonlinear estimates, including weighted Sobolev type estimates, weighted trace estimates, weighted Hardy inequality, and a weighted fractional chain rule and a weighted Leibniz rule. In Section 3, we present a version of Morawetz type local energy estimates for linear wave equations with variable C^1 coefficients, as well as estimates with fractional regularity. In Sections 4 and 5, by an iteration argument, we prove local existence and uniqueness for $n = 3$ and $n \geq 4$. In Section 6, we show persistence of regularity for weak solutions when the initial data have higher regularity, as well as continuous dependence on the data. Next, in Sections 7 and 8, we present the proof of almost global existence and global existence for $n = 3$ and $n \geq 4$ when the initial data are small. Finally, in the appendix, we establish the fundamental Morawetz type estimates, by an elementary multiplier approach with carefully chosen multipliers.

1.3. Notations

We close this section by listing some notations.

• $\mathcal{F}(f)$ and $\widehat{f}$ denote the Fourier transform of f; $D = \sqrt{-\Delta} := \mathcal{F}^{-1}|\xi|\mathcal{F}$; and $P_j = \phi(2^{-j}D)$ is the (homogeneous) Littlewood–Paley projection on the space variable, $j \in \mathbb{Z}$.
• $r = |x|$, $\langle r\rangle = \sqrt{2+r^2}$, $\partial = (\partial_t, \nabla_x) = (\partial_t, \nabla)$, $\tilde{\partial} u = (\partial u, u/r)$, $|\nabla^k u| = \sum_{|\gamma|=k} |\nabla^\gamma u|$ for multi-indices γ.
• $L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space, and $L^p_r(\mathbb{R}^+) = L^p(\mathbb{R}^+, r^{n-1}dr)$.
• $L^p_r L^q_\omega$ is the Banach space defined by the norm

$$\|f\|_{L^p_r L^q_\omega} = \big\| \|f(r\omega)\|_{L^q_\omega} \big\|_{L^p_r}.$$

• H^s, $\dot{H}^s$ (resp. $B^s_{p,q}$, $\dot{B}^s_{p,q}$) are the usual inhomogeneous and homogeneous Sobolev (resp. Besov) spaces on $\mathbb{R}^n$.
• With parameters $\mu, \mu_1 \in (0,1)$ and $T \in (0,\infty)$, we define

$$\begin{aligned}\|u\|_{LE_T} &= \|\partial u\|_{L^\infty_t L^2_x} + \|r^{-(1-\mu)/2}\langle r\rangle^{-(\mu+\mu_1)/2}\partial u\|_{L^2_{t,x}} \\ &\quad + \langle T\rangle^{-\mu/2}\|r^{-(1-\mu)/2}\partial u\|_{L^2_{t,x}} + (\ln\langle T\rangle)^{-1/2}\|r^{-(1-\mu)/2}\langle r\rangle^{-\mu/2}\partial u\|_{L^2_{t,x}},\end{aligned} \tag{1.8}$$

for functions on $[0,T]\times\mathbb{R}^n$. In the limit case $T = \infty$, we set

$$\|u\|_{LE} = \|\partial u\|_{L^\infty_t L^2_x} + \|r^{-(1-\mu)/2}\langle r\rangle^{-(\mu+\mu_1)/2}\partial u\|_{L^2_{t,x}} + \sup_{T>0}\|u\|_{LE_T}.$$

In addition, for fixed $\mu \in (0,1)$,

$$\|u\|_{X_T} := \|u\|_{L_t^\infty L_x^2} + T^{-\mu/2}\|r^{-(1-\mu)/2}u\|_{L^2_{t,x}}, \tag{1.9}$$

$$\|F\|_{X_T^*} := \inf_{F=F_1+F_2}(\|F_1\|_{L_t^1 L_x^2} + T^{\mu/2}\|r^{(1-\mu)/2}F_2\|_{L^2_{t,x}}). \tag{1.10}$$

For $q \in [1,\infty]$, we introduce the Besov version as follows:

$$\|u\|_{X_{T,q}} := \|u\|_{L_t^\infty \dot{B}^0_{2,q}} + T^{-\mu/2}\|r^{-(1-\mu)/2}P_j u\|_{\ell_j^q L^2_{t,x}}.$$

2. Sobolev type and nonlinear estimates

In this section, we recall and prove various basic estimates to be used.

2.1. *Weighted Sobolev type estimates*

We will use the following version of weighted Sobolev estimates, which essentially are consequences of the well-known trace estimates.

Lemma 2.1 (Trace estimates). *Let $n \geq 2$ and $s \in [0, n/2)$. Then*

$$\|r^{(n-1)/2}u\|_{L_r^\infty L_\omega^2} \lesssim \|u\|_{\dot{B}^{1/2}_{2,1}}, \quad \|r^{n/2-s}u\|_{L_r^\infty H_\omega^{s-1/2}} \lesssim \|u\|_{\dot{H}^s}, \quad s > 1/2, \tag{2.1}$$

$$\|r^{n(1/2-1/p)-s}f\|_{L_r^p L_\omega^2} \lesssim \|f\|_{\dot{H}^s}, \quad 2 \leq p < \infty,\ 1/2 - 1/p \leq s < n/2. \tag{2.2}$$

The estimate (2.1) is well-known: see, e.g., [11, (1.3), (1.7)] and references therein. The inequality (2.2) with $s = 1/2 - 1/p$ is due to [27]; see also [15] for an alternative proof using real interpolation and (2.1).

We shall also use the following weighted variant of trace estimates.

Proposition 2.2 (Weighted trace estimates). *Let $n \geq 2$, $\alpha \in (1/2, n/2)$ and $\beta \in (\alpha - n/2, n/2)$. Then*

$$\|r^{n/2-\alpha+\beta}P_j u\|_{l_j^2 L_r^\infty H_\omega^{\alpha-1/2}} \lesssim \|r^\beta D^\alpha u\|_{L^2}, \tag{2.3}$$

$$\|r^{n/2-\alpha+\beta}u\|_{L_r^\infty H_\omega^{\alpha-1/2-}} \lesssim \|r^\beta 2^{j\alpha}P_j u\|_{l_j^\infty L^2}. \tag{2.4}$$

In addition,

$$\|r^{n(1/2-1/p)-\alpha+\beta}u\|_{L_r^p L_\omega^2} \lesssim \|r^\beta D^\alpha u\|_{L^2} \tag{2.5}$$

for any $p \in [2,\infty]$, $\alpha \in (1/2 - 1/p, n/2)$ and $\beta \in (\alpha - n/2, n/2)$.

Proof. We essentially follow [50, Lemma 4.2], where (2.3) was proven for $\alpha \in (1/2, 1]$ and $n \geq 3$. Recall that we have the following weighted Littlewood–Paley square-function estimate:

$$\|wP_j f\|_{L^p \ell_j^2} \simeq \|wf\|_{L^p}, \quad w^p \in A_p,\ f \in L^p(w^p dx),\ p \in (1,\infty). \tag{2.6}$$

As $r^{2\beta} \in A_2$ if and only if $|\beta| < n/2$, we get

$$\|r^\beta 2^{j\alpha} P_j u\|_{\ell_j^2 L^2} \simeq \|r^\beta D^\alpha u\|_{L^2}, \quad \beta \in (-n/2, n/2). \tag{2.7}$$

Based on this estimate, we observe that, by rescaling, interpolation and frequency localization, the proof of (2.3) and (2.4) can be reduced to proving

$$\|u(\omega)\|_{H_\omega^{\alpha-1/2}(\mathbb{S}^{n-1})} \lesssim \|r^\beta \nabla^k u\|_{L^2}^{\alpha/k} \|r^\beta u\|_{L^2}^{1-\alpha/k}, \quad \alpha \in [1/2, n/2), \tag{2.8}$$

where $\beta \in (\alpha - n/2, n/2)$, $k \in (\alpha, \alpha+1]$.

For the proof of (2.8), we recall the weighted Hardy–Littlewood–Sobolev estimates of Stein–Weiss:

$$\|r^{\beta-\alpha} u\|_2 \lesssim \|r^\beta D^\alpha u\|_2, \quad \alpha \in (0, n), \ \beta \in (\alpha - n/2, n/2). \tag{2.9}$$

Then for any $\alpha \in (0, n)$ and $\beta \in (\alpha - n/2, n/2)$, we have

$$\|r^{\beta-\alpha} u\|_2 \lesssim \|r^\beta D^\alpha u\|_2 \lesssim \|r^\beta \nabla^k u\|_2^{\alpha/k} \|r^\beta u\|_2^{1-\alpha/k} \tag{2.10}$$

if $\alpha < k$. Moreover, if $k \in (\alpha, \alpha+1] \cap \mathbb{N}$, we have

$$\|r^{\beta-j} u\|_{L^2} \lesssim \|r^\beta \nabla^j u\|_{L^2}, \quad \forall j < k, \tag{2.11}$$

for such α, β. Let ϕ be a cutoff function of $B_2 \setminus B_{1/2}$ which equals 1 for $|x| = 1$. We deduce from (2.1) that for $\alpha \in [1/2, n/2)$ and $\beta \in (\alpha - n/2, n/2)$,

$$\begin{aligned}
\|u(\omega)\|_{H_\omega^{\alpha-1/2}} &\lesssim \|\nabla^k(\phi u)\|_{L^2}^{\alpha/k} \|\phi u\|_{L^2}^{1-\alpha/k} \\
&\lesssim \Big(\sum_{j<k} \|r^{\beta-j} \nabla^{k-j} u\|_{L^2}\Big)^{\alpha/k} \|r^\beta u\|_{L^2}^{1-\alpha/k} + \|r^{\beta-\alpha} u\|_{L^2} \\
&\lesssim \|r^\beta \nabla^k u\|_{L^2}^{\alpha/k} \|r^\beta u\|_{L^2}^{1-\alpha/k},
\end{aligned}$$

where we have used (2.10) and (2.11). This gives (2.8), and so (2.3) and (2.4).

Finally, (2.5) follows directly from interpolation between (2.9), (2.3) and (2.4). ∎

2.2. *Weighted fractional chain rule*

When dealing with nonlinear problems, it is natural to introduce a weighted fractional chain rule and a weighted Leibniz rule. We first present the following generalized version of the weighted fractional chain rule of Hidano–Jiang–Lee–Wang [12], which could be viewed as a transition from Sobolev type norm to Besov type norm, as well as a transition from space variables to space-time variables. We recall the Muckenhoupt A_p class of weight functions: by definition,

$$w \in A_1 \iff \mathcal{M}w(x) \le C w(x), \text{a.e. } x \in \mathbb{R}^n,$$

$$w \in A_p \ (1 < p < \infty) \iff \Big(\int_Q w(x)\,dx\Big)\Big(\int_Q w(x)^{1-p'}\,dx\Big)^{p-1} \le C|Q|^p, \quad \forall \text{ cubes } Q,$$

where $\mathcal{M}w(x) = \sup_{r>0} r^{-n} \int_{B_r(x)} w(y)\,dy$ denotes the Hardy–Littlewood maximal function. See, e.g., [39, §2.5.2].

Theorem 2.3 (Weighted fractional chain rule). *Let $s \in (0,1)$, $\lambda \geq 1$, $q, q_1, q_2 \in (1,\infty)$ and $p, p_1, p_2 \in [1,\infty]$ with*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{2.12}$$

Assume $F : \mathbb{R}^k \to \mathbb{R}^l$ is a C^1 map satisfying $F(0) = 0$ and

$$|F'(\tau v + (1-\tau)w)| \leq \mu(\tau)|G(v) + G(w)| \tag{2.13}$$

with $G > 0$ and $\mu \in L^1([0,1])$. If $(w_1 w_2)^q \in A_q$, $w_1^{q_1} \in A_{q_1}$ and $w_2^{q_2} \in A_{q_2}$, then

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell^\lambda_{j\in\mathbb{Z}} L^p_t L^q_x} \lesssim \|w_1 2^{js} P_j u\|_{\ell^\lambda_{j\in\mathbb{Z}} L^{p_1}_t L^{q_1}_x} \|w_2 G(u)\|_{L^{p_2}_t L^{q_2}_x} \tag{2.14}$$

for any $[0,T) \times \mathbb{R}^n \ni (t,x) \mapsto u(t,x) \in \mathbb{R}^k$. In addition, when $q_2 = \infty$ and $q \in (1,\infty)$, if $w_1^q, (w_1 w_2)^q \in A_q$ and $w_2^{-1} \in A_1$, we have

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell^\lambda_{j\in\mathbb{Z}} L^p_t L^q_x} \lesssim \|w_1 2^{js} P_j u\|_{\ell^\lambda_{j\in\mathbb{Z}} L^{p_1}_t L^q_x} \|w_2 G(u)\|_{L^{p_2}_t L^\infty_x}. \tag{2.15}$$

For comparison, we recall here that the estimates obtained from [12, Theorem 1.2] can be stated as follows:

$$\|w_1 w_2 D^s F(u)\|_{L^q} \lesssim \|w_1 D^s u\|_{L^{q_1}} \|w_2 G(u)\|_{L^{q_2}}, \tag{2.16}$$

$$\|w_1 w_2 D^s F(u)\|_{L^q} \lesssim \|w_1 D^s u\|_{L^q} \|w_2 G(u)\|_{L^\infty}. \tag{2.17}$$

Proof of Theorem 2.3. The proof uses similar arguments to those for the estimates obtained from [12, Theorem 1.2]. First, recall that, by repeating essentially the argument of Taylor [49, (5.6), p. 112], we can obtain

$$|P_j F(u)(x)| \lesssim \sum_{k\in\mathbb{Z}} \min(1, 2^{k-j}) \big(\mathcal{M}(P_k u)(x)\mathcal{M}(H)(x) + \mathcal{M}(H P_k u)(x)\big), \tag{2.18}$$

where $H(x) := G(u(x))$.

By (2.18), we know that

$$\begin{aligned}
&\|w_1 w_2 2^{js} P_j F(u)\|_{\ell^\lambda_{j\in\mathbb{Z}} L^p_t L^q_x} \\
&\quad \lesssim \big\|w_1 w_2 2^{js} \min(1, 2^{k-j}) \big(\mathcal{M}(P_k u)\mathcal{M}(H) + \mathcal{M}(H P_k u)\big)\big\|_{\ell^\lambda_{j\in\mathbb{Z}} L^p_t L^q_x \ell^1_k} \\
&\quad \lesssim \big\|w_1 w_2 2^{ks} \min(2^{(j-k)s}, 2^{(k-j)(1-s)}) \big(\mathcal{M}(P_k u)\mathcal{M}(H) + \mathcal{M}(H P_k u)\big)\big\|_{\ell^r_{j\in\mathbb{Z}} \ell^1_k L^p_t L^q_x} \\
&\quad \lesssim \big\|w_1 w_2 2^{ks} \big(\mathcal{M}(P_k u)\mathcal{M}(H) + \mathcal{M}(H P_k u)\big)\big\|_{\ell^r_{k\in\mathbb{Z}} L^p_t L^q_x}
\end{aligned}$$

where we have used Young's inequality with the assumption $s \in (0,1)$ in the last inequality.

By applying Minkowski's and Hölder's inequalities to the last expression we have

$$\begin{aligned}
&\|w_1 w_2 2^{js} P_j F(u)\|_{\ell^\lambda_{j\in\mathbb{Z}} L^p_t L^q_x} \\
&\quad\lesssim \|w_1 w_2 2^{ks} \mathcal{M}(P_k u)\mathcal{M}(H)\|_{\ell^\lambda_{k\in\mathbb{Z}} L^p_t L^q_x} + \|w_1 w_2 2^{ks} \mathcal{M}(H P_k u))\|_{\ell^\lambda_{k\in\mathbb{Z}} L^p_t L^q_x} \\
&\quad\lesssim \|w_2 \mathcal{M}(H)\|_{L^{p_2}_t L^{q_2}_x} \|w_1 2^{ks} \mathcal{M}(P_k u)\|_{\ell^\lambda_{k\in\mathbb{Z}} L^{p_1}_t L^{q_1}_x} + \|w_1 w_2 2^{ks} H P_k u\|_{\ell^\lambda_{k\in\mathbb{Z}} L^p_t L^q_x}
\end{aligned}$$

for any $q \in (1,\infty)$, $p, p_1, p_2 \in [1,\infty]$ and $q_1, q_2 \in (1,\infty]$ satisfying (2.12). In the last term above we have used a weighted Hardy–Littlewood inequality for $(w_1 w_2)^q \in A_q$ with $q \in (1,\infty)$.

If $q_2 < \infty$, recall that we have assumed $w_1^{q_1} \in A_{q_1}$ and $w_2^{q_2} \in A_{q_2}$; applying Hölder's inequality and the weighted Hardy–Littlewood inequality again, we obtain

$$\|w_1 w_2 2^{js} P_j F(u)\|_{\ell^\lambda_{j\in\mathbb{Z}} L^p_t L^q_x} \lesssim \|w_2 H\|_{L^{p_2}_t L^{q_2}_x} \|w_1 2^{ks} P_k u\|_{\ell^\lambda_{k\in\mathbb{Z}} L^{p_1}_t L^{q_1}_x},$$

which gives the desired inequality.

For the the remaining case $q_2 = \infty$, a similar argument yields (2.15) if we recall the weighted L^∞ estimate of [12, (2.17)]:

$$\|w^{-1}\mathcal{M}(H)\|_{L^\infty} \lesssim \|w^{-1} H\|_{L^\infty}, \quad \forall w \in A_1. \tag{2.19}$$

This completes the proof. ∎

2.3. *Weighted fractional Leibniz rule*

A closely related and useful result is the weighted fractional Leibniz rule.

Theorem 2.4 (Weighted fractional Leibniz rule). *Let $s > 0$, $q_0, q_1, q_2 \in (1,\infty)$, $p_1, p_2 \in (1,\infty]$ and $s_j \in [1,\infty]$ be such that*

$$\frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{p_1} = \frac{1}{q_2} + \frac{1}{p_2}, \qquad \frac{1}{s_0} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_3} + \frac{1}{s_4}.$$

Suppose the time-independent weight functions w_i satisfy $w_0 = w_1 z_1 = w_2 z_2 > 0$, $w_j^{q_j} \in A_{q_j}$, $z_j^{p_j} \in A_{p_j}$ when $p_j < \infty$, and $z_j^{-1} \in A_1$ when $p_j = \infty$. Then

$$\begin{aligned}
\|w_0 2^{js} P_j(uv)\|_{\ell^\lambda_j L^{s_0}_t L^{q_0}_x} &\lesssim \|w_1 2^{js} P_j u\|_{\ell^\lambda_j L^{s_1}_t L^{q_1}_x} \|z_1 v\|_{L^{s_2}_t L^{p_1}_x} \\
&\quad + \|w_2 2^{js} P_j v\|_{\ell^\lambda_j L^{s_3}_t L^{q_2}_x} \|z_2 u\|_{L^{s_4}_t L^{p_2}_x},
\end{aligned} \tag{2.20}$$

which also yields (for time-independent functions)

$$\|w_0 2^{js} P_j(uv)\|_{\ell^\lambda_j L^{q_0}_x} \lesssim \|w_1 2^{js} P_j u\|_{\ell^\lambda_j L^{q_1}_x} \|z_1 v\|_{L^{p_1}_x} + \|w_2 2^{js} P_j v\|_{\ell^\lambda_j L^{q_2}_x} \|z_2 u\|_{L^{p_2}_x}. \tag{2.21}$$

We remark that the weighted fractional Leibniz rule

$$\|w_0 D^s(uv)\|_{L^{q_0}} \lesssim \|w_1 D^s u\|_{L^{q_1}} \|z_1 v\|_{L^{p_1}} + \|w_2 D^s v\|_{L^{q_2}} \|z_2 u\|_{L^{p_2}} \tag{2.22}$$

with $q_j, p_j \in (1,\infty)$ has been obtained by Cruz–Uribe and Naibo [5] and D'Ancona [6]. However, in view of applications, the results with $p_j = \infty$ seem to be more desirable.

Proof of Theorem 2.4. The proof follows from a standard paraproduct argument and we present only the proof of (2.20). In view of $u = \sum P_j u$, $v = \sum P_j v$, we introduce the paraproduct and decompose uv as follows:

$$T_u v = \sum_{j-k>N} P_k u P_j v, \quad uv = T_u v + T_v u + R(u,v),$$

where N is chosen such that $P_k u P_j v$ has spectral localization in the annulus of radius $\sim 2^j$. The estimates for the $T_v u$ and $T_u v$ are easy:

$$\begin{aligned}\|w_0 2^{js} P_j (T_v u)\|_{L_t^{s_0} L_x^{q_0}} &\lesssim \sum_{|l-j|\lesssim N} \|w_1 2^{ls} P_l u\|_{L_t^{s_1} L_x^{q_1}} \|z_1 P_{<l-N} v\|_{L_t^{s_2} L_x^{p_1}}\\ &\lesssim \sum_{|l-j|\lesssim N} \|w_1 2^{ls} P_l u\|_{L_t^{s_1} L_x^{q_1}} \|z_1 v\|_{L_t^{s_2} L_x^{p_1}},\end{aligned}$$

and so is (2.20) for $T_u v + T_v u$, where we have applied (2.19) when $p_1 = \infty$, as well as the facts that

$$\|P_{<l-N} v\|_{L^p(w dx)} \le C \|v\|_{L^p(w dx)}, \quad w \in A_p,\ p \in (1,\infty), \tag{2.23}$$

and $|P_{<l-N} v| \lesssim \mathcal{M}(v)$.

It remains to control $R(u,v) = \sum_{|l-k|\le N} P_l u P_k v$, for which we have

$$P_j R(u,v) = P_j \Big(\sum_{|l-k|\le N,\, j-k\lesssim N} P_k u P_l v \Big).$$

Then it follows that

$$\begin{aligned}&\|w_0 2^{js} P_j R(u,v)\|_{\ell_j^\lambda L_t^{s_0} L_x^{q_0}}\\ &\quad\lesssim \Big\| \|w_1 2^{js} P_k u\|_{L_t^{s_1} L_x^{q_1}} \|z_1 P_l v\|_{L_t^{s_2} L_x^{p_1}} \Big\|_{\ell_j^\lambda \ell^1_{k\ge j-CN} \ell^1_{|l-k|\le N}}\\ &\quad\lesssim \|w_1 2^{js} P_k u\|_{\ell_j^\lambda \ell^1_{k\ge j-CN} L_t^{s_1} L_x^{q_1}} \|z_1 v\|_{L_t^{s_2} L_x^{p_1}}\\ &\quad\lesssim \|2^{(j-k)s} w_1 2^{ks} P_k u\|_{\ell_j^\lambda \ell^1_{k\ge j-CN} L_t^{s_1} L_x^{q_1}} \|z_1 v\|_{L_t^{s_2} L_x^{p_1}}\\ &\quad\lesssim \|w_1 2^{ks} P_k u\|_{\ell_k^\lambda L_t^{s_1} L_x^{q_1}} \|z_1 v\|_{L_t^{s_2} L_x^{p_1}},\end{aligned}$$

where we have used Young's inequality in the last inequality, and the assumption $s > 0$. ∎

We shall encounter the following weight functions, which are known to be A_p weights [12, Lemma 2.5].

Lemma 2.5. *Let* $w(x) = r^{-1+2\delta_1} \langle r\rangle^{-2\delta_1-2\delta_2}$ *with* $0 \le 1-2\delta_1 \le 1+2\delta_2 < n$. *Then* $w \in A_p(\mathbb{R}^n)$ *for any* $p \in [1,\infty)$.

As a corollary of the weighted fractional Leibniz rule (Theorem 2.4), together with a weighted variant of the trace estimates (Proposition 2.2), we obtain the following inequality which will be frequently used.

Proposition 2.6. *Let $n \geq 3$, $\mu \in (0,1)$ and $|\theta| \leq (n-2)/2+\mu$. Then*

$$\|r^{(1-\mu)/2} D^\theta (fg)\|_{L^2} \lesssim \|r^{-(1-\mu)/2} D^\theta f\|_{L^2} \|g\|_{\dot{H}^{(n-2)/2+\mu}} \tag{2.24}$$

whenever f, g are either spherically symmetric or first-order derivatives of spherically symmetric functions. Moreover, for any $q \in [1,\infty]$ and non-endpoint θ, i.e., $|\theta| < (n-2)/2+\mu$, we can obtain the following estimates by interpolation:

$$\|r^{(1-\mu)/2} 2^{j\theta} P_j (fg)\|_{\ell^q_j L^2_t L^2} \lesssim \|r^{-(1-\mu)/2} 2^{j\theta} P_j f\|_{\ell^q_j L^2_t L^2} \|g\|_{L^\infty_t \dot{H}^{(n-2)/2+\mu}}. \tag{2.25}$$

Proof. First, we notice that it suffices to prove the result with $\theta = (n-2)/2+\mu$, by duality and complex interpolation, if we recall the well-known fact that $r^\alpha \in A_2$ if and only if $|\alpha| < n$, and so $\|r^{\alpha/2} D^\theta f\|_{L^2} \simeq \|r^{\alpha/2} 2^{j\theta} P_j f\|_{\ell^2_j L^2}$.

