JEMS

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Disproving Hooley's conjecture

Abstract. Define $G(x; q)$ to be the variance of primes $p \le x$ in the arithmetic progressions modulo q, weighted by $\log p$. In analogy with his q-analogue of Selberg's upper bound on the variance of primes in intervals, Hooley conjectured that as soon as q tends to infinity and $x \ge q$, we have the upper bound $G(x; q) \ll x \log q$. This conjecture was proven true over function fields by Keating and Rudnick, using equidistribution results of Katz. In this paper we show that the upper bound does not hold in general, and that $G(x; q)$ can be much larger than $x \log q$ for values of q which are $\asymp \log \log x$. This implies that a conjecture of the first author on the range of validity of Hooley's conjecture is essentially best possible.

Keywords. Variance of primes in progressions, Hooley's conjecture, primes in arithmetic progressions, zeros of Dirichlet L-functions

1. Introduction and statement of results

For $x > q \ge 3$, we define the variance

$$G(x; q) := \sum_{\substack{a \bmod q \\ (a,q)=1}} \left| \sum_{\substack{p \le x \\ p \equiv a \bmod q}} \log p - \frac{x}{\phi(q)} \right|^2,$$

as well as the closely related (and perhaps slightly more natural)

$$V_\Lambda(x; q) := \sum_{\substack{a \bmod q \\ (a,q)=1}} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{\substack{n \le x \\ (n,q)=1}} \Lambda(n) \right|^2. \tag{1}$$

Since the pioneering work of Barban, Davenport and Halberstam [2, 7], the study of this variance has seen a long line of developments, and continues to be an active research

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topic. To cite a few of the numerous papers written over the years, we mention [11,12,16, 17,20,29,31,35,45], as well as Hooley's series of 19 research papers and 2 survey papers (see for instance [18,19,21–23]). We also have the recent works [1,3,5,27,30,43], which explore sparse averages over q, as well as analogues for number fields and function fields.

The quantitites $G(x;q)$ and $V_\Lambda(x;q)$ are q-analogues of the variance

$$V_\Lambda(x,\delta) := \frac{1}{\log x} \int_1^x \frac{(\psi(t+\delta t) - \psi(t) - \delta t)^2}{t} \frac{dt}{t}, \tag{2}$$

which was first bounded under RH by Selberg [41] in connection with the distribution of primes in almost all intervals. (Note that $V_\Lambda(x,\delta)$ is normalized by the factor $\log x$, which is the total logarithmic measure of the interval $[1,x]$; we feel that it would be more natural for $G(x;q)$ to also be normalized by $\phi(q)$, which is the counting measure of the set of invertible residues modulo q. However, we retain the current definition to be consistent with the literature.) Goldston and Montgomery [15, Theorem 1] extended this result by showing under RH that uniformly for all $0 < \delta \le 1$,

$$V_\Lambda(x,\delta) \ll \delta \log(2\delta^{-1}).$$

In this direction, we mention the works of Gallagher [14] and Montgomery–Soundararajan [32] on the conjectural Poisson distribution of $\psi(t+\delta t) - \psi(t) - \delta t$. Hooley [21, Theorem 1] was interested in a q-analogue of Selberg's result, and he proved under GRH that uniformly for $q \le x$,

$$\frac{1}{\log T} \int_2^T G(t;q) \frac{dt}{t} \ll \log q. \tag{3}$$

In light of the heuristic correspondence between δ^{-1} and $\phi(q)$, this result of Hooley is an averaged analogue of the Selberg/Goldston–Montgomery bounds. In this analogy, the role of the average over t in (2) is played by the sum over $a \bmod q$ in (1). In other words, the left-hand side of (3) contains a double average, and one might think that averaging only once as in (2) could be sufficient for the upper bound (3) to hold. This is the content of Hooley's conjecture [23, p. 217], [21, (2)], which states that as soon as q tends to infinity and $x \ge q$, we have the upper bound

$$G(x;q) \ll x \log q. \tag{4}$$

More generally, one expects primes to approximately follow a Poisson distribution, which becomes Gaussian as soon as each arithmetic progression modulo q contains infinitely many primes on average, that is, as soon as $\phi(q) = o(x/\log x)$ (this is again analogous to the aforementioned conjectures of Gallagher [14] and Montgomery–Soundararajan [32]; see [8, Conjecture 1.9] for a precise conjecture). This heuristic is consistent with the asymptotic $V_\Lambda(x;q) \sim x \log q$.

In a seminal paper, Littlewood [28] established the (unconditional) oscillation result $\psi(x,(\frac{-1}{\cdot})) = \Omega_\pm(x^{1/2} \log\log\log x)$. This result was extended by Davidoff (unpublished) to all real Dirichlet characters. It turns out that Davidoff's result is already sufficient to

disprove Hooley's conjecture. Indeed, for each modulus q, one can select a real character χ_q and an arbitrarily large real number x_q such that $\psi(x_q, \chi_q) \gg_q x_q^{1/2} \log\log\log x_q$. No matter what the implied constant is, and for arbitrarily large (absolute) $M \geq 1$, the value x_q can be chosen large enough so that $\psi(x_q, \chi_q) \geq x_q^{1/2} M \phi(q)^{1/2} (\log q)^{1/2}$; consequently, Parseval's identity

$$V_\Lambda(x; q) = \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} |\psi(x, \chi)|^2$$

results in the bound $V_\Lambda(x_q; q) \geq M^2 x_q \log q$. A similar calculation would give the same result for $G(x_q; q)$, contradicting the conjecture (4). Note that this argument does not give an effective rate of growth of x_q in terms of q.

In a recent paper [9], the first author conjectured that Hooley's upper bound (4) (as well as the corresponding lower bound) holds in the range $(\log\log x)^{1+\varepsilon} \leq q \leq x$, and suggested that the constant 1 in the exponent of $\log\log x$ is best possible. This belief is based on estimates on the large deviations of the limiting distribution of $e^{-y} V_\Lambda(e^y; q)$, under GRH and a linear independence hypothesis on the zeros of $L(s, \chi)$; the resulting heuristic suggests that $\phi(q) = \log\log x$ is a transition point for $V(x; q)$.

The main result of the current paper is an unconditional proof that this range is best possible—in other words, that Hooley's conjectured upper bound (3) is false in the range $q = o(\log\log x)$.

Theorem 1.1. *Fix any sufficiently large positive real number M. There exists an infinite sequence of pairs (q_j, x_j), with $q_j \asymp \frac{1}{M} \log\log x_j$ (with an absolute implied constant) both tending to infinity, for which*

$$G(x_j; q_j) \geq M x_j \log q_j. \tag{5}$$

The same statement holds for $V_\Lambda(x_j; q_j)$ in place of $G(x_j; q_j)$.

For fixed moduli q, our argument allows us to extend Davidoff's result from real characters to complex characters.

Theorem 1.2. *For any fixed nonprincipal character χ mod q,*

$$\Re(\theta(x, \chi)) = \Omega_-(x^{1/2} \log\log\log x). \tag{6}$$

Moreover, for any fixed modulus $q \geq 3$,

$$G(x; q) = \Omega(x(\log\log\log x)^2). \tag{7}$$

The same oscillation results hold with $\psi(x, \chi)$ and $V_\Lambda(x; q)$ in place of $\theta(x, \chi)$ and $G(x; q)$, respectively. The sequences of x-values implied by these oscillation results depend upon χ or q, respectively; the implied Ω-constants, however, are absolute.

The size of the large values of $G(x; q)$ and $V_\Lambda(x; q)$ exhibited in this paper are highly dependent on the relative sizes of q and x. We now state our main technical result which

makes this dependence explicit. It will be useful to consider functions $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$h(x) \text{ is increasing to infinity, and } h(e^{y^A}) \ll_{A,h} h(e^y) \text{ for every } A > 1, \qquad (8)$$

the prototypical example of which is $h(x) = \max\{\log\log x, 1\}$.

Theorem 1.3. *Let $h(x)$ be a function satisfying* (8), *and let $\varepsilon > 0$. If*

$$\frac{\varepsilon \log\log x}{\log\log\log x} \leq h(x) \leq \log\log x \quad \text{for } x \geq e^3, \qquad (9)$$

then for a positive proportion of moduli q, there exist associated values x_q such that $q \asymp h(x_q)$ and

$$G(x_q; q) \gg_\varepsilon x_q \log q \cdot \frac{\log\log x_q}{q}. \qquad (10)$$

In particular, when $\delta > 0$ is sufficiently small, the bound (4) *cannot hold in any range of q that satisfies $q < \delta \log\log x$.*

*   If on the other hand

$$h(x) \leq \varepsilon \frac{\log\log x}{\log\log\log x} \quad \text{for } x \geq e^3,$$

then for a positive proportion of moduli q, there exist associated values x_q such that $q \asymp h(x_q)$ and

$$G(x_q; q) \geq \left(\frac{1}{4} - \varepsilon\right) x_q \cdot (\log q + \log\log\log x_q)^2. \qquad (11)$$

The same statements hold for $V_\Lambda(x_q; q)$ in place of $G(x_q, q)$.

