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The boundary of linear subvarieties

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Abstract. We describe the boundary of linear subvarieties in the moduli space of multi-scale differentials. Linear subvarieties are algebraic subvarieties of strata of (possibly) meromorphic differentials that in local period coordinates are given by linear equations. The main example of such are affine invariant submanifolds, that is, closures of $\mathrm{SL}(2,\mathbb{R})$-orbits. We prove that the boundary of any linear subvariety is again given by linear equations in *generalized period coordinates* of the boundary. Our main technical tool is an asymptotic analysis of periods near the boundary of the moduli space of multi-scale differentials which yields further techniques and results of independent interest.

Keywords. Moduli spaces, abelian differentials, flat surfaces, affine invariant submanifolds

1. Introduction

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^n \mu_i = 2g - 2$. The stratum $\mathcal{H}(\mu)$ is the moduli space consisting of pairs (X, ω) where X is a Riemann surface of genus g and ω is a meromorphic differential with multiplicities of zeroes and poles prescribed by μ. The projectivized stratum $\mathbb{P}\mathcal{H}(\mu)$ is the quotient of $\mathcal{H}(\mu)$ by $\mathbb{C}^*$, where $\mathbb{C}^*$ acts on a differential by rescaling. Strata have a natural linear structure, i.e. a set of coordinates, distinguished up to the action of the linear group, called period coordinates, such that the transition functions are linear. A special class of subvarieties of strata is given by linear subvarieties.

Definition 1.1. A $(\mathbb{C})$-*linear subvariety* M is an irreducible algebraic subvariety of a stratum $\mathcal{H}(\mu)$ that, at any point, is given by a finite union of linear subspaces in local period coordinates.

A particularly important class of linear subvarieties are *affine invariant submanifolds*. Those are linear subvarieties in strata of holomorphic differentials where the linear subspaces are defined over the real numbers. By a combination of [7] and [8], affine invariant submanifolds are exactly orbit closures of the natural $\mathrm{SL}(2,\mathbb{R})$-action.

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Linear subvarieties in projectivized strata are usually not compact. For example, affine invariant submanifolds are never compact since one can use cylinder deformations to degenerate to a stable curve.

Recently in [2] the authors constructed a modular compactification $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ of the projectivized stratum $\mathbb{P}\mathcal{H}(\mu)$, the *moduli space of projectivized multi-scale differentials*. The goal of this paper is to study the boundary of a linear subvariety in $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$.

The boundary $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu) \setminus \mathbb{P}\mathcal{H}(\mu)$ parametrizes *multi-scale differentials*, i.e. stable curves together with a collection of meromorphic differential forms on the irreducible components, subject to several technical conditions, which we recall in Section 2.4. Furthermore, the boundary decomposes into a union of open boundary strata, each of which possesses a natural linear structure induced by *generalized period coordinates*. We will explain the structure of the boundary in more detail in Section 2.7.

For technical reasons we work with the "unprojectivized" *moduli space of multi-scale differentials* $\Xi\mathcal{M}_{g,n}(\mu)$. The group $\mathbb{C}^*$ acts on $\Xi\mathcal{M}_{g,n}(\mu)$ by rescaling and $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu) = \Xi\mathcal{M}_{g,n}(\mu)/\mathbb{C}^*$ is the quotient. Our main result is as follows.

Theorem 1.2 (Main theorem). *Let $M \subseteq \mathcal{H}(\mu)$ be a $\mathbb{C}$-linear subvariety. Then the intersection of the closure $\bar{M} \subseteq \Xi\mathcal{M}_{g,n}(\mu)$ with any open boundary stratum $D_{\overline{\Gamma}}$ of the moduli space $\Xi\mathcal{M}_{g,n}(\mu)$ of multi-scale differentials is a* levelwise *linear subvariety, for the natural linear structure on the boundary stratum $D_{\overline{\Gamma}} \subset \partial\Xi\mathcal{M}_{g,n}(\mu)$.*

Furthermore, the linear equations for $\partial M \cap D_{\overline{\Gamma}}$ are explicitly *computable from the linear equations for M near the boundary.*

For this statement, we recall that the irreducible components of stable curves in the boundary of $\Xi\mathcal{M}_{g,n}(\mu)$ are stratified by levels, depending on the vanishing order of the differential on each component along one-parameter families. By a levelwise linear subvariety we mean that each linear equation only relates periods of the differential along curves contained in the same level.

This version of the main theorem is only a preliminary qualitative result. In the course of the paper we state several more precise versions. Once we define the linear structure of the boundary, we can make a more precise, but still qualitative statement, given in Theorem 2.3. Later, in Sections 7, 8 and 10, we will be able to determine the explicit equations defining the boundary $\partial M \cap D_{\overline{\Gamma}}$ provided we know the linear equations defining M at a point near the boundary. In Proposition 8.2 we give an explicit formula in local coordinates, while Proposition 10.1 gives a coordinate-free description of the linear equations defining ∂M.

Our main technical tool is a detailed asymptotic analysis of the behavior of periods near the boundary of $\Xi\mathcal{M}_{g,n}(\mu)$. When integrating differentials over cycles passing through nodes of the limiting stable curve, the period might diverge logarithmically. In particular, periods do not extend as holomorphic functions to the boundary $\partial\Xi\mathcal{M}_{g,n}(\mu)$, but they do extend after subtracting their logarithmic divergences. We call the resulting functions *log periods*. The first part of the paper is devoted to properly defining log periods and computing their limits at the boundary.

Log periods can also be viewed naturally in the context of Hodge theory. Stated in this language, the extension of log periods can be seen as an analogue of Schmid's nilpotent orbit theorem [13] in the flat setting. We discuss the relations to Hodge theory in Section 7. See also [6] for similar discussions.

The linear equations of the boundary

We now explain how to obtain the linear equations defining the intersection of a boundary stratum with the closure of a linear subvariety M from the linear equations defining M near the boundary.

Let (X, η) be a multi-scale differential in the boundary of $M \cap D_{\overline{\Gamma}}$. The stable curve X has two types of nodes: *vertical* nodes connect irreducible components of different levels, while *horizontal* nodes connect components of the same level. Near the boundary stratum $D_{\overline{\Gamma}}$ of $\Xi\mathcal{M}_{g,n}(\mu)$, every smooth surface can be cut by simple closed curves into subsurfaces of different levels. The subsurface of level i specializes to the irreducible components of X of level i under degeneration.

In a period chart inside $\mathcal{H}(\mu)$, a linear equation for M is a homology class $F = \sum_l A_l[\gamma_l]$ where the collection $\{\gamma_l\}$ is a suitable basis for relative homology, called a $\overline{\Gamma}$-*adapted* homology basis. We define the notion of $\overline{\Gamma}$-adapted basis in Section 4.5. Roughly speaking one starts by choosing a homology basis for each subsurface of level i and extends those to a basis on the whole surface by only passing through lower levels.

We say F is of *top level at most* i if it can be represented by a sum of paths, each of which is completely contained in the subsurface of level $\leq i$. To obtain the equations for the boundary proceed as follows. Start with the defining equations $F_1, \ldots, F_k$ for M, written in terms of a $\overline{\Gamma}$-adapted bases and put into reduced row echelon form. Then for each F_l repeat the following steps.

(1) Determine the top level $\top(F_l)$ of F_l.

(2) If the equation F_l crosses horizontal nodes of level $\top(F_l)$, delete it.

(3) Otherwise, restrict F_l to each irreducible component of X of level $\top(F_l)$. The resulting cycle then defines an equation for $\partial M \cap D_{\overline{\Gamma}}$.

We describe the restriction procedure more explicitly in Section 4. The collection of linear equations obtained in this way are then the linear equations defining the boundary of M. In Section 9 we give an explicit example illustrating the above process.

Potential applications

The main theorem gives a novel tool to study the classification problem for affine invariant submanifolds. Let $M \subseteq \mathcal{H}(\mu)$ be an affine invariant submanifold. Then by Theorem 1.2 the intersection of ∂M with *any* boundary stratum is a lower-dimensional linear subvariety. One can now try to iterate this process inductively. A useful consequence of Theorem 1.2 for this approach is the following corollary.

Corollary 1.3. *If the linear equations for M are defined over a field $K \subseteq \mathbb{C}$, then the linear equations ∂M intersected with any open boundary stratum of $\Xi\mathcal{M}_{g,n}(\mu)$ are defined over a subfield of K.*

Another consequence of the proof of Theorem 1.2 are restrictions on the possible linear equation defining M inside $\mathcal{H}(\mu)$ arising from considerations of invariance under monodromy. The precise statement is given in Remark 7.7. This should be compared to the cylinder deformation theorem [16, Thm. 5.1] which also restricts the possible linear equations, albeit in a slightly different language and thus the results are not directly comparable. In [5] we investigate in detail the relation of our approach to cylinder deformation results, together with applications to describing the geometry and combinatorics of possible degenerations of affine invariant manifolds. More precisely, we show that the cylinder deformation theorem for affine invariant submanifold is a direct consequence of algebraicity. Furthermore, we determine the explicit analytic equations for the closure of linear subvarieties in plumbing coordinates in a neighborhood of the boundary, rather than just describing the boundary.

Algebraicity

We stress that our setup only works for algebraic subvarieties that are locally given by finitely many linear subspaces, and does not apply to merely analytic subvarieties. By [8] affine invariant submanifolds, i.e. analytic subvarieties given by subspaces defined over the *real* numbers, are always algebraic. On the other hand, [3] have communicated to us an example of an analytic subvariety of a meromorphic stratum which is locally defined by linear equations with rational coefficients, which is not algebraic.

Algebraicity is only used once in the argument, in Section 7.3, where we use the classical fact that the Euclidean closure of an algebraic variety in an algebraic compactification coincides with the Zariski closure and in particular is an analytic variety.

Afterwards, we use the fact that every boundary point of an analytic variety is the limit of a holomorphic one-parameter family, and not just of some sequence. This will ultimately allow us to avoid the cautionary example from [4, Section 4] and take limits of linear equations. We will discuss the cautionary example in more detail in Remark 7.11.

1.1. Relationship to previous work

Degenerations of affine invariant submanifolds have been considered in [4, 12]. If we consider a family of differentials inside the Hodge bundle, the limit on a stable curve is a collection of differentials on each irreducible component with at most simple poles at the nodes and opposite residues at each node. In [4,12] the authors consider a partial compactification $\widetilde{\mathcal{H}}(\mu)$ of $\mathcal{H}(\mu)$ which is constructed by removing all nodes and filling them in with marked points, and contracting all components where the differential vanishes. Thus they only consider the top level part of a limit multi-scale differential in the boundary. Each differential $(X_\infty, \omega_\infty) \in \widetilde{\mathcal{H}}(\mu)$ is contained in a stratum $\mathcal{H}(\omega_\infty)$ of possibly disconnected differentials with at most simple poles. The resulting partial compactification is called "WYSIWYG" compactification because only the parts of the limit are considered that are represented by flat surfaces of positive area. The following is a description of the boundary of an affine invariant submanifold inside $\widetilde{\mathcal{H}}(\mu)$.

Theorem 1.4 ([4, Thm. 1.2]). *Let M be an affine invariant submanifold and $(X_\infty, \omega_\infty) \in \widetilde{\mathcal{H}}(\mu)$ with no simple poles. The intersection of the boundary $\partial M \subseteq \widetilde{\mathcal{H}}(\mu)$ with the stratum $\mathcal{H}(\omega_\infty) \subseteq \widetilde{\mathcal{H}}(\mu)$ is an algebraic variety, locally given by finitely many subspaces in the period coordinates of $\mathcal{H}(\omega_\infty)$. Furthermore, assume that a sequence (X_n, ω_n) of points of M converges to $(X_\infty, \omega_\infty)$. After removing finitely many terms, the sequence (X_n, ω_n) may be partitioned into finitely many subsequences such that for each subsequence the tangent space to a branch of $\partial M \cap \mathcal{H}(\omega_\infty)$ at $(X_\infty, \omega_\infty)$, inside $\widetilde{\mathcal{H}}(\mu)$, is equal to the intersection of the tangent space of a branch of M at (X_n, ω_n) and the tangent space of $\mathcal{H}(\omega_\infty)$, for n sufficiently large. Here we use the fact that, since $(X_\infty, \omega_\infty)$ has no simple poles, the tangent space to $\mathcal{H}(\omega_\infty)$ is naturally a subspace of $\mathcal{H}(\mu)$ at (X_n, ω_n).*

Theorem 1.2 should then be seen as an analogue of Theorem 1.4 for the moduli space of multi-scale differentials. Roughly speaking, Theorem 1.4 says that the boundary of an affine invariant submanifold is given by linear equations on all components of the limit where the differential does not vanish. After suitable rescaling, the limits become non-zero on the remaining components, and we show that, after rescaling, the whole boundary is given by linear equations.

There exists a forgetful map $p : \Xi\mathcal{M}_{g,n}(\mu) \to \widetilde{\mathcal{H}}(\mu)$ by sending a multi-scale differential to its top level piece. In Section 11 we will see that our results quickly imply Theorem 1.4. The crucial observation is that p has compact fibers. In the presence of simple poles and multiple levels, the description of the tangent space to the boundary in $\Xi\mathcal{M}_{g,n}(\mu)$ is much more involved than Theorem 1.4, and the complete description is given by Proposition 10.1.

The proofs in [4, 12] use the theory of cylinder deformations and thus only work for affine invariant submanifolds in strata of holomorphic differentials, our results on the other hand work for linear subvarieties with arbitrary coefficients and in meromorphic strata, provided that they are algebraic. In particular, Theorem 1.4 is true for arbitrary linear subvarieties in meromorphic strata.

1.2. Outline of the proof

The proof of Theorem 1.2 can be roughly divided into two parts. The first part is to determine a set of linear equations that are satisfied by any boundary point of ∂M. Afterwards we need to show that every point on the boundary satisfying those linear equations is indeed in $\bar{M}$.

After choosing a homology basis $\{\gamma_1, \ldots, \gamma_d\}$, M can locally near $x_0 \in M$ be written as the zero locus of $k_0 = \operatorname{codim}_{\mathcal{H}(\mu)}(M)$ linear equations. In particular, we can find a matrix $A = (A_{kl})_{kl}$ in reduced row echelon form such that

$$M = \left\{ (X, \omega) \in \mathcal{H}(\mu) : \sum_{l=1}^{d} A_{kl} \int_{\gamma_l} \omega = 0 \text{ for } k = 1, \ldots, k_0 \right\}.$$

Naïvely one would now take the limit of these equations as ω approaches the boundary of $\Xi\mathcal{M}_{g,n}(\mu)$, but the periods $\int_{\gamma_l} \omega$, which are locally holomorphic functions on $\mathcal{H}(\mu)$, cannot be extended holomorphically to the boundary. Firstly, due to monodromy one cannot continuously extend the cycles $[\gamma_l]$ to a whole neighborhood of the boundary and secondly, along a sequence converging to the boundary the period might diverge. In Section 5 we thus study the asymptotic behavior of periods as they approach the boundary of $\Xi\mathcal{M}_{g,n}(\mu)$. The main result of that section, Theorem 5.2, says that after subtracting suitable, explicitly given, multivalued, logarithmic terms, the period $\int_\gamma \omega$ becomes monodromy-invariant and extends holomorphically to $\Xi\mathcal{M}_{g,n}(\mu)$. The resulting extended "periods" are called *log periods*. In Theorem 5.2 we additionally compute the limit of the log periods at the boundary $\partial\Xi\mathcal{M}_{g,n}(\mu)$.

We can now describe our strategy to produce linear equations satisfied on the boundary of M, which is the content of Section 7. Let $b_0 \in \partial M$ be a boundary point of the linear subvariety M contained in an open boundary stratum. We can choose a one-parameter family $f : \Delta \to \bar{M}$ which is generically contained in M and such that $f(0) = b_0$. Since M is a linear subvariety, the linear equations are invariant under the monodromy of the Gauss–Manin connection, and this forces the linear equations to be of a special form; see Proposition 7.6.

The special form of the linear equations together with the explicit formula for the limit of log periods then immediately implies that, at least along one-parameter families, we can take the limit of the linear equations defining M. Thus we get necessary linear equations satisfied on ∂M. The precise statement is Corollary 7.9.

In Section 8 we then show that the linear equations obtained in Section 7 are actually the defining equations for the boundary ∂M intersected with an open boundary stratum. On $\Xi\mathcal{M}_{g,n}(\mu)$ the linear equations for M cannot be extended to the boundary, even if rewritten in log periods, but on a suitable cover of $\Xi\mathcal{M}_{g,n}(\mu)$, which we call the *log period space* LPS, they do extend. The proof of Theorem 1.2 is then obtained by a detailed analysis of the extended linear equations on LPS. For technical reasons, instead of a single cover LPS, we need to consider a countable collection LPS_σ of such. The indexing set corresponds roughly to different monodromies of the periods along one-parameter families.

As a result of the proof of Theorem 1.2 we obtain an explicit formula, in local coordinates, for how obtain linear equations for ∂M intersected with an open boundary stratum, given the linear equations for M. In Section 10 we interpret these results in a coordinate-free way by constructing natural maps in relative homology relating the tangent spaces of the stratum $\mathcal{H}(\mu)$ and the boundary $D_{\overline{\Gamma}}$.

In Section 11 we apply the results of Section 10 to prove Theorem 1.4.

2. Basic setup and notation

2.1. Setup for families

We fix a stratum $\mathcal{H}(\mu)$ of meromorphic differentials with

$$\mu = (\mu_1, \dots, \mu_r, \mu_{r+1}, \dots, \mu_{r+s}) \in \mathbb{Z}^{r+s}, \quad r + s = n,$$

where

$$\sum_{i=1}^{r+s} \mu_i = 2g - 2, \quad \mu_1 \geq \dots \geq \mu_r \geq 0 > \mu_{r+1} \geq \dots \geq \mu_{r+s}.$$

Our setup for families of differentials is as follows. A *family of differentials* $(\pi : \mathcal{X} \to B, \omega, \mathcal{S})$ is a family $(\pi : \mathcal{X} \to B, \mathcal{S})$ of pointed stable curves with sections $\mathcal{S} = (\mathcal{S}_1, \dots, \mathcal{S}_{r+s})$ over the base B, together with a section ω of $\omega_{\mathcal{X}/B}(-\sum_{i=r+1}^{r+s} \mu_i \mathcal{S}_i)$ defined on the complement of the nodes. By abuse of notation we will sometimes denote the family of differentials by ω. For generic $b \in B$ we require $\operatorname{ord}_{\mathcal{S}_i(b)} \omega_b = \mu_i$ and additionally require ω to have no other zeroes or poles outside of the nodes. A family of flat surfaces *of type* μ is a family of differentials where all fibers X_b are smooth and $\omega_b \in \mathcal{H}(\mu)$ for all $b \in B$.

We will often write

$$\mathcal{Z} = (\mathcal{Z}_1 := \mathcal{S}_1, \dots, \mathcal{Z}_r := \mathcal{S}_r), \quad \mathcal{P} = (\mathcal{P}_1 := \mathcal{S}_{r+1}, \dots, \mathcal{P}_s := \mathcal{S}_{r+s})$$

for the *zero* and *pole* sections, respectively.

For equisingular families $(\mathcal{X}, \omega)$ of differentials we let $(\widetilde{\mathcal{X}}, \omega)$ be the associated family which is obtained by fiberwise normalization. Here the differential on $\widetilde{\mathcal{X}}$ is simply the pullback of ω from X and by abuse of notation we denote it also by ω. In this case we let $\mathcal{Q}_e^{\pm}$ be the sections of the preimages of the nodes on $\widetilde{\mathcal{X}}$.

We usually consider families over a smooth base $B = (\Delta^*)^d \times \Delta^e$ for non-negative integers d, e, most of the time arising as the complement of a simple normal crossing divisor. Our convention is that Δ^k is a polydisk in $\mathbb{C}^k$ centered at the origin of sufficiently small radius, to be chosen, and possibly further shrunk.

The moduli space of multi-scale differentials. We now start recalling the moduli space of multi-scale differentials $\Xi\mathcal{M}_{g,n}(\mu)$ and its projectivized version $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ constructed in [2].

The main features of interest for us are:

- $\Xi\mathcal{M}_{g,n}(\mu)$ and $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ are smooth algebraic orbifolds and their respective boundaries $\partial\Xi\mathcal{M}_{g,n}(\mu)$, $\partial\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ are normal crossing divisors;
- the boundary has a modular interpretation in terms of multi-scale differentials and assigned prong-matchings, which we will recall next;
- $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ is compact.

Orbifold structure. The moduli space $\Xi\mathcal{M}_{g,n}(\mu)$ of multi-scale differentials and its projectivization $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ are smooth, algebraic DM-stacks. All our results are true for linear algebraic substacks of $\mathcal{H}(\mu)$. We usually omit the stack structure and only work with the underlying varieties.

2.2. Enhanced level graphs

To describe the boundary of $\Xi\mathcal{M}_{g,n}(\mu)$, we need to add additional decorations to the dual graph of a stable curve. Our setup mostly follows the conventions from [1], where we simplify some conventions to focus on the features that are important to us, avoiding some of the more technical notions. A *level graph* $\overline{\Gamma} = (\Gamma, \ell)$ is a stable graph $\Gamma = (V, E, H)$ with half-edges H corresponding to marked points of the stable curve, together with a total *pre*-order on the vertices V defined by a *level function*

$$\ell : V \to L^\bullet(\overline{\Gamma})$$

where $L^\bullet(\overline{\Gamma}) := \{0, -1, \dots, -\ell(\overline{\Gamma})\}$ is the set of levels. Following the convention of [2] we write $L(\overline{\Gamma}) := L^\bullet(\overline{\Gamma}) \setminus \{0\}$, and refer to it as the set of *lower levels*. An edge is called *horizontal* if it joins vertices of the same level, and *vertical* otherwise. We let $E^{\text{ver}}, E^{\text{hor}} \subseteq E$ be the sets of all vertical and horizontal edges, respectively. An *enhancement* is an assignment of an integer $\kappa_e \geq 0$, called the *number of prongs*, to each edge e, so that $\kappa_e = 0$ if and only if $e \in E^{\text{hor}}$. If an edge e joins the vertices v and v' such that $\ell(v) \geq \ell(v')$ then we let $\ell(e+)$ be the level of v and similarly $\ell(e-)$ the level of v'. Furthermore, we set $v(e+) := v$ and $v(e-) := v'$. At horizontal nodes we make a random choice. Similarly, for a half-edge h we let $v(h)$ be the vertex connected to h, and $\ell(h)$ the level of $v(h)$.

We let $\overline{\Gamma}_{(\leq i)}$ be the restriction of $\overline{\Gamma}$ to levels at most i, i.e. we remove all vertices from $\overline{\Gamma}$ with levels above i and all edges and half-edges adjacent to those vertices. The restrictions $\overline{\Gamma}_{(i)}, \overline{\Gamma}_{(>i)}$ are defined similarly.

For later use we define

$$a_i := \operatorname{lcm}(\kappa_e), \quad m_{e,i} := a_i/\kappa_e \tag{2.1}$$

where the lcm is taken over all edges connecting $\Gamma_{(\leq i)}$ and $\Gamma_{(>i)}$, and $m_{e,i}$ is defined for any edge e such that $\ell(e+) > i \geq \ell(e-)$.

2.3. Stable curves and level graphs

Let $\overline{\Gamma}$ be an enhanced level graph and (X, S) be a stable curve with marked points S and dual graph Γ. Usually we omit the marked points in our notation. We denote by X_v the irreducible component of X corresponding to $v \in V$. Similarly we let $X_{(i)}$ be the subcurve consisting of all irreducible components of level i. We refer to $X_{(0)}$ as the *top level* of X. There are analogous definitions for the subcurve $X_{(\leq i)}$ consisting of components of level $\leq i$, for $X_{(\geq i)}$, and $X_{(>i)}$. For each node e let q_e^+ and q_e^- be the preimages of the node that are contained in $X_{v(e+)}$ and $X_{v(e-)}$, respectively.

Let $S = Z \cup P$ be the marked points, partitioned into marked zeroes and poles.
On the normalization $\widetilde{X}$ of X we define

$$\begin{aligned}
\widetilde{Z} &:= Z \cup \{q_e^+ : e \in E^{\mathrm{ver}}\}, \\
\widetilde{P} &:= P \cup \{q_e^- : e \in E^{\mathrm{ver}}\} \cup \{q_e^\pm : e \in E^{\mathrm{hor}}\}, \\
\widetilde{S} &:= \widetilde{Z} \cup \widetilde{P}.
\end{aligned}$$

We denote by $\widetilde{X}_{(i)}$ the normalization of $X_{(i)}$ and consider it as a possibly disconnected curve with marked points $\widetilde{S}_{(i)}$ where

$$\widetilde{Z}_{(i)} := \widetilde{Z} \cap X_{(i)}, \quad \widetilde{P}_{(i)} := \widetilde{P} \cap X_{(i)}, \quad \widetilde{S}_{(i)} := \widetilde{S} \cap X_{(i)}. \tag{2.2}$$

We define $\widetilde{Z}_v, \widetilde{P}_v, \widetilde{S}_v$ on the normalization $\widetilde{X}_v$ of X_v analogously.