By (2.21) of Theorem 2.4 with $\lambda = q_0 = 2$, we have

$$\|r^{(1-\mu)/2} D^\theta (fg)\|_{L^2} \lesssim \|r^{-(1-\mu)/2} D^\theta f\|_{L^2} \|r^{1-\mu} g\|_{L^\infty} + \|D^\theta g\|_{L^2} \|r^{(1-\mu)/2} f\|_{L^\infty}$$

provided that

$$r^{-(1-\mu)/2}, r^{\mu-1} \in A_1, \quad r^{\pm(1-\mu)} \in A_2,$$

which is true as $\mu \in (1-n, 1)$. By the symmetry assumption, we have

$$\|r^{1-\mu} g\|_{L^\infty} \lesssim \|r^{1-\mu} g\|_{L^\infty_r L^2_\omega} \lesssim \|D^\theta g\|_{L^2},$$
$$\|r^{(1-\mu)/2} f\|_{L^\infty} \lesssim \|r^{(1-\mu)/2} f\|_{L^\infty_r L^2_\omega} \lesssim \|r^{-(1-\mu)/2} D^\theta f\|_{L^2},$$

where we have used Lemma 2.1 and (2.5) of Proposition 2.2. This gives us (2.24) with $\theta = (n-2)/2+\mu$, completing the proof. ∎

2.4. *Inhomogeneous weight*

We will need the following weighted Hardy type estimate with inhomogeneous weight.

Lemma 2.7 (Weighted Hardy inequality). *Let $0 \leq \alpha \leq \beta < n/2 - s$ and $s \geq 0$. Then*

$$\|r^{-\alpha-s} \langle r\rangle^{-\beta+\alpha} u\|_{L^2} \lesssim \|r^{-\alpha} \langle r\rangle^{-\beta+\alpha} D^s u\|_{L^2}.$$

Proof. The proof is inspired by [39, §9.3]. By Lemma 2.5, the conditions are sufficient to ensure

$$(r^{-\alpha-s} \langle r\rangle^{-\beta+\alpha})^2 \in A_2, \quad \forall s \in [0, n/2).$$

By Littlewood–Paley theory, we can reduce the proof to the case of $u = P_k u$ with $k \in \mathbb{Z}$, that is, we want to show uniform boundedness of the following operators on L^2:

$$T_k = 2^{-ks} r^{-\alpha-s} \langle r\rangle^{-\beta+\alpha} P_k r^\alpha \langle r\rangle^{\beta-\alpha}.$$

This is equivalent to the uniform boundedness of $T_k^* T_k$ with kernel

$$K_k(x,y) = \int 2^{-2ks} w(x)\phi_k(x-z)|z|^{-2s} w(z)^{-2}\phi_k(z-y)w(y)\,dz$$

where we set $w(x) = |x|^\alpha \langle x\rangle^{\beta-\alpha}$ and $\phi_k(x) = 2^{kn}\phi(2^k x)$ with $\phi \in \mathcal{S}(\mathbb{R}^n)$. As $K_k(x,y) = K_k(y,x)$, by Schur's test we need only prove the uniform boundedness of

$$K_k(x,y) \in L_x^\infty L_y^1. \tag{2.26}$$

We will divide the proof into three cases: (i) $|y| \lesssim |z|$, (ii) $|y| \gg |z|$ and $|z| \gg 2^{-k}$, (iii) $|y| \gg |z|$ and $|z| \lesssim 2^{-k}$.

Case (i): $|y| \lesssim |z|$. In this case, we have $w(y) \lesssim w(z)$ and so

$$\int |K_k(x,y)|\,dy \lesssim \int 2^{-2ks} w(x)|\phi_k(x-z)|\,|z|^{-2s} w(z)^{-1} dz. \tag{2.27}$$

We consider first the subcase $|z| \gtrsim 2^{-k}$, when we have $|2^k z|^{-2s} \lesssim 1$, as $s \geq 0$. If $|x| \lesssim |z|$, we have $w(x) \lesssim w(z)$, and so

$$\int |K_k(x,y)|\,dy \lesssim \int |\phi_k(x-z)|\,dz \lesssim 1.$$

On the other hand, if $|x| \gg |z|$, we know that

$$|\phi_k(x-z)| \lesssim 2^{kn}\langle 2^k x\rangle^{-N},$$

and thus

$$\begin{aligned}\int |K_k(x,y)|\,dy &\lesssim \int w(x)|\phi_k(x-z)|w(z)^{-1}\,dz \\ &\lesssim \langle 2^k x\rangle^{-N} w(x) 2^{kn} \int_{|x|\gg|z|\gtrsim 2^{-k}} w(z)^{-1}\,dz.\end{aligned}$$

If $|x| \lesssim 1$, we have $w(x) \simeq |x|^\alpha$ and

$$\int_{|x|\gg|z|\gtrsim 2^{-k}} w(z)^{-1}\,dz \lesssim |x|^{n-\alpha} \lesssim |x|^n w(x)^{-1},$$

while for $|x| \gg 1$, $w(x) \simeq |x|^\beta$ and so

$$\int_{|x|\gg|z|\gtrsim 2^{-k}} w(z)^{-1}\,dz \lesssim |x|^{n-\beta} \lesssim |x|^n w(x)^{-1}.$$

In conclusion, we get (2.26) for $|x| \gg |z| \gtrsim 2^{-k}$:

$$\int |K_k(x,y)|\,dy \lesssim \langle 2^k x\rangle^{-N}(2^k|x|)^n \lesssim 1.$$

Next, we consider another subcase: $|z| \ll 2^{-k}$, when we have

$$|\phi_k(x-z)| \lesssim 2^{kn}\langle 2^k x\rangle^{-N},$$

and we need to control $I := \int |z|^{-2s} w(z)^{-1}\, dz$. If $k \geq 0$, we have $|z|^{-2s} w(z)^{-1} \simeq |z|^{-2s-\alpha}$ and so $I \lesssim 2^{-k(n-2s-\alpha)}$. Then

$$\int |K_k(x,y)|\, dy \lesssim \int 2^{-2ks} w(x) |\phi_k(x-z)|\, |z|^{-2s} w(z)^{-1}\, dz \lesssim 2^{k\alpha}\langle 2^k x\rangle^{-N} w(x) \lesssim 1.$$

For $k < 0$, we have $I \lesssim 1 + 2^{-k(n-2s-\beta)} \lesssim 2^{-k(n-2s-\beta)}$ and we get (2.26) similarly.

Case (ii): $|y| \gg |z|$ **and** $|z| \gg 2^{-k}$. First, if $|z| \gg \max(1, 2^{-k})$, we have $w(y) \simeq |y|^\beta$, and

$$\int |\phi_k(z-y)| w(y)\, dy \lesssim \int 2^{kn}\langle 2^k y\rangle^{-N} |y|^\beta\, dy \lesssim 2^{-k\beta} \lesssim |z|^\beta \simeq w(z).$$

Thus we get (2.27), which has been proven to be bounded.

On the other hand, if $2^{-k} \ll |z| \lesssim \max(1, 2^{-k})$, we have $k > 0$, $2^{-k} \ll |z| \lesssim 1$ and $w(z) \simeq |z|^\alpha$. Then

$$\int |\phi_k(z-y)| w(y)\, dy \lesssim \int_{|y|\geq 1} + \int_{|z|\leq |y|\leq 1} |\phi_k(z-y)| w(y)\, dy \lesssim 2^{k(n-N)} + 2^{-k\alpha} \lesssim |z|^\alpha,$$

which also gives (2.27).

Case (iii): $|y| \gg |z|$ **and** $|z| \lesssim 2^{-k}$. In this case, we have

$$|\phi_k(x-z)| \lesssim 2^{kn}\langle 2^k x\rangle^{-N}, \quad |\phi_k(z-y)| \lesssim 2^{kn}\langle 2^k y\rangle^{-N}.$$

Consider first the case $k \geq 0$; we then have $w(z) \simeq |z|^\alpha$ and

$$\begin{aligned}\int |K_k(x,y)|\, dy &\lesssim \int_{|z|\ll|y|} 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}\langle 2^k y\rangle^{-N} |z|^{-2s} w(z)^{-2} w(y)\, dz\, dy \\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N} \int \langle 2^k y\rangle^{-N} |y|^{n-2(s+\alpha)} w(y)\, dy \\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N} 2^{k(2s+\alpha-2n)} \lesssim w(2^k x)\langle 2^k x\rangle^{-N} \lesssim 1.\end{aligned}$$

For $k < 0$, we consider three subcases: $|z| \geq 1$, $|y| \leq 1$, and $|z| < 1 < |y|$.

When $|z| \geq 1$, we get

$$\begin{aligned}\int |K_k(x,y)|\, dy &\lesssim \int_{1\leq|z|\ll|y|} 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}\langle 2^k y\rangle^{-N} |z|^{-2(s+\beta)} |y|^\beta\, dz\, dy \\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N} \int \langle 2^k y\rangle^{-N} |y|^{n-2s-\beta}\, dy \\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N} 2^{k(2s+\beta-2n)} \lesssim w(2^k x)\langle 2^k x\rangle^{-N} \lesssim 1,\end{aligned}$$

where we have used the assumption that $2(s+\beta) < n$.

On the other hand, if $|y| \le 1$, we have $\langle 2^k y\rangle \simeq 1$ and so

$$\begin{aligned}\int |K_k(x,y)|\,dy &\lesssim \int_{|z|\ll|y|\le 1} 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}|z|^{-2(s+\alpha)}|y|^{\alpha}\,dz\,dy\\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}\int_{|y|\le 1}|y|^{n-2s-\alpha}\,dy\\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}\lesssim 1,\end{aligned}$$

where we have used the fact that $-2(s-n)\ge\beta\ge\alpha$.

Finally, when $|z|<1<|y|$, we see that

$$\begin{aligned}\int |K_k(x,y)|dy &\lesssim \int_{|z|<1<|y|} 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}\langle 2^k y\rangle^{-N}|z|^{-2(s+\alpha)}|y|^{\beta}\,dz\,dy\\ &\lesssim 2^{-2k(s-n)} w(x)\langle 2^k x\rangle^{-N}\int_{|y|\ge 1}\langle 2^k y\rangle^{-N}|y|^{\beta}\,dy\\ &\lesssim 2^{k(n-2s-\beta)} w(x)\langle 2^k x\rangle^{-N}\lesssim 1,\end{aligned}$$

where we have used the assumption that $n-2s-\beta\ge\beta\ge\alpha$ in the last inequality. This completes the proof. ■

Based on Theorems 2.3 and 2.4, we obtain a weighted fractional chain rule with higher regularity. For simplicity and future reference, we present the result with the inhomogeneous weight $r^{-\alpha}\langle r\rangle^{-(\beta-\alpha)}$ as in Lemma 2.5.

Proposition 2.8 (Weighted fractional chain rule, higher regularity). *Let $\theta\in\mathbb{R}_+$, $k=[\theta]\in[0,n/2)$ and $0\le\alpha\le\beta<n/2-k$. Then*

$$\|r^{-\alpha}\langle r\rangle^{-(\beta-\alpha)}D^{\theta}f(u)\|_{L^2_x}\lesssim_f C\Big(\max_{j\le k}\|r^j\nabla^j u\|_{L^\infty_x}\Big)\|r^{-\alpha}\langle r\rangle^{-(\beta-\alpha)}D^{\theta}u\|_{L^2_x}$$

for any $f\in C^\infty$.

Proof. Let $w:=r^{-\alpha}\langle r\rangle^{-(\beta-\alpha)}$. By Lemma 2.5, the assumptions on α,β ensure

$$w^2, r^{-2k}w^2\in A_2,\quad r^{-j}\in A_1,\quad \forall j\in[0,k].$$

The case $k=0$ follows directly from Theorem 2.3. In the following, we assume $k\ge 1$.

Letting $\theta=k+\tau$ with $\tau\in[0,1)$ and $k\ge 1$, we have

$$\begin{aligned}\|wD^{\theta}f(u)\|_{L^2_x}&\lesssim\|w\nabla^k D^{\tau}f(u)\|_{L^2_x}\\ &\lesssim\sum_{|\sum\beta_l|=k,\,|\beta_l|\ge 1}\Big\|wD^{\tau}\Big(f^{(j)}(u)\prod_{l=1}^{j}\nabla^{\beta_l}u\Big)\Big\|_{L^2_x}.\end{aligned}$$

For each term, we know from Theorems 2.3 and 2.4 that

$$
\begin{aligned}
&\Big\|wD^\tau\Big(f^{(j)}(u)\prod_{l=1}^{j}\nabla^{\beta_l}u\Big)\Big\|_{L^2_x}\\
&\quad\lesssim \|r^{-k}wD^\tau(f^{(j)}(u)-f^{(j)}(0))\|_{L^2_x}\Big\|r^k\prod_{l=1}^{j}\nabla^{\beta_l}u\Big\|_{L^\infty_x}\\
&\qquad+\sum C(f,\|u\|_{L^\infty_x})\|r^{-(k-|\beta_{l_0}|)}wD^\tau\nabla^{\beta_{l_0}}u\|_{L^2_x}\Big\|r^{k-|\beta_{l_0}|}\prod_{l\neq l_0}\nabla^{\beta_l}u\Big\|_{L^\infty_x}\\
&\quad\lesssim C(f,\|u\|_{L^\infty_x})\|r^{-k}wD^\tau u\|_{L^2_x}\prod_{l}\|r^{|\beta_l|}\nabla^{\beta_l}u\|_{L^\infty_x}\\
&\qquad+\sum_{l_0}C(f,\|u\|_{L^\infty_x})\|wD^{k+\tau}u\|_{L^2_x}\prod_{l\neq l_0}\|r^{|\beta_l|}\nabla^{\beta_l}u\|_{L^\infty_x}\\
&\quad\lesssim C\Big(f,\max_{j\le k}\|r^j\nabla^j u\|_{L^\infty_x}\Big)\|wD^\theta u\|_{L^2_x},
\end{aligned}
$$

where we have also used the weighted Hardy inequality (Lemma 2.7) in the last two inequalities. ■

Lemma 2.9 (Weighted trace estimate). *Let $n\ge 2$ and $0\le\alpha\le\beta\le(n-1)/2$. Then, for any $p\in[2,\infty)$,*

$$\|r^{-\alpha+(n-1)(1/2-1/p)}\langle r\rangle^{\alpha-\beta}\phi\|_{L^p_rL^2_\omega}\lesssim\|r^{-\alpha}\langle r\rangle^{\alpha-\beta}D^{1/2-1/p}\phi\|_{L^2}.$$

Proof. Let $w=r^{-\alpha}\langle r\rangle^{\alpha-\beta}$ with $w^2\in A_2$. As before, by interpolation, we need only prove the endpoint case:

$$\|r^{(n-1)/2}w\phi\|_{L^\infty_rL^2_\omega}\lesssim\|wP_j2^{j/2}\phi\|_{l^1_jL^2},$$

which follows from

$$\|r^{(n-1)/2}w\phi\|^2_{L^\infty_rL^2_\omega}\lesssim\|w\phi\|_{L^2}\|w\nabla\phi\|_{L^2}.$$

The proof is elementary, by observing that $r^{n-1}w^2$ is essentially increasing:

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}}w(R)^2R^{n-1}\phi(R\omega)^2\,d\omega&=\int_{\mathbb{S}^{n-1}}\int_R^\infty w(R)^2R^{n-1}\partial_r\phi(r\omega)^2\,dr\,d\omega\\
&\lesssim\int_{\mathbb{R}^n}w^2|\phi\phi'|\,dx\lesssim\|w\phi\|_{L^2}\|w\nabla\phi\|_{L^2},
\end{aligned}
$$

which completes the proof. ■

3. Morawetz type local energy estimates

In this section, we present a version of Morawetz type local energy estimates, involving fractional derivatives, for linear wave equations with small, variable C^1 coefficients. It is this version of local energy estimates that makes it possible to decrease the regularity requirement for quasilinear wave equations.

Similar estimates for linear wave equations with small C^2 coefficients are well-known (see Metcalfe–Tataru [35]). There the authors employ the paradifferential calculus and the positive commutator method to obtain a microlocal version of local energy estimates. However, it is well-known that for applications to quasilinear problems, the C^2 requirement is simply too strong for a problem with low regularity. In particular, in the current setting, we are working with the regularity level $s < 2$ (in the most physical related case of $n = 3$) and the most we can require is a local energy estimate with $C^{1,\alpha}$ ($\alpha \le 1/2$) metric, even in the spherically symmetric case.

Here, we present a certain weaker but still strong enough variant of Morawetz type local energy estimates, which applies to linear wave equations with small C^1 coefficients. The approach is remarkably simple, relying basically on the multiplier method with a well-chosen multiplier, and interpolation, without involving paradifferential calculus. The multiplier method has been well-developed in Metcalfe–Sogge [34] and Hidano–Wang–Yokoyama [14, Section 2] (with more general weights, which we will mainly follow) for small perturbations of $\Box$. As we shall see, the Morawetz type local energy estimates we shall use are also closely related to the KSS estimates, which appear first in Keel–Smith–Sogge [20].

Let $T \in (0, \infty]$, $S_T = [0, T) \times \mathbb{R}^n$, and let $h^{\alpha\beta} \in C^1(S_T)$ with $h^{\alpha\beta} = h^{\beta\alpha}$, $0 \le \alpha, \beta \le n$, satisfy the following uniform hyperbolicity condition:

$$\delta_0(\delta^{jk}) < (h^{jk}(t,x)) < \delta_0^{-1}(\delta^{jk}), \quad h^{00} = -1, \quad |h^{0j}| \le \delta_0^{-1}, \tag{3.1}$$

for some $\delta_0 \in (0, 1)$. Set $\tilde{h}^{\alpha\beta} = h^{\alpha\beta} - m^{\alpha\beta}$ where $m^{\alpha\beta}$ is the flat Minkowski metric component, $(m^{\alpha\beta}) = \mathrm{Diag}(-1, 1, 1, \dots, 1)$. Consider the linear wave equation with variable coefficients (with the summation convention for repeated upper and lower indices)

$$\Box_h u := (h^{\alpha\beta}(t,x)\partial_\alpha\partial_\beta)u = F(t,x) \quad \text{in } (0,T) \times \mathbb{R}^n, \tag{3.2}$$

with the initial data

$$u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1. \tag{3.3}$$

Theorem 3.1 (Morawetz type estimates). *Let $n \ge 3$ and $\mu \in (0,1)$ and consider the initial value problem (3.2)–(3.3) with $h^{0j} = 0$, $h^{jk} \in C^\infty(S_{T_0})$ satisfying (3.1) and*

$$\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}([0,T_0)\times\mathbb{R}^n)} \le \delta_0^{-1}. \tag{3.4}$$

Then there exist $\delta_1 \in (0, \min(\delta_0, T_0))$ and $C_0 \ge 1$ such that for any $T \in (0, \delta_1]$, we have

$$\|\tilde{\partial} u\|_{X_T} \le C_0(\|(\nabla u_0, u_1)\|_{L^2(\mathbb{R}^n)} + \|F\|_{X_T^*}), \tag{3.5}$$

where the X_T and X_T^* norms are defined in (1.9) and (1.10).

3.1. Morawetz type estimates for small perturbations of fixed background

We begin with standard energy estimates.

Lemma 3.2 (Energy estimates). *For any solutions $u \in C^\infty([0,T); C_0^\infty(\mathbb{R}^n))$ to the uniformly hyperbolic equation (3.2) in S_T, let $e^0 = (h^{jk}u_j u_k - h^{00}(u_t)^2)/2 \simeq |u_t|^2 + |\nabla u|^2$. Then there exists a uniform constant C depending only on δ and n such that, for $E(t) = \int_{\mathbb{R}^n} e^0 dx$,*

$$\left|\frac{d}{dt}E(t)\right| \le C \int_{\mathbb{R}^n} (|F|\,|u_t| + |\partial h|\,|e^0|)\,dx. \tag{3.6}$$

Proof. The result is classical, and follows from a multiplier argument:

$$\begin{aligned}
&u_t(h^{\alpha\beta}(t,x)\partial_\alpha\partial_\beta)u\\
&\quad= \partial_\alpha(h^{\alpha\beta}u_\beta u_t) - \partial_\alpha(h^{\alpha\beta})u_\beta u_t - (\partial_\alpha\partial_t u)h^{\alpha\beta}u_\beta\\
&\quad= \partial_\alpha(h^{\alpha\beta}u_\beta u_t) - \partial_\alpha(h^{\alpha\beta})u_\beta u_t - \partial_t\left(\frac{h^{\alpha\beta}u_\beta u_\alpha}{2}\right) + \frac{1}{2}(\partial_t h^{\alpha\beta})u_\beta u_\alpha\\
&\quad= \partial_\alpha E^\alpha + R,
\end{aligned}$$

where

$$E^0 = -\frac{h^{\alpha\beta}u_\alpha u_\beta}{2} + h^{0\beta}u_\beta u_t = \frac{h^{00}(u_t)^2 - h^{jk}u_j u_k}{2}, \quad E^j = h^{j\beta}u_\beta u_t,$$

and $R = -\partial_\alpha(h^{\alpha\beta})u_\beta u_t + (\partial_t h^{\alpha\beta})u_\alpha u_\beta/2$. We observe that $e^0 = -E^0 \simeq u_t^2 + |\nabla u|^2$ as long as we have (3.1), which gives us (3.6) in view of the divergence theorem. ∎

Lemma 3.3 (Morawetz type estimates for small perturbation of Minkowski). *Let $n \ge 3$, $\mu \in (0,1)$ and consider the initial value problem (3.2)–(3.3) for $h^{\alpha\beta} \in C^\infty(S_T)$ satisfying (3.1). Then there exist $\delta \in (0,\delta_0)$ and $C \ge 1$ such that for any $T > 0$ with*

$$\|r^{1-\mu}\partial h^{\alpha\beta}\|_{L^\infty_{t,x}([0,T]\times\mathbb{R}^n)} \le \delta T^{-\mu}, \quad \|\tilde{h}^{\alpha\beta}\|_{L^\infty(S_T)} \le \delta, \tag{3.7}$$

we have

$$\|\tilde{\partial} u\|_{X_T} \le C(\|(\nabla u_0, u_1)\|_{L^2(\mathbb{R}^n)} + \|F\|_{X_T^*}). \tag{3.8}$$

To prove this result, we need the following fundamental Morawetz type estimates, which follow from the elementary multiplier approach with carefully chosen multipliers. We leave the tedious proof to the appendix.

Theorem 3.4 (Morawetz type estimates, multiplier approach). *Let $n \ge 3$, $\mu \in (0,1)$ and consider the initial value problem (3.2)–(3.3) for $h^{\alpha\beta} \in C^\infty(S_T)$ satisfying (3.1). Then there exists $C \ge 1$, independent of $T \in (0,\infty)$, such that*

$$\begin{aligned}
&\|\tilde{\partial} u\|^2_{X_T} \le C\|(\nabla u_0, u_1)\|^2_{L^2(\mathbb{R}^n)}\\
&\qquad + C\int_0^T \int_{\mathbb{R}^n} |\tilde{\partial} u|\left(|F| + |\partial u|\left(|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}(r+T)^\mu}\right)\right) dx\,dt
\end{aligned} \tag{3.9}$$

for any solution $u \in C^\infty([0,T); C_0^\infty(\mathbb{R}^n))$ *to* (3.2)–(3.3) *with* $F \in C^\infty([0,T); C_0^\infty)$. *In addition, for* $T \in (0,\infty]$, *we have*

$$\|u\|_{LE_T}^2 \le C\|(\nabla u_0, u_1)\|_{L^2(\mathbb{R}^n)}^2 + C\int_0^T\int_{\mathbb{R}^n} |\tilde{\partial} u|\left(|F| + |\partial u|\left(|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}\langle r\rangle^\mu}\right)\right) dx\, dt. \tag{3.10}$$

With the use of (3.9) and the Cauchy–Schwarz inequality, Lemma 3.3 follows directly from the assumption

$$|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}(r+T)^\mu} \ll r^{\mu-1}T^{-\mu}.$$

3.2. *Proof of Theorem* 3.1

With the help of Lemmas 3.2 and 3.3, we are ready to present the proof of Theorem 3.1.

Let $\delta_1 > 0$, to be determined. First, without loss of generality, we may assume that the speed of propagation does not exceed δ_0^{-1}, and then for any $x_0 \in \mathbb{R}^n \setminus B_{\delta_1}$, the solution u in

$$\Lambda_{\delta_1}(x_0) = \{(t,x) : t \in [0,\delta_1],\ |x - x_0| < 2\delta_1 + \delta_0^{-1}(\delta_1 - t)\}$$

depends only on h, F in $\Lambda_{\delta_1}(x_0)$, and the data in $B_{(2+\delta_0^{-1})\delta_1}(x_0)$.

To apply Lemma 3.3, we need an estimate of the perturbation in $\Lambda_{\delta_1}(x_0)$. Let $x(s) = x_0 + s(x - x_0)$ with $s \in [0,1]$. We have either $\inf|x(s)| \ge |x_0|/2$ or $\inf|x(s)| \le |x_0|/2$. In the second case, there exists $s_0 \in [0,1]$ such that $|x(s_0)| = \inf|x(s)| \le |x_0|/2$. Then we have $x - x_0 \perp x(s_0)$, $|x - x_0| \ge |x - x(s_0)| \ge |x_0|/2$ and

$$|x(s)|^2 = |x(s_0)|^2 + (s-s_0)^2|x-x_0|^2 \ge (s-s_0)^2|x-x_0|^2 \ge (s-s_0)^2|x_0|^2/4.$$

Notice that in the first case, we also have $|x(s)| \ge |x_0|/2 \ge s|x_0|/2$, and we see that in either case,

$$|x(s)| \ge |s - s_0|\,|x_0|/2 \tag{3.11}$$

for some $s_0 \in [0,1]$.