Note that the right hand side of (11) is always $\gg x_q(\log q)^2$, contradicting (4) for large enough q. Moreover, taking for instance $h(x) = \log\log\log x$ results in the even starker quantitative contradiction to Hooley's conjecture $G(x_q, q) \gg x_q q^2$.

Remark 1.4. Under GRH, the generalized Riemann hypothesis for Dirichlet L-functions, oscillation results similar to Theorem 1.3 hold for *all* moduli q. Indeed, we will show in Theorem 3.9 that for any function h satisfying (8) and $h(x) \leq \delta \log\log x$ with $\delta > 0$ small enough, there exists a sequence $\{x_q\}_{q \geq 1}$ satisfying $\phi(q) \asymp h(x_q)$ and having the property that $G(x_q; q)/(x_q \log q)$ tends to infinity as $q \to \infty$.

Our method also produces an oscillation result in a wider range, namely for

$$\frac{\log\log x}{\log\log\log x} \leq q \leq (\log x)^\delta$$

with δ small enough. Indeed, $G(x; q)$ can be as large as $(x \log\log x)(\log q)/\phi(q)$ (as can be seen by combining Theorem 3.9 with Proposition 2.3). However, sharper oscillation results are obtained for this range in [8] for a weighted variant of $V_\Lambda(x; q)$ and for all higher even moments.

We recall that Hooley conjectured that the estimate (4) holds, as soon as q tends to infinity with $q \leq x$, based on his average result (3). A natural question to ask here would be whether one can replace the t-average (3) with a more classical q-average, that is, whether as soon as $Q \to \infty$ and $Q \leq x$, we might have

$$\frac{1}{Q} \sum_{q \leq Q} G(x; q) \ll x \log Q. \tag{12}$$

As it turns out, this assertion is also false.

Theorem 1.5. *Let $\varepsilon > 0$ be small enough, and let $Q: \mathbb{R}_{\geq 0} \to \mathbb{N}$ be a monotonic function with the property* (8) *and satisfying $Q(x) \leq \varepsilon (\log \log x)^{1/2}/(\log \log \log x)^{\frac{1}{2}}$. Then we have the oscillation result*

$$\frac{1}{Q(x)} \sum_{Q(x) < q \leq 2Q(x)} G(x; q) = \Omega(x (\log \log \log x)^2),$$

and the same statement holds for $V_\Lambda(x; q)$ in place of $G(x; q)$.

Let us briefly describe the tools used in the proofs of Theorems 1.2, 1.3, and 1.5.

• The first step, which is carried out in Section 2, is to show that the upper bound (4) (in certain ranges) implies GRH. In other words, since our goal is to disprove (4), we will be able to assume GRH for the rest of the paper. More precisely, if $L(s, \chi)$ has a non-trivial zero $\rho_\chi = \Theta_\chi + i\gamma_\chi$ off the critical line, then $|\theta(x, \chi) - \mathbf{1}_{\chi=\chi_0} x|$ can be as large as $x^{\Theta_\chi - \varepsilon}$ by Landau's theorem, and then one can apply Parseval's identity (13). (Here $\mathbf{1}_{\chi=\chi_0}$ equals 1 if $\chi = \chi_0$ and 0 otherwise.) This works well for fixed moduli q (as in Theorem 1.2 and Davidoff's result); however, one needs to modify this approach to have a result which is uniform in the range $q \leq x^{o(1)}$ (for Theorems 1.1 and 1.3). To achieve this, we combine the identity (13) with positivity and the fact that large values of $|\theta(x, \chi) - \mathbf{1}_{\chi=\chi_0} x|$ translate to large values of $|\theta(x, \chi') - \mathbf{1}_{\chi'=\chi_0} x|$ for all χ' induced by χ (of conductor at most x).

Let us use GRH(χ) to denote the generalized Riemann hypothesis for a specific Dirichlet L-function $L(s, \chi)$. If q_e is the least modulus for which a character χ_e exists such that GRH(χ_e) is false, then χ_e will induce a character modulo every multiple of q_e whose associated Dirichlet L-function also violates GRH. As a result, we will deduce (see Proposition 2.3) that $G(x; q)$ can be as large as $x^{2\Theta_{\chi_e} - \varepsilon}/\phi(q)$; since Θ_{χ_e} is independent of q and x, this deduction will violate the conjecture (4) in the range $q \leq x^{o(1)}$ for a positive proportion of moduli q. In other words, the conjecture (4) in any range of the form $q \asymp x^{o(1)}$ is stronger than GRH; indeed, the validity of (4) for $q \asymp x^\delta$ implies the zero-free strip $\Re(s) > 1/2 + \delta/2$ for all Dirichlet L-functions modulo q (see Proposition 2.2).

A difficulty arises in this approach when one is looking for a result which holds for many values of q. Indeed, if one uses the oscillations of $\theta(x, \chi_e) - \mathbf{1}_{\chi=\chi_0} x$ to create large values of $G(x; q)$, say on a sequence x_j, then the condition $q \asymp h(x_q)$ will force q to be in a set which is possibly thin, since Landau's theorem alone does not give the rate of growth of x_j. To circumvent this possible issue, we apply a refined oscillation result of

Kaczorowski and Pintz [26], which gives a rate of growth for x_j; however, their main theorem requires the assumption that $L(\sigma, \chi_e) \neq 0$ for $1/2 \leq \sigma < 1$. Fortunately, for our purposes it is sufficient to apply a weaker result, Lemma 2.1, which as we will show can be proven unconditionally.

• In the second step, which is more intricate, we assume that GRH holds. Our general strategy in Section 3 is to apply the explicit formula and to synchronize the summands using homogeneous Diophantine approximation, an approach that hearkens back to Littlewood's work, modified as in [33, Theorem 15.11] and [37, Lemma 2.4]. However, working uniformly in q poses significant new challenges. Indeed, several approximations used in previous arguments of this kind translate to error terms which are too large in the current context.

In order to circumvent these issues, we need to significantly refine this approach, resulting in more complicated formulas. In particular, quite early in the argument we need to apply results of Murty [34] (see also Hughes and Rudnick [24]) and Selberg [42] on multiplicities of zeros of $L(s, \chi)$ (see Lemma 3.1). We then compute the average of $\psi(e^y, \chi)$ in suitable short intervals, which are determined by an application of homogeneous Diophantine approximation, to synchronize the frequencies $y\gamma_\chi/2\pi$ mod 1 simultaneously for all zeros $\rho_\chi = 1/2 + i\gamma_\chi$, with height at most T, of all $L(s, \chi)$ with χ mod q. Interestingly, in certain ranges, rather than synchronizing an unbounded number of frequencies for a single character, we synchronize a large enough but bounded number of frequencies for each character modulo q. This step forces the value of $\log x$ to be as large as $\exp(c\phi(q)T \log(qT))$ for some constant $c > 0$, which explains the range $q \leq \log \log x / \log \log \log x$ in the second part of Theorem 1.3.

We also localize the large values fairly precisely in Theorem 1.3, that is, we obtain two-sided bounds on x in terms of q and T. The key observation here is that we can exploit the almost-periodicity of $\psi(e^y, \chi)$ as in [37, Section 2.2] by finding *many* values of n in the Diophantine approximation step, which will force one of these values to be $\geq \exp(\frac{c}{3}\phi(q)T \log(qT))$. Once this is done, the last step is to estimate the resulting sums using the Riemann–von Mangoldt formula and an evaluation of the average log-conductor [10, Proposition 3.3], which yields the second part of Theorem 1.3.

In order to obtain the full range in Theorem 1.3 (that is, $q \leq \varepsilon \log \log x$), this approach needs to be further modified. Indeed, the fact that we are synchronizing the frequencies $y\gamma_\chi/2\pi$ mod 1 for *all* characters χ mod q forces x to be as large as $\exp(q^{O(q)})$. To reduce this bound, we instead synchronize a *subset* of characters modulo q, resulting in the weaker oscillation result (10), which is still strong enough to contradict the upper bound (4). This approach introduces additional difficulties in the estimation of the average of the log-conductor, which are overcome by applying recent statistical results [8, Lemma 3.1] and [9, Lemma 3.2] (see Lemmas 3.2 and 3.6 below).

To summarize, in Section 2 we establish propositions that imply our main theorems when GRH is false, while in Section 3 we prove more delicate results that imply our main theorems when GRH is true. In particular, our main technical result is Theorem 3.9, after which we deduce Theorems 1.1, 1.2, 1.3, and 1.5.

