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No exceptional words for Bernoulli percolation

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Abstract. Benjamini and Kesten introduced in 1995 the problem of embedding infinite binary sequences into a Bernoulli percolation configuration, known as *percolation of words*. We give a positive answer to their Open Problem 2: almost surely, all words are seen for site percolation on \mathbb{Z}^3 with parameter $p = 1/2$. We also extend this result in various directions, proving the same result on \mathbb{Z}^d , $d \geq 3$, for any value $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, and for restrictions to slabs. Finally, we provide an explicit estimate on the probability to find all words starting from a finite box.

Keywords. Percolation, percolation of words, dynamic renormalization, oriented percolation

1. Introduction

Percolation of words

We consider Bernoulli site percolation on the hypercubic lattice \mathbb{Z}^d , $d \geq 3$, with parameter $p \in (0, 1)$. It is obtained by “coloring” at random the vertices of the lattice (with colors 0 or 1): each vertex has state 1 with probability p , and 0 with probability $1 - p$, independently of the other vertices. We denote by \mathbb{P}_p the corresponding probability measure (we refer the reader to Section 2.1 for detailed notation regarding Bernoulli percolation, as well as the questions discussed below). This process displays a phase transition for the existence of an infinite connected component of 1’s, at a certain critical value $p_c^{\text{site}}(\mathbb{Z}^d) \in (0, 1)$ of the parameter p . It is known from [2] that the percolation threshold satisfies $p_c^{\text{site}}(\mathbb{Z}^d) < 1/2$, so that at $p = 1/2$, infinite connected components of both colors coexist almost surely.

In the present paper, we study the problem known as *percolation of words*, introduced by Benjamini and Kesten [1] in 1995. It pertains to embedding infinite binary words,

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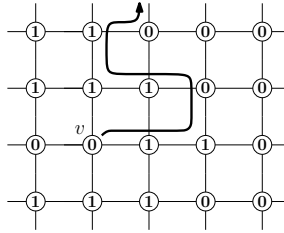


Fig. 1. An embedding of the word $\xi = (0, 1, 1, 0, 1, 1, 1, 0, \dots)$ into Bernoulli site percolation on the square lattice \mathbb{Z}^2 , starting from a given vertex v ; we say that the word ξ is seen from v .

i.e. sequences of the form $\xi = (\xi_0, \xi_1, \dots)$ where each $\xi_i \in \{0, 1\}$, into the percolation configuration (see Figure 1 for an illustration).

In particular, embedding the constant words $(0, 0, \dots)$ and $(1, 1, \dots)$ is equivalent to finding infinite connected components of 0’s and 1’s, respectively. Another special case, called *AB percolation*, was introduced earlier by Mai and Halley [14], motivated by phenomena of polymerization and gelation. In this model, the lattice is randomly populated by “particles” of two types, and neighboring particles of opposite types are bonded together (see [17] for a survey). Studying the existence of infinite AB clusters amounts to asking whether the alternating word $(1, 0, 1, 0, \dots)$ can be embedded. Somewhat surprisingly, it is possible on the triangular lattice at $p = 1/2$, as shown by Wierman and Appel [18], even though the constant words cannot be embedded (actually, they prove it for all p in the interval $(1 - p_c^{\text{site}}(\mathbb{Z}^2), p_c^{\text{site}}(\mathbb{Z}^2))$).

What makes percolation of words particularly challenging to study is its lack of monotonicity. In contrast to usual Bernoulli percolation (i.e. the case of constant words), where one considers events such as “there exists an infinite connected component of 1’s”, the events studied in percolation of words are neither increasing nor decreasing in general. For example one can think of the event, from AB percolation, of finding the alternating word $(1, 0, 1, 0, \dots)$. In particular, the probability that a given non-constant word can be embedded has no clear monotonicity in p , and it could even be the case (although not expected) that the set of p for which this probability is positive is not an interval.

Main result: no exceptional words

Before stating our main result, we need to give a few definitions specific to the study of percolation of words (the more standard definitions of percolation are presented in Section 2). We denote by

$$\Xi := \{0, 1\}^{\mathbb{N}}$$

(where $\mathbb{N} = \{0, 1, \dots\}$) the set of all infinite words. For a vertex $v \in \mathbb{Z}^d$ and a site percolation configuration ω on \mathbb{Z}^d , we say that a word $\xi \in \Xi$ is *seen from* v in ω if there exists an infinite self-avoiding path $v = v_0 \sim v_1 \sim \dots$ such that

$$\omega_{v_i} = \xi_i \quad \text{for all } i \geq 0. \tag{1.1}$$

We also introduce the corresponding random set of words $S(v) := \{\xi \in \Xi : \xi \text{ is seen from } v\}$, as well as, for $V \subseteq \mathbb{Z}^d$,

$$S_V := \bigcup_{v \in V} S(v) = \{\xi \in \Xi : \xi \text{ is seen from at least one vertex } v \in V\}.$$

Following the notation of [1], we write $S_\infty := S_{\mathbb{Z}^d}$; it is the set of words that can be seen somewhere on the lattice.

Benjamini and Kesten [1] proved that¹ for every $\xi \in \Xi$, $\mathbb{P}_{1/2}(\xi \in S_\infty) = 1$. This follows from Wierman’s coupling, as explained in Section 2.3. Combined with Fubini’s theorem, it implies the following property which says, roughly speaking, that the “uniform random word” can be seen at $p = 1/2$. For the product measure $\mu_{1/2} := (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes \mathbb{N}}$ on Ξ , we have

$$\mathbb{P}_{1/2}(\mu_{1/2}(S_\infty) = 1) = 1. \tag{1.2}$$

In other words, $\mathbb{P}_{1/2}$ -a.s., all “typical” $\xi \in \Xi$ can be seen.

This naturally leads to ask whether there could be some exceptional² words that cannot be seen, i.e. whether $S_\infty \neq \Xi$ or $S_\infty = \Xi$ ($\mathbb{P}_{1/2}$ -a.s.). This question was stated as Open Problem 2 in [1], and it remained widely open since then. Our main result gives a complete answer to it.

Theorem 1. For $d = 3$,

$$\mathbb{P}_{1/2}(S_\infty = \Xi) = 1. \tag{1.3}$$

We prove in fact a stronger statement; see Theorem 5. In particular, it is possible to see all words in a (sufficiently thick) slab. Moreover, our result holds not only for $p = 1/2$, but also for all $d \geq 3$ and $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, i.e. in the regime where we know that two infinite connected components, of 0’s and 1’s, coexist. Finally, we establish a quantitative result, namely a precise estimate on the probability that all words can be seen starting from a large ball.

Loosely speaking, we establish Theorem 5 by constructing all the words simultaneously, thanks to a renormalization argument. We want to emphasize that our proof uses properties of \mathbb{Z}^d which are not really specific to this lattice, so we expect it to be quite robust.

¹The questions discussed here give rise to some non-trivial measurability issues, addressed in [1, Section 2]. For the reader’s convenience, we provide in Section 2.2 a quick summary of the properties relevant for the present paper.

²The terminology “exceptional word” is inspired from the work [16] on exceptional times for critical dynamical percolation. Dynamical percolation is a process indexed by a continuous time parameter $t \in \mathbb{R}$. At criticality on the triangular lattice, it is known that at every fixed time there is no percolation, and the authors study the existence of possible exceptional times when percolation occurs. Here, the situation is analogous, every fixed word can be seen a.s. and we investigate the eventuality of exceptional words that could not be seen. Finally, we mention that in Section 7 of [1], a tree is constructed on which exceptional words do exist: for all $\xi \in \Xi$, $\mathbb{P}_{1/2}(\xi \in S_\infty) = 1$, but $\mathbb{P}_{1/2}(S_\infty = \Xi) = 0$ (see [1, (7.5) and (7.14)]).

Previous results

We conclude this introduction by mentioning previous works on percolation of words, and related questions.

First, it was shown in [1, Theorem 1] that (1.3) holds for all $d \geq 10$. A few years later, Kesten, Sidoravicius and Zhang [11] proved that (1.3) also holds for the close-packed graph \mathbb{Z}_{cp}^2 of the square lattice, obtained from \mathbb{Z}^2 by adding diagonal edges to each face (for this graph, $p_c^{\text{site}}(\mathbb{Z}_{\text{cp}}^2) < 1/2$).

In addition, it was proved in [1, Theorem 1] that for all $d \geq 40$, all words can be seen from the neighbors of one single vertex, i.e.

$$\mathbb{P}_{1/2}\left(\exists v \in \mathbb{Z}^d : \bigcup_{v' \sim v} S(v') = \Xi\right) = 1,$$

and the same was proved in [11] for \mathbb{Z}_{cp}^2 . Of course, one cannot hope to see all words $\xi \in \Xi$ from one single vertex v , since the value of ξ_0 then has to coincide with ω_v . Note that because of this issue, a slightly different convention is adopted in [1]: the condition in (1.1) is required to hold only starting from $i = 1$.

For percolation of words on \mathbb{Z}^d , $d \geq 3$, the following result (weaker than (1.3)) was established in [9, Theorem 2], using a renormalization argument. Let Ξ_M stand for the set of words such that all runs of consecutive 0's or 1's have length at least M (we allow in particular such words to be ultimately constant), then for all $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$,

$$\exists M = M(d, p) \text{ such that } \mathbb{P}_p(\Xi_M \subseteq S_\infty) = 1.$$

Finally, the case of site percolation on the triangular lattice \mathbb{T} at $p = p_c^{\text{site}}(\mathbb{T}) = 1/2$ was studied in [10]. In this case, the monochromatic words $(0, 0, \dots)$ and $(1, 1, \dots)$ cannot be seen, since percolation does not occur, so we have in particular $\mathbb{P}_{1/2}(S_\infty = \Xi) = 0$. However, one can show that (1.2) still holds in this case, i.e. *almost all* words can be seen:

$$\mathbb{P}_{1/2}(\mu_{1/2}(S_\infty) = 1) = 1.$$

This result answered Open Problem 1 in [1]. In this case where $p = 1/2$ is critical, one cannot use Wierman's coupling directly, since the lower bound that it provides is simply 0. Much more work is required for this result than in the case of \mathbb{Z}^d ($d \geq 3$), where $p = 1/2$ is supercritical.

Organization of the paper

In Section 2, we first set notation, and then we discuss preliminary properties. In particular, we present Wierman's coupling, which is instrumental in the proof of Theorem 1, in combination with a renormalization argument. In Section 3, we state and prove a stronger and quantitative version of Theorem 1; see Theorem 5. This proof uses several auxiliary results, which are established in subsequent sections. We then proceed in detail with the renormalization procedure in Section 4. Finally, the last two sections are devoted to proving two inputs used for Theorem 5, namely a strengthened version of Wierman's coupling

(Section 5), and a “propagation” lemma for oriented percolation (Section 6). Finally, we present in Section 7 some open problems that arise naturally from this work.

