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The simple type conjecture for mod 2 Seiberg–Witten invariants

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Abstract. We prove that, under a simple condition on the cohomology ring, every closed 4-manifold has mod 2 Seiberg–Witten simple type. This result shows that there exists a large class of topological 4-manifolds such that all smooth structures have mod 2 simple type, and yet some have non-vanishing (mod 2) Seiberg–Witten invariants. As corollaries, we obtain adjunction inequalities and show that, under a mild topological condition, every geometrically simply connected closed 4-manifold has the vanishing mod 2 Seiberg–Witten invariant for at least one orientation.

Keywords. 4-manifolds, Bauer–Furuta invariants, handle decompositions

1. Introduction

The Seiberg–Witten invariant [27] of a smooth 4-manifold has played a significant role in the study of 4-manifolds over the past 25 years, and has produced many striking applications to low-dimensional topology. Although the invariants have been computed for various 4-manifolds, in general it still seems out of reach to compute. The simple type conjecture, posed in 1990s, states a fundamental constraint on the invariant (e.g. [17, Conjecture 1.6.2]).

Conjecture 1.1 (Simple type conjecture). *Every closed, connected, oriented, smooth 4-manifold with $b_2^+ > 1$ has Seiberg–Witten simple type.*

Here a closed, connected, oriented, smooth 4-manifold X with $b_2^+ > 1$ is called of *Seiberg–Witten simple type* if the (integer valued) Seiberg–Witten invariant $SW_X(\mathfrak{s})$ (see [17, 19]) of a spin^c structure $\mathfrak{s}$ on X is zero whenever the virtual dimension $d_X(\mathfrak{s})$ of

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the Seiberg–Witten moduli space for $\mathfrak{s}$ is non-zero. We note that $d_X(\mathfrak{s}) = \frac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X))$, where χ and σ respectively denote the Euler characteristic and the signature. Due to [28], the conjecture is equivalent to the following: if $SW_X(\mathfrak{s}) \neq 0$, then $c_1(\mathfrak{s})$ is the first Chern class of an almost complex structure on X.

In the case where $b_2^+ - b_1 \equiv 0 \pmod 2$, $SW_X(\mathfrak{s}) = 0$ by the definition, and hence the conjecture is trivial. In the case where $b_2^+ - b_1 \equiv 1 \pmod 2$, the conjecture has been proved for many smooth 4-manifolds under smooth restrictions such as the existence of a symplectic structure [23]. However, the conjecture remains open for general smooth structures on any topological 4-manifold.

In this paper, we discuss the mod 2 version of the conjecture. A closed, connected, oriented, smooth 4-manifold X with $b_2^+ > 1$ will be called of *mod 2 Seiberg–Witten simple type* if $SW_X(\mathfrak{s}) \equiv 0 \pmod 2$ whenever $d_X(\mathfrak{s}) \neq 0$. Here we prove the mod 2 simple type conjecture under a simple condition on the cohomology ring.

Theorem 1.2. *Let X be a closed, connected, oriented, smooth 4-manifold with $b_2^+ - b_1 > 1$ and $b_2^+ - b_1 \equiv 3 \pmod 4$, and let $\{\delta_1, \dots, \delta_k\}$ be a generating set of $H^1(X;\mathbb{Z})$. If each cup product $\delta_i \cup \delta_j$ is either torsion or divisible by 2, then X has mod 2 Seiberg–Witten simple type.*

This result gives the first examples of topological 4-manifolds such that all smooth structures are of mod 2 Seiberg–Witten simple type, and yet some have non-vanishing (mod 2) Seiberg–Witten invariants. As is easily seen, this theorem provides a large class of such topological 4-manifolds.