2. Hooley's conjecture and GRH

The goal of this section is to show that Hooley's conjecture (4) in any range of the form $q \asymp x^{o(1)}$ is stronger than GRH; as a result, we will be able to assume GRH in subsequent sections. We will use the classical notation

$$\psi(x;q,a) := \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n), \quad \theta(x;q,a) := \sum_{\substack{p \leq x \\ p \equiv a \bmod q}} \log p,$$

and for a Dirichlet character $\chi \bmod q$,

$$\psi(x,\chi) := \sum_{n \leq x} \chi(n)\Lambda(n), \quad \theta(x,\chi) := \sum_{p \leq x} \chi(p) \log p.$$

We record the Parseval identities

$$V_\Lambda(x;q) = \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} |\psi(x,\chi)|^2, \quad G(x;q) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} |\theta(x,\chi) - \mathbf{1}_{\chi=\chi_0}x|^2. \quad (13)$$

Our first step is to see that if $\chi \bmod q$ is a character for which $L(s,\chi)$ does not satisfy the Riemann Hypothesis, then $|\psi(x,\chi)|$ and $|\theta(x,\chi) - \mathbf{1}_{\chi=\chi_0}x|$ have large values. This will follow from a result of Kaczorowski and Pintz [26] which we will adapt in order to obtain an unconditional statement. One should keep in mind that this will give no information about uniformity in q. We also mention that for $\chi = \chi_0$, we have the more precise results of Pintz [36] and Schlage-Puchta [38].

Lemma 2.1. *Fix $\varepsilon > 0$, let $q \geq 1$, and let χ be a character modulo q. Define $\Theta_\chi \geq 1/2$ to be the supremum of the real parts of the zeros of $L(s,\chi)$. Then, for every large enough X (in terms of χ and ε), there exists $x \in [X^{1-\varepsilon}, X]$ such that*

$$\Re(\psi(x,\chi)) - \mathbf{1}_{\chi=\chi_0}x < -x^{\Theta_\chi - \varepsilon}.$$

Proof. Suppose first that $L(\Theta_\chi, \chi) \neq 0$. Then the claim follows from setting $f(x) = \Re(\psi(x,\chi) - \mathbf{1}_{\chi=\chi_0}\mathbf{1}_{x\geq 1}x)$ and applying [26, Theorem 1]. Indeed,

$$\int_0^\infty f(x)x^{-s-1}dx = -\frac{1}{2s}\left(\frac{L'(s,\chi)}{L(s,\chi)} + \frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})}\right) - \frac{\mathbf{1}_{\chi=\chi_0}}{s-1},$$

which is regular in the half-plane $\Re(s) > \Theta_\chi$ but not in any half-plane of the form $\Re(s) > \Theta_\chi - \varepsilon$. Here we use the fact that the residues of $L'(s,\chi)/L(s,\chi)$ are nonnegative, since $L(s,\chi) - \mathbf{1}_{\chi=\chi_0}/(s-1)$ is entire (in other words, the poles of $L'(s,\chi)/L(s,\chi)$ and $L'(s,\overline{\chi})/L(s,\overline{\chi})$ cannot cancel each other).

Suppose otherwise that $L(\Theta_\chi, \chi) = 0$, and thus $\chi \neq \chi_0$. The explicit formula [33, Theorems 12.5 and 12.10] implies that for $T \geq 1$,

$$\psi(x,\chi) = - \sum_{\substack{\rho_\chi \\ |\Im(\rho_\chi)| \leq T}} \frac{x^{\rho_\chi}}{\rho_\chi} + O\left(\log(qx) + \frac{x(\log(qxT))^2}{T}\right).$$

We deduce that for any $0 < T_1 < T_2$,

$$\frac{1}{T_2-T_1} \int_{T_1}^{T_2} e^{-\Theta_\chi t} \psi(e^t, \chi) \, dt = - \sum_{\rho_\chi \neq \Theta_\chi} \frac{e^{T_2(\rho_\chi-\Theta_\chi)}-e^{T_1(\rho_\chi-\Theta_\chi)}}{\rho_\chi(\rho_\chi-\Theta_\chi)(T_2-T_1)} - \frac{\mathrm{ord}_{s=\Theta_\chi} L(s,\chi)}{\Theta_\chi}$$
$$+ O(e^{-\Theta_\chi T_1} \log(qT_2)).$$

Taking $T_1 = (1-\varepsilon)\log X$ and $T_2 = \log X$, and noting that the infinite sum over zeros in the last equation converges absolutely, we deduce that for large enough X there exists $x \in [X^{1-\varepsilon}, X]$ for which

$$x^{-\Theta_\chi} \Re(\psi(x,\chi)) < -\frac{\mathrm{ord}_{s=\Theta_\chi} L(s,\chi)}{2\Theta_\chi}.$$

The claim follows.   ∎

With this oscillation result in hand, we will deduce that in certain ranges, the upper bound (4) is stronger than GRH. This is made precise in the following proposition. The goal here is to overcome the uniformity problems caused by the fact that Θ_χ depends on χ in Lemma 2.1. This will be done by noticing that for many moduli q, characters of small conductor occur in the sums in equation (13).

Proposition 2.2. *Fix $0 < \delta < 1$, and assume that equation* (4) *holds in the range $x^\delta \leq q \leq 2x^\delta$. Then every Dirichlet L-function is nonvanishing in the half-plane $\Re(s) > 1/2 + \delta/2$. If one replaces $G(x;q)$ with $V_\Lambda(x;q)$ in* (4)*, then the same half-plane is zero-free for all Dirichlet L-functions corresponding to nonprincipal characters.*

Proof. We prove the contrapositive. Suppose that there exists a primitive Dirichlet L-function $L(s, \chi_e)$ of conductor $q_e \geq 1$ that has a zero with real part $\beta_e > 1/2 + \delta/2$. By Lemma 2.1, for every $0 < \varepsilon < \beta_e$ there exists an increasing sequence $\{x_j\}_{j\geq 1}$ tending to infinity such that

$$|\theta(x_j, \chi_e) - \mathbf{1}_{\chi=\chi_0} x_j| \geq x_j^{\beta_e-\varepsilon}.$$

We may assume that $x_j > q_e^{1/\delta}$ for each $j \geq 1$, so that any interval of length x_j^δ contains a multiple of q_e. For each $j \geq 1$, choose an integer $x_j^\delta \leq q_j \leq 2x_j^\delta$ that is a multiple of q_e, and let χ_j be the character modulo q_j induced by χ_e. Note that

$$|\theta(x_j, \chi_e) - \theta(x_j, \chi_j)| \leq \log q_j \ll_\delta \log x_j, \tag{14}$$

and hence for j large enough in terms of χ_e and ε,

$$|\theta(x_j, \chi_j) - \mathbf{1}_{\chi=\chi_0} x_j| \geq x_j^{\beta_e-\varepsilon}/2 \tag{15}$$

as well. Consequently, when j is large enough we have

$$G(x_j; q_j) = \frac{1}{\phi(q_j)} \sum_{\chi \bmod q_j} |\theta(x_j, \chi) - \mathbf{1}_{\chi=\chi_0} x_j|^2$$
$$\geq \frac{|\theta(x_j, \chi_j) - \mathbf{1}_{\chi=\chi_0} x_j|^2}{q_j} \geq \frac{x_j^{2(\beta_e-\varepsilon)}}{8x_j^\delta} = \frac{x_j^{2\beta_e-\delta-2\varepsilon}}{8}, \tag{16}$$

so that

$$\frac{G(x_j; q_j)}{x_j \log q_j} \gg \frac{x_j^{2\beta_e - 1 - \delta - 2\varepsilon}}{\log q_j} \gg \frac{x_j^{2\beta_e - 1 - \delta - 2\varepsilon}}{\delta \log x_j}. \tag{17}$$

By assumption, $2\beta_e - 1 - \delta > 0$, and so the exponent $2\beta_e - 1 - \delta - 2\varepsilon$ is positive as long as ε is chosen small enough. Therefore

$$\lim_{j \to \infty} \frac{G(x_j; q_j)}{x_j \log q_j} = \infty,$$

contradicting (4). The proof is identical for $V_\Lambda(x; q)$.   ∎

We now adapt the arguments in the proof of Proposition 2.2 to prove a proposition that is more suitable for the proof of Theorem 1.1.

Proposition 2.3. *Assume that GRH is false. Then there exists an absolute constant $\delta > 0$ with the following property. Let $h(x)$ be an increasing function tending to infinity such that $h(x) = o(x^\delta)$ as $x \to \infty$. For a positive proportion of moduli q, there exist associated values x_q such that $h(x_q^{1-\delta}) \le q \le h(x_q)$ and*

$$G(x_q; q) \ge x_q^{1+\delta}.$$

If $GRH(\chi)$ is false for some nonprincipal character χ, then the same lower bound holds with $V_\Lambda(x_q; q)$ in place of $G(x_q; q)$.