In view of (3.4) and (3.11), the perturbation of h in $\Lambda_{\delta_1}(x_0)$ can be controlled as follows:

$$\begin{aligned}
|h(t,x) - h(0,x_0)| &\le \left|\int_0^1 \nabla h(t,x(s))\cdot(x-x_0)\, ds\right| + t\|\partial_t h(s,x_0)\|_{L^\infty(s\in[0,t])}\\
&\le \left(|x-x_0|\int_0^1 |x(s)|^{\mu-1}\, ds + t|x_0|^{\mu-1}\right)\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}([0,\delta_1]\times\mathbb{R}^n)}\\
&\lesssim (|t| + |x-x_0|)|x_0|^{\mu-1}\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}([0,\delta_1]\times\mathbb{R}^n)}\\
&\lesssim \delta_0^{-1}\delta_1^{\mu}\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}([0,\delta_1]\times\mathbb{R}^n)}.
\end{aligned}$$

Thus, $h^{\alpha\beta}$ can be viewed as a small perturbation of $h^{\alpha\beta}(0,x_0)$ in $\Lambda_{\delta_1}(x_0)$ when $\delta_1 \ll 1$. If $h^{\alpha\beta}(0,x_0) = m^{\alpha\beta}$, we can apply Lemma 3.3 in $\Lambda_{\delta_1}(x_0)$.

In general, as h^{jk} are uniformly elliptic, there exists a linear transform $M:\mathbb{R}^n \to \mathbb{R}^n$ such that in the new coordinates $h^{jk}(0,x_0)$ reduces to the Euclidean metric. Suppose that in the new coordinates, $y = Mx$, we have

$$H^{\alpha\beta}(t,y)\partial_\alpha\partial_\beta u = F$$

and $H^{00} = -1$, $H^{0j} = 0$, and $H^{jk}(t, Mx_0) = \delta^{jk}$. Notice that there exists a uniform $C > 0$ such that

$$\|r^{1-\mu}\partial H\|_{L^\infty_{t,y}([0,T]\times\mathbb{R}^n)} \le C\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}([0,T]\times\mathbb{R}^n)}.$$

Thus, when $\delta_1 \ll 1$, we have the following variant of (3.7) with $T \le \delta_1$:

$$\|r^{1-\mu}\partial H^{\alpha\beta}\|_{L^\infty_{t,y}([0,\delta_1]\times\mathbb{R}^n)} \le \delta\delta_1^{-\mu}, \quad \|H^{\alpha\beta} - m^{\alpha\beta}\|_{L^\infty(M\Lambda_{\delta_1}(x_0))} \le \delta, \tag{3.12}$$

from which we deduce (3.8) in $M\Lambda_{\delta_1}(x_0)$, with $T \le \delta_1$, from Lemma 3.3. Transforming back to the original variable we obtain, for some uniform $C > 0$,

$$\|\chi_{\Lambda_{\delta_1}(x_0)}\tilde{\partial} u\|_{X_T} \le C\big(\|(\nabla u_0, u_1)\|_{L^2(B_{(2+\delta_0^{-1})\delta_1}(x_0))} + \|\chi_{\Lambda_{\delta_1}(x_0)}F\|_{X_T^*}\big) \tag{3.13}$$

for any $|x_0| \ge \delta_1 \ge T$.

Finally, we choose $\{z_j\}_{j=1}^\infty \ni x_0$ so that $\bigcup_j \Lambda_{\delta_1}(z_j) = S_{\delta_1}$ and the $\Lambda_{\delta_1}(z_j)$ satisfy the finite overlapping property. Then we conclude (3.5) from (3.13).

3.3. *Local energy estimates with fractional regularity, $\theta \in [0,1]$*

Based on Theorem 3.1, we obtain the following local energy estimates with fractional regularity.

Proposition 3.5 (Local energy estimates with positive regularity). *Let $n \ge 3$ and $\mu \in (0,1)$, and let $h \in C^1$ with $h^{0j} = 0$ satisfy (3.1) and (3.4). Then there exist $\delta_2 \in (0,\delta_1]$ and a constant $C_1 > 4C_0$ such that for any $T \in (0,\delta_1]$ with*

$$T^\mu\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}([0,T]\times\mathbb{R}^n)} \le \delta_2, \tag{3.14}$$

and solutions u to (3.2) with data (u_0,u_1), we have

$$\|\tilde{\partial} D^\theta u\|_{X_T} \le C_1(\|(\nabla u_0, u_1)\|_{\dot H^\theta} + \|D^\theta F\|_{X_T^*}), \quad \forall\theta \in [0,1], \tag{3.15}$$

$$\|\partial D^{1/2} u\|_{X_{T,1}} \le C_1\big(\|(\nabla u_0, u_1)\|_{\dot B^{1/2}_{2,1}} + T^{\mu/2}\|r^{(1-\mu)/2}2^{j/2}P_jF\|_{\ell^1_j L^2_{t,x}}\big). \tag{3.16}$$

Proof. First, by approximation, we can assume $h \in C^\infty$, $u, F \in C_t^\infty C_0^\infty$ so that we can apply Theorem 3.1.

Let us begin by proving a higher order estimate of (3.5). Applying the spatial derivative ∂_j to equation (3.2), we get

$$(-\partial_t^2 + \Delta + \tilde{h}^{mk}\partial_m\partial_k)\partial_j u = \partial_j F(t,x) - (\partial_j h^{mk})\partial_m\partial_k u. \tag{3.17}$$

By (3.5), we see that

$$\|\tilde{\partial}\nabla u\|_{X_T} \lesssim \|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} + \|\nabla F\|_{X_T^*} + T^{\mu/2}\|r^{(1-\mu)/2}(\nabla h)\nabla^2 u\|_{L^2_{t,x}}$$
$$\lesssim \|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} + \|\nabla F\|_{X_T^*} + T^{\mu}\|r^{1-\mu}\nabla h\|_{L^\infty}\|\partial\nabla u\|_{X_T}.$$

In view of (3.14) for some $\delta_2 \ll 1$, we can absorb the last term and have

$$\|\tilde{\partial}\nabla u\|_{X_T} \lesssim \|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} + \|\nabla F\|_{X_T^*}. \tag{3.18}$$

Notice that all of the weights occurring in $\|\tilde{\partial}u\|_{X_T}$ and X_T^* are among the functions $w = r^{-(1-\mu)/2}, r^{-(3-\mu)/2}$ and their reciprocals, which share the property that $w^2 \in A_2$. Based on this fact, we know (2.6) holds for $p = 2$, and so complex interpolation holds for the weighted Sobolev space of fractional order (see e.g. [4, Theorem 6.4.3], [36, Lemma 4.6] for similar results):

$$[\dot{H}^0_w, \dot{H}^1_w]_\theta = \dot{H}^\theta_w, \quad \theta \in [0,1], \quad \|f\|_{\dot{H}^\theta_w} := \|wD^\theta f\|_{L^2}. \tag{3.19}$$

With the help of (2.6), we see that (3.18) gives us (3.15) with $\theta = 1$. As (3.5) is just (3.15) with $\theta = 0$, the general estimate (3.15) with $\theta \in [0,1]$ follows in view of (3.19). Finally, with the help of (3.15), (2.6), and real interpolation with $\theta = 1/2$, we obtain the estimate (3.16). ∎

Using basically the same argument, from Theorem 3.4 we also get the following local energy estimates with fractional regularity, for a small perturbation of Minkowski.

Proposition 3.6. *Let $n \ge 3$ and $\mu \in (0,1)$ and let $h \in C^1$ satisfy (3.1). There exists a constant $C > 1$ such that if*

$$\|(r^{1-\mu}\partial h, \tilde{h})\|_{L^\infty_{t,x}([0,T]\times B_1)} + (\ln\langle T\rangle)\|(r\partial h, \tilde{h})\|_{L^\infty_{t,x}([0,T]\times B_1^c)} \le 1/C, \tag{3.20}$$

then for any weak solutions u to (3.2) with data (u_0, u_1), we have

$$\|D^\theta u\|_{LE_T} \le C\left(\|(\nabla u_0, u_1)\|_{\dot{H}^\theta} + (\ln\langle T\rangle)^{1/2}\|r^{(1-\mu)/2}\langle r\rangle^{\mu/2}D^\theta F\|_{L^2_{t,x}}\right) \tag{3.21}$$

for any $\theta \in [0,1]$. Similarly, if instead of (3.20), we assume

$$\|\langle r\rangle^{\mu_1}(r^{1-\mu}\langle r\rangle^{\mu}\partial h, \tilde{h})\|_{L^\infty_{t,x}([0,\infty)\times\mathbb{R}^n)} \le 1/C, \tag{3.22}$$

then we have

$$\|D^\theta \phi\|_{LE} \le C\left(\|(\nabla u_0, u_1)\|_{\dot{H}^\theta} + \|r^{(1-\mu)/2}\langle r\rangle^{(\mu+\mu_1)/2}D^\theta F\|_{L^2_{t,x}}\right), \quad \forall\theta \in [0,1]. \tag{3.23}$$

3.4. Local energy estimates with negative regularity

It is well-known that quasilinear problems suffer loss of regularity, which naturally occurs when we try to prove the convergence of the Picard iteration series. More precisely, we will need to control some term like $(g(u) - g(v))\Delta v$, for which one standard way to bypass is to prove the convergence in a weaker topology. One typical choice will be the standard energy norm, for which we are led to the requirement $s \ge 2$ for the regularity. In this sense, to break the regularity barrier 2 (for dimension 3), it is very natural to consider energy type estimates with negative regularity. To obtain such estimates, as we have limited regularity for h, it is natural to work with equations in divergence form.

Proposition 3.7 (Local energy estimates with negative regularity). *Let $n \ge 3$ and $\mu \in (0, 1/2]$, and let $h^{\alpha\beta} \in C^1$ with $h^{0j} = 0$ satisfy (3.1) and (3.4). Then there exist $\delta_3 \in (0, \delta_2]$ and a constant $C > 0$ such that for any $T \in (0, \delta_1]$ with*

$$T^{\mu} \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}([0,T]\times\mathbb{R}^n)} \le \delta_3, \tag{3.24}$$

we have

$$\| \partial D^{-\theta} u \|_{X_T} \le C \| D^{-\theta} F \|_{X_T^*}, \quad \forall \theta \in [0, 1], \tag{3.25}$$

for any weak solutions u to

$$(\partial_\alpha h^{\alpha\beta} \partial_\beta) u = F \tag{3.26}$$

with vanishing data. Here the X_T and X_T^* norms are defined in (1.9)–(1.10). In addition, if $\theta \in [(4-n)/2 - \mu, 1] \cap [0, 1]$, and h^{jk} are spherically symmetric, then

$$\| \partial D^{-\theta} u \|_{X_T} \lesssim \| \partial u(0) \|_{\dot H^{-\theta}} + T^{\mu} \| h \|_{L^\infty_t \dot H^{(n-2)/2+\mu}} \| \partial u(0) \|_{\dot H^{1-\theta}} + \| D^{-\theta} F \|_{X_T^*} \tag{3.27}$$

for any spherically symmetric weak solutions u to (3.26).

Remark 3.8. As is clear from the local energy estimate (3.5), when $\theta = 0$, the second term on the right of (3.27) is not necessary. We do not know, however, if it is necessary to have such a term for general θ.

Proof of Proposition 3.7. First, we observe that the local energy estimate (3.5) applies also to the wave operator in divergence form

$$\partial_\alpha h^{\alpha\beta}(t, x) \partial_\beta,$$

as the difference of these two operators is just a term like $(\partial_\alpha h^{\alpha\beta})\partial_\beta$, which could be absorbed into the left hand side by (3.24) with small δ_3, and gives us (3.25) with $\theta = 0$, which in particular yields

$$\| D u \|_{X_T} \lesssim \| F \|_{X_T^*}.$$

By duality, we obtain

$$\| \nabla D^{-1} u \|_{X_T} \lesssim \| u \|_{X_T} \lesssim \| D^{-1} F \|_{X_T^*}. \tag{3.28}$$

By interpolation, to prove (3.25), it remains to estimate $\partial_t u$ with $\theta = 1$, for which we shall also argue by duality. Observe that the difference between $(\Box + \partial_j \tilde h^{jk} \partial_k) u$ and $(\Box + \partial_j \partial_k \tilde h^{jk}) u$ is given by $\partial_j((\partial_k \tilde h^{jk}) u) = \partial_j((\partial_k h^{jk}) u)$, which is an admissible error term thanks to (3.4) and (3.28), as we have

$$\| D^{-1} \partial_j ((\partial_k h^{jk}) u) \|_{X_T^*} \lesssim \| (\partial_k h^{jk}) u) \|_{X_T^*} \lesssim T^{\mu} \| r^{1-\mu} \partial h \|_{L^\infty_{t,x}} \| u \|_{X_T} \lesssim \| D^{-1} F \|_{X_T^*}.$$

The proof is thus reduced to obtaining an estimate for $(\Box + \partial_j\partial_k\tilde{h}^{jk})u = F$. Recall that, for any $G \in C_t^\infty C_0^\infty$ with $\|DG\|_{X_T^*} \le 1$, we have

$$\|\partial_j\partial_k w\|_{X_T} + \|D\partial_t w\|_{X_T} \lesssim \|D\partial w\|_{X_T} \lesssim \|DG\|_{X_T^*} \le 1$$

for any solutions to $(\Box + \tilde{h}^{jk}\partial_j\partial_k)w = G$ with vanishing data at time $t = T$, which follows directly from the estimate (3.18). Now, for the sake of duality, we observe that

$$\frac{d}{dt}\int_{\mathbb{R}^n}(w_t u_t + \nabla w\cdot\nabla u - u\tilde{h}^{jk}\partial_j\partial_k w)\,dx + \int_{\mathbb{R}^n}\left(w_t F + u_t G - u(\partial_t h^{jk})\partial_j\partial_k w\right)dx = 0,$$

and so

$$\int_{S_T} G\partial_t u\,dt\,dx = \int_{S_T}((\partial_t h^{jk})u\partial_j\partial_k w - F\partial_t w)\,dt\,dx.$$

Then, thanks to (3.24) and (3.25) with $\nabla D^{-\theta}u$, we obtain

$$\begin{aligned}\left|\int_{S_T} G\partial_t u\,dt\,dx\right| &\le \|D\partial_t w\|_{X_T}\|D^{-1}F\|_{X_T^*} + T^\mu\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}}\|u\|_{X_T}\|\partial_j\partial_k w\|_{X_T}\\ &\lesssim \|D^{-1}F\|_{X_T^*} + T^\mu\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}}\|\nabla D^{-1}u\|_{X_T}\\ &\lesssim \|D^{-1}F\|_{X_T^*},\end{aligned}$$

which, by duality, gives the desired estimate

$$\|D^{-1}\partial_t u\|_{X_T} \lesssim \|D^{-1}F\|_{X_T^*},$$

and completes the proof of (3.25).

Finally, we prove the homogeneous estimates, for $(\partial_\alpha h^{\alpha\beta}\partial_\beta)u = 0$. For this purpose, we introduce the homogeneous solution for the standard d'Alembertian $\Box w = 0$, $w(0) = u(0)$, $w_t(0) = u_t(0)$. Then it follows from (3.5) and $[\Box, D^\theta] = 0$ that

$$\|D^\theta\partial w\|_{X_T} \le C_0\|\partial u(0)\|_{\dot{H}^\theta}$$

for any $\theta \in \mathbb{R}$. Next, we want to estimate the difference $v = u - w$, for which we observe that $v(0) = \partial_t v(0) = 0$ and

$$(\partial_\alpha h^{\alpha\beta}\partial_\beta)v = -\partial_j(\tilde{h}^{jk}\partial_k w).$$

Applying (3.25) to v, we obtain

$$\begin{aligned}\|D^{-\theta}\partial v\|_{X_T} &\lesssim T^{\mu/2}\|r^{(1-\mu)/2}D^{-\theta}\partial_j(\tilde{h}^{jk}\partial_k w)\|_{L^2_{t,x}}\\ &\lesssim T^{\mu/2}\|r^{(1-\mu)/2}D^{1-\theta}(\tilde{h}\nabla w)\|_{L^2_{t,x}}\\ &\lesssim T^\mu\|\tilde{h}\|_{L^\infty_t\dot{H}^{(n-2)/2+\mu}}\|D^{1-\theta}\nabla w\|_{X_T}\\ &\lesssim T^\mu\|h\|_{L^\infty_t\dot{H}^{(n-2)/2+\mu}}\|\partial u(0)\|_{\dot{H}^{1-\theta}},\end{aligned}$$

provided that $\theta \in [(4-n)/2-\mu, 1] \cap [0, 1]$ so that $|1-\theta| \le (n-2)/2+\mu$ and $\theta \in [0, 1]$, where we have used Lemma 2.6 in the third inequality, since h and u are assumed to be spherically symmetric. Combining all these estimates, we obtain (3.27), and this completes the proof. ■

3.5. Local energy estimates with high regularity: radial case

Considering spherically symmetric equations and solutions, we have the following version of local energy estimates with high regularity.

Proposition 3.9 (Local energy estimates with high regularity). *Let $n \ge 3$, $\mu \in (0,1)$ and $h(t,x) = h(t,|x|) \in (\delta_0 - 1, \delta_0^{-1} - 1)$, and consider radial solutions ϕ for*

$$(-\partial_t^2 + \Delta + h\Delta)\phi = F, \quad \phi(0) = u_0, \, \phi(0) = u_1. \tag{3.29}$$

Then there exists $\delta > 0$ such that

$$\|\tilde{\partial} D^\theta \phi\|_{X_T} \lesssim \|D^\theta(\nabla u_0, u_1)\|_{L^2} + \|D^\theta F\|_{X_T^*}, \quad \theta \in \big[0, [n/2]\big], \tag{3.30}$$

$$\|\partial D^{n/2-1}\phi\|_{X_{T,1}} \lesssim \|(\nabla u_0, u_1)\|_{\dot{B}^{n/2-1}_{2,1}} + T^{\mu/2}\|r^{(1-\mu)/2}2^{j(n/2-1)}P_j F\|_{\ell^1_j L^2_{t,x}}, \tag{3.31}$$

for any classical solutions ϕ to (3.29) provided that

$$T^\mu \|\partial h\|_{L^\infty_t \dot{H}^{n/2-1+\mu}} \le \delta. \tag{3.32}$$

In addition, when (3.32) is satisfied, for $k = 1 + [n/2]$ we have

$$\|\tilde{\partial}\nabla^k \phi\|_{X_T} \lesssim \|\nabla^k(\nabla u_0, u_1)\|_{L^2} + \|\nabla^k F\|_{X_T^*} + T^\mu \|\nabla^k \phi\|_{X_T}\|h\|_{L^\infty_t \dot{H}^{(n+2)/2+\mu}} \tag{3.33}$$

if n is odd or $\mu > 1/2$, and

$$\|\tilde{\partial}\nabla^k \phi\|_{X_T} \lesssim \|\nabla^k(\nabla u_0, u_1)\|_{L^2} + \|\nabla^k F\|_{X_T^*} + T^\mu \|h\|_{L^\infty_t \dot{H}^{k+1}}\|D^{n/2+\mu}\phi\|_{X_T} \tag{3.34}$$

if $\mu > 1/2$. Similarly, when $n \ge 4$, $\mu \in (0, 1/2)$ and $\mu_1 \in (0, \mu]$, there exists $\delta' > 0$ such that

$$\|D^\theta \phi\|_{LE_T} \lesssim \|D^\theta(\nabla u_0, u_1)\|_{L^2} + \|r^{(1-\mu)/2}\langle r\rangle^{(\mu+\mu_1)/2} D^\theta F\|_{L^2_{t,x}}, \quad \theta \in \big[0, [(n-1)/2]\big], \tag{3.35}$$

for any classical solutions to (3.29) provided that

$$\|h\|_{L^\infty_{t,x}} + \|\partial h\|_{L^\infty_t \dot{H}^{n/2-1+\mu}} + \|\partial h\|_{L^\infty_t \dot{H}^{n/2-1-\mu_1}} \le \delta'. \tag{3.36}$$

Proof. As in Proposition 3.5, the proof of (3.30) and (3.31) can be reduced to the proof of (3.30) with D^θ replaced by ∇^k with $k \in \mathbb{N}$. When $\theta = 0, 1$, this has been proven in (3.15) of Proposition 3.5, by recalling the trace estimates

$$\|r^{1-\mu}\partial h\|_{L^\infty_{t,x}} \lesssim \|\partial h\|_{L^\infty_t \dot{H}^{n/2-1+\mu}}.$$

The general case follows from a similar strategy. By applying ∇^α with $|\alpha| = k \ge 2$, we have

$$(-\partial_t^2 + \Delta + h\Delta)\nabla^\alpha \phi = \nabla^\alpha F + [h, \nabla^\alpha]\Delta\phi = \nabla^\alpha F + \sum_{j=1}^{k} \mathcal{O}(|\nabla^j h||\nabla^{k-j}\Delta\phi|).$$

First, if $1 \le j \le k-1$, we have $n/2+\mu-j \in (1/2, n/2)$, and

$$\begin{aligned}\|(\nabla^j h)\nabla^{k-j}\Delta\phi\|_{X_T^*} &\lesssim T^{\mu/2}\|r^{j-\mu}\nabla^j h\|_{L^\infty}\|r^{\mu/2-j+1/2}\nabla^{k-j}\Delta\phi\|_{L^2}\\ &\lesssim T^{\mu/2}\|h\|_{L_t^\infty \dot H^{n/2+\mu}}\|r^{-(1-\mu)/2}\nabla^{k-1}\Delta\phi\|_{L^2},\end{aligned}$$

where we have used (2.9) and the trace estimate. For the term with $j=k$, we proceed similarly if $n/2+\mu-k \in (1/2, n/2)$, that is, $k<(n-1)/2+\mu$. Thus all these commutator terms can be absorbed into the left side, in view of (3.32), which proves (3.30) with $0 \le k < (n-1)/2+\mu$.

The remaining case, $(n-1)/2+\mu \le k \le [n/2]$, only happens when n is even and $k=n/2$ with $\mu \in (0,1/2]$. Then we have $k-1 \ge 1$, and

$$\begin{aligned}\|(\nabla^k h)\Delta\phi\|_{X_T^*} &\lesssim T^{\mu/2}\|r^{-\mu}\nabla^k h\|_{L_t^\infty L^2}\|r^{(1+\mu)/2}\Delta\phi\|_{L_t^2 L^\infty}\\ &\lesssim T^{\mu/2}\|h\|_{L_t^\infty \dot H^{k+\mu}}\|r^{-(1-\mu)/2}\nabla^{k-1}\Delta\phi\|_{L^2},\end{aligned}$$

where we have used (2.4) and Hardy's inequality. This gives (3.30) with $k=n/2$.

We now turn to the proof of (3.33) and (3.34), in which case we have $(n-1)/2+\mu \le k < (n+1)/2+\mu$. Notice that we still have $n/2+\mu-j \in (1/2, n/2)$ for $1 \le j \le k-1$ and, as before, these commutator terms are good terms. For the case $j=k$, we have $(n+2)/2+\mu-k > 1/2$ and so

$$\begin{aligned}\|(\nabla^k h)\Delta\phi\|_{X_T^*} &\lesssim T^{\mu/2}\|r^{1-\mu+k-2}\nabla^k h\|_{L_{t,x}^\infty}\|r^{-(1-\mu)/2+2-k}\Delta\phi\|_{L_{t,x}^2}\\ &\lesssim T^{\mu/2}\|h\|_{L_t^\infty \dot H^{(n+2)/2+\mu}}\|r^{-(1-\mu)/2}\nabla^k\phi\|_{L^2}\\ &\lesssim T^{\mu}\|h\|_{L_t^\infty \dot H^{(n+2)/2+\mu}}\|\nabla^k\phi\|_{X_T},\end{aligned}$$

which gives us (3.33). Similarly, for (3.34), the term with $j=k$ can be controlled as follows:

$$\begin{aligned}\|(\nabla^k h)\Delta\phi\|_{X_T^*} &\lesssim T^{\mu/2}\|r^{n/2-1}\nabla^k h\|_{L_{t,x}^\infty}\|r^{(1-\mu)/2+1-n/2}\Delta\phi\|_{L_{t,x}^2}\\ &\lesssim T^{\mu/2}\|h\|_{L_t^\infty \dot H^{k+1}}\|r^{-(1-\mu)/2}D^{n/2+\mu}\phi\|_{L^2}\\ &\lesssim T^{\mu}\|h\|_{L_t^\infty \dot H^{k+1}}\|D^{n/2+\mu}\phi\|_{X_T},\end{aligned}$$

where we have used $n/2+\mu \ge 2$ and (2.9), which completes the proof of (3.34).