Corollary 1.3. *Let X be a closed, connected, oriented, smooth 4-manifold with $b_2^+ - b_1 > 1$ and $b_2^+ - b_1 \equiv 3 \pmod 4$. Suppose that the cohomology ring is isomorphic to that of a connected sum of (possibly more than two) closed oriented 4-manifolds, each with either $b_1 \leq 1$ or $b_2 = 0$. Then X has mod 2 Seiberg–Witten simple type.*

Corollary 1.4. *Every closed, connected, oriented, smooth 4-manifold with $b_2^+ > 1$, $b_2^+ - b_1 \equiv 3 \pmod 4$ and $b_1 \leq 1$ has mod 2 Seiberg–Witten simple type.*

We note that there are many 4-manifolds with non-vanishing mod 2 Seiberg–Witten invariants satisfying the assumption of Corollary 1.3; see, for example, [1, 2, 8, 22, 25, 26, 30]. Also, the normal connected sum formula [18, Corollary 3.3] provides many 4-manifolds with non-vanishing mod 2 Seiberg–Witten invariants for which the mod 2 simple type conjecture is difficult to prove (without our results).

Remark 1.5. The $b_1 = 0$ case of Corollary 1.4 can be alternatively derived from results of Bauer and Furuta [4, Corollary 3.6 and Theorem 3.7]. We note that the proofs of these results are homotopy-theoretic and hence very different from ours.

We give simple applications of these results. We first discuss adjunction inequalities. A second cohomology class K of a 4-manifold X will be called a *mod 2 Seiberg–Witten basic class* if there exists a spin^c structure $\mathfrak{s}$ on X satisfying $K = c_1(\mathfrak{s})$ and $SW_X(\mathfrak{s}) \not\equiv 0 \pmod 2$. Here we assume that every immersed sphere intersects itself only at transverse double points. Due to the generalized adjunction formula of Fintushel and Stern [7], Theorem 1.2 implies the following adjunction inequality for immersed spheres.

Theorem 1.6. *Let X be a closed, connected, oriented, smooth 4-manifold satisfying the assumption of Theorem 1.2. Suppose that a second homology class α is represented by an immersed sphere having exactly p_+ positive double points and p_- negative double points. If $p_+ > 0$ and $\alpha \cdot \alpha < 0$, then any mod 2 Seiberg–Witten basic class K satisfies*

$$|\langle K, \alpha \rangle| + \alpha \cdot \alpha \leq 2p_+ - 2.$$

We note that this theorem holds for any 4-manifold satisfying the assumption of Corollary 1.3. For embedded surfaces, Corollary 1.4 implies the following adjunction inequality due to the generalized adjunction formula of Ozsváth and Szabó [20].

Theorem 1.7. *Let X be a closed, connected, oriented, smooth 4-manifold with $b_2^+ > 1$, $b_2^+ - b_1 \equiv 3 \pmod 4$ and $b_1 \leq 1$. Suppose that a second homology class α is represented by a smoothly embedded, closed, oriented surface of genus g. If $g > 0$ and $\alpha \cdot \alpha < 0$, then any mod 2 Seiberg–Witten basic class K satisfies*

$$|\langle K, \alpha \rangle| + \alpha \cdot \alpha \leq 2g - 2.$$

We next discuss the following conjecture, which states that the choice of an orientation of a 4-manifold imposes a strong constraint on the Seiberg–Witten invariant.

Conjecture 1.8 (cf. Kotschick [13]; see [5]). *Every simply connected, closed, oriented, smooth 4-manifold with $b_2^+ > 1$ and $b_2^- > 1$ has the vanishing Seiberg–Witten invariant for at least one orientation.*

Kotschick [14] proved this conjecture for a large class of complex surfaces (see also [5]). We remark that this conjecture has counterexamples if we remove the simply connected condition (e.g. the 4-torus). To state our result, let us recall that a compact, connected, smooth manifold is called *geometrically simply connected* if it admits a handle decomposition without 1-handles. We note that a geometrically simply connected manifold is simply connected. Also, we say that a 4-manifold X has the *vanishing mod 2 Seiberg–Witten invariant* if $SW_X(\mathfrak{s}) \equiv 0 \pmod 2$ for any spin^c structure $\mathfrak{s}$ on X. In [29], the third author showed that every geometrically simply connected, closed 4-manifold with $b_2^+ \not\equiv 1$ and $b_2^- \not\equiv 1 \pmod 4$ admits no symplectic structure for at least one orientation. Improving this result, Corollary 1.4 implies the mod 2 version of Conjecture 1.8 under a mild condition.