Proof. Fix a modulus $q_e \ge 1$ for which there exists an associated primitive character χ_e such that $L(s, \chi_e)$ has a zero with real part $\beta_e > 1/2$. Fix a positive number $\varepsilon < \beta_e - 1/2$, and choose a positive number

$$\delta < \beta_e - 1/2 - \varepsilon \le 1/2.$$

Now let q be any large enough multiple of q_e; the set of such moduli q has positive (though ineffective) density in $\mathbb{N}$. By Lemma 2.1, there exists $x_q \in [h^{-1}(q), h^{-1}(q)^{\frac{1}{1-\delta}}]$ such that

$$|\theta(x_q, \chi_e)| > x_q^{\beta_e - \varepsilon}.$$

Note that this implies that $q \in [h(x_q^{1-\delta}), h(x_q)]$. Denote by χ_q the character modulo q induced by χ_e. Then the calculations in equations (14) through (16) apply exactly to this situation; we conclude that

$$G(x_q; q) \gg x_q^{2\beta_e - \delta - 2\varepsilon},$$

and the right-hand side is eventually larger than $x_j^{1+\delta}$ by our choice of δ, establishing the asserted lower bound. The proof for $V_\Lambda(x; q)$ is identical as long as χ_e is nonprincipal.   ∎

To end this section, we further adapt Proposition 2.2 with the aim of proving Theorem 1.2. This situation is much easier since there are no uniformity issues (that is, q is fixed).

Proposition 2.4. *Fix $q \geq 1$, and assume that there exists a character χ_e modulo q such that $GRH(\chi_e)$ is false. Then there exists a sequence $\{x_i\}_{i \geq 1}$, depending on q, such that for each $\varepsilon > 0$,*

$$G(x_i; q) \geq \frac{1}{\phi(q)} |\theta(x_i, \chi_e) - \mathbf{1}_{\chi_e = \chi_0} x_i|^2 \gg_{\varepsilon, q} x_i^{2\Theta_{\chi_e} - \varepsilon}.$$

Here, Θ_{χ_e} is the supremum of real parts of zeros of $L(s, \chi_e)$. Similarly for $V_\Lambda(x; q)$ and $\psi(x, \chi)$.

Proof. This follows at once from equation (13) and Lemma 2.1. ∎

3. Explicit formulas and homogeneous Diophantine approximation

The goal of this section is to show that GRH implies Theorem 1.3. Our goal will be to synchronize the arguments of the summands in the explicit formula, but only for a subset $\mathcal{F}_q$ of the set $\mathcal{X}_q$ of characters modulo q.

Throughout, γ_χ denotes the imaginary part of a nontrivial zero of $L(s, \chi)$, and q_χ denotes the conductor of χ. We let $\|t\|$ denote the distance from t to the nearest integer, and we use the shorthand $\log_2 t = \log \log t$ and $\log_3 t = \log \log \log t$.

Lemma 3.1. *Let $q \geq 3$ be an integer. If χ mod q is a nonprincipal character, then assuming $GRH(\chi)$ we have the bound*

$$\operatorname{ord}_{s=1/2} L(s, \chi) \ll \frac{\log q}{\log_2 q}.$$

Moreover, if $GRH(\chi)$ is true for all nonprincipal χ mod q, then

$$\sum_{\chi \bmod q} \operatorname{ord}_{s=1/2} L(s, \chi) \leq \left(\frac{1}{2} + o_{q \to \infty}(1) \right) \phi(q).$$

Proof. The first bound is due to Selberg (see [42] or [25, Proposition 5.21]; see also [4]). As for the second, it was first proven by Murty [34, p. 436] (see also [24]). ∎

We will also need to control the conductors of the characters in $\mathcal{F}_q$. We define $\Phi_q = \#\mathcal{F}_q$. We will require $\mathcal{F}_q$ to have the property that

$$\chi \in \mathcal{F}_q \quad \text{if and only if} \quad \overline{\chi} \in \mathcal{F}_q. \tag{18}$$

Lemma 3.2. *Let $w(q) > 0$ be any function tending to zero as $q \to \infty$. For each $q \geq 3$, there exists a subset $\mathcal{F}_q \subset \mathcal{X}_q$ of the set of characters modulo q of cardinality*

$$\Phi_q = \#\mathcal{F}_q \geq \phi(q)(1 - O(w(q)^2)), \tag{19}$$

having the property (18), such that $\log q_\chi = \log q + O(w(q)^{-1} \log_2 q)$ for each character $\chi \in \mathcal{F}_q$.

Proof. By [8, Lemma 3.1], we have the estimate

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} (\log q_\chi - \log q)^2 \ll (\log_2 q)^2. \tag{20}$$

Combined with [10, Proposition 3.3] and Chebyshev's inequality, this yields

$$\frac{1}{\phi(q)} \#\{\chi \bmod q \colon |\log q_\chi - \log q| > w(q)^{-1} \log_2 q\} \ll w(q)^2.$$

Since $q_{\overline{\chi}} = q_\chi$, the characters not included in the above set have the property (18), and the desired estimate (19) follows. ∎

We are now ready to estimate the logarithmic averages of $\psi(x, \chi)$ and $\theta(x, \chi)$ over a short interval. By carefully choosing this interval to synchronize the frequencies in the explicit formula, we will ultimately create large values of $G(x; q)$ and $V_\Lambda(x; q)$. The Riemann–von Mangoldt formula

$$N(T, \chi) := \#\{\rho_\chi \colon |\Im(\rho_\chi)| \le T\} = \frac{T}{\pi} \log\left(\frac{q_\chi T}{2\pi e}\right) + O(\log(qT)) \tag{21}$$

for $T \ge 2$ will be central in our analysis. The next lemma records some estimates that follow easily from this asymptotic formula and partial summation.

Lemma 3.3. *For any real parameters y, $0 < \delta < 1$ and $T \ge \delta^{-1}$,*

$$\sum_{|\gamma_\chi|>T} \frac{e^{iy\gamma_\chi}}{\rho_\chi^2} \left(\frac{i \sin(\delta\gamma_\chi)}{\delta} + \frac{\cos(\delta\gamma_\chi)}{2}\right) \ll \frac{1}{\delta} \sum_{|\gamma_\chi|>T} \frac{1}{\gamma_\chi^2} \ll \frac{\log qT}{\delta T},$$

$$\max\left\{\sum_{0 \le \gamma_\chi \le T} \frac{\sin^2(\delta\gamma_\chi)}{\delta|\rho_\chi|^4}, \sum_{0 \le \gamma_\chi \le T} \frac{\gamma_\chi |\sin(\delta\gamma_\chi)\cos(\delta\gamma_\chi)|}{|\rho_\chi|^4}\right\} \ll \delta \sum_{0 \le \gamma_\chi \le T} \frac{1}{\gamma_\chi^2} \ll \delta \log q.$$

The next lemma is a careful evaluation of the averages of $\psi(e^t, \chi) - \mathbf{1}_{\chi=\chi_0} e^t$ and $\theta(e^t, \chi) - \mathbf{1}_{\chi=\chi_0} e^t$ over a short interval. The specific interval will be chosen later using homogeneous Diophantine approximation, and will contain a large value of those functions.

Lemma 3.4. *Let $q \ge 1$ be an integer, and let $\mathcal{F}_q$ be a set of characters modulo q with the property (18) such that $GRH(\chi)$ is true for all $\chi \in \mathcal{F}_q$. Moreover, let $0 < \delta < 1$, $T \ge \delta^{-1}$, and $M \ge 1$ be real parameters, and define*

$$R_\delta(y) := \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \sum_{\chi \in \mathcal{F}_q} (\psi(e^t, \chi) - \mathbf{1}_{\chi=\chi_0} e^t) \, dt,$$

$$S_\delta(y) := \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \sum_{\chi \in \mathcal{F}_q} (\theta(e^t, \chi) - \mathbf{1}_{\chi=\chi_0} e^t) \, dt.$$