2.4. *Multi-scale differentials*

The boundary of $\Xi\mathcal{M}_{g,n}(\mu)$ can be described in terms of multi-scale differentials. A *multi-scale differential* $(X, S, \boldsymbol{\eta})$ compatible with an enhanced dual graph $\overline{\Gamma}$ is a stable curve (X, S) and a collection $\boldsymbol{\eta} = (\eta_v)_{v \in V}$ of meromorphic differentials on the normalization $\widetilde{X}_v$ of each irreducible component X_v satisfying

- **(Prescribed vanishing)** Each differential η_v is non-zero, and has no zeroes or poles outside $\widetilde{S}_v$. Moreover, the order of vanishing at the marked point S_k is μ_k.
- **(Matching orders)** For every node e we have

$$\begin{aligned}
\operatorname{ord}_{q_e^+} \eta_{v(e+)} &= \kappa_e - 1, \\
\operatorname{ord}_{q_e^-} \eta_{v(e-)} &= -\kappa_e - 1.
\end{aligned}$$

- **(Matching residues at horizontal nodes)** At horizontal nodes $e \in E^{\mathrm{hor}}$, we have

$$\operatorname{res}_{q_e^+} \eta_{v(e+)} + \operatorname{res}_{q_e^-} \eta_{v(e-)} = 0.$$

- **(Global residue condition)** For every level i and every connected component Y of $X_{(>i)}$ that does not contain a marked point with a prescribed pole, i.e. such that $P \cap Y = \emptyset$, the following condition holds. Let $\{e_1, \ldots, e_b\}$ denote the set of all nodes where Y intersects $X_{(i)}$. Then

$$\sum_{j=1}^{b} \operatorname{res}_{q_{e_j}^-} \eta_{v(e_j-)} = 0.$$

Instead of grouping the differentials by irreducible components, it is often useful to group them level by level. In this case we write $\boldsymbol{\eta} = (\eta_{(i)})_{i \in L^\bullet(\overline{\Gamma})}$. We usually omit the marked points S in the notation, since they are already encoded as the zeroes and poles of $\boldsymbol{\eta}$ away from the nodes.

2.5. *The structure of the boundary*

The boundary components of the moduli space of multi-scale differentials $\Xi\mathcal{M}_{g,n}(\mu)$ are indexed by the discrete data of enhanced level graphs. The *open* boundary stratum corresponding to the enhanced level graph $\overline{\Gamma}$ is denoted by $D_{\overline{\Gamma}}$. A point of $D_{\overline{\Gamma}} \subseteq \partial\Xi\mathcal{M}_{g,n}(\mu)$ corresponds to a pair $(X, \boldsymbol{\eta})$ where X is a stable curve with dual graph Γ, and $\boldsymbol{\eta}$ is a multi-scale differential compatible with $\overline{\Gamma}$. Additionally there needs to be a choice of a prong-matching at every vertical node. Since we only work locally, we do not need to keep track of the prong-matching, and refer to [2, Section 5.4] for a proper discussion.

Two multi-scale differentials $(X, \boldsymbol{\eta})$ and $(X', \boldsymbol{\eta}')$ correspond to the same boundary point of $D_{\overline{\Gamma}}$ if they are related by the action of the level-rotation torus which acts simultaneously on the different levels by rescaling and on the prong-matchings; we refer to [2, Section 6] for the precise definitions. For our purposes we can again mostly ignore the action: near a boundary point $(X, \boldsymbol{\eta})$ with a chosen prong-matching, the boundary component $D_{\overline{\Gamma}}$ can be parametrized by a small neighborhood in the space of multi-scale differentials compatible with $\overline{\Gamma}$, considered up to scaling each differential $\boldsymbol{\eta}_{(i)}$ on lower levels by an arbitrary non-zero complex number, one complex number for each level.

Remark 2.1. Suppose $(X, \boldsymbol{\eta})$ and $(X, \boldsymbol{\eta}')$ are two multi-scale differentials with the same underlying differential but different prong-matchings. Since $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ have the same periods it seems interesting to ask: if $(X, \boldsymbol{\eta})$ is contained in the boundary of a linear subvariety, is the same true for $(X, \boldsymbol{\eta}')$ and furthermore are the linear equations the same? Our methods are purely local inside the moduli space of multi-scale differentials, i.e. they only allow us to describe the linear subvariety in a small neighborhood of $(X, \boldsymbol{\eta})$ which might not contain $(X, \boldsymbol{\eta}')$, and thus do not seem to be applicable to this question.

2.6. *Local coordinates on the boundary*

It is classically known that the stratum $\mathcal{H}(\mu)$ has local coordinates given by the relative cohomology $H^1(X \setminus P, Z; \mathbb{C})$. The boundary stratum $D_{\overline{\Gamma}}$ has a similar local description, which we now discuss.

The prescribed vanishing and matching orders conditions for multi-scale differentials imply that a multi-scale differential $\boldsymbol{\eta}$ is contained in the product of strata $\prod_{v\in V} \mathcal{H}(\mu_v)$, where each μ_v is completely determined by μ and the enhanced level graph $\overline{\Gamma}$. Thus the space of multi-scale differentials, i.e. unprojectivized and without a choice of prong-matchings, can be identified with the subspace $\prod_{v\in V} \mathcal{H}(\mu_v)^{\mathrm{GRC}}$ of $\prod_{v\in V} \mathcal{H}(\mu_v)$, constrained by the matching residues at horizontal nodes, as well as the global residue conditions. To describe the boundary component $D_{\overline{\Gamma}}$, we need to additionally projectivize the differential on lower levels, and choose the prong-matchings. This causes the stratum $D_{\overline{\Gamma}}$ to be a cover of $\prod_{v\in V} \mathcal{H}(\mu_v)^{\mathrm{GRC}}$, suitably projectivized. We can use this to describe local coordinates on $D_{\overline{\Gamma}}$. For every level i we set

$$\mathrm{H}^1_{(i)}(X; \mathbb{Z}) := H^1(\widetilde{X}_{(i)} \setminus \widetilde{P}_{(i)}, \widetilde{Z}_{(i)}; \mathbb{Z}), \tag{2.3}$$

where $\widetilde{P}_{(i)}, \widetilde{Z}_{(i)}$ are defined in (2.2). We sometimes simply write $H^1_{(i)}(X)$ instead of $H^1_{(i)}(X;\mathbb{Z})$ and also write $H^1_{(i)}(X;\mathbb{C}) := H^1_{(i)}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. We additionally denote by $H^1_{(i)}(X)^{\mathrm{GRC}} \subseteq H^1_{(i)}(X)$ the subspace satisfying the global residue and matching residue conditions at horizontal nodes. We will revisit the global residue condition in Section 4.2. See in particular (4.3) for an explicit definition of $H^1_{(i)}(X)^{\mathrm{GRC}}$. The boundary stratum $D_{\overline{\Gamma}}$ then has *local projective coordinates* given by

$$H^1(X,\overline{\Gamma}) := H^1_{(0)}(X;\mathbb{C}) \times \prod_{i \in L(\overline{\Gamma})} \mathbb{P}(H^1_{(i)}(X;\mathbb{C})^{\mathrm{GRC}}).$$

By this we mean that, after choosing local coordinates on each projective space, we get local coordinates on $D_{\overline{\Gamma}}$. Note that this statement is only meaningful because the transition functions in those coordinates are given by projective linear maps. We will discuss the transition functions in more detail in Section 2.7. We refer to those coordinates as *generalized period coordinates*. Similarly, the boundary $\mathbb{P}D_{\overline{\Gamma}}$ of $\mathbb{P}\,\Xi\mathcal{M}_{g,n}(\mu)$ has projective local coordinates given by $\mathbb{P}(H^1(X,\overline{\Gamma}))$, where additionally the homology $H^1_{(0)}(X)$ of the top level is projectivized.

Let $U \subseteq D_{\overline{\Gamma}}$ be such a generalized period chart centered at $b_0 = (X_{b_0}, \boldsymbol{\eta}_{b_0})$. Then over U there exists an equisingular family

$$(\mathcal{X} \to U, \boldsymbol{\eta}) \tag{2.4}$$

of stable curves with dual graph Γ, where η is the multi-scale differential determined by generalized period coordinates. We define $(\mathcal{X}_{(i)} \to U, \boldsymbol{\eta}_{(i)})$ to be the potentially disconnected family consisting only of irreducible components of level i. From time to time it will be useful to consider the fiberwise normalization

$$(\tilde{\mathcal{X}} \to U, \boldsymbol{\eta}), \tag{2.5}$$

which is a family of smooth, possibly disconnected, Riemann surfaces, where we make a choice of marking of the preimages of all nodes. Notice that while a point in U only parametrizes an equivalence class of multi-scale differentials, choosing local charts on each projective space $\mathbb{P}(H^1_{(i)}(X)^{\mathrm{GRC}})$ allows us to choose for each $u \in U$ a representative (X_u, η_u), varying holomorphically in u.

Convention 2.2. From now on, $b_0 \in D_{\overline{\Gamma}}$ will denote a boundary point, chosen once and for all, in a neighborhood of which in $\Xi\mathcal{M}_{g,n}(\mu)$ we will perform all of our constructions and computations. We will usually write $(X, \boldsymbol{\eta})$ instead of $(X_{b_0}, \boldsymbol{\eta}_{b_0})$. Furthermore, from now on, X always denotes a stable curve contained in $D_{\overline{\Gamma}}$ and Σ a smooth curve in $\mathcal{H}(\mu)$.

2.7. *The linear structure of the boundary*

After choosing a homology basis on $X_{(i)}$ for each i, the changes of coordinates for $D_{\overline{\Gamma}}$ are given by linear transformations in $\mathrm{GL}(\overline{\Gamma}) := \mathrm{GL}(d_1, \mathbb{Z}) \times \prod_{i \in L(\overline{\Gamma})} \mathrm{PGL}(d_i, \mathbb{Z})$ where $d_i := \dim H^1_{(i)}(X)^{\mathrm{GRC}}$. Thus $D_{\overline{\Gamma}}$ possesses a $\mathrm{GL}(\overline{\Gamma})$-structure or what we call a *levelwise linear structure*. We call a subvariety of $D_{\overline{\Gamma}}$ *levelwise linear* if locally in generalized period coordinates it equals a finite union of products

$$V_0 \times \prod_{i \in L(\overline{\Gamma})} \mathbb{P}(V_i) \subseteq \mathrm{H}^1(X, \overline{\Gamma})$$

where $V_i \subseteq \mathrm{H}^1_{(i)}(X; \mathbb{C})$ are some linear subspaces. Since on $\mathbb{P}D_{\overline{\Gamma}}$ also the top level is projectivized, $\mathbb{P}D_{\overline{\Gamma}}$ admits a $\prod_{i \in L^\bullet(\overline{\Gamma})} \mathrm{PGL}(d_i, \mathbb{Z})$-structure, i.e. coordinate changes live in $\prod_{i \in L^\bullet(\overline{\Gamma})} \mathrm{PGL}(d_i, \mathbb{Z})$, and a subvariety is *levelwise linear* if locally it is given by $\prod_{i \in L^\bullet(\overline{\Gamma})} \mathbb{P}(V_i)$. Note that if a subvariety $M \subseteq D_{\overline{\Gamma}}$ is levelwise linear, the same is true for its image $\mathbb{P}M \subseteq \mathbb{P}D_{\overline{\Gamma}}$. We can now give a more precise, though still qualitative, version of our main result, Theorem 1.2.

Theorem 2.3 (Main theorem, levelwise version). *Let $M \subseteq \mathcal{H}(\mu)$ be a linear subvariety. For each open boundary component $D_{\overline{\Gamma}} \subseteq \Xi\mathcal{M}_{g,n}(\mu)$ the intersection $\partial M \cap D_{\overline{\Gamma}}$ is a levelwise linear subvariety of $D_{\overline{\Gamma}}$. The same is true for the projectivization $\mathbb{P}M \subseteq \mathbb{P}\mathcal{H}(\mu)$.*

We stress that this statement already greatly restricts the possible linear equations of $\partial M \cap D_{\overline{\Gamma}}$, since each linear equation only involves periods contained in the same level.

In Section 10 we will describe how to relate the levelwise linear structure on the boundary stratum $D_{\overline{\Gamma}}$ to the linear structure on the stratum $\mathcal{H}(\mu)$.

Remark 2.4. We stress that all levelwise projectivizations above are taken with respect to the standard action of $\mathbb{C}^{L(\overline{\Gamma})}$ on $\prod_{i \in L(\overline{\Gamma})} H^1_{(i)}(X)$, not to be confused with the triangular action which will be introduced in (2.10), following [2, (11.1)].

2.8. *The model domain*

We now recall the local structure of the moduli space of multi-scale differentials near the open boundary stratum $D_{\overline{\Gamma}}$. In [2, Section 8], the authors first introduce an auxiliary space, the model domain, and then show later in [2, Section 10] that it is locally biholomorphic to $\Xi\mathcal{M}_{g,n}(\mu)$. Local coordinates of the model domain $\Xi\mathcal{M}_{g,n}(\mu)$ near $b_0 \in D_{\overline{\Gamma}}$ can be given by

$$B := U \times \Delta^{\ell(\overline{\Gamma})-1} \times \Delta^{|E^{\mathrm{hor}}|} \tag{2.6}$$

where $U \subseteq D_{\overline{\Gamma}}$ denotes a generalized period chart.

Convention 2.5. From now on, unless stated otherwise, $U \subseteq D_{\overline{\Gamma}}$ will always refer to a generalized period chart in $D_{\overline{\Gamma}}$ centered at b_0, which we allow to be further shrunk as needed. Furthermore, we often implicitly identify U with $U \times (0, \ldots, 0) \subseteq B$.

We call B the *local model domain* and denote its coordinates by $b = (\boldsymbol{\eta}, t, h)$ with *scaling parameters* $t = (t_i)_{i \in L(\overline{\Gamma})}$, *horizontal node parameters* $h = (h_e)_{e \in E^{\mathrm{hor}}}$, and $\boldsymbol{\eta}$ a multi-scale differential. We omit the stable curve X from the notation.

Notation 2.6. Throughout the text we denote

$$N := \ell(\overline{\Gamma}) - 1 + |E^{\mathrm{hor}}|, \quad M := \dim U = \dim D_{\overline{\Gamma}}. \tag{2.7}$$

In particular, $\dim B = N + M$.

Note that our notation here differs from [2] where N denotes the number of levels, not including the count of horizontal nodes. We recall that we denote the number of levels by $\ell(\overline{\Gamma})$.

Let $p : B \to U$ denote the projection onto the first factor. On B we consider the pullback family $(p^*\mathcal{X} \to B, \eta)$ where $(\mathcal{X} \to U)$ is the family from (2.4), which we call the *model family*. We usually omit the projection map p and only write

$$(\mathcal{X} \to B, \boldsymbol{\eta}) \tag{2.8}$$

for the model family.

We will explain the role of the parameters t_i and h_e more precisely in Section 2.9. For the moment we only define, following [2],

$$t_{\lceil i \rceil} := \prod_{k=i}^{-1} t_k^{a_k}\,, \tag{2.9}$$

where the exponents a_k are defined in (2.1), and define the triangular action of t on $\boldsymbol{\eta}$ by

$$t \star \boldsymbol{\eta} := (t_{\lceil i \rceil} \boldsymbol{\eta}_{(i)})_{i \in L^{\bullet}(\overline{\Gamma})}. \tag{2.10}$$

We also define the *plumbing parameters*

$$s_e := \begin{cases} \prod_{i=\ell(e-)}^{\ell(e+)-1} t_i^{m_{e,i}} & \text{at vertical nodes,} \\ h_e & \text{at horizontal nodes.} \end{cases} \tag{2.11}$$

where the exponents $m_{e,i}$ are defined in (2.1). Note that at vertical nodes we have the relation

$$t_{\lceil \ell(e-) \rceil} = s_e^{\kappa_e} t_{\lceil \ell(e+) \rceil}. \tag{2.12}$$

We define the (local) *boundary* $D \subseteq B$ as the normal crossing divisor given by the equations

$$D := \Big\{ \prod_{i \in L(\overline{\Gamma})} t_i \cdot \prod_{e \in E^{\mathrm{hor}}} h_e = 0 \Big\}. \tag{2.13}$$

The boundary component $U \simeq D_{\overline{\Gamma}} \cap B = U \times (0, \ldots, 0) \subseteq D$ is called *the most degenerate boundary stratum*, while the complement $D \setminus D_{\overline{\Gamma}}$ corresponds to *partial undegenerations* of $\overline{\Gamma}$. We will not need the precise definition of undegenerations, and instead refer the reader to [2].

2.9. *The universal family of multi-scale differentials*

In [2, Section 10] the authors use plumbing to construct the *universal family* $(\mathcal{Y} \to B, \omega)$ *of multi-scale differentials* over the base B defined in (2.6). We refer the reader to [2, Section 11] for the precise definition of families of multi-scale differentials. For our purposes we only need the following properties of the universal family $\mathcal{Y}$:

(1) For any $b \in B \setminus D$ the differential ω_b is a flat surface in the stratum $\mathcal{H}(\mu)$.

(2) On the most degenerate stratum, i.e. for any $b \in U \times (0, \dots, 0)$, the differential ω_b is a multi-scale differential in $D_{\overline{\Gamma}}$.

(3) There exist families of unions of disks $\widetilde{\mathcal{U}} \subseteq \mathcal{Y}$, $\mathcal{U} \subseteq \mathcal{X}$ containing $\widetilde{S}$, and a biholomorphism

$$\Psi : \mathcal{Y} \setminus \widetilde{\mathcal{U}} \simeq \mathcal{X} \setminus \mathcal{U}. \tag{2.14}$$

Near a marked point, $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ are homeomorphic to a disk, while at nodes they are homeomorphic to a union of two disks intersecting at the node.

(4) Suppose $K \subseteq \mathcal{Y} \setminus \widetilde{\mathcal{U}}$ is compact and $\Psi(K) \subseteq \mathcal{X}_{(i)}$. Then

$$\lim_{t,h \to 0} \frac{1}{t_{\lceil i \rceil}} \omega(b)|_K = \boldsymbol{\eta}_{(i)}$$

uniformly, where $b = (\boldsymbol{\eta}, t, h)$ and in the limit all t_i and h_e go to zero. In other words, as b approaches a boundary point, $\eta \in D_{\overline{\Gamma}}$, on the i-level $\omega_{(i)}(b)$, rescaled by $t_{\lceil i \rceil}$, converges uniformly to $\boldsymbol{\eta}_{(i)}$, away from the nodes and marked points.

(5) Along the most degenerate boundary stratum $D_{\overline{\Gamma}}$ the map Ψ extends to an isomorphism

$$\mathcal{Y}|_{D_{\overline{\Gamma}}} \simeq \mathcal{X}|_{D_{\overline{\Gamma}}}.$$

3. Constructing the universal family of $\Xi\mathcal{M}_{g,n}(\mu)$

In this section we outline the construction of the universal family $\mathcal{Y}$. We follow [2, Section 10] in notation and setup, but we only highlight the features of the construction necessary for our discussion.

3.1. *Modification differentials*

For a multi-scale differential, the residues match at horizontal nodes, while at vertical nodes the multi-scale differential is holomorphic at q_e^+ and has a pole at q_e^-. On the other hand, for the plumbing construction in Section 3.2 it will be important to have differentials with matching residues at every node. The solution, as found in [1], is as follows. It is a consequence of the global residue condition that we can add a "small" differential ξ to $\boldsymbol{\eta}$ such that the residues of $t \star \boldsymbol{\eta} + \xi$ match at all nodes. The precise definition is as follows (see also [2, Def. 9.1]). A family of *modifying differentials* ξ for the model family $(\mathcal{X} \to B, \boldsymbol{\eta})$ is a family of meromorphic differentials $(\mathcal{X} \to B, \xi)$ with $\xi = (\xi_v)_{v \in V}$ such that

- ξ is holomorphic except for possible simple poles along nodal and polar sections of $\boldsymbol{\eta}$, and we allow ξ to have residues at horizontal nodes;
- $\xi_{(i)}$ is divisible by $t_{\lceil i-1 \rceil}$ for each $i \in L(\overline{\Gamma})$, and $\xi_{(-\ell(\overline{\Gamma}))} \equiv 0$;
- $t \star \boldsymbol{\eta} + \xi$ has matching residues at all nodes.

3.2. *Plumbing setup*

In [2, Section 10] the authors introduced a plumbing setup for multi-scale differentials which we will now recall and use subsequently. We will work on a polydisk $B_\varepsilon \subseteq B$ of radius $\varepsilon = \varepsilon(b_0) > 0$. We define the standard annulus in $\mathbb{C}$:

$$A_{\delta_1,\delta_2} := \{\delta_1 < |z| < \delta_2\}.$$

For $\delta = \delta(b_0) > 0$ to be determined, we define the *standard plumbing fixture* to be

$$\mathbb{V}_e := \{(b,u,v) \in B_\varepsilon \times \Delta_\delta^2 : uv = s_e(b)\}$$

where $s_e(b)$ is defined by (2.11). We consider $\mathbb{V}_e \to B_\varepsilon$ as a family over B_ε with fibers $(\mathbb{V}_e)_b$. We equip $\mathbb{V}_e$ with the relative one-form Ω_e given by

$$\Omega_e := (t_{\lceil \ell(e+)\rceil} u^{\kappa_e} - r'_e)\frac{du}{u} = -(t_{\lceil \ell(e-)\rceil} v^{-\kappa_e} + r'_e)\frac{dv}{v}$$

with residue r'_e to be determined. We also consider the families of disjoint annuli $\mathcal{A}_e^+, \mathcal{A}_e^- \subseteq \mathbb{V}_e$ given by

$$\mathcal{A}_e^+ := \{(b,u,v) : \delta/R < |u| < \delta\}, \quad \mathcal{A}_e^- := \{(b,u,v) : \delta/R < |v| < \delta\},$$

for some constant $R > 0$.

Definition 3.1. For $b \in B \setminus D$, we define the *vanishing cycle* $\lambda_e \subseteq (\mathbb{V}_e)_b$ to be the standard generator of the fundamental group of the annulus $(\mathbb{V}_e)_b$ in u-coordinates, represented by a path encircling the origin once with counterclockwise orientation.

Our convention for the orientation on λ_e has the following interpretation: If one chooses a tangent vector $\vec{v}$ on the curve pointing from lower levels to higher levels, then the frame $(\vec{u}, \vec{v})$ is positively oriented where $\vec{u}$ is the tangent vector along λ_e, as seen in Figure 1. At horizontal vanishing cycles the orientation depends on the random choice of $e+$ and $e-$.

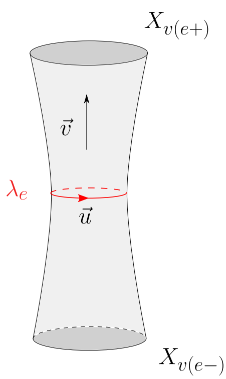


Fig. 1. Orientation on vanishing cycles.

We stress that we consider λ_e as an actual path and not just a homology class. If no confusion is possible we will not distinguish between λ_e and its class.

For each marked zero $\mathcal{Z}_k$ of order m_k, we define a family of disks, equipped with a relative one-form $\Omega_{\mathcal{Z}_k}$, by

$$\mathbb{D}_{\mathcal{Z}_k} := B_\varepsilon \times \Delta_\delta, \quad \Omega_{\mathcal{Z}_k} := z^{m_k} dz.$$

We define a family of annuli $\mathcal{A}_{\mathcal{Z}_k} \subseteq \mathbb{D}_{\mathcal{Z}_k}$ by

$$\mathcal{A}_{\mathcal{Z}_k} := B_\varepsilon \times A_{\delta/R,\delta}.$$

3.3. *Standard form coordinates for multi-scale differentials*

The idea of the plumbing construction for the universal family $\mathcal{Y}$ is to find local coordinates near marked zeroes and nodal sections in which the families $(\mathcal{X}, t \star \eta)$ and $(\mathcal{X}, t \star \eta + \xi)$ have a simple form. The difference between the two families is that the order of vanishing at the nodes and marked points is constant for the first family $(\mathcal{X}, t \star \eta)$ but it can jump for the second family $(\mathcal{X}, t \star \eta + \xi)$. In [2] the authors introduce the following solution. For the family $(\mathcal{X}, t \star \eta)$ we can find local coordinates near each nodal section and each marked point, in which the differential has a simple form, while for the second family $(\mathcal{X}, t \star \eta + \xi)$ this is only possible on an annulus such that the disk bounded by it contains the marked zeroes and the nodes. The results are as follows. We begin with the family $(\mathcal{X}, t \star \eta)$.

Theorem 3.2 ([2, Thm. 4.1]). *There exists a constant $\delta_1 > 0$ such that for each edge e and for each marked zero $\mathcal{Z}_k$ of Γ there are families of conformal maps of disks*

$$\phi_e^+ : B_\varepsilon \times \Delta_{\delta_1} \to \mathcal{X}_{\ell(e+)}, \quad \phi_e^- : B_\varepsilon \times \Delta_{\delta_1} \to \mathcal{X}_{\ell(e-)}, \quad \phi_{\mathcal{Z}_k} : B_\varepsilon \times \Delta_{\delta_1} \to \mathcal{X}_{\ell(\mathcal{Z}_k)}$$

such that the following properties are satisfied:

(1) *The restrictions of these maps to $B_\varepsilon \times 0$ coincide with the nodal sections $\mathcal{Q}_{e^+}, \mathcal{Q}_{e^-}$ and the marked sections $\mathcal{Z}_k$, respectively.*

(2) *The pullback of $t \star \eta$ has standard form, that is,*

$$(\phi_e^+)^*(t \star \eta) = t_{\lceil \ell(e+) \rceil}\big(z^{\kappa_e} - \mathrm{res}_{q_e^-}(t \star \eta)\big)\frac{dz}{z},$$
$$(\phi_e^-)^*(t \star \eta) = -t_{\lceil \ell(e-) \rceil}\big(z^{-\kappa_e} + \mathrm{res}_{q_e^-}(t \star \eta)\big)\frac{dz}{z},$$
$$(\phi_{\mathcal{Z}_k})^*(t \star \eta) = t_{\lceil \ell(\mathcal{Z}_k) \rceil} z^{m_k} dz.$$

The next result concerns the family $(\mathcal{X}, t \star \eta + \xi)$.