Theorem 1.9. *Every geometrically simply connected, closed, oriented, smooth 4-manifold with $b_2^+ \not\equiv 1$ and $b_2^- \not\equiv 1 \pmod 4$ has the vanishing mod 2 Seiberg–Witten invariant for at least one orientation.*

We note that many simply connected, closed 4-manifolds including a large class of complex surfaces are geometrically simply connected (see [8, 29]). If this theorem does not hold without the condition “geometrically”, then it guarantees the existence of a counterexample to a long-standing open problem whether every simply connected, closed, smooth 4-manifold is geometrically simply connected [12, Problem 4.18]. For background on this problem, we refer to [29]. In fact, we prove this theorem under a more general condition, which holds for many 4-manifolds including non-simply connected ones. Furthermore, this condition is much easier to verify. See Theorem 2.7.

2. Proofs

2.1. *Mod 2 Seiberg–Witten simple type*

For a closed, connected, oriented 4-manifold X, let $[X]$ denote the fundamental class of X. We first prove the following theorem.

Theorem 2.1. *Let X be a closed, connected, oriented, smooth 4-manifold with $b_2^+ - b_1 > 1$ and $b_2^+ - b_1 \equiv 3 \pmod 4$, and let $\{\delta_1, \dots, \delta_k\}$ be a generating set of $H^1(X;\mathbb{Z})$. Suppose that a spin^c structure $\mathfrak{s}$ on X satisfies the following conditions:*

- $SW_X(\mathfrak{s}) \equiv 1 \pmod 2$.
- $\langle c_1(\mathfrak{s}) \cup \delta_i \cup \delta_j, [X]\rangle \equiv 0 \pmod 4$ *for any i, j.*

Then $d_X(\mathfrak{s}) = 0$.

Proof. Suppose, to the contrary, that $d_X(\mathfrak{s}) \neq 0$. Put $K = c_1(\mathfrak{s})$. Due to the assumption $SW_X(\mathfrak{s}) \neq 0$, it follows from the definition of $SW_X(\mathfrak{s})$ that $d_X(\mathfrak{s}) = 2n$ for some positive integer n (e.g. [19, Section 2.3]). Let X_n be the 4-manifold $X \# n\overline{\mathbb{CP}^2}$, and let K_n be its second cohomology class defined by $K_n = K + 3E_1 + \cdots + 3E_n$, where each E_i denotes the Poincaré dual of the second homology class e_i of the i-th $\overline{\mathbb{CP}^2}$ represented by the exceptional sphere. By the blow-up formula [7, 19], we have a spin^c structure $\mathfrak{s}_n$ on X_n satisfying $c_1(\mathfrak{s}_n) = K_n$ and $SW_{X_n}(\mathfrak{s}_n) = SW_X(\mathfrak{s}) \equiv 1 \pmod 2$. We note that $d_{X_n}(\mathfrak{s}_n) = 0$. Hence, we see that $(X_n, \mathfrak{s}_n)$ is BF admissible in the sense of Ishida and Sasahira [11, Definition 2], due to the assumption on $(X, \mathfrak{s})$. (We remark that the "mod 2" non-vanishing condition is a part of the definition of BF admissibility.) One can also check that $(K3, \mathfrak{t})$ is BF admissible, where $(K3, \mathfrak{t})$ denotes the $K3$ surface equipped with a spin^c structure $\mathfrak{t}$ with $c_1(\mathfrak{t}) = 0$. By a result of Ishida and Sasahira [11, Theorem A] (see also [11, Theorem 23 and Proposition 14]) on the Bauer–Furuta invariant [4], we see that K_n is a Bauer–Furuta basic class of $Z := X_n \# K3$, and thus a monopole class of Z due to [10, Proposition 6].