Finally, we treat (3.35), for which we follow a similar strategy, by reducing it to ∇^k with $k=[(n-1)/2]$. First, for $1 \le j \le [n/2]-1$, we notice that $n/2-\mu_1-j$ and $n/2+\mu-j$ are in $(1/2, n/2)$, and so

$$\begin{aligned}&\|r^{(1-\mu)/2}\langle r\rangle^{(\mu+\mu_1)/2}(\nabla^j h)\nabla^{k-j}\Delta\phi\|_{L_{t,x}^2}\\ &\quad\lesssim \|r^{j-\mu}\langle r\rangle^{\mu+\mu_1}\nabla^j h\|_{L^\infty}\|r^{-(1-\mu)/2-(j-1)}\langle r\rangle^{-(\mu+\mu_1)/2}\nabla^{k-j}\Delta\phi\|_{L^2}\\ &\quad\lesssim \|h\|_{L_t^\infty \dot H^{n/2+\mu}\cap L_t^\infty \dot H^{n/2-\mu_1}}\|r^{-(1-\mu)/2}\langle r\rangle^{-(\mu+\mu_1)/2}\nabla^{k-1}\Delta\phi\|_{L^2},\end{aligned}$$

where we have used Lemma 2.7. For the remaining terms with $j > [n/2] - 1$, we see that n is odd, $j = k = (n-1)/2$ and so $n/2 + \mu - k = 1/2 + \mu$, $n/2 - \mu_1 - k = 1/2 - \mu_1$. Notice that (2.2) gives

$$\|r^{(n-1)(1/2-\mu_1)}h\|_{L^{1/\mu_1}} \lesssim \|h\|_{\dot H^{1/2-\mu_1}}, \quad \|r^{(n-1)(1/2-\mu_1)-\mu-\mu_1}h\|_{L^{1/\mu_1}} \lesssim \|h\|_{\dot H^{1/2+\mu}},$$

that is,

$$\|r^{(n-1)(1/2-\mu_1)-\mu-\mu_1}\langle r\rangle^{\mu+\mu_1}h\|_{L^{1/\mu_1}} \lesssim \|h\|_{\dot H^{1/2+\mu}} + \|h\|_{\dot H^{1/2-\mu_1}}. \tag{3.37}$$

With the help of (3.37) we obtain, for $1/q = 1/2 - \mu_1$,

$$\begin{aligned}
&\|r^{(1-\mu)/2}\langle r\rangle^{(\mu+\mu_1)/2}(\nabla^k h)\Delta\phi\|_{L^2_{t,x}} \\
&\quad\lesssim \|r^{(n-1)(1/2-\mu_1)-\mu-\mu_1}\langle r\rangle^{\mu+\mu_1}\nabla^k h\|_{L^\infty_t L^{(1)/\mu_1}} \\
&\qquad\times \|r^{-(1-\mu)/2-(n-3)/2+n\mu_1}\langle r\rangle^{-(\mu+\mu_1)/2}\Delta\phi\|_{L^q} \\
&\quad\lesssim \|h\|_{L^\infty_t \dot H^{n/2+\mu}\cap L^\infty_t \dot H^{n/2-\mu_1}}\|r^{-(1-\mu)/2-(n-3)/2+\mu_1}\langle r\rangle^{-(\mu+\mu_1)/2}D^{\mu_1}\Delta\phi\|_{L^2} \\
&\quad\lesssim \|h\|_{L^\infty_t \dot H^{n/2+\mu}\cap L^\infty_t \dot H^{n/2-\mu_1}}\|r^{-(1-\mu)/2}\langle r\rangle^{-(\mu+\mu_1)/2}D^{(n-3)/2}\Delta\phi\|_{L^2},
\end{aligned}$$

where we have used Lemma 2.9, Lemma 2.7 and the assumption $n \ge 4$, so that we have $(n-3)/2 \ge \mu_1$. This gives (3.35). ■

4. Local existence and uniqueness for dimension 3

With the help of Propositions 3.5 and 3.7, we are able to prove the local existence and uniqueness part of Theorem 1.1.

4.1. Approximate solutions

First, we fix a spherically symmetric function $\rho \in C_0^\infty(\mathbb{R}^n)$ which equals 1 near the origin and $\int_{\mathbb{R}^3}\rho(x)\,dx = 1$, and set $\rho_k(x) = 2^{3k}\rho(2^k x)$. Using ρ, we define a standard sequence of C^∞ spherically symmetric functions approximating (u_0, u_1),

$$u_0^{(k)}(x) = \rho_k * u_0(x), \quad u_1^{(k)}(x) = \rho_k * u_1(x), k \ge 3. \tag{4.1}$$

It is clear that

$$\begin{aligned}
\|(\nabla u_0^{(k)}, u_1^{(k)})\|_{\dot H^\theta_{\text{rad}}} &\le \|(\nabla u_0, u_1)\|_{\dot H^\theta_{\text{rad}}}, \quad \forall\theta\in\mathbb{R}, \\
\|(\nabla u_0^{(k)}, u_1^{(k)})\|_{\dot B^\theta_{2,1}} &\le \|(\nabla u_0, u_1)\|_{\dot B^\theta_{2,1}}, \quad \forall\theta\in\mathbb{R}.
\end{aligned}$$

Since $(u_0, u_1) \in H^s_{\text{rad}} \times (H^{s-1}_{\text{rad}} \cap \dot H^{s_0-1}_{\text{rad}})$ with $s \in (3/2, 2]$ and $s_0 \in [2-s, s-1]$, we have

$$\lim_{k\to\infty}(\|u_0^{(k)} - u_0\|_{H^s_{\text{rad}}} + \|u_1^{(k)} - u_1\|_{H^{s-1}_{\text{rad}}\cap\dot H^{s_0-1}_{\text{rad}}}) = 0.$$

In addition, for any $\theta \in [s_0 - 3, s - 1)$,

$$\sum_{k=3}^{\infty} (\|\nabla u_0^{(k)} - \nabla u_0^{(k+1)}\|_{\dot H^\theta(\mathbb{R}^3)} + \|u_1^{(k)} - u_1^{(k+1)}\|_{\dot H^\theta(\mathbb{R}^3)}) < \infty. \tag{4.2}$$

Indeed, we can easily check this by using the fact that $\|\rho_k * \varphi - \varphi\|_{L^2} \le C 2^{-\theta k}\|\varphi\|_{\dot H^\theta}$ for any $\theta \in [0, 2]$. Moreover, there exists a subsequence $\{j_k\}$ such that

$$\|u_0^{(j_k)} - u_0^{(j_k+1)}\|_{\dot H^s(\mathbb{R}^3)} + \|u_1^{(j_k)} - u_1^{(j_k+1)}\|_{\dot H^{s-1}(\mathbb{R}^3)} \le 2^{-k}, \tag{4.3}$$

and we also have (4.2) for $(u_0^{(j_k)}, u_1^{(j_k)})$ with $\theta = s - 1$. Furthermore, we can cut off the data so that they are compactly supported smooth functions, while all of the above properties remain valid (with possibly larger constants). We still denote the sequence (after cut-off) as $(u_0^{(j_k)}, u_1^{(j_k)})$.

With $(u_0^{(j_k)}, u_1^{(j_k)})$ as data, we use a standard iteration to define a sequence of approximate solutions. Let $F(u) = a(u)u_t^2 + b(u)|\nabla u|^2$, $u_2 \equiv 0$ and define u_k $(k \ge 3)$ recursively by solving

$$\begin{cases} \Box u_k + g(u_{k-1})\Delta u_k = F(u_{k-1}), & (t, x) \in (0, T) \times \mathbb{R}^3, \\ u_k(0, \cdot) = u_0^{(j_k)}, \quad \partial_t u_k(0, \cdot) = u_1^{(j_k)}. \end{cases} \tag{4.4}$$

By Proposition 3.5, together with a standard existence, uniqueness and regularity theorem, we will see that there exists some uniform $T(u_0, u_1) \in (0, \infty)$ such that for all $k \ge 2$, u_k is well-defined and C^∞ on S_T, spherically symmetric, and satisfies

$$\|\partial u_k\|_{L^\infty \dot H^\theta} \le \|\partial D^\theta u_k\|_{X_T} \le 2C_1\|(\nabla u_0^{(j_k)}, u_1^{(j_k)})\|_{\dot H^\theta}, \quad \forall \theta \in [0, 1], \tag{4.5}$$

$$\|\partial u_k\|_{L_t^\infty \dot B^{1/2}_{2,1}} \le \|\tilde\partial D^{1/2} u_k\|_{X_{T,1}} \le 2C_1\|(\nabla u_0, u_1)\|_{\dot B^{1/2}_{2,1}} = 2C_1\varepsilon_c. \tag{4.6}$$

4.2. *Uniform boundedness of u_k*

In this subsection, we prove the uniform boundedness of the sequence, that is, (4.5) and (4.6).

Lemma 4.1. *Let $s \in (3/2, 2]$, $\varepsilon_s := \|(\nabla_x u_0, u_1)\|_{\dot H^{s-1}}$ and set $s = 3/2 + \mu$. Then there exists $c = c(g, a, b, \varepsilon_c)$ such that the spherically symmetric functions $u_k \in C^\infty \cap CH^\theta \cap C^1 H^{\theta-1}$ are well-defined on S_T for any $k \ge 2$, $\theta \ge 3$ and enjoy the uniform bounds (4.5) and (4.6) for any $T \in (0, T_0]$ with $T_0 = \min(\delta_1, c(g, a, b, \varepsilon_c)\varepsilon_s^{-1/\mu})$.*

Proof. The proof proceeds by induction. First, the result is trivial for $k = 2$. Then we make the inductive assumption that for some $m \ge 3$ we have for any $2 \le k \le m - 1$, $u_k \in C^\infty \cap CH^\theta \cap C^1 H^{\theta-1}$ for any $\theta \ge 3$ with the bounds (4.5)–(4.6) satisfied.

Recalling the Sobolev inequality

$$\|\phi\|_{L^\infty} \le C\|\phi\|_{\dot B^{3/2}_{2,1}},$$

in view of (4.6) for u_{m-1} we see that

$$\|u_{m-1}\|_{L^\infty_{t,x}} \le C\|u_{m-1}\|_{L^\infty_t \dot B^{3/2}_{2,1}} \le 2CC_1\|(\nabla u_0, u_1)\|_{\dot B^{1/2}_{2,1}} = 2CC_1\varepsilon_c. \tag{4.7}$$

As $u_{m-1} \in C^\infty \cap CH^\theta \cap C^1 H^{\theta-1}$ for any $\theta \ge 3$ with the bounds (4.5) and (4.6), we see that $F(u_{m-1}) \in L^1([0,T]; H^{\theta-1})$ and $g(u_{m-1}) \in C^\infty$. Based on this information, we see from the classical local existence theorem that (4.4) is solvable with solution u_m well-defined, smooth in $[0,T]\times\mathbb{R}^n$ and with $u_m \in CH^\theta \cap C^1 H^{\theta-1}$ for any $\theta \ge 3$.

To apply Proposition 3.5 for u_m, we need to check (3.14) for $h^{jj} = g(u_{m-1})$ and $h^{\alpha\beta} = 0$ with $\alpha \ne \beta$. As $\mu \in (0, 1/2]$ and u_{m-1} is spherically symmetric, by (2.1) we have

$$\|r^{1-\mu}\partial g(u_{m-1})\|_{L^\infty} \le \|r^{1-\mu} g'(u_{m-1})\partial u_{m-1}\|_{L^\infty} \lesssim \|\partial u_{m-1}\|_{L^\infty \dot H^{1/2+\mu}_{\mathrm{rad}}} \lesssim \varepsilon_s, \tag{4.8}$$

where we have used (4.5) and (4.7) for u_{m-1}. Here we notice that the implicit constant may depend on g and ε_c through $\|g'(u_{m-1})\|_{L^\infty}$. Thus, with

$$T_0 = c(g,\varepsilon_c)\varepsilon_s^{-1/\mu} \tag{4.9}$$

for some small constant c, which may depend on ε_c and g, we have (3.14) for $g(u_{m-1})$ and can apply Proposition 3.5 for u_m. In conclusion, for $\theta \in [0,1]$ and $T \in (0, \delta_1]$ we get

$$\|\partial D^\theta u_m\|_{X_T} \le C_1\big(\|(\nabla u_0^{(jm)}, u_1^{(jm)})\|_{\dot H^\theta} + T^{\mu/2}\|r^{(1-\mu)/2} D^\theta F(u_{m-1})\|_{L^2_{t,x}}\big), \tag{4.10}$$

$$\|\partial D^{1/2} u_m\|_{X_{T,1}} \le C_1(\|(\nabla u_0, u_1)\|_{\dot B^{1/2}_{2,1}} + T^{\mu/2}\|r^{(1-\mu)/2} 2^{j/2} P_j F(u_{m-1})\|_{\ell^1_j L^2_{t,x}}). \tag{4.11}$$

To control the right hand side, we will exploit Lemma 2.6, the weighted fractional chain rule (Theorem 2.3), as well as the weighted fractional Leibniz rule (Theorem 2.4). Without loss of generality, we assume $b = 0$ and write

$$F(u) = a(u)u_t^2 = \tilde a(u)u_t^2 + a(0)u_t^2 =: F_1(u) + F_2(u). \tag{4.12}$$

We first give an estimate for $F_2(u)$ for any $\theta \in [0,1]$:

$$\begin{aligned}\|r^{(1-\mu)/2}2^{j\theta}P_j F_2(u)\|_{\ell^q_j L^2_{t,x}} &\lesssim \|r^{-(1-\mu)/2}2^{j\theta}P_j u_t\|_{\ell^q_j L^2_{t,x}}\|r^{1-\mu}u_t\|_{L^\infty_{t,x}}\\ &\lesssim \|r^{-(1-\mu)/2}2^{j\theta}P_j u_t\|_{\ell^q_j L^2_{t,x}}\|u_t\|_{L^\infty_t \dot H^{1/2+\mu}}\end{aligned}$$

by Lemma 2.5, (2.20) and (2.1). For $F_1(u)$, we use a similar argument, applying (2.20), (2.15), and (2.1) to obtain, for $\theta \in [0,1]$,

$$\begin{aligned}&\|r^{(1-\mu)/2}2^{j\theta}P_j F_1(u)\|_{\ell^q_j L^2_{t,x}}\\ &\lesssim \|\tilde a(u)\|_{L^\infty}\|r^{(1-\mu)/2}2^{j\theta}P_j(u_t^2)\|_{\ell^q_j L^2_{t,x}} + \|r^{-(3-\mu)/2}2^{j\theta}P_j\tilde a(u)\|_{\ell^q_j L^2_{t,x}}\|r^{2-\mu}u_t^2\|_{L^\infty}\\ &\lesssim C(\|\partial u\|_{L^\infty_t \dot B^{1/2}_{2,1}})\|u_t\|_{L^\infty_t \dot H^{1/2+\mu}}\\ &\quad\times(\|r^{-(1-\mu)/2}2^{j\theta}P_j u_t\|_{\ell^q_j L^2_{t,x}} + \|r^{-(3-\mu)/2}2^{j\theta}P_j u\|_{\ell^q_j L^2_{t,x}}).\end{aligned}$$

For the last term in the last inequality, we recall that

$$\|r^{-(3-\mu)/2}2^{j\theta}P_j u\|_{\ell^2_j L^2_{t,x}} \simeq \|r^{-(3-\mu)/2}D^\theta u\|_{L^2_{t,x}} \lesssim \|r^{-(1-\mu)/2}D^{1+\theta}u\|_{L^2_{t,x}}$$
$$\simeq \|r^{-(1-\mu)/2}D^\theta \nabla u\|_{L^2_{t,x}}$$

in the case of $q = 2$ and $\theta \in [0, 1]$, which follows directly from the weighted Hardy–Littlewood–Sobolev inequality. The general result for q with non-endpoint $\theta \in (0, 1)$ follows then by real interpolation, that is, we have

$$\|r^{-(3-\mu)/2}2^{j\theta}P_j u\|_{\ell^q_j L^2_{t,x}} \lesssim \|r^{-(1-\mu)/2}2^{j\theta}P_j \nabla u\|_{\ell^q_j L^2_{t,x}}.$$

In conclusion, we have proved that, for $q = 2$, $\theta \in [0, 1]$ or $q = 1$, $\theta = 1/2$,

$$\|r^{(1-\mu)/2}2^{j\theta}P_j F(u)\|_{\ell^q_j L^2_{t,x}} \le C(a, \|\partial u\|_{L^\infty_t \dot{B}^{1/2}_{2,1}})T^{\mu/2}\|\partial D^\theta u\|_{X_{T,q}}\|\partial u\|_{L^\infty_t \dot{H}^{1/2+\mu}}, \tag{4.13}$$

from which we get

$$\|\partial D^\theta u_m\|_{X_{T,q}} \le 2C_1\|(\nabla u_0^{(j_m)}, u_1^{(j_m)})\|_{\dot{B}^\theta_{2,q}}$$

provided that $T \le \min(\delta_1, T_0)$ where T_0 is given in (4.9) with possibly smaller $c = c(g, a, b, \varepsilon_c) > 0$ such that $4C_1^2 C(a, 2C_1\varepsilon_c)T^\mu \le 1$. This completes the proof by induction. ∎

To prove convergence of the approximate solutions when $s < 2$, we will also require bounds for the solutions in a Sobolev space of negative order.

Lemma 4.2. *Under the same assumption as in Lemma 4.1, let* $(u_0, u_1) \in H^s_{\text{rad}}(\mathbb{R}^3) \times (H^{s-1}_{\text{rad}} \cap \dot{H}^{s_0-1}_{\text{rad}})(\mathbb{R}^3)$ *with* $s_0 \in [2-s, s-1]$. *Then there exist* $c = c(g, a, b, \varepsilon_c) \in (0, 1)$ *and* $C > 0$ *such that for any* $\theta \in [s_0, s-1]$, *we have*

$$\|D^{\theta-1}\partial u_k\|_{X_T} \le C\varepsilon_\theta + CT^\mu \varepsilon_{\theta+1}\varepsilon_{s-1}, \quad \forall k \ge 2, \tag{4.14}$$

provided that

$$T \le \min(\delta_1, c\varepsilon_s^{-1/\mu}). \tag{4.15}$$

In particular, with $\theta = s-1$, *we have*

$$\|\partial u_k\|_{L^\infty \dot{H}^{s-2}} \le \|D^{s-2}\partial u_k\|_{X_T} \le 2C\|(\nabla u_0, u_1)\|_{\dot{H}^{s-2}}. \tag{4.16}$$

Proof. As in the proof of Lemma 4.1, we proceed by induction. First, the result is trivial for $k = 2$. Then we make the inductive assumption that for some $m \ge 3$, (4.14) is satisfied by u_k for any $2 \le k \le m-1$.

To apply Proposition 3.7, we write equation (4.4) of u_k in the equivalent divergence form for $(t, x) \in (0, T) \times \mathbb{R}^3$:

$$\begin{cases} \Box u_k + \nabla\cdot(g(u_{k-1})\nabla u_k) = \nabla(g(u_{k-1}))\cdot\nabla u_k + F(u_{k-1}), \\ u_k(0,\cdot) = u_0^{(j_k)}, \quad \partial_t u_k(0,\cdot) = u_1^{(j_k)}. \end{cases} \tag{4.17}$$

From (4.7) and (4.8) in the proof of Lemma 4.1, we see that (3.24) is satisfied and we can apply Proposition 3.7 to obtain, for $\theta \in [1/2-\mu, 1/2+\mu]$,

$$\begin{aligned}
&\|D^{-\theta}\partial u_k\|_{X_T}\\
&\lesssim \|\partial u_k(0)\|_{\dot H^{-\theta}} + T^{\mu}\|g(u_{k-1})\|_{L_t^\infty \dot H^{1/2+\mu}}\|\partial u_k(0)\|_{\dot H^{1-\theta}}\\
&\quad + T^{\mu/2}\|r^{(1-\mu)/2}D^{-\theta}(\nabla(g(u_{k-1}))\cdot\nabla u_k)\|_{L^2_{t,x}} + T^{\mu/2}\|r^{(1-\mu)/2}D^{-\theta}F(u_{k-1})\|_{L^2_{t,x}}\\
&\lesssim \|\partial u_k(0)\|_{\dot H^{-\theta}} + C(g,\|u_{k-1}\|_{L^\infty})T^{\mu}\|u_{k-1}\|_{L_t^\infty \dot H^{1/2+\mu}}\|\partial u_k(0)\|_{\dot H^{1-\theta}}\\
&\quad + T^{\mu}\|D^{-\theta}\partial(u_k,u_{k-1})\|_{X_T}\left\|\big(\nabla(g(u_{k-1})),(a(u_{k-1}),b(u_{k-1}))\partial u_{k-1}\big)\right\|_{L_t^\infty \dot H^{1/2+\mu}},
\end{aligned}$$

where, in the last inequality, we have used Proposition 2.6 and fractional chain rule based on the fact that $g(0)=0$. To control the last term, as $\nabla(g(u_{k-1})) = g'(u_{k-1})\nabla u_{k-1}$, we see that all terms are of the form of $f(u)\partial u$, for which we can use the classical fractional Leibniz rule and chain rule to conclude that

$$\begin{aligned}
\|f(u)\partial u\|_{\dot H^{1/2+\mu}} &\lesssim |f(0)|\,\|\partial u\|_{\dot H^{1/2+\mu}} + \|\tilde f(u)\|_{L^\infty}\|\partial u\|_{\dot H^{1/2+\mu}}\\
&\quad + \|\tilde f(u)\|_{\dot W^{1/2+\mu,6}}\|\partial u\|_{L^3}\\
&\lesssim C(f,\|u\|_{\dot B^{1/2}_{2,1}})\|\partial u\|_{\dot H^{1/2+\mu}},
\end{aligned}\tag{4.18}$$

where $\tilde f(u) = f(u) - f(0)$.

In view of the boundedness (4.5) and (4.6), for $\theta \in [1/2-\mu, 1/2+\mu]$ we see that

$$\|D^{-\theta}\partial u_k\|_{X_T} \le \frac{C}{2}\varepsilon_{1-\theta} + C(g,a,b,\varepsilon_c)T^{\mu}\varepsilon_s\big(\varepsilon_{2-\theta} + \|D^{-\theta}\partial(u_k,u_{k-1})\|_{X_T}\big).$$

Thus, for T satisfying (4.15) with sufficiently small c, by the inductive assumption we get

$$\|D^{-\theta}\partial u_k\|_{X_T} \le C\varepsilon_{1-\theta} + C(g,a,b,\varepsilon_c)T^{\mu}\varepsilon_{s-1}\varepsilon_{2-\theta},$$

which completes the proof. ■

4.3. *Convergence in* $C\dot H^{s_0}_{\mathrm{rad}}$

In this subsection, we show that the approximate solutions are convergent in the weaker topology $C\dot H^{s_0}$, so that the desired solution of the quasilinear problem is given by the limit.

Lemma 4.3. *Under the same assumption as in Lemma 4.1, let* $(u_0,u_1) \in H^s_{\mathrm{rad}} \times (H^{s-1}_{\mathrm{rad}} \cap \dot H^{s_0-1}_{\mathrm{rad}})$ *with* $s_0 \in [2-s, s-1]$, *and* $\{u_k\}_{k\ge 2}$ *be the approximate solutions defined in (4.4), or equivalently (4.17), which satisfy the bounds (4.5), (4.6) and (4.14) for any* $k \ge 2$. *Then there exists* $c = c(g,a,b,\varepsilon_c) \in (0,1)$ *such that for any* T *with*

$$T \le \min(\delta_1, c(\varepsilon_s + \varepsilon_{s-1})^{-1/\mu}),\tag{4.19}$$

$\{u_k\}$ is a Cauchy sequence in $C([0,T];\dot H^{s_0})\cap C^{0,1}([0,T];\dot H^{s_0-1})$ with

$$\sum_{k\ge 3}\|D^{s_0-1}\partial(u_{k+1}-u_k)\|_{X_T}\lesssim \varepsilon_{s_0}+\varepsilon_{s_0+1}+\sum_{k\ge 3}\|\partial(u_{k+1}-u_k)(0)\|_{\dot H^{s_0-1}\cap\dot H^{s_0}}. \tag{4.20}$$

Here the right hand side is bounded because of (4.2)–(4.3).