Theorem 3.3 ([2, Thm. 10.4]). *For any $R > 1$, there exist constants $\varepsilon, \delta > 0$ such that for each edge e and each marked zero $\mathcal{Z}_k$ of Γ there are families of conformal maps of annuli*

$$\upsilon_e^+ : B_\varepsilon \times A_{\delta/R,\delta} \to \mathcal{X}_{\ell(e+)}, \quad \upsilon_e^- : B_\varepsilon \times A_{\delta/R,\delta} \to \mathcal{X}_{\ell(e-)}, \quad \upsilon_{\mathcal{Z}_k} : B_\varepsilon \times A_{\delta/R,\delta} \to \mathcal{X}_{\ell(\mathcal{Z}_k)}$$

such that the following properties are satisfied:

(1) *The images of* $\upsilon_e^+, \upsilon_e^-, \upsilon_{\mathcal{Z}_k}$ *are families of annuli* $\mathcal{B}_e^+, \mathcal{B}_e^-, \mathcal{B}_{\mathcal{Z}_k}$ *not containing any zeroes of* $(\mathcal{X}, t \star \boldsymbol{\eta} + \xi)$. *The families of annuli bound families of unions of disks* $\mathcal{U}_e^+, \mathcal{U}_e^-, \mathcal{U}_{\mathcal{Z}_k}$ *containing the nodal sections* $\mathcal{Q}_e^+$, $\mathcal{Q}_e^-$ *and the section* $\mathcal{Z}_k$, *respectively.*

(2) *The pullback of* $t \star \boldsymbol{\eta} + \xi$ *has standard form, that is,*

$$(\upsilon_e^+)^*(t \star \boldsymbol{\eta} + \xi) = t_{\lceil \ell(e+) \rceil}\big(z^{\kappa_e} - \operatorname{res}_{q_e^-}(t \star \boldsymbol{\eta} + \xi)\big)\frac{dz}{z},$$
$$(\upsilon_e^-)^*(t \star \boldsymbol{\eta} + \xi) = -t_{\lceil \ell(e-) \rceil}\big(z^{-\kappa_e} + \operatorname{res}_{q_e^-}(t \star \boldsymbol{\eta} + \xi)\big)\frac{dz}{z},$$
$$(\upsilon_{\mathcal{Z}_k})^*(t \star \boldsymbol{\eta} + \xi) = t_{\lceil \ell(\mathcal{Z}_k) \rceil} z^{m_k}\, dz.$$

(3) *The holomorphic maps* $\upsilon_e^+, \upsilon_e^-, \upsilon_{\mathcal{Z}_k}$ *agree with the corresponding maps* $\phi_e^+, \phi_e^-, \phi_{\mathcal{Z}_k}$ *on the most degenerate boundary stratum, i.e. on* $(U \times (0, \dots, 0)) \times A_{\delta/R,\delta}$.

Definition 3.4. By a slight abuse of notation, we refer to the family of coordinates given by ϕ_e^+ as ϕ_e^+*-coordinates*, and similarly for $\phi_e^-, \phi_h, \upsilon_e^+, \upsilon_e^-, \upsilon_{\mathcal{Z}_k}$. If we do not want to specify whether we refer to a preimage of a node or a marked point, we simply write ϕ or υ.

The maps $\upsilon_e^+, \upsilon_e^-, \upsilon_{\mathcal{Z}_k}$ are not determined uniquely. Following [2], note that the maps can be specified uniquely by choosing base points near the marked zeroes and nodes. We choose sections $\varsigma_e^+, \varsigma_e^-, \varsigma_{\mathcal{Z}_k} : B \to \mathcal{Y}$ such that the image is contained in a chart centered at q_e^+, q_e^-, Z_k respectively. Fix $p_0 := \delta/\sqrt{R} \in A_{\delta/R,\delta}$ as the base point of the annulus. Then by [2, Thm. 4.1] there exist unique $\upsilon_e^+, \upsilon_e^-, \upsilon_h$ such that

$$\upsilon_e^+(b, p_0) = \varsigma_{e+}(b), \quad \upsilon_e^-(b, p_0) = \varsigma_{e-}(b), \quad \upsilon_{\mathcal{Z}_k}(b, p_0) = \varsigma_{\mathcal{Z}_k}(b),$$

for any $b \in U \times (0, \dots, 0)$.

Convention 3.5. From now on, we fix once and for all a choice of nearby sections ς.

3.4. The plumbing construction

For each node e and each marked zero $\mathcal{Z}_k$ we define conformal isomorphisms $\Upsilon_e^\pm : \mathcal{A}_e^\pm \to \mathcal{B}_e^\pm$ and $\Upsilon_{\mathcal{Z}_k} : \mathcal{A}_{\mathcal{Z}_k} \to \mathcal{B}_{\mathcal{Z}_k}$ by

$$\Upsilon_e^+(b, u, v) := \upsilon_e^+(b, u), \quad \Upsilon_e^-(b, u, v) := \upsilon_e^-(b, v), \quad \Upsilon_{\mathcal{Z}_k}(b, z) := \upsilon_{\mathcal{Z}_k}(b, z).$$

We define $\mathcal{Y}$ to be the family obtained by removing the disks $\mathcal{U}_e^\pm$, $\mathcal{U}_{\mathcal{Z}_k}$ from $\mathcal{X}$ and attaching $\mathbb{V}_e$ and $\mathbb{D}_{\mathcal{Z}_k}$ by identifying the $\mathcal{A}$- and $\mathcal{B}$-annuli via the Υ-gluing maps. Since $t \star \boldsymbol{\eta} + \xi$ and $\Omega^\pm, \Omega_{\mathcal{Z}_k}$ are identified via Υ, the family $\mathcal{Y}$ inherits a relative one-form ω.

We denote by $\mathcal{U}$ the union of $\mathcal{U}_e^+, \mathcal{U}_e^-, \mathcal{U}_{\mathcal{Z}_k}$ over all nodes and marked points, and similarly we denote by $\widetilde{\mathcal{U}}$ the union of the families of disks $B_\varepsilon \times A_{\delta/R,\delta} \subseteq \mathcal{Y}$ over all marked points and nodes. The families $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ are exactly the families of disks from Section 2.9, (3).

We have thus locally described the universal family $(\mathcal{Y} \to B, \omega)$. It will be needed to compare the periods of ω and the limit multi-scale differential $\boldsymbol{\eta}$ in Theorem 5.2. We will in particular need the particular form of υ-coordinates to analyze what happens in a neighborhood of the nodes.

4. Level filtrations

In this section we introduce various notions of level for paths and homology classes. This will be necessary since the asymptotics of periods $\int_\gamma \omega$ are governed by the level of γ.

We now introduce the setup for the rest of this section. We let Σ be a topological surface homeomorphic to surfaces in $\mathcal{H}(\mu)$, which is obtained from a nodal Riemann surface using the plumbing construction from Section 3. In Definition 3.1 we have defined the vanishing cycles on an annulus. Using the plumbing maps from Theorem 3.3 we can pull back λ_e from the annulus to Σ. By abuse of notation we denote the resulting curves again by λ_e. We define $\Lambda := \{\lambda_e : e \in E\} \subseteq \Sigma$ considered as a multicurve, and then topologically a stable curve in $D_{\overline{\Gamma}}$ is obtained by pinching Λ. As before, we fix a stable curve $X \in D_{\overline{\Gamma}}$.

4.1. *Thickenings of vanishing cycles*

For each vanishing cycle λ_e, let λ_e° be a small open neighborhood of λ_e that deformation retracts onto λ_e, and denote by $\Lambda^\circ \subseteq \Sigma$ the union of all such thickenings.

We decompose

$$\Lambda = \Lambda^{\mathrm{ver}} \sqcup \Lambda^{\mathrm{hor}}$$

into vanishing cycles corresponding to vertical and horizontal nodes, respectively, and further decompose

$$\Lambda^{\mathrm{ver}} = \bigsqcup_{i \in L^\bullet(\overline{\Gamma})} \Lambda^{\mathrm{ver}}_{(i)}, \quad \Lambda^{\mathrm{hor}} = \bigsqcup_{i \in L^\bullet(\overline{\Gamma})} \Lambda^{\mathrm{hor}}_{(i)}$$

where

$$\Lambda^{\mathrm{ver}}_{(i)} := \{\lambda_e : e \in E^{\mathrm{ver}},\ \ell(e+) = i,\ \ell(e-) < i\},$$
$$\Lambda^{\mathrm{hor}}_{(i)} := \{\lambda_e : e \in E^{\mathrm{hor}},\ \ell(e+) = \ell(e-) = i\}.$$

In words, $\Lambda^{\mathrm{ver}}_{(i)}$ consists of vertical vanishing cycles connecting $\Sigma_{(i)}$ to lower levels and $\Lambda^{\mathrm{hor}}_{(i)}$ consists of horizontal vanishing cycles contained in $\Sigma_{(i)}$.

For each edge $e \in E$ the boundary $\partial\lambda_e^\circ$ consists of two boundary circles, $\lambda_e^+ \sqcup \lambda_e^-$, with $\lambda_e^+ \subset \Sigma_{(\ell(e+))}$ and $\lambda_e^- \subset \Sigma_{(\ell(e-))}$. At horizontal nodes we randomly choose which boundary component is denoted λ_e^+. We need to be careful about choosing orientations for $\lambda_e^\pm$. Our convention is that λ_e^+ has the same orientation as λ_e, while λ_e^- has the opposite orientation. We write

$$\Lambda^+ := \{\lambda_e^+ : e \in E^{\mathrm{ver}}\}, \quad \Lambda^- := \{\lambda_e^- : e \in E^{\mathrm{ver}}\} \sqcup \{\lambda_e^\pm : e \in E^{\mathrm{hor}}\}.$$

We define analogues of $\Lambda_{(i)}$ and Λ^{ver} for Λ° and $\Lambda^\pm$. For example, $\Lambda^{+,\mathrm{ver}}_{(i)}$ consists of λ^+_e for all vertical nodes e with $\ell(e+) = i$ and $\ell(e-) < i$, while $\Lambda^{\circ,\mathrm{ver}}_{(>i)}$ consists of λ°_e for all vertical nodes with $\ell(e+) > i$ and $\ell(e-) \le i$.

Figure 2 illustrates the definitions.

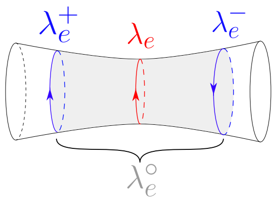


Fig. 2. Thickenings of vanishing cycles.

We now introduce several different ways of filtering the Riemann surface Σ by levels. If we remove all vertical vanishing cycles from X, then the remaining surface decomposes into a disjoint union of surfaces of a fixed level, i.e.

$$\Sigma \setminus \Lambda^{\circ,\mathrm{ver}} = \bigsqcup_{i \in L^\bullet(\overline{\Gamma})} \Sigma_{(i)}.$$

Each of the resulting subcurves $\Sigma_{(i)}$ is a connected compact surface potentially with boundary. We note that we remove $\Lambda^{\circ,\mathrm{ver}}$ instead of just Λ^{ver} since we want the result to be compact. If instead of removing all vertical vanishing cycles, we only remove vertical vanishing cycles that cross the i-th level transition, i.e. exactly the vanishing cycles in $\Lambda^{\circ,\mathrm{ver}}_{(\ge i)}$, then we decompose Σ into two surfaces: the part of Σ that is at least of level i and the part of Σ below level i. We write

$$\Sigma \setminus \Lambda^{\circ,\mathrm{ver}}_{(\ge i)} =: \Sigma_{(\ge i)} \sqcup \Sigma_{(<i)}.$$

So far we have only removed the vertical vanishing cycles but later we will also need to remove the horizontal ones. We thus define

$$\Sigma^{\mathrm{cut}}_{(i)} := \Sigma_{(i)} \setminus \Lambda^{\circ,\mathrm{hor}} \tag{4.1}$$

to be the surface obtained from $\Sigma_{(i)}$ by cutting along all horizontal vanishing cycles. Note that the resulting surface is usually a disconnected compact surface with boundary. See Figure 3 for an example illustrating the different ways of filtering Σ.

4.2. *Global residue condition revisited*

We later want to compare period coordinates on $\mathcal{H}(\mu)$ and generalized period coordinates on $D_{\overline{\Gamma}}$. In the terminology of this section, we have

$$H^1_{(i)}(X;\mathbb{C}) = H^1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{+,\mathrm{ver}}_{(i)}; \mathbb{C})$$

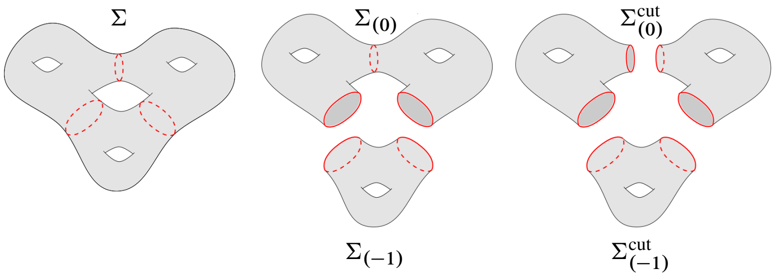


Fig. 3. The different ways of filtering Σ by level.

where $\mathrm{H}^1_{(i)}(X)$ is defined in (2.3). Thus local coordinates on $D_{\overline{\Gamma}}$ and $\mathcal{H}(\mu)$ are given by $\bigoplus_{i\in L^\bullet(\overline{\Gamma})} H^1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{+,\mathrm{ver}}_{(i)}; \mathbb{Z})^{\mathrm{GRC}}$ and $H^1(\Sigma \setminus Z, P; \mathbb{C})$, respectively.

Instead of working with cohomology we phrase everything in this section in terms of homology, where we think of a linear subspace of cohomology as being the annihilator of a subspace in homology. For the rest of the section we follow the convention that, unless stated otherwise, homology is taken with $\mathbb{Z}$-coefficients. We first need to set up various spaces modeling residue conditions on multi-scale differentials. Let Y be a connected component of $X_{(>i)}$ with $P \cap Y = \emptyset$, and denote by $\{e_1, \dots, e_b\}$ the set of all nodes where Y intersects $X_{(i)}$. Then we define

$$\lambda_Y := \sum_{k=1}^{b} \lambda_{e_k} \in H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)}),$$

and we denote by $\mathrm{GRC}^{\mathrm{ver}}_{(i)} \subseteq H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)})$ the linear span

$$\mathrm{GRC}^{\mathrm{ver}}_{(i)} := \langle \lambda_Y : Y \text{ a connected component of } X_{(>i)} \text{ with } P \cap Y = \emptyset \rangle_{\mathbb{C}}. \tag{4.2}$$

In words, $\mathrm{GRC}^{\mathrm{ver}}_{(i)}$ is the span of all the equations defining the global residue conditions of level i. We stress that this does not include the matching residue conditions at horizontal nodes at level i. We analogously define $\mathrm{GRC}^{\mathrm{ver,cut}}_{(i)} \subseteq H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)})$ by the same cycles λ_Y, now considered as elements of $H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)})$.

To include the matching residue condition at horizontal nodes, we let

$$\mathrm{MRH}_{(i)} := \langle \lambda^+ - \lambda^- : \lambda \in \Lambda^{\mathrm{hor}}_{(i)} \rangle_{\mathbb{C}} \subseteq H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)}).$$

and finally we define

$$\mathrm{GRC}_{(i)} := \mathrm{GRC}^{\mathrm{ver,cut}}_{(i)} + \mathrm{MRH}_{(i)} \subseteq H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)}),$$
$$\mathrm{GRC} := \bigoplus_{i\in L^\bullet(\overline{\Gamma})} \mathrm{GRC}_{(i)}\,.$$

In particular, GRC consists exactly of all global residue equations including the matching residue condition at horizontal nodes. We obtain the following description for

$H^1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{+,\mathrm{ver}})^{\mathrm{GRC}}$, which is exactly the subspace in cohomology satisfying all global residue and matching residue conditions:

$$H^1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{+,\mathrm{ver}})^{\mathrm{GRC}} = \mathrm{Ann}(\mathrm{GRC}_{(i)}) \\ \simeq (H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{+,\mathrm{ver}})/\mathrm{GRC}_{(i)})^*. \tag{4.3}$$

4.3. *Level and vertical filtration*

In this section we define the concept of level for homology classes. Additionally, we introduce two filtrations $L_\bullet$ and $W_\bullet$ of $H_1(\Sigma \setminus P, Z)$. Roughly speaking, L_i will consist of all cycles which can be represented by paths supported in the subsurface $\Sigma_{(\le i)}$ of level at most i, while cycles $W_i \subseteq L_i$ can additionally be represented by paths disjoint from the horizontal vanishing cycles of level i.

The motivation for introducing W_i is that it will come with a surjective linear map

$$f_i : W_i \to H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)},$$

the *specialization morphism*, with kernel L_{i-1}, and we will thus have

$$\bigoplus_{i \in L^\bullet(\overline{\Gamma})} W_i/L_{i-1} \simeq \bigoplus_{i \in L^\bullet(\overline{\Gamma})} H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)}\,.$$

This will allow us to compare the local coordinates on $\mathcal{H}(\mu)$ and on $D_{\overline{\Gamma}}$.

We start by describing the specialization morphism in words. Any class $[\gamma]$ in W_i can be represented by a collection of smooth curves in $\Sigma_{(\le i)}$ which are all disjoint from the horizontal vanishing cycles. Suppose for simplicity that $[\gamma]$ can be represented by a single curve α in $\Sigma_{(\le i)}$, disjoint from the vanishing cycles. We then restrict α to the subsurface $\Sigma_{(i)}$ of level i, i.e. we remove all the parts of α that go into lower levels. The result is a path α' in $\Sigma_{(i)}$ but since α is disjoint from all horizontal vanishing cycles, α' is actually contained in $\Sigma_{(i)}^{\mathrm{cut}}$. We then define

$$f_i([\gamma]) = \alpha'.$$

The rest of the section is concerned with making the definition of f_i precise and showing that it is well-defined in homology. The specialization morphism is illustrated in Figure 4.

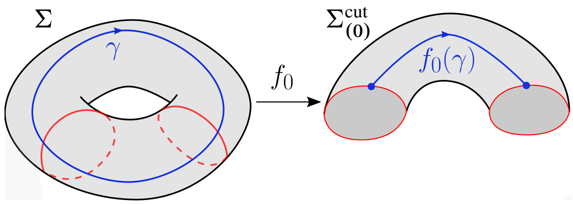


Fig. 4. The specialization morphism f_0.

To simplify notation, for the rest of this section we adopt the convention that for $B, A \subseteq C$ the relative homology $H_1(B, A)$ is always to be understood as $H_1(B, A \cap B)$ (which is simply equal to $H_1(B, A)$ if $A \subseteq B$).

Definition 4.1. The inclusion $\Sigma_{(\leq i)} \subseteq \Sigma$ induces a map

$$v_i : H_1(\Sigma_{(\leq i)} \setminus P, Z) \to H_1(\Sigma \setminus P, Z)\,,$$

and we define the *level filtration* $L_\bullet$ by

$$L_i := \operatorname{Im}(v_i) \subseteq H_1(\Sigma \setminus P, Z)\,.$$

By naturality v_{i-1} factors over the natural map

$$H_1(\Sigma_{(\leq i-1)} \setminus P, Z) \to H_1(\Sigma_{(\leq i)} \setminus P, Z)\,,$$

and thus $L_{i-1} \subseteq L_i$. We say a cycle $[\gamma] \in H_1(\Sigma \setminus P, Z)$ is of *top level* i if $[\gamma] \in L_i \setminus L_{i-1}$, and we then write

$$\top([\gamma]) = i\,.$$

We let

$$(\Lambda^{\mathrm{hor}}_{(i)})^{\perp} := \{[\gamma] \in H_1(\Sigma \setminus P, Z) : \langle [\gamma], \lambda \rangle = 0 \ \forall \lambda \in \Lambda^{\mathrm{hor}}_{(i)}\}$$

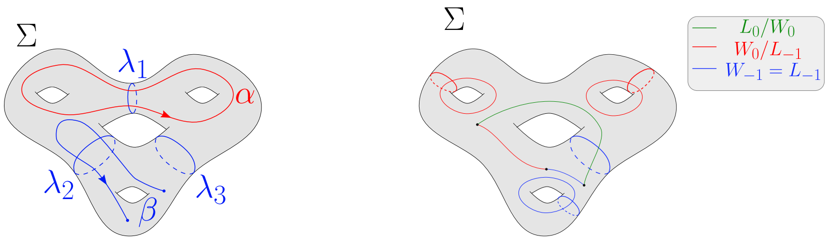
where $\langle \cdot, \cdot \rangle : H_1(\Sigma \setminus P, Z) \times H_1(\Sigma \setminus Z, P) \to \mathbb{Z}$ denotes the algebraic intersection pairing. We then define the *vertical filtration* $W_\bullet$ by

$$W_i := L_i \cap (\Lambda^{\mathrm{hor}}_{(i)})^{\perp} \subseteq L_i\,.$$

By construction every cycle in L_{i-1} can be represented by a collection of paths contained in $\Sigma_{(\leq i-1)}$. Since $\Sigma_{(\leq i-1)}$ is disjoint from all horizontal vanishing cycles, of level i, we have in particular

$$L_i \supseteq W_i \supseteq L_{i-1}\,.$$

Example 4.2. To demonstrate some of the features of the level and vertical filtration we consider the example from Figure 4. Here λ_1 is a horizontal vanishing cycle and λ_2 and λ_3 are vertical vanishing cycles separating Σ into $\Sigma_{(0)}$ and $\Sigma_{(-1)}$. Since α is a path completely contained in $\Sigma_{(0)}$ and is not homologous to any path contained in $\Sigma_{(-1)}$, we have $[\alpha] \in L_0$. Furthermore, α has zero intersection number with λ_1 and thus $[\alpha]$ is contained in the vertical filtration W_0. On the other hand, the path β intersects $\Sigma_{(0)}$ but is homologous to a path completely contained in $\Sigma_{(-1)}$, thus $[\beta] \in L_{-1}$. Since there are no horizontal vanishing cycles in $\Sigma_{(-1)}$, the -1-th piece of the level filtration coincides with the -1-th piece of the vertical filtration, i.e. $L_{-1} = W_{-1}$. Thus $[\beta] \in W_{-1} = L_{-1}$. All three vanishing cycles λ_1, λ_2 and λ_3 are homologous (up to orientation), thus $\lambda_i \in L_{-1}$ for $i = 1, 2, 3$. In Figure 5 (b) we exhibit an explicit basis for each graded piece of the filtration $L_{-1} = W_{-1} \subseteq W_0 \subseteq L_0$.



(a) Cautionary examples (b) A basis for $L_\bullet$ and $W_\bullet$

Fig. 5. The level filtration $L_\bullet$ and the vertical filtration $W_\bullet$.

The specialization morphism

Our goal is now to make the construction of the *specialization morphism*

$$f_i : W_i \to H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)}$$

precise. As a technical step we first construct an auxiliary map

$$g_i : L_i \to H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})$$

which is defined on the whole level filtration L_i and not just the vertical filtration W_i. Both f_i and g_i are basically restriction maps: Given a path of level i, the map g_i simply restricts the path to the i-th level, i.e. the result is a path in $\Sigma_{(i)}$. The map f_i is similar; the difference is that if a path is disjoint from all horizontal vanishing cycles, then its restriction is actually contained in $\Sigma_{(i)}^{\mathrm{cut}} = \Sigma_{(i)} \setminus \Lambda_{(i)}^{\mathrm{hor}}$. There is an ambiguity in how a cycle in W_i can be represented by a collection of curves disjoint from the vanishing cycles. It turns out that the ambiguity is an element of $\mathrm{GRC}_{(i)}$ and thus f_i will give a well-defined map to $H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)}$ and not to $H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})$.

We now start the description of f_i and g_i. We first describe then on the level of paths and then show that they give well-defined maps on homology.

Let $[\gamma] \in L_i$. By the definition of the level filtration we can write $[\gamma] = \sum_k a_k \gamma_k$ where γ_k are simple smooth curves contained in $\Sigma_{(\le i)}$. For each γ_k we let $\gamma_k' := \gamma_k|_{\Sigma_{(i)}}$ be the restriction to $\Sigma_{(i)}$ considered as a relative cycle with boundary in $Z \cup \Lambda_{(i)}^{\mathrm{ver},+}$ and then define

$$g_i([\gamma]) := \sum_k a_k[\gamma_k'] \in H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+}).$$

Now that we have constructed g_i we need to define an auxiliary map h_i, and afterwards we will define f_i as the composition of h_i and g_i. Set

$$\widetilde{W}_i := (\Lambda_{(i)}^{\mathrm{hor}})^\perp \subseteq H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+}).$$

Note that in particular $g_i(W_i) \subseteq \widetilde{W}_i$. We are now going to define a map

$$h_i : \widetilde{W}_i \to H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)}$$

as follows.

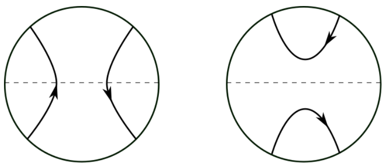


Fig. 6. A band move.

Let $[\gamma] \in \widetilde{W}_i$. Write $[\gamma] = \sum_k c_k \alpha_k$ as a sum of smooth simple curves. Since the intersection number with any horizontal vanishing cycle is zero, we can make the collection $\{\alpha_k\}$ of curves disjoint from any horizontal vanishing cycle of level i by a series of *band moves*, as depicted in Figure 6 or [10]. Thus we can write $[\gamma] = \sum_k c_k \alpha''_k$ where α''_k is a collection of smooth simple curves in $\Sigma^{\mathrm{cut}}_{(i)}$. We then define

$$h_i([\gamma]) := \sum_k c_k [\alpha''_k] \in H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)})/\mathrm{GRC}_{(i)},$$

and finally set

$$f_i := h_i \circ g_i|_{W_i}.$$

The maps f_i are not well-defined as maps to $H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)})$, since different choices of band moves can differ by multiples of the vanishing cycles, as seen in Figure 7, but we will see that f_i is well-defined as a map to $H_1(\Sigma^{\mathrm{cut}}_{(i)} \setminus P, Z \cup \Lambda^{\mathrm{ver},+}_{(i)})/\mathrm{GRC}_{(i)}$. See Figure 8 for an illustration of the map f_i.

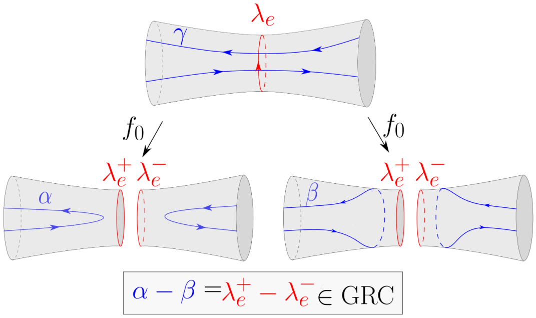


Fig. 7. The ambiguity of band moves.