2. Preliminaries

2.1. Notation

Let $d \geq 3$. We work with the hypercubic lattice \mathbb{Z}^d , whose vertices are the points with integer coordinates, and two vertices are connected by an edge if and only if they are at Euclidean distance 1 from each other. Write $u \sim v$ if u and v are two neighbors.

Bernoulli site percolation on \mathbb{Z}^d , with parameter $p \in (0, 1)$, can be represented as follows. A percolation configuration is of the form $(\omega_v)_{v \in \mathbb{Z}^d}$, where each ω_v (the state of vertex v) is either 0 or 1. We denote by $\Omega := \{0, 1\}^{\mathbb{Z}^d}$ the set of configurations, and we equip it with the cylindrical σ -algebra \mathcal{F} . We consider on Ω the product measure $\mathbb{P}_p := (p\delta_1 + (1 - p)\delta_0)^{\otimes \mathbb{Z}^d}$, under which vertices have independently state 0 or 1, with respective probabilities $1 - p$ and p . This random coloring defines a partition of the lattice: the vertices can be grouped into connected components, or *clusters*, of 0-vertices and of 1-vertices.

As the parameter p varies, Bernoulli site percolation displays a phase transition, i.e. a major change of macroscopic behavior, at a certain critical value $p_c = p_c^{\text{site}}(\mathbb{Z}^d) \in (0, 1)$. For all $p < p_c$, there is \mathbb{P}_p -almost surely no infinite cluster of 1-vertices, while for all $p > p_c$, there is \mathbb{P}_p -almost surely such an infinite cluster, which moreover turns out to be unique. As mentioned in the introduction, we know from [2] that $p_c^{\text{site}}(\mathbb{Z}^d) < 1/2$. We thus consider a value $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, for which infinite clusters of both colors coexist almost surely, and we can assume without loss of generality that $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1/2]$.

Recall that we denote by $\Xi := \{0, 1\}^{\mathbb{N}}$ the set of infinite words, and (for $v \in \mathbb{Z}^d$ and $V \subseteq \mathbb{Z}^d$) recall the notations $S(v)$, S_V , $S_\infty (\subseteq \Xi)$ from the introduction. For $\ell \geq 1$, we also introduce the set $\Xi_\ell := \{0, 1\}^\ell$ of finite words $\xi = (\xi_0, \dots, \xi_{\ell-1})$ of length ℓ . For $\xi \in \Xi$, the finite subword between indices i and j ($0 \leq i \leq j$) is denoted by $\xi_{[i,j]} := (\xi_i, \dots, \xi_j) (\in \Xi_{j-i+1})$.

For two vertices $v, v' \in \mathbb{Z}^d$ and an infinite word $\xi \in \Xi$, we denote by $v \overset{\xi}{\rightsquigarrow} v'$ the event that there exists $n \geq 1$ and a self-avoiding path $v = v_0 \sim v_1 \sim \dots \sim v_{n-1} = v'$ from v to v' along which the beginning of ξ is seen, i.e. such that $\omega_{v_i} = \xi_i$ for all $i \in \{0, \dots, n - 1\}$. Note that when ξ is the infinite monochromatic word $\mathbb{1} := (1, 1, \dots)$, $v \overset{\mathbb{1}}{\rightsquigarrow} v'$ refers to the usual 1-connectedness between vertices. For $\xi \in \Xi$ and $v \in \mathbb{Z}^d$, we often use the notation $v \overset{\xi}{\rightsquigarrow} \infty$ for the event that ξ is seen from v (i.e. $\xi \in S(v)$). The percolation probability is denoted, as usual, by $\theta(p) := \mathbb{P}_p(0 \overset{\mathbb{1}}{\rightsquigarrow} \infty)$.

We also need the following notion of “exact” word-connectedness. For a finite word $\xi \in \Xi_\ell$ (for some $\ell \geq 1$), we denote by $v \overset{\xi}{\rightarrow} v'$ the event that there exists a self-avoiding path $v = v_0 \sim v_1 \sim \dots \sim v_{\ell-1} = v'$ of length ℓ along which ξ is seen.

2.2. Measurability properties

In this section, we recall some useful measurability properties that were established in [1]. The most important one is $\{S_\infty = \Xi\} \in \mathcal{F}$, from [1, Proposition 2]. A zero-one law thus holds for this event, by ergodicity.

Proving this result requires some work, and it is obtained in [1] from an application of the Baire category theorem. This proof actually yields the following stronger property. If $S_\infty = \Xi$, i.e. all words are seen somewhere, then there exists a finite subset of vertices $F \subseteq \mathbb{Z}^d$ such that all words are seen from F :

$$\{S_\infty = \Xi\} = \bigcup_{\substack{F \subseteq \mathbb{Z}^d \\ |F| < \infty}} \{S_F = \Xi\} \tag{2.1}$$

(see [1, Remark 1]).

For values $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, the constructions used for Theorem 1 do not provide a “constructive” proof of measurability for $\{S_\infty = \Xi\}$. However, they do show that there exists a measurable subset of $\{S_\infty = \Xi\}$, which has full probability, and on which a finite subset of vertices F as in (2.1) exists (as well as a quantitative statement, see Theorem 5 below).

Remark 2. Note that the following (easier to establish) property holds as well. For every $\xi \in \Xi$, $\{\xi \in S(v)\} \in \mathcal{F}$ for all $v \in \mathbb{Z}^d$, so $\{\xi \in S_\infty\} \in \mathcal{F}$. Hence, a zero-one law follows readily, again by ergodicity:

$$\text{for all } \xi \in \Xi, \quad \mathbb{P}_p(\xi \in S_\infty) \in \{0, 1\} \tag{2.2}$$

(see [1, Proposition 3]).

2.3. Wierman’s coupling

We are going to use a construction due to Wierman [17] (see also [13]) to translate high 1-connectedness in the supercritical regime into high “ ξ -connectedness”. It produces the basic building blocks of our renormalization strategy, taking care of reading words at the microscopic level. A simple way of stating it is the following: for every $\xi \in \Xi$,

$$\mathbb{P}_p(0 \overset{\xi}{\rightsquigarrow} \infty) \geq \mathbb{P}_p(0 \overset{1}{\rightsquigarrow} \infty) = \theta(p) > 0$$

(recall that we assume $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1/2)$). Hence, we can use (2.2) to get

$$\text{for all } \xi \in \Xi, \quad \mathbb{P}_p(\xi \in S_\infty) = 1. \tag{2.3}$$

In a similar way as in the introduction, this implies, using Fubini’s theorem, that for every $q \in (0, 1)$, \mathbb{P}_p -a.s., μ_q -almost every word $\xi \in \Xi$ can be seen, where μ_q is the product measure $((1 - q)\delta_0 + q\delta_1)^{\otimes \mathbb{N}}$ on Ξ .

Remark 3. The property (2.3) does not imply, in general, that $\mathbb{P}_p(S_\infty = \Xi) = 1$. A counter-example is provided by [1, Theorem 5], on a well-chosen tree (see in particular [1, (7.5) and (7.14)]).

The construction of Wierman’s coupling uses an iterative exploration of the 1-cluster of a given vertex. For any fixed $\xi \in \Xi$, it constructs two coupled configurations $(\omega, \tilde{\omega})$ such that $\omega, \tilde{\omega} \sim \mathbb{P}_p$, and having the property that for every $y \in \mathbb{Z}^d$,

$$\omega \in \{0 \overset{1}{\rightsquigarrow} y\} \text{ implies } \tilde{\omega} \in \{0 \overset{\xi}{\rightsquigarrow} y\}. \tag{2.4}$$

We will actually use a generalization of this construction, where we replace the single vertex 0 by a set S equipped with a set of words $(\xi^{(x)})_{x \in S} \in \Xi^S$.

Lemma 4. *Let $\Lambda \subseteq \mathbb{Z}^d$, $S \subseteq \Lambda$, and $(\xi^{(x)})_{x \in S} \in \Xi^S$. There exist two coupled random configurations $(\omega, \tilde{\omega})$ such that $\omega, \tilde{\omega} \sim \mathbb{P}_p$, and having the following property:*

$$\text{for all } y \in \Lambda, \quad \omega \in \{\exists x \in S : x \overset{1}{\rightsquigarrow} y \text{ in } \Lambda\} \text{ implies } \tilde{\omega} \in \{\exists x \in S : x \overset{\xi^{(x)}}{\rightsquigarrow} y \text{ in } \Lambda\}.$$

We postpone the proof of Lemma 4 to Section 5.

3. Main result

In this section we state our main contribution, Theorem 5, and we prove it assuming an auxiliary result, Proposition 7, that will be established in Section 4.

3.1. Statement

Theorem 5. *Let $d \geq 3$, and $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$.*

(1) *We have*

$$\mathbb{P}_p(S_\infty = \Xi) = 1. \tag{3.1}$$

Moreover, there exists $l (= l(d, p))$ such that \mathbb{P}_p -a.s., all words can be seen in the slab $\mathbb{Z}^2 \times [0, l]^{d-2}$.

(2) *The following quantitative statement holds, where $B_m(0) := [-m, m]^d$ is the ball with radius m (for the norm $\|\cdot\|_\infty$) around 0 in \mathbb{Z}^d . There exists a constant $c_0 = c_0(d, p) > 0$ such that*

$$\text{for all } m \geq 1, \quad \mathbb{P}_p(S_{B_m(0)} = \Xi) \geq 1 - e^{-c_0 m^{d-1}}. \tag{3.2}$$

Note that (3.1) follows immediately from (3.2) by letting $m \rightarrow \infty$. Also, an analogous upper bound clearly holds in (3.2), since it is known to be the order of magnitude (up to a different choice of c_0) for $\mathbb{P}_p(B_m(0) \overset{1}{\rightsquigarrow} \infty)$.

Remark 6. There is nothing special about our using two colors. For example, let $I \geq 3$ and consider the process (still in dimension $d \geq 3$) where the vertices are colored independently $0, \dots, I - 1$, with respective probabilities $p_0, \dots, p_{I-1} \in [0, 1]$ (satisfying $p_0 + \dots + p_{I-1} = 1$). If all these parameters are chosen to be strictly larger than $p_c^{\text{site}}(\mathbb{Z}^d)$ (which is possible if d is large enough), then Wierman’s coupling can easily

be adapted, and a result analogous to Theorem 5 holds. These parameters $(p_i)_{0 \leq i \leq I-1}$ do not even need to be space-homogeneous: if they are uniformly bounded from below by $p_c^{\text{site}}(\mathbb{Z}^d) + \varepsilon$ for some $\varepsilon > 0$, then the same arguments work. Finally, let us mention that the methods in our proof of Theorem 5 could also be used to construct point-to-point ξ -connections.

