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Global Schrödinger map flows to Kähler manifolds with small data in critical Sobolev spaces: Energy critical case

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Abstract. In this paper and the companion work [J. Funct. Anal. 281 (2021)], we prove that the Schrödinger map flows from $\mathbb{R}^d$ with $d \geq 2$ to compact Kähler manifolds with small initial data in critical Sobolev spaces are global. The main difficulty compared with the constant sectional curvature case is that the gauged equation now is not self-contained due to the curvature part. Our main idea is to use a novel bootstrap-iteration scheme to reduce the gauged equation to an approximate constant curvature system in finite times of iteration. This paper together with the companion work solves the open problem raised by Tataru.

Keywords. Landau–Lifshitz equation, global regularity, Schrödinger map

Contents

1. Introduction

Let $(\mathcal{N}, J, h)$ be a Kähler manifold. The *Schrödinger map flow* (SMF) on Euclidean space is a map $u : \mathbb{R} \times \mathbb{R}^d \to \mathcal{N}$ which satisfies

$$\begin{cases} u_t = J(\sum_{i=1}^d \nabla_i \partial_i u), \\ u\restriction_{t=0} = u_0, \end{cases} \tag{1.1}$$

where ∇ denotes the induced covariant derivative in the pullback bundle $u^* T\mathcal{N}$.

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Assume that $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^N$. Then (1.1) can be formulated as

$$\begin{cases} u_t = JP_u^{\mathcal{N}}(\Delta_{\mathbb{R}^d} u), \\ u\restriction_{t=0} = u_0, \end{cases} \tag{1.2}$$

where $P_u^{\mathcal{N}}$ denotes the orthogonal projection from $\mathbb{R}^N$ onto $T_u\mathcal{N}$.

(1.1) plays a fundamental role in solid-state physics and is usually to referred to as the Landau–Lifshitz flow in physics literature. The various forms of SMF are commonly used in micromagnetics to model the effects of a magnetic filed on ferromagnetic materials (e.g. [20]). In the $d = 1$ and $d = 2$ case with $\mathcal{N} = \mathbb{S}^2$, SMF is referred to as the ferromagnetic chain equation and the continuous isotropic Heisenberg spin model respectively (e.g. [42]).

The Schrödinger map flow can be viewed as a Hamiltonian flow on an infinite-dimensional symplectic manifold; see Ding [8]. One of the conservation laws of SMF is conservation of energy defined by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\partial_x u|^2\, dx.$$

And SMF has the scaling invariance property: $u(t,x) \mapsto u(\lambda^2 t, \lambda x)$. Thus $d = 2$ is the energy critical case. In the case $\mathcal{N} = \mathbb{S}^2$, SMF has mass as another conserved quantity:

$$M(u) = \frac{1}{2}\int_{\mathbb{R}^d} |u - P|^2\, dx \quad \text{if } \|u_0 - P\|_{L^2_x} < \infty \text{ for some } P \in \mathbb{S}^2.$$

However, the mass is not conserved for a general target $\mathcal{N}$.

We recall the following non-exhaustive list of works on Cauchy problems. The local well-posedness theory of Schrödinger map flows was developed by Sulem–Sulem–Bardos [32], Ding–Wang [9] and McGahagan [24]. The global well-posedness theory was started by Chang–Shatah–Uhlenbeck [7] and Nahmod–Stefanov–Uhlenbeck [28]. And the $d = 1$ case with general targets was studied by Rodnianski–Rubinstein–Staffilani [30]. Global existence for small data in critical Besov spaces was proved by Ionescu–Kenig [14] and Bejenaru [1] independently. The small data global well-posedness theory in critical Sobolev spaces was developed by Bejenaru–Ionescu–Kenig [2] for high dimensions $d \geq 4$. The dimension 2 case, which is energy critical, was studied by Bejenaru–Ionescu–Kenig–Tataru [5] where the global well-posedness theory for small data in critical Sobolev spaces was established for $\mathcal{N} = \mathbb{S}^2$ with $d \geq 2$. And Dodson–Smith [10] studied the conditional global regularity problem for $d = 2$.

The stationary solutions of SMF are harmonic maps. So far, the dynamical behavior of SMF near harmonic maps is partly known in the equivariant case with $d = 2$, $\mathcal{N} = \mathbb{S}^2$. The works of Gustafson–Kang–Tsai–Nakanishi [11, 12] studied asymptotic stability vs. wind oscillating near harmonic maps in high equivariant classes. Bejenaru–Tataru [6] studied global stability vs. instability of harmonic maps for 1-equivariant 2D SMF. The type II blowup solutions were constructed by Merle–Raphaël–Rodnianski [25] and Perelman [29] for 1-equivariant 2D SMF. And the below-threshold conjecture was verified for equivariant SMF from $\mathbb{R}^2$ into $\mathbb{S}^2$ or $\mathbb{H}^2$ by Bejenaru–Ionescu–Kenig–Tataru [3, 4].

All the above mentioned global well-posedness results for SMF with $d \geq 2$ are for targets $\mathbb{S}^2$ or $\mathbb{H}^2$. Tataru raised the question of small data global well-posedness in critical Sobolev spaces for general compact Kähler targets in the survey report [18]. This work, which deals with the energy critical case $d = 2$, together with the companion work [21], solves this problem.

1.1. Main results

Before stating our main results, we introduce some notations. For geometric PDEs, it is convenient to work in both intrinsic and extrinsic Sobolev spaces. For smooth maps from $\mathbb{R}^d \to \mathcal{N}$ the intrinsic norms are defined by

$$\|u\|^p_{\mathrm{W}^{k,p}} := \sum_{j=1}^{k} \|\nabla^j u\|^p_{L^p_x(\mathbb{R}^d)},$$

where ∇ denotes the induced covariant derivative in $u^* T\mathcal{N}$.

Given a point $Q \in \mathcal{N}$, we define the extrinsic Sobolev space H^k_Q by

$$H^k_Q := \{u : \mathbb{R}^d \to \mathbb{R}^N \mid u(x) \in \mathcal{N} \text{ a.e. in } \mathbb{R}^d, \|u - Q\|_{H^k(\mathbb{R}^d)} < \infty\},$$

equipped with the metric $d_Q(f, g) = \|f - g\|_{H^k}$. Define

$$\mathcal{H}_Q := \bigcap_{k=1}^{\infty} H^k_Q.$$

Our main results are the following.