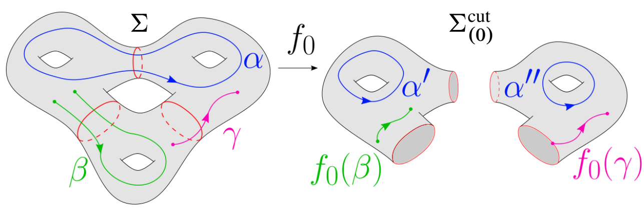


Fig. 8. The map f_0.

Proposition 4.3. *The linear maps*

$$g_i : L_i \to H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+}),$$
$$f_i : W_i \to H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)}$$

are well-defined and surjective. Furthermore,

$$\ker f_i = \ker g_i = L_{i-1}.$$

Proof. We start with the map g_i. From the long exact sequence of the triple

$$Z \subseteq (\Sigma_{(\le i-1)} \setminus P) \cup Z \subseteq (\Sigma_{(\le i)} \setminus P) \cup Z,$$

we obtain an exact sequence

$$\begin{array}{ccccc} H_1(\Sigma_{(\le i-1)} \setminus P, Z) & \xrightarrow{\nu_i} & H_1(\Sigma_{(\le i)} \setminus P, Z) & \longrightarrow & H_1(\Sigma_{(\le i)} \setminus P, (\Sigma_{(\le i-1)} \setminus P) \cup Z) \\ & & & & \big\downarrow \simeq \\ & & & & H_1(\Sigma_{(i)} \setminus P, \Lambda_{(i)}^{\mathrm{ver},+} \cup Z) \end{array}$$

where the vertical isomorphism is induced by excising $\Sigma_{(\le i-1)} \setminus \Lambda_{(i)}^{\mathrm{ver}}$.

Since the excision map is defined via barycentric subdivision, it follows that for a simple smooth curve α the composition

$$H_1(\Sigma_{(\le i)} \setminus P, Z) \to H_1(\Sigma_{(i)} \setminus P, \Lambda_{(i)}^{\mathrm{ver},+} \cup Z)$$

is given by the restriction α' to $\Sigma_{(i)}$ and thus the map coincides with g_i. From the exact sequence we then obtain $\ker g_i = L_{i-1}$. To see that g_i is surjective, take any smooth closed curve γ representing a class in $H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})$. By [1, Lemma 3.9] we can connect the boundary points of γ in $\Lambda_{(i)}^{\mathrm{ver},+}$ to marked zeroes in $\Sigma_{(\le i)}$ by only passing through levels below i and thus creating a g_i-preimage for γ. This proves all claims about g_i; it remains to prove the analogous statements for f_i.

To show that f_i is well-defined, it is enough to show that h_i is well-defined. Let $[\gamma] \in \widetilde{W}_i$ and represent $[\gamma] = \sum_k c_k\alpha_k = \sum_l d_l\beta_l \in H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})$ in two different ways by collections of smooth simple curves contained in $\Sigma_{(i)}^{\mathrm{cut}}$. We let $\alpha = \sum_k c_k\alpha_k$ and $\beta = \sum_l d_l\beta_l$ be the associated cohomology classes in $H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})$ considered as relative cohomology classes in $\Sigma_{(i)}^{\mathrm{cut}}$. We want to show that $\alpha - \beta \in \mathrm{GRC}_{(i)}$.

We will apply the relative version of Mayer–Vietoris. We set

$$A := \Sigma_{(i)} \setminus (\Lambda_{(i)}^{\mathrm{hor}} \cup P) = \Sigma_{(i)}^{\mathrm{cut}} \setminus P, \quad B := \Lambda_{(i)}^{\mathrm{hor},\circ}.$$

In particular, we have $A \cap B = \Lambda_{(i)}^{\mathrm{hor},\pm}$ and $A \cup B = \Sigma_{(i)} \setminus P$. We need the following part of the Mayer–Vietoris sequence:

$$H_1(\Lambda_{(i)}^{\mathrm{hor},\pm}) \xrightarrow{(\iota_*,\iota'_*)} H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+}) \oplus H_1(\Lambda_{(i)}^{\mathrm{hor},\circ}) \xrightarrow{k_*-l_*} H_1(\Sigma_{(i)} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})$$

where $\iota : A \cap B \to A$, $\iota' : A \cap B \to B$, $k : A \to A \cup B$ and $l : B \to A \cup B$ are inclusions.

By construction $(\alpha - \beta, 0) \in H_1(\Sigma_{(i)}^{\text{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\text{ver},+}) \oplus H_1(\Lambda_{(i)}^{\text{hor},\circ})$ lies in the kernel of $k_* - l_*$ and thus in the image of (ι_*, ι'_*). Note that we have $(\iota_*, \iota'_*)(a\lambda^+ + b\lambda^-) = (a\lambda^+ + b\lambda^-, (a+b)\lambda)$. We conclude that $\alpha - \beta = a(\lambda^+ - \lambda^-) \in \text{GRC}_{(i)}$ and thus h_i and f_i are well-defined.

Observe that h_i fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{W}_i & \xleftarrow{\quad p_i \quad} & H_1(\Sigma_{(i)}^{\text{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\text{ver},+}) \\ & {\scriptstyle h_i} \searrow \qquad \swarrow {\scriptstyle q_i} & \\ & H_1(\Sigma_{(i)}^{\text{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\text{ver},+})/\text{GRC}_{(i)} & \end{array}$$

where p_i is the natural map induced by the inclusion $\Sigma_{(i)}^{\text{cut}} \subseteq \Sigma_{(i)}$ and q_i is the natural quotient map. Thus h_i is surjective and $\ker h_i = p_i(\text{GRC}_{(i)})$.

Since g_i and h_i are surjective, so is $f_i = h_i \circ g_i$. It remains to show that $\ker f_i = L_{i-1}$.

We define $G_{(i)} \subseteq W_i$ to be the subspace generated by λ_Y as in (4.2). Then $g_i(G_{(i)}) = p_i(\text{GRC}_{(i)}) = \ker h_i$ and thus $\ker f_i = G_{(i)} + \ker g_i = G_{(i)} + L_{i-1}$. We claim that $G_{(i)} \subseteq L_{i-1}$. This can be seen as follows. Let Y be a connected component of $X_{(>i)}$ with $P \cap Y = \emptyset$, and denote by $\{e_1, \ldots, e_b\}$ the set of all nodes where Y intersects $X_{(i)}$ and additionally by $\{e_{b+1}, \ldots, e_c\}$ the nodes where Y intersects $X_{(<i)}$. Then $\sum_{k=1}^{c} \lambda_{e_k}$ is zero in $H_1(\Sigma \setminus P, Z)$ since this collection of vanishing cycles is separating, and thus

$$\lambda_Y = \sum_{k=1}^{b} \lambda_{e_k} = - \sum_{k=b+1}^{c} \lambda_{e_k} \in L_{i-1}. \qquad \blacksquare$$

4.4. *Top level*

So far we have defined the level of cycles $[\gamma] \in H_1(\Sigma \setminus P, Z)$ but it will be convenient to be able to talk about the level of paths. There has to be some care when comparing the level of a path and of its homology class.

Definition 4.4. For a collection of curves γ on Σ we define its (*top*) *level* to be the largest i such that $\gamma \cap \Sigma_{(i)} \neq \emptyset$ and then write $\top(\gamma) = i$. Note that this is only well-defined as long as none of the curves are contained in Λ°. We then define the *top level restriction* $\gamma_\top$ to be the intersection of γ with $\Sigma_{(\top(\gamma))}$. By considering $\Sigma_{(\top(\gamma))}$ as a subsurface of X we can also define the level of a collection of curves on the stable curve X. In this case we define $\gamma_\top$ to be the restriction of γ to $X_{(\top(\gamma))}$.

The following example shows that one has to be cautious when comparing the level of a path and a homology class.

Example 4.5 (Tilted cherry). Figure 9 depicts a smooth genus 3 curve with vanishing cycles corresponding to a tilted cherry level graph. The two vanishing cycles λ_1 and λ_2 are homologous and thus $\top([\lambda_1]) = \top([\lambda_2]) = -2$. On the other hand, both γ and γ' are simple closed curves representing $[\lambda_1]$, but $\top(\gamma) = 0$, $\top(\gamma') = -1$.

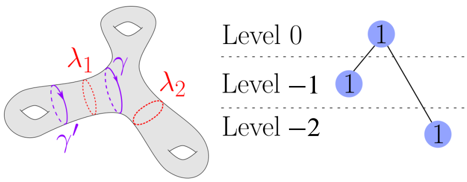


Fig. 9. A smooth genus 3 curve and the tilted cherry level graph.

The example shows that even if a cycle is represented by a simple curve, we cannot necessarily read off the level of the cycle from a path representing it. But since every cycle $[\gamma] \in L_i$ can be represented by a collection of paths supported on $X_{(\leq i)}$ we have

$$\top([\gamma]) = \inf\{\top(\gamma) : \gamma \text{ is a collection of simple smooth curves representing } [\gamma]\}.$$

4.5. An adapted homology basis

In this section we construct a homology basis suited to analyzing linear equations. Roughly speaking, we only want to consider paths that cross different levels as little as possible. This will allow us later to compare the local coordinates on $\mathcal{H}(\mu)$ and on $D_{\overline{\Gamma}}$. For the remainder of this section we let Σ be a topological surface in $\mathcal{H}(\mu)$ and X a stable curve in $D_{\overline{\Gamma}}$.

Definition 4.6. We say a cycle $[\gamma] \in H_1(\Sigma \setminus P, Z)$ *crosses* a node $e \in E(\overline{\Gamma})$ if $\langle [\gamma], \lambda_e \rangle \neq 0$. A cycle $[\gamma]$ is called a *horizontal-crossing cycle* if it crosses some horizontal node at level $\top([\gamma])$, and *non-horizontal* otherwise. Note that non-horizontal cycles are allowed to cross horizontal nodes below the top level. Similarly, for $[\gamma] \in H_1(X \setminus P, Z)$ we say that $[\gamma]$ crosses e, is a horizontal-crossing cycle or is non-horizontal if the same is true for some lift of $[\gamma]$ to $H_1(\Sigma \setminus P, Z)$.

If $[\gamma] \in H_1(\Sigma \setminus P, Z)$ has top level i and is a horizontal-crossing cycle, then $[\gamma] \in L_i \setminus W_i$. On the other hand, if $[\gamma]$ is non-horizontal, then $[\gamma] \in W_i \setminus L_{i-1}$.

Example 4.7. We consider the dual graph in Figure 10 with two components of top level and three horizontal nodes. The diamond indicates a marked pole. The cycle γ_1 crosses e_1, but γ_2 is non-horizontal, since it can be deformed away from e_2.

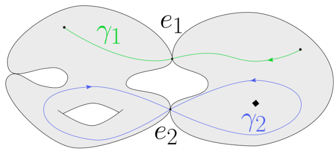


Fig. 10. Crossing and non-crossing curves.

Definition 4.8. A basis $\{\gamma_1, \dots, \gamma_n\}$ of $H_1(\Sigma \setminus P, Z)$ is called $\overline{\Gamma}$-*adapted* if there exists a partition

$$\{\gamma_1, \dots, \gamma_n\} = \bigsqcup_{i \in L^\bullet(\overline{\Gamma})} (\{\alpha_1^{(i)}, \dots, \alpha_{n(i)}^{(i)}\} \sqcup \{\delta_e^{(i)} : e \in E_{(i)}^{\text{hor}}\})$$

into horizontal-crossing cycles $\delta_e^{(i)} \in L_i \setminus W_i$ and non-horizontal cycles $\alpha_k^{(i)} \in W_i \setminus L_{i-1}$ such that

$$\begin{aligned} L_i &= \Big\langle \bigsqcup_{j \le i} \{\alpha_1^{(j)}, \dots, \alpha_{n(j)}^{(j)}\} \sqcup \{\delta_e^{(j)} : e \in E_{(j)}^{\text{hor}}\} \Big\rangle_{\mathbb{C}}, \\ L_i / W_i &\simeq \langle \delta_e^{(i)} : e \in E_{(i)}^{\text{hor}} \rangle_{\mathbb{C}}, \\ W_i / L_{i-1} &\simeq \langle \alpha_1^{(i)}, \dots, \alpha_{n(i)}^{(i)} \rangle_{\mathbb{C}}, \end{aligned}$$

and additionally

$$\langle \delta_e^{(i)}, \lambda_{e'} \rangle = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } e \in E_{(i)}^{\text{hor}}, e' \in E^{\text{hor}}.$$

As a first remark, we note that the definition of $\overline{\Gamma}$-adapted basis only depends on the level graph and not on the enhancement. The basic statement is the existence of $\overline{\Gamma}$-adapted bases.

Proposition 4.9. *For every enhanced level graph $\overline{\Gamma}$ there exists a $\overline{\Gamma}$-adapted homology basis.*

Proof. We claim that the natural map

$$\rho_i : L_i \to \mathbb{C}^{|E_{(i)}^{\text{hor}}|}, \quad [\gamma] \mapsto (\langle [\gamma], \lambda_e \rangle)_{e \in E_{(i)}^{\text{hor}}},$$

is surjective and thus

$$L_i / W_i \simeq \mathbb{C}^{|E_{(i)}^{\text{hor}}|}.$$

Assuming this, the existence of a $\overline{\Gamma}$-adapted basis can now be seen as follows. We have the filtration

$$H_1(\Sigma \setminus P, Z) \supseteq L_0 \supseteq W_0 \supseteq \cdots \supseteq L_{\ell(\overline{\Gamma})} \supseteq W_{\ell(\overline{\Gamma})}$$

with graded pieces

$$L_i / W_i \simeq \mathbb{C}^{|E_{(i)}^{\text{hor}}|}, \quad W_i / L_{i-1} \simeq H_1(\Sigma_{(i)}^{\text{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\text{ver},+}) / \text{GRC}_{(i)}\,.$$

We are now going to construct a $\overline{\Gamma}$-adapted basis inductively by lifting a basis from each graded piece. We start by choosing a basis $\{\tilde{\alpha}_1^{(i)}, \dots, \tilde{\alpha}_{n(i)}^{(i)}\}$ for $H_1(\Sigma_{(i)}^{\text{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\text{ver},+}) / \text{GRC}_{(i)}$ and then let $\alpha_k^{(i)}$ be a preimage of $\tilde{\alpha}_k^{(i)}$ under the specialization map f_i. Afterwards we let $\{\delta_e^{(i)}\}$ be ρ_i-preimages of the unit basis in $\mathbb{C}^{|E_i^{\text{hor}}|}$.

It thus remains to prove the surjectivity of the map ρ_i. For this we construct explicit cycles $\delta_e^{(i)}$ with

$$\langle \delta_e^{(i)}, \lambda_{e'} \rangle = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } e' \in E^{\mathrm{hor}}.$$

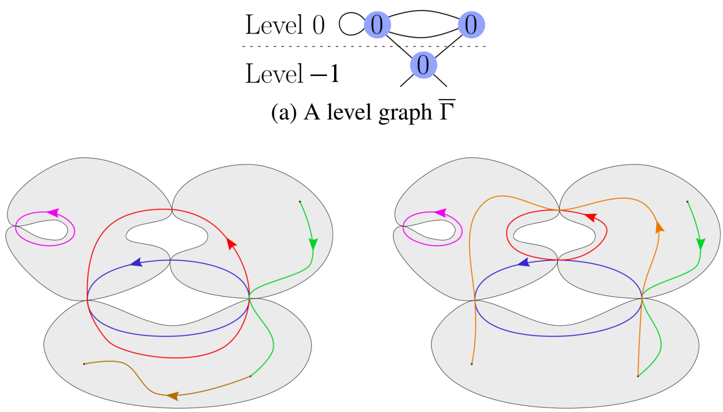
We fix a horizontal node $e \in E_{(i)}^{\mathrm{hor}}$. If both $v(e+)$ and $v(e-)$ are local minima for the level order, then they contain a marked point which is not a pole or a preimage of any node (see [1, Lemma 3.9]), and we can then connect these marked points by a path that goes through the node e once and does not cross any other horizontal nodes. On the other hand, if for example $v(e+)$ is not a local minimum then consider a path γ' in the dual graph connecting $v(e+)$ to a local minimum v' by only passing through vertical edges connecting to levels below i. Then by the same argument as before, $X_{v'}$ contains a marked point P_+ as above. We can run the same argument for $v(e-)$ and find a marked point P_-. By embedding the dual graph into X we can represent γ' by a path $\delta_e^{(i)}$ in X connecting P_+ and P_-. By construction, $\delta_e^{(i)}$ intersects λ_e once and is disjoint from all other horizontal vanishing cycles. ■

The motivation for introducing $\overline{\Gamma}$-adapted bases is that this allows relating the coordinates on the boundary $D_{\overline{\Gamma}}$ to the coordinates on the open stratum $\mathcal{H}(\mu)$, which we recall are given by $\bigoplus_{i \in L^\bullet(\overline{\Gamma})} H^1_{(i)}(X)^{\mathrm{GRC}}$ and $H^1(\Sigma \setminus P, Z)$, respectively. We now fix a $\overline{\Gamma}$-adapted basis $\{\gamma_1, \ldots, \gamma_n\}$, once and for all.

Example 4.10. We now illustrate this definition with some examples. Figure 11 depicts a level graph $\overline{\Gamma}$ and two different homology bases for $H_1(X \setminus P, Z)$. The numbers inside the vertices denote the genus of the corresponding irreducible component of the stable curve. Note that the decorations κ_e and the zero orders at the marked points are irrelevant for our discussion (since the notion of a $\overline{\Gamma}$-adapted basis only depends on the level graph and not on the choice of an enhancement) and thus we omit them. The stable curves are degenerations of genus 3 curves and the multi-scale differentials live in the codimension 4 boundary stratum $D_{\overline{\Gamma}}$. The homology basis of Figure 11 (b) provides an example of a $\overline{\Gamma}$-adapted basis while the basis in Figure 11 (c) violates the definition in two ways. Firstly, every horizontal node is crossed by multiple basis elements and secondly, all paths have top level 0, thus when restricting to the top level they are linearly dependent. In particular, the top level restrictions together with the vanishing cycles generate $H_1(X_{(0)} \setminus P_{(0)}, Z_{(0)})$ but fail to generate $H_1(X_{(1)} \setminus P_{(1)}, Z_{(1)})$.

The second example, as depicted in Figure 12, is a degeneration of a genus 7 curve in a codimension 4 boundary component of $\Xi\mathcal{M}_{g,n}(\mu)$. Bullets and diamonds represent marked zeroes and marked poles, respectively. We omit the orientation of paths. In this slightly more complicated example one can see all the features of a $\overline{\Gamma}$-adapted basis. First we start by choosing a basis for the vertical filtration $W_{-1} = \langle \alpha_1^{(-1)}, \alpha_2^{(-1)} \rangle$ and extend this to a basis of $L_{-1} = \langle \delta_1^{(-1)}, \delta_2^{(-1)} \rangle \oplus W_{-1}$ by adding paths that cross the horizontal nodes in level -1. On the top level we have

$$W_0 = L_{-1} \oplus \langle \alpha_1^{(0)}, \ldots, \alpha_8^{(0)} \rangle.$$

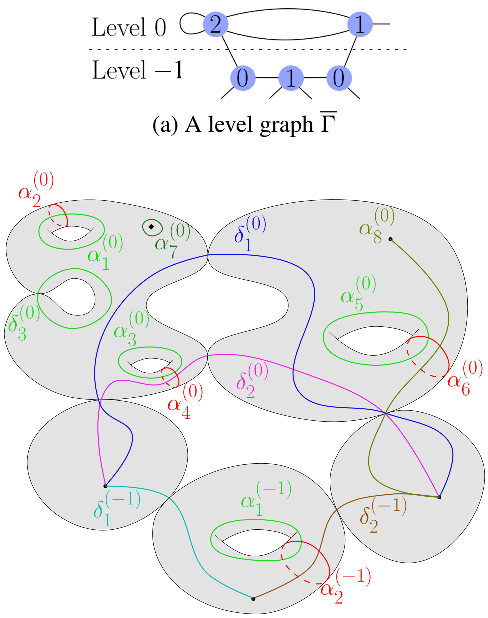


(a) A level graph $\overline{\Gamma}$

(b) A $\overline{\Gamma}$-adapted basis

(c) A non-$\overline{\Gamma}$-adapted basis

Fig. 11. A level graph $\overline{\Gamma}$ and an example and a non-example of $\overline{\Gamma}$-adapted bases.



(a) A level graph $\overline{\Gamma}$

(b) A $\overline{\Gamma}$-adapted basis

Fig. 12. An example of a $\overline{\Gamma}$-adapted basis.

Here we started by choosing a symplectic basis $\{\alpha_1^{(0)}, \dots, \alpha_6^{(0)}\}$ on each irreducible component and then extended it by paths encircling marked nodes, in this case only $\alpha_7^{(0)}$, as well as paths connecting marked zeroes, in this case $\alpha_8^{(0)}$. Finally, for the vertical filtration L_0 one has to add paths crossing horizontal nodes in level 0. In this case we have

$$L_0 = W_0 \oplus \langle \delta_1^{(0)}, \delta_2^{(0)}, \delta_3^{(0)} \rangle.$$

5. Log periods

In this section we only work with the universal family of multi-scale differentials $(\mathcal{Y} \to B, \omega)$. Our goal is to study the asymptotics of relative periods as we approach the boundary D of the local model domain B. When integrating differential forms over cycles, we will not usually distinguish between a cycle $[\gamma]$ and a representative γ.

5.1. The definition of log periods

We fix a cycle $[\gamma] \in H_1(X \setminus P, Z)$ represented by a path γ. We want to investigate the behavior of the periods of $\omega(b)$ as b approaches the boundary. Thus we need a way of deforming the cycle $[\gamma]$ from $X = X_{b_0} = Y_{b_0}$ to nearby fibers of $\mathcal{Y} \to B$. In Section 5.2 below we will give an explicit construction of a continuous family of cycles $[\gamma(b)] \in H_1(Y_b \setminus P_b, Z_b)$ deforming $[\gamma]$. The family of cycles $[\gamma(b)]$ is only well-defined locally. By a process analogous to analytic continuation, it can be considered as a multivalued family with multiple branches; the values of different branches differ by integral multiples of the vanishing cycles λ_e.

In Section 5.5, we perform a second construction, making the family of cycles $[\gamma(b)]$ invariant under monodromy, thus obtaining a family $[\hat{\gamma}(b)]$ of relative cycles well-defined on $B \setminus D$. By repeating the process for all elements of a basis $\{\gamma_1, \dots, \gamma_n\}$ of homology we construct a family of bases for $H_1(X_b \setminus P_b, Z_b; \mathbb{C})$ for all $b \in B \setminus D$ that varies continuously over $B \setminus D$. In algebro-geometric terms, we construct an explicit frame for the dual of the Deligne extension of the local system of relative cohomology (see for example [14, Section 3]). Postponing the explicit constructions of $[\gamma(b)]$ and $[\hat{\gamma}(b)]$ for now, we are able to define log periods. We define the *vanishing cycle period*

$$r_e(b) := \frac{1}{2\pi i} \int_{\lambda_e} \omega(b).$$

Definition 5.1. We define the *log period* $\psi_\gamma : B \setminus D \to \mathbb{C}$ of ω along γ by

$$\psi_\gamma(b) := \frac{1}{t_{\lceil \top(\gamma) \rceil}} \int_{\hat{\gamma}(b)} \omega(b) = \frac{1}{t_{\lceil \top(\gamma) \rceil}} \left[\int_{\gamma(b)} \omega(b) - \sum_{e \in E} \langle \gamma, \lambda_e \rangle r_e(b) \ln(s_e) \right]$$

where $[\gamma(b)]$ and $[\hat{\gamma}(b)]$ are the families of cycles that will be constructed in Sections 5.2 and 5.5, respectively, and $\langle \cdot, \cdot \rangle$ denotes the intersection pairing. Recall that the plumbing parameters s_e were defined in (2.11).

Several comments are in order. Heuristically, the scaling factor $1/t_{\lceil i \rceil}$ comes from the fact that on $\mathcal{X}_{(i)}$, at least away from the nodes, the differential ω behaves like $(t \star \boldsymbol{\eta})_{(i)} = t_{\lceil \top(\gamma) \rceil} \boldsymbol{\eta}$ since the contribution from the modification differential ξ is small. The logarithm $\ln(s_e)$ is to be understood as follows. One starts by choosing a branch of the logarithm for each coordinate t_i and h_e at some base point $x_0 \in B \setminus D_{\Gamma}$. Next, we extend the branches via analytic continuation. By requiring that $\ln(s_e) = \sum_{i=\ell(e-)}^{\ell(e+)-1} a_i \ln(t_i)$ we then define branches for all parameters s_e. This of course only defines a multivalued function but later on we will see that ψ_γ is single-valued, where we recall that the deformation $\gamma(b)$ is also multivalued, and this multivaluedness will cancel out the multivaluedness of $\ln(s_e)$. The idea is that $\int_{\gamma(b)} \omega$ and $\ln(s_e)$ behave similarly under analytic continuation and thus their difference is single-valued. We stress that there is not a unique function ψ_γ but a countable collection of such depending on the choices of representatives γ on the fixed based surface and branches of the logarithms, but once those initial choices are made, ψ_γ is a well-defined and single-valued function on $B \setminus D$. From now on we always fix such an initial choice.

The following is the main result of this section.

Theorem 5.2. *For any homology class $[\gamma] \in H_1(X \setminus P, Z)$, the log period ψ_γ is single-valued and extends to an analytic function on B. Furthermore, the limit of ψ_γ at $b = (\eta, 0, \dots, 0) \in D_{\overline{\Gamma}}$ is*

$$\psi_\gamma(b) = \int_{\gamma(b)_\top} \mathrm{Hol}(\boldsymbol{\eta}) - \sum_{e \in E^{\mathrm{hor}}} \langle \gamma_\top, \lambda_e \rangle \operatorname{res}_{q_e^-}(\boldsymbol{\eta}) c_e$$

where c_e is a constant and the holomorphic part $\int_{\gamma(b)} \mathrm{Hol}(\boldsymbol{\eta})$ *will be defined in* (5.2). *The constants c_e only depend on choices of the normal coordinates and branches of the logarithm.*

While we postpone the definition of $\mathrm{Hol}(\omega(b))$ in general to Section 5.6, we mention here a special case. If $b \in D_{\overline{\Gamma}}$ and γ is non-horizontal, then our definition will yield

$$\int_\gamma \mathrm{Hol}(\omega(b)) = \int_\gamma \boldsymbol{\eta}.$$

Recall that non-horizontal cycles were defined in Definition 4.6. We thus obtain the following corollary.