(a) *For all $y \ge 0$,*

$$S_\delta(y) = R_\delta(y) + O(e^{y/2} \Phi_q). \tag{22}$$

(b) *If $n \in \mathbb{N}$ satisfies $\|n\gamma_\chi\delta/2\pi\| < M^{-1}$ for each $0 \le \gamma_\chi \le T$ with $\chi \in \mathcal{F}_q$, then $y = (n+1)\delta$ has the property that*

$$R_\delta(y) =$$
$$-e^{y/2} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \left(\frac{2\gamma_\chi \sin(\delta\gamma_\chi)(\gamma_\chi \sin(\delta\gamma_\chi)+\cos(\delta\gamma_\chi))}{\delta|\rho_\chi|^4} + \left(\frac{1}{4}-\gamma_\chi^2\right)\frac{\cos^2(\delta\gamma_\chi)}{|\rho_\chi|^4} \right)$$
$$+ O\left(e^{y/2}\Phi_q \left(\frac{\log(qT)}{\delta T} + \min\left(\frac{\phi(q)}{\Phi_q}, \frac{\log q}{\log_2 q}\right) + \frac{y\log q}{e^{y/2}} \right. \right.$$
$$\left. \left. + \frac{\log(qT)\log T}{M} + \delta \log q \right) \right). \quad (23)$$

Proof. We begin by noting that part (a) is a direct consequence of the bound

$$|\psi(x,\chi) - \theta(x,\chi)| \le \sum_{\substack{p^k \le x \\ k \ge 2}} \log p \ll x^{1/2}.$$

As for part (b), we can transfer the question to primitive characters by noting that if χ^* denotes the primitive character inducing χ, then

$$\sum_{\chi \in \mathcal{F}_q} \psi(e^t,\chi) - \sum_{\chi \in \mathcal{F}_q} \psi(e^t,\chi^*) \ll \Phi_q t \log q.$$

Moreover, $L(s,\chi)$ and $L(s,\chi^*)$ have the same zeros on the critical line. Hence, the explicit formula [33, Theorems 12.5 and 12.10] gives that for $y \ge 1$ and $S \ge 1$,

$$R_\delta(y) = -\frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \sum_{\chi \in \mathcal{F}_q} \sum_{\substack{\rho_\chi \\ |\Im(\rho_\chi)| \le S}} \frac{e^{t\rho_\chi}}{\rho_\chi} \, dt + O\left(\Phi_q y \log q + \frac{\Phi_q e^y (\log(qe^y S))^2}{S} \right)$$

$$= -\frac{1}{2\delta} \sum_{\chi \in \mathcal{F}_q} \sum_{\substack{\rho_\chi \\ |\Im(\rho_\chi)| \le S}} \frac{e^{y\rho_\chi}}{\rho_\chi^2}(e^{\delta\rho_\chi} - e^{-\delta\rho_\chi}) + O\left(\Phi_q y \log q + \frac{\Phi_q e^y (\log(qe^y S))^2}{S} \right)$$

$$= -\frac{1}{2\delta} \sum_{\chi \in \mathcal{F}_q} \sum_{\rho_\chi} \frac{e^{y\rho_\chi}}{\rho_\chi^2}(e^{\delta\rho_\chi} - e^{-\delta\rho_\chi}) + O(\Phi_q y \log q)$$

after taking $S \to \infty$. Using $e^{\pm\delta\rho_\chi} = e^{\pm i\delta\gamma_\chi}(1 \pm \delta/2 + O(\delta^2))$ and truncating the infinite sum over zeros using Lemma 3.3, we deduce the estimate

$$R_\delta(y)$$
$$= -e^{y/2} \sum_{\chi \in \mathcal{F}_q} \sum_{\rho_\chi} \frac{e^{iy\gamma_\chi}}{\rho_\chi^2} \left(\frac{i\sin(\delta\gamma_\chi)}{\delta} + \frac{\cos(\delta\gamma_\chi)}{2} \right) + O(\Phi_q y \log q + \Phi_q \delta e^{y/2} \log q)$$

$$= -e^{y/2} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \left(\frac{i\sin(\delta\gamma_\chi)}{\delta}\left(\frac{e^{iy\gamma_\chi}}{\rho_\chi^2} - \frac{e^{-iy\gamma_\chi}}{\overline{\rho_\chi}^2}\right) + \frac{\cos(\delta\gamma_\chi)}{2}\left(\frac{e^{iy\gamma_\chi}}{\rho_\chi^2} + \frac{e^{-iy\gamma_\chi}}{\overline{\rho_\chi}^2}\right) \right)$$
$$+ O\left(e^{y/2}\frac{\Phi_q \log(qT)}{\delta T} + e^{y/2}\min\left(\phi(q), \Phi_q \frac{\log q}{\log_2 q}\right) + \Phi_q y \log q + \Phi_q \delta e^{y/2} \log q \right), \quad (24)$$

where we have grouped conjugate zeros together, using Lemma 3.1 to bound the contribution from possible zeros at $s = 1/2$ for which this grouping is erroneous.

We now apply the hypothesis that $y = (n + 1)\delta$ with $\|n\gamma_\chi\delta/2\pi\| < M^{-1}$ for each $0 \le \gamma_\chi \le T$ and $\chi \in \mathcal{F}_q$. It follows that $e^{\pm iy\gamma_\chi} = e^{\pm i\delta\gamma_\chi}(1 + O(M^{-1}))$, and thus the main term of (24) equals

$$
-e^{y/2} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \left(\frac{i\sin(\delta\gamma_\chi)}{\delta} \left(\frac{e^{i\delta\gamma_\chi}}{\rho_\chi^2} - \frac{e^{-i\delta\gamma_\chi}}{\overline{\rho}_\chi^2} \right) + \frac{\cos(\delta\gamma_\chi)}{2} \left(\frac{e^{i\delta\gamma_\chi}}{\rho_\chi^2} + \frac{e^{-i\delta\gamma_\chi}}{\overline{\rho}_\chi^2} \right) \right)
$$
$$
+ O\left(\frac{e^{y/2}}{M} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \frac{1}{|\rho_\chi|} \right)
$$
$$
= -e^{y/2} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \left\{ \frac{-2\sin(\delta\gamma_\chi)}{\delta|\rho_\chi|^4} \left(\left(\frac{1}{4} - \gamma_\chi^2 \right) \sin(\delta\gamma_\chi) - \gamma_\chi \cos(\delta\gamma_\chi) \right) \right.
$$
$$
\left. + \frac{\cos(\delta\gamma_\chi)}{|\rho_\chi|^4} \left(\left(\frac{1}{4} - \gamma_\chi^2 \right) \cos(\delta\gamma_\chi) + \gamma_\chi \sin(\delta\gamma_\chi) \right) \right\} + O\left(\Phi_q \frac{e^{y/2}}{M} \log(qT) \log T \right)
$$
$$
= -e^{y/2} \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \left(\frac{2\gamma_\chi \sin(\delta\gamma_\chi)(\gamma_\chi \sin(\delta\gamma_\chi) + \cos(\delta\gamma_\chi))}{\delta|\rho_\chi|^4} + \left(\frac{1}{4} - \gamma_\chi^2 \right) \frac{\cos^2(\delta\gamma_\chi)}{|\rho_\chi|^4} \right)
$$
$$
+ O\left(\Phi_q \frac{e^{y/2}}{M} \log(qT) \log T + \Phi_q \delta e^{y/2} \log q \right)
$$

by Lemma 3.3. ∎

Now that we have expressed averages of $\theta(x, \chi)$ and $\psi(x, \chi)$ in suitable short intervals in terms of sums over zeros, our strategy is to estimate the sums in (23) using the Riemann–von Mangoldt formula (21).

Lemma 3.5. *Let $q \ge 1$ be an integer, and let $\mathcal{F}_q$ be a set of characters modulo q with the property* (18) *such that GRH(χ) is true for all $\chi \in \mathcal{F}_q$. For any $0 < \delta \le e^{-1}$ and $T \ge e\delta^{-1}$,*

$$
\delta \sum_{\chi \in \mathcal{F}_q} \sum_{0 \le \gamma_\chi \le T} \frac{2\gamma_\chi^4}{|\rho_\chi|^4} \frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} = \frac{\Phi_q}{2} \log(q\delta^{-1})
$$
$$
+ O\left(\delta\Phi_q \log(qT) \log T + \delta^{1/2}\Phi_q \log(q\delta^{-1}) + (E_q + \Phi_q)\log(T\delta) + \frac{\Phi_q \log(qT)}{\delta T} \right),
$$
$$(25)$$

where

$$
E_q := \sum_{\chi \in \mathcal{F}_q} (\log q - \log q_\chi). \tag{26}
$$

Proof. We will see that the main contribution in the sum on the left-hand side of (25) comes from zeros γ_χ of intermediate size. We can therefore discard low-lying zeros as follows:

$$\delta \sum_{\chi \in \mathcal{F}_q} \sum_{0 \leq \gamma_\chi \leq T} \frac{2\gamma_\chi^4}{|\rho_\chi|^4} \frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2}$$

$$= \delta \sum_{\chi \in \mathcal{F}_q} \sum_{\delta^{-1/2} < \gamma_\chi \leq T} \frac{2\gamma_\chi^4}{|\rho_\chi|^4} \frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} + O\Big(\delta \sum_{\chi \in \mathcal{F}_q} N(\delta^{-1/2}, \chi)\Big)$$