3.2. Proof of Theorem 5

For some given $\delta > 0$ and $k \geq 1$, which we explain how to choose in Proposition 7 below, we write

$$C = C(k) := |[-k, k]^d| = (2k + 1)^d. \tag{3.3}$$

In the slab

$$S_h := \mathbb{Z}^2 \times (0, hk] \times (-k, k]^{d-3} \subseteq \mathbb{Z}^d \quad (h \geq 2),$$

we consider the finite box

$$\Lambda_n := [-kn, kn]^2 \times (0, hk] \times (-k, k]^{d-3} \quad (n \geq 1),$$

as well as its inner vertex boundary $\partial_S \Lambda_n$ with respect to the slab, i.e.

$$\partial_S \Lambda_n := \{x \in \Lambda_n : x \sim y \text{ for some } y \in S_h \setminus \Lambda_n\} \tag{3.4}$$

(see Figure 2).

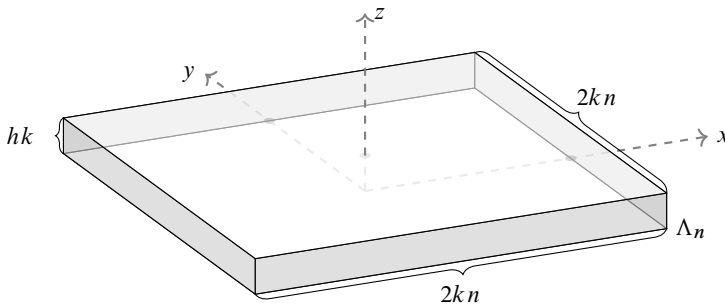


Fig. 2. Illustration of the box Λ_n for $d = 3$, with $\partial_S \Lambda_n$ marked in transparent gray.

For any given $n > m \geq 1$ and $\xi \in \mathfrak{E}$, we introduce the event

$$E_m^n(\xi) := \left\{ \begin{array}{l} \exists T \subseteq \partial_S \Lambda_n, \text{ with } |T| \geq 8\delta \cdot |\partial_S \Lambda_n|, \text{ such that for all } y \in T, \\ \text{there exists } x \in \partial_S \Lambda_{m+1} \text{ with } x \xrightarrow{\xi_{[0,tx]}} y \text{ for some } t_x \leq Cn \end{array} \right\}. \tag{3.5}$$

When $n = m$, we define $E_m^n(\xi)$ to be the whole probability space.

The constants δ and k are chosen to ensure the existence of microscopic connections as described in Proposition 7 below, and we think of them as fixed. The role of h (thickness of the slab) is explained by the computation in the proof of Theorem 5 below. We denote by \mathbb{P}_p^h the Bernoulli site percolation measure on the slab S_h .

Proposition 7. *For every $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, there exist $\delta, c > 0$ (small enough) and $k \geq 1$ (large enough) such that for all $n \geq m \geq 1, h \geq 20$ and $\xi \in \Xi$,*

$$\mathbb{P}_p^h(E_m^n(\xi) \setminus E_m^{2n}(\xi)) \leq e^{-chn}. \tag{3.6}$$

Proposition 7 is really the main technical result of this paper, and its proof is postponed to the next section. Assuming its validity, our main result follows easily, as we explain now.

Proof of Theorem 5. Pick k and δ as in Proposition 7 and fix some integer m . For $n = m$, the event $E_m^n(\xi)$ was defined to be the whole space, therefore

$$\mathbb{P}_p^h(E_m^n(\xi)) = 1.$$

We now observe that for every word $\xi \in \Xi$, the event $\bigcap_{j \geq 1} E_m^{2^j m}(\xi)$ implies in particular that $\partial_S \Lambda_{m+1} \xrightarrow{\xi} \partial_S \Lambda_{2^j m}$ for all $j \geq 1$, from which it follows (as one can easily convince oneself, using a diagonal argument) that $\partial_S \Lambda_{m+1} \xrightarrow{\xi} \infty$. We can thus write

$$\begin{aligned} \mathbb{P}_p^h(\exists \xi \in \Xi \text{ such that } \partial_S \Lambda_{m+1} \xrightarrow{\xi} \infty) &\leq \mathbb{P}_p^h\left(\bigcup_{n \geq m} \bigcup_{\xi \in \Xi_{2Cn}} (E_m^n(\xi) \setminus E_m^{2n}(\xi))\right) \\ &\leq \sum_{n \geq m} \sum_{\xi \in \Xi_{2Cn}} \mathbb{P}_p^h(E_m^n(\xi) \setminus E_m^{2n}(\xi)), \end{aligned}$$

where we have used the fact that the event $E_m^n(\xi) \setminus E_m^{2n}(\xi)$ only depends on the first $2Cn$ coordinates of the word ξ (this is a key observation in order to apply the union bound). Now, it follows from Proposition 7 that

$$\mathbb{P}_p^h(\exists \xi \in \Xi \text{ such that } \partial_S \Lambda_{m+1} \xrightarrow{\xi} \infty) \leq \sum_{n \geq m} 2^{2Cn} \cdot e^{-chn}, \tag{3.7}$$

and we can thus choose h_0 large enough so that the latter probability is at most e^{-m} . By taking the $m \rightarrow \infty$ limit, we obtain part (1) of the statement, with $l = h_0 k$.

Let us now turn to part (2), the quantitative statement. For that, it is enough to observe that by “slicing up” the box $B_m(0)$, we can find, for some constant $c' > 0$, at least $c' m^{d-2}$ disjoint slabs, of height $h_0 k$, whose intersection with this box is a graph isomorphic to Λ_m . Hence, we have these many independent attempts to see all words starting from $B_m(0)$ in a slab. Since each of these attempts fails with probability at most e^{-m} , from the proof of part (1), we obtain the desired lower bound (3.2). ■

4. Proof of Proposition 7

In this section, we prove Proposition 7, which was used in the proof of Theorem 5. For that, we use two auxiliary results that are established later: Lemma 4 and Proposition 10, proved in Sections 5 and 6 respectively.

4.1. Renormalization setting

We will use a dynamic renormalization procedure similar to the standard one presented in [6, Section 7.2]. Before describing this process, we introduce the necessary geometric framework. For a given word $\xi \in \Xi$, the idea is to construct a “ ξ -cluster” in the slab \mathbb{S}_h by propagating “seeds”, i.e. sets which have a reasonably good probability to be ξ -connected to infinity, from boxes to boxes (in an oriented way) using local connections. Let k be a fixed even integer.

For $h \geq 1$, we define the (oriented) renormalization graph $\overrightarrow{\text{Slab}}_h = (V, \overrightarrow{E})$, where

$$V := \left\{ v = (v_1, v_2, v_3) \in \mathbb{Z}^3 : 0 < v_3 < h, v_1 \in 2\mathbb{Z}, \frac{v_1}{2} + v_2 \in 2\mathbb{Z} \text{ and } \frac{v_1}{2} + v_3 \in 2\mathbb{Z} \right\} \tag{4.1}$$

($\subset \mathbb{Z}^2 \times (0, h)$), equipped with the oriented edges

$$\overrightarrow{uv} \in \overrightarrow{E} \quad \text{if and only if} \quad v \in u + \{(2, \varepsilon, \varepsilon') : \varepsilon, \varepsilon' \in \{\pm 1\}\} \tag{4.2}$$

(see Figure 3). We explore an oriented site percolation on this graph, that will later be coupled with our microscopic process.

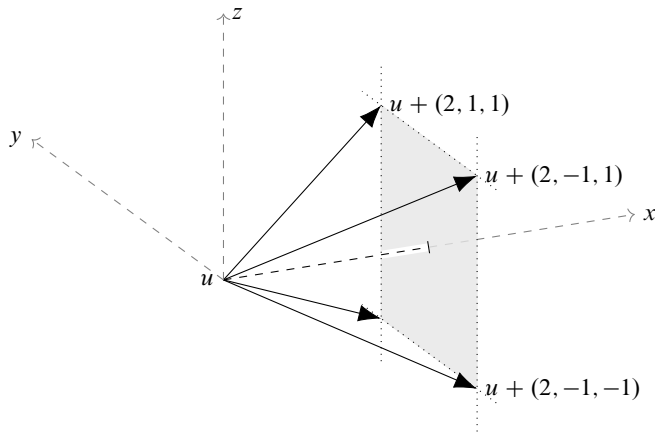


Fig. 3. We consider the renormalization graph $\overrightarrow{\text{Slab}}_h$, where each vertex u is connected to the (at most four) vertices $u + \{(2, \pm 1, \pm 1)\}$.

For $U \subseteq V$, we define

$$U^+ := \bigcup_{u \in U} \{v \in V : uv \in \overrightarrow{E}\} \quad \text{and} \quad \partial^+ U := U^+ \setminus U. \tag{4.3}$$

Intuitively speaking, we see the graph $\overrightarrow{\text{Slab}}_h$ as a renormalized version of the slab \mathbb{S}_h ($= \mathbb{Z}^2 \times (0, hk] \times (-k, k]^{d-3}$) introduced in Section 3.2, scaled by a factor $\frac{1}{2k}$. This intuitive correspondence will be made clear through (4.4) and the exploration process defined underneath it; see also Figure 4. We call $\overrightarrow{\text{Slab}}_h$ and \mathbb{S}_h the *macroscopic* and *microscopic*

slabs, respectively. Note that the macroscopic graph is oriented, while the microscopic graph is not. In order to distinguish between the two graphs, we use the following typographic convention. For vertices in \mathbb{S}_h , we write x, y, z, \dots , and subsets of vertices are denoted by A, B, C, \dots ; for vertices in $\overrightarrow{\text{Slab}}_h$, we write u, v, w, \dots , and subsets of vertices are denoted by A, B, C, \dots .

In order to carry out the renormalization procedure described at the beginning of the section, we now define *boxes*, *seeds*, and *good local connections*.

Boxes: With every vertex $u \in V$ in the macroscopic slab, we associate two subsets of the microscopic slab, a box B^u and a face F^u , defined as follows. Set

$$B := (-k, k]^d, \quad F := \{-k\} \times (-k, k]^{d-1},$$

and for every $u \in V$,

$$B^u := ku + B \quad \text{and} \quad F^u := ku + F \tag{4.4}$$

(we see ku as an element of \mathbb{Z}^d by identifying \mathbb{Z}^3 and $\mathbb{Z}^3 \times \{0_{\mathbb{Z}^{d-3}}\}$). By definition, F^u is disjoint from B^u . Note also that the boxes B_u are disjoint from each other for different values of u , as illustrated in Figure 4.