Proof. If we set $w_k=u_{k+1}-u_k$ then w_k satisfies

$$\begin{aligned}\Box w_k+\nabla\cdot(g(u_k)\nabla w_k)&=\nabla\cdot((g(u_{k-1})-g(u_k))\nabla u_k)\\&\quad+\nabla(g(u_k))\cdot\nabla u_{k+1}-\nabla(g(u_{k-1}))\cdot\nabla u_k+F(u_k)-F(u_{k-1})=:G.\end{aligned}$$

As we see from the proof of Lemma 4.1, (3.4) is satisfied for $h=g(u_k)$ and we can apply Proposition 3.7 with $\theta\ge 1/2-\mu=2-s$ to obtain

$$\|D^{-\theta}\partial w_k\|_{X_T}\lesssim\|\partial w_k(0)\|_{\dot H^{-\theta}}+T^{\mu}\|g(u_k)\|_{L_t^\infty\dot H^{s-1}}\|\partial w_k(0)\|_{\dot H^{1-\theta}}+\|D^{-\theta}G\|_{X_T^*}. \tag{4.21}$$

For the term involving $g(u_k)$, we know from Theorem 2.3, $g(0)=0$, (4.6) and (4.16) that

$$\|g(u_k)\|_{L_t^\infty\dot H^{s-1}}\le C(g,\varepsilon_c)\|u_k\|_{L_t^\infty\dot H^{s-1}}\lesssim\|(\nabla u_0,u_1)\|_{\dot H^{s-2}}=\varepsilon_{s-1}. \tag{4.22}$$

The main part of the proof is to deal with G. We will write it as a combination of favorable terms and deal with each term separately. For this purpose, we denote $G_1=\nabla\cdot((g(u_{k-1})-g(u_k))\nabla u_k)$ and $G_2=F(u_k)-F(u_{k-1})$. Then

$$\begin{aligned}G-G_1-G_2&=g'(u_k)\nabla u_k\cdot\nabla u_{k+1}-g'(u_{k-1})\nabla u_{k-1}\cdot\nabla u_k\\&=g'(u_k)\nabla u_k\cdot\nabla w_k+g'(u_k)\nabla w_{k-1}\cdot\nabla u_k+(g'(u_k)-g'(u_{k-1}))\nabla u_{k-1}\cdot\nabla u_k\\&=:G_3+G_4+G_5.\end{aligned}$$

For G_j with $j\ge 2$, we observe that they fall into the following two categories:

$$\begin{aligned}\tilde G_2&=(f(u_k)-f(u_{k-1}))\partial(u_{k-1},u_k)\partial(u_{k-1},u_k),\\ \tilde G_3&=f(u_k)\partial(u_{k-1},u_k)\partial(w_{k-1},w_k).\end{aligned}$$

For all these terms, we claim that for any $\theta\in[2-s,s-1]$,

$$\|D^{-\theta}G\|_{X_T^*}\le C(g,a,b,\varepsilon_c)T^{\mu}\varepsilon_s\|D^{-\theta}\partial(w_{k-1},w_k)\|_{X_T}. \tag{4.23}$$

Before presenting the proof of (4.23), we apply it to prove (4.20). Actually, by (4.21) and (4.22), we have

$$\|D^{-\theta}\partial w_k\|_{X_T}\lesssim\|\partial w_k(0)\|_{\dot H^{-\theta}}+T^{\mu}\varepsilon_{s-1}\|\partial w_k(0)\|_{\dot H^{1-\theta}}+T^{\mu}\varepsilon_s\|D^{-\theta}\partial(w_{k-1},w_k)\|_{X_T}$$

for any $\theta\in[2-s,s-1]$, where the implicit constant may depend on g,a,b,ε_c. Let $\theta=1-s_0$. Then for any T satisfying (4.19) with sufficiently small c, we have

$$\|D^{s_0-1}\partial w_k\|_{X_T}\le C(\|\partial w_k(0)\|_{\dot H^{s_0-1}}+\|\partial w_k(0)\|_{\dot H^{s_0}})+\tfrac14\|D^{s_0-1}\partial(w_{k-1},w_k)\|_{X_T},$$

and so

$$\|D^{s_0-1}\partial w_k\|_{X_T} \le 2C(\|\partial w_k(0)\|_{\dot H^{s_0-1}} + \|\partial w_k(0)\|_{\dot H^{s_0}}) + \tfrac{1}{2}\|D^{s_0-1}\partial w_{k-1}\|_{X_T}$$

for any $k \ge 3$. However, recalling that $w_2 = u_3 - u_2 = u_3$, we know from (4.14) and (4.19) that

$$\|D^{s_0-1}\partial w_2\|_{X_T} \le C(\varepsilon_{s_0} + T^{\mu}\varepsilon_{s_0+1}\varepsilon_{s-1}) \le C(\varepsilon_{s_0} + \varepsilon_{s_0+1}).$$

Thus an iteration argument shows that $\sum \|D^{s_0-1}\partial w_k\|_{X_T}$ is convergent and we have (4.20). Actually, for any $j \in [3,\infty)$, for a finite summation from 3 to j, we have

$$\sum_{k=3}^{j} \|D^{s_0-1}\partial w_k\|_{X_T} \le 2C \sum_{k=3}^{j} \|\partial w_k(0)\|_{\dot H^{s_0-1}\cap \dot H^{s_0}} + \sum_{k=2}^{j-1} \frac{1}{2}\|D^{s_0-1}\partial w_k\|_{X_T},$$

and so

$$\begin{aligned}\sum_{k=3}^{j} \|D^{s_0-1}\partial w_k\|_{X_T} &\le 4C \sum_{k=3}^{j} \|\partial w_k(0)\|_{\dot H^{s_0-1}\cap \dot H^{s_0}} + \|D^{s_0-1}\partial w_2\|_{X_T}\\ &\le 4C \sum_{k=3}^{j} \|\partial w_k(0)\|_{\dot H^{s_0-1}\cap \dot H^{s_0}} + C(\varepsilon_{s_0} + \varepsilon_{s_0+1}).\end{aligned}$$

Letting j go to ∞, we obtain (4.20).

It remains to prove (4.23), for which we separately handle three terms, G_1, $\tilde G_2$ and $\tilde G_3$.

(i) First term: $G_1 = \nabla \cdot ((g(u_{k-1}) - g(u_k))\nabla u_k)$. We see that

$$\begin{aligned}\|D^{-\theta}G_1\|_{X_T^*} &\le T^{\mu/2}\|r^{(1-\mu)/2}D^{-\theta}G_1\|_{L^2_{t,x}}\\ &\lesssim T^{\mu/2}\|r^{(1-\mu)/2}D^{1-\theta}((g(u_{k-1}) - g(u_k))\nabla u_k)\|_{L^2_{t,x}}\\ &\lesssim T^{\mu/2}\|r^{-(1-\mu)/2}D^{1-\theta}(g(u_{k-1}) - g(u_k))\|_{L^2_{t,x}}\|\nabla u_k\|_{L^\infty_t \dot H^{s-1}}\end{aligned}$$

where, as $\theta \in [2-s, s]$, in the last inequality we have used Proposition 2.6 with $|1-\theta| \le s-1$. To control the term involving $g(u_{k-1}) - g(u_k)$, we observe that

$$g(u) - g(v) = \int_0^1 g'(v + \lambda(u-v))(u-v)\,d\lambda$$

and so

$$\begin{aligned}&\|r^{-(1-\mu)/2}D^{1-\theta}(g'(v+\lambda(u-v))(u-v))\|_{L^2_{t,x}}\\ &\quad\lesssim \|r^{-(1-\mu)/2}D^{1-\theta}(u-v)\|_{L^2_{t,x}}\|g'(v+\lambda(u-v))\|_{L^\infty_{t,x}}\\ &\qquad + \|r^{-(1-\mu)/2}(u-v)\|_{L^2_t L^{6/(1+2\theta)}_x}\|D^{1-\theta}g'(v+\lambda(u-v))\|_{L^\infty_t L^{3/(1-\theta)}_x}\\ &\quad\lesssim \|g'(v+\lambda(u-v))\|_{L^\infty_t \dot W^{1-\theta,3/(1-\theta)}_x \cap L^\infty_{t,x}}\|r^{-(1-\mu)/2}D^{1-\theta}(u-v)\|_{L^2_{t,x}}\\ &\quad\lesssim C(g, \|(u,v)\|_{L^\infty_t \dot B^{3/2}_{2,1}})T^{\mu/2}\|D^{1-\theta}(u-v)\|_{X_T},\end{aligned}$$

where we have used Theorem 2.4 and (2.5) in the first and second inequalities, for $\theta \in [0,1]$. In summary, we have proved that

$$\|r^{-(1-\mu)/2}D^{1-\theta}(g(u)-g(v))\|_{L^2_{t,x}} \le C(g,\|(u,v)\|_{L^\infty_t \dot B^{3/2}_{2,1}})T^{\mu/2}\|D^{1-\theta}(u-v)\|_{X_T}, \tag{4.24}$$

which gives

$$\|D^{-\theta}G_1\|_{X_T^*} \le C(g,\varepsilon_c)T^\mu\|D^{1-\theta}w_{k-1}\|_{X_T}\|\nabla u_k\|_{L^\infty_t \dot H^{s-1}}. \tag{4.25}$$

(ii) Second category of terms: $\tilde G_2$. Recall that

$$\tilde G_2 = (f(u_k)-f(u_{k-1}))\partial(u_{k-1},u_k)\partial(u_{k-1},u_k).$$

Let us present the proof for the typical term $\tilde G_2 = (f(u_k)-f(u_{k-1}))\partial y\partial z$, for which we know that, as $\theta \in [0, s-1]$, $\|D^{-\theta}\tilde G_2\|_{X_T^*}$ is bounded by

$$\begin{aligned}
&T^{\mu/2}\|r^{-(1-\mu)/2}D^{-\theta}((f(u_k)-f(u_{k-1}))\partial y)\|_{L^2_{t,x}}\|\partial z\|_{L^\infty \dot H^{s-1}}\\
&\quad\lesssim T^{\mu/2}\|r^{-(1-\mu)/2+\theta}((f(u_k)-f(u_{k-1}))\partial y)\|_{L^2_{t,x}}\|\partial z\|_{L^\infty \dot H^{s-1}}\\
&\quad\lesssim T^{\mu/2}\|r^{-(1-\mu)/2-1+\theta}(f(u_k)-f(u_{k-1}))\|_{L^2_{t,x}}\|r\partial y\|_{L^\infty_{t,x}}\|\partial z\|_{L^\infty \dot H^{s-1}}\\
&\quad\lesssim C(f,\|(u_{k-1},u_k)\|_{L^\infty_{t,x}})T^{\mu/2}\|r^{-(1-\mu)/2-1+\theta}w_{k-1}\|_{L^2_{t,x}}\|\partial y\|_{L^\infty_t \dot B^{1/2}_{2,1}}\|\partial z\|_{L^\infty \dot H^{s-1}}\\
&\quad\lesssim C(f,\|(u_{k-1},u_k)\|_{L^\infty_{t,x}})T^{\mu/2}\|r^{-(1-\mu)/2}D^{1-\theta}w_{k-1}\|_{L^2_{t,x}}\|\partial y\|_{L^\infty_t \dot B^{1/2}_{2,1}}\|\partial z\|_{L^\infty \dot H^{s-1}}.
\end{aligned}$$

That is, we have

$$\|D^{-\theta}\tilde G_2\|_{X_T^*} \le C(f,\varepsilon_c)T^\mu\|D^{1-\theta}w_{k-1}\|_{X_T}\|\partial(u_k,u_{k-1})\|_{L^\infty_t \dot H^{s-1}}. \tag{4.26}$$

(iii) Third category of terms: $\tilde G_3 = f(u_k)\partial(u_{k-1},u_k)\partial(w_{k-1},w_k)$. In this case, with the help of Proposition 2.6, we see that $\|D^{-\theta}\tilde G_3\|_{X_T^*}$ is bounded by

$$T^{\mu/2}\|r^{-(1-\mu)/2}D^{-\theta}\partial(w_{k-1},w_k)\|_{L^2_{t,x}}\|f(u_k)\partial(u_{k-1},u_k)\|_{L^\infty \dot H^{s-1}}.$$

Similar to the proof of (4.18) in Lemma 4.2, we find that

$$\|f(u_k)\partial(u_{k-1},u_k)\|_{L^\infty_t \dot H^{s-1}} \lesssim C(f,\|\partial(u_{k-1},u_k)\|_{L^\infty_t \dot B^{1/2}_{2,1}})\|\partial(u_{k-1},u_k)\|_{L^\infty_t \dot H^{s-1}},$$

and so

$$\|D^{-\theta}\tilde G_3\|_{X_T^*} \le C(f,\varepsilon_c)T^\mu\|D^{-\theta}\partial(w_{k-1},w_k)\|_{X_T}\|\partial(u_k,u_{k-1})\|_{L^\infty_t \dot H^{s-1}}. \tag{4.27}$$

In summary, in view of (4.25)–(4.27), as well as the uniform bounds (4.5) and (4.6), we have completed the proof of (4.23) and Lemma 4.3. ∎

4.4. Local well-posedness in H^s

Equipped with Lemmas 4.1–4.3, we are ready to prove the (unconditional) local well-posedness.

Lemma 4.4. *Let $n = 3$, $s = 3/2 + \mu \in (3/2, 2]$ and $s_0 \in [2 - s, s - 1]$. Consider the initial value problem* (1.1)–(1.2) *with $(u_0, u_1) \in H^s_{\mathrm{rad}} \times (H^{s-1}_{\mathrm{rad}} \cap \dot H^{s_0-1}_{\mathrm{rad}})$. Then, for any T satisfying* (4.19)*, there exists a unique weak solution*

$$u \in L^\infty_t H^s_{\mathrm{rad}} \cap C^{0,1}_t H^{s-1}_{\mathrm{rad}} \cap C_t \dot H^{s_0} \cap C^{0,1}_t \dot H^{s_0-1} \tag{4.28}$$

in $[0, T] \times \mathbb{R}^3$. Moreover, there exists $C_2 > 0$ such that $\partial u \in C([0, T]; \dot H^{\theta-1})$ for any $\theta \in [s_0, s)$, and

$$\|\partial u\|_{L^\infty \dot H^\theta} \le \|\partial D^\theta u\|_{X_T} \le C_2 \|(\nabla u_0, u_1)\|_{\dot H^\theta}, \quad \forall \theta \in [0, s-1], \tag{4.29}$$

$$\|\partial u\|_{L^\infty_t \dot B^{1/2}_{2,1}} \le \|\partial D^{1/2} u\|_{X_{T,1}} \le C_2 \|(\nabla u_0, u_1)\|_{\dot B^{1/2}_{2,1}}, \tag{4.30}$$

$$\|\partial u\|_{L^\infty \dot H^{\theta-1}} \le \|D^{\theta-1} \partial u\|_{X_T} \le C_2(\varepsilon_\theta + \varepsilon_{s-1}), \quad \forall \theta \in [s_0, s-1]. \tag{4.31}$$

Proof. By Lemmas 4.1 and 4.2, the approximate solutions u_k are well-defined and satisfy the bounds (4.5), (4.6) and (4.14). Moreover, Lemma 4.3 tells us that $\{u_k\}$ is a Cauchy sequence in the space $C([0, T]; \dot H^{s_0}) \cap C^{0,1}([0, T]; \dot H^{s_0-1})$; we denote its limit by $u \in C([0, T]; \dot H^{s_0}) \cap C^{0,1}([0, T]; \dot H^{s_0-1})$. By Helly's selection theorem, a subsequence of $\{u_k\}$ is weak star convergent to u in $L^\infty([0, T]; \dot H^s_{\mathrm{rad}}) \cap C^{0,1}([0, T]; \dot H^{s-1}_{\mathrm{rad}})$, proving (4.28). Then ∂u_k converges to ∂u in $C([0, T]; \dot H^{\theta-1})$ for any $\theta \in [s_0, s)$, which follows directly, by interpolation, from the boundedness for $\theta = s$ and continuity for $\theta = s_0$. Consequently, in view of the definition (4.4) of u_k, u is the desired weak solution for the initial value problem (1.1)–(1.2), and satisfies the bounds (4.29)–(4.31).

It remains to prove unconditional uniqueness. Suppose there is a solution

$$v \in L^\infty_t H^s_{\mathrm{rad}} \cap C^{0,1}_t H^{s-1}_{\mathrm{rad}} \cap C_t \dot H^{s_0}, \quad \partial_t v \in C_t \dot H^{s_0-1}, \tag{4.32}$$

in $[0, T_1] \times \mathbb{R}^3$ for the initial value problem (1.1)–(1.2), for some $T_1 \in (0, T]$. The key observation here is that by (4.32), we have

$$D^{s_0-1} \partial v \in L^\infty H^1,$$

and so by Hardy's inequality,

$$\begin{aligned} \|r^{-(1-\mu)/2} D^{s_0-1} \partial v\|_{L^2([0,T_2]\times\mathbb{R}^3)} &\lesssim_T \|D^{s_0-1} \partial v\|_{L^\infty([0,T_2];\dot H^{(1-\mu)/2})} \\ &\lesssim \|D^{s_0-1} \partial v\|_{L^\infty([0,T_2];H^1)} \end{aligned}$$

for any $T_2 \in (0, T_1]$. In other words, $D^{s_0-1} \partial v \in X_{T_2}$ for any $T_2 \in (0, T_1]$.

As in the proof of Lemma 4.3, we set $w = u - v$ with $D^{s_0-1} \partial w \in X_{T_2}$, and write the equation for w as

$$\begin{aligned} -\partial_t^2 w + \Delta w + \nabla \cdot (g(u) \nabla w) &= \nabla \cdot ((g(v) - g(u)) \nabla v) \\ &\quad + \nabla(g(u)) \cdot \nabla u - \nabla(g(v)) \cdot \nabla v + F(u) - F(v) =: G(u, w), \end{aligned}$$

together with $w(0, x) = 0$, $\partial_t w(0, x) = 0$.

As u is constructed as the limit of u_k, we can apply Proposition 3.7 with $\theta = 1 - s_0$ to the wave operator $-\partial_t^2 + \Delta + \nabla \cdot g(u)\nabla$. That is, we have

$$\|D^{s_0-1}\partial w\|_{X_T} \lesssim \|D^{s_0-1}G\|_{X_T^*}. \tag{4.33}$$

With the help of (4.33), applied to $w = u - v$, together with a similar proof to that for (4.25)–(4.27), we get

$$\begin{aligned}\|D^{s_0-1}\partial w\|_{X_T} &\lesssim \|D^{s_0-1}G(u,w)\|_{X_T^*} \\ &\lesssim C(g,a,b,\|\partial(u,v)\|_{L^\infty_{T_1}\dot B^{1/2}_{2,1}})T^\mu\|D^{s_0-1}\partial w\|_{X_T}\|\partial(u,v)\|_{L^\infty_{T_1}\dot H^{s-1}} \\ &\lesssim T^\mu\|D^{s_0-1}\partial w\|_{X_T}.\end{aligned}$$

Thus, for $T_2 \in (0, T_1]$ sufficiently small, we see that $\|D^{s_0-1}\partial w\|_{X_{T_2}} = 0$ and so $w \equiv 0$ in $[0, T_2] \times \mathbb{R}^3$, in view of $w(0, x) = 0$. After a simple iteration argument, this proves that $u \equiv v$ in $[0, T_1] \times \mathbb{R}^3$, which completes the proof of unconditional uniqueness. ■

5. High-dimensional well-posedness

Let $n \geq 4$, $s = n/2 + \mu$ with μ as in (1.4), and ε_s, ε_c be as in (1.7). In this section, we prove the existence and uniqueness part of Theorem 1.3, following a similar approach to Section 4.

5.1. *Approximate solutions*

As in Section 4.1, we can construct a sequence of spherically symmetric, compactly supported, smooth functions $(u_0^{(k)}, u_1^{(k)}) \to (u_0, u_1)$ in $H^s_{\text{rad}} \times H^{s-1}_{\text{rad}}$ such that

$$\|(\nabla u_0^{(k)}, u_1^{(k)})\|_{\dot B^\theta_{2,q}} \leq C_{\theta,q}\|(\nabla u_0, u_1)\|_{\dot B^\theta_{2,q}}, \quad \forall\theta \in \mathbb{R},\ q \in [1,\infty], \tag{5.1}$$

$$\|\nabla u_0^{(k)} - \nabla u_0^{(k+1)}\|_{H^s_{\text{rad}}(\mathbb{R}^n)} + \|u_1^{(k)} - u_1^{(k+1)}\|_{H^{s-1}_{\text{rad}}(\mathbb{R}^n)} \leq 2^{-k}. \tag{5.2}$$

Let $F(u) = a(u)u_t^2 + b(u)|\nabla u|^2$, $u_2 \equiv 0$ and define u_k $(k \geq 3)$ recursively by solving

$$\begin{cases}\Box u_k + g(u_{k-1})\Delta u_k = F(u_{k-1}), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u_k(0,\cdot) = u_0^{(k)}, & \partial_t u_k(0,\cdot) = u_1^{(k)}.\end{cases} \tag{5.3}$$

5.2. *Uniform boundedness of* u_k

Let C be the implicit constant in the estimates of Proposition 3.9. We claim that we have uniform bounds

$$\|\tilde\partial D^\theta u_k\|_{X_T} \leq 2C\|D^\theta(\nabla u_0, u_1)\|_{L^2}, \quad \theta \in [0, s-1], \tag{5.4}$$

$$\|\partial D^{n/2-1}u_k\|_{X_{T,1}} \leq 2C\|(\nabla u_0, u_1)\|_{\dot B^{n/2-1}_{2,1}}, \tag{5.5}$$

for any $T > 0$ satisfying
$$T^{\mu} f(C\varepsilon_c)\varepsilon_s \le c \tag{5.6}$$
for some increasing function f and constants $c \ll 1 \ll C$.

We prove the bounds by induction. They are trivially true when $k = 2$. Assuming that for some $m \ge 2$, they are true for any $k \le m$, for $h = g(u_m)$ we have
$$T^{\mu}\|\partial h\|_{L_t^\infty \dot H^{n/2-1+\mu}} \le C(\varepsilon_c)T^{\mu}\varepsilon_s \le \delta, \tag{5.7}$$
and so the requirement (3.32) of Proposition 3.9 is satisfied.

From Proposition 3.9, we know that
$$\begin{aligned}
\|\tilde\partial D^{\theta} u_{m+1}\|_{X_T} &\le C\|D^{\theta}(\nabla u_0, u_1)\|_{L^2} + C\|D^{\theta}F(u_m)\|_{X_T^*}, \quad \theta \in \big[0, [n/2]\big],\\
\|\partial D^{n/2-1} u_{m+1}\|_{X_{T,1}} &\le C\|(\nabla u_0, u_1)\|_{\dot B^{n/2-1}_{2,1}}\\
&\quad + CT^{\mu/2}\|r^{(1-\mu)/2}2^{j(n/2-1)}P_j F(u_m)\|_{\ell^1_j L^2_{t,x}}.
\end{aligned}$$

To control the nonlinear term, we apply Proposition 2.6 to obtain, for a sample term $F(u) = a(u)u_t^2$,
$$\begin{aligned}
\|D^{\theta}F(u)\|_{X_T^*} &\lesssim T^{\mu/2}\|r^{-(1-\mu)/2}D^{\theta}\partial u\|_{L^2_{t,x}}\|a(u)\partial u\|_{L_t^\infty \dot H^{s-1}}\\
&\lesssim T^{\mu}\|\tilde\partial D^{\theta}u\|_{X_T}\ \tilde a(\|u\|_{L^\infty \dot B^{n/2}_{2,1}})\|\partial u\|_{L_t^\infty \dot H^{s-1}}
\end{aligned}$$
whenever $\theta \in [0, s-1]$, where, since $s - 1 < n/2$, we have used the following well-known consequence of the fractional Leibniz rule and chain rule:
$$\|(a(u)-a(0))v\|_{\dot H^{s-1}} \lesssim \|a(u)-a(0)\|_{\dot B^{n/2}_{2,1}}\|v\|_{\dot H^{s-1}} \lesssim C(\|u\|_{L^\infty})\|u\|_{\dot B^{n/2}_{2,1}}\|v\|_{\dot H^{s-1}}.$$
Similarly, by (2.25), we have
$$\begin{aligned}
\|r^{(1-\mu)/2}2^{j(n/2-1)}P_j F(u)\|_{\ell^1_j L^2_{t,x}} &\lesssim \|r^{-(1-\mu)/2}2^{j(n/2-1)}P_j \partial u\|_{\ell^1_j L^2_{t,x}}\|a(u)\partial u\|_{L_t^\infty \dot H^{s-1}}\\
&\lesssim T^{\mu/2}\|\partial D^{n/2-1}u\|_{X_{T,1}}\tilde a(\|u\|_{L^\infty \dot B^{n/2}_{2,1}})\|\partial u\|_{L_t^\infty \dot H^{s-1}}.
\end{aligned}$$
From the induction assumption and (5.6), we get (5.4) and (5.5) for $k = m+1$ if we set $c > 0$ to be sufficiently small. This completes the proof by induction.

5.3. *Convergence in $C\dot H^1 \cap C^{0,1}L^2$*

Let $w_k = u_{k+1} - u_k$. Then
$$\Box w_k + g(u_k)\Delta w_k = (g(u_{k-1}) - g(u_k))\Delta u_k + F(u_k) - F(u_{k-1}).$$
Thus, by Theorem 3.1,
$$\|\partial w_k\|_{X_T} \lesssim \|\partial w_k(0)\|_{L^2} + \|(g(u_{k-1}) - g(u_k))\Delta u_k\|_{X_T^*} + \|F(u_k) - F(u_{k-1})\|_{X_T^*}.$$

Notice that $\|(g(u_{k-1})-g(u_k))\Delta u_k\|_{X_T^*}$ is controlled by

$$T^{\mu/2}C(\varepsilon_c)\|r^{n/2-1}w_{k-1}\|_{L^\infty}\|r^{-(1-\mu)/2-(n/2-2+\mu)}\Delta u_k\|_{L^2}$$
$$\lesssim T^{\mu/2}C(\varepsilon_c)\|\partial w_{k-1}\|_{L^\infty L^2}\|r^{-(1-\mu)/2}D^s u_k\|_{L^2}\lesssim T^{\mu}\varepsilon_s C(\varepsilon_c)\|\partial w_{k-1}\|_{X_T}.$$

Similarly, for the sample term $F(u)=a(u)u_t^2$, we have

$$\begin{aligned}
&\|F(u_k)-F(u_{k-1})\|_{X_T^*}\\
&\quad\lesssim T^{\mu/2}\|r^{n/2-1}(a(u_k)-a(u_{k-1}))\|_{L^\infty}\|r\partial u_k\|_{L^\infty}\|r^{(1-\mu)/2-n/2}\partial u_k\|_{L^2}\\
&\qquad + C_1(\varepsilon_c)T^{\mu/2}\|r^{1-\mu}\partial(u_{k-1},u_k)\|_{L^\infty}\|r^{-(1-\mu)/2}\partial w_{k-1}\|_{L^2}\\
&\quad\lesssim T^{\mu}\varepsilon_s\tilde{C}(\varepsilon_c)\|\partial w_{k-1}\|_{X_T}.
\end{aligned}$$

By letting c in (5.6) be even smaller, we can conclude that

$$\|\partial w_k\|_{X_T}\le C\|\partial w_k(0)\|_{L^2}+\tfrac{1}{2}\|\partial w_{k-1}\|_{X_T},$$

which yields convergence in $C\dot{H}^1\cap C^{0,1}L^2$, thanks to (5.2).

5.4. *Local well-posedness*

Let $u\in CH^1\cap C^{0,1}L^2$ be the limit of u_k. By weak star compactness, we have $\partial u\in L^\infty H^{s-1}$ and ∂u_k is convergent to ∂u in $C([0,T];\dot{H}^{\theta-1})$ for any $\theta\in[1,s)$. Then, in view of the definition (5.3) of u_k, it is clear that u is a weak solution for the initial value problem (1.1)–(1.2).

Unconditional uniqueness follows from a similar argument to that in Lemma 4.4, and we omit the proof.

6. Persistence of regularity

In this section, we show persistence of regularity for weak solutions when the initial data have higher regularity, as well as continuous dependence on the data.