Theorem 1.1. *Let $d = 2$, let $\mathcal{N}$ be a $2n$-dimensional compact Kähler manifold which is isometrically embedded into $\mathbb{R}^N$, and let $Q \in \mathcal{N}$ be a given point. There exists a sufficiently small constant $\epsilon_* > 0$ such that if $u_0 \in \mathcal{H}_Q$ satisfies*

$$\|\partial_x u_0\|_{L^2_x} \leq \epsilon_*, \tag{1.3}$$

then (1.1) with initial data u_0 evolves into a global unique solution $u \in C(\mathbb{R}; \mathcal{H}_Q)$. Moreover, as $|t| \to \infty$ the solution u converges to the constant map Q in the sense that

$$\lim_{|t|\to\infty} \|u(t) - Q\|_{L^\infty_x} = 0. \tag{1.4}$$

Furthermore, in the energy space, we also have

$$\lim_{t\to\infty} \Big\| u(t) - \sum_{j=1}^{n} \Re(e^{it\Delta} h^j_+) - \sum_{j=1}^{n} \Im(e^{it\Delta} g^j_+) \Big\|_{\dot{H}^1_x} = 0 \tag{1.5}$$

for some functions $h^j_+, g^j_+ : \mathbb{R}^2 \to \mathbb{C}^N$ belonging to $\dot{H}^1$ with $j = 1, \ldots, n$.

Remark 1.1. The asymptotic behaviors (1.4) and (1.5) are new for SMF. The analogous result of (1.4) for wave maps was obtained in part VII of Tao [38]. A similar result to (1.5) was recently obtained by the author [22] in the setting of SMF on hyperbolic planes. One can see that (1.5) is natural by checking the trivial target $\mathcal{N} = \mathbb{R}^{2n}$; see [22, Remark 1.1] for instance.

We also prove uniform bounds and well-posedness results analogous to those of [5].

Theorem 1.2. *Let $d = 2$ and $\sigma_1 \geq 0$. Let $\mathcal{N}$ be a compact Kähler manifold which is isometrically embedded into $\mathbb{R}^N$, and let $Q \in \mathcal{N}$ be a given point. There exists a sufficiently small constant $\epsilon_{\sigma_1} > 0$ depending only on σ_1 such that the global solution $u = S_Q(t)u_0 \in C(\mathbb{R}; \mathcal{H}_Q)$ constructed in Theorem 1.1 satisfies the uniform bounds*

$$\sup_{t\in\mathbb{R}} \|u(t) - Q\|_{H_x^{\sigma+1}} \leq C_\sigma(\|u_0 - Q\|_{H_x^{\sigma+1}}), \quad \forall\sigma \in [0, \sigma_1]. \tag{1.6}$$

In addition, for any $\sigma \in [0, \sigma_1]$, the operator S_Q admits a continuous extension

$$S_Q : \mathfrak{B}^\sigma_{\epsilon_{\sigma_1}} \to C(\mathbb{R}; H^{\sigma+1}),$$

where we denote

$$\mathfrak{B}^\sigma_\epsilon := \{f \in H_Q^{\sigma+1} : \|f - Q\|_{\dot{H}^1} \leq \epsilon\}.$$

Remark 1.2. Theorem 1.1 holds for $d \geq 3$ as well. This is proved in the companion work [21]. The main proof in higher dimensions uses ideas of this work and some additional ingredients on heat flows. We will explain this issue at the end of the introduction.

1.2. Caloric gauge and heat flows

For dispersive geometric PDEs, especially for critical problems, it is important to choose suitable gauges and function spaces adapted to the structure of nonlinearities (e.g. null structure). Most of these tools were developed in the study of wave map equations; see for instance [16,17,19,27,33,34,39,40]. In this work, we will use Tao's caloric gauge and function spaces developed in [5, 13]. As observed in [5, 35], the caloric gauge is essential for eliminating bad frequency interactions in dimension 2 compared with Coulomb gauge. For convenience, we briefly recall the definition of caloric gauge.

First, let us recall the moving frame dependent quantities and some related identities; see [26] and [30] for more extensive expositions. Let Greek indices run in $\{1, \ldots, n\}$. Let Roman indices run in $\{1, \ldots, 2n\}$ or $\{1, \ldots, d\}$ according to the context. Denote $\overline{\beta} = \beta + n$ for $\beta \in \{1, \ldots, n\}$.

Let $\mathcal{N}$ be a $2n$-dimensional compact Kähler manifold. Since $\mathbb{R}^2 \times [-T, T]$ is contractible, there must exist global orthonormal frames for $u^*(T\mathcal{N})$. Using the complex structure one can assume the orthonormal frames are of the form

$$\mathbf{E} := \{e_1(t, x), Je_1(t, x), \ldots, e_n(t, x), Je_n(t, x)\}. \tag{1.7}$$

Let $\psi_i = (\psi_i^1, \psi_i^{\bar 1}, \dots, \psi_i^n, \psi_i^{\bar n})$ for $i = 0, 1, 2$ be the components of $\partial_{t,x} u$ in the frame $\mathbf{E}$:

$$\psi_i^\alpha := \langle \partial_i u, e_\alpha \rangle, \quad \psi_i^{\bar\alpha} := \langle \partial_i u, J e_\alpha \rangle. \tag{1.8}$$

We always use 0 to represent t in subscripts. The isomorphism of $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ induces a $\mathbb{C}^n$-valued function defined by $\phi_i^\beta := \psi_i^\beta + \sqrt{-1}\,\psi_i^{\bar\beta}$ with $\beta = 1, \dots, n$. Conversely, given a function $\phi : [-T, T] \times \mathbb{R}^2 \to \mathbb{C}^n$, we associate with it a section $\phi\mathbf{E}$ of the bundle $u^*(T\mathcal{N})$ via

$$\phi \leftrightarrow \phi\mathbf{E} := \Re(\phi^\beta) e_\beta + \Im(\phi^\beta) J e_\beta, \tag{1.9}$$

where $(\phi^1, \dots, \phi^n)$ are the components of ϕ. Then u induces a covariant derivative on the trivial complex vector bundle over the base manifold $[-T, T] \times \mathbb{R}^d$ with fiber $\mathbb{C}^n$, defined by