Now let α be a second homology class of the $K3$ surface represented by a smoothly embedded, closed, oriented surface of genus $g > 1$ satisfying $\alpha \cdot \alpha = 2g - 2$. As is easily seen, there are many examples of such α (e.g. [9, Theorem 1.1]). We note that the class $\alpha - e_1$ of the 4-manifold Z is represented by a closed surface of genus g with non-negative self-intersection number. Applying the adjunction inequality of Kronheimer [15, p. 53] to Z, we obtain the inequality

$$\langle K_n, \alpha - e_1\rangle + (\alpha - e_1) \cdot (\alpha - e_1) \leq 2g - 2.$$

Since the left side is $3 + (2g - 3) = 2g$, the above inequality gives a contradiction. ∎

Remark 2.2. (1) The role of the $K3$ surface in this proof can be replaced by any closed, connected, oriented, smooth 4-manifold Y with $b_2^+ \equiv 3 \pmod 4$ and $b_1 = 0$ satisfying the following conditions: (i) Y has a mod 2 Seiberg–Witten basic class; (ii) Y has a smoothly embedded, closed surface of genus $g > 1$ satisfying $\alpha \cdot \alpha = 2g - 2$. This can be easily checked by using the adjunction inequality [21]. We remark that there are many examples of such Y.

(2) We used a connected sum formula for the Bauer–Furuta invariant to obtain a restriction on the smooth structure of a connected summand. A similar idea was used by the third author [29] to impose constraints on geometrically simply connected 4-manifolds and, more generally, on 4-manifolds admitting a non-torsion second homology class represented by a 2-handle neighborhood. We remark that the $b_1 = 0$ condition of [29, Theorem 2.4] can be relaxed to conditions similar to Theorem 1.2 (and hence Corollaries 1.3 and 1.4) of this paper without changing the proof, except that the connected sum formula of [11] is used instead of the formula of [3].

Proof of Theorem 1.2. We note that $(\delta_i \cup \delta_j) \cup (\delta_i \cup \delta_j) = 0$ for any i, j (see also the proof of Lemma 2.3). It is thus easy to see that every spin^c structure $\mathfrak{s}$ on X satisfies the condition $\langle c_1(\mathfrak{s}) \cup \delta_i \cup \delta_j, [X]\rangle \equiv 0 \pmod 4$ for any i, j, since $c_1(\mathfrak{s})$ is characteristic, and any $\delta_i \cup \delta_j$ is either torsion or divisible by 2. Hence Theorem 1.2 follows from Theorem 2.1. ∎

The lemma below will be used to prove Corollaries 1.3 and 1.4.

Lemma 2.3. *Let X be a closed, connected, oriented, smooth 4-manifold with $b_1 \le 1$. Then for any classes γ, δ of $H^1(X;\mathbb{Z})$, the cup product $\gamma \cup \delta$ is zero.*

Proof. By the universal coefficient theorem, we see that $H^1(X;\mathbb{Z})$ has no torsion. Due to the assumption $b_1(X) \le 1$, it suffices to prove $\gamma \cup \gamma = 0$ for any class γ of $H^1(X;\mathbb{Z})$. We note that the Poincaré dual $PD(\gamma)$ is represented by a closed oriented codimension 1 submanifold of X having a trivial normal bundle. This implies that $PD(\gamma \cup \gamma)$ is represented by the empty set, showing $\gamma \cup \gamma = 0$. ∎

Proof of Corollaries 1.3 and 1.4. We note that any 4-manifold with $b_2^+ > 1$, $b_2^+ - b_1 \equiv 3 \pmod 4$ and $b_1 \le 1$ satisfies $b_2^+ - b_1 > 1$. Hence, by the above lemma, Corollary 1.4 follows from Theorem 1.2. Therefore, we easily see that Corollary 1.3 also follows from Theorem 1.2. ∎