Seeds: For $\delta \in (0, 1)$, a δ -seed for $u = (u_1, u_2, u_3)$ is a pair (S, t) where

- $S \subseteq F^u$ with $|S| \geq \delta|F^u|$,
- $t = (t_x)_{x \in S}$ is a collection of nonnegative integers such that $t_x \leq Cu_1$ for every $x \in S$. Informally speaking, one can think of t_x as encoding the time at which we arrived at x , when reading the word ξ .

We write \mathcal{S}_δ^u for the set of all δ -seeds for u .

Intuitively, we will use seeds as follows. Starting from a given subset $L \subseteq \mathbb{Z}^d$, we want to explore the vertices that can be reached while reading the word ξ , which we do by examining the boxes B^u one after the other.

Assume that at some given step, the exploration reaches the box B^u . At that moment, we can read the word ξ from L to every point in a set S on the face F^u without using B^u , and we now want to extend the exploration inside this box. A *seed* is simply a set S that is sufficiently large to ensure that the exploration has a good probability to continue within that box.

Notice that knowing the set S alone is not sufficient if one wants to continue the exploration in B^u . When we read the word ξ from L to a vertex x in S , one needs to keep track of where we stand along ξ , i.e. at which index t_x , when the exploration reaches x . We bring the reader’s attention to the condition $t_x \leq Cu_1$ in the definition of a seed, which will play an important role: we want to keep track of the length of the path connecting L to x , so that we can control the “entropy” created by the words.

Good local connections: As explained above, we want to continue the exploration inside a given box B^u . We say that this step succeeds when the event below occurs. Let $\delta \in (0, \frac{1}{64000})$. Given a word $\xi \in \Xi$ and a δ -seed (S^u, t^u) for u , we define the “good event”

$$\begin{aligned}
 &G_\xi^u(S^u, t^u) \\
 &:= \bigcap_{v \in \{u\}^+} \{ \exists (S^v, t^v) \in \mathcal{S}_{64000\delta}^v : \forall y \in S^v, \exists x \in S^u \text{ with } x \xrightarrow{\xi_{[u_x^u, t_y^v]}} y \text{ inside } F^u \cup B^u \}
 \end{aligned}
 \tag{4.5}$$

(see Figure 4 for an illustration of the boxes involved).

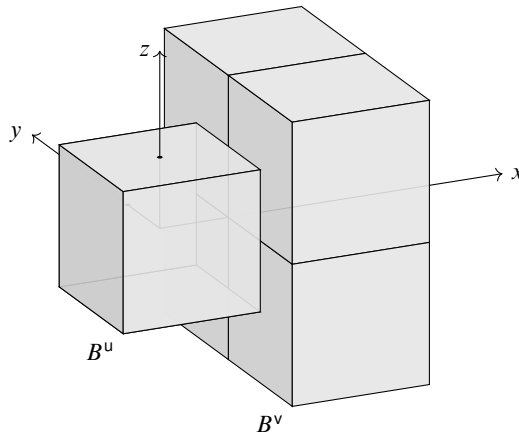


Fig. 4. Example of a box B^u together with its neighbors $\{B^v\}$, $v \in \{u\}^+$. The corresponding faces F^u and $(F^v)_{v \in \{u\}^+}$ are colored in darker gray.

The following proposition ensures that such local connections occur with sufficiently high probability, provided the scaling constant k is chosen large enough.

Proposition 8. *Let $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$ and $\zeta < 1$. There exists $0 < \delta(d, p) < \frac{1}{64000}$ small enough, and $k = k(d, p, \zeta) \geq 1$ large enough, such that for all $\xi \in \Xi$, $u \in V$ and $(S, t) \in \mathcal{S}_\delta^u$,*

$$\mathbb{P}_p(G_\xi^u(S, t)) \geq \zeta.
 \tag{4.6}$$

We emphasize that the choice of δ in the proposition above does not depend on ζ . This is important later on, to ensure that there is no circularity in our choice of parameters.

Proof of Proposition 8. We first observe from Wierman’s coupling (Lemma 4) that we can restrict ourselves to $\xi = \mathbb{1}$, i.e. it is sufficient to show the following: for some $\delta > 0$ and $k \geq 1$, respectively small and large enough,

$$\mathbb{P}_p \left(S \text{ is 1-connected within } B^u \text{ to at least } 64000\delta |F^v| \text{ vertices} \right. \\
 \left. \text{in each of the faces } F^v, v \in \{u\}^+ \right) \geq \zeta.
 \tag{4.7}$$

Indeed, this allows one to produce $(S^v, t^v) \in \mathcal{S}_{64000\delta}^v$, since (once k is chosen) the condition on t^v is automatically satisfied, using $|B^u| \leq C$ (recall that $C = C(k)$ was defined in (3.3)).

This result follows immediately from classical properties of percolation theory, but we give a brief sketch of proof for the sake of completeness. By symmetry, it suffices to prove (4.7) for every seed S and for a single face F^v , instead of for all faces at once. Fix a face F^v with $v \in \{u\}^+$, let \tilde{F} be the intersection of F^v with B^u (which has side length k), and let \hat{F} be the middle third of \tilde{F} , that is, the vertices of \tilde{F} that are within a distance at least $k/3$ from its four edges.

Note that for every vertex x in \hat{F} , the ball $B(x, k/3) \cap B^u$ is isomorphic to the intersection of $B(0, k/3)$ with the half-space $\mathbb{Z}^+ \times \mathbb{Z}^{d-1}$. It is a well-known fact that the critical parameters for site percolation on this half-space and on the full space \mathbb{Z}^d coincide (see [6, Theorem (7.2) (b)]).

Let $\theta > 0$ be the \mathbb{P}_p -probability that 0 is connected to infinity in the half-space $\mathbb{Z}^+ \times \mathbb{Z}^{d-1}$. By mixing and a second moment estimate, we have

$$\mathbb{P}_p \left(\begin{array}{l} \text{at least } \theta|\hat{F}|/2 \text{ points in } \hat{F} \text{ are} \\ \text{1-connected to distance } k/3 \text{ within } B^u \end{array} \right) \xrightarrow[k \rightarrow \infty]{} 1. \tag{4.8}$$

We now fix $\delta > 0$ such that

$$64000\delta|F| \leq \frac{\theta}{2}|\hat{F}|. \tag{4.9}$$

Observe that this choice is independent of ζ (but it depends on d and p). For this choice, (4.8) immediately implies

$$\mathbb{P}_p \left(\begin{array}{l} \text{at least } 64000\delta|F| \text{ points in } \hat{F} \text{ are} \\ \text{1-connected to distance } k/3 \text{ within } B^u \end{array} \right) \xrightarrow[k \rightarrow \infty]{} 1. \tag{4.10}$$

Similarly, we can prove that

$$\inf_{\substack{S \subseteq F^u \\ |S| \geq \delta|F^u|}} \mathbb{P}_p \left(\begin{array}{l} \text{some point in } S \text{ is 1-connected} \\ \text{to distance } \sqrt{k} \text{ within } B^u \cup F^u \end{array} \right) \xrightarrow[k \rightarrow \infty]{} 1. \tag{4.11}$$

We finally make use of [15, Theorem 3.1] to conclude that in a given box with side length k , the existence of two distinct 1-connected components of size at least \sqrt{k} , as above, has a vanishing probability as $k \rightarrow \infty$. This completes the proof of Proposition 8. ■

4.2. Exploration of an oriented percolation in the macroscopic slab

Let $n \geq 3$ (for simplicity, n is always assumed to be of the form $n = 4n' + 3$ for some $n' \in \mathbb{N}$, and h is always assumed to be even). Define

$$\begin{aligned} B_n &:= V \cap ((n, 2n) \times (-2n, 2n) \times (0, h)), \\ L_n &:= V \cap (\{n + 1\} \times [-n, n] \times (0, h)), \\ R_n &:= V \cap (\{2n - 1\} \times [-n, n] \times (0, h)), \end{aligned} \tag{4.12}$$

as depicted in Figure 5.

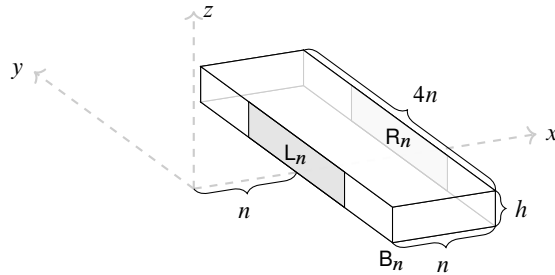


Fig. 5. Illustration of the sets B_n , L_n and R_n .

We also choose an arbitrary ordering of B_n ; to fix ideas, we can decide to order the vertices in the lexicographic way, which ensures that $u = (u_1, u_2, u_3)$ is smaller than $v = (v_1, v_2, v_3)$ when $u_1 < v_1$. It will be simpler to think in terms of this particular ordering since it gives rise to reasoning “layer by layer”, even though the particular choice of an ordering is immaterial for the correctness of the proof.

Fix a set $S \subseteq L_n$, and let

$$S = U_0 \subseteq U_1 \subseteq \dots \quad \text{and} \quad \emptyset = V_0 \subseteq V_1 \subseteq \dots$$

be two growing sequences of subsets of B_n . Such sequences will appear when we explore all the (oriented) clusters touching S : the sequence (U_i) corresponds to the exploration of the open vertices in these clusters, while the sequence (V_i) explores their boundaries. The sequences arising in this context are defined as follows, where at each step we add at most one vertex to U_i (collecting the open vertices connected to S) or V_i (the closed boundary). We say that the sequence $(X_i)_{i \geq 0} = (U_i, V_i)_{i \geq 0}$ is an *exploration sequence from S* (in B_n) if for every $i \geq 0$,

$$X_{i+1} = \begin{cases} (U_i, V_i) & \text{if } \partial^+ U_i \cap B_n \subseteq V_i, \\ (U_i \cup \{z_i\}, V_i) \text{ or } (U_i, V_i \cup \{z_i\}) & \text{otherwise,} \end{cases} \tag{4.13}$$

where z_i is the minimum vertex that belongs to $\partial^+ U_i \cap B_n$, but not to V_i .

Note that the exploration terminates in finite time (since we explore a finite graph), so the growing sequence $(U_i)_{i \geq 0}$ is ultimately constant; we denote by U_∞ its end value.