$$D_i \varphi^\beta = \partial_i \varphi^\beta + \sum_{\alpha=1}^n \big([A_i]_\alpha^\beta + \sqrt{-1}\,[A_i]_\alpha^{\bar\beta}\big)\varphi^\alpha,$$

where the induced connection coefficient matrices are defined by

$$[A_i]_q^p := \langle \nabla_i e_p, e_q \rangle.$$

Schematically we write $D_i = \partial_i + A_i$. Recall the torsion free identity and the commutator identity

$$D_i \phi_j = D_j \phi_i, \tag{1.10}$$

$$[D_i, D_j]\varphi = (\partial_i A_j - \partial_j A_i + [A_i, A_j])\varphi \;\leftrightarrow\; \mathbf{R}(\partial_i u, \partial_j u)(\varphi\mathbf{E}), \tag{1.11}$$

where $\mathbf{R}$ is the curvature rensor. Schematically, we write $[D_i, D_j] = \mathcal{R}(\phi_i, \phi_j)$. With the notations above, (1.1) can be written as

$$\phi_t = \sqrt{-1} \sum_{i=1}^2 D_i \phi_i. \tag{1.12}$$

In [31] it is proved that the heat flow with initial data $u(t, x)$ below threshold energy converges to Q as $s \to \infty$ in the topology of $C([-T, T]; C_x^\infty)$. Tao's caloric gauge is defined as follows:

Definition 1.1. Let $u : [-T, T] \times \mathbb{R}^2 \to \mathcal{N}$ be a solution of (1.1) in $C([-T, T]; \mathcal{H}_Q)$. For a given orthonormal frame $E^\infty := \{e_1^\infty, Je_1^\infty, \dots, e_n^\infty, Je_n^\infty\}$ for $T_Q\mathcal{N}$, a *caloric gauge* is a tuple consisting of a map $v : \mathbb{R}^+ \times [-T, T] \times \mathbb{R}^2 \to \mathcal{N}$ and orthonormal frames $\mathbf{E}(v(s, t, x)) := \{e_1, Je_1, \dots, e_n, Je_n\}$ such that

$$\begin{cases} \partial_s v = \sum_{i=1}^2 \nabla_i \partial_i v, \\ v(0, t, x) = u(t, x), \end{cases} \tag{1.13}$$

and

$$\begin{cases} \nabla_s e_k = 0, \quad k = 1, \dots, n, \\ \lim_{s\to\infty} e_k = e_k^\infty. \end{cases} \tag{1.14}$$

Denote

$$\mathcal{H}_Q(T) := C([-T,T];\mathcal{H}_Q).$$

Proposition 1.1. *Let $u \in \mathcal{H}_Q(T)$ solve SMF with $u_0 \in \mathcal{H}_Q$. For any fixed frame $E^\infty := \{e_k^\infty, Je_k^\infty\}_{k=1}^n$ for $T_Q\mathcal{N}$, there exists a unique corresponding caloric gauge as defined in Definition 1.1. Moreover, for $i = 1,2$ and $p,q = 1,\dots,2n$,*

$$\lim_{s\to\infty}[A_i]_p^q(s,t,x) = 0, \quad \lim_{s\to\infty}[A_t]_p^q(s,t,x) = 0.$$

In particular, for $i = 1,2$ and $s > 0$,

$$[A_i]_q^p(s,t,x) = -\int_s^\infty \langle \mathbf{R}(\partial_s v(\tilde{s})), \partial_i v(\tilde{s}) e_p, e_q\rangle \, d\tilde{s},$$
$$[A_t]_q^p(s,t,x) = -\int_s^\infty \langle \mathbf{R}(\partial_s v(\tilde{s}), \partial_t v(\tilde{s})) e_p, e_q\rangle \, d\tilde{s}.$$

Proof. The proof is standard (see e.g. [31]). The only new issue here is the complex structure J. But this will not cause any trouble since J commutes with ∇_s. ■

Given $u \in \mathcal{H}_Q(T)$ which solves (1.1), let $v : \mathbb{R}^+ \times [-T,T] \times \mathbb{R}^2 \to \mathcal{N}$ be the solution to (1.13). Let $\{e_\alpha, Je_\alpha\}_{\alpha=1}^n$ be the corresponding caloric gauge. Define the *heat tension field* ϕ_s to be

$$\phi_s^\alpha := \langle \partial_s v, e_\alpha\rangle + \sqrt{-1}\,\langle \partial_s v, Je_\alpha\rangle, \quad \alpha = 1,\dots,n,$$

and the *differential fields* to be

$$\phi_i^\alpha := \langle \partial_i v, e_\alpha\rangle + \sqrt{-1}\,\langle \partial_i v, Je_\alpha\rangle, \quad \alpha = 1,\dots,n,$$

where $i = 1,2$ refers to the variable x_i, $i = 1,2$, and $i = 0$ refers to the variable t.

Lemma 1.1. *The heat tension field ϕ_s satisfies*

$$\phi_s = \sum_{j=1}^2 D_j\phi_j. \tag{1.15}$$

The differential fields $\{\phi_i\}_{i=1}^2$ along the heat flow satisfy

$$\partial_s\phi_i = \sum_{j=1}^2 D_jD_j\phi_i + \sum_{j=1}^2 \mathcal{R}(\phi_i,\phi_j)\phi_j. \tag{1.16}$$

And when $s = 0$, along the Schrödinger flow direction, $\{\phi_i\}_{i=1}^2$ satisfy

$$-\sqrt{-1}\,D_t\phi_i = \sum_{j=1}^2 D_jD_j\phi_i + \sum_{j=1}^2 \mathcal{R}(\phi_i,\phi_j)\phi_j. \tag{1.17}$$

Notations. Let $\mathbb{Z}_+ = \{1,2,\dots\}$ and $\mathbb{N} = \{0,1,2,\dots\}$. We apply the notation $X \lesssim Y$ whenever there exists some constant $C > 0$ so that $X \le CY$. Similarly, we will use $X \sim Y$

if $X \lesssim Y \lesssim X$. We sometimes drop the integral variable in the integration if no confusion occurs. And we closely follow the notations of [5] for the reader's convenience.

Let $\mathcal{F}$ denote the Fourier transformation in $\mathbb{R}^2$.