2.2. *Adjunction inequalities*

Proof of Theorem 1.6. Suppose, to the contrary, that $|\langle K, \alpha\rangle| + \alpha \cdot \alpha > 2p_+ - 2$ for some $K = c_1(\mathfrak{s})$ and α. Then the generalized adjunction formula of Fintushel and Stern [7, Theorem 1.3] shows that $K' = K + 2\epsilon PD(\alpha)$ is a mod 2 Seiberg–Witten basic class, where $\epsilon = \pm 1$ is the sign of $\langle K, \alpha\rangle$. It is straightforward to see that $K' = c_1(\mathfrak{s}')$ satisfies

$$d_X(\mathfrak{s}') = d_X(\mathfrak{s}) + |\langle K, \alpha\rangle| + \alpha \cdot \alpha > 0.$$

Since X is of mod 2 Seiberg–Witten simple type due to Theorem 1.2, this is a contradiction. ∎

Ozsváth and Szabó [20] introduced the Seiberg–Witten invariant of the form

$$SW_{X,\mathfrak{s}} \colon \mathbb{A}(X) \to \mathbb{Z}$$

for a spin^c structure $\mathfrak{s}$, where $\mathbb{A}(X) = \bigwedge H_1(X;\mathbb{Z}) \otimes \mathbb{Z}[U]$, $H_1(X;\mathbb{Z})$ has grading 1 and U is a degree 2 generator (cf. [24]). This function and the integer valued invariant have the relation

$$SW_{X,\mathfrak{s}}(U^{d_X(\mathfrak{s})/2}) = SW_X(\mathfrak{s})$$

when $d_X(\mathfrak{s})$ is non-negative and even. Let Σ be a smoothly embedded, closed, oriented surface of genus g representing α. For such a surface Σ, they defined the class $\xi(\Sigma) \in \mathbb{A}(X)$ by

$$\xi(\Sigma) = \prod_{i=1}^{g} (U - A_i \cdot B_i),$$

where $\{A_i, B_i\}_{i=1}^{g}$ are the images in $H_1(X;\mathbb{Z})$ of a standard symplectic basis for $H_1(\Sigma;\mathbb{Z})$.

Proof of Theorem 1.7. Suppose $|\langle K, \alpha\rangle| + \alpha \cdot \alpha > 2g - 2$. Then, by [20, Theorem 1.3], the relation

$$SW_{X,\mathfrak{s}+\epsilon\alpha}(\xi(\epsilon\Sigma)U^m) = SW_{X,\mathfrak{s}}(1)$$

holds, where $\epsilon = \pm 1$ is the sign of $\langle K, \alpha\rangle$ and $2m = |\langle K, \alpha\rangle| + \alpha \cdot \alpha - 2g$. Since $\xi(\Sigma) = U^g$ when $b_1(X) \leq 1$, we obtain

$$SW_X(\mathfrak{s} + \epsilon\alpha) = SW_{X,\mathfrak{s}+\epsilon\alpha}(U^{m+g}) = SW_{X,\mathfrak{s}}(1) = SW_X(\mathfrak{s}) \underset{(2)}{\equiv} 1.$$

On the other hand, $d_X(\mathfrak{s} + \epsilon\alpha) = d_X(\mathfrak{s}) + |\langle K, \alpha\rangle| + \alpha \cdot \alpha > 0$. This contradicts Corollary 1.4. ∎

Remark 2.4. Conjecture 1.1 was originally posed for 4-manifolds with $b_1 = 0$ in relation to Witten's conjecture [6, 27] on the relationship between the Donaldson and Seiberg–Witten invariants, and was later extended to the case of arbitrary b_1 in the literature. Ozsváth and Szabó [20] gave a stronger version of the simple type condition. They call X of simple type when the function $SW_{X,\mathfrak{s}}$ is identically zero if $d_X(\mathfrak{s}) \neq 0$. Taubes [24, Proof of Proposition 2.2] proved that every closed symplectic 4-manifold X with $b_2^+(X) > 1$ is of simple type in this strong sense (see also [20, Remark 3.3]). However, in contrast to the case of (ordinary) simple type, there are many 4-manifolds which are not of strong simple type. For instance, it follows from the surgery formula of Ozsváth and Szabó [21, Proposition 2.2] that a connected sum $X \# (S^1 \times S^3)$ is such an example when X has a non-vanishing integer Seiberg–Witten invariant.