$$= \delta \sum_{\chi \in \mathcal{F}_q} \sum_{\delta^{-1/2} < \gamma_\chi \leq T} \Big(2 + O\Big(\frac{1}{\gamma_\chi^2}\Big)\Big) \frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} + O(\Phi_q \delta^{1/2} \log(q\delta^{-1/2}))$$

$$= 2\delta \sum_{\chi \in \mathcal{F}_q} \sum_{\delta^{-1/2} < \gamma_\chi \leq T} \frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} + O(\Phi_q \delta^{3/2} \log(q\delta^{-1/2}) + \Phi_q \delta^{\frac{1}{2}} \log(q\delta^{-1})),$$

and the first error term is smaller than the second. Define

$$N(t, \mathcal{F}_q) := \sum_{\chi \in \mathcal{F}_q} N(t, \chi) = \sum_{\chi \in \mathcal{F}_q} \Big(\frac{t}{\pi} \log\Big(\frac{q_\chi t}{2\pi e}\Big) + O(\log(qt))\Big)$$

$$= \sum_{\chi \in \mathcal{F}_q} \frac{t}{\pi} \log(qt) - \sum_{\chi \in \mathcal{F}_q} \frac{t}{\pi}(\log q - \log q_\chi) + O\Big(\sum_{\chi \in \mathcal{F}_q}(t + \log(qt))\Big)$$

$$= \Phi_q \frac{t}{\pi} \log(qt) + O(E_q t + \Phi_q(t + \log(qt)))$$

by the asymptotic formula (21). Notice that counting only zeros above the real axis would yield $\frac{1}{2}N(t, \mathcal{F}_q)$ in place of $N(t, \mathcal{F}_q)$ thanks to the property (18) and the functional equation for Dirichlet L-functions. We may now compute

$$2\delta \sum_{\chi \in \mathcal{F}_q} \sum_{\delta^{-1/2} < \gamma_\chi \leq T} \frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} = 2\delta \int_{\delta^{-1/2}}^T \frac{\sin^2(\delta t)}{(\delta t)^2} d\big(\tfrac{1}{2}N(t, \mathcal{F}_q)\big)$$

$$= \delta N(t, \mathcal{F}_q) \frac{\sin^2(\delta t)}{(\delta t)^2}\Big|_{\delta^{-1/2}}^T - \delta \int_{\delta^{-1/2}}^T \Big(\frac{2\sin(\delta t)\cos(\delta t)}{\delta t^2} - \frac{2\sin^2(\delta t)}{\delta^2 t^3}\Big) N(t, \mathcal{F}_q)\, dt$$

$$= -\delta \int_{\delta^{-1/2}}^T 2\Big(\frac{\sin(2\delta t)}{2\delta t^2} - \frac{\sin^2(\delta t)}{\delta^2 t^3}\Big) N(t, \mathcal{F}_q)\, dt$$

$$+ O\Big(\delta^{1/2}\Phi_q \log(q\delta^{-1/2}) + \frac{\Phi_q \log(qT)}{\delta T}\Big)$$

$$= -\frac{\Phi_q \delta}{\pi} \int_{\delta^{-1/2}}^T 2\Big(\frac{\sin(2\delta t)}{2\delta t^2} - \frac{\sin^2(\delta t)}{\delta^2 t^3}\Big) t \log(qt)\, dt$$

$$+ O\Big(\Phi_q(\delta \log(qT)\log T + \log(T\delta)) + E_q \log(T\delta)$$

$$\hspace{4cm} + \delta^{1/2}\Phi_q \log(q\delta^{-1}) + \frac{\Phi_q \log(qT)}{\delta T}\Big), \quad (27)$$

using $\frac{\sin u}{u} \ll \min\{1, 1/|u|\}$. Finally, integrating by parts in the other direction, the main term in this expression equals

$$-\frac{\Phi_q \delta}{\pi} t \log(qt) \frac{\sin^2(\delta t)}{(\delta t)^2}\Big|_{\delta^{-1/2}}^T + \frac{\Phi_q \delta}{\pi} \int_{\delta^{-1/2}}^T \frac{\sin^2(\delta t)}{(\delta t)^2}(\log(qt) + 1)\, dt$$

$$= \frac{\Phi_q}{\pi} \int_0^\infty \frac{\sin^2 u}{u^2}(\log(q\delta^{-1}u) + 1)\, du + O\Big(\delta^{1/2}\Phi_q \log(q\delta^{-1/2}) + \frac{\Phi_q \log(qT)}{\delta T}\Big)$$

$$= \frac{\Phi_q}{2} \log(q\delta^{-1}) + O\Big(\delta^{1/2}\Phi_q \log(q\delta^{-1}) + \frac{\Phi_q \log(qT)}{\delta T} + \Phi_q\Big)$$

by the evaluation $\int_0^\infty \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$, which is a particular case of the identity

$$\widehat{\frac{\sin^2(2\pi x)}{(2\pi x)^2}}(\xi) = \begin{cases} \frac{1}{2} - \frac{|\xi|}{4} & \text{if } |\xi| \leq 2, \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Now that we have evaluated some of the main terms in (23), we can deduce a precise estimate of the values attained by $R_\delta(y)$ and $S_\delta(y)$ in Lemma 3.4.

Lemma 3.6. *Let $q \geq 1$ be an integer, and let $\mathcal{F}_q$ be a set of characters modulo q with the property* (18) *such that $GRH(\chi)$ is true for all $\chi \in \mathcal{F}_q$. Let $\delta > 0$ be small enough (in absolute terms), $T \geq e\delta^{-1}$ and $M \geq 1$. If $n \in \mathbb{N}$ satisfies $\|n\gamma_\chi\delta/(2\pi)\| < M^{-1}$ for each $0 \leq \gamma_\chi \leq T$ with $\chi \in \mathcal{F}_q$, then $y = (n+1)\delta$ has the property that*

$$R_\delta(y) = -\frac{e^{y/2}\Phi_q \log(q^2\delta^{-1})}{2} + O\left(e^{y/2}\Phi_q\left(\frac{\log(qT)}{\delta T} + \left(\frac{E_q}{\Phi_q} + 1\right)\log(T\delta)\right.\right.$$
$$\left.\left. + \frac{y\log q}{e^{y/2}} + \frac{\log(qT)\log T}{M} + \delta^{1/2}\log(qT) + \frac{\log q}{\log_2 q}\right)\right),$$

where E_q was defined in (26).

Proof. We first truncate part of the sum in Lemma 3.4, in order to expand $\sin(\delta\gamma_\chi)$ and $\cos(\delta\gamma_\chi)$ into Taylor series. Doing so, we see that the main term in the estimate (23) equals

$$-e^{y/2}\sum_{\chi\in\mathcal{F}_q}\sum_{0\leq\gamma_\chi\leq\delta^{-1/2}}\left(\frac{2\gamma_\chi\sin(\delta\gamma_\chi)\cos(\delta\gamma_\chi)}{\delta|\rho_\chi|^4} + \left(\frac{1}{4}-\gamma_\chi^2\right)\frac{\cos^2(\delta\gamma_\chi)}{|\rho_\chi|^4}\right)$$

$$-e^{y/2}\delta\sum_{\chi\in\mathcal{F}_q}\sum_{0\leq\gamma_\chi\leq T}\frac{2\gamma_\chi^4}{|\rho_\chi|^4}\frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} + O\big(e^{y/2}\delta^{1/2}\Phi_q\log(q\delta^{-1})\big)$$

$$= -e^{y/2}\delta\sum_{\chi\in\mathcal{F}_q}\sum_{0\leq\gamma_\chi\leq T}\frac{2\gamma_\chi^4}{|\rho_\chi|^4}\frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} - e^{y/2}\sum_{\chi\in\mathcal{F}_q}\sum_{0\leq\gamma_\chi\leq\delta^{-1/2}}\frac{1/4+\gamma_\chi^2}{|\rho_\chi|^4}$$
$$+ O\big(e^{y/2}\delta^{1/2}\Phi_q\log(q\delta^{-1})\big)$$

$$= -e^{y/2}\delta\sum_{\chi\in\mathcal{F}_q}\sum_{\gamma_\chi\geq 0}\frac{2\gamma_\chi^4}{|\rho_\chi|^4}\frac{\sin^2(\delta\gamma_\chi)}{(\delta\gamma_\chi)^2} - e^{y/2}\sum_{\chi\in\mathcal{F}_q}\sum_{\gamma_\chi\geq 0}\frac{1}{|\rho_\chi|^2}$$
$$+ O\left(e^{y/2}\delta^{1/2}\Phi_q\log(q\delta^{-1}) + e^{y/2}\Phi_q\frac{\log(qT)}{\delta T}\right). \qquad (28)$$