Lemma 9. *For every $\delta > 0$, there exist $\zeta = \zeta(\delta) < 1$ and $c = c(\delta) > 0$ such that the following holds. Let $h \geq 20$. Assume that*

$$S \subseteq L_n \quad \text{with} \quad |S| \geq \delta |L_n|,$$

and that $(X_i)_{i \geq 0} = (U_i, V_i)_{i \geq 0}$ is a random exploration sequence from S in B_n satisfying: for each $i \geq 0$,

$$\mathbb{P}(V_{i+1} = V_i \mid X_0, \dots, X_i) \geq \zeta \quad \text{a.s.} \tag{4.14}$$

Then

$$\mathbb{P}(|U_\infty \cap R_n| \geq \frac{1}{1000} |R_n|) \geq 1 - e^{-chn}. \tag{4.15}$$

Before proving this lemma, let us state an auxiliary result which is proved in Section 6. Write P_γ for the site percolation measure³ in the slab $\overrightarrow{\text{Slab}}_h$ with parameter γ : independently, each vertex is open with probability γ , closed with probability $1 - \gamma$. For $A \subset V$ and $v \in V$, we write $A \rightarrow v$ if there exists an open oriented path of $\overrightarrow{\text{Slab}}_h$ from a vertex of A to v . The notation above corresponds to the existence of paths in standard oriented site percolation, and it should not be confused with the notation $x \xrightarrow{\xi} y$ introduced at the end of Section 2.1 and corresponding to percolation of words.

Proposition 10. *For every $\delta \in (0, 1)$, there exist $\gamma \in (0, 1)$ and $c > 0$ such that the following holds. For all $h \geq 20$ and n large enough, if $S \subseteq L_n$ satisfies $|S| \geq \delta|L_n|$, then for oriented site percolation, we have*

$$P_\gamma(|\{v \in R_n : S \rightarrow v\}| \leq \frac{1}{1000}|R_n|) \leq e^{-chn}.$$

We postpone the proof of the above proposition to Section 6.

We now write out in detail the proof of Lemma 9, assuming Proposition 10. We provide it for the sake of completeness, and also to indicate how the parameters involved in the exploration are chosen precisely.⁴ However, a reader with experience in this type of argument can safely skip the proof.

Proof of Lemma 9. We follow closely [8, proof of Lemma 1]. Let $(Y_v)_{v \in B_n}$ be i.i.d. random variables with uniform distribution on $[0, 1]$. We construct an exploration sequence $X' = (U', V')$ with the same law as the one in the statement of the lemma, in such a way that all vertices v ultimately in $V' = V'_\infty$ satisfy $Y_v > \zeta$ (these vertices are called “red”). Define

$$X'_0 = (U'_0, V'_0) := (S, \emptyset),$$

and declare all the vertices of S to be “green”. Let z be the minimum vertex that belongs to $\partial^+ U_0$ (notice that z is always well-defined at this first step if $n \geq 1$). Then define $\zeta_z = \mathbb{P}(X_1 = (S \cup \{z\}, \emptyset) \mid X_0 = (S, \emptyset))$ and set

$$X'_1 = \begin{cases} (S \cup \{z\}, \emptyset) & \text{if } Y_z \leq \zeta_z, \\ (S, \{z\}) & \text{if } Y_z > \zeta_z. \end{cases} \tag{4.16}$$

Declare the vertex z to be “green” in the first case, and “red” in the second case. The conditional probability ζ_z above is defined in such a way that (X'_0, X'_1) has exactly the same law as the first two steps of the random exploration sequence of the lemma.

³The parameter γ mentioned in this informal description corresponds later to the value ζ appearing in Proposition 8. More precisely, we let ζ be the value $\gamma(\delta)$ from Proposition 10, where $\delta = \delta(d, p)$ is produced by Proposition 8, and we then apply Proposition 8 with this value of ζ to obtain the value of the scaling constant k .

⁴In the following proof, we derive Lemma 9 by applying our general result for oriented percolation, Proposition 10, with the same parameter δ . This produces a value $\gamma = \gamma(\delta)$, which is then exactly ζ in the conclusion of Lemma 9.

We proceed iteratively. Assume that $X'_0 = (U'_0, V'_0), \dots, X'_i = (U'_i, V'_i)$ have been constructed. Let z be the minimum vertex that belongs to $\partial^+ U'_i \setminus V'_i$ (if $\partial^+ U'_i \setminus V'_i$ is empty then we set $X'_{i+1} = (U'_{i+1}, V'_{i+1}) = (U'_i, V'_i)$). In case z is well-defined, consider $\zeta_z = \mathbb{P}(X_{i+1} = (U_i \cup \{z\}, V_i) \mid X_0 = X'_0, \dots, X_i = X'_i)$ and set

$$X'_{i+1} = \begin{cases} (U'_i \cup \{z\}, V'_i) & \text{if } Y_z \leq \zeta_z, \\ (U'_i, V'_i \cup \{z\}) & \text{if } Y_z > \zeta_z. \end{cases} \tag{4.17}$$

As in the first step, declare the vertex z to be “green” in the first case and “red” in the second case.

Once the algorithm above terminates (when $X'_{i+1} = X'_i$), declare all the vertices that have not been explored to be “green”.

Since at each step, the uniform random variable Y_z is independent of the previous steps, the process X' has the same law as X . Furthermore, since for every explored vertex z we have $\zeta_z \geq \zeta$ (by hypothesis), we see that the set of green vertices dominates a Bernoulli site percolation with parameter ζ . By Proposition 10, if ζ is close enough to 1, there exist more than $\frac{1}{1000} |R_n|$ vertices of $|R_n|$ that can be reached from S by a green oriented path with probability larger than $1 - e^{-chn}$. By definition of the exploration sequence, all these vertices have been reached by the exploration sequence. This implies that

$$|U'_\infty \cap R_n| \geq \frac{1}{1000} |R_n| \tag{4.18}$$

with probability larger than $1 - e^{-chn}$. ■

4.3. Proof of Proposition 7

Fix $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$. Let $\delta = \delta(d, p)$ be as in Proposition 8, and $\zeta \in (0, 1)$ associated with it by Lemma 9. Finally, let k be as in Proposition 8 (applied with p and our choice of ζ).

Consider some $n \geq 3$. It is convenient to assume that $n = 4n' + 3$ for some $n' \in \mathbb{N}$. There are two ways to see why one can make this assumption. Either one observes that after having proved Proposition 7 along this subsequence we can obtain the full proposition by slightly changing δ and giving the points in T a second chance to advance towards the next layer. Another way to see why this assumption is sufficient is to observe that the proof of Theorem 1 uses Proposition 7 along a very sparse subsequence that could have been chosen to coincide with the integers of the form above.

Having this assumption, we define the following subsets of the microscopic graph S_h :

$$\begin{aligned} B_n &:= \mathbb{Z}^d \cap ([kn, 2kn] \times (-2kn, 2kn] \times (0, hk] \times (-k, k]^{d-3}), \\ L_n &:= \mathbb{Z}^d \cap (\{kn\} \times (-kn, kn] \times (0, hk] \times (-k, k]^{d-3}), \\ R_n &:= \mathbb{Z}^d \cap (\{2kn\} \times (-kn - 3k, kn + 3k] \times (0, hk] \times (-k, k]^{d-3}). \end{aligned} \tag{4.19}$$

A picture of the above sets would look very similar to that of the macroscopic B_n , L_n and R_n in Figure 5. Notice that

$$\bigcup_{u \in B_n} B^u \subset B_n, \quad L_n \subset \bigcup_{u \in L_n} F^u \quad \text{and} \quad \bigcup_{u \in \partial^+ R_n} F^u \subset R_n \tag{4.20}$$

(recall the notations B^u and F^u from (4.4)). Let $\xi \in \Xi$. We first claim that for every fixed set $T \subseteq L_n$ with $|T| \geq 2\delta|L_n|$, for all $t : T \rightarrow [0, Cn]$, we have

$$\mathbb{P}_p \left(\begin{array}{l} \exists T' \subseteq R_n \text{ with } |T'| \geq 8\delta \cdot |\partial_{\mathbb{S}} \Lambda_{2n}|, \exists t' : T' \rightarrow [0, 2Cn] \text{ s.t.} \\ \text{for all } y \in T', \exists x \in T \text{ with } x \xrightarrow{\xi_{[t_x, t'_y]}} y \text{ in } B_n \end{array} \right) \geq 1 - e^{-chn}. \tag{4.21}$$

Before proving (4.21), let us explain how it yields Proposition 7. Assume that $E_m^n(\xi)$ occurs and consider a set T as in (3.5). Partition T into four disjoint subsets along the left, right, top and bottom parts of $\partial_{\mathbb{S}} \Lambda_n$ (the boundary of Λ_n with respect to the slab \mathbb{S}_h , defined in (3.4)). By symmetry, we may assume for example that the right part, denoted by T_r , has size larger than $|T|/4 \geq 2\delta|L_n|$. Finally, using independence, we can deduce from (4.21) that with probability larger than $1 - e^{-chn}$, the set T_r gives rise to a set T' in $\partial_{\mathbb{S}} \Lambda_{2n}$ of size larger than $8\delta|\partial_{\mathbb{S}} \Lambda_{2n}|$. This yields

$$\mathbb{P}_p(E_m^{2n}(\xi) \mid E_m^n(\xi)) \geq 1 - e^{-chn}, \tag{4.22}$$

which completes the proof of Proposition 7. The remainder of this section is devoted to proving (4.21).

We first set some notation. Let $T \subseteq L_n$ with $|T| \geq 2\delta|L_n|$, and $t : T \rightarrow [0, Cn]$. Introduce

$$T := \{u \in L_n : |T \cap F^u| \geq \delta|F|\}. \tag{4.23}$$

Using that T has density at least 2δ in L_n , the second inclusion in (4.20) implies that T has density at least δ in L_n , i.e.

$$|T| \geq \delta|L_n|. \tag{4.24}$$

Let ω be a configuration of site percolation with parameter p in B_n . We want to define a random exploration process in the macroscopic box B_n that will correspond to a ξ -exploration in the microscopic box B_n . For $U \subseteq B_n$, set $B^U := \bigcup_{u \in U} B^u$. For $v = (v_1, v_2, v_3) \in U^+$, define

$$S^v(U) := \bigcup_{t \leq Cv_1} \bigcup_{x \in T} \{y \in F^v : x \xrightarrow{\xi_{[t_x, t]}} y \text{ in } B^U\}, \tag{4.25}$$

and for every $y \in S^v(U)$,

$$t_y^v(U) := \min \{t \leq C(n + v_1) : \exists x \in T \text{ with } x \xrightarrow{\xi_{[t_x, t]}} y \text{ in } B^U\}. \tag{4.26}$$

We now construct a random exploration sequence $X_0 = (U_0, V_0), X_1 = (U_1, V_1) \dots$ in B_n that is measurable with respect to the microscopic percolation configuration ω , and such that for each $i \geq 0$,

$$\text{for all } v \in U_i^+, \quad (S^v(U_i), t^v(U_i)) \text{ is a } \delta\text{-seed for } v. \tag{4.27}$$

First, define

$$U_0 := \{u \in T : G_\xi^u(T \cap F^u, t|_{T \cap F^u}) \text{ occurs}\} \quad \text{and} \quad V_0 := \emptyset \tag{4.28}$$

(here we use that $(T \cap F^u, t|_{T \cap F^u})$ is a δ -seed for u , from the definition of T in (4.23)). It follows from the definition (4.5) that (4.27) is satisfied for $i = 0$.