Let $\chi : \mathbb{R} \to [0,1]$ be a given smooth even function which is supported in $\{z \in \mathbb{R} : |z| \le 8/5\}$ and equal to 1 for $\{z \in \mathbb{R} : |z| \le 5/4\}$. Define $\chi_k(z) := \chi(z/2^k) - \chi(z/2^{k-1})$, $k \in \mathbb{Z}$. The Littlewood–Paley projection operators with Fourier multiplier $\eta \mapsto \chi_k(|\eta|)$ are denoted by $P_k, k \in \mathbb{Z}$. For $I \subset R$, let $\chi_I := \sum_{i\in I} \chi_i(|\xi|)$. The low frequency cutoff operator with Fourier multiplier $\eta \mapsto \chi_{(-\infty,k]}(|\xi|)$ is denoted by $P_{\le k}$, and the high frequency cutoff is defined by $P_{\ge k} := I - P_{\le k}$.

Given $\mathbf{e} \in \mathbb{S}^1$, $k \in \mathbb{Z}$, denote by $P_{k,\mathbf{e}}$ the operator with Fourier multiplier $\xi \mapsto \chi_k(\xi \cdot \mathbf{e})$.

The Riemannian curvature tensor on $\mathcal{N}$ is denoted by $\mathbf{R}$. The covariant derivative on $\mathcal{N}$ is denoted by $\widetilde{\nabla}$, and we denote ∇ the induced covariant derivative on $u^*T\mathcal{N}$. The metric tensor of $\mathcal{N}$ is denoted by $\langle \cdot, \cdot \rangle$. Let E be a Riemannian manifold with connection ∇, and $\mathbb{T}$ be a $(0,r)$ type tensor. For $k, r \in \mathbb{Z}_+$, we define the $(0, r+k)$ type tensor $\nabla^k \mathbb{T}$ by

$$\nabla^k \mathbb{T}(X_1, \ldots, X_k; Y_1, \ldots, Y_r) := (\nabla_{X_k}(\nabla^{k-1}\mathbb{T}))(X_1, \ldots, X_{k-1}; Y_1, \ldots, Y_r)$$

for any tangent vector fields $X_1, \ldots, X_k, Y_1, \ldots, Y_r$ on E.

1.3. Function spaces built in [5]

We recall the spaces developed by Bejenaru–Ionescu–Kenig–Tataru [5]. Given a unit vector $\mathbf{e} \in \mathbb{S}^1$ we denote its orthogonal complement in $\mathbb{R}^2$ by $\mathbf{e}^\perp$. The *lateral space* $L^{p,q}_{\mathbf{e}}$ is defined by the norm

$$\|f\|_{L^{p,q}_{\mathbf{e}}} := \left(\int_{\mathbb{R}} \left(\int_{\mathbf{e}^\perp \times \mathbb{R}} |f(t, x_1\mathbf{e} + x')|^q \, dx' \, dt \right)^{p/q} dx_1 \right)^{1/p},$$

with standard modifications when either $p = \infty$ or $q = \infty$. And for any given $\lambda \in \mathbb{R}$, $W \subset \mathbb{R}$, we define the spaces $L^{p,q}_{\mathbf{e},\lambda}$ and $L^{p,q}_{\mathbf{e},W}$ with norms

$$\|f\|_{L^{p,q}_{\mathbf{e},\lambda}} := \|G_{\lambda\mathbf{e}}(f)\|_{L^{p,q}_{\mathbf{w}}}, \quad \|f\|_{L^{p,q}_{\mathbf{e},W}} := \inf_{f=\sum_{\lambda\in W} f_\lambda} \sum_{\lambda \in W} \|f_\lambda\|_{L^{p,q}_{\mathbf{e},\lambda}},$$

where $G_{\mathbf{a}}$ denotes the Galilean transform:

$$G_{\mathbf{a}}(f)(t,x) := e^{-\frac{1}{2} i x \cdot \mathbf{a}} e^{-\frac{i}{4} t |\mathbf{a}|^2} f(x + t\mathbf{a}, t).$$

We remark that the lateral space was studied by Linares–Ponce [23], Kenig–Ponce–Vega [15] and Ionescu–Kenig [14].

We now recall the main dyadic function spaces $N_k(T), F_k(T), G_k(T)$. Given $T \in \mathbb{R}^+$ and $k \in \mathbb{Z}$, let $I_k := \{\eta \in \mathbb{R}^2 : 2^{k-1} \le |\eta| \le 2^{k+1}\}$ and

$$L^2_k(T) := \{g \in L^2([-T,T] \times \mathbb{R}^2) : \mathcal{F}g(t,\eta) \text{ is supported in } \mathbb{R} \times I_k\}. \tag{1.18}$$

Given $\mathcal{L} \in \mathbb{Z}_+$, $T \in (0, 2^{2\mathcal{L}}]$, and $k \in \mathbb{Z}$, define

$$W_k := \{\lambda \in [-2^{2k}, 2^{2k}] : 2^{k+2\mathcal{L}}\lambda \in \mathbb{Z}\}. \tag{1.19}$$

The $N_k(T), F_k(T), G_k(T)$ spaces are Banach spaces of functions in $L^2_k(T)$ for which the associated norms are finite:

$$\|g\|_{F_k^0(T)} := \|g\|_{L_t^\infty L_x^2} + 2^{-k/2}\|g\|_{L_x^4 L_t^\infty} + \|g\|_{L^4} + 2^{-k/2} \sup_{\mathbf{e}\in\mathbb{S}^1} \|g\|_{L^{2,\infty}_{\mathbf{e},W_{k+40}}},$$

$$\|g\|_{F_k(T)} := \inf_{j\in\mathbb{Z}_+, n_1,\dots,n_j\in\mathbb{N}} \inf_{g=g_{n_1}+\cdots+g_{n_j}} \sum_{l=1}^{j} 2^{n_l}\|g_{n_l}\|_{F^0_{k+n_l}},$$

$$\|g\|_{G_k(T)} := \|g\|_{F_k^0} + 2^{-k/6}\sup_{\mathbf{e}\in\mathbb{S}^1}\|g\|_{L_{\mathbf{e}}^{3,6}} + 2^{k/6}\sup_{|k-j|\le 20}\sup_{\mathbf{e}\in\mathbb{S}^1}\|P_{j,\mathbf{e}}g\|_{L_{\mathbf{e}}^{6,3}}$$
$$+ 2^{k/2}\sup_{|k-j|\le 20}\sup_{\mathbf{e}\in\mathbb{S}^1}\sup_{|\lambda|<2^{k-40}}\|P_{j,\mathbf{e}}g\|_{L^{\infty,2}_{\mathbf{e},\lambda}},$$