The second double sum can be treated by using the functional equation (with Lemma 3.1 accounting for possible zeros at $s = 1/2$), and applying [10, Lemma 3.5] and the Littlewood bound $L'(1,\chi)/L(1,\chi) \ll \log_2 q$; we obtain

$$\sum_{\chi\in\mathcal{F}_q}\sum_{\gamma_\chi\geq 0}\frac{1}{|\rho_\chi|^2} = \frac{1}{2}\sum_{\chi\in\mathcal{F}_q}\sum_{\gamma_\chi}\frac{1}{|\rho_\chi|^2} + O\left(\frac{\Phi_q\log q}{\log_2 q}\right)$$
$$= \frac{1}{2}\sum_{\chi\in\mathcal{F}_q}\log q_\chi + O\left(\frac{\Phi_q\log q}{\log_2 q}\right) = \frac{\Phi_q\log q}{2} + O\left(\frac{\Phi_q\log q}{\log_2 q} + E_q\right).$$

Combining this evaluation with Lemma 3.5, we find that the sum of the first two terms in (28) equals

$$
- e^{y/2} \frac{\Phi_q \log(q^2 \delta^{-1})}{2} + O\bigg(e^{y/2} \Phi_q \bigg(\delta \log(qT) \log T + \delta^{1/2} \log(q\delta^{-1})
$$
$$
+ \frac{\log(qT)}{\delta T} + \bigg(\frac{E_q}{\Phi_q} + 1 \bigg) \log(T\delta) + \frac{\log q}{\log_2 q} \bigg) \bigg). \quad \blacksquare
$$

Now that we know exactly how large $R_\delta(y)$ and $S_\delta(y)$ can be, it is time to understand more precisely the set of y which are admissible. In particular, it is important for us to localize these values in terms of the modulus q. This will be done using a counting argument, inspired by the proof of [37, Lemma 2.4].

Lemma 3.7. *Let $\Lambda = \{\lambda_1, \dots, \lambda_k\}$ be a set of real numbers. For any positive integers M and N,*

$$
\#\{n \le N : \|n\lambda\| \le M^{-1} \text{ for all } \lambda \in \Lambda\} \ge \frac{N}{M^k} - 1.
$$

Proof. Consider the integer multiples $n\mathbf{v}$, with $1 \le n \le N$, of the vector $\mathbf{v} = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k/\mathbb{Z}^k$. If we divide $\mathbb{R}^k/\mathbb{Z}^k$ into M^k cubes of side length M^{-1}, then one of these cubes will contain $s \ge N/M^k$ multiples of $\mathbf{v}$. If the integers producing these multiples are $m_1 < \cdots < m_s$, then we have

$$
\{n \le N : \|n\lambda\| \le M^{-1} \text{ for all } \lambda \in \Lambda\} \supset \{m_2 - m_1, m_3 - m_1, \dots, m_s - m_1\},
$$

and the cardinality of the right-hand side is at least $N/M^k - 1$. $\quad \blacksquare$

Proposition 3.8. *Fix $\varepsilon > 0$ sufficiently small, and let $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ be two functions such that f is minorized by a large enough constant and $(\varepsilon^{-1} \log q)/\phi(q) \le g(q) \le \log q$. If GRH is true, then for each sufficiently large q there exists x_q satisfying*

$$
\log_2 x_q \asymp_\varepsilon \phi(q) f(q) g(q) \bigg(1 + \frac{\log(f(q))}{\log q} \bigg)
$$

with the property that

$$
\frac{G(x_q; q)}{x_q} \ge \bigg(\frac{1}{4} - 2\varepsilon \bigg) \frac{g(q)(\log(q^2 f(q)))^2}{\log q}.
$$

Under the slightly weaker assumption that GRH(χ) is true for every nonprincipal character χ, the same statement holds with $G(x_q; q)$ replaced by $V_\Lambda(x_q; q)$.

Proof. We apply Lemma 3.2 with $w(q) = (\log_2 q)^{-1}$. We deduce the existence of a set $\mathcal{G}_q$ of characters modulo q for which $|\mathcal{G}_q| \ge \phi(q)(1 - K(\log_2 q)^{-2})$, where $K > 0$ is an absolute constant, such that $\log q_\chi = \log q + O((\log_2 q)^2)$ for each $\chi \in \mathcal{G}_q$. We can assume that all elements of $\mathcal{G}_q$ are complex characters, since there are $\ll 2^{\omega(q)} \ll \sqrt{q}$ real characters modulo q. Since $q_\chi = q_{\overline{\chi}}$, we can also assume that $\mathcal{G}_q$ has the property (18).

We extract a subset $\mathcal{F}_q \subset \mathcal{G}_q$ of cardinality

$$\Phi_q = 2\left\lfloor \frac{\phi(q)g(q)}{2\log q}(1 - K(\log_2 q)^{-2}) \right\rfloor$$

for which (18) holds, where the right-hand side is at least 2 when q is sufficiently large. By Lemma 3.2, the error term (26) then satisfies the bound

$$E_q \ll \Phi_q(\log_2 q)^2.$$

We now apply Lemma 3.7 to the set $\Lambda = \{0 \leq \gamma_\chi \leq T : \chi \in \mathcal{F}_q\}$. For T large enough, it follows from equation (21) that the set S of values of $n \leq N$ for which $\|n\delta\gamma_\chi/(2\pi)\| < (2\pi M)^{-1}$ has at least $NM^{(-1+o(1))\Phi_q T \log(qT)/2\pi} - 1$ elements. Taking $N = M^{\Phi_q T \log(qT)/\pi}$, we find that $S \cap [N^{1/3}, N] \neq \emptyset$. Then, Lemma 3.6 implies that for $y = \delta(n + 1)$ with $n \in S$,

$$S_\delta(y) = -\frac{e^{y/2}\Phi_q \log(q^2\delta^{-1})}{2} + O\left(e^{y/2}\Phi_q\left(\frac{\log(qT)}{\delta T} + (\log_2 q)^2 \log(\delta T)\right.\right.$$
$$\left.\left. + \frac{y\log q}{e^{y/2}} + \frac{\log(qT)\log T}{M} + \delta^{1/2}\log(qT) + \frac{\log q}{\log_2 q}\right)\right).$$

If $C = C(\varepsilon) > 0$ is large enough, then picking $T = C/\delta$, $M = C\log T$ and $0 < \delta < C^{-2}$ will result, for q and y large enough, in the bound

$$S_\delta(y) \leq -e^{y/2}\Phi_q \log(q^2\delta^{-1})\left(\frac{1-3\varepsilon}{4}\right)^{1/2}. \tag{29}$$

Now $S_\delta(y)$ is the average of the function $\sum_{\chi \in \mathcal{F}_q}(\theta(e^t, \chi) - \mathbf{1}_{\chi=\chi_0}e^t)$ over the short interval $e^t \in [e^{y-\delta}, e^{y+\delta}]$, and hence this function itself has such a large negative value in that interval. In other words, there exists a value $x = e^y(1 + O(\delta))$ such that

$$\sum_{\chi \in \mathcal{F}_q}(\theta(x, \chi) - \mathbf{1}_{\chi=\chi_0}x) \leq -x^{1/2}\Phi_q \log(q^2\delta^{-1})\left(\frac{1-2\varepsilon}{4}\right)^{1/2}, \tag{30}$$

since C is large enough in terms of ε. Using positivity in (13) and applying the Cauchy-Schwarz inequality, we obtain

$$G(x; q) \geq \frac{1}{\phi(q)}\sum_{\chi \in \mathcal{F}_q}|\theta(x, \chi) - \mathbf{1}_{\chi=\chi_0}x|^2 \geq \frac{1}{\phi(q)\Phi_q}\left|\sum_{\chi \in \mathcal{F}_q}(\theta(x, \chi) - \mathbf{1}_{\chi=\chi_0}x)\right|^2$$
$$\geq x\frac{\Phi_q}{\phi(q)}(\log(q^2\delta^{-1}))^2\left(\frac{1-2\varepsilon}{4}\right). \tag{31}$$

Since $y = \delta(n + 1)$ with $n \in [N^{1/3}, N]$, it follows that the associated x satisfies

$$\log_2 x \asymp_\varepsilon \delta^{-1} \log_2(\delta^{-1})\Phi_q \log(q\delta^{-1}). \tag{32}$$

The result follows from taking $\delta = f(q)^{-1}\log_2 f(q)$.

The proof is identical for $V_\Lambda(x; q)$. ∎

We are now ready to prove our main technical theorem, at which point we will be able to deduce Theorems 1.1, 1.3, 1.2, and 1.5.