Corollary 5.3. *If γ is* non-*horizontal, then*

$$\psi_\gamma(\boldsymbol{\eta}, 0, 0) = \int_{\gamma_\top} \boldsymbol{\eta}.$$

Remark 5.4. We will see in Section 7.2 that Theorem 5.2 can be seen as a version of Schmid's nilpotent orbit theorem [13] for flat surfaces with the following difference in the setup. Instead of a whole basis for stable differentials we only have a single multi-scale differential and instead of absolute homology we integrate over relative homology.

Comparing log periods and perturbed periods

In [2] the authors introduce a coordinate system on B given by so-called *perturbed period coordinates*. Perturbed periods come in two different types, depending on whether γ crosses any horizontal nodes or not. If γ only crosses vertical nodes, one truncates γ_k at the nearby section ς of such a node and the perturbed period along γ is obtained by integrating $\omega(b)$ over the truncation of γ. In particular, the perturbed period forgets about the period inside the plumbing cylinder. We use log periods in this paper because it is easier to compare them, rather than perturbed periods, quantitatively to the actual periods $\int_{\gamma(b)} \omega(b)$. The downside of using log periods is that the collection of all log periods over a relative homology basis does not give local coordinates on B, since one cannot recover the plumbing parameters h_e at horizontal nodes.

The rest of the section is devoted to the setup for and the proof of Theorem 5.2.

5.2. *Deforming cycles to the universal family*

We now describe the construction of the family of cycles $[\gamma(b)]$. We first explain the construction at the level of paths.

The construction proceeds in two steps. In the first step we deform γ from X to nearby curves X_b of the *model family*. In the second step, we parse through the explicit construction of the universal family in Section 3.4 to deform the cycles to $\mathcal{Y}$.

We now start with the first step. First, lift γ to a path on the normalization $\tilde{X}$. Since the family of normalizations $(\tilde{\mathcal{X}} \to B, \boldsymbol{\eta})$ is a family of (possibly disconnected) smooth Riemann surfaces, we can, after possibly shrinking B, find a C^∞-trivialization of $(\tilde{\mathcal{X}} \to B, \boldsymbol{\eta})$, by Ehresmann's lemma. Furthermore, we can choose the trivialization so that it identifies the marked points and nodes. Via the trivialization we construct a family of paths on $\tilde{\mathcal{X}}$ deforming γ, which we still denote by γ. Since we chose a trivialization that preserves the nodes, the family of paths γ descends to $\mathcal{X}$. By abuse of notation we denote this new family of paths on $\mathcal{X}$ also by γ.

Suppose we now start with the homology cycle $[\gamma] \in H_1(X \setminus P, Z)$ represented by the original path γ. By deforming γ as above and then taking the associated homology class, we get a family of cycles in $H_1(X_b \setminus P_b, Z_b)$ for all $b \in B$ deforming $[\gamma]$. Note that the cycles still live in the appropriate relative homology since we chose the trivializations to preserve the marked points.

5.3. *Thin and thick part of γ*

We now prepare for the second step of the construction. Again we work with actual paths first. For every node or marked point that γ goes through, we modify γ through a homotopy such that, locally in a ϕ-coordinate neighborhood of b_0, where we recall ϕ-coordinates from Definition 3.4, the path γ coincides with the straight line from p_0 to the origin. By choosing the trivialization from the first step appropriately, we can achieve this for the whole family of paths γ over B. Afterwards, we define the *thick part* γ^{thick} of γ to be the path contained in $\mathcal{X} \setminus \mathcal{U}$, obtained by truncating γ at the nearby sections $\varsigma(b_0)$. The remaining part of γ, given by the straight lines from $p_0 = \varsigma(b_0)$ to the origin in ϕ-coordinates, is denoted by γ^{thin} and is called the *thin part* of γ. By construction, γ is the collection of disjoint paths γ^{thick} and γ^{thin}. See Figure 13 for an example.

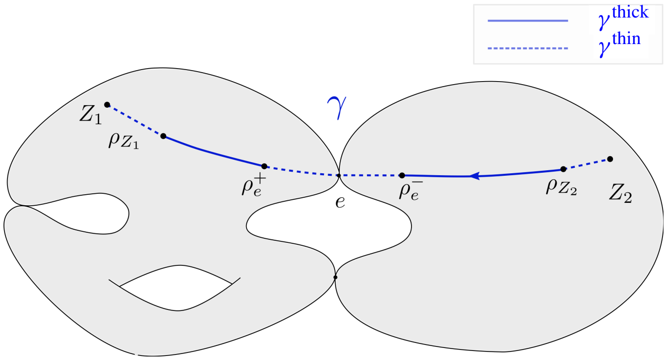


Fig. 13. The separation of γ into γ^{thick} and γ^{thin}

5.4. Second construction step

We now proceed with the construction of $\gamma(b)$. Recall that so far we have constructed a family γ of paths on $\mathcal{X}$. We will define the thick part $\gamma(b)^{\text{thick}}$ and the thin part $\gamma(b)^{\text{thin}}$ separately and finally let $\gamma(b)$ be the composition of $\gamma(b)^{\text{thick}}$ and $\gamma(b)^{\text{thin}}$. The thick part $\gamma(b)^{\text{thick}}$ is simply

$$\gamma(b)^{\text{thick}} := \Psi^{-1}(\gamma^{\text{thick}}),$$

where Ψ was defined in (2.14). The construction now differs near nodes and near marked points. We focus on a marked point $\mathcal{Z}_k$ first. In $\upsilon_{\mathcal{Z}_k}$-coordinates the endpoint of $\gamma(b)^{\text{thick}}$ is $(\upsilon_{h,b})^{-1}(\varsigma_k(b_0))$. We denote by $\gamma(b)_k^{\text{thin}}$ the straight line from $(\upsilon_{k,b})^{-1}(\varsigma_h(b_0))$ to the base point $(\upsilon_{k,b_0})^{-1}(\varsigma_h(b_0)) = p_0$, followed by the straight line from p_0 to the origin.

At nodes the construction is more involved. As before we can connect $(\upsilon_{e,b}^+)^{-1}(\varsigma_e^+(b_0))$ to p_0 via a straight line in υ_e^+-coordinates and similarly $(\upsilon_{e,b}^-)^{-1}(\varsigma_e^-(b_0))$ to p_0 via a straight line in υ_e^--coordinates. We denote by p_0^+ and p_0^- the images of p_0 in $\mathcal{A}_e^+$ and $\mathcal{A}_e^-$ respectively. To finish the construction it remains to connect p_0^+ and p_0^- on the plumbing fixture $\mathbb{V}_e$.

At $b \in B$, we identify $(\mathbb{V}_e)_b := \{(u, v) \in \Delta_\delta : uv = s_e\}$ with the annulus $A := \{u : \delta/|s_e| \le |u| \le \delta\}$ in u-coordinates. Under this identification we have

$$p_0^+ = \delta/\sqrt{R}, \quad p_0^- = s_e\sqrt{R}/\delta.$$

We divide the annulus A into finitely many sectors Sec_l, each with a chosen base point x_l. Suppose $s_e\sqrt{R}/\delta \in \mathrm{Sec}_l$. We then choose a path from $\delta/\sqrt{R}$ to x_l and connect x_l to $s_e\sqrt{R}/\delta$ via a straight line, as depicted in Figure 14.

The resulting construction is continuous on Sec_l but depends on the choice of a path from $\delta/\sqrt{R}$ to x_l; different choices differ by a multiple of the vanishing cycle λ_e. We can make all choices in such a way that the construction is continuous on the intersection of two sectors but has monodromy if we try to extend it to all of A. We have thus constructed a (multivalued) path $\gamma(b)_e^{\text{thin}}$ from p_0^+ to p_0^-. We let γ^{thin} be the composition of $\gamma(b)_e^{\text{thin}}$

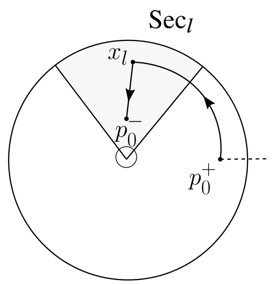


Fig. 14. The path $\gamma(b)_e^{\text{thin}}$ in u-coordinates.

and $\gamma(b)_k^{\text{thin}}$ for all nodes and marked zeroes $\mathcal{S}_k$ crossed through by γ. We finally let $\gamma(b)$ be the composition of $\gamma(b)^{\text{thick}}$ and $\gamma(b)^{\text{thin}}$.

We stress that while γ^{thick} and γ^{thin} are paths on the model family $\mathcal{X}$, the paths $\gamma(b)^{\text{thin}}$, $\gamma(b)^{\text{thick}}$, and thus also $\gamma(b)$, are on the universal family $\mathcal{Y}$.

The family $\gamma(b)$ is multivalued, with different branches differing by integral multiples of the vanishing cycles. Furthermore, if γ and γ' are homologous on X but differ by multiples of the vanishing cycles, then $\gamma(b)$ and $\gamma'(b)$ yield different branches of the same multivalued function. Thus we can define a multivalued family of cycles $[\gamma(b)]$, but which branch is picked out depends on the choice of a representative γ for $[\gamma]$ on X.

5.5. *Monodromy-invariant cycles*

Due to monodromy the family $[\gamma(b)] \in H_1(X_b \setminus P_b, Z_b; \mathbb{Z})$ is not well-defined on all of $B \setminus D$. By subtracting suitable logarithmic terms we are going to construct a new family of cycles which will be monodromy-invariant. We now choose, once and for all, branches of logarithms for t_i and h_e locally near a base point $x_0 \in B \setminus D$, and then define branches for s_e at vertical nodes, locally near x_0, via

$$\ln(s_e) := \sum_{i=\ell(e-)}^{\ell(e+)-1} a_i \ln(t_i)$$

where a_i was defined in (2.1).

To make the family $[\gamma(b)]$ monodromy-invariant we set

$$[\hat{\gamma}(b)] := [\gamma(b)] - \frac{1}{2\pi i} \sum_{e\in E} \langle \gamma, \lambda_e \rangle \ln(s_e)[\lambda_e] \in H_1(X_b \setminus P_b, Z_b; \mathbb{C}). \tag{5.1}$$

We call $[\hat{\gamma}(b)]$ the *invariant cycle associated to* γ since $[\hat{\gamma}(b)]$ is invariant under analytic continuation along any path in $\pi_1(B \setminus D, x_0)$. The invariant cycle is well-defined on $B \setminus D$ but not unique, since both $[\gamma(b)]$ as well as the branches of $\ln(s_e)$ involve certain choices. From now on we fix one set of those choices.

5.6. *Holomorphic part of a period*

We can now define the holomorphic part of the period which appeared in the statement of Theorem 5.2. We recall that $b = (\boldsymbol{\eta}, t, h)$. We set

$$\int_\gamma \mathrm{Hol}(\omega(b)) := \int_{\gamma^{\mathrm{thick}}} (t \star \boldsymbol{\eta} + \xi) + \int_{\gamma^{\mathrm{thin}}(b)} (t \star \boldsymbol{\eta} + \xi)^{\mathrm{hol}} \tag{5.2}$$

where $(t \star \boldsymbol{\eta} + \xi)^{\mathrm{hol}}$ is the holomorphic part of the Laurent expansion of $t \star \boldsymbol{\eta} + \xi$ of η in $\phi_e^{\pm}$-coordinates near the nodes. We stress that we are not defining a differential $\mathrm{Hol}(\omega(b))$ but only the expression $\int_\gamma \mathrm{Hol}(\omega(b))$. We define $\int_\gamma \mathrm{Hol}(\boldsymbol{\eta})$ in the same way with $\int_{\gamma^{\mathrm{thin}}(b)} (t \star \boldsymbol{\eta} + \xi)^{\mathrm{hol}}$ replaced by $\int_{\gamma^{\mathrm{thin}}} \boldsymbol{\eta}$.

Note that in particular $\int_{\gamma^{\mathrm{thin}}} \boldsymbol{\eta}^{\mathrm{hol}} = \int_{\gamma^{\mathrm{thin}}} \boldsymbol{\eta}$ if γ is non-horizontal and we thus obtain Corollary 5.3.

We have now defined all the objects appearing in Theorem 5.2 and can thus proceed with the proof.

Proof of Theorem 5.2. We first show that the log period is indeed single-valued. Recall that both $\int_{\gamma(b)} \omega(b)$ and $r_e(b) \ln(s_e)$ are multivalued; analytic continuation along a path encircling the origin k_e times counter-clockwise in s_e-coordinates changes $\gamma(b)$ to $\gamma(b) + k_e \langle \gamma(b), \lambda_e \rangle \lambda_e$, where λ_e is the vanishing cycle of the node e. Thus both $\int_{\gamma(b)} \omega(b)$ and $r_e(b) \langle \gamma(b), \lambda_e \rangle \ln(s_e)$ change under such an analytic continuation by the addition of

$$k_e \int_{\lambda_e} \omega(b) = k_e r_e(b) \langle \gamma(b), \lambda_e \rangle,$$

and in particular their difference is single-valued. Our goal is to compare the periods of $\int_\gamma \eta$ and $\int_{\gamma(b)} \omega(b)$. For this we need to use the plumbing construction of ω reviewed in Section 3.4. We split the period over γ into the thick and thin part, i.e.

$$\int_{\gamma(b)} \omega(b) = \int_{\gamma(b)^{\mathrm{thick}}} \omega(b) + \int_{\gamma(b)^{\mathrm{thin}}} \omega(b).$$

Over the thick part we have

$$\int_{\gamma(b)^{\mathrm{thick}}} \omega(b) = \int_{\gamma^{\mathrm{thick}}} (t \star \boldsymbol{\eta} + \xi(b))$$

and thus

$$\lim_{t,h \mapsto 0} \frac{1}{t_{\lceil \top(\gamma) \rceil}} \int_{\gamma(b)^{\mathrm{thick}}} \omega(b) = \int_{\gamma_\top^{\mathrm{thick}}} \boldsymbol{\eta},$$

since on $\mathcal{X}_{(i)}$ the modification differential ξ is divisible by $t_{\lceil i-1 \rceil}$. It remains to compute the integral over $\gamma(b)^{\mathrm{thin}}$.

We discuss the situation at vertical nodes, horizontal nodes and marked zeroes separately. We start with the case of vertical nodes. We recall from the construction that, in this case, $\gamma(b)_e^{\mathrm{thin}}$ consists of two parts, the straight line from $\upsilon_b^{-1}(\varsigma(b))$ to $p_0^{\pm}$ and then a chosen path from $p_0^+ = \delta/\sqrt{R}$ to $p_0^- = s_e\sqrt{R}/\delta$. We analyze both parts separately.

Near a vertical node e, in u-coordinates, we have

$$\begin{aligned}\int_{s_e\sqrt{R}/\delta}^{\delta/\sqrt{R}}(t\star\boldsymbol{\eta}+\xi) &= \int_{s_e\sqrt{R}/\delta}^{\delta/\sqrt{R}}(t_{\lceil\ell(e+)\rceil}u^{\kappa_e}-r_e(b))\,\frac{du}{u}\\ &= t_{\lceil\ell(e+)\rceil}\cdot\frac{(\delta/\sqrt{R})^{\kappa_e}-(s_e\sqrt{R}/\delta)^{\kappa_e}}{\kappa_e}\\ &\quad -r_e(b)[\ln(\delta/\sqrt{R})-\ln(s_e\sqrt{R}/\delta)].\end{aligned}$$

Note that there exist integers α_e such that

$$\ln(s_e\sqrt{R}/\delta)=\ln(s_e)-\ln(\delta/\sqrt{R})+2\pi i\alpha_e.$$

We thus define $c_e:=2\pi i\alpha_e-2\ln(\delta/\sqrt{R})$. Additionally, we compute

$$\int_{\gamma_{e+}^{\mathrm{thin}}}\boldsymbol{\eta}=\int_{\gamma_{e+}^{\mathrm{thin}}}\eta^{\mathrm{hol}}=\int_0^{\delta/\sqrt{R}}u^{\kappa_e-1}du=\frac{(\delta/\sqrt{R})^{\kappa_e}}{\kappa_e}.$$

Finally, we need to estimate the period along the straight line segment from $\upsilon_b^{-1}(\varsigma(b))$ to p_0. Recall from Section 3.3 that $\upsilon_{b_0}^{-1}(\varsigma(b_0))=p_0$ for all multi-scale differentials $\boldsymbol{\eta}$ and thus

$$\lim_{t,h\to 0}\upsilon^{-1}_{(\boldsymbol{\eta},t,h)}(\varsigma(\boldsymbol{\eta},t,h))=p_0.$$

We conclude that

$$\int_{p_0}^{\upsilon^{-1}_{(\boldsymbol{\eta},t,h)}(\varsigma(\boldsymbol{\eta},t,h))}(t\star\boldsymbol{\eta}+\xi)=O(t_{\lceil\ell(e+)\rceil}(t+h)).$$

The notation $O(t_{\lceil i\rceil}(t+h))$ here means that the left-hand side is analytic in $b=(t,h,\boldsymbol{\eta})$ and every monomial in the power series expansion is divisible by $t_{\lceil i\rceil}t_i$ or $t_{\lceil i\rceil}h_e$ for some i or e.

Putting everything together, we conclude that at vertical nodes,

$$\int_{\gamma(b)_e^{\mathrm{thin}}}\omega(b)=t_{\lceil\ell(e+)\rceil}\int_{\gamma_{e+}^{\mathrm{thin}}}\boldsymbol{\eta}^{\mathrm{hol}}+r_e(b)\ln(s_e)+O(t_{\lceil\ell(e+)\rceil}(t+h))$$

where we have used $r_e(b)=O(t_{\lceil\ell(e+)\rceil}(t+h))$.

We now turn to the case of a marked zero $\mathcal{Z}_k$. In this case $\gamma(b)_k^{\mathrm{thin}}$ consists of the straight line from $(\upsilon_{k,b})^{-1}(\varsigma_h(b_0))$ to $(\upsilon_{k,b_0})^{-1}(\varsigma_h(b_0))=p_0$ combined with the straight line from p_0 to the origin.

As before we have

$$\int_{p_0}^{\upsilon^{-1}_{(\boldsymbol{\eta},t,h)}(\varsigma(\boldsymbol{\eta},t,h))}(t\star\boldsymbol{\eta}+\xi)=O(t_{\lceil\ell(e+)\rceil}(t+h)).$$

And furthermore

$$\int_0^{p_0}(t\star\boldsymbol{\eta}+\xi)=\int_0^{p_0}(t_{\lceil\ell(e+)\rceil}u^{\kappa_e-1})\,du=t_{\lceil\ell(e+)\rceil}\int_{\gamma_k^{\mathrm{thin}}}\boldsymbol{\eta}=t_{\lceil\ell(e+)\rceil}\int_{\gamma_k^{\mathrm{thin}}}\boldsymbol{\eta}^{\mathrm{hol}}.$$

Finally, we address the case of horizontal nodes. In that case we have $\eta^{\mathrm{hol}} = 0$ in ϕ_e-coordinates and

$$\int_{s_e\sqrt{R}/\delta}^{\delta/\sqrt{R}} (t \star \eta + \xi) = \int_{s_e\sqrt{R}/\delta}^{\delta/\sqrt{R}} -r_e(b)\,\frac{du}{u}$$
$$= r_e(b)\ln(s_e) + O(t_{\lceil \ell(e+)\rceil}(t+h)). \qquad \blacksquare$$

6. Setup for one-parameter families

Our method for studying the boundary of a linear subvariety is based on the observation that every point in the boundary can be approached along a holomorphic one-parameter family. This will enable us to do computations in one-parameter families, which is more useful for our purposes since it allows controling the relative growth rates of the parameters t_i and h_e. We first collect some simple facts about one-parameter families.

6.1. Short arcs

See [11] for an introduction to this circle of ideas.

Definition 6.1. A (*complex*) *analytic arc* on a complex analytic space X is a holomorphic map $f : \Delta \to X$. Given a subset $Z \subseteq X$, a *short arc* on (X, Z) is an analytic arc with $f^{-1}(Z) = \{0\}$. We say an analytic arc f *connects two points* x and y if both are contained in the image of f. Similarly we say f passes through x if x is contained in the image. Furthermore, we say a short arc f is *smooth* if $f(\Delta^*) \subseteq X_{\mathrm{reg}}$, where X_{reg} denotes the smooth locus of X.

Unless stated otherwise, we denote the coordinate on Δ by z. Recall that we do not specify the radius of Δ in order to lighten the notation. The following is a simple consequence of the ideas developed by Winkelmann [15].

Lemma 6.2. *Let X be an irreducible complex analytic space and let $Z \subseteq X$ be a complex analytic subspace. Then for any pair of points $x \in X \setminus Z$, $z \in Z$ there exists a short arc $f : \Delta \to X$ on (X, Z) connecting x and z. Furthermore, if $x \in X_{\mathrm{reg}} \setminus Z$, then there exists a smooth such arc f.*

Proof. By [15, Thm. 5] there exists a holomorphic map $f : \Delta \to X$ passing through x and z. Since $f^{-1}(Z)$ is a proper subspace, after possibly shrinking Δ, we can assume that $f^{-1}(Z)$ is a finite set. We can choose a Jordan curve in Δ such that z and x are in its interior component while all other points of $f^{-1}(Z)$ lie in the exterior component. By the Riemann mapping theorem the interior component is biholomorphic to Δ. For the second claim we proceed similarly. Again, after shrinking, we can assume that the preimage $f^{-1}((X \setminus Z)_{\mathrm{sing}})$ of the singular locus is finite. Again we choose a Jordan curve containing z and x in its interior and all singular points in its exterior. $\blacksquare$

6.2. Log periods along one-parameter families

Along a one-parameter family f we immediately get the following slight improvement of Theorem 5.2. We define σ_e to be the order of vanishing of $s_e \circ f$ at $z = 0$.

Corollary 6.3. *Let $f : \Delta \to B$ be a one-parameter family of differentials with multi-scale limit $f(0) = b_0 = (X, \boldsymbol{\eta})$ and let $\gamma \in H_1(X \setminus P, Z)$. The log period ψ_γ^f along f defined by*

$$\psi_\gamma^f(z) := \frac{1}{t_{\lceil \top(\gamma) \rceil}} \Big[\int_{\gamma(f(z))} \omega(f(z)) - \Big(\sum_{e \in E} \langle \gamma, \lambda_e \rangle r_e(f(z)) \sigma_e \Big) \ln(z) \Big]$$

is single-valued, analytic in a neighborhood of the origin, and satisfies

$$\psi_\gamma^f(0) = \int_{\gamma_\top} \mathrm{Hol}(\boldsymbol{\eta}) + \sum_{e \in E} \langle \gamma_\top, \lambda_e \rangle \operatorname{res}_{q_e^-}(\boldsymbol{\eta}) \sigma_e c'_e.$$

Here c'_e are constants and $\gamma_\top$ is the restriction of γ to its top level.

Proof. Along f we can write $s_e(f(z)) = z^{\sigma_e} e^{g_e(z)}$ for some analytic function g_e. Then for each e there exists an integer k_e such that

$$\ln(s_e) = \sigma_e \ln(z) + g(z) + 2\pi i k_e.$$

We thus compute

$$\psi_\gamma(f(z)) = \psi_\gamma^f(z) - \sum_{e \in E} \langle \gamma, \lambda_e \rangle \frac{r_e(f(z))}{t_{\lceil \top(\gamma) \rceil}} (g(z) + 2\pi i k_e)$$

and then the result follows directly from Theorem 5.2 by setting $c'_e := c_e + g(0) + 2\pi i k_e$. ∎

We note that since b_0 is contained in the most degenerate stratum $D_{\overline{\Gamma}} \subseteq D$, all integers σ_e are strictly positive.

The usefulness of log periods ψ_γ^f along f stems from the fact that the logarithmic divergence now only depends on one variable z. Thus in order to get sufficient control over the divergence of ψ_γ^f on the punctured disk Δ^*, we only need to control one expression $\sum_{e \in E(\overline{\Gamma})} \langle \gamma_\top, \lambda_e \rangle r_e(f(z)) \sigma_e$.

7. Monodromy of complex linear varieties

7.1. The Gauss–Manin connection

In this subsection we let $(\pi : \mathcal{T} \to A, \omega)$ be an arbitrary family of flat surfaces over an arbitrary smooth base A.

We let $\mathbb{L}$ be the local system, or equivalently the vector bundle with flat connection, of relative cohomology over A, with fiber $\mathbb{L}_a \simeq \mathrm{H}^1(T_a \setminus \mathcal{P}_a, \mathcal{Z}_a; \mathbb{Z})$. More explicitly,

$$\mathbb{L} := \mathrm{R}^1(\pi|_{\mathcal{T} \setminus \mathcal{P}})_* j_! \underline{\mathbb{Z}}$$

where $j : \mathcal{Z} \to \mathcal{T} \setminus \mathcal{P}$ is the inclusion of the reduced zero-divisor of ω. For any given pair of points $a, a' \in A$ choose a path γ connecting them. The Gauss–Manin connection associated to $\mathbb{L}$ allows us to identify different fibers $\mathbb{L}_a \simeq \mathbb{L}_{a'}$ via parallel transport along γ. For the convenience of the reader, we recall the details. On any contractible neighborhood $W \subseteq A$ we can trivialize $\mathbb{L}$ and thus identify $\mathbb{L}_a \simeq \mathbb{L}_{a'}$ for all $a, a' \in W$. Let $\gamma : [0, 1] \to A$ be a path and let $V \subseteq \mathbb{L}_{\gamma(0)}$ be a subspace. By covering $\gamma([0, 1])$ with finitely many contractible neighborhoods, we get an induced isomorphism $\phi_\gamma : \mathbb{L}_{\gamma(0)} \simeq \mathbb{L}_{\gamma(1)}$ and we define

$$\mathrm{GM}_\gamma(V) := \phi_\gamma(V) \subseteq \mathbb{L}_{\gamma(1)},$$

which only depends on the homotopy class of γ and not on how it is covered by contractible neighborhoods.