$$\|g\|_{N_k(T)} := \inf_{g=g_1+g_2+g_3+g_4} \|g_1\|_{L^{4/3}} + 2^{k/6}\|g_2\|_{L^{3/2,6/5}_{\mathbf{e}_1}} + 2^{k/6}\|g_3\|_{L^{3/2,6/5}_{\mathbf{e}_2}}$$
$$+ 2^{-k/2}\sup_{\mathbf{e}\in\mathbb{S}^1}\|g_4\|_{L^{1,2}_{\mathbf{e},W_{k-40}}},$$

where $\{\mathbf{e}_1, \mathbf{e}_2\} \subset \mathbb{S}^1$ is the standard basis of $\mathbb{R}^2$. The G_k, F_k spaces were built by Bejenaru–Ionescu–Kenig–Tataru [5].

Recall also the refined space for $F_k(T)$: Let $S_k^\omega(T)$ denote the normed space of functions in $L^2_k(T)$ for which

$$\|g\|_{S_k^\omega(T)} := 2^{k\omega}(\|g\|_{L_t^\infty L_x^{2_\omega}} + \|g\|_{L_t^4 L_x^{p_\omega^*}} + 2^{-k/2}\|g\|_{L_x^{p_\omega^*} L_t^\infty})$$

is finite, where the exponents 2_ω and p^*_ω are defined via

$$\frac{1}{2_\omega} - \frac{1}{2} = \frac{1}{p^*_\omega} - \frac{1}{4} = \frac{\omega}{2}.$$

1.4. Overview of the proof

Main difficulty for general targets. The new difficulty arising in the case of general targets is to control the curvature terms in frequency localized spaces. Since the curvature term relates to the map itself, it cannot be written in a self-contained form in terms of differential fields and heat tension fields $\{\phi_x, \phi_s\}$. Thus directly working with the moving frame dependent quantities may lead to losing control of curvature terms, which is much more serious when frequency interactions are considered. In the wave map setting, the general targets case was solved by Tataru [41] using Tao's micro-local gauge and Tataru's function spaces. It is important that the wave map equation is semilinear in the extrinsic form, and the micro-local gauge adapts to the extrinsic formulation well. However, for SMF, on the one hand, since the extrinsic form equation is quasilinear, one has to use the intrinsic formulation to obtain a semilinear equation. On the other hand, the intrinsic form is not a self-contained system where the curvature term is not determined by differential fields. The two conflicting sides of the problem make solving SMF for general targets challenging.

Outline of proof for $d = 2$. Let us sketch the proof in the $d = 2$ case. The whole proof is divided into ten steps. Given $\delta > 0$, let $\{a_k\}_{k\in\mathbb{Z}}$ be a positive sequence; we call it a *frequency envelope of order δ* if

$$\sum_{k\in\mathbb{Z}} a_k^2 < \infty, \quad \text{and} \quad a_j \le 2^{\delta|l-j|} a_l,\ \forall j, l \in \mathbb{Z}.$$

We call the frequency envelope $\{a_k\}$ an *ϵ-envelope* if it additionally satisfies

$$\sum_{k\in\mathbb{Z}} a_k^2 \le \epsilon^2.$$

Step 1. Tracking $L_t^\infty L_x^2$ bounds along the heat flow direction. Recall the extrinsic formulations of heat flows: Assume that the target manifold $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^N$; then the heat flow equation can be formulated as

$$\partial_s v^l - \Delta v^l = \sum_{a=1}^{2} \sum_{i,j=1}^{N} S_{ij}^l \partial_a v^i \partial_a v^j, \quad l = 1, \dots, N, \tag{1.20}$$

where $S = \{S_{ij}^l\}$ denotes the second fundamental form of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^N$. For $u \in \mathcal{H}_Q(T)$, define

$$\gamma_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k' + k'} \|P_{k'} u\|_{L_t^\infty L_x^2}, \quad \sigma \ge 0,\ \delta = \tfrac{1}{800}.$$

Denote by $\{\gamma_k\}$ the frequency envelope for the energy norm, i.e. $\gamma_k = \gamma_k(0)$. The first result of Step 1 is stated for $\sigma \in [0, \frac{5}{4}]$:

Proposition 1.2. *Assume that $u \in \mathcal{H}_Q(T)$ satisfies*

$$\|\partial_x u\|_{L_t^\infty L_x^2} = \epsilon_1 \ll 1, \tag{1.21}$$

and let $v(s, t, x)$ be the solution of the heat flow (1.20) with initial data $u(t, x)$. Then

$$\sup_{s\ge 0} (1 + s2^{2k})^{31} 2^k \|P_k v\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} \gamma_k(\sigma)$$

for all $\sigma \in [0, \frac{99}{100}]$, $k \in \mathbb{Z}$. Moreover, for any $\sigma \in [\frac{99}{100}, \frac{5}{4}]$, and $k \in \mathbb{Z}$, we have

$$\sup_{s\ge 0} (1 + s2^{2k})^{30} 2^{\sigma k + k} \|P_k v\|_{L_t^\infty L_x^2} \lesssim \gamma_k(\sigma) + \gamma_k(\sigma - 3/8)\gamma_k(3/8).$$

Remark 1.3. The power of $1 + s2^{2k}$ in Proposition 1.2 can be chosen to be any $M \in \mathbb{Z}_+$ if we additionally assume that ϵ_1 is sufficiently small depending on M; see Proposition 1.3 below.

The second result of this step is bounds of $2^{(\sigma+1)k} \|P_k v\|_{L_t^\infty L_x^2}$ for $\sigma \in [0, 1 + j/4]$:

Proposition 1.3 (j-th iteration). *Let $j \in \mathbb{N}$ and $M \in \mathbb{Z}_+$. Assume that $u \in \mathcal{H}_Q(T)$ satisfies (1.21) with ϵ_1 sufficiently small depending on j, M. Let $v(s,t,x)$ be the solution of the heat flow (1.20) with initial data $u(t,x)$. Then for $\sigma \in [0, 1 + j/4]$ and any $k \in \mathbb{Z}$, v satisfies*

$$\sup_{s\in[0,\infty)} (1 + s2^{2k})^M 2^{k+\sigma k} \|P_k v\|_{L_t^\infty L_x^2} \lesssim_M \gamma_k^{(j)}(\sigma), \tag{1.22}$$

where $\{\gamma_k^{(j)}(\sigma)\}$ are defined in (3.18)–(3.20).