2.3. A vanishing theorem for mod 2 *Seiberg–Witten invariants*

We recall a definition and a lemma given in [29].

Definition 2.5. Let α be a second homology class of a smooth 4-manifold X. We say that α is represented by a 2-*handle neighborhood* if X has a codimension 0 submanifold W satisfying the following conditions:

- The submanifold W is diffeomorphic to a 4-manifold obtained from the 4-ball by attaching a single 2-handle. (This submanifold will be called a 2-*handle neighborhood*.)
- α is the image of a generator of $H_2(W;\mathbb{Z}) \cong \mathbb{Z}$ by the inclusion induced homomorphism $H_2(W;\mathbb{Z}) \to H_2(X;\mathbb{Z})$.

Lemma 2.6 ([29, Lemma 3.1]). *Every second homology class of a geometrically simply connected, compact, smooth* 4*-manifold is represented by a* 2*-handle neighborhood.*

For an oriented 4-manifold X, let $\overline{X}$ denote the 4-manifold X equipped with the reverse orientation. We show the following vanishing theorem for mod 2 Seiberg–Witten invariants.

Theorem 2.7. *Let X be a closed, connected, oriented, smooth* 4*-manifold satisfying* $b_2^+ - b_1 > 1$, $b_2^- - b_1 > 1$, $b_2^+ - b_1 \not\equiv 1$ *and* $b_2^- - b_1 \not\equiv 1 \pmod 4$, *and let* $\{\delta_1, \dots, \delta_k\}$ *be a generating set of* $H^1(X;\mathbb{Z})$. *Suppose that each cup product* $\delta_i \cup \delta_j$ *is either torsion or divisible by* 2*. If X admits a non-torsion second homology class represented by a* 2*-handle neighborhood, then X or $\overline{X}$ has the vanishing mod* 2 *Seiberg–Witten invariant.*

We note that the existence of a non-torsion second homology class represented by a 2-handle neighborhood is much easier to verify than the geometrically simply connected condition, since it is often not necessary to decompose an entire 4-manifold into a handlebody. Indeed, many closed 4-manifolds including non-simply-connected ones admit such second homology classes (see [29, Section 3] for more background on such 4-manifolds).

The proof of this theorem relies on the following result.

Theorem 2.8 ([29]). *Let X be a* 4*-manifold satisfying the assumption of Theorem* 2.7*. Then at least one of the following properties holds:*

- *Every spin^c structure $\mathfrak{s}$ with $d_X(\mathfrak{s}) = 0$ satisfies $SW_X(\mathfrak{s}) \equiv 0 \pmod 2$.*
- *Every spin^c structure $\mathfrak{s}$ with $d_{\overline{X}}(\mathfrak{s}) = 0$ satisfies $SW_{\overline{X}}(\mathfrak{s}) \equiv 0 \pmod 2$.*

This theorem is implicit in the proof of [29, Theorem 2.4], which states that any 4-manifold with $b_1 = 0$ satisfying the assumption of Theorem 2.7 admits no symplectic structure for at least one orientation. The proof of the $b_1 = 0$ case of Theorem 2.8 is identical with the proof of [29, Theorem 2.4], and the proof of the general case is also identical, except that the connected sum formula of [11] is used instead of the formula of [3].

Proof of Theorem 2.7. This is straightforward from Theorems 2.8 and 1.2. ■

Proof of Theorem 1.9. This is straightforward from Theorem 2.7 and Lemma 2.6. ■

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