In Sections 4 and 5, for data in $H^s\times H^{s-1}$, with additional requirement in $\dot{H}^{s_0-1}$ for initial velocity when $n=3$, we have constructed solutions in H^s where

$$s\in\begin{cases}(n/2,(n+1)/2], & n \text{ odd},\\ (n/2,(n+2)/2), & n \text{ even}.\end{cases}$$

Recall that the classical energy argument shows local well-posedness in H^{s_1} for any $s_1>n/2+1$, together with persistence of higher regularity. Keeping this fact in mind, we need only prove persistence of regularity in H^{s_1} with $s_1=[(n+4)/2]$.

6.1. Persistence of regularity: a weaker version

First, we prove a weaker version of persistence of regularity, when the data has slightly better regularity $s_2 = n/2 + \mu_2$, $(u_0, u_1) \in \dot{H}^{s_2} \times \dot{H}^{s_2-1}$, with

$$\mu_2 \in \begin{cases} (\mu, 1/2], & n \text{ odd}, \\ (\mu, (\mu+1)/2), & n \text{ even}, \end{cases}$$

where $\mu = s - n/2$.

Fixing $T < T_*$, we have a uniform bound on $\|\partial u\|_{L_t^\infty \dot{H}^{s-1} \cap L_t^\infty \dot{B}_{2,1}^{n/2-1}([0,T]\times\mathbb{R}^n)}$. With $\delta_3 > 0$ to be determined, by dividing $[0, T]$ into finitely many small, disjoint, adjacent intervals I_j, we have $|I_j|^\mu \|\partial u\|_{L_t^\infty \dot{H}^{s-1}(I_j\times\mathbb{R}^n)} \le \delta_3$, so that

$$|I_j|^\mu \|\partial g(u)\|_{L_t^\infty \dot{H}^{s-1}(I_j\times\mathbb{R}^n)} \le \delta \tag{6.1}$$

for each $I_j = [T_j, T_{j+1}]$, with $\Delta T_j = |I_j|$, where δ is as in (3.32) of Proposition 3.9. In addition, possibly shrinking I_j, we can apply the iteration argument in I_j to obtain a uniform bound in H^s for data $(u(T_j), \partial_t u(T_j))$ at $t = T_j$.

By recasting the iteration argument for local well-posedness on I_j we obtain, for the iterative C^∞ sequence u_k on I_j,

$$\|\partial u_k\|_{L_t^\infty \dot{H}^{s-1}(I_j\times\mathbb{R}^n)} \le C_j \|\partial u(T_j)\|_{\dot{H}^{s-1}}, \quad \|\partial u_k\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1}} \le C_j \|\partial u(T_j)\|_{\dot{B}_{2,1}^{n/2-1}},$$
$$\lim_{k\to\infty} \|\partial(u_k - u)\|_{L_t^\infty L^2(I_j\times\mathbb{R}^n)} = 0.$$

Reasoning by induction on j, assume that

$$\|\partial u_k(T_j)\|_{\dot{H}^{s_2-1}} \le C\|\partial u(T_j)\|_{\dot{H}^{s_2-1}} \le \tilde{C}_j \|\partial u(0)\|_{\dot{H}^{s_2-1}}. \tag{6.2}$$

Applying Proposition 3.9 with $\theta = s_2 - 1$, we have

$$\|D^\theta u_{k+1}\|_{\widetilde{LE}_{I_j,\mu}} := \|\tilde{\partial} D^\theta u_{k+1}\|_{X_{\Delta T_j}(I_j)} \lesssim \|\partial u(T_j)\|_{\dot{H}^\theta} + \|D^\theta F(u_k)\|_{X^*_{\Delta T_j}(I_j)}. \tag{6.3}$$

For the nonlinear term, we have

Lemma 6.1. *Let n be odd or $\mu_2 < (\mu + 1)/2$, and $F(u) = a(u)u_t^2 + b(u)|\nabla u|^2$. Then for radial functions u,*

$$\|D^{s_2-1} F(u)\|_{X^*_{\Delta T_j}(I_j)} \lesssim C(\|\partial u\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1}})|I_j|^\mu \|D^{s_2-1} u\|_{\widetilde{LE}_{I_j,\mu}} \|\partial u\|_{L_t^\infty \dot{H}^{s-1}}. \tag{6.4}$$

Proof. As in (4.12), without loss of generality, we deal with $F_1(u)$ and $F_2(u) = u_t^2$. For $F_2(u) = u_t^2$, we have

$$\begin{aligned}\|r^{(1-\mu)/2} D^\theta F_2(u)\|_{L^2_{t,x}} &\lesssim \|r^{-(1-\mu)/2} D^\theta u_t\|_{L^2_{t,x}} \|r^{1-\mu} u_t\|_{L^\infty_{t,x}} \\ &\lesssim |I_j|^{\mu/2} \|D^\theta u\|_{\widetilde{LE}_{I_j,\mu}} \|\partial u\|_{L_t^\infty \dot{H}^{n/2-1+\mu}}\end{aligned}$$

by Theorem 2.4 and (2.1). Concerning the other term $F_1(u) = \tilde{a}(u)u_t^2 = \tilde{a}(u)F_2(u)$, with $\tilde{a}(0) = 0$, from Theorem 2.4 we get

$$\begin{aligned}
&\|r^{(1-\mu)/2}D^\theta F_1(u)\|_{L^2_{t,x}} \\
&\quad\lesssim \|\tilde{a}(u)\|_{L^\infty_{t,x}}\|r^{(1-\mu)/2}D^\theta F_2(u)\|_{L^2_{t,x}} + \|r^{-(3-\mu)/2}D^\theta \tilde{a}(u)\|_{L^2_{t,x}}\|r^{2-\mu}F_2(u)\|_{L^\infty_{t,x}} \\
&\quad\lesssim \|r^{1-\mu}\partial u\|_{L^\infty_{t,x}} \\
&\qquad\times (\|r^{-(1-\mu)/2}D^\theta \partial u\|_{L^2_{t,x}} C(\|u\|_{L^\infty_{t,x}}) + \|r^{-(3-\mu)/2}D^\theta \tilde{a}(u)\|_{L^2_{t,x}}\|r\partial u\|_{L^\infty_{t,x}}) \\
&\quad\lesssim \tilde{C}(\|\partial u\|_{L^\infty_t \dot{B}^{n/2-1}_{2,1}})|I_j|^{(\mu)/2}\|D^\theta u\|_{\widetilde{LE}_{I_j,\mu}}\|\partial u\|_{L^\infty_t \dot{H}^{n/2-1+\mu}},
\end{aligned}$$

where we have used Theorem 2.3 with $\theta = s_2 - 1 \in (0, 1]$ when $n = 3$, in the last inequality.

For odd $n \geq 5$, the inequality still holds for $\mu_2 < 1/2$. Actually, as $\theta = s_2 - 1$ with $k = [\theta] = (n-3)/2 \geq 1$, with $\alpha = -(3-\mu)/2$ we see that

$$\alpha < n/2, \quad k - \alpha < n/2,$$

and so we apply Proposition 2.8, together with Lemma 2.1, to obtain

$$\begin{aligned}
\|r^{-(3-\mu)/2}D^\theta \tilde{a}(u)\|_{L^2_{t,x}} &\lesssim C\left(\max_{j\leq k}\|r^j\nabla^j u\|_{L^\infty_{t,x}}\right)\|r^{-(3-\mu)/2}D^\theta u\|_{L^2_{t,x}} \\
&\lesssim C(\|\partial u\|_{L^\infty_t \dot{B}^{n/2-1}_{2,1}})\|r^{-(3-\mu)/2}D^\theta u\|_{L^2_{t,x}}. \qquad (6.5)
\end{aligned}$$

Alternatively, when $\mu_2 = 1/2$ and so $\theta = (n-1)/2 \geq 2$, we can estimate directly as follows:

$$\begin{aligned}
\|r^{-(3-\mu)/2}D^\theta \tilde{a}(u)\|_{L^2_{t,x}} &\lesssim \sum_{|\sum\beta_l|=\theta, |\beta_1|\geq|\beta_j|\geq 1}\left\|r^{-(3-\mu)/2}\prod_{l=1}^j \nabla^{\beta_l}u\right\|_{L^2_{t,x}} \\
&\lesssim \sum_{1\leq|\beta_1|\leq\theta}\|r^{-(3-\mu)/2+|\beta_1|-\theta}\nabla^{\beta_1}u\|_{L^2_{t,x}}\prod_{l=2}^j\|r^{|\beta_l|}\nabla^{\beta_l}u\|_{L^\infty_{t,x}} \\
&\lesssim C(\|\partial u\|_{L^\infty_t \dot{B}^{n/2-1}_{2,1}})\|r^{-(3-\mu)/2}D^\theta u\|_{L^2_{t,x}}.
\end{aligned}$$

For n even, we have $k = [\theta] = (n-2)/2$, $\tau = \theta - k$ and $n/2 < k - \alpha < n/2 + 1$. Let $q, p \in (2, \infty)$ be such that $1/q + 1/p = 1/2$. A similar argument to that for Proposition 2.8 gives the desired bound, except the following term:

$$\sum_{|\sum\beta_l|=k, |\beta_l|\geq 1}\|r^{\tau-n/q}D^\tau(\tilde{a}^{(j)}(u) - \tilde{a}^{(j)}(0))\|_{L^q_x}\left\|r^{\alpha-\tau+n/q}\prod_{l=1}^j\nabla^{\beta_l}u\right\|_{L^p_x}.$$

As $-n < \tau q - n < n(q-1)$, we have $r^{\tau q-n} \in A_q$ and so

$$\begin{aligned}
\|r^{\tau-n/q}D^\tau(\tilde{a}^{(j)}(u) - \tilde{a}^{(j)}(0))\|_{L^q_x} &\lesssim C(\|\partial u\|_{\dot{B}^{n/2-1}_{2,1}})\|r^{\tau-n/q}D^\tau u\|_{L^q_x} \\
&\lesssim C(\|\partial u\|_{\dot{B}^{n/2-1}_{2,1}})\|u\|_{\dot{B}^{n/2}_{2,1}},
\end{aligned}$$

where we have used Theorem 2.3 and Lemma 2.1. For the other term, we let p be sufficiently close to 2 such that $\theta - \beta_1 \in (1/2 - 1/p, n/2)$. Because of the assumption that $\alpha - \mu_2 + 2 = (\mu + 1)/2 - \mu_2 > 0$, we also have

$$\alpha < n/2, \quad \alpha - (\theta - |\beta_1|) > -n/2,$$

and thus we can apply (2.5) to obtain

$$\begin{aligned}\left\| r^{\alpha-\tau+n/q} \prod_{l=1}^{j} \nabla^{\beta_l} u \right\|_{L_x^p} &\le \| r^{\alpha-\theta+n/q+|\beta_1|} \nabla^{\beta_1} u \|_{L_x^p} \left\| r^{k-|\beta_1|} \prod_{l=2}^{j} \nabla^{\beta_l} u \right\|_{L_x^\infty} \\ &\lesssim \| r^\alpha D^\theta u \|_{L^2} \| u \|_{\dot{B}^{n/2}_{2,1}}^{j-1}.\end{aligned}$$

Thus, we still have (6.5), which completes the proof. ■

In view of (6.3) and Lemma 6.1, we have

$$\| D^{s_2-1} u_{k+1} \|_{\widetilde{LE}_{I_j,\mu}} \lesssim \| \partial u(T_j) \|_{\dot{H}^{s_2-1}} + |I_j|^\mu \| D^{s_2-1} u_k \|_{\widetilde{LE}_{I_j,\mu}} \| \partial u_k \|_{L_t^\infty \dot{H}^{s-1}}$$

for any $k \ge 2$. Then, with $\delta_3 > 0$ sufficiently small, we deduce a uniform bound

$$\| D^{s_2-1} u_{k+1} \|_{\widetilde{LE}_{I_j,\mu}} \lesssim \| \partial u(T_j) \|_{\dot{H}^{s_2-1}}$$

for any $k \ge 2$, which, combined with the induction assumption (6.2), gives us the desired bound

$$\| D^{s_2-1} u \|_{\widetilde{LE}_{I_j,\mu}} \lesssim \| \partial u(T_j) \|_{\dot{H}^{s_2-1}} \lesssim \| \partial u(0) \|_{\dot{H}^{s_2-1}}.$$

As (6.2) is trivial when $T_j = 0$, by induction (6.2) holds for any j and thus

$$\| D^{s_2-1} u \|_{\widetilde{LE}_{T,\mu}} \lesssim \| \partial u(0) \|_{\dot{H}^{s_2-1}}.$$

This completes the proof of $\partial u \in L_t^\infty \dot{H}^{s_2-1}([0,T] \times \mathbb{R}^n)$. As it is true for any $T < T_*$, we see that $\partial u \in L_{\mathrm{loc}}^\infty \dot{H}^{s_2-1}([0,T_*) \times \mathbb{R}^n)$.

Note also that for n even, the result can be iterated to show that for any $s_2 \in (s, n/2+1)$, we have persistence of regularity.

6.2. *Persistence of regularity for n odd*

Now we prove persistence of regularity to H^{s_1} with $s_1 = [(n+4)/2]$. Let us begin with the case of n odd, when $s_1 = (n+3)/2$.

As we see from Section 6.1, we can assume we have an H^k solution, where $k = (n+1)/2 = [(n+2)/2]$ and $\mu = 1/2$. Also, it suffices to prove

$$\| \partial u \|_{L^\infty \dot{H}^k([0,T] \times \mathbb{R}^n)} \lesssim 1 + \| \partial u(0) \|_{\dot{H}^k} \tag{6.6}$$

for any T such that

$$T^{1/2}\|\nabla^k u\|_{X_T} \lesssim T^{1/2}\|D^{k-1}u\|_{\widetilde{LE}_{T,1/2}} \ll 1,$$
$$\|u\|_{L_t^\infty H^k([0,T]\times\mathbb{R}^n)} + \|\partial_t u\|_{L_t^\infty H^{k-1}([0,T]\times\mathbb{R}^n)} \lesssim 1.$$

For simplicity, we will just illustrate the proof for solutions, rather than approximate solutions.

By (3.33), we have

$$\|\tilde{\partial}\nabla^k u\|_{X_T} \lesssim \|\partial u(0)\|_{\dot{H}^k} + \|\nabla^k F\|_{X_T^*} + T^{1/2}\|\nabla^k u\|_{X_T}\|g(u)\|_{L_t^\infty \dot{H}^{k+1}}. \tag{6.7}$$

The classical Schauder estimates yield

$$\|g(u)\|_{L_t^\infty H^{k+1}} \lesssim C(g, \|u\|_{L^\infty})\|u\|_{L_t^\infty H^{k+1}},$$

which shows that the last term on the right of (6.7) is admissible.

Then, to finish the proof of (6.6), we need only prove a nonlinear estimate for the nonlinear term $\|\nabla^k F\|_{X_T^*}$, which is provided by the following

Lemma 6.2. *Let n be odd and $k = (n+1)/2$, and let $F(u) = a(u)u_t^2 + b(u)|\nabla u|^2$. Then for radial functions u,*

$$\|r^{1/4}\nabla^k F(u)\|_{L^2_{t,x}} \lesssim C(\|\partial u\|_{L_t^\infty H^{k-1}})\|r^{-1/4}\nabla^k \partial u\|_{L^2}\|\partial u\|_{L_t^\infty H^{k-1}}. \tag{6.8}$$

Proof. First, when there are no derivatives acting on $a(u)$ or $b(u)$, we need only control

$$\begin{aligned}\|r^{1/4}\nabla^k(\partial u)^2\|_{L^2_{t,x}} &\lesssim \|r^{-1/4}\nabla^k\partial u\|_{L^2_{t,x}}\|r^{1/2}\partial u\|_{L^\infty_{t,x}}\\ &\lesssim \|r^{-1/4}\nabla^k\partial u\|_{L^2_{t,x}}\|\partial u\|_{L_t^\infty \dot{H}^{k-1}},\end{aligned}$$

by Theorem 2.4 and (2.1).

For the remaining case, thanks to the uniform boundedness of u, we are reduced to controling

$$\left\| r^{1/4}\prod_{j=1}^{l}\nabla^{\alpha_j}\partial u\right\|_{L^2_x}$$

where $l \ge 3$ and $\sum|\alpha_j| = k+2-l$. Without loss of generality, we assume $|\alpha_j|$ is non-increasing. Notice then that

$$k+2-l = \sum|\alpha_j| \ge 2|\alpha_2|, \quad \text{so} \quad |\alpha_2| \le \frac{n-1}{4}, \quad \text{so} \quad |\alpha_2| \le \frac{n-3}{2},$$

where we have used the fact that $|\alpha_2|$ must be an integer. Then we see from (2.1) that

$$\|r^{|\alpha_j|+1/2}\nabla^{\alpha_j}\partial u\|_{L^\infty_{t,x}} + \|r^{|\alpha_j|+1}\nabla^{\alpha_j}\partial u\|_{L^\infty_{t,x}} \lesssim \|\partial u\|_{L_t^\infty H^{k-1}}$$

for any $j \ge 2$.

When $|\alpha_1| \geq 1$, we have $-1/4 - k + |\alpha_1| > -n/2$, and

$$
\begin{aligned}
\|r^{1/4} \prod_{j=1}^{l} \nabla^{\alpha_j} \partial u\|_{L_x^2} &\lesssim \|r^{1/4+1/2-\sum_{j\geq 2}(|\alpha_j|+1)} \nabla^{\alpha_1} \partial u\|_{L^2} \\
&\quad \times \left\| r^{1/2+|\alpha_2|} \nabla^{\alpha_2} \partial u\|_{L^\infty} \prod_{j=3}^{l} \|r^{1+|\alpha_j|} \nabla^{\alpha_j} \partial u \right\|_{L^\infty} \\
&\lesssim \|\partial u\|_{L_t^\infty H^{k-1}}^{l-1} \|r^{-1/4-k+|\alpha_1|} \nabla^{\alpha_1} \partial u\|_{L^2} \\
&\lesssim \|\partial u\|_{L_t^\infty H^{k-1}}^{l-1} \|r^{-1/4} \nabla^{k} \partial u\|_{L^2},
\end{aligned}
$$

by (2.9), while for $|\alpha_1| = 0$, we have $l = k+2$, $|\alpha_j| = 0$, and

$$
\begin{aligned}
\left\| r^{1/4} \prod_{j=1}^{l} \nabla^{\alpha_j} \partial u \right\|_{L_x^2} &\lesssim \left\| r^{1/4} \prod_{j=1}^{k+2} \partial u \right\|_{L_x^2} \\
&\lesssim \|\partial u\|_{L_t^\infty H^{k-1}}^{k+1} \|r^{1/4-(k+1)/2} \langle r \rangle^{-(k+1)/2} \partial u\|_{L^2} \\
&\lesssim \|\partial u\|_{L_t^\infty H^{k-1}}^{k+1} \|r^{1/4-k} \partial u\|_{L^2} \\
&\lesssim \|\partial u\|_{L_t^\infty H^{k-1}}^{k+1} \|r^{-1/4} D^{k} \partial u\|_{L^2}.
\end{aligned}
$$

This completes the proof. ∎

6.3. *Persistence of regularity for n even*

When n is even, we use a similar argument. Here $s_1 = [(n+4)/2] = n/2+2$, $k = n/2+1$ and $\mu \in (1/2, 1)$. We need only prove

$$
\|\partial u\|_{L^\infty \dot{H}^k([0,T]\times\mathbb{R}^n)} \lesssim 1 + \|\partial u(0)\|_{\dot{H}^k}, \tag{6.9}
$$

for any $T \ll 1$ such that

$$
T^{\mu} \|D^{n/2-1+\mu} \partial u\|_{X_T} \ll 1, \quad \|(u, \partial u)\|_{L^\infty H^{n/2-1+\mu}} + \|\partial u\|_{X_T} \lesssim 1.
$$

By (3.34), we have

$$
\|\tilde{\partial} \nabla^k u\|_{X_T} \lesssim \|\partial u(0)\|_{\dot{H}^k} + \|\nabla^k F\|_{X_T^*} + T^{\mu} \|g(u)\|_{L_t^\infty \dot{H}^{k+1}} \|D^{n/2+\mu} u\|_{X_T}, \tag{6.10}
$$

where, as before, the last term is admissible, thanks to Schauder estimates.

As for the nonlinear term, we have the following estimate, which is sufficient to conclude the proof of (6.9).

Lemma 6.3. *Let $n \geq 4$ be even, $k = n/2+1$, $\mu = 2/3$, and $F(u) = a(u)u_t^2 + b(u)|\nabla u|^2$. Then for radial functions u,*

$$
\|\nabla^k F\|_{X_T^*} \lesssim C(\|\partial u\|_{L_t^\infty H^{n/2-1+\mu}}) T^{\mu} (\|\partial u\|_{X_T} + \|D^k \partial u\|_{X_T}) \|\partial u\|_{L_t^\infty H^{n/2-1+\mu}}. \tag{6.11}
$$

Proof. The proof is similar to that of Lemma 6.2. First, when there are no derivatives acting on $a(u)$ or $b(u)$, we need only control

$$\|\nabla^k(\partial u)^2\|_{X_T^*} \lesssim T^{\mu/2}\|r^{-(1-\mu)/2}\nabla^k\partial u\|_{L^2_{t,x}}\|r^{1-\mu}\partial u\|_{L^\infty_{t,x}} \lesssim T^{\mu}\|\nabla^k\partial u\|_{X_T}\|\partial u\|_{L^\infty_t\dot H^{n/2-1+\mu}},$$

by Theorem 2.4 and (2.1).

For the remaining case, thanks to the uniform boundedness of u, we are reduced to controling

$$\left\|r^{(1-\mu)/2}\prod_{j=1}^{l}\nabla^{\alpha_j}\partial u\right\|_{L^2_x}$$

where $l \ge 3$, $\sum|\alpha_j| = k+2-l$ and $|\alpha_j|$ is nonincreasing.

When $|\alpha_2| = 0$, we see from (2.1) that

$$\|r^{1-\mu}\partial u\|_{L^\infty_{t,x}} \lesssim \|\partial u\|_{L^\infty_t\dot H^{n/2-1+\mu}}$$

for any $j \ge 2$. As $3 \le l \le k+2$, $|\alpha_1| = k+2-l$, $k-\mu(l-2) \in [(1-\mu)k, k]$ and $(1-\mu)/2-(l-1)(1-\mu) > -n/2$, we have

$$\begin{aligned}\left\|r^{(1-\mu)/2}\prod_{j=1}^{l}\nabla^{\alpha_j}\partial u\right\|_{L^2_x} &\lesssim \left\|r^{(1-\mu)/2-(l-1)(1-\mu)}\nabla^{\alpha_1}\partial u\|_{L^2_x}\prod_{j=2}^{l}\|r^{1-\mu}\partial u\right\|_{L^\infty_x}\\ &\lesssim \|r^{-(1-\mu)/2}D^{(l-2)(1-\mu)}\nabla^{\alpha_1}\partial u\|_{L^2_x}\|\partial u\|^{l-1}_{\dot H^{n/2-1+\mu}}\\ &\lesssim \|r^{-(1-\mu)/2}D^{k-\mu(l-2)}\partial u\|_{L^2_x}\|\partial u\|^{l-1}_{\dot H^{n/2-1+\mu}}.\end{aligned}$$

On the other hand, if $|\alpha_1| = |\alpha_2| = 1$, as $\mu > 1/2$ we have $n/2+\mu-1-|\alpha_2| > 1/2$, and so

$$\|r^{|\alpha_j|+1-\mu}\nabla^{\alpha_j}\partial u\|_{L^\infty_x} \lesssim \|\partial u\|_{\dot H^{n/2-1+\mu}}$$

for any $j \ge 2$. Thus

$$\begin{aligned}&\left\|r^{(1-\mu)/2}\prod_{j=1}^{l}\nabla^{\alpha_j}\partial u\right\|_{L^2_x}\\ &\lesssim \left\|r^{(1-\mu)/2-(1-\mu)(l-1)-(k+2-l-|\alpha_1|)}\nabla^{\alpha_1}\partial u\right\|_{L^2_x}\prod_{j=2}^{l}\|r^{|\alpha_j|+1-\mu}\nabla^{\alpha_j}\partial u\|_{L^\infty_x}\\ &\lesssim \|r^{-(1-\mu)/2+\mu(l-2)-n/2}\nabla\partial u\|_{L^2_x}\|\partial u\|^{l-1}_{\dot H^{n/2-1+\mu}}\\ &\lesssim \|r^{-(1-\mu)/2}D^{k-\mu(l-2)}\partial u\|_{L^2_x}\|\partial u\|^{l-1}_{\dot H^{n/2-1+\mu}},\end{aligned}$$

where in the last inequality we have used the fact that $\mu(l-2) \ge \mu > (1-\mu)/2$, thanks to $\mu \in (1/2, 1)$, so that we can apply (2.9).