Now, fix $i \geq 0$ and assume that $X_0 = (U_0, V_0), \dots, X_i = (U_i, V_i)$ have been constructed in such a way that (4.27) is satisfied for some i . We define X_{i+1} as follows. If $\partial^+ U_i \cap B_n \subseteq V_i$, then set $X_{i+1} = X_i$. Otherwise let v be the minimum vertex that belongs to $\partial^+ U_i \cap B_n$ but not to V_i . By (4.27), the pair $(S^v, t^v) = (S^v(U_i), t^v(U_i))$ is a seed for v , and we define

$$(U_{i+1}, V_{i+1}) = \begin{cases} (U_i \cup \{v\}, V_i) & \text{if } G_\xi^v(S^v, t^v) \text{ occurs,} \\ (U_i, V_i \cup \{v\}) & \text{otherwise.} \end{cases} \tag{4.29}$$

First, by Hoeffding’s inequality, we have

$$\mathbb{P}_p(|U_0| \geq \frac{\delta}{2}|L_n|) \geq 1 - e^{-chn} \tag{4.30}$$

for some constant $c > 0$. Using independence and Proposition 8, we also obtain

$$\mathbb{P}_p(V_{i+1} = V_i \mid X_0, \dots, X_i) \geq \zeta \quad \text{a.s.} \tag{4.31}$$

The two inequalities above, combined with Lemma 9, imply that

$$\mathbb{P}_p(|U_\infty \cap R_n| \geq \frac{1}{1000}|R_n|) \geq 1 - 2e^{-chn}. \tag{4.32}$$

When the exploration terminates, the property (4.27) is still satisfied. Therefore, on the event $|U_\infty \cap R_n| \geq \frac{1}{1000}|R_n|$, the set

$$T' = \bigcup_{v \in U_\infty^+ \cap \partial^+ R_n} S^v(U_\infty) \tag{4.33}$$

satisfies

- $|T'| \geq 64000\delta|F| \cdot |U_\infty^+ \cap \partial^+ R_n| \geq 64\delta|F| \cdot |R_n| = 64\delta|R_n| \geq 8\delta|\partial_S \Lambda_{2n}|$,
- for each y in T' , there exist $t'_y \leq 2Cn$ and $x \in T$ such that

$$x \xrightarrow{\xi_{[t_x, t'_y]}} y \quad \text{in } B_n.$$

We can thus deduce (4.21) from (4.32), which completes the proof.

5. Proof of Wierman’s coupling

Proof of Lemma 4. In the configuration ω , we are going to explore the set of vertices which are 1-connected to S in Λ , as well as the immediate neighborhood of this set. During this exploration, we grow (in a Markovian way) a spanning forest of this explored set, such that at all times, it contains exactly one tree for each vertex $x \in S$, as illustrated in Figure 6. For this purpose, we introduce an auxiliary sequence of i.i.d. random variables $(U_t)_{t \geq 0}$, each uniformly distributed on $[0, 1]$.

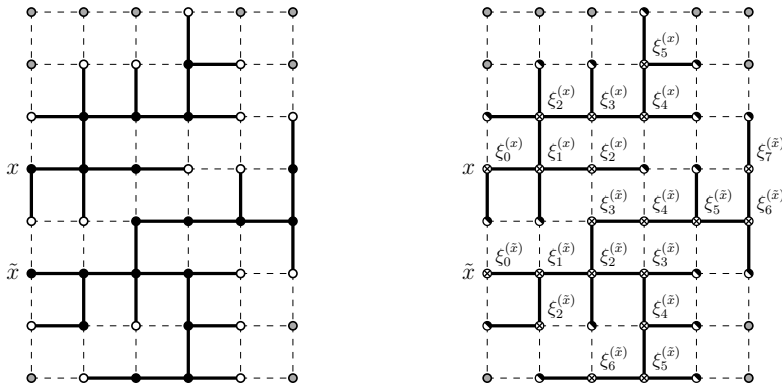


Fig. 6. This figure shows the coupling between two configurations ω (left) and $\tilde{\omega}$ (right) used in the proof of Lemma 4. Each of these configurations is distributed as Bernoulli site percolation with parameter $p \leq 1/2$. Starting an exploration procedure from the vertices in $S = \{x, \tilde{x}\}$, we produce a spanning forest of the 1-cluster of S , together with its outer boundary; its branches are used to read $\mathbb{1}$ on ω , and $(\xi^{(x)}, \xi^{(\tilde{x})})$ on $\tilde{\omega}$. The 0-sites in ω discovered during the exploration procedure may be 0 or 1 in $\tilde{\omega}$, and they are not used to read $\xi^{(x)}$ or $\xi^{(\tilde{x})}$. The grey vertices remain undiscovered.

We start by fixing an arbitrary order on the vertices of Λ . We then let the first forest \mathcal{F}_0 in our construction consist simply of the vertices in S , without any edges. Having defined \mathcal{F}_t for some time t , we inductively add one extra vertex and one extra edge to \mathcal{F}_t in order to obtain \mathcal{F}_{t+1} . For this, we pick the minimum unexplored vertex y' that neighbors a 1-vertex of \mathcal{F}_t , and we let y be the minimum 1-vertex in \mathcal{F}_t neighboring y' .

We now add the vertex y' and the edge (y, y') to \mathcal{F}_{t+1} as follows: let d be the distance between y and S within the forest, and set

$$(\omega_{y'}, \tilde{\omega}_{y'}) = \begin{cases} (1, \xi_{d+1}^{(x)}) & \text{w.p. } p, \\ (0, 0) & \text{w.p. } 1 - 2p \text{ if } \xi_{d+1}^{(x)} = 0, \text{ and w.p. } 1 - p \text{ if } \xi_{d+1}^{(x)} = 1, \\ (0, 1) & \text{w.p. } p \text{ if } \xi_{d+1}^{(x)} = 0 \end{cases} \tag{5.1}$$

(the third option does not arise when $\xi_{d+1}^{(x)} = 1$). More specifically, we let $(\omega_{y'}, \tilde{\omega}_{y'}) = (1, \xi_{d+1}^{(x)})$ if $U_t \leq p$, $(\omega_{y'}, \tilde{\omega}_{y'}) = (0, 1)$ if $U_t > 1 - p$ and $\xi_{d+1}^{(x)} = 0$, and $(\omega_{y'}, \tilde{\omega}_{y'}) = (0, 0)$ in all other cases (so that the choice of $(\omega_{y'}, \tilde{\omega}_{y'})$ is made independently of everything

observed so far). When the exploration procedure stops, we simply draw independently the remaining undiscovered vertices, e.g. by following the order on the vertices of Λ : if y is the vertex examined in step t , we let $\omega_y = \tilde{\omega}_y = 1$ if $U_t \leq p$, and 0 otherwise.

By construction, all the sites y which are 1-connected to S are $\xi^{(x)}$ -connected to some $x \in S$ in $\tilde{\omega}$, by following the same branch of the spanning forest, which ensures that the property in the statement holds true. Note that the sites which are discovered and have state 0 in ω are not used here to observe $(\xi^{(x)})_{x \in S} \in \Xi$. ■

6. Oriented percolation lemma

In this section, we establish Proposition 10, which is a result about oriented percolation on the slab $\overrightarrow{\text{Slab}}_h$. Before proving it, we establish an auxiliary lemma for planar oriented percolation, i.e. in dimension $d = 1 + 1$ (for more details on the topic, see e.g. [4]). We consider the oriented graph G (illustrated in Figure 7) with vertex set

$$V = \{(x, y) \in \mathbb{Z}^2 : x \in 2\mathbb{Z}, x/2 + y \in 2\mathbb{Z}\}, \tag{6.1}$$

and (oriented) edge set

$$E = \{(u, u + (2, e)) : u \in V, e = \pm 1\}. \tag{6.2}$$

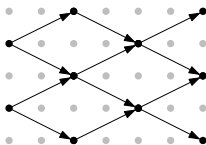


Fig. 7. A portion of the graph $G = (V, E)$.

This graph, which is isomorphic to the usual oriented square lattice, can also be seen as the projection of $\overrightarrow{\text{Slab}}_h$ onto its first two coordinates. This will be convenient later because we will see G as a subgraph of the oriented slab $\overrightarrow{\text{Slab}}_h$.

Consider the Bernoulli site percolation measure on G with parameter $0 \leq \gamma \leq 1$. Each vertex in V is open with probability γ and closed with probability $1 - \gamma$, independently of the other vertices. Let n be an even integer. For a set of vertices $A \subseteq \{0\} \times \mathbb{Z}$ and $m \geq 5n$, define

$$\xi_m^A := \{y \in [-n, n] : A \rightarrow (m, y)\}, \tag{6.3}$$

where $A \rightarrow v$ means that there exists an open oriented path from a vertex $u \in A$ to v . Notice that the quantity above satisfies the following monotonicity property:

$$\text{If } A \subseteq B, \text{ then } \xi_m^A \subseteq \xi_m^B. \tag{6.4}$$

Lemma 11 ($d = 1 + 1$). For any $0 < \delta < \frac{1}{10}$, there exists $\gamma = \gamma(\delta) \in (0, 1)$ large enough such that for every $n \geq 2$ even, for every set of vertices $S \subseteq \{0\} \times [-n, n]$ with $|S| \geq \delta n$, and for all $m \geq 5n$ divisible by four, ξ_m^S stochastically dominates the product measure on $\{-n, -n + 2, \dots, n\}$ with parameter $1/2$.

Proof. Let $\varepsilon > 0$. By [12, Theorem 1.1], we can choose γ close enough to 1 such that

$$\xi_m^{\mathbb{Z}} \text{ stochastically dominates the product measure with parameter } 1 - \varepsilon, \tag{6.5}$$

where we use the notation $\xi_m^{\mathbb{Z}} = \xi_m^{\{0\} \times \mathbb{Z}}$. The main idea here is to use the stochastic domination above, and replace \mathbb{Z} by S . To achieve this, we construct an increasing event G such that

- (i) G occurs with probability exponentially close to 1,
- (ii) on the event G , we have $\xi_m^S = \xi_m^{\mathbb{Z}}$.