Step 2. Pretreating curvature terms. The curvature part in the master equation (1.17) can be schematically written as

$$\Re[\mathcal{R}(\phi_i, \phi_j)\phi_j]^\alpha = \sum_{1\le j_0, j_1, j_2 \le 2n} \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_\alpha\rangle \psi_i^{i_0}\psi_j^{i_1}\psi_j^{i_2},$$

$$\Im[\mathcal{R}(\phi_i, \phi_j)\phi_j]^\alpha = \sum_{1\le j_0, j_1, j_2 \le 2n} \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{\overline{\alpha}}\rangle \psi_i^{i_0}\psi_j^{i_1}\psi_j^{i_2}.$$

With abuse of notation, we denote

$$\mathcal{G} = \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle$$

for any given indices $j_0, \dots, j_3 \in \{1, \dots, 2n\}$. And let $\phi_i \diamond \phi_j$ denote the linear combinations of multiplications of real and imaginary parts of ϕ_i, ϕ_j, i.e. $\sum_{ij} c_{ij}\phi_i^{\pm}\phi_j^{\pm}$, where we denote $\phi_j^+ = \Re\phi_j$, $\phi_j^- = \Im\phi_j$. Then the master equation (1.17) is schematically written as

$$-\sqrt{-1}\, D_t\phi_i = \sum_{j=1}^{2} D_j D_j \phi_i + \sum_{j=1}^{2} \mathcal{G}\phi_i \diamond \phi_j \diamond \phi_j. \tag{1.23}$$

Moreover, the connection coefficients in D_j also depend on curvatures, and with abuse of notation can be schematically written as

$$[A_j]_q^p(s) = \int_s^\infty \phi_s \diamond \phi_j \mathcal{G}\, ds'.$$

We shall perform a dynamical separation of $\mathcal{G}$. In fact, by the caloric condition $\nabla_s e_j = 0$ and twice dynamic separation, $\mathcal{G}$ can be decomposed into

$$\begin{aligned}
\mathcal{G}(s) &= \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle(s) \\
&= \Gamma^\infty - \Gamma_l^{\infty,(1)} \int_s^\infty \psi_s^l(\widetilde{s})\, d\widetilde{s} \\
&\quad - \int_s^\infty \psi_s^l(\widetilde{s}) \left(\int_{\widetilde{s}}^\infty \psi_s^p(s')(\widetilde{\nabla}^2\mathbf{R})(e_l, e_p; e_{j_0}, \dots, e_{j_3}) ds' \right) d\widetilde{s} \\
&= \Gamma^\infty + \mathcal{U}_{00} + \mathcal{U}_{01} + \mathcal{U}_I + \mathcal{U}_{II},
\end{aligned}$$

where we define

$$\Gamma^\infty := \lim_{s\to\infty} \mathcal{G}(s), \quad \Gamma_l^{\infty,(1)} := \lim_{s\to\infty} (\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \dots, e_{j_3}) \quad \textbf{(constant limit part)},$$

$$\mathcal{U}_{00} := -\Gamma_l^\infty \int_s^\infty \sum_{i=1}^{2} (\partial_i \psi_i)^l\, ds' \quad \textbf{(first order terms)},$$

and **quadratic terms** by

$$\mathcal{U}_{01} := -\int_s^\infty \sum_{i=1}^2 (\partial_i \psi_i)\big((\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \dots, e_{j_3}) - \Gamma_l^{\infty,(1)}\big)\, ds',$$

$$\mathcal{U}_I := -\Gamma_l^{\infty,(1)} \int_s^\infty \sum_{i=1}^2 (A_i \psi_i)^l\, ds',$$

$$\mathcal{U}_{II} := -\int_s^\infty \sum_{i=1}^2 (A_i \psi_i)^l(\widetilde{s})\bigg(\int_{\widetilde{s}}^\infty \psi_s^p(s')(\widetilde{\nabla}^2\mathbf{R})(e_l, e_p; e_{j_0}, \dots, e_{j_3})\, ds'\bigg)\, d\widetilde{s},$$

$$= -\int_s^\infty \sum_{i=1}^2 (A_i \psi_i)^l(\widetilde{s})\big((\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \dots, e_{j_3}) - \Gamma_l^{\infty,(1)}\big)\, d\widetilde{s}.$$

Here, with abuse of notation, A_j denotes the $2n \times 2n$ real-valued matrix with elements $\{[A_j]_q^p\}_{p,q=1}^{2n}$. The constant limit part and the first order terms will be dominated by frequency envelopes of $\{\phi_i\}$, while the bounds of quadratic terms essentially rely on a delicate bootstrap on the term

$$\widetilde{\mathcal{G}}_l^{(1)} := (\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \dots, e_{j_3}) - \Gamma_l^{\infty,(1)}.$$

We also need higher order derivatives of $\mathcal{G}$. Given $k \in \mathbb{N}$, let

$$\mathcal{G}_{l_1,\dots,l_k}^{(k)} := (\widetilde{\nabla}^{(k)}\mathbf{R})(e_{l_1}, \dots, e_{l_k}; e_{j_0}, \dots, e_{j_3}),$$
$$\widetilde{\mathcal{G}}_{l_1,\dots,l_k}^{(k)} := \mathcal{G}_{l_1,\dots,l_k}^{(k)} - \Gamma_{l_1,\dots,l_k}^{\infty,(k)},$$

where we denote

$$\Gamma_{l_1,\dots,l_k}^{\infty,(k)} := \lim_{s\to\infty} \mathcal{G}_{l_1,\dots,l_k}^{(k)}(s).$$

Similarly, we perform a dynamical separation of frames. In fact, let $\mathcal{P}$ be the isometric embedding of $\mathcal{N}$ into $\mathbb{R}^N$, and let $\{e_l\}_{l=1}^{2n}$ be the caloric gauge built in Proposition 1.1. With abuse of notation, we denote

$$[d\mathcal{P}]^{(k)} := (\mathbf{D}^k d\mathcal{P})(\underbrace{e, \dots, e}_{k}; e),$$

$$[\widetilde{d\mathcal{P}}]^{(k)} := [d\mathcal{P}]^{(k)} - \lim_{s\to\infty} [d\mathcal{P}]^{(k)}.$$

Step 3. Tracking $L^4 \cap L_t^\infty L_x^2$ bounds for curvature terms and frames along heat direction.