It remains to consider the case $|\alpha_1| \ge 2$. Then

$$-(1-\mu)/2-(k-|\alpha_1|) > -n/2,$$

and so

$$\|r^{-(1-\mu)/2-(k-|\alpha_1|)}\nabla^{\alpha_1}\partial u\|_{L^2_x} \lesssim \|r^{-(1-\mu)/2}\nabla^k\partial u\|_{L^2_x}.$$

Also, noticing that

$$|\alpha_2| \le k+2-l-|\alpha_1| \le k+2-3-2 = n/2-2,$$

we see from (2.1) that

$$\|r^{|\alpha_2|+1-\mu}\nabla^{\alpha_2}\partial u\|_{L^\infty_x} \lesssim \|\partial u\|_{\dot H^{n/2-1+\mu}}, \quad \|r^{|\alpha_j|+1}\nabla^{\alpha_j}\partial u\|_{L^\infty_x} \lesssim \|\partial u\|_{\dot H^{n/2-1}}$$

for any $j \ge 3$. Thus

$$\begin{aligned}
&\Big\| r^{(1-\mu)/2}\prod_{j=1}^{l}\nabla^{\alpha_j}\partial u\Big\|_{L^2_x} \\
&\quad\lesssim \|r^{-(1-\mu)/2-(k-|\alpha_1|)}\nabla^{\alpha_1}\partial u\|_{L^2_x}\|r^{|\alpha_2|+1-\mu}\nabla^{\alpha_2}\partial u\|_{L^\infty_x}\prod_{j=3}^{l}\|r^{|\alpha_j|+1}\nabla^{\alpha_j}\partial u\|_{L^\infty_x} \\
&\quad\lesssim \|r^{-(1-\mu)/2}\nabla^k\partial u\|_{L^2_x}\|\partial u\|_{\dot H^{n/2-1+\mu}}\|\partial u\|^{l-2}_{\dot H^{n/2-1}} \\
&\quad\lesssim \|r^{-(1-\mu)/2}D^k\partial u\|_{L^2_x}\|\partial u\|^{l-1}_{H^{n/2-1+\mu}}.
\end{aligned}$$

This completes the proof. ∎

6.4. *Continuous dependence*

The continuous dependence property is essentially included in the proofs of convergence of the approximate solutions (Lemma 4.3), and unconditional uniqueness.

Let $T_* > 0$ be the lifespan of the solution u with data (u_0, u_1). Fix $T < T_*$ and $s_1 \in (s_c, s)$. We have a uniform bound on $\|\partial u\|_{L^\infty_t H^{s-1}([0,T]\times\mathbb{R}^n)}$. When $n = 3$, as $s_0 < s-1$, without loss of generality, we can assume that $s_0 = s_1 - 1$ and also that we have a uniform bound on $\|\partial u\|_{L^\infty_t \dot H^{s_1-2}}$. As the proof for $n \ge 4$ is relatively easier, we present only the proof for $n = 3$.

6.4.1. Short time continuity. Before proving the full continuous dependence property, we present a result on short time continuous dependence, from data with regularity τ_0 to solution with regularity τ_1, with $s \ge \tau_0 > \tau_1 \ge s_1$. Suppose $\|\partial u(0)\|_{\dot H^{s-1}\cap \dot H^{s_1-2}} \le M < \infty$ and $\tau_0 - \tau_1 \ge \varepsilon > 0$. We would like to find $T > 0$ with the following property: for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that whenever $\|(\nabla(u_0 - v_0), u_1 - v_1)\|_{\dot H^{\tau_0-1}\cap \dot H^{s_1-2}} \le \delta$, the corresponding solution $v \in L^\infty H^{\tau_0} \times C^{0,1}_t H^{\tau_0-1}$ is well-defined in $[0,T]\times\mathbb{R}^3$ and

$$\|\partial(u-v)\|_{L^\infty(\dot H^{\tau_1-1}\cap \dot H^{s_1-2})} \le \varepsilon.$$

Here, T can be chosen to be independent of the specific choices of τ_0, τ_1.

First, by assuming $\delta \le 1$, we can always assume $\|\partial v(0)\|_{\dot H^{\tau_0-1}\cap \dot H^{s_1-2}} \le M+1<\infty$. Hence, we find that

$$(\varepsilon_{\tau_0}+\varepsilon_{\tau_0-1})^{-1/(\tau_0-s_c)} \ge (2(M+1))^{-1/(s_1-s_c)} > 0,$$

and thus, in view of Lemma 4.4 and (4.19), the corresponding solution $v \in L^\infty H^{\tau_0} \times C_t^{0,1}H^{\tau_0-1} \cap C_t \dot H^{s_1-1} \cap C_t^{0,1}\dot H^{s_1-2}$ is well-defined in $[0,T_4]\times \mathbb{R}^3$, together with a uniform bound in $L^\infty \dot H^{\tau_0} \times C_t^{0,1}\dot H^{\tau_0-1}$, where

$$T_4 := \min(\delta_4, c(2(M+1))^{-1/(s_1-s_c)}, T_0). \tag{6.12}$$

We need to estimate

$$\|\partial(u-v)\|_{L^\infty(\dot H^{\tau_1-1}\cap \dot H^{s_1-2})}$$

in terms of the norm of $\partial(u-v)(0)$. For this purpose, we first estimate $\|\partial(u-v)\|_{L^\infty \dot H^{s_1-2}}$. Let $w=u-v$ and $\mu_0=\tau_0-s_c$. Then

$$\begin{aligned}\Box w + \nabla\cdot(g(u)\nabla w) &= \nabla\cdot((g(v)-g(u))\nabla v)\\ &\quad + \nabla(g(u))\cdot\nabla u - \nabla(g(v))\cdot\nabla v + F(u)-F(v) =: G(u,w),\end{aligned}$$

with $w(0,x)=u_0-v_0$ and $\partial_t w(0,x)=u_1-v_1$. Noticing that $T^{\mu_0}\|g(u)\|_{L_t^\infty \dot H^{\tau_0}} \lesssim 1$, by Proposition 3.7 and arguing as in Lemma 4.3 we find that for any $R\in(0,T_4]$,

$$\begin{aligned}&\|D^{s_1-2}\partial w\|_{X_R}\\ &\quad\lesssim \|\partial w(0)\|_{\dot H^{s_1-2}} + R^{\mu_0}\|g(u)\|_{L_R^\infty \dot H^{\tau_0-1}}\|\partial w_j(0)\|_{\dot H^{s_1-1}} + \|D^{s_1-2}G(u,w)\|_{X_R^*}\\ &\quad\lesssim \|\partial w(0)\|_{\dot H^{s_1-2}\cap \dot H^{s_1-1}}\\ &\qquad + C(g,a,b,\|\partial(u,v)\|_{L^\infty_{t\in[0,T]}\dot B^{s_c-1}_{2,1}})R^{\mu_0}\|D^{s_1-2}\partial w\|_{X_R}\|\partial(u,v)\|_{L^\infty_{t\in[0,T]}\dot H^{\tau_0-1}}\\ &\quad\lesssim \|\partial w(0)\|_{\dot H^{s_1-2}\cap \dot H^{s_1-1}} + R^{\mu_0}\|D^{s_1-2}\partial w\|_{X_R},\end{aligned}$$

where the implicit constant is independent of $R\in(0,T_4]$. Thus by choosing R small enough, we obtain

$$\|\partial w\|_{L^\infty([0,R];\dot H^{s_1-2})} \lesssim \|D^{s_1-2}\partial w\|_{X_R} \lesssim \|\partial w(0)\|_{\dot H^{s_1-2}\cap \dot H^{s_1-1}}.$$

Iterating this argument finitely many times ($\sim T_4/R$), we obtain

$$\|\partial w\|_{L^\infty([0,T_4];\dot H^{s_1-2})} \lesssim \|\partial w(0)\|_{\dot H^{s_1-2}\cap \dot H^{s_1-1}}. \tag{6.13}$$

Combining this with the uniform bounds (like that in Lemma 4.4),

$$\|\partial w\|_{L^\infty([0,T_4];\dot H^{\tau_0-1})} \le \|\partial u\|_{L^\infty([0,T_4];\dot H^{\tau_0-1})} + \|\partial v\|_{L^\infty([0,T_4];\dot H^{\tau_0-1})} \le 2C_2(M+1),$$

we obtain, for any $t\in[0,T_4]$,

$$\|\partial w(t)\|_{\dot H^{\tau_1-1}} \le \|\partial w(t)\|^{1-\theta}_{\dot H^{\tau_0-1}}\|\partial w(t)\|^{\theta}_{\dot H^{s_1-2}} \lesssim \|\partial w(0)\|^{\theta}_{\dot H^{s_1-2}\cap \dot H^{s_1-1}}, \tag{6.14}$$

where $\tau_0(1-\theta)+(s_1-1)\theta=\tau_1$.

6.4.2. Long time continuity. With short time continuity available, it is easy to deduce long time continuity. Actually, as $T < T_*$, there exists $M < \infty$ such that

$$\|\partial u\|_{L^\infty([0,T];\dot{H}^{s-1}\cap \dot{H}^{s_1-2})} \le M < \infty.$$

For fixed $s_1 \in (s_c, s)$, we have a uniform T_4 such that we have short time continuity in any interval of length less than T_4, around the solution u. Thus, we divide $[0, T]$ into finitely many, say N, adjacent intervals $\{I_j\}_{j=1}^N$ with $|I_j| < T_4$, $I_j = [t_{j-1}, t_j]$, $t_0 = 0$, $t_N = T$.

Let $\tau_j = s - j(s - s_1)/N$. We apply short time continuity, from t_{j-1} with regularity τ_{j-1}, to solution with regularity τ_j in the interval I_j. Combining the results in those intervals, we obtain long time continuity.

7. Three-dimensional almost global existence with small data

In this section, when $n = 3$, we show that the lower bound of the lifespan, available from local results, can be improved to almost global existence (Theorem 1.4). Without loss of generality, we assume that $s = 3/2 + \mu$ with $\mu \in (0, 1/2]$ and the solution lies in $CH^s \cap C^1 H^{s-1}$.

Let $I \subset J := [0, T_*(u_0, u_1))$ be the subset such that for any $T \in I$, we have

$$\|D^{s-1}u\|_{LE_T} \le 10C_3\varepsilon_s, \quad \|u\|_{LE_T} \le 10C_3\varepsilon_1, \tag{7.1}$$

where C_3 denotes the constant in (3.21). It is clear that I is nonempty and closed in J. By a bootstrap argument, to show existence up to time $\exp(c_1/(\varepsilon_1 + \varepsilon_s))$, we need only show that (7.1) holds for $5C_3$ instead of $10C_3$, for any $T \in I \cap [0, \exp(c_1/(\varepsilon_1 + \varepsilon_s))]$, provided that $\varepsilon_1 + \varepsilon_s < \delta$ for some sufficiently small $\delta > 0$.

By Sobolev embedding, we see that

$$\|u\|_{L^\infty_{t,x}(S_T)} \le C\|\nabla u\|_{L^\infty H^{s-1}(S_T)} \le 10CC_3(\varepsilon_1 + \varepsilon_s) \lesssim 1,$$

$g'(u) = \mathcal{O}(1)$, and so

$$\|g(u)\|_{L^\infty(S_T)} \le \left\| u \int_0^1 g'(\lambda u)\, d\lambda \right\|_{L^\infty(S_T)} \lesssim \varepsilon_1 + \varepsilon_s.$$

Moreover, we have

$$\|r^{1-\mu}\partial g(u)\|_{L^\infty} \lesssim \|r^{1-\mu}\partial u\|_{L^\infty} \lesssim \|\partial u\|_{L^\infty \dot{H}^{s-1}} \lesssim \varepsilon_s, \tag{7.2}$$

$$\|r\partial g(u)\|_{L^\infty} \lesssim \|r\partial u\|_{L^\infty} \lesssim \|\partial u\|_{L^\infty_t H^{s-1}} \lesssim \varepsilon_1 + \varepsilon_s. \tag{7.3}$$

From these estimates, we see that (3.20) is satisfied when $T \le \exp(c/\varepsilon_c)$ with $c + \varepsilon_1 + \varepsilon_s \ll 1$.

Recalling that u is constructed through approximation from $C_t^\infty C_c^\infty$ solutions of approximate equations, Proposition 3.6 applies to u as well, which gives us

$$\|D^{s-1}u\|_{LE_T} \le C_3\varepsilon_s + C_3(\ln\langle T\rangle)^{1/2}\|r^{(1-\mu)/2}\langle r\rangle^{\mu/2}D^{s-1}F(u)\|_{L^2_{t,x}}, \tag{7.4}$$

$$\|u\|_{LE_T} \le C_3\varepsilon_1 + C_3(\ln\langle T\rangle)^{1/2}\|r^{(1-\mu)/2}\langle r\rangle^{\mu/2}F(u)\|_{L^2_{t,x}}. \tag{7.5}$$

When $F(u) = a(u)u_t^2$, in view of (7.2) and (7.3) we have

$$\|r^{(1-\mu)/2}\langle r\rangle^{\mu/2}F(u)\|_{L^2_{t,x}} \lesssim \|r^{-(1-\mu)/2}\langle r\rangle^{-\mu/2}u_t\|_{L^2_{t,x}}\|r^{1-\mu}\langle r\rangle^{\mu}a(u)u_t\|_{L^\infty_{t,x}}$$
$$\lesssim (\ln\langle T\rangle)^{1/2}\varepsilon_1(\varepsilon_1+\varepsilon_s).$$

For the term with $D^{s-1} = D^{1/2+\mu}$, by Theorems 2.4 and 2.3 together with Lemma 2.5 we have

$$\begin{aligned}
&\|r^{(1-\mu)/2}\langle r\rangle^{\mu/2}D^{s-1}F(u)\|_{L^2_{t,x}} \\
&\quad\lesssim \|r^{-(1-\mu)/2}\langle r\rangle^{-\mu/2}D^{s-1}u_t\|_{L^2_{t,x}}\|r^{1-\mu}\langle r\rangle^{\mu}(|a(u)|+|a(0)|)u_t\|_{L^\infty_{t,x}} \\
&\qquad+\|r^{-3(1-\mu)/2}\langle r\rangle^{-\mu/2}D^{s-1}(a(u)-a(0))\|_{L^2_{t,x}}\|r^{2(1-\mu)}\langle r\rangle^{\mu}u_t^2\|_{L^\infty_{t,x}} \\
&\quad\lesssim \|r^{-(1-\mu)/2}\langle r\rangle^{-\mu/2}D^{s-1}u_t\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_t H^{s-1}} \\
&\qquad+\|r^{-(1-\mu)/2}\langle r\rangle^{-\mu/2}r^{\mu-1}D^{s-1}u\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_t \dot H^{s-1}}\|\partial u\|_{L^\infty_t H^{s-1}} \\
&\quad\lesssim (\ln\langle T\rangle)^{1/2}\varepsilon_s(\varepsilon_1+\varepsilon_s)+\|r^{-(1-\mu)/2}\langle r\rangle^{-\mu/2}D^{3/2}u\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_t \dot H^{s-1}} \\
&\quad\lesssim (\ln\langle T\rangle)^{1/2}\varepsilon_s(\varepsilon_1+\varepsilon_s),
\end{aligned}$$

where in the second to last inequality, we have used Lemma 2.7.

Then, combining this with (7.4) and (7.5), we arrive at

$$\|D^{\lambda-1}u\|_{LE_T} \le C_3\varepsilon_\lambda + C(\varepsilon_1+\varepsilon_s)\varepsilon_\lambda\ln\langle T\rangle, \quad \lambda = 1, s,$$

and so

$$\|D^{s-1}u\|_{LE_T} \le 5C_3\varepsilon_s, \quad \|u\|_{LE_T} \le 5C_3\varepsilon_1,$$

for any $T \in I \cap [0, \exp(c_1/(\varepsilon_1+\varepsilon_s))]$, where $c_1 = \min(c, 1/(4C))$.

8. High-dimensional global well-posedness

In this section, we show that when $\varepsilon_s + \varepsilon_1$ is small enough, the lower bound of the lifespan can be improved to global existence when $n \ge 4$.

For any $s > s_c$, there exists $\mu \in (0, 1/3)$ such that $s > s_c + \mu$. Without loss of generality, we assume $s = s_c + \mu$ and the solution lies in $CH^s \cap C^1H^{s-1}$. Let $I \subset J := [0, T_*(u_0, u_1))$ be the subset such that for any $T \in I$, we have

$$\|D^{s-1}u\|_{LE_T} \le 10C\varepsilon_s, \quad \|u\|_{LE_T} \le 10C\varepsilon_1, \tag{8.1}$$

where C is the constant in (3.35) of Proposition 3.9. It is clear that I is nonempty and closed in J. By a bootstrap argument, to show global existence, we need only show that (8.1) holds for $5C$ instead of $10C$, for any $T \in I$, provided that $\varepsilon_1 + \varepsilon_s < \delta$ for some sufficiently small $\delta > 0$.

By Sobolev embedding, we see that

$$\|u\|_{L^\infty_{t,x}(S_T)} \lesssim \|\partial u\|_{L^\infty H^{s-1}(S_T)} \lesssim \varepsilon_1 + \varepsilon_s,$$

and $g'(u) = \mathcal{O}(1)$, and so

$$\|\langle r\rangle^{\mu_1} g(u)\|_{L^\infty(S_T)} \lesssim \|\langle r\rangle^{\mu_1} u\|_{L^\infty(S_T)} \lesssim \|u\|_{L^\infty(\dot H^{n/2-\mu_1}\cap \dot H^{n/2+\mu_1})(S_T)} \lesssim \varepsilon_1 + \varepsilon_s,$$

provided that $\mu_1 \le \mu$. Moreover, we have

$$\|r^{1-\mu}\partial g(u)\|_{L^\infty} \lesssim \|r^{1-\mu}\partial u\|_{L^\infty} \lesssim \|\partial u\|_{L^\infty \dot H^{s-1}_{\text{rad}}} \lesssim \varepsilon_s, \tag{8.2}$$

$$\|r^{1+\mu_1}\partial g(u)\|_{L^\infty} \lesssim \|r^{1+\mu_1}\partial u\|_{L^\infty} \lesssim \|\partial u\|_{L^\infty_t \dot H^{n/2-1-\mu_1}} \lesssim \varepsilon_1 + \varepsilon_s, \tag{8.3}$$

and for any $0 \le j \le [(n-1)/2]$,

$$\|r^j \nabla^j u\|_{L^\infty_x} \lesssim \|u\|_{\dot B^{n/2}_{2,1}} \lesssim \varepsilon_1 + \varepsilon_s\ .$$

From these estimates, we see that (3.36) is satisfied when $\varepsilon_1 + \varepsilon_s \ll 1$.

Recalling that u is constructed through approximation from $C^\infty_t C^\infty_c$ solutions of approximate equations, (3.35) applies for u as well, which gives us

$$\|u\|_{LE_T} \le C\varepsilon_1 + C\|wF(u)\|_{L^2_{t,x}}, \tag{8.4}$$

and in the case of odd n,

$$\|D^{s-1}u\|_{LE_T} \le C\varepsilon_s + C\|wD^{s-1}F(u)\|_{L^2_{t,x}}, \tag{8.5}$$

where we set $w = r^{(1-\mu)/2}\langle r\rangle^{(\mu+\mu_1)/2}$. We claim that the following variant of (8.5):

$$\|D^{s-1}u\|_{LE_T} \le C\varepsilon_s + C\|wD^{s-1}F(u)\|_{L^2_{t,x}} + \tilde C\varepsilon_s(\varepsilon_1+\varepsilon_s) \tag{8.6}$$

for some $\tilde C$, applies for even n as well. Before presenting the proof of (8.6), let us use it to conclude the global existence.

First, we have

$$\begin{aligned}\|wF(u)\|_{L^2_{t,x}} &\lesssim \|w^{-1}\partial u\|_{L^2_{t,x}}\|w^2(|a(u)|+|b(u)|)\partial u\|_{L^\infty_{t,x}}\\ &\lesssim \|w^{-1}\partial u\|_{L^2_{t,x}} C(\|u\|_{L^\infty_{t,x}})\|\partial u\|_{L^\infty_t H^{s-1}} \lesssim (\varepsilon_1+\varepsilon_s)\varepsilon_1.\end{aligned}$$

Concerning the part with D^{s-1}, when $F(u) = a(u)u_t^2 = (a(0)+\tilde a(u))u_t^2$ and n is odd, by Theorem 2.4, Proposition 2.8 with $[s-1] = k = (n-3)/2$ and $k + (1-\mu) + (1+\mu_1)/2 < n/2$, together with Lemma 2.5, we see that $\|wD^{s-1}F(u)\|_{L^2_{t,x}}$ is controlled by

$$\begin{aligned}&\|w^{-1}D^{s-1}u_t\|_{L^2_{t,x}}\|w^2u_t\|_{L^\infty_{t,x}} + \|r^{-(1-\mu)}w^{-1}D^{s-1}\tilde a(u)\|_{L^2_{t,x}}\|r^{1-\mu}wu_t^2\|_{L^\infty_{t,x}}\\ &\quad\lesssim \|D^{s-1}u\|_{LE_T}\|\partial u\|_{L^\infty_t H^{s-1}} + \|r^{-(1-\mu)}w^{-1}D^{s-1}u\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_t \dot H^{s-1}}\\ &\quad\lesssim \|D^{s-1}u\|_{LE_T}\|\partial u\|_{L^\infty_t H^{s-1}} + \|w^{-1}D^{s-\mu}u\|_{L^2_{t,x}}\|\partial u\|_{L^\infty_t \dot H^{s-1}} \lesssim (\varepsilon_1+\varepsilon_s)\varepsilon_s,\end{aligned}$$

where in the second inequality, we have used Lemma 2.7.

When n is even, we have $[s-1] = n/2 - 1$, and we can apply Proposition 2.8 only if $\mu > (1+\mu_1)/2$. For the remaining case $0 < \mu \le (1+\mu_1)/2$, we notice that $1 - \mu + (1+\mu_1)/2 < n/2$, and we can apply Lemma 2.7 to obtain

$$\|w^{-1} r^{\mu-1} D^{s-1} \tilde{a}(u)\|_{L^2_{t,x}} \lesssim \|w^{-1} D^{s-\mu} \tilde{a}(u)\|_{L^2_{t,x}} \lesssim \|w^{-1} \nabla^{n/2} \tilde{a}(u)\|_{L^2_{t,x}}.$$

Noticing that

$$\begin{aligned} |\nabla^{n/2} \tilde{a}(u)| &\lesssim \sum_{|\sum \beta_l| = n/2, |\beta_1| \ge |\beta_l| \ge 1} \prod_{l=1}^{j} |\nabla^{\beta_l} u| \\ &\lesssim \sum_{|\beta_l| < n/2, l \ge 2} r^{|\beta_1| - n/2} |\nabla^{\beta_1} u| \prod_{l=2}^{j} |r^{|\beta_l|} \nabla^{\beta_l} u|, \end{aligned}$$

we get

$$\|w^{-1} \nabla^{n/2} \tilde{a}(u)\|_{L^2_{t,x}} \lesssim \sum_{1 \le j \le n/2} \|w^{-1} r^{j-n/2} \nabla^j u\|_{L^2_{t,x}} \lesssim \|w^{-1} D^{n/2} u\|_{L^2_{t,x}} \lesssim \varepsilon_1 + \varepsilon_s, \tag{8.7}$$

and thus we have the same estimate as for the odd spatial dimension.

Then, combining the above with (8.4) and (8.5), we arrive at

$$\|D^{s-1} u\|_{LE_T} \le C \varepsilon_s + \tilde{C} \varepsilon_s (\varepsilon_1 + \varepsilon_s), \quad \|u\|_{LE_T} \le C \varepsilon_1 + \tilde{C} \varepsilon_1 (\varepsilon_1 + \varepsilon_s).$$

Consequently, with $\varepsilon_1 + \varepsilon_s \ll 1$, we have

$$\|D^{s-1} u\|_{LE_T} \le 2C \varepsilon_s, \quad \|u\|_{LE_T} \le 2C \varepsilon_1,$$

for any $T \in I$, which shows that (8.1) holds for $2C$. By continuity, we see that $T_* = \infty$ and this completes the proof.