These properties together with (6.5) easily conclude the proof (the argument is presented at the end of the proof). Let us now construct the event G . Set

$$S' := S \cap (\{0\} \times [-n + \frac{\delta}{4}n, n - \frac{\delta}{4}n])$$

and note that since $|S| \geq \delta n$, we have $|S'| \geq \frac{\delta}{2}n$. Let G be the event (illustrated in Figure 8) that

- $S' \rightarrow v$ for some $v \in \{m\} \times \mathbb{Z}$,
- there is an open oriented path from $\{0\} \times [-n, -n + \frac{\delta}{4}n]$ to $\{m\} \times [n, \infty)$,
- there is an open oriented path from $\{0\} \times [n - \frac{\delta}{4}n, n]$ to $\{m\} \times (-\infty, -n]$.

The first item occurs with probability larger than $1 - \varepsilon^{\delta n/2}$. To see this, consider the set X of vertices $u \in \{0\} \times \mathbb{Z}$ that are connected to some $v \in \{m\} \times \mathbb{Z}$. Then use the stochastic domination (6.5) and symmetry to show that X stochastically dominates a product measure with parameter $1 - \varepsilon$. The second and third items can be proved to occur with probability larger than $1 - \frac{1}{3}4^{-n}$ by choosing γ close enough to 1. This follows from choosing the density of open vertices close enough to 1 in the proof of [5, Theorem 4].

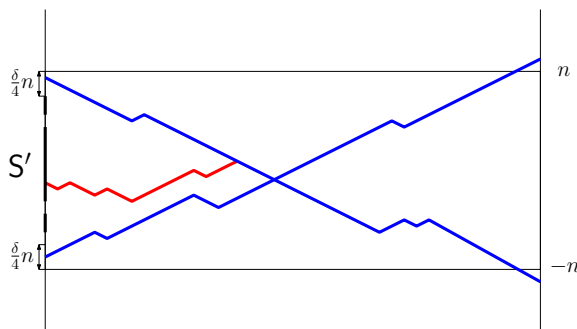


Fig. 8. Schematic representation of the event G .

Alternatively, this can also be proved directly using a Peierls-type argument. Therefore, if $\varepsilon > 0$ is chosen small enough, we obtain

$$\mathbb{P}(G) \geq 1 - 4^{-n}, \tag{6.6}$$

which establishes item (i) above.

Now, assume that the event G occurs, and consider a vertex v on $\{m\} \times [-n, n]$ that can be reached by an open path starting on $\{0\} \times \mathbb{Z}$. Then, from the three oriented paths that appear in the definition of G , one can produce an open oriented path from S' to v . In other words, when G occurs, we have $\xi_m^{\mathbb{Z}} \subseteq \xi_m^{S'}$. Since $S' \subseteq S$, the monotonicity property (6.4) directly implies that

$$\xi_m^{\mathbb{Z}} \subseteq \xi_m^S. \tag{6.7}$$

By monotonicity again, the reverse inclusion also holds, which establishes item (ii). Once we know the stochastic domination (6.5) and the existence of an increasing event G satisfying (i) and (ii) above, the proof of the lemma follows from a general stochastic domination argument that we detail now.

We consider an auxiliary product random variable $\eta = (\eta_{-n}, \eta_{-n+2}, \dots, \eta_n)$ independent of the percolation configuration, where the η_i 's are i.i.d. Bernoulli random variables with parameter $3/4$. In order to prove the lemma, it is enough to show that ξ_m^S stochastically dominates $\eta \cdot \xi_m^{\mathbb{Z}}$ (where $\eta_1 \cdot \eta_2$ denotes the coordinatewise product of η_1 and η_2). Let $f : \{0, 1\}^{\{-n, -n+2, \dots, n\}} \rightarrow \mathbb{R}_+$ be a non-decreasing function such that $f(\mathbf{0}) = 0$. We have

$$\begin{aligned} \mathbb{E}[f(\xi_m^S)] &\stackrel{f \geq 0}{\geq} \mathbb{E}[f(\xi_m^S)\mathbf{1}_G] \\ &\stackrel{(ii)}{=} \mathbb{E}[f(\xi_m^{\mathbb{Z}})\mathbf{1}_G] \\ &\stackrel{\text{FKG}}{\geq} \mathbb{E}[f(\xi_m^{\mathbb{Z}})]\mathbb{P}(G) \\ &\stackrel{(6.6)}{\geq} \mathbb{E}[f(\xi_m^{\mathbb{Z}})]\mathbb{P}(\eta \neq \mathbf{0}) \\ &\stackrel{\text{indep.}}{=} \mathbb{E}[f(\xi_m^{\mathbb{Z}})\mathbf{1}_{\eta \neq \mathbf{0}}] \\ &\stackrel{f(\mathbf{0})=0}{\geq} \mathbb{E}[f(\eta \cdot \xi_m^{\mathbb{Z}})]. \end{aligned}$$

This proves the desired stochastic domination and concludes the proof. ■

Before moving to the proof of Proposition 10, we use the planar result above to deduce a useful statement for oriented percolation on the slab. Fix $h \geq 20$, and consider Bernoulli site percolation with parameter $0 \leq \gamma < 1$ on the slab $\overrightarrow{\text{Slab}}_h = (V, \overrightarrow{E})$ (where $V \subseteq \mathbb{Z}^2 \times [0, h]$), as defined in (4.1) and (4.2). For any given $n \geq m \geq h$, we also define the sets

$$\begin{aligned} B_{n,m} &:= V \cap ((n, 2n) \times (-2m, 2m) \times (0, h)), \\ L_{n,m} &:= V \cap (\{n + 1\} \times [-m, m] \times (0, h)), \\ R_{n,m} &:= V \cap (\{2n - 1\} \times [-m, m] \times (0, h)). \end{aligned} \tag{6.8}$$

We extend the definition to non-integers m by setting $B_{n,m} := B_{n,\lceil m \rceil}$ (and similarly for $L_{n,m}$ and $R_{n,m}$). Note that these sets are similar to those defined in (4.12), except that they are shorter in the y -direction. For $A \subset B \subset V$ and $v \in B$, we say that $A \rightarrow v$ in B if there exists an open oriented path of $\overrightarrow{\text{Slab}}_h$ that stays in B from a vertex of A to v .

Lemma 12. *For every $\delta > 0$, there exists $\gamma = \gamma(\delta) \in (0, 1)$ (large enough) such that for every $n \geq h \geq 20$ and $S \subseteq L_{n,n/h}$ with $|S| \geq \delta n$, we have*

$$\mathbb{P}(N > \frac{1}{20}n) \geq 1 - e^{-cn} \tag{6.9}$$

for some $c > 0$, where N is the number of vertices $u \in R_{n,n/h}$ such that $S \rightarrow u$ in $B_{n,n/h}$.

Proof of Lemma 12. It is enough to provide a proper graph embedding, or more precisely to find an injective mapping $f : V \cap ([0, n - 2] \times [-n/2, n/2]) \rightarrow B_{n,n/h}$ satisfying

- (a) there are at least $\frac{\delta}{10}n$ vertices $u \in \{0\} \times [-n/6, n/6]$ such that $f(u) \in S$,
- (b) for every vertex $u \in \{n - 2\} \times [-n/6, n/6]$, we have $f(u) \in R_{n,n/h}$,
- (c) if (u, v) is an edge on the oriented graph (V, E) , then $(f(u), f(v))$ is an edge in (V, \overrightarrow{E}) .

Once we have such an embedding at our disposal, we can simply use Lemma 11 (combined with Hoeffding’s inequality).

The construction of the embedding f follows from an “accordion-like” folding of the rectangle, and then finding a translate of this accordion along the y -direction such that property (a) above occurs. Although the reader can be convinced of the existence of such a function f by looking at Figure 9, we provide now an explicit construction. For simplicity, we assume that h and $n + 1$ are both even (a similar construction can be given in the other cases). We let

$$f(a, b) := (n + 1 + a, f_2(a, b), f_3(a, b)), \tag{6.10}$$

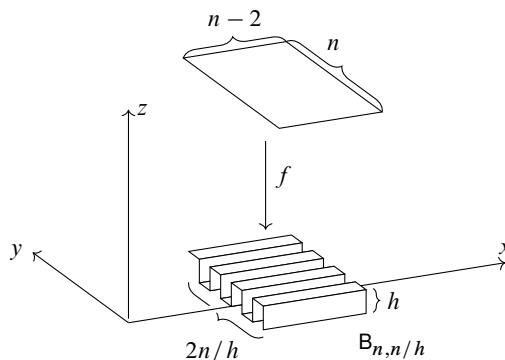


Fig. 9. This figure depicts, roughly, the embedding f used in the proof of Lemma 12.

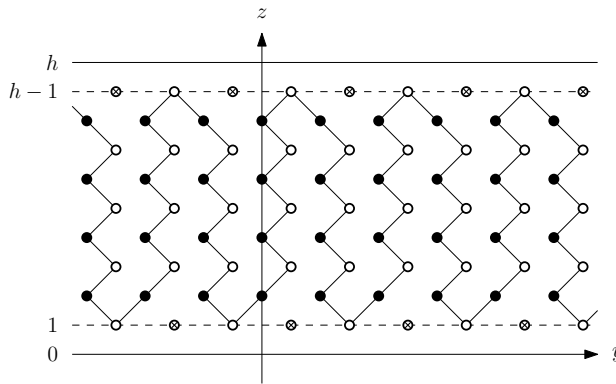


Fig. 10. This figure shows a “microscopic” view of the embedding f into $B_{n,n/h}$. Black and white vertices lie on layers where $x \equiv 0 \pmod{4}$ and $x \equiv 2 \pmod{4}$, respectively. The white vertices marked with a cross do not belong to the image of f (but this is immaterial for the argument).

where f_2 and f_3 are defined as follows. If $a \pmod{4} = 0$, b ranges in $[-n/2, n/2] \cap (2\mathbb{Z})$. Write $b = i \cdot 2(h - 2) + j$ with $i \in \mathbb{Z}$ and $j \in [0, 2(h - 2)) \cap (2\mathbb{Z})$. We let

$$f_2(a, b) = 2 \cdot \left\lfloor \frac{b}{h - 2} \right\rfloor \quad \text{and} \quad f_3(a, b) = \tilde{f}_3(j),$$

where $\tilde{f}_3(j) = 2 + j$ for $j = 0, \dots, h - 4$, and $\tilde{f}_3(j) = 2(h - 2) - j$ for $j = h - 2, \dots, 2h - 6$. If $a \pmod{4} = 2$, b ranges in $[-n/2, n/2] \cap (1 + 2\mathbb{Z})$. Write $b = i \cdot 2(h - 2) + j$ with $i \in \mathbb{Z}$ and $j \in [0, 2(h - 2)) \cap (1 + 2\mathbb{Z})$. We let

$$f_2(a, b) = 1 + 2 \cdot \left\lfloor \frac{b}{h - 2} \right\rfloor \quad \text{and} \quad f_3(a, b) = \tilde{\tilde{f}}_3(j),$$

where $\tilde{\tilde{f}}_3(j) = 2 + j$ for $j = 1, \dots, h - 3$, and $\tilde{\tilde{f}}_3(j) = 2(h - 2) - j$ for $j = h - 1, \dots, 2h - 5$. This specific choice of f is depicted in Figure 10. Note that the image of f “catches” all vertices in $L_{n,n/h}$, and all translates of this image by at most $\pm \frac{2}{3} \frac{n}{h}$ along the y -direction are contained in $B_{n,n/h}$. ■

We are now in a position to prove the main proposition of this section.