7.2. *Hodge-theoretic description of log periods*

We now describe the monodromy action on the relative cohomology near the boundary of $\Xi\mathcal{M}_{g,n}(\mu)$. As a byproduct, we get a more conceptual definition of log periods. In this subsection we work with the local universal family $(\mathcal{Y} \to B, \omega)$ of multi-scale differentials. For the remainder of this subsection only, we relabel the local coordinates on B. We set

$$(z_1, \ldots, z_M, z_{M+1}, \ldots, z_{N+M}) := b = (\boldsymbol{\eta}, t, h)$$

where we recall from (2.7) that $N = \ell(\overline{\Gamma}) - 1 + |E^{\mathrm{hor}}|$ and $M = \dim U$.

The boundary D of B in these coordinates is then $D = (\prod_{k=M+1}^{M+N} z_k = 0)$. The universal family $(\mathcal{Y}, \omega)$ over $B \setminus D = \Delta^M \times (\Delta^*)^N$ is a family of flat surfaces contained in $\mathcal{H}(\mu)$. We now restrict the local system $\mathbb{L}$ from Section 7.1 to $B \setminus D$ with associated monodromy action

$$\mathbb{Z}^N \simeq \pi_1(B \setminus D, x_0) \to \mathrm{GL}(\mathrm{H}^1_{(i)}(X_{x_0} \setminus \mathcal{P}_{x_0}, \mathcal{Z}_{x_0}; \mathbb{Z}))$$

for some base point x_0.

Convention 7.1. From now on, $x_0 \in B \setminus D$ always denotes a base point in $\mathcal{H}(\mu)$ with corresponding fiber (X_{x_0}, ω_{x_0}).

Let T_k be the monodromy operator, i.e. the image under the monodromy action, of the standard generator of $\pi_1(B \setminus D, x_0)$ encircling the origin once in the coordinate z_k and constant otherwise. We sometimes write T_i or T_e instead of T_k if $z_k = t_i$ or $z_k = s_e$. The monodromy action can be computed explicitly from the construction of $\Xi\mathcal{M}_{g,n}(\mu)$ in Section 3. We have

$$T_e([\gamma]) = [\gamma] + \langle \gamma, \lambda_e \rangle [\lambda_e], \tag{7.1}$$

i.e. T_e acts as a Dehn twist along λ_e. Similarly,

$$T_i([\gamma]) = [\gamma] + \sum_{e \in E^{\mathrm{ver}},\, \ell(e-) \leq i < \ell(e+)} m_{e,i} \, \langle \gamma, \lambda_e \rangle [\lambda_e], \tag{7.2}$$

i.e. T_i acts as a multitwist along all curves λ_e with $\ell(e-) \le i < \ell(e+)$ where the multiplicities $m_{e,i}$ were defined in (2.1). Note that in particular $(I - T_k)^2 = 0$ for all k.

We set

$$N_k := -\log T_k = I - T_k. \tag{7.3}$$

We choose a basis $\{\gamma_1(b), \dots, \gamma_n(b)\}$ of $H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0})$ where $d := \dim H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0}) = \dim \mathcal{H}(\mu)$. Due to the multivaluedness of $\gamma_k(b)$, there are two ways of defining a relative period map, which we now explain. We choose one of the branches $\gamma_k(b)$ near x_0, as explained in Section 5.4.

Definition 7.2. Locally in a period chart W around $x_0 \in B \setminus D$ we can define $\varphi : W \to \mathbb{C}^d$ by

$$\varphi(b) := \left(\int_{\gamma_k(b)} \omega \right)_{k=1}^{d}.$$

Note that φ does depend on the choice of branches for $\gamma_k(b)$. We cannot extend φ to all of $B \setminus D$ due to the monodromy action but we still have the following analogue. The fundamental group $\mathbb{Z}^N \simeq \pi_1(B \setminus D, x_0)$ acts on $\mathbb{C}^d \simeq H^1(X_{x_0} \setminus P_{x_0}, Z_{x_0})$ by

$$(m_1, \dots, m_N) \cdot v = T_1^{m_1} \circ \dots \circ T_N^{m_N}(v)$$

and we denote by $\pi : \mathbb{C}^d \to \mathbb{C}^d/\mathbb{Z}^N$ the quotient by the monodromy action. On $B \setminus D$ we define the relative log period map $\phi : B \setminus D \to \mathbb{C}^d/\mathbb{Z}^N$ by setting $\phi := \pi \circ \varphi$. Note that ϕ does not depend on the choice of branches for $\gamma_k(b)$, since different branches of log periods differ exactly by the monodromy action for some path $\gamma \in \pi_1(B \setminus D, x_0)$.

Via the universal cover

$$\begin{aligned} \tilde{\pi} : \Delta^M \times \mathbb{H}^N &\to \Delta^M \times (\Delta^*)^N, \\ (w_1, \dots, w_{N+M}) &\mapsto (w_1, \dots, w_M, e^{2\pi i w_{M+1}}, \dots, e^{2\pi i w_{M+N}}), \end{aligned}$$

we obtain a lifting $\tilde{\phi}$ of ϕ that fits in the following commutative diagram:

$$\begin{array}{ccc} \Delta^M \times \mathbb{H}^N & \xrightarrow{\tilde{\phi}} & \mathbb{C}^d \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Delta^M \times (\Delta^*)^N & \xrightarrow{\phi} & \mathbb{C}^d/\mathbb{Z}^N \end{array}$$

The map

$$\tilde{\psi} : \Delta^M \times \mathbb{H}^N \to \mathbb{C}^d, \quad (w_1, \dots, w_{N+M}) \mapsto e^{-\sum_{k=M+1}^{M+N} w_k N_{k-M}} \tilde{\phi}(w),$$

is $\mathbb{Z}^N$-invariant and thus descends to a map $\psi : \Delta^M \times (\Delta^*)^N \to \mathbb{C}^d$.

Proposition 7.3. *The map $\psi : \Delta^M \times (\Delta^*)^N \to \mathbb{C}^d$ is, up to rescaling of each component by the scaling parameters of the top level of the corresponding curve, given by log periods. More precisely,*

$$\psi(w) = \left(t_{\lceil \top(\gamma_k) \rceil} \psi_{\gamma_k}(w)\right)_{k=1}^{d} \quad \textit{for all } w \in \Delta^M \times (\Delta^*)^N.$$

Proof. In (7.1) and (7.2) we described the action of T_i and T_e on homology. Thus, if we locally write $w_e = \frac{1}{2\pi i}\ln(s_e)$, $w_i = \frac{1}{2\pi i}\ln(t_i)$, we can compute

$$e^{-\sum_k w_k N_k}\tilde{\psi}(w) = \left(\int_{\gamma_k(w)} \omega - \frac{1}{2\pi i}\sum_{e\in E}\langle \gamma, \lambda_e\rangle \left(\int_{\lambda_e}\omega\right)\ln(s_e)\right)_{k=1}^{d}.$$

Now compare this with Definition 5.1, where we defined log periods. ■

7.3. *Setup for complex linear varieties*

Let $M \subseteq \mathcal{H}(\mu)$ be a linear subvariety. Since M is algebraic, the Euclidean closure $\bar{M} \subseteq \Xi\mathcal{M}_{g,n}(\mu)$ is an algebraic variety. This uses the algebraicity of $\Xi\mathcal{M}_{g,n}(\mu)$ [2, Thm. 1.3]. We stress that this is the *only* time where we use algebraicity of M. From now on we assume that our chosen base point b_0 is contained in $\partial M \cap D_{\overline{\Gamma}}$. Locally near b_0, the variety $\bar{M}$ has finitely many irreducible components. Note that this only uses the fact $\bar{M}$ is an analytic variety, i.e. we do not have to use algebraicity a second time.

Assumption 7.4. For now we will assume that $\bar{M}$ is irreducible near b_0 and will work under this assumption. In Section 8.4 we explain how to extend the results to the general case.

Note that a linear subvariety is only near a smooth point defined by a single linear subspace. Near singular points it looks like a union of multiple linear subspaces. For example, affine invariant submanifolds are manifolds immersed in a stratum and the points of self-intersections correspond exactly to the singular locus.

We choose $x_0 \in M_{\text{reg}} \cap B$ in the smooth locus M_{reg} of M. In a local period chart near x_0, the variety M coincides with a linear subspace $V \subseteq H^1(X_{x_0} \setminus P_{x_0}, Z_{x_0})$. By abuse of notation we do not distinguish the subspace V from the analytic subvariety it defines in a period chart. We let $\{\gamma'_1, \ldots, \gamma'_{d'}\}$ be a basis of $H_1(X \setminus P, Z)$ where $d' = \dim H_1(X \setminus P, Z)$ and choose a $\overline{\Gamma}$-adapted basis $\{\gamma_1(b), \ldots, \gamma_d(b)\}$ of $H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0})$ such that each cycle is a deformation of either γ'_k for some k as described in Section 5.2, a vanishing cycle or a horizontal-crossing cycle. In coordinates given by the $\overline{\Gamma}$-adapted basis we can write

$$V = \{A \cdot \varphi(b) = 0\}$$

where $A = (A_{kl})_{k=1,\ldots,\operatorname{codim}(M),\, l=1,\ldots,d}$ and $\varphi(b) := (\int_{\gamma_l(b)}\omega(b))_{l=1}^{d}$. To make our computations easier, we will always assume that the matrix A is in *reduced row echelon* form. This will be useful in two ways: this determines the matrix A uniquely, and allows us to read off the rank of A easily, for computations in Section 8. We consider linear equations on $H^1(X_{x_0} \setminus P_{x_0}, Z_{x_0})$ as elements of the dual and thus as homology classes.

We let

$$\ell(l) := \top([\gamma_l]), \quad \ell(k) := \max_{l=1,\ldots,d}\{\ell(l) : A_{kl} \neq 0\}.$$

Let $A^{(i)} = (A^{(i)}_{kl})_{kl}$ denote the matrix obtained from A by defining

$$A^{(i)}_{kl} := \begin{cases} A_{kl} & \text{if } \ell(k) = \ell(l) = i, \\ 0 & \text{otherwise,} \end{cases}$$

and deleting all zero rows. In words, $A^{(i)}$ collects all the linear equations of top level i and restricts them to the subsurface $X_{(i)}$, i.e. forgets about all terms in the linear equations corresponding to cycles of levels below i. Furthermore, let $A^{(i),\mathrm{ver}}$ denote the submatrix of $A^{(i)}$ only containing rows corresponding to non-horizontal equations. We refer to $A^{(i)}$ as *i-th level equations* and to $A^{(i),\mathrm{ver}}$ as *vertical i-th level* equations.

From now on, x_0 denotes a point in M and b_0 a point in $\partial M \cap D_{\overline{\Gamma}}$. If not stated otherwise, we denote by f a short arc on $(B, B \cap D_{\overline{\Gamma}})$ connecting x_0 and b_0. We recall that these notions were defined in Section 6.1. We also denote by $z_0 \in \Delta^*$ an f-preimage of x_0, i.e. $f(z_0) = x_0$.

Definition 7.5. A short arc f on $(B, B \cap D_{\overline{\Gamma}})$ is called an *M-disk* if $f(\Delta^*) \subseteq M_{\mathrm{reg}}$.

7.4. Monodromy along arcs

Let f be as above. Then the monodromy of the local system $f^*(\mathbb{L}|_{B\setminus D})$ can be described directly as follows. For every level i and every horizontal node e we define σ_i and σ_e to be the orders of vanishing of t_i and h_e, respectively, as functions of z. At vertical nodes we set

$$\sigma_e := \sum_{i=\ell(e-)}^{\ell(e+)-1} m_{e,i}\sigma_i.$$

Thus σ_e is defined for all nodes $e \in E$ as the vanishing order of $s_e \circ f$ at $z = 0$. We call the tuple

$$\sigma_f := ((\sigma_i)_{i\in L(\overline{\Gamma})}, (\sigma_e)_{e\in E^{\mathrm{hor}}}) \in \mathbb{Z}^N \tag{7.4}$$

the *monodromy type* of f. We let T_f be the monodromy of the standard generator on Δ^* and denote $N_f = I - T_f$ its *monodromy logarithm*. We have the explicit equation

$$N_f = \sum_{i\in L(\overline{\Gamma})} \sigma_i N_i + \sum_{e\in E^{\mathrm{hor}}} \sigma_e N_e \tag{7.5}$$

where the monodromy logarithms N_k are defined in Section 7.2. In particular, the monodromy action on the homology $H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0})$ is completely determined by the monodromy type.

We now study one-parameter families of differentials contained in a linear subvariety. For an arc $f : \Delta \to B$ the monodromy of $f^*(\mathbb{L})$ is controlled completely by the monodromy type σ_f. Along a one-parameter family contained in a linear subvariety, the monodromy acts trivially on the defining subspaces and this forces the linear equations for V to be of a special type. That is precisely the content of the next proposition.

Proposition 7.6. *Let f be an M-disk. Then*

$$N_f(V) \subseteq V,$$

i.e. the linear subspace defining M is invariant under the monodromy logarithm.

Proof. Let $z_0 \in \Delta^*$ be a preimage of x_0 under f and $y \in \Delta^*$ an arbitrary point. Furthermore, we choose a path $\gamma : [0,1] \to \Delta^*$ starting at z_0 and ending at y. We let

$$S := \{t \in [0,1] : \exists \text{ open period chart } W \ni f(\gamma(t)) \text{ such that} \operatorname{GM}_{\gamma(t)}(V) \cap W = M \cap W\}.$$

Note that by abuse of notation we do not distinguish between the vector space $\operatorname{GM}_{\gamma(t)}(V)$ and the analytic variety it defines in a small period chart around $\gamma(t)$. A period chart here is any open contractible subset $W \subseteq \mathcal{H}(\mu)$ such that periods are injective. For any point $y \in S$, let W be an open period chart as in the definition of S; then $W \subseteq S$. Thus S is open and it is also non-empty since $z_0 \in S$. Let $(t_l)_l$ be a sequence in S converging to t. After passing to a subsequence we can assume that the whole segment $f(\gamma([t_1,t]))$ lies in a contractible period chart W around $f(\gamma(t))$. Furthermore, we can choose W such that the analytic variety $\operatorname{GM}_{\gamma(t)}(V) \cap W$ is irreducible. By assumption there exists a contractible period chart $W_1 \subseteq W$ containing $f(\gamma(t_1))$ such that

$$\operatorname{GM}_{\gamma(t)}(V) \cap W_1 = \operatorname{GM}_{\gamma(t_1)}(V) \cap W_1 = M \cap W_1$$

where the first equality follows since $\operatorname{GM}_{\gamma(t)}(V)$ and $\operatorname{GM}_{\gamma(t_1)}$ are obtained from each other via parallel transport along $\gamma|_{[t_1,t]}$ and thus both vector spaces define the same analytic variety. Since both $\operatorname{GM}_{\gamma(t)}(V) \cap W$ and $M \cap W$ are irreducible it follows that we have equality, $\operatorname{GM}_{\gamma(t)}(V) \cap W = M \cap W$.

Let γ' be another path connecting z_0 and y. We then have

$$\operatorname{GM}_{\gamma}(V) \cap W = M \cap W = \operatorname{GM}_{\gamma'}(V) \cap W$$

and thus $\operatorname{GM}_{\gamma}(V) = \operatorname{GM}_{\gamma'}(V)$.

The second statement follows by choosing a loop γ starting at x_0. Since $\operatorname{GM}_{\gamma}(V) = V$, the monodromy operator $T_f = I - N_f$ sends V to itself and thus $N_f(V) \subseteq V$. ∎

Remark 7.7. Proposition 7.6 should be seen as a type of cylinder deformation theorem (see [16, Thm. 5.1]) in the sense that it constrains the possible linear equations of complex linear varieties. In period coordinates the equations for M are

$$\sum_{l=1}^{d} A_{kl} \int_{\gamma_l(b)} \omega(b) = 0 \quad \text{for } k = 1, \dots, \operatorname{codim}(M), \tag{7.6}$$

and the condition $N_f(V) \subseteq V$ can be written as

$$\sum_{l=1}^{d} A_{kl} \Big(\sum_{e \in E} \langle \gamma_l, \lambda_e \rangle \sigma_e r_e(b) \Big) = 0 \quad \text{for } k = 1, \dots, \operatorname{codim}(M). \tag{7.7}$$

Thus every linear equation for M forces an additional relation between the vanishing cycle periods. Furthermore, note that the coefficients of (7.7) involve the monodromy type of f. In particular, the monodromy type of M-disks is not arbitrary. This is the main motivation for the complicated construction of the log period space LPS_σ in Section 8.

Remark 7.8. In Section 9 we give an example of how one linear equation forces an additional one. In a subsequent work [5] we study this phenomenon in more detail and obtain several more restrictions among the linear equations. As a consequence we are able to determine the explicit analytic equations defining $\bar{M}$ in a neighborhood of a boundary point, instead of only the defining equations of ∂M as we do in Theorem 1.2. In *loc.cit.* we heavily use the results from this paper. If one could compute the analytic equations by other means this would potentially give a much quicker proof of Theorem 1.2, avoiding the technical difficulties of the log period space in Section 8. In the special case of linear subvarieties defined over the real numbers, in [5, Theorem 1.9] we use the restrictions on the linear equations to reprove Wright's cylinder deformation theorem [16].

Along M-disks we can rewrite the linear equations cutting out V in period coordinates as linear equations in log periods, and this will allow us to take the limit of the linear equations as z goes to zero and to obtain necessary linear equations that are satisfied on the boundary ∂M.

Corollary 7.9. *Let f be an M-disk. Then the boundary point $b_0 = f(0)$ lies in the linear subvariety of $U \subseteq D_{\overline{\Gamma}}$ locally defined by the equations*

$$A^{(i)} \cdot \psi^f(0) = 0 \quad \textit{for every } i \in L^\bullet(\overline{\Gamma}), \tag{7.8}$$

where $\psi^f(0) := (\psi^f_{\gamma_k}(0))_{k=1}^n$ is the vector of log periods.

Proof. Locally near x_0 we know that

$$\begin{aligned} 0 &= \sum_{l=1}^{d} A_{kl} \int_{\gamma_l(z)} \omega = \sum_{l=1}^{d} A_{kl} \left(\int_{\gamma_l(z)} \omega - \sum_{e\in E} \langle \gamma_l, \lambda_e\rangle \sigma_e r_e(b) \ln(z) \right) \\ &= \sum_{l=1}^{d} A_{kl} \cdot t_{\lceil \ell(l) \rceil} \psi^f_{\gamma_l}(z), \end{aligned}$$

where the second equality follows from (7.7). After rescaling each equation by $1/t_{\lceil \ell(k)\rceil}$, it follows that the function

$$\sum_{l=1}^{d} A_{kl} \frac{t_{\lceil \ell(l)\rceil}}{t_{\lceil \ell(k)\rceil}} \psi^f_{\gamma_l}(z)$$

is identically zero on Δ^*. We then take the limit as $z \to 0$. Since

$$\lim_{z\to 0} \frac{t_{\lceil \ell(l)\rceil}}{t_{\lceil \ell(k)\rceil}} = \begin{cases} 1 & \text{if } \ell(l) = \ell(k), \\ 0 & \text{if } \ell(l) < \ell(k), \end{cases}$$

we conclude that

$$\lim_{z\to 0}\sum_{l=1}^{d} A_{kl}\frac{t_{\lceil \ell(l)\rceil}}{t_{\lceil \ell(k)\rceil}}\psi^f_{\gamma_l}(z) = \sum_{l=1} A^{(\ell(k))}_{kl}\psi^f_{\gamma_l}(0). \qquad \blacksquare$$

Remark 7.10. Equation (7.8) depends not only on the limit point b_0, but also on the short arc f. On the other hand, if we restrict ourselves to the vertical equations, we can write

$$A^{(i),\mathrm{ver}}\cdot\psi^f(0) = \Bigg(\sum_{l:\,\ell(l)=\ell(k)=i} A^{(i)}_{kl}\int_{(\gamma_l)_\top}\eta\Bigg)_k = 0 \tag{7.9}$$

where the index k runs only over non-horizontal equations. Note that (7.9) is independent of f and only depends on the limit point b_0. The goal of the next section is to show that given any boundary point $b_0 \in D_{\overline{\Gamma}}$ satisfying (7.9), we can choose a short arc f such that (7.8) is satisfied along f, that is, showing that b_0 lies in ∂M and thus proving sufficiency of the linear equations which were shown above to be necessary in Corollary 7.9.

Remark 7.11 (Avoiding the cautionary example). In [4, Section 4] the authors give an example of a continuous family $f:[0,\varepsilon_0)\to \Xi\mathcal{M}_{g,n}(\mu)$ that satisfies certain linear equations for $t\in(0,\varepsilon_0)$ such that the limit at $t=0$ does not satisfy the limit of the equations, which is in stark contrast to Corollary 7.9. The limit (X_0,ω_0) is a multi-scale differential which contains two horizontal nodes such that their plumbing parameters along f behave like e^{-1/t^2} and thus are not real-analytic at $t=0$. The proof of Corollary 7.9 breaks down since one cannot find suitable rescaling parameters $t_{\lceil i\rceil}$. On the other hand, for families that extend real-analytically to the boundary an analogue of Corollary 7.9 holds, since all periods and plumbing parameters asymptotically grow like a power of the base parameter and are thus comparable to each other.

8. The defining equations on the boundary

This section contains the proof of Theorem 1.2. The setup of this section is the same as in Section 7.3. We recall that given any boundary point $b_0\in\partial M\cap D_{\overline{\Gamma}}$, we need to show that in a small neighborhood $U\subseteq D_{\overline{\Gamma}}$ of b_0 the subvariety $\partial M\cap U$ of U is defined by linear equations in generalized period coordinates, as introduced in Section 2.6. In (7.9) we have found a collection of necessary equations satisfied by $\partial M\cap D_{\overline{\Gamma}}\subset D_{\overline{\Gamma}}$ in a neighborhood of b_0, and our goal is now to show that these equations define $\partial M\cap D_{\overline{\Gamma}}$, i.e. any point in $D_{\overline{\Gamma}}$ near b_0 satisfying them is indeed contained in ∂M.

Definition 8.1. We define $V^{\lim}$ to be the subvariety of $U\subseteq D_{\overline{\Gamma}}$ defined by (7.9), that is,

$$V^{\lim} := \big(A^{(i),\mathrm{ver}}\cdot\varphi^{\mathrm{ver}}(\boldsymbol{\eta}) = 0,\ i\in L^\bullet(\overline{\Gamma})\big)$$

where $\varphi^{\mathrm{ver}}(\boldsymbol{\eta}) := (\int_{(\gamma_l)_\top}\boldsymbol{\eta})_l$.

Recall that geometrically this means we take the equations defining M, restrict them to each level subsurface of the stable curve, and forget about all horizontal-crossing equations.

The following proposition says precisely that $\partial M \cap D_{\overline{\Gamma}}$ is defined by the linear equations V^{lim}, i.e. satisfying the linear equations defining V^{lim} are both necessary and sufficient conditions for a point of $D_{\overline{\Gamma}}$ to be contained in $\partial M \cap D_{\overline{\Gamma}}$.

Proposition 8.2. *After possibly shrinking U, we have*

$$\partial M \cap U = V^{\mathrm{lim}}.$$

For now we only prove the inclusion $\partial M \cap U \subseteq V^{\mathrm{lim}}$, which follows readily from Corollary 7.9. The proof of the remaining inclusion $V^{\mathrm{lim}} \subseteq \partial M \cap U$ is the core argument, which we will give in Section 8.2.

Proof of $\partial M \cap U \subseteq V^{\mathrm{lim}}$. Let $b_0 \in \partial M \cap U$. By Lemma 6.2 there exists an M-disk connecting b_0 and x_0. By Corollary 7.9 the limit $f(0) = b_0$ satisfies

$$A^{(i)} \cdot \psi_\gamma^f(0) = 0,$$

which in particular implies

$$A^{(i),\mathrm{ver}} \cdot \varphi^{\mathrm{ver}}(\boldsymbol{\eta}) = 0$$

as explained in Remark 7.10. ∎

8.1. *Proof of the main theorem*

Assuming the proof of Proposition 8.2 for now, we show how to finish the proof of our main theorem.

Proof of Theorem 1.2. We stress that at the moment we still work under the additional assumption that ∂M is locally irreducible near b_0. The general case will be handled in Section 8.4.

We recall our setup for convenience. Let $b_0 \in \partial M \cap D_{\overline{\Gamma}}$ and $U \subseteq D_{\overline{\Gamma}}$ a period chart containing b_0. To finish the proof we need to exhibit linear equations defining ∂M in a neighborhood of b_0. The content of Proposition 8.2 is exactly that $\partial M \cap U$ is defined by the linear equations defining V^{lim}. ∎

8.2. *The log period space*

Our goal is now to show the remaining inclusion $\partial M \cap U \supseteq V^{\mathrm{lim}}$, after possibly further shrinking U. For this we need a new concept, the log period space, which we now motivate. We have already seen in Proposition 7.6 and Remark 7.7 that along one-parameter families the monodromy type of a short arc is restricted by the linear equations for V. Instead of working on $\Xi\mathcal{M}_{g,n}(\mu)$, where the monodromy around the boundary is unrestricted, we will thus work on a suitable cover LPS_σ, the *log period space*. On U the linear equations defining M are only well-defined in a small period chart, and they do not extend to a whole neighborhood of the boundary due to monodromy. But LPS_σ will be defined in such a way that the linear equations extend to a whole neighborhood of the boundary and thus define a subvariety $\widetilde{V}_\sigma \subseteq \mathrm{LPS}_\sigma$. By studying the limiting behavior of the equations for $\widetilde{V}_\sigma$ explicitly, we will be able to prove the inclusion above. We remark that there is not just one log period space, but rather a collection $(\widetilde{V}_\sigma \subseteq \mathrm{LPS}_\sigma)_{\sigma\in\Sigma}$ indexed by the set Σ of possible vanishing orders of coordinates t_i and h_e along one-parameter families. In contrast to $\Xi\mathcal{M}_{g,n}(\mu)$, on LPS_σ the vanishing of the plumbing parameters s_e is controlled by a single parameter z, and the discrete data σ controls how fast each plumbing parameter tends to zero. Thus LPS_σ has monodromy properties similar to a holomorphic arc. Before giving the (technical) definition of LPS_σ, we state those of its properties that we need, and then demonstrate how LPS_σ can be used to finish the proof of Proposition 8.2.