Proposition 1.4. *Let $u \in \mathcal{H}_Q(T)$ be a solution of SMF. Denote by $v(s,t,x)$ the solution to the heat flow with initial data $u(t,x)$, and denote by $\{\phi_i\}_{i=0}^2$ the corresponding differential fields under the caloric gauge. Assume that $\{\beta_k(\sigma)\}$ is a frequency envelope of order δ such that for all $i = 1, 2$ and $k \in \mathbb{Z}$,*

$$2^{\sigma k}\|\phi_i\restriction_{s=0}\|_{L_t^\infty L_x^2 \cap L_{t,x}^4} \le \beta_k(\sigma). \tag{1.24}$$

- *There exists a sufficiently small constant* $\epsilon > 0$ *such that if*

$$\sum_{k\in\mathbb{Z}} |\beta_k(0)|^2 < \epsilon, \tag{1.25}$$

then for any $m \in \mathbb{N}$, $\sigma \in [0, \frac{99}{100}]$, $s \in [2^{2j-1}, 2^{2j+1})$ *and* $j, k \in \mathbb{Z}$,

$$\begin{aligned}
\|P_k \widetilde{\mathcal{G}}^{(m)}\|_{L^4 \cap L_t^\infty L_x^2} &\lesssim_m 2^{-\sigma k - k}\beta_k(\sigma)(1+s2^{2k})^{-30},\\
\|P_k \phi_s\|_{L^4 \cap L_t^\infty L_x^2} &\lesssim 2^{-\sigma k + k}\\
&\quad \times \Big[1_{k+j\ge 0}(1+s2^{2k})^{-30}\beta_k(\sigma) + 1_{k+j\le 0}\sum_{k\le l\le -j}\beta_l(\sigma)\beta_l\Big],\\
\|P_k(\widetilde{[d\mathcal{P}]}^{(m)})\|_{L_t^\infty L_x^2 \cap L^4} &\lesssim_m \beta_k(\sigma)(1+s2^{2k})^{-29}2^{-\sigma k - k},\\
\|P_k A_i\|_{L_t^\infty L_x^2} &\lesssim \beta_{k,s}(\sigma)(1+s2^{2k})^{-27}2^{-\sigma k}.
\end{aligned}$$

- *Furthermore, given* $j, M \in \mathbb{Z}_+$, *if* $\{\beta_k(\sigma)\}$ *is a frequency envelope of order* $\frac{1}{2^j}\delta$, *then similar results hold for* $\sigma \in [0, 1 + j/4]$ *and* ϵ *sufficiently small depending only on* $j, M \in \mathbb{Z}_+$. *In particular, for any* $m \in \mathbb{N}$, $k \in \mathbb{Z}$ *and* $\sigma \in [0, 1 + j/4]$, *one has*

$$\begin{aligned}
(1+s2^{2k})^{M+2}2^{\sigma k + k}\|P_k \widetilde{\mathcal{G}}^{(m)}\|_{L^4 \cap L_t^\infty L_x^2} &\lesssim_{m,M} \beta_k^{(j)}(\sigma),\\
(1+s2^{2k})^{M+1}2^{\sigma k + k}\|P_k(\widetilde{[d\mathcal{P}]}^{(m)})\|_{L_t^\infty L_x^2 \cap L_{t,x}^4} &\lesssim_{m,M} \beta_k^{(j)}(\sigma),\\
(1+s2^{2k})^{M}2^{\sigma k}\|P_k A_i\|_{L_t^\infty L_x^2} &\lesssim_M \beta_{k,s}^{(j)}(\sigma).
\end{aligned}$$

Remark 1.4. The $\{\beta_k^{(j)}\}$ and $\{\beta_{k,s}^{(j)}(\sigma)\}$ are defined in (3.18)–(3.20) below. Proposition 1.4 will be proved in Sections 3.5 and 3.6.

Step 4.1. $F_k \cap S_k^{1/2}$ **bounds for connections along the heat direction.** In this step, we prove

Lemma 1.2. *Given* $\sigma \in [0, \frac{99}{100}]$, *let* $\{h_k(\sigma)\}$ *be frequency envelopes defined by*

$$h_k(\sigma) := \sup_{k'\in\mathbb{Z},\, j=1,2} 2^{-\delta|k-k'|}2^{\sigma k'}(1+s2^{2k'})^4\|P_{k'}\phi_j\|_{F_{k'}(T)}. \tag{1.26}$$

Let $\{b_k\}$ *be an* ε*-frequency envelope. Assume that for any* $k, j \in \mathbb{Z}$ *and* $s \in [2^{2j-1}, 2^{2j+1})$,

$$2^{k/2}\|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L_x^4 L_t^\infty(T)} \le \varepsilon^{-1/4}h_k[(1+s2^{2k})^{-20}1_{j+k\ge 0} + 1_{j+k\le 0}2^{\delta|k+j|}]. \tag{1.27}$$

Then, if $\varepsilon > 0$ *is sufficiently small, for* $\sigma \in [0, \frac{99}{100}]$ *one has*

$$\|P_k A_i(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim h_{k,s}(\sigma)2^{-\sigma k}(1+s2^{2k})^{-4}.$$

For the proof, see the proof of Lemma 4.1.