8.1. *(8.6) for even spatial dimension*

In the case of even n with $s = n/2 + \mu \ge 2$, we can apply (3.35) with $\theta = s - 2$ to the equation of ∇u,

$$(\Box + g(u)\Delta) \nabla u = \nabla F(u) - (\nabla g(u)) \Delta u,$$

which gives

$$\|D^{s-1} u\|_{LE_T} \lesssim \varepsilon_s + \|w D^{s-1} F(u)\|_{L^2_{t,x}} + \|w D^{s-2} ((\nabla g(u)) \Delta u)\|_{L^2_{t,x}}. \tag{8.8}$$

When $n \ge 6$, we have $n/2 - 2 - \mu_1 > 1/2$ so that

$$\|w^2 r \Delta u\|_{L^\infty_x} \lesssim \|\Delta u\|_{H^{s-2}_{\mathrm{rad}}} \lesssim \varepsilon_1 + \varepsilon_s, \quad \|r^{1-\mu} \nabla u\|_{L^\infty_{t,x}} \lesssim \varepsilon_s.$$

Moreover, by Lemma 2.7, as $2 - \mu + (1+\mu_1)/2 < n/2$, we have the following estimate similar to (8.7):

$$\|w^{-1} r^{\mu-2} D^{s-2} g'(u)\|_{L^2_{t,x}} \lesssim \|w^{-1} D^{n/2} g'(u)\|_{L^2_{t,x}} \lesssim \|w^{-1} D^{n/2} u\|_{L^2_{t,x}} \lesssim \varepsilon_1 + \varepsilon_s,$$

which gives

$$\|w^{-1}r^{\mu-2}D^{s-2}g'(u)\|_{L^2_{t,x}} \lesssim \varepsilon_1 + \varepsilon_s. \tag{8.9}$$

Then, by Theorem 2.4,

$$\begin{aligned}\|wD^{s-2}((\nabla g(u))\Delta u)\|_{L^2_{t,x}} &\lesssim \|w^{-1}D^{s-2}\Delta u\|_{L^2_{t,x}}\|w^2 g'(u)\nabla u\|_{L^\infty_{t,x}}\\ &\quad + \|w^{-1}r^{-1}D^{s-2}\nabla u\|_{L^2_{t,x}}\|w^2 r g'(u)\Delta u\|_{L^\infty_{t,x}}\\ &\quad + \|w^{-1}r^{\mu-2}D^{s-2}g'(u)\|_{L^2_{t,x}}\|w^2 r^{2-\mu}\nabla u\Delta u\|_{L^\infty_{t,x}}\\ &\lesssim \varepsilon_s(\varepsilon_1+\varepsilon_s).\end{aligned}$$

In the case of $n=4$, we have $s=2+\mu$. Let $q=2/(1-2\mu)$ so that $1/q+\mu=1/2$, and

$$\|r^{3\mu}\Delta u\|_{L^\infty_t L^q_x} \lesssim \|D^\mu \Delta u\|_{L^\infty_t L^2_x} \lesssim \varepsilon_s.$$

Moreover, we claim that

$$\|wr^{-3\mu}D^\mu \nabla g(u)\|_{L^2_t L^{1/\mu}_x} \lesssim \varepsilon_1 + \varepsilon_s, \tag{8.10}$$

thus

$$\begin{aligned}\|wD^\mu((\nabla g(u))\Delta u)\|_{L^2_{t,x}} &\lesssim \|w^{-1}D^\mu\Delta u\|_{L^2_{t,x}}\|w^2\nabla g(u)\|_{L^\infty_{t,x}}\\ &\quad + \|wr^{-3\mu}D^\mu\nabla g(u)\|_{L^2_t L^{1/\mu}_x}\|r^{3\mu}\Delta u\|_{L^\infty_t L^q_x}\\ &\lesssim \varepsilon_s(\varepsilon_1+\varepsilon_s).\end{aligned}$$

It remains to prove the claim (8.10). Actually, noticing that

$$wr^{-3\mu} = r^{1-4\mu}\langle r\rangle^{\mu+\mu_1}w^{-1} \lesssim r^{3(1/2-\mu)}(w^{-1}r^{-1/2-\mu} + w^{-1}r^{\mu_1-1/2}),$$

an application of Lemma 2.9 gives

$$\begin{aligned}&\|wr^{-3\mu}D^\mu\nabla g(u)\|_{L^2_t L^{1/\mu}_x}\\ &\qquad \lesssim \|w^{-1}r^{-1/2-\mu}D^{1/2}\nabla g(u)\|_{L^2_{t,x}} + \|w^{-1}r^{\mu_1-1/2}D^{1/2}\nabla g(u)\|_{L^2_{t,x}},\end{aligned}$$

where we have used the assumption $\mu \le 1/3$ to ensure $-(1+\mu_1)/2 - 1/2 - \mu \ge -(n-1)/2$. The second term on the right can be controlled by using Proposition 2.8 and Lemma 2.7:

$$\begin{aligned}\|w^{-1}r^{-1/2+\mu_1}D^{1/2}\nabla g(u)\|_{L^2_{t,x}} &\lesssim \|w^{-1}r^{-1/2+\mu_1}D^{3/2}u\|_{L^2_{t,x}}\\ &\lesssim \|w^{-1}D^{2-\mu_1}u\|_{L^2_{t,x}} \lesssim \varepsilon_1+\varepsilon_s.\end{aligned}$$

For the first term on the right, we use Lemma 2.7 to obtain

$$\begin{aligned}\|w^{-1}r^{-1/2-\mu}D^{1/2}\nabla g(u)\|_{L^2_{t,x}} &\lesssim \|w^{-1}r^{-\mu}\Delta g(u)\|_{L^2_{t,x}}\\ &\lesssim \|w^{-1}r^{-\mu}\Delta u\|_{L^2_{t,x}} + \|w^{-1}r^{-1}\nabla u\|_{L^2_{t,x}}\|r^{1-\mu}\nabla u\|_{L^\infty_{t,x}}\\ &\lesssim \varepsilon_1+\varepsilon_s.\end{aligned}$$

9. Appendix: Proof of Morawetz type estimates of Theorem 3.4

In this section, we prove the fundamental Morawetz type estimates (Theorem 3.4). Let $S_T = [0, T) \times \mathbb{R}^n$ with $n \geq 3$. We consider the linear wave equation (3.2), that is,

$$\Box_h u := (-\partial_t^2 + \Delta + \tilde{h}^{\alpha\beta}(t,x)\partial_\alpha\partial_\beta)u = F, \tag{9.1}$$

where we assume $\tilde{h}^{\alpha\beta} = h^{\alpha\beta} - m^{\alpha\beta}$, $h^{\alpha\beta} = h^{\beta\alpha}$, $\tilde{h}^{00} = 0$ and $\Box_h$ is uniformly hyperbolic in the sense of (3.1).

Lemma 9.1 (Morawetz type estimates). *Let $f = f(r)$ be any fixed differentiable function. For any solutions $u \in C^\infty([0,T]; C_0^\infty(\mathbb{R}^n))$ to equation (9.1) in S_T with (3.1) and $n \geq 3$, we have*

$$(fXu)(h^{\alpha\beta}\partial_\alpha\partial_\beta u) = \partial_\gamma P_h^\gamma - Q, \tag{9.2}$$

where $Xu = (\partial_r + \frac{n-1}{2r})u$, $P_h^0 = fh^{0\beta}\partial_\beta u Xu$, $P_h^j = \mathcal{O}((|f| + |rf'| + |f\tilde{h}|)|\tilde{\partial}u|^2)$,

$$\begin{aligned} Q &= Q_0 + \mathcal{O}\big((|f\tilde{h}|/r + |\partial(f\tilde{h})|)|\partial u|\,|\tilde{\partial}u|\big), \\ Q_0 &= \frac{2f - rf'}{r}\frac{|\not\nabla u|^2}{2} + f'\frac{|\partial_r u|^2 + |\partial_t u|^2}{2} - \frac{n-1}{4}\Delta\left(\frac{f}{r}\right)u^2, \end{aligned} \tag{9.3}$$

and $|\not\nabla u|^2 = |\nabla u|^2 - |\partial_r u|^2$.

This essentially comes from multiplying the wave equation by $f(r)\big(\partial_r + \frac{n-1}{2r}\big)u$ and a tedious calculation with integration by parts. See e.g. [34, pp. 199–200], [14, p. 273, (2.10)–(2.11)]. Typically, f is chosen to be a differentiable function satisfying

$$f \leq 1, \quad 2f \geq rf'(r) \geq 0, \quad -\Delta(f/r) \geq 0, \tag{9.4}$$

which ensures that Q_0 is positive semidefinite. In the literature, some of the typical choices are $f = 1$ [38], $1 - (3+r)^{-\delta}$ ($\delta > 0$) [45], $r/(R+r)$ [34, 43], $(r/(R+r))^\mu$ ($\mu \in (0,1)$) [13, 14].

9.1. Details: general case

Let $\omega^j = \omega_j = x^j/r$. As $\partial_j = \omega_j\partial_r + \not\nabla_j$, $\partial_r = \omega^j\partial_j$, $2X = \partial_r - \partial_r^* = \omega^j\partial_j + \partial_j\omega^j = 2\omega^j\partial_j + (n-1)/r$, we have $[X, \partial_t] = 0$ and

$$[X, \partial_k] = [\omega^j, \partial_k]\partial_j + \frac{n-1}{2}\left[\frac{1}{r}, \partial_k\right] = -\frac{\delta_k^j - \omega^j\omega_k}{r}\partial_j + \frac{n-1}{2r^2}\omega_k = \frac{1}{r}(-\partial_k + \omega_k X).$$

Noticing that

$$\begin{aligned} \partial_\alpha\partial_\beta u Xu &= \partial_\alpha(\partial_\beta u Xu) - \partial_\beta u \partial_\alpha Xu \\ &= \partial_\alpha(\partial_\beta u Xu) - \partial_\beta u[\partial_\alpha, X]u - \partial_\beta u X\partial_\alpha u, \end{aligned}$$

as $u_{\alpha\beta} = u_{\beta\alpha}$ we obtain

$$2\partial_\alpha\partial_\beta uXu = \partial_\alpha(\partial_\beta uXu) + \partial_\beta(\partial_\alpha uXu) - \partial_\beta u[\partial_\alpha, X]u - \partial_\alpha u[\partial_\beta, X]u \\ - \partial_\beta uX\partial_\alpha u - \partial_\alpha uX\partial_\beta u.$$

Noticing also that $\partial_j(\omega^j FG) = FXG + GXF$, we get

$$2\partial_\alpha\partial_\beta uXu = \partial_\alpha(\partial_\beta uXu)+\partial_\beta(\partial_\alpha uXu)-\partial_\beta u[\partial_\alpha, X]u-\partial_\alpha u[\partial_\beta, X]u-\partial_j(\omega^j\partial_\beta u\partial_\alpha u).$$

To be specific, we have

$$\begin{aligned}
\partial_t^2 uXu &= \partial_t(\partial_t uXu) - \partial_j(\omega^j u_t^2/2),\\
2\partial_t\partial_j uXu &= \partial_t(\partial_j uXu) + \partial_j(\partial_t uXu) - \partial_k(\omega^k\partial_t u\partial_j u) + \frac{u_t(-\partial_k + \omega_k X)u}{r},\\
2\partial_j\partial_k uXu &= \partial_j(\partial_k uXu) + \partial_k(\partial_j uXu) - \partial_m(\omega^m\partial_j u\partial_k u)\\
&\quad + \frac{\partial_j u(-\partial_k + \omega_k X)u + \partial_k u(-\partial_j + \omega_j X)u}{r}.
\end{aligned}$$

In summary, we have

$$(\partial_\alpha\partial_\beta u)Xu = \partial_\gamma P^\gamma_{\alpha\beta} + Q_{\alpha\beta}, \tag{9.5}$$

with $P^\gamma_{\alpha\beta} = \mathcal{O}(|\partial u|\,|\tilde{\partial} u|)$ and $rQ_{\alpha\beta} = \mathcal{O}(|\partial u|\,|\tilde{\partial} u|)$.

9.2. *Details: general multiplier*

By (9.5), when we multiply fXu by $\Box u$, we get

$$\partial^\alpha\partial_\alpha ufXu = \partial_\gamma(fm^{\alpha\beta}P^\gamma_{\alpha\beta}) - f'\omega_k m^{\alpha\beta}P^k_{\alpha\beta} + fm^{\alpha\beta}Q_{\alpha\beta},$$

where

$$\begin{aligned}
m^{\alpha\beta}P^0_{\alpha\beta} &= -P^0_{00} + \sum_j P^0_{jj} = -\partial_t uXu,\\
m^{\alpha\beta}P^k_{\alpha\beta} &= -P^k_{00} + \sum_j P^k_{jj} = \omega^k\frac{u_t^2}{2} + \sum_j\left(\delta^k_j u_j Xu - \frac{1}{2}\omega^k u_j^2\right)\\
&= \omega^k\frac{u_t^2 - |\nabla u|^2}{2} + u_k Xu,\\
m^{\alpha\beta}Q_{\alpha\beta} &= -Q_{00} + \sum_j Q_{jj} = \sum_j \partial_j u\frac{1}{r}(-\partial_j + \omega_j X)u\\
&= \frac{1}{r}(-|\nabla u|^2 + u_r^2) + \frac{n-1}{2r^2}u\partial_r u.
\end{aligned}$$

Noticing that $\frac{u\partial_r u}{r^2} = \frac{1}{r^2}\partial_r u^2 = -\frac{1}{2}\nabla r^{-1}\cdot\nabla u^2$ and

$$2f\frac{u\partial_r u}{r^2} = -\nabla\frac{f}{r}\cdot\nabla u^2 + \frac{f'}{r}\partial_r u^2 = -\nabla\cdot\left(u^2\nabla\frac{f}{r}\right) + u^2\Delta\left(\frac{f}{r}\right) + \frac{f'}{r}\partial_r u^2,$$

we obtain

$$-f'\omega_k m^{\alpha\beta}P^k_{\alpha\beta} = -f'\left(\frac{u_t^2-|\nabla u|^2}{2}+u_r Xu\right),$$

and

$$fm^{\alpha\beta}Q_{\alpha\beta} = -\frac{f}{r}|\nabla\!\!\!/ u|^2+\frac{n-1}{4}\left(\Delta\left(\frac{f}{r}\right)u^2+\frac{f'}{r}\partial_r u^2\right)+\partial_j F^j$$

with $F^j = \mathcal{O}((|f|+|rf'|)r^{-2}u^2)$.

In summary, we have

$$\begin{aligned}(\partial^\alpha\partial_\alpha u)fXu &= \partial_\gamma(fm^{\alpha\beta}P^\gamma_{\alpha\beta}) - f'\omega_k m^{\alpha\beta}P^k_{\alpha\beta} + fm^{\alpha\beta}Q_{\alpha\beta}\\ &= \partial_\gamma\tilde{P}^\gamma - \frac{f}{r}|\nabla\!\!\!/ u|^2 + \frac{n-1}{4}\Delta\left(\frac{f}{r}\right)u^2 - f'\left(\frac{u_t^2-|\nabla u|^2}{2}+u_r^2\right)\\ &= \partial_\gamma\tilde{P}^\gamma - f'\frac{u_t^2+u_r^2}{2}+\frac{n-1}{4}\Delta\left(\frac{f}{r}\right)u^2 - \frac{(2f-rf')|\nabla\!\!\!/ u|^2}{2r}\\ &= \partial_\gamma\tilde{P}^\gamma - Q_0,\end{aligned}$$

where $\tilde{P}^j = \mathcal{O}((|f|+|rf'|)|\tilde{\partial}u|^2)$ and $\tilde{P}^0 = -fu_t Xu$.

For perturbation, we have

$$f\tilde{h}^{\alpha\beta}(\partial_\alpha\partial_\beta u)Xu = \partial_\gamma(f\tilde{h}^{\alpha\beta}P^\gamma_{\alpha\beta}) - P^\gamma_{\alpha\beta}\partial_\gamma(f\tilde{h}^{\alpha\beta}) + f\tilde{h}^{\alpha\beta}Q_{\alpha\beta}. \tag{9.6}$$

In summary, we have obtained (9.2).

9.3. *Choice of multiplier function f*

To prove the Morawetz type estimates (Lemma 3.4), we will choose two kinds of multiplier functions f, with parameter $R>0$,

$$f = \frac{r}{R+r}, \tag{9.7}$$

$$f = \left(\frac{r}{R+r}\right)^\mu = \left(1-\frac{R}{R+r}\right)^\mu, \quad \mu\in(0,1). \tag{9.8}$$

Of course, (9.7) can be viewed as the limit case of (9.8) when $\mu=1$.

Now we do the calculation for f given in (9.8) with $\mu\in(0,1]$. We first notice that

$$f'(r) = \mu\left(\frac{r}{R+r}\right)^{\mu-1}\frac{R}{(R+r)^2} = \mu\frac{Rr^{\mu-1}}{(R+r)^{\mu+1}} \ge 0, \tag{9.9}$$

$$\frac{f(r)}{r} - f'(r) = \frac{r^{\mu-1}}{(R+r)^\mu}\left(1-\frac{\mu R}{R+r}\right) \ge 0, \tag{9.10}$$

$$\frac{2f(r)-rf'(r)}{r} \ge \frac{f(r)}{r} \ge f'(r). \tag{9.11}$$

In order to compute $-\Delta(f(r)/r)$, we recall that

$$-\Delta\left(\frac{f(r)}{r}\right) = -r^{1-n}\partial_r\left(r^{n-1}\partial_r\frac{f(r)}{r}\right) = r^{1-n}\partial_r\left(r^{n-2}\left(\frac{f(r)}{r}-f'(r)\right)\right).$$

Using this identity and (9.10), we see that $-\Delta(f(r)/r)$ equals

$$
\begin{aligned}
r^{1-n}\partial_r&\left(\frac{r^{n+\mu-3}}{(R+r)^\mu}\left(1-\frac{\mu R}{R+r}\right)\right)\\
&=\left(\frac{(n+\mu-3)r^{-3+\mu}}{(R+r)^\mu}-\frac{\mu r^{-2+\mu}}{(R+r)^{\mu+1}}\right)\left(1-\frac{\mu R}{R+r}\right)+\frac{\mu R r^{-2+\mu}}{(R+r)^{\mu+2}}\\
&=\frac{r^{-3+\mu}}{(R+r)^\mu}\left((n+\mu-3)-\frac{\mu r}{R+r}\right)\left(1-\frac{\mu R}{R+r}\right)+\frac{\mu R r^{-2+\mu}}{(R+r)^{\mu+2}}\\
&=\frac{r^{-3+\mu}}{(R+r)^\mu}\left(n-3+\frac{\mu R}{R+r}\right)\left(1-\frac{\mu R}{R+r}\right)+\frac{\mu R r^{-2+\mu}}{(R+r)^{\mu+2}},
\end{aligned}
$$

from which we see that, as $n \geq 3$,

$$
-\Delta\left(\frac{f(r)}{r}\right) \geq (1-\mu)\frac{\mu R^2 r^{-3+\mu}}{(R+r)^{\mu+2}}+\frac{\mu R r^{-2+\mu}}{(R+r)^{\mu+2}} \geq 0. \tag{9.12}
$$

In summary, when $\mu \in (0,1)$ we see that Q_0 from (9.3) is nonnegative and has the following lower bound for $r \leq R$:

$$
Q_0 \geq f'\frac{|\partial u|^2}{2}-\frac{n-1}{4}\Delta\left(\frac{f}{r}\right)u^2 \gtrsim_\mu \frac{|\tilde{\partial} u|^2}{R^\mu r^{1-\mu}}, \tag{9.13}
$$

where the implicit constant depends only on n and $\mu \in (0,1)$, and in particular is independent of $R > 0$. On the other hand, when $\mu = 1$, Q_0 is still nonnegative and has the following lower bound for $R/2 \leq r \leq R$:

$$
Q_0 \geq f'\frac{|\partial u|^2}{2}-\frac{n-1}{4}\Delta\left(\frac{f}{r}\right)u^2 \geq \frac{1}{8R}|\partial u|^2+\frac{n-1}{32}\frac{1}{R^2 r}u^2 \gtrsim \frac{|\tilde{\partial} u|^2}{r}. \tag{9.14}
$$

9.4. *Proof of Morawetz type estimates*

Equipped with Lemmas 3.2 and 9.1, together with the observations (9.13)–(9.14), we can prove the Morawetz type estimates of Lemma 3.4.

Let us begin with the proof of (3.10). First, applying (9.13) with $R = 1$, and (9.14) with $R \geq 1$, that is, using $f = \left(\frac{r}{1+r}\right)^\mu$ and $f = \frac{r}{R+r}$ with $R \geq 1$, we get

$$
\begin{aligned}
\int_{r\leq 1}&\frac{|\tilde{\partial} u|^2}{r^{1-\mu}}\,dx\,dt+\sup_{R\geq 1}\int_{R/2\leq r\leq R}\frac{|\tilde{\partial} u|^2}{r}\,dx\,dt\\
&\lesssim \sup_{f=(\frac{r}{1+r})^\mu,\frac{r}{R+r},R\geq 1}\int_{S_T} Q_0\,dx\,dt\\
&\lesssim \sup_{f=(\frac{r}{1+r})^\mu,\frac{r}{R+r},R\geq 1}\left(-\int_{S_T} fF\left(\partial_r+\frac{n-1}{2r}\right)u\,dx\,dt+\int_{\mathbb{R}^n} P_h^0(t,\cdot)\,dx\Big|_{t=0}^{T}\right.\\
&\qquad\left.+\int_{S_T}(Q_0-Q)\,dx\,dt\right)
\end{aligned}
$$

$$\lesssim \int_{S_T} |F\tilde{\partial}u|\,dx\,dt + \int_{\mathbb{R}^n} |\partial u(T)|\,|\tilde{\partial}u(T)|\,dx + \int_{\mathbb{R}^n} |\partial u(0)|\,|\tilde{\partial}u(0)|\,dx$$

$$+ \sup_{f=(\frac{r}{1+r})^\mu, \frac{r}{R+r}, R\geq 1} \int_{S_T} \left(|f\partial h| + \frac{|f\tilde{h}|}{r}\right)|\partial u|\,|\tilde{\partial}u|\,dx\,dt$$

$$\lesssim \int_{S_T} |F\tilde{\partial}u|\,dx\,dt + \|\tilde{\partial}u(t)\|^2_{L^\infty L^2(S_T)} + \int_{S_T} \left(|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}\langle r\rangle^\mu}\right)|\partial u|\,|\tilde{\partial}u|\,dx\,dt,$$

where we have used (9.2) in the second inequality, and the facts $|f| \leq 1$, $0 \leq f' \leq f/r$,

$$|Q - Q_0| \lesssim \left(|f\partial h| + |f'\tilde{h}| + \frac{|f\tilde{h}|}{r}\right)|\partial u|\,|\tilde{\partial}u|$$
$$\lesssim \left(|f\partial h| + \frac{|f\tilde{h}|}{r}\right)|\partial u|\,|\tilde{\partial}u|,$$

and $|P^0| \lesssim |\partial u|\,|\tilde{\partial}u|$ in the third inequality. By Lemma 3.2 and Hardy's inequality,

$$\|u\|^2_{X_1} := \int_{r\leq 1} \frac{|\tilde{\partial}u|^2}{r^{1-\mu}}\,dx\,dt + \sup_{R\geq 1} \int_{R/2\leq r\leq R} \frac{|\tilde{\partial}u|^2}{r}\,dx\,dt + \|\partial u(t)\|^2_{L^\infty L^2(S_T)}$$
$$\lesssim \int_{S_T} |F\tilde{\partial}u|\,dx\,dt + \|\partial u(0)\|^2_{L^2}$$
$$+ \int_{S_T} \left(|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}\langle r\rangle^\mu}\right)|\partial u|\,|\tilde{\partial}u|\,dx\,dt.$$

Thus to get (3.10), we need only show that

$$\|u\|_{LE_T} \lesssim \|u\|_{X_1}, \tag{9.15}$$

which essentially follows from a standard argument of Keel–Smith–Sogge [20]. Here, for completeness, we write down the proof. The first and second terms are trivial to control. For the remaining two terms, with $\alpha \in [0, \mu]$, we have

$$\|r^{-(1-\mu)/2}\langle r\rangle^{-\alpha/2}\tilde{\partial}u\|^2_{L^2_{t,x}(S_T)}$$
$$\lesssim \|r^{(\mu-1)/2}\tilde{\partial}u\|^2_{L^2(r\leq 1)} + \sum_{0\leq j\leq \ln\langle T\rangle} \|r^{(\mu-1-\alpha)/2}\tilde{\partial}u\|^2_{L^2(r\simeq 2^j)} + \|r^{(\mu-1-\alpha)/2}\tilde{\partial}u\|^2_{L^2(r\geq\langle T\rangle)}$$
$$\lesssim \|u\|^2_{X_1} + \sum_{0\leq j\leq \ln\langle T\rangle} 2^{j(\mu-\alpha)}\|r^{-(1/2)}\tilde{\partial}u\|^2_{L^2(r\simeq 2^j)} + \langle T\rangle^{\mu-1-\alpha}\|\tilde{\partial}u\|^2_{L^2_{t,x}(r\geq\langle T\rangle)}$$
$$\lesssim \sum_{0\leq j\leq \ln\langle T\rangle} 2^{j(\mu-\alpha)}\|u\|_{X_1} + \langle T\rangle^{\mu-\alpha}\|\tilde{\partial}u\|_{L^\infty_t L^2_x} \lesssim C_\alpha(T)\|u\|^2_{X_1},$$

where

$$C_\alpha(T) = \begin{cases} \ln\langle T\rangle, & \alpha = \mu, \\ \langle T\rangle^{\mu-\alpha}, & \alpha \in [0, \mu). \end{cases}$$

This completes the proof of (9.15), and so of (3.10).

Turning to the proof of (3.9), we will use $f = (\frac{r}{T+r})^\mu \le 1$. Applying (9.13), we get as before

$$\begin{aligned}\int_{r\le T} &\frac{|\tilde{\partial} u|^2}{T^\mu r^{1-\mu}}\, dx\, dt \\ &\lesssim \int_{S_T} (|F| + |f\tilde{\partial}\tilde{h}|\,|\partial u|)|\tilde{\partial} u|\, dx\, dt + \|\tilde{\partial} u(t)\|^2_{L^\infty L^2(S_T)} \\ &\lesssim \int_{S_T} \Big(|F| + \Big(|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}(T+r)^\mu}\Big)|\partial u|\Big)|\tilde{\partial} u|\, dx\, dt + \|\tilde{\partial} u(t)\|^2_{L^\infty L^2(S_T)}.\end{aligned}$$

Together with Lemma 3.2, we see that

$$\begin{aligned}\|u\|^2_{X_2} &:= \int_{r\le T} \frac{|\tilde{\partial} u|^2}{T^\mu r^{1-\mu}}\, dx\, dt + \|\partial u(t)\|^2_{L^\infty L^2(S_T)} \\ &\lesssim \int_{S_T} \Big(|F| + \Big(|\partial h| + \frac{|\tilde{h}|}{r^{1-\mu}(T+r)^\mu}\Big)|\partial u|\Big)|\tilde{\partial} u|\, dx\, dt + \|\partial u(0)\|^2_{L^2}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|r^{-(1-\mu)/2}\tilde{\partial} u\|_{L^2_{t,x}} &\lesssim \|r^{-(1-\mu)/2}\tilde{\partial} u\|_{L^2(|x|\le T)} + \|r^{-(1-\mu)/2}\tilde{\partial} u\|_{L^2(|x|\ge T)} \\ &\lesssim \|r^{-(1-\mu)/2}\tilde{\partial} u\|_{L^2(|x|\le T)} + T^{-(1-\mu)/2}\|\tilde{\partial} u\|_{L^2_{t,x}(|x|\ge T)} \\ &\lesssim \|r^{-(1-\mu)/2}\tilde{\partial} u\|_{L^2(|x|\le T)} + T^{\mu/2}\|\tilde{\partial} u\|_{L^\infty_t L^2_x} \lesssim T^{\mu/2}\|u\|_{X_2},\end{aligned}$$

which gives (3.9).

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