Proposition 8.3. *There exists a collection $(\widetilde{V}_\sigma \subseteq \mathrm{LPS}_\sigma)_{\sigma\in\Sigma}$ of varieties with maps $\pi_\sigma : \mathrm{LPS}_\sigma \to B$ such that*

(1) *every M-disk f can be lifted to a short arc $\tilde{f} : \Delta \to \widetilde{V}_\sigma$ on $(\widetilde{V}_\sigma, \widetilde{V}_\sigma \cap \pi_\sigma^{-1}(D_{\overline{\Gamma}}))$ for some $\sigma \in \Sigma$;*

(2) *for every short arc $\tilde{f}$ on $(\widetilde{V}_\sigma, \widetilde{V}_\sigma \cap \pi_\sigma^{-1}(D_{\overline{\Gamma}}))$ passing through some preimage of x_0 under π_σ, the composition $\pi_\sigma \circ \tilde{f}$ is an M-disk;*

(3) *$\widetilde{V}_\sigma$ is smooth at any point of the preimage $\pi_\sigma^{-1}(D_{\overline{\Gamma}})$;*

(4) *the restriction $\pi_\sigma|_{\widetilde{V}_\sigma \cap \pi_\sigma^{-1}(D_{\overline{\Gamma}})}$ is open, and $\pi_\sigma(\widetilde{V}_\sigma) \cap U \subseteq V^{\mathrm{lim}}$.*

Assuming the above proposition for now, we can prove the other containment in Proposition 8.2, finishing its proof, and thus also the proof of our main theorem.

Proof of Proposition 8.2. We prove the containment $\partial M \cap U \supseteq V^{\mathrm{lim}}$. Choose an M-disk f_0 connecting b_0 and x_0. By (1) there exists a lift $\tilde{f}_0$ to a short arc on $\widetilde{V}_\sigma$ for some σ. Let $\tilde{b}_0, \tilde{x}_0$ be some π_σ-preimages of b_0, x_0 contained in $\tilde{f}_0(\Delta)$, respectively. Let Z be the irreducible component of $\widetilde{V}_\sigma$ containing $\tilde{f}_0(\Delta^*)$. Since $\widetilde{V}_\sigma$ is smooth at $\tilde{b}_0$ by (3), only one irreducible component of $\widetilde{V}_\sigma$ passes through $\tilde{b}_0$ and thus there exists an open neighborhood $W \subseteq \widetilde{V}_\sigma$ of $\tilde{b}_0$ contained in Z. We define $U_{b_0} := \pi_\sigma(W \cap \pi_\sigma^{-1}(U)) = \pi_\sigma(W) \cap U$, and note that U_{b_0} is an open neighborhood of b_0 by (4). It remains to show that

$$\partial M \cap U_{b_0} \supseteq V^{\mathrm{lim}} \cap U_{b_0}.$$

By definition of U_{b_0}, for any point $z \in V^{\mathrm{lim}} \cap U_{b_0}$ there exists a π_σ-preimage $\tilde{z} \in W \subseteq Z$ of z. Since Z is irreducible, there exists a short arc on $(Z, Z \cap \pi_\sigma^{-1}(D_{\overline{\Gamma}}))$ connecting $\tilde{z}$ and $\tilde{x}_0$. Composing with π_σ yields an M-disk connecting z and x_0 by (2). By the definition of M-disks, this shows $z \in \partial M$. ∎

8.3. The construction of LPS_σ

We now start constructing the log period spaces LPS_σ. In this section we write an element $\sigma \in \mathbb{Z}^N$ as

$$\sigma = ((\sigma_i)_{i\in L(\overline{\Gamma})}, (\sigma_e)_{e\in E^{\mathrm{hor}}}).$$

We consider the positive cone

$$\mathcal{C} := \{\sigma \in \mathbb{Z}^N : \sigma_i > 0,\, \sigma_e > 0\} \subseteq \mathbb{Z}^N.$$

In analogy to the monodromy logarithm N_f of a short arc (see (7.5)), for any $\sigma \in \mathcal{C}$ we define the associated *monodromy logarithm*

$$N_\sigma := \sum_{i \in L(\overline{\Gamma})} \sigma_i N_i + \sum_{e \in E^{\mathrm{hor}}} \sigma_e N_e\,.$$

Additionally, we define the *V-preserving cone* to be the set of those monodromy logarithms that preserve V:

$$\Sigma := \{\sigma \in \mathcal{C} : N_\sigma(V) \subseteq V\}.$$

This Σ will be the index set for LPS_σ stipulated in the proposition above. By Proposition 7.6 we have $\sigma_f \in \mathcal{C}_V$ for any M-disk f, where σ_f is defined in Section 7.4.

Our construction of LPS_σ proceeds in two steps. First we define a covering space $\pi_\sigma : \mathrm{LPS}_\sigma^\circ \to B \setminus D$, and then construct LPS_σ by adding suitable limit points to $\mathrm{LPS}_\sigma^\circ$ such that the map extends to a holomorphic map $\pi_\sigma : \mathrm{LPS}_\sigma \to B$.

We start by describing $\mathrm{LPS}_\sigma^\circ$. For any $\sigma \in \Sigma$, we let $\pi_\sigma : \mathrm{LPS}_\sigma^\circ \to B \setminus D$ be the covering of $B \setminus D$ corresponding to the cyclic subgroup $\langle\sigma\rangle \subseteq \mathbb{Z}^N = \pi_1(B \setminus D)$. Denote coordinates on $\Delta^* \times \mathbb{C}^{N-1} \times \Delta^M$ by

$$\tilde{b} = \big(z, \nu = ((\nu_i)_{i \in L(\overline{\Gamma})}, \chi = (\chi_e)_{e \in E^{\mathrm{hor}}}), \boldsymbol{\eta}\big).$$

If $\overline{\Gamma}$ has at least two levels, we set $\nu_{\ell(\overline{\Gamma})} = 0$. On the other hand, if $\ell(\overline{\Gamma}) = 0$, we choose one horizontal node $e_0 \in E^{\mathrm{hor}}$ and set $\chi_{e_0} = 0$. This notation will simplify the following formulas.

Explicitly, we can describe $\mathrm{LPS}_\sigma^\circ \subseteq \Delta^* \times \mathbb{C}^{N-1} \times \Delta^M$ as the domain (that is, open connected subset) given by

$$\mathrm{LPS}_\sigma^\circ := \Big\{(z, \nu, \chi, \boldsymbol{\eta}) : \operatorname{Im} \nu_i > \tfrac{\sigma_i}{2\pi} \log|z|,\ \operatorname{Im} \chi_e > \tfrac{\sigma_e}{2\pi} \log|z|\Big\}.$$

Note that $\mathrm{LPS}_\sigma^\circ$ is diffeomorphic to $\Delta^* \times \mathbb{H}^{N-1} \times \Delta^M$, since the conditions on the imaginary parts define a family of smoothly varying horizontal half-planes over the punctured disk, and thus in particular $\pi_1(\mathrm{LPS}_\sigma^\circ) \simeq \mathbb{Z}$.

The covering map

$$\pi_\sigma : \mathrm{LPS}_\sigma^\circ \to \Delta^* \times (\Delta^*)^{N-1} \times \Delta^M = B \setminus D$$

is explicitly given by

$$z = z, \quad t_i = z^{\sigma_i} e^{2\pi i \nu_i}, \quad h_e = z^{\sigma_e} e^{2\pi i \chi_e}, \quad \boldsymbol{\eta} = \boldsymbol{\eta}.$$

Additionally, the universal cover $\mathbb{H} \times \mathbb{H}^{N-1} \times \Delta^M \to \mathrm{LPS}_\sigma^\circ$ is given by

$$z = e^{2\pi i \tau}, \quad t_i = \alpha_i - \sigma_i \tau, \quad h_e = \beta_e - \sigma_e \tau, \quad \boldsymbol{\eta} = \boldsymbol{\eta}$$

where $(\tau, (\alpha_i), (\beta_e), \eta)$ are the coordinates on $\mathbb{H} \times \mathbb{H}^{N-1} \times \Delta^M$. At horizontal nodes we set $s_e(\tilde{b}) := h_e(\tilde{b}) = z^{\sigma_e} e^{2\pi i \chi_e}$ and we are now going to also define functions $s_e : \mathrm{LPS}^\circ_\sigma \to \mathbb{C}$ at vertical nodes. For any vertical node e we define

$$\sigma_e := \sum_{i=\ell(e-)}^{\ell(e+)-1} m_{e,i}\sigma_i, \quad \chi_e := \sum_{i=\ell(e-)}^{\ell(e+)-1} m_{e,i}\nu_i, \tag{8.1}$$

$$s_e(\tilde{b}) := z^{\sigma_e} e^{2\pi i \chi_e}. \tag{8.2}$$

Here σ_e and χ_e are so defined that the relation

$$s_e = \prod_{\ell(e-)}^{\ell(e+)-1} t_i^{m_{e,i}}$$

is satisfied, where $m_{e,i}$ was defined by (2.1). The fact that N_σ preserves V is then equivalent to

$$\sum_{l=1}^{d} A_{kl} \sum_{e\in E} \langle \gamma_l, \lambda_e\rangle r_e(b)\sigma_e = 0 \quad \text{for all } k = 1, \ldots, \operatorname{codim}(M).$$

Note that this follows from (7.7) together with (8.1).

We let finally

$$\mathrm{LPS}_\sigma := \mathrm{LPS}^\circ_\sigma \sqcup (\{0\} \times \mathbb{C}^{N-1} \times \Delta^M) \subseteq \Delta \times \mathbb{C}^{N-1} \times \Delta^M.$$

Observe that $\mathrm{LPS}_\sigma = \operatorname{int}(\overline{\mathrm{LPS}^\circ_\sigma}) \subseteq \Delta \times \mathbb{C}^{N-1} \times \Delta^M$ and thus LPS_σ is open.

Remark 8.4. The space $\mathrm{LPS}^\circ_\sigma$ can be seen as a family of products of horizontal half-planes $\{\operatorname{Im} z > c(b)\}$ parametrized over the punctured disk with $\lim_{b\to 0} c(b) = -\infty$. Each half-plane becomes a copy of $\mathbb{C}$ in the limit $b \to 0$ and taking the interior closure of $\mathrm{LPS}^\circ_\sigma$ fills in the limiting copies of $\mathbb{C}$.

Furthermore, since $\pi_\sigma : \mathrm{LPS}^\circ_\sigma \to B \setminus D$ is the restriction of a holomorphic map $\mathbb{C}^{N+M} \to \mathbb{C}^{N+M}$, it extends to a holomorphic map of the closures, $\mathrm{LPS}_\sigma \to B$, which we still denote π_σ. The *boundary* $\tilde{D}$ of LPS_σ is

$$\tilde{D} := \{z = 0\} = \pi_\sigma^{-1}(B \cap D_{\overline{\Gamma}}) = \mathrm{LPS}_\sigma \setminus \mathrm{LPS}^\circ_\sigma \subseteq \mathrm{LPS}_\sigma.$$

Arc log periods

Now that we have explicitly described the log period space LPS_σ, we describe a variant of log periods which is suitably adapted to LPS_σ.

Definition 8.5. We define the *arc log period* $\psi^\Delta_\gamma : \mathrm{LPS}_\sigma \to \mathbb{C}$ by

$$\psi^\Delta_\gamma(\tilde{b}) := \frac{1}{t_{\lceil\top(\gamma)\rceil}} \left[\int_{\gamma(\tilde{b})} \omega(\tilde{b}) - \sum_{e\in E} \langle \gamma, \lambda_e\rangle r_e(\tilde{b})\sigma_e \ln(z) \right]$$

where σ_e is defined by (8.1).

As in the case of one-parameter families in Corollary 6.3, we can use the asymptotics of log periods from Theorem 5.2 to obtain the limit of arc log periods at the boundary $\widetilde{D}$.

Proposition 8.6. *The arc log period $\psi_\gamma^\Delta : \mathrm{LPS}_\sigma \to \mathbb{C}$ is single-valued and analytic. Furthermore,*

$$\psi_\gamma^\Delta(0, \nu, \chi, \boldsymbol{\eta}) = \left[\int_{\gamma_\top} \mathrm{Hol}(\boldsymbol{\eta}) + \sum_{e \in E} \langle \gamma_\top, \lambda_e \rangle \operatorname{res}_{q_e^+}(\boldsymbol{\eta})(\chi_e + \tilde{c}_e) \right]$$

where $\tilde{c}_e$ are certain constants, depending only on the choice of normal form coordinates and branches of logarithms.

Proof. We write $b = \pi_\sigma(\tilde{b})$ for the rest of the proof. For all nodes e, there exist integers k_e' such that

$$\ln(s_e(\tilde{b})) = \sigma_e \ln(z) + \chi_e + 2\pi i k_e'$$

by (8.1). We thus have

$$\begin{aligned} \psi_\gamma(b) &= \frac{1}{t_{\lceil \top(\gamma) \rceil}} \left[\int_\gamma \omega(b) - \sum_{e \in E} \langle \gamma, \lambda_e \rangle r_e(b) \ln(s_e) \right] \\ &= \psi_\gamma^\Delta(\tilde{b}) - \sum_{e \in E} \langle \gamma, \lambda_e \rangle \frac{r_e(b)}{t_{\lceil \top(\gamma) \rceil}} (\chi_e + 2\pi i k_e'). \end{aligned}$$

Thus the result follows from Theorem 5.2 with $\tilde{c}_e := c_e + 2\pi i k_e'$. ∎

The subvariety $\widetilde{V}_\sigma$

We now come to the definition of $\widetilde{V}_\sigma \subseteq \mathrm{LPS}_\sigma$. On the stratum we can only define the linear equations defining M in a small period chart. Due to monodromy, periods do not extend as holomorphic functions to the boundary $\partial \Xi \mathcal{M}_{g,n}(\mu)$. On the other hand, we have seen that log periods do extend to $\Xi \mathcal{M}_{g,n}(\mu)$. Thus naïvely one would try to convert the linear equations defining M into equations involving log periods. The naïve idea does not work since the logarithmic divergences do not cancel out. The space LPS_σ is constructed in such a way that the logarithmic divergences cancel out, and thus we will be able to rewrite linear equations in period coordinates as equations in arc log periods. We let $A' = (A'_{kl})_{1 \le k \le \mathrm{codim}(M),\, 1 \le l \le d}$ be the matrix with

$$A'_{kl} := \frac{t_{\lceil \ell(l) \rceil}}{t_{\lceil \ell(k) \rceil}} A_{kl}$$

being the equations for V, suitably rescaled, and define

$$\widetilde{V}_\sigma := \{\tilde{b} \in \mathrm{LPS}_\sigma : A' \cdot \psi^\Delta(\tilde{b}) = 0\} \subseteq \mathrm{LPS}_\sigma \tag{8.3}$$

where $\psi^\Delta := (\psi_{\gamma_k}^\Delta(\tilde{b}))_k$. The rescaling factors in the definition of A'_{kl} are motivated by the proof of Corollary 7.9.

The next result says that, over a period chart, $\widetilde{V}_\sigma$ is just the π_σ-preimage of V.

Proposition 8.7. *For any sufficiently small period chart $W \subseteq B \setminus D$ containing x_0 we have*

$$\pi_\sigma(\widetilde{V}_\sigma) \cap W = V \cap W, \tag{8.4}$$

$$\pi_\sigma^{-1}(V \cap W) = \widetilde{V}_\sigma \cap \pi_\sigma^{-1}(W). \tag{8.5}$$

Proof. For any $\tilde{b} \in \pi_\sigma^{-1}(W)$ we compute

$$\sum_{l=1}^{d} A_{kl} t_{\lceil \ell(l) \rceil} \psi_{\gamma_l}^{\Delta}(\tilde{b}) = \sum_{l=1}^{d} A_{kl} \Big[\int_{\gamma_l} \omega - \sum_{e \in E} \langle \gamma_l, \lambda_e \rangle \sigma_e r_e(b) \ln(z) \Big].$$

Since the matrix A are the defining linear equations for the linear subvariety M near x_0, it follows from Proposition 7.6 or equivalently (7.7) that

$$\sum_{l=1}^{d} A_{kl} \cdot \Big(\sum_{e \in E} \langle \gamma_l, \lambda_e \rangle \sigma_e r_e(b) \Big) = 0 \quad \text{for } k = 1, \dots, \operatorname{codim}(M).$$

Thus $\pi_\sigma^{-1}(V \cap W) = \widetilde{V}_\sigma \cap \pi_\sigma^{-1}(W)$ and the first claim follows since W is contained in the image of π_σ. ■

We now study the limiting behavior of the equations defining $\widetilde{V}_\sigma$ on the boundary $\widetilde{D}$ of LPS_σ, for arbitrary σ.

Consider one of the defining equations $\sum_{l=1}^{d} A'_{kl} \psi_{\gamma_l}^{\Delta} = 0$ of $\widetilde{V}_\sigma$ and restrict it to $\widetilde{D} = \{z = 0\}$. In the limit $z \to 0$ only the arc log periods $\psi_{\gamma_l}^{\Delta}$ with $\ell(l) = \ell(k)$ contribute. Thus, using Proposition 8.6 the equations for $\widetilde{V}_\sigma \cap \widetilde{D}$ can be written as

$$\sum_{\{l : \ell(l) = \ell(k)\}} A_{kl} \Big[\int_{(\gamma_l)_\top} \mathrm{Hol}(\eta) + \sum_{e \in E} \langle (\gamma_l)_\top, \lambda_e \rangle \operatorname{res}_e(\eta)(\chi_e + \tilde{c}_e) \Big] = 0. \tag{8.6}$$

Thus on the boundary $\widetilde{D}$, the equations for $\widetilde{V}_\sigma$ and for $\pi_\sigma^{-1}(V^{\lim})$ coincide except for the equations involving horizontal nodes. As a corollary of this discussion we obtain

Corollary 8.8. *The image $\pi_\sigma(\widetilde{V}_\sigma \cap \widetilde{D})$ is contained in $V^{\lim}$.*

The following step is crucial in the proof of property (3) of Proposition 8.3.

Proposition 8.9. *For any $\tilde{b}_0 \in \widetilde{V}_\sigma \cap \widetilde{D}$, the subvarieties $\widetilde{V}_\sigma$ and $\widetilde{V}_\sigma \cap \widetilde{D}$ are smooth at $\tilde{b}_0$ and furthermore the restriction*

$$\pi_\sigma|_{\widetilde{V}_\sigma \cap \widetilde{D}} : \widetilde{V}_\sigma \cap \widetilde{D} \to V^{\lim}$$

is a submersion at $\tilde{b}_0$.

Remark 8.10. This proposition is the key technical component of the proof of Theorem 1.2. The proof uses both the asymptotic analysis for log periods and the notion of $\overline{\Gamma}$-adapted basis in a crucial way.

Proof of Proposition 8.9. For the rest of the proof we write $\boldsymbol{\ell} = \ell(\overline{\Gamma})$. We start with the smoothness of $\widetilde{V}_\sigma$. We choose a $\overline{\Gamma}$-adapted basis

$$\{\gamma_1, \ldots, \gamma_d\} = \{\delta_1^{(0)}, \ldots, \delta_{c(0)}^{(0)}, \alpha_1^{(0)}, \ldots, \alpha_{d(0)}^{(0)}, \ldots, \delta_1^{(\boldsymbol{\ell})}, \ldots, \delta_{c(L)}^{(\boldsymbol{\ell})}, \alpha_1^{(\boldsymbol{\ell})}, \ldots, \alpha_{d(\boldsymbol{\ell})}^{(\boldsymbol{\ell})}\},$$

where we recall that $\alpha_1^{(i)}, \ldots, \alpha_{d(i)}^{(i)}$ are non-horizontal cycles of level i and $\delta_1^{(i)}, \ldots, \delta_{c(i)}^{(i)}$ are horizontal-crossing cycles of level i. We will write $\chi_l^{(i)}$ instead of χ_e where e is the unique horizontal edge crossed by $\delta_l^{(i)}$. For each level i we order the $\overline{\Gamma}$-adapted basis in such a way that $\int_{(\alpha_{d(i)}^i)_\top} \boldsymbol{\eta} \neq 0$. Additionally, if $\overline{\Gamma}$ has only one level, we also arrange that $\delta_{d(0)}^{(0)}$ crosses only the horizontal node e_0, where χ_{e_0} is the omitted coordinate on LPS_σ.

Let $F_1, \ldots, F_{\mathrm{codim}(M)}$ be the defining equations for $\widetilde{V}_\sigma$ in LPS_σ, considered as functions on LPS_σ. Our goal is to show that the Jacobian matrix of the $F_1, \ldots, F_{\mathrm{codim}(M)}$ has full rank with respect to a suitable coordinate system on LPS_σ. For this we recall that LPS_σ has different coordinates, depending on whether $\overline{\Gamma}$ has only one or multiple levels.

In the case of only one level we can describe a coordinate system as follows. We choose a horizontal edge e_0 such that the coordinate χ_{e_0} is omitted. Then z, χ_e for $e \in E^{\mathrm{hor}} \setminus \{e_0\}$ and $\int_{(\gamma_l^{(0)})_\top} \boldsymbol{\eta}$ for $l = 1, \ldots, d(0)$ are coordinates on LPS_σ.

On the other hand, if $\overline{\Gamma}$ has multiple levels, coordinates on LPS_σ are given by z, ν_i for $i = 1, \ldots, \boldsymbol{\ell} - 1$; χ_e for $e \in E^{\mathrm{hor}}$; $\int_{(\alpha_l^{(i)})_\top} \boldsymbol{\eta}$ for $i = 0, \ldots, \boldsymbol{\ell}$ and $l = 1, \ldots, d(i) - 1$; and $\int_{(\alpha_{d(0)}^{(0)})_\top} \boldsymbol{\eta}$.

We are now going to compute the Jacobian with respect to the coordinate systems just described. According to (8.3) we can write

$$F_k = \sum_{l=1}^{d} A_{kl} \frac{t_{\lceil \ell(l) \rceil}}{t_{\lceil \ell(k) \rceil}} \psi_{\gamma_l}^{\Delta},$$

where A_{kl} are the coefficients of the linear equations defining V. We assume, as always, that the matrix $A = (A_{kl})$ is in reduced row echelon form and we denote by $A_{kp(k)}$ the pivot of the k-th row. Each pivot $A_{kp(k)}$ corresponds to some element $\gamma_{p(k)}$ of the $\overline{\Gamma}$-adapted basis. For the rest of the proof we write $u := \mathrm{codim}(M)$ and we let $F_1, \ldots, F_{u'}$ be the linear equations such that $\gamma_{p(k)}$ is a horizontal-crossing cycle, and $F_{u'+1}, \ldots, F_u$ the remaining equations. In the former case, we let $e(k)$ be the unique horizontal edge crossed by $\gamma_{p(k)}$.

We now distinguish the two cases described above. First we assume that $\overline{\Gamma}$ has more than one level.

For every $1 \leq k \leq u'$ we can write, using Proposition 8.6,

$$F_k = r_{e(k)}(\chi_{e(k)} + \tilde{c}_{e(k)}) + h_k(\tilde{b}) + z g_k(\tilde{b})$$

where g_k, h_k are analytic. Furthermore, by inspecting Proposition 8.6 closely we see that

$$h_k(\tilde{b}) = h_k\left(\chi_{p(k)+1}^{(\ell(k))}, \ldots, \chi_{c(k)}^{(\ell(k))}, \int_{(\alpha_1^{(\ell(k))})_\top} \boldsymbol{\eta}, \ldots, \int_{(\alpha_{d(k)}^{(\ell(k))})_\top} \boldsymbol{\eta}\right).$$

Similarly, the remaining equations $F_{u'+1}, \dots, F_u$ can be written near $\tilde{b}_0$ as

$$F_k = \int_{(\alpha^{(\ell(k))}_{p(k)})_\top} \eta + h_k\left(\int_{(\alpha^{(\ell(k)}_{p(k)+1})_\top} \eta, \dots, \int_{(\alpha^{(\ell(k))}_{d(k)})_\top} \eta\right) + z g_k(\widetilde{b})$$

where again h_k and g_k are analytic.

We note that in this case $p(k) \neq d(i)$ since otherwise $\int_{(\alpha^{(i)}_{d(i)})_\top} \eta = 0$, which can be seen by taking the limit of the equations at b_0. Thus for any equation F_k, $k > u'$, the pivot $\int_{(\alpha^{(\ell(k))}_{p(k)})_\top} \eta$ is a coordinate on LPS_σ. Thus the submatrix of the Jacobian corresponding to $\chi_{e(k)}$ for $k = 1, \dots, u'$ and $\int_{(\alpha^{(\ell(k))}_{p(k)})_\top} \eta$ for $k = u'+1, \dots, u$ has full rank at $\tilde{b}_0$ and therefore $\widetilde{V}_\sigma$ is smooth at $\tilde{b}_0$. Smoothness of $\widetilde{V}_\sigma \cap \widetilde{D}$ follows similarly, by noting that additionally z is one of the coordinates on LPS_σ.

The argument is very similar in the second case where $\overline{\Gamma}$ has only one level, with some care needed to make sure everything works out well for the omitted coordinate $\chi_{e_0} = \chi^{(0)}_{c(0)}$. We recall that we ordered the $\overline{\Gamma}$-adapted basis in such a way that $\chi_{e(i)}$ is the omitted coordinate on LPS_σ. We claim that $\delta^{(0)}_{c(0)}$ does not correspond to any of the pivots $p(k)$, since otherwise we would have $\mathrm{res}_{e_0}(\eta) = 0$ by Remark 7.7, which is impossible. Thus, as before, each pivot corresponds to a coordinate on LPS_σ, and thus the Jacobian has full rank.