Theorem 3.9. *Assume GRH, and fix $\varepsilon > 0$ small enough.*

(a) *If $h(x)$ is an increasing function satisfying*

$$\varepsilon \frac{\log_2 x}{\log_3 x} \leq h(x) \leq (\log x)^{\varepsilon^2/3} \quad \text{for all } x \geq e^3, \tag{33}$$

then for all moduli q there exist associated values x_q satisfying

$$h(\exp((\log x_q)^{c_1 \varepsilon^{-1}})) \leq \phi(q) \leq h(\exp((\log x_q)^{c_2 \varepsilon^{-1}}))$$

such that

$$G(x_q; q) \gg_\varepsilon x_q \log q \cdot \frac{\log_2 x_q}{\phi(q)}.$$

(b) *If $h(x)$ is a function with the property (8) and satisfying*

$$h(x) \leq \varepsilon \frac{\log_2 x}{\log_3 x} \quad \text{for all } x \geq e^3, \tag{34}$$

then for all sufficiently large moduli q there exist associated values x_q satisfying $\phi(q) \asymp_\varepsilon h(x_q)$ such that

$$G(x_q; q) \geq \left(\frac{1}{4} - \varepsilon\right) x_q (\log q + \log_3 x_q)^2.$$

These results hold with $V_\Lambda(x_q; q)$ in place of $G(x_q; q)$, under the weaker assumption that GRH(χ) is true for every nonprincipal character χ.

Proof. Under the condition (33), we apply Proposition 3.8 with $f(q)$ equal to a sufficiently large absolute constant, and with $g(q) = \varepsilon \log_2(h^{-1}(\phi(q)))/(2\phi(q))$. Note that the inequality $h(\exp(q^{2\varepsilon^{-1}\phi(q)})) \geq \phi(q)$ holds for q large enough, and thus $g(q) \leq \log q$. Moreover, $h(\exp(q^{2\varepsilon^{-2}})) \leq q^{2/3} \leq \phi(q)$, and hence $g(q) \geq (\varepsilon^{-1} \log q)/\phi(q)$. Therefore, the hypotheses of Proposition 3.8 are satisfied. We deduce the existence of a sequence $\{x_q\}_{q \geq q_0}$ such that $\log_2 x_q \asymp \phi(q)g(q)$ and

$$\frac{G(x_q; q)}{x_q} \gg g(q) \log q \asymp \frac{\log_2 x_q \cdot \log q}{\phi(q)},$$

establishing part (a).

On the other hand, under the conditions (8) and (34), we make the choices $g(q) = \log q$ and

$$f(q) := \begin{cases} \dfrac{\log_2(h^{-1}(\phi(q)))}{\phi(q) \log q} & \text{if } \log_2(h^{-1}(\phi(q))) \leq q^2, \\[4mm] \dfrac{\log_2(h^{-1}(\phi(q)))}{\phi(q) \log_3(h^{-1}(\phi(q)))} & \text{otherwise.} \end{cases}$$

The condition (34) ensures that $f(q)$ is minorized by a large enough positive constant when q is sufficiently large. Moreover, one can check that

$$\phi(q) f(q) \log q \cdot \left(1 + \frac{\log(f(q))}{\log q} \right) \asymp \log_2(h^{-1}(\phi(q))).$$

Hence, Proposition 3.8 (applied with $\varepsilon/4$ in place of 2ε) yields a real number x_q satisfying $\log_2 x_q \asymp \log_2(h^{-1}(\phi(q)))$ with the property that if $\log_2(h^{-1}(\phi(q))) > q^2$, then

$$
\begin{aligned}
\frac{G(x_q; q)}{x_q} &\geq \left(\frac{1}{4} - \frac{\varepsilon}{2} \right) (\log(q^2 f(q)))^2 \\
&\geq \left(\frac{1}{4} - \frac{\varepsilon}{2} \right) \left(\log \left(\frac{q \log_2(h^{-1}(\phi(q)))}{\log_3(h^{-1}(\phi(q)))} \right) \right)^2 \\
&\geq \left(\frac{1}{4} - \varepsilon \right) (\log(q \log_2(h^{-1}(\phi(q)))))^2
\end{aligned}
$$

when q is sufficiently large, and similarly when $\log_2(h^{-1}(\phi(q))) \leq q^2$. Moreover, the estimate $\log_2 x_q \asymp \log_2(h^{-1}(\phi(q)))$ combined with the property (8) implies that $\phi(q) \asymp h(x_q)$, establishing part (b). ∎

Proof of Theorems 1.1 and 1.3. We prove Theorem 1.3 which implies Theorem 1.1. If we assume that GRH is false, then the desired result for $G(x; q)$ follows from Proposition 2.3. On the other hand, if we assume that GRH holds, then the desired result follows from applying Theorem 3.9, which holds for all moduli q, and then restricting to the positive proportion of moduli q that satisfy $\phi(q) \geq \frac{1}{2} q$, say. (The constant $\frac{1}{2}$ is unimportant here; any constant less than 1 suffices, since we know [40, Theorem 1, §8] (see also [39, Section 5]) that the limiting distribution function $\phi(q)/q$ is strictly increasing on $(0, 1)$.)

The proof is similar for $V_\Lambda(x; q)$, and the Riemann hypothesis for principal characters χ_0 is never needed (see (13)). ∎

Proof of Theorem 1.2. If GRH(χ) is false, then the desired result for $\theta(x, \chi)$ follows from Proposition 2.4. If GRH(χ) is true, then we argue analogously to the proof of Proposition 3.8.

Take $\mathcal{F}_q = \{\chi, \overline{\chi}\}$ in Lemma 3.6, as well as $T = C/\delta$, $M = C \log T$ and $\delta < C^{-2}$ with C large enough. Take moreover $N = M^{T \log(qT)/\pi}$ in Lemma 3.7. Hence, there exists $n \in [N^{1/3}, N]$ such that $y = (n + 1)\delta$ has the property that

$$S_\delta(y) \leq -\frac{1}{2} e^{y/2} \log(\delta^{-1}).$$

Since

$$\log_2 x = \log y \asymp_q \delta^{-1} \log(\delta^{-1}) \log_2(\delta^{-1}),$$

we have $e^{y/2} \log(\delta^{-1}) \asymp_q x^{1/2} \log_3 x$ and the lower bound (6) follows. The proof of the lower bound (7) is similar, this time taking $\mathcal{F}_q$ to be the set of all characters modulo q and applying (31). ∎

Proof of Theorem 1.5. If $Q(x)$ is bounded, and thus is eventually constant, $Q(x) = Q_0$, the result follows from the bound

$$\sum_{Q_0 < q \leq 2Q_0} G(x;q) \geq G(x;2Q_0)$$

and Theorem 1.2.

We now assume that $Q(x)$ tends to infinity. If GRH is false, we let $\chi_e \bmod q_e$ be a primitive character for which $L(s, \chi_e)$ has a nontrivial zero off the critical line. Then for x large enough the interval $(Q(x), 2Q(x)]$ will contain a multiple q_j of q_e. Hence, if $\chi_j \bmod q_j$ is the character induced by χ_e, we have

$$\sum_{Q(x) < q \leq 2Q(x)} G(x;q) \geq \frac{|\theta(x, \chi_j) - \mathbf{1}_{\chi_j = \chi_0}|^2}{\phi(q)},$$

and the rest of the proof proceeds as in the proof of Proposition 2.2.

If GRH is true, then we argue as in the proof of Proposition 3.8. We apply Lemma 3.7 to the set $\Lambda = \{0 \leq \gamma_\chi \leq T : \chi \bmod q,\ Q < q \leq 2Q\}$. Taking $N = M^{2Q^2 T \log(QT)/\pi}$, we see that the set S of values of $n \leq N$ for which $\|n\delta_Q \gamma_\chi/(2\pi)\| < (2\pi M)^{-1}$ has at least one element exceeding $N^{1/9}$. Then we set $T = C/\delta_Q$, $M = C \log T$ and $\delta_Q \leq C^{-2}$ with C large enough in Lemma 3.6, and find that for $y = \delta_Q(n + 1)$ with $n \in S$,

$$S_{\delta_Q}(y) \leq -\left(\frac{1}{2} - \varepsilon\right) e^{y/2} \phi(q) \log(q^2 \delta_Q^{-1}).$$

Hence, as in (31), for each large enough Q there exists x_Q such that

$$\frac{1}{Q} \sum_{Q < q \leq 2Q} G(x_Q; q) \gg \frac{x_Q}{Q} \sum_{Q < q \leq 2Q} (\log(q^2 \delta_Q^{-1}))^2 \gg x_Q (\log(Q^2 \delta_Q^{-1}))^2,$$

and for which

$$\log_2 x_Q \asymp Q^2 \delta_Q^{-1} \log(Q \delta_Q^{-1}).$$

The rest of the proof proceeds as in the proof of Theorem 3.9. ∎

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