Proof of Proposition 10. We start the proof by making some assumptions that do not reduce the generality of the result. First suppose that the side length of the rectangle is a multiple of $40h$ (this way, all the fractions of n appearing in the proof will be integers).

Observe that the starting set L_n can be written as the union

$$L_n = I_{-5} \cup \dots \cup I_4, \tag{6.11}$$

where $I_i := \bigvee \cap (\{n\} \times [in/5, in/5 + n/5] \times [0, h])$. Since S has at least δnh points, we deduce that the intersection of S with at least one of the I_i ’s contains at least

$\delta nh/10$ points. Without loss of generality, we assume that

$$|\mathbb{S} \cap I_1| \geq \delta nh/10. \tag{6.12}$$

We are going to construct the connections described in Proposition 10 in two steps. Intuitively speaking, we show that the infection spreads from left to right in two steps: first until the middle of the box ($x = 3n/2$), where the infection becomes evenly distributed across the y -direction; then from the middle to the end ($x = 2n$), where we show that it spreads across the z -direction. Let us state precisely these two steps of the proof. We first choose a collection $J_1, \dots, J_{h/30}$ of rectangles of the form

$$J_i = \mathbb{V} \cap (\{3n/2\} \times [j_i - n/h, j_i + n/h] \times [0, h])$$

such that the mutual distance between different J_i 's is strictly larger than $2n/h$. We say that some set J_i is *bad* if

$$|\{u \in J_i : \mathbb{S} \rightarrow u\}| \leq \frac{\delta}{100} |J_i|. \tag{6.13}$$

In order to establish Proposition 10, we claim that for some constant $c > 0$ (depending on δ)

$$P_\gamma(\text{at least } \frac{h}{60} \text{ sets } J_i \text{ are bad}) \leq e^{-chn} \tag{6.14}$$

(this claim will be established at the end of the proof). In the final step of the proof, we assume that at least $\frac{h}{60}$ sets J_i are good. For each good J_i , we consider the set S_i (with density at least $\frac{\delta}{100}$ in J_i) of vertices in J_i which can be reached from \mathbb{S} by an open oriented path, and we work in the box $K_i = [3n/2, n] \times [j_i - 2n/h, j_i + 2n/h] \times [0, h]$. Using Lemma 12, we can show that with probability larger than $1 - e^{-cn}$, we can reach at least $\frac{1}{20}n$ points in \mathbb{R}_n with oriented paths starting in S_i and staying in K_i . Since the boxes K_i are disjoint, this construction is successful for at least $\frac{h}{100}$ indices i , with probability larger than $1 - e^{-c'hn}$ (by Hoeffding's inequality). Therefore,

$$P_\gamma\left(|\{v \in \mathbb{R}_n : \mathbb{S} \rightarrow v\}| \geq \frac{h}{100} \cdot \frac{1}{20}n\right) \geq 1 - e^{-chn} - e^{-c'hn}, \tag{6.15}$$

which implies Proposition 10.

Let us now prove (6.14). Assume for simplicity that h is odd. We split the set I_1 into its rows, $I_1 = \bigcup_{i=0,2,\dots,h-1} C_i$, where C_i is defined as the set of points in I_1 with $z = i$. We can bound the size of \mathbb{S} intersected with I_1 as follows:

$$|\mathbb{S} \cap I_1| \leq n \cdot \left| \left\{ i : |\mathbb{S} \cap C_i| \geq \frac{\delta n}{20} \right\} \right| + \frac{\delta nh}{20}. \tag{6.16}$$

Since the left-hand side is larger than $\delta nh/10$ (see (6.12)), we deduce

$$\left| \left\{ i : |\mathbb{S} \cap C_i| \geq \frac{\delta n}{20} \right\} \right| \geq \frac{\delta h}{20}. \tag{6.17}$$

We denote the above set of good columns by $G \subseteq \{0, 2, \dots, h - 1\}$. For each good column $C_i, i \in G$, we restrict our oriented percolation to the sheet that is isomorphic to the

oriented square lattice, and is contained in $\mathbb{Z} \times \mathbb{Z} \times \{i, i + 1\}$. This restriction can only reduce the set of open paths and it has two advantages: distinct sheets become independent, and it allows us to use the planar result of Lemma 11.

Consider the set $S' = V \cap (\{3n/2\} \times [0, n/10] \times G)$. Applying Lemma 11 produces γ for which

$$\{u \in S' : S \xrightarrow{B_n} u\} \text{ stochastically dominates the product measure on } S' \text{ w.p. } 1/2.$$

This implies that for each i , since $|J_i \cap S'| \geq \frac{\delta}{20}|J_i \cap V|$, we have

$$P(J_i \text{ is bad}) \leq P(\text{Bin}(\frac{\delta}{20}|J_i \cap V|, \frac{1}{2}) \leq \frac{\delta}{100}|J_i \cap V|) \leq e^{-c_0 n}$$

for some constant $c_0 > 0$ (using again Hoeffding’s inequality). Hence,

$$\begin{aligned} P(\text{at least } \frac{h}{60} \text{ sets } J_i \text{ are bad}) &\leq 2^{h/30} \sup_{\substack{D \subseteq \{0, \dots, h/30\} \\ |D| \geq \frac{h}{60}}} P(J_i \text{ is bad for all } i \in D) \\ &\leq 2^{h/30} (e^{-c_0 n})^{h/60}, \end{aligned}$$

which establishes (6.14). ■

7. Open problems

We state two open problems that follow naturally from the results presented in this work. The first problem is related to local uniqueness for percolation of words.

Problem 13. For percolation of words on \mathbb{Z}^d , $d \geq 3$, at $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, consider the following event. For $n \geq 1$, $U_n := \{\text{there exist } x, y \in B_n(0) \text{ and } \xi \in \Xi \text{ for which } x \xrightarrow{\xi} \infty \text{ and } y \xrightarrow{\xi} \infty, \text{ but there does not exist any way to see } \xi \text{ starting from } x \text{ and } y \text{ that “coalesces” before exiting } B_{2n}(0)\}$. More precisely, we require x, y and ξ to satisfy: if γ and γ' are two infinite self-avoiding paths from x and y , respectively, along which ξ is seen, then γ and γ' do not “meet” (i.e. intersect each other at the same time) before exiting $B_{2n}(0)$. What is the rate of decay of $\mathbb{P}_p(U_n)$ as $n \rightarrow \infty$?

Percolation of words can be generalized to higher-dimensional objects, namely Lipschitz embeddings of d -dimensional structures into \mathbb{Z}^D , as previously studied in [3, 7]. Let $2 \leq d \leq D$, and write $\Omega_d = \{0, 1\}^{\mathbb{Z}^d}$ for the set of d -dimensional site percolation configurations. For $M \geq 1$, we say that a configuration $\eta \in \Omega_d$ can be M -embedded into $\omega \in \Omega_D$ if there exists an injection $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^D$ such that

$$\forall x \in \mathbb{Z}^d \quad \omega_{f(x)} = \eta_x, \tag{7.1}$$

$$\forall x, y \in \mathbb{Z}^d \quad \|f(x) - f(y)\|_1 \leq M \|x - y\|_1. \tag{7.2}$$

In [7, Section 1.3], the authors consider the following question. For any given $p \in (0, 1)$ and $\eta \in \Omega_d$, is it possible to M -embed the configuration η into a random ω , obtained

from Bernoulli site percolation with parameter p on \mathbb{Z}^D ? In the spirit of our work, we suggest the following problem. For $M \geq 1$, define $p_c(d, D, M) \in [0, 1]$ as the infimum of all p such that the constant configuration $\mathbb{1}_d \in \Omega_d$ can be M -embedded into ω with positive probability. A consequence of [7, Proposition 4 (b)] is that for all $d < D$, there exists $M \geq 1$ such that $p_c(d, D, M) < 1/2$.

Problem 14. Let $2 \leq d < D$, $M \geq 1$, and assume that $p_c(d, D, M) < 1/2$. For any fixed $p \in (p_c(d, D, M), 1 - p_c(d, D, M))$, is it the case that every configuration $\eta \in \Omega_d$ can be embedded into a p -percolation ω on \mathbb{Z}^D ?

Let us also mention that the construction in [1, Section 4] shows the following. For any fixed $p \in (0, 1)$, there exists $d = d(p)$ large enough such that in \mathbb{Z}^d , \mathbb{P}_p -a.s., all words can be seen from the neighbors of a single vertex. Our proofs show that for all $d \geq 3$ and $p \in (p_c^{\text{site}}(\mathbb{Z}^d), 1 - p_c^{\text{site}}(\mathbb{Z}^d))$, there exists $n = n(d, p)$ large enough such that \mathbb{P}_p -a.s., there exists a ball $B_n(v)$ ($v \in \mathbb{Z}^d$) of radius n from which all words can be seen. However, the value of n produced by the proof, using a renormalization procedure, is typically very large, and we cannot say anything about what the optimal n should be. For example, it leaves completely open the question of whether $d(1/2) = 3$, i.e. whether all words can already be seen from the neighbors of a single vertex on \mathbb{Z}^3 , at $p = 1/2$.

For more open questions and comments regarding percolation of words, the reader can consult the classification conjecture of de Lima and Sidoravicius, stated in [9, Section 1.2]. In the present paper, we completely verified part II (c) of this conjecture in the case of \mathbb{Z}^d , $d \geq 3$. Also, since Wierman's coupling is fully general, our proof suggests that II (c) should be valid for a rather wide family of graphs, on which the renormalization procedure can be carried out.

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