We now come to the final claim that $\pi_\sigma|_{\widetilde{V}_\sigma \cap \widetilde{D}} : \widetilde{V}_\sigma \cap \widetilde{D} \to V^{\lim}$ is a submersion at $\tilde{b}_0$. Let $\Omega \subseteq \{1, \dots, d\}$ be the set of all non-pivotal rows and let $\Omega' \subseteq \Omega$ be the non-pivots corresponding to cross cycles $\delta^{(i)}_e$. In particular, we can then use the periods $\{\int_{(\gamma_l)_\top} \eta\}$ for $l \in \Omega \setminus \Omega'$ together with χ_e for $e \in \Omega'$ and ν_i for $i \in L(\overline{\Gamma})$ as local coordinates on $\widetilde{V}_\sigma \cap \widetilde{D}$.

Similarly, we can use $\int_{(\gamma_l)_\top} \eta$ for all $l \in \Omega \setminus \Omega'$, as coordinates on $V^{\lim}$, and thus π_σ is a submersion near $\tilde{b}_0$. ■

The proof of Proposition 8.3

We now have all the necessary ingredients for the proof of Proposition 8.3; it is a matter of summarizing what we have proved so far.

Proof of Proposition 8.3. We have seen in Proposition 8.9 that $\widetilde{V}_\sigma$ is smooth at any point of $\pi_\sigma^{-1}(D_{\overline{\Gamma}})$, thus proving (3). Furthermore, Proposition 8.9 also shows that $\pi_\sigma|_{\widetilde{V}_\sigma \cap \pi_\sigma^{-1}(D_{\overline{\Gamma}})}$ is open and maps into $V^{\lim}$ by Corollary 8.8, and we have thus proved (4).

We now address the lifting properties of short arcs, (1) and (2). Let $f : \Delta \to B$ be a short arc with $\sigma_f = \sigma$. Since $f_*(\pi_1(\Delta^*, x_0)) = \langle \sigma_f \rangle$, there exists a lift $\tilde{f}^\circ : \Delta^* \to \mathrm{LPS}_\sigma$. More explicitly, we can write

$$t_i = z^{\sigma_i} e^{2\pi i \nu_i(z)}, \quad s_e = z^{\sigma_e} e^{2\pi i \chi_e(z)}, \quad \eta = \eta(z), \tag{8.7}$$

where ν_i and χ_e are holomorphic. In particular, ν_i and χ_e are holomorphic at $z = 0$. Recall that on LPS_σ there exists either a level i with $\nu_i = 0$ or a horizontal node e with

$\chi_e = 0$. We assume that $\nu_i = 0$ for some i; the other case can be treated analogously. After a change of coordinates $z \mapsto z e^{2\pi i \nu_i(z)/\sigma_i}$ we can arrange that $\nu_i(z) = 0$ and then define

$$\tilde{f}^\circ(z) := \big(z, (\nu_i(z))_i, (\chi_e(z))_e, \boldsymbol{\eta}\big).$$

Since f is holomorphic at the origin, $\tilde{f}^\circ$ extends to a short arc $\tilde{f} : \Delta \to B$. Now suppose f is an M-disk and W a period chart containing x_0. Then $f(\Delta) \cap W \subseteq V \cap W$ and by (8.5) the lift $\tilde{f}$ maps into $\widetilde{V}_\sigma$, thus showing (1).

Similarly, if $g : \Delta \to \widetilde{V}_\sigma$ is a short arc on $(\widetilde{V}_\sigma, \widetilde{V}_\sigma \cap \widetilde{D})$ passing through a preimage of x_0 on, then by (8.5) the composition $\pi_\sigma \circ g$ is an M-disk. This proves (2). ■

8.4. *Multiple components*

So far we have assumed that $\overline{M}$ is locally irreducible near b_0. In general we can write $\overline{M} = \bigcup_\alpha M_\alpha$ locally near b_0 where M_α are the finitely many, local irreducible components of $\overline{M}$. For every α we choose a base point x_α and a subspace V_α such that M_α coincides with V_α near x_α. We can then apply Proposition 8.2 to each irreducible component M_α and thus obtain

$$\partial M \cap U = \bigcup_\alpha V_\alpha^{\lim}$$

for a suitable neighborhood U. In particular, ∂M is defined by a finite union of linear subspaces at any boundary point $b_0 \in \partial M \cap D_{\overline{\Gamma}}$ and this finishes the proof of Theorem 1.2 for the case of multiple components.

9. An example

We now demonstrate how to obtain the linear equations on the boundary from the linear equations on a nearby smooth surface in an example. We stress that we do not claim that there exists an actual linear subvariety which is locally defined by those linear equations; the example is only hypothetical. In Figure 15 we see a smooth genus 7 curve Σ, just chosen sufficiently complicated to illustrate all possible phenomena. We consider the degeneration X obtained by simultaneously pinching the cycles λ_i, $i = 1, \ldots, 6$, with normalization $\widetilde{X} \to X$. By abuse of notation we denote homology cycles on X and Σ with the same name. The level structure on $\overline{\Gamma}$ can be seen in Figure 16.

We note that the image of the vanishing cycles in $H_1(\Sigma \setminus P, Z)$ is generated by $\langle \lambda_1, \lambda_2 \rangle$. Furthermore, the images of $\{\alpha, \gamma_1, \gamma_2, \delta_1, \delta_2, \gamma_3, \gamma_4, \beta\}$ on the stable curve X can be extended to a $\overline{\Gamma}$-adapted basis. The advantage of using a $\overline{\Gamma}$-adapted basis is that we can read off the equations directly. We assume that at all vertical nodes the number of prongs is one, i.e. $\kappa_e = 1$.

Suppose $M \subseteq \mathcal{H}(\mu)$ were a linear subvariety which locally near Σ is given by the linear equations

$$\int_\alpha \omega + \int_{\gamma_2} \omega + 3 \int_\beta \omega = 0, \tag{9.1}$$

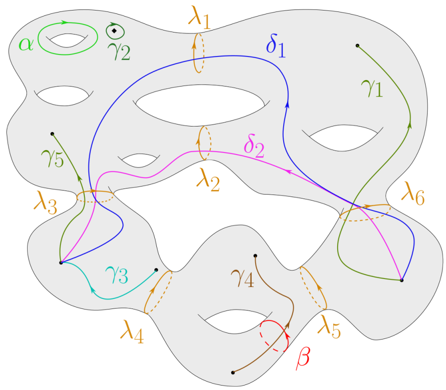


Fig. 15. A smooth genus 7 curve.

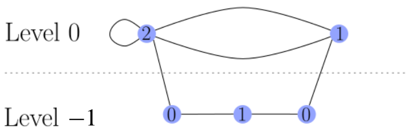


Fig. 16. The level graph $\overline{\Gamma}$.

$$\int_{\gamma_1} \omega + \int_{\gamma_5} \omega = 0, \tag{9.2}$$

$$3\int_{\delta_1} \omega - 5\int_{\delta_2} \omega = 0, \tag{9.3}$$

$$3\int_{\lambda_1} \omega - 10\int_{\lambda_2} \omega = 0, \tag{9.4}$$

$$\int_{\gamma_3} \omega = \int_{\gamma_4} \omega. \tag{9.5}$$

Before describing the linear equations of $\partial M \cap D_{\overline{\Gamma}}$ we describe the implications of Proposition 7.6 in this case. Since (9.3) crosses the horizontal vanishing cycles λ_1 and λ_2 as well as the vertical vanishing cycles λ_3 and λ_6, there has to be an additional equation of the form

$$3\left(m_1\int_{\lambda_1} \omega + m_3\int_{\lambda_3} \omega + m_3\int_{\lambda_6} \omega\right) - 5\left(m_2\int_{\lambda_2} \omega + m_3\int_{\lambda_3} \omega + m_3\int_{\lambda_6} \omega\right) \tag{9.6}$$

for some positive integers m_1, m_2 and m_3. Here we use the fact that the number of prongs at each of λ_3 and λ_6 is 1, thus the coefficient m_3 is the same for both. Note that $\lambda_3 + \lambda_6 = 0$ since the sum of the two vanishing cycles is separating. We decide, for the sake of an example, that $m_1 = 1, m_2 = 2$ and thus (9.6) reduces to (9.4). Similarly, the equation

$\int_{\gamma_1} \omega + \int_{\gamma_5} \omega = 0$ forces a linear equation

$$m_3 \int_{\lambda_3} \omega + m_3 \int_{\lambda_6} \omega = 0.$$

Since $\lambda_6 = -\lambda_3$ this equation is vacuously true and thus does not impose an additional constraint.

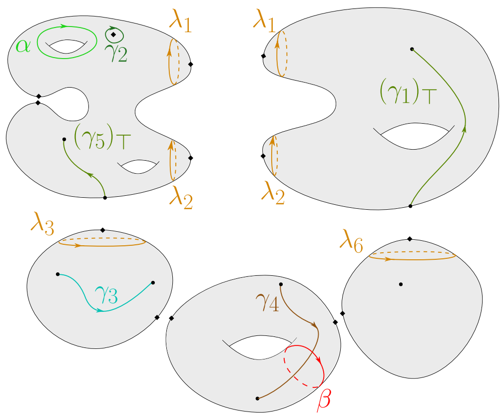


Fig. 17. The normalization $\widetilde{X}$.

We now describe the linear equations defining $\partial M \cap D_{\overline{\Gamma}}$ near X. For each of the defining equations $F \in H_1(\Sigma \setminus P, Z)$ for M we repeat the following steps.

(1) Determine the top level $\top(F)$ and write $F = \sum_k d_k \gamma_k$ in a $\overline{\Gamma}$-adapted basis.

(2) If the equation F is a horizontal-crossing cycle, delete it.

(3) Otherwise, restrict F to its top level $\top(F)$, i.e. we consider $f_{\top(F)}(F)$ in the language of Section 4. The resulting cycle then defines an equation for the boundary $\partial M \cap D_{\overline{\Gamma}}$ at level $\top(F)$.

First, since (9.3) crosses the horizontal vanishing cycles δ_1 and δ_2, we omit it from the equations of $\partial M \cap D_{\overline{\Gamma}}$. We then restrict all remaining equations to their respective level. Equations (9.1), (9.2) and (9.4) are of level 0, while (9.5) is of level -1. When restricting (9.1) we loose $\int_\beta \omega$ since $\ell(\beta) = -1$. We thus arrive at the following equations:

$$\int_\alpha \eta + \int_{\gamma_2} \eta = 0, \tag{9.7}$$

$$\int_{(\gamma_1)_\top} \eta - \int_{(\gamma_5)_\top} \eta = 0, \tag{9.8}$$

$$3 \int_{\lambda_1} \eta - 10 \int_{\lambda_2} \eta = 0, \tag{9.9}$$

$$\int_{\gamma_3} \eta = \int_{\gamma_4} \eta. \tag{9.10}$$

We remark that the global residue condition for $\overline{\Gamma}$ imposes the additional constraint $\int_{\lambda_3} \boldsymbol{\eta} = \int_{\lambda_6} \boldsymbol{\eta}$, which does not come from the linear equations for M but simply from the fact that λ_3 and λ_6 are homologous.

10. Comparing the linear structures

We now give a coordinate-free way of interpreting the linear equations for ∂M on the boundary, which will give the final and most precise formulation of Theorem 1.2. Again we suppose that $\bar{M}$ is locally irreducible near b_0 and for the general case we just apply the results on each local irreducible component separately. We still use the same setup as in Section 7.3. In the course of the proof of Theorem 1.2 we described how to obtain the subspace defining ∂M in terms of defining equations.

Suppose $T_{x_0} M = V \subseteq H^1(X_{x_0} \setminus P_{x_0}, Z_{x_0}; \mathbb{C})$ is the linear subspace defining M near x_0. We consider an equation in $H^1(\Sigma \setminus P, Z; \mathbb{C})$ as an element of the dual $H_1(\Sigma \setminus P, Z; \mathbb{C})$. Suppose

$$F = \sum_l A_l[\gamma_l] \in H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0}; \mathbb{C})$$

is an equation vanishing on V of level i. If F is a horizontal-crossing cycle we want to ignore F, and otherwise we restrict it to its top level to obtain an equation vanishing on ∂M. In order to makes this precise, we need to recall some of the setup from Section 4.3. The only difference is that now we exclusively work with homology and cohomology with $\mathbb{C}$-coefficients; to lighten notation all spaces here are simply obtained by tensoring their analogue from Section 4.3 by $\mathbb{C}$. For example, the vertical filtration

$$W_i \subseteq H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0}; \mathbb{C})$$

consists of paths of level at most i that do not intersect any horizontal vanishing cycles of level i. We denote by $a_i : W_i \to H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0}; \mathbb{C})$ the inclusion. The specialization map

$$f_i : W_i \to H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+})/\mathrm{GRC}_{(i)}$$

is obtained by restricting a path in W_i to the surface $\Sigma_{(i)}^{\mathrm{cut}}$. We recall that $\Sigma_{(i)}^{\mathrm{cut}}$ consists only of the i-th level piece of Σ and is furthermore cut along all horizontal vanishing cycles of level i.

The following proposition is now a matter of rephrasing what we have proved so far in the language of Section 4.3. We denote by f_i^* and a_i^* the dual maps of f_i and a_i, respectively.

Proposition 10.1. *Let M be a linear subvariety, $b_0 \in \partial M \cap D_{\overline{\Gamma}}$ and $x_0 \in M_{\mathrm{reg}} \cap (B \setminus D)$ with $T_{x_0} M = V$. Then $\partial M \cap D_{\overline{\Gamma}}$ is near b_0 defined by the linear subspace*

$$p\Big(\prod_{i \in L^\bullet(\overline{\Gamma})} (f_i^*)^{-1}(a_i^*(V)) \Big) \subseteq \mathrm{H}^1_{(0)}(X; \mathbb{C}) \times \prod_{i \in L(\overline{\Gamma})} \mathbb{P}(\mathrm{H}^1_{(i)}(X; \mathbb{C})^{\mathrm{GRC}}),$$

where $p : \prod_{i \in L^\bullet(\overline{\Gamma})} \mathrm{H}^1_{(i)}(X; \mathbb{C})^{\mathrm{GRC}} \to \mathrm{H}^1_{(0)}(X) \times \prod_{i \in L(\overline{\Gamma})} \mathbb{P}(\mathrm{H}^1_{(i)}(X; \mathbb{C})^{\mathrm{GRC}})$ is the natural quotient map.

Proof. We write $V = \mathrm{Ann}(E)$ where $E \subseteq H_1(X_{x_0} \setminus P_{x_0}, Z_{x_0}; \mathbb{C})$ are the linear equations defining V. More explicitly, $E = (H^1(X_{x_0} \setminus P_{x_0}, Z_{x_0}; \mathbb{C})/V)^*$.

The restriction $E \cap W_i$ corresponds to all equations with top level $\leq i$ which do not pass through any horizontal nodes, and $f_i(E \cap W_i)$ is then generated by their restrictions to level i. Note that $f_i(E \cap W_i) \subseteq H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+}; \mathbb{C})/\mathrm{GRC}_{(i)}$ and

$$\begin{aligned}\mathrm{Ann}(f_i(E \cap W_i)) &\subseteq (H_1(\Sigma_{(i)}^{\mathrm{cut}} \setminus P, Z \cup \Lambda_{(i)}^{\mathrm{ver},+}; \mathbb{C})/\,\mathrm{GRC}_{(i)})^* \\ &= H^1(\Sigma_{(i)}^{\mathrm{cut}}, Z \cup \Lambda_{(i)}^{+,\mathrm{ver}}; \mathbb{C})^{\mathrm{GRC}} = \mathrm{H}^1_{(i)}(X; \mathbb{C})^{\mathrm{GRC}}.\end{aligned}$$

By Proposition 8.2 and the explicit form of (7.9), we see that the defining equations for ∂M near x_0 on the i-th level are exactly given by $f_i(E \cap W_i)$. In particular, ∂M is given by the linear subspace $p(\prod_{i \in L^\bullet(\overline{\Gamma})} \mathrm{Ann}(f_i(E \cap W_i)))$.

It now follows from properties of the annihilator that

$$\begin{aligned}\mathrm{Ann}(f_i(E \cap W_i)) &= \mathrm{Ann}(f_i(a_i^{-1}(E))) = (f_i^*)^{-1}(\mathrm{Ann}(a_i^{-1}(E))) \\ &= (f_i^*)^{-1}(a_i^*(V)).\end{aligned}$$

■

Remark 10.2. We now explain how Proposition 10.1 relates to the results of [12]. Since f_i is surjective, its dual f_i^* is injective, and we can identify $(f_i^*)^{-1}(a_i^*(V))$ with its image under f_i^* inside W_i^*. We obtain

$$(f_i^*)^{-1}(a_i^*(V)) \simeq a_i^*(V) \cap \mathrm{Im}(f_i^*) = a_i^*(V) \cap \mathrm{Ann}(\ker f_i) \subseteq W_i^*.$$

In the special case where $\mathcal{H}(\mu)$ is a stratum of holomorphic differentials and $\overline{\Gamma}$ has no horizontal nodes, we have $H_1(\Sigma \setminus P, Z) = L_0 = W_0$ and $L_1 = W_1 = \ker f_0$. In this case the tangent space for the top level part only is given by

$$(f_0^*)^{-1}(V) \simeq V \cap \mathrm{Ann}(W_1).$$

This coincides with the tangent space description of [12, Thm. 2.9]. We stress that the space of vanishing cycles in [12] coincides with W_1 in our notation and should not be confused with the collection of vanishing cycles Λ.

11. The boundary in the "WYSIWYG" partial compactification

We now show how to quickly deduce Theorem 1.4 from Proposition 10.1, re-proving one of the main results of [4].

First we recall some of the setup. We work on a fixed stratum $\mathcal{H}(\mu)$ which is allowed to be meromorphic (note that in [4] only holomorphic strata are considered). For a multi-scale differential $(X, \boldsymbol{\eta})$ we define $\pi(X, \boldsymbol{\eta}) = (X_{(0)}, \omega_{(0)})$ to be the top level projection and recall the partial compactification

$$\widetilde{\mathcal{H}}(\mu) := \Xi\mathcal{M}_{g,n}(\mu)/(\pi(X, \boldsymbol{\eta})) \simeq \pi(X', \boldsymbol{\eta}')).$$

with the natural projection map

$$p : \Xi\mathcal{M}_{g,n}(\mu) \to \widetilde{\mathcal{H}}(\mu).$$

The fibers of p are compact, being given by a cover of a finite union $\bigcup_{\mu'} \mathbb{P}\Xi\mathcal{M}_{g',n'}(\mu')$.

Let $M \subseteq \mathcal{H}(\mu)$ be a linear subvariety, potentially defined over the complex numbers. We now fix a boundary point $(X_\infty, \omega_\infty) \in p(\partial M)$ in a stratum $\mathcal{H}(\omega_\infty)$. Our goal is to show that $p(\partial M) \cap \mathcal{H}(\omega_\infty)$ is a finite union of linear subvarieties. While Chen and Wright [4, Thm. 1.1] show that $\widetilde{\mathcal{H}}(\mu)$ is not a complex analytic space and in general p is only continuous, the restriction $p|_{p^{-1}(\mathcal{H}(\omega_\infty))} : p^{-1}(\mathcal{H}(\omega_\infty)) \to \mathcal{H}(\omega_\infty)$ is in fact algebraic and proper. Thus $p(\partial M) \cap \mathcal{H}(\omega_\infty)$ is algebraic. Furthermore, we have computed at each point x of $p^{-1}(X_\infty, \omega_\infty) \cap (\partial M \cap D_{\overline{\Gamma}})$ the linear equations defining $\partial M \cap D_{\overline{\Gamma}}$. We now want to show that there are only finitely many different linear equations for the top level when we vary the point x over $p^{-1}(X_\infty, \omega_\infty) \cap \partial M$. While so far we have worked on each open boundary component $D_{\overline{\Gamma}}$ separately, p maps different boundary strata into $\mathcal{H}(\omega_\infty)$ and thus [4, Thm. 1.2] does not just follow as part of the statements we proved. Before we can proceed with the proof we thus need some preparation.

Given $(X, \eta) \in p^{-1}(X_\infty, \omega_\infty) \cap D_{\overline{\Gamma}}$ our first goal is to determine which nearby boundary strata map into $p^{-1}(X_\infty, \omega_\infty)$. Let $B \subseteq \Xi\mathcal{M}_{g,n}(\mu)$ be a small chart containing (X, η).

We define

$$D_{-1} := \{t_{-1} = 0\} \subseteq B$$

where t_{-1} was defined in Section 2.8. Thus D_{-1} is a union of boundary strata, corresponding to undegenerations of $\overline{\Gamma}$ with the same top level graph as $\overline{\Gamma}$. Since on $\widetilde{\mathcal{H}}(\mu)$ all zeroes and preimages of nodes are marked, we have the following observation.

Lemma 11.1. *Let $(X, \eta) \in B \cap p^{-1}(X_\infty, \omega_\infty)$ as above. Then*

$$B \cap p^{-1}(X_\infty, \omega_\infty) \subseteq D_{-1}.$$

We need to study the asymptotic of log periods on D_{-1}. While the general asymptotic for undegenerations is complicated, for paths of top level 0 there is a formula similar to Theorem 5.2. We only state it in the case of curves not crossing horizontal nodes but it can be adapted in general.

Lemma 11.2. *Let γ be a path of top level 0 not crossing any horizontal nodes. Then*

$$\psi_\gamma(b) = \int_{\gamma(b)_\top} \eta \quad \textit{for all } b \in D_{-1}.$$

Note that this shows that the top level equations for $\partial M \cap D_{\overline{\Gamma}}$ are constant on D_{-1}, at least along a local irreducible component of ∂M.

Proof of Lemma 11.2. The proof of Theorem 5.2 works almost verbatim, the only difference being that $\upsilon_{b_0}^{-1}(\varsigma_e(b_0)) = p_0$ is only true for nodes with $\ell(e+) = 0$ since the modifying differential on the top level vanishes.

Thus at nodes with $\ell(e+) = 0$ we have

$$\int_{p_0}^{\upsilon^{-1}_{(\eta,t,h)}(\varsigma(\eta,t,h))} \eta = O(t_{-1}).$$

On lower levels, we only know that $\upsilon_{b_0}^{-1}(\varsigma_e(b_0))$ is bounded, but the integrand is divisible by t_{-1} and thus

$$\int_{p_0}^{\upsilon^{-1}_{(\eta,t,h)}(\varsigma(\eta,t,h))} (t \star \eta + \xi) = O(t_{-1})$$

as well. ∎

We have now assembled all the tools to give an independent proof of Theorem 1.4.

Proof of Theorem 1.4. Let $(X_\infty, \omega_\infty) \in \partial M \cap \widetilde{\mathcal{H}}(\mu)$. For each $y \in W := p^{-1}(X_\infty, \omega_\infty)$ we let $\overline{\Gamma}(y)$ be the associated enhanced level graph. If $y \in \bar{M}$ we choose U_y such that $U_y \cap \bar{M}$ has only finitely many irreducible components and $U_y \cap D_{\overline{\Gamma}}$ satisfies the conditions of Proposition 8.2 for each local irreducible component.

We write $\partial M \cap D_{\overline{\Gamma}} \cap U_y = \bigcup_\beta M_\beta(y)$ as a union of its local irreducible components. We choose arbitrary points $z_\beta(y)$ in $M_\beta(y) \cap \widetilde{\mathcal{H}}(\mu)$ and let $V_\beta(y) = T_{z_\beta(y)} M_\beta(y)$ be the corresponding tangent space. Note that so far in Proposition 8.2 we assumed that $z_\alpha(y)$ is a smooth point of M. We now claim that this assumption can be removed as follows.

In a small period chart around $z_\beta(y)$ we can choose a smooth point of $M_\beta(y)$ and then transport the linear subspace for the given branch via the Gauss–Manin connection to $z_\beta(y)$.

We set $V_\beta^{(0)}(y) := (f_0^*)^{-1}(\alpha_0)^*(V_\beta(y))$. In the case ω_∞ has no simple poles, by Remark 10.2, we can identify $V_\beta^{(0)}(y)$ with the intersection of the tangent space to $M_\beta(y)$ at $z_\beta(y)$ and the tangent space to $\mathcal{H}(\omega_\infty)$.

We can assume that U_{y_l} is a product $U'_{y_l} \times U''_{y_l}$ of polydisks where $U'_{y_l} \subseteq H_1(X_{(0)} \setminus P_{(0)}, Z_{(0)})$ is a polydisk in the coordinates of the top level. By compactness of W we can cover $W \cap \partial M$ by finitely many $U_{y_1}, \dots, U_{y_l}$ with $y_k \in \partial M$. We set $U := \bigcap_{k=1}^l U'_{y_k}$.

We claim that

$$\partial M \cap \mathcal{H}(\omega_\infty) \cap U = \bigcup_{k=1}^{l} \bigcup_\beta V_\beta^{(0)}(y_k) \cap U.$$

Let (x_n) be a sequence in M converging to $(X'_\infty, \omega'_\infty) \in U \cap p(\partial M)$. After removing finitely many elements, the sequence is contained in $\bigcup_{k=1}^l U_{y_k}$. We partition it into subsequences $(x_n^{(k)})$ contained in U_{y_k}. After passing to a subsequence we can assume that $(x_n^{(k)})$ converges to $x^{(k)} \in \partial M \cap U_{y_k}$. Now if $x^{(k)}$ lies in the same open boundary component $D_{\overline{\Gamma}}$ as y_k, it follows from Proposition 10.1 that $p(x^{(k)}) \in V_\beta^{(0)}(y_k))$ for some β. If x lies in an undegeneration, then it follows first from Lemma 11.1 that $x^{(l)} \in D_{-1}$, and then the claim follows from Lemma 11.2.

On the other hand, if $x^{(k)} \in V_\beta^{(0)}(y_k) \cap U$, then we construct a multi-scale differential (X, η) by gluing $x^{(k)}$ and the lower level parts of y_k. Since the equations are levelwise, it follows that (X, ω) is contained in $U_{y_k} \cap \partial M$, and thus there exists a sequence (x_n) in M converging to $(X, \boldsymbol{\eta})$ and by continuity (x_n) converges to $x^{(k)}$ in $\widetilde{\mathcal{H}}(\mu)$. This proves the first claim.

The second claim follows since we can choose $z_\beta(y)$ arbitrarily inside the branch $M_\beta(y)$ and thus we can take $z_\beta(y) = x_n$. ■

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