Step 4.2. F_k **bounds along the heat direction without assuming (1.27).**

Lemma 1.3. *Let* $\{b_k\}$ *be an* ε*-frequency envelope. Given* $\sigma \in [0, \frac{99}{100}]$*, suppose that* $\{b_k(\sigma)\}$ *are also frequency envelopes, and* $\{h_k(\sigma)\}$ *are the frequency envelopes defined by* (1.26)*. Assume that for* $i = 1, 2$*,*

$$\begin{aligned} \|P_k\phi_i\restriction_{s=0}\|_{F_k(T)} &\le b_k(\sigma)2^{-\sigma k}, & \sigma \in \big[0, \tfrac{99}{100}\big],\\ \|P_k\phi_t\restriction_{s=0}\|_{L^4_{t,x}} &\lesssim b_k(\sigma)2^{-(\sigma-1)k}, & \sigma \in \big[0, \tfrac{99}{100}\big],\\ \|P_k\phi_i(s)\|_{F_k(T)} &\lesssim \varepsilon^{-1/2} b_k(1+s2^{2k})^{-4}. \end{aligned}$$

Then, if $\varepsilon > 0$ *is sufficiently small, for* $\sigma \in [0, \frac{99}{100}]$*,* $i = 1, 2$*, one has*

(B1) $\|P_k A_i(s)\|_{F_k(T)\cap S^{1/2}_k} \lesssim h_{k,s}(\sigma)2^{-\sigma k}(1+s2^{2k})^{-4}$,

$$\text{(B2)} \quad \begin{cases} \|P_k\phi_i(s)\|_{F_k(T)} \lesssim b_k(\sigma)2^{-\sigma k}(1+s2^{2k})^{-4},\\ \|P_k A_i\restriction_{s=0}\|_{L^4_{t,x}} \lesssim b_k(\sigma)2^{-\sigma k}, \quad i = 1, 2,\\ \|P_k\phi_t(s)\|_{L^4_{t,x}} \lesssim b_k(\sigma)2^{-(\sigma-1)k}(1+2^{2k}s)^{-2}, \end{cases}$$

$$\text{(B3)} \quad \begin{cases} \|P_k A_t\restriction_{s=0}\|_{L^2_{t,x}} \lesssim \varepsilon b_k(\sigma)2^{-\sigma k},\\ \|P_k A_t\restriction_{s=0}\|_{L^2_{t,x}} \lesssim \varepsilon^2. \end{cases}$$

Remark 1.5. In the proof of Lemma 1.3, we first assume (1.27) and apply Lemma 1.2 to obtain all the estimates stated in (B2); see Lemmas 4.3, 4.5, 4.6. Then combining theses estimates and Proposition 1.4, we improve (1.27) by dropping $\varepsilon^{-1/4}$ on the RHS of (1.27), and thus close the bootstrap assumption of (1.27); see Lemma 4.7. Then Lemma 1.2, i.e. (B1), holds without assuming (1.27). And (B3) is proved as Lemma 4.8.

Step 5. G_k **bounds along the SMF direction for** $\sigma \in [0, \frac{99}{100}]$**.**

Proposition 1.5. *Assume that* $\sigma \in [0, \frac{99}{100}]$*. Given any* $\mathcal{L} \in \mathbb{Z}_+$*, assume that* $T \in (0, 2^{2\mathcal{L}}]$*. Let* ϵ_0 *be a sufficiently small constant. Assume that* $\{c_k\}$ *is an* ϵ_0*-frequency envelope of order* δ*, and let* $\{c_k(\sigma)\}$ *be another frequency envelope of order* δ*. Let* $u \in \mathcal{H}_Q(T)$ *be the solution to SMF with initial data* u_0 *which satisfies*

$$\|P_k\nabla u_0\|_{L^2_x} \le c_k, \quad \|P_k\nabla u_0\|_{L^2_x} \le c_k(\sigma)2^{-\sigma k}.$$

Denote by $\{\phi_i\}$ *the corresponding differential fields of the heat flow initiated from* u*. Suppose also that at the initial time* $s = 0$*,*

$$\|P_k\phi_i\|_{G_k(T)} \le \epsilon_0^{-1/2} c_k.$$

Then, when $s = 0$*, for all* $i = 1, 2$ *and* $k \in \mathbb{Z}$ *we have*

$$\|P_k\phi_i\|_{G_k(T)} \lesssim c_k, \quad \|P_k\phi_i\|_{G_k(T)} \lesssim c_k(\sigma)2^{-\sigma k}.$$

Proposition 1.5 is proved in Section 5.

Step 6. Improved F_k bounds of $P_k\widetilde{\mathscr{G}}^{(1)}$ once.

Lemma 1.4. *Let $u\in\mathscr{H}_Q(T)$ solve SMF with data u_0. Let $\{c_k\}$ be an ϵ_0-frequency envelope of order $\frac{1}{2}\delta$. Given any $\sigma\in[0,\frac{99}{100}]$, let $\{c_k(\sigma)\}$ be another frequency envelope of order δ such that*

$$\|P_k\nabla u_0\|_{L^2_x}\le c_k,\quad \|P_k\nabla u_0\|_{L^2_x}\le c_k(\sigma)2^{-\sigma k}.$$

Then for ϵ_0 sufficiently small,

$$2^{k/2}\|P_k\widetilde{\mathscr{G}}^{(1)}\|_{L^4_tL^\infty_x}\lesssim c_k(\sigma)2^{-\sigma k}[(1+2^{2k+2k_0})^{-20}1_{k+k_0\ge0}+1_{k+k_0\le0}2^{\delta|k+k_0|}], \tag{1.28}$$

$$\|P_k\widetilde{\mathscr{G}}\|_{F_k(T)}\lesssim 2^{-\sigma k}c_k(\sigma)(1+2^{k+k_0})^{-7}[1_{k+k_0\ge0}2^{k_0}+1_{k+k_0\le0}2^{-k}], \tag{1.29}$$

for any $\sigma\in[0,\frac{99}{100}]$, $k,k_0\in\mathbb{Z}$ and $s\in[2^{2k_0-1},2^{2k_0+1})$.

The proof of (1.28) is given in Lemma 6.1, and (1.29) follows from Corollary 4.1.

Step 7. Improved $F_k\cap S_k^{1/2}$ bounds of $\{P_kA_j\}_{j=0}^2$ and parabolic estimates for $\{\phi_j\}_{j=0}^2$ with $\sigma\in[0,\frac{5}{4}]$.

Lemma 1.5. *Let $u\in\mathscr{H}_Q(T)$ be the solution to SMF with initial data $u_0\in\mathscr{H}_Q$. Define a frequency envelope $\{c^{(1)}(\sigma)\}$ as in Definition 6.1. Given any $\sigma\in[0,\frac{5}{4}]$, let $\{b_k(\sigma)\}$ be a frequency envelope of order δ, and assume that*

$$b_k(\sigma)\lesssim c_k^{(1)}(\sigma),\quad \forall\sigma\in\left[0,\tfrac{99}{100}\right].$$

Assume also that for $i=1,2$,

$$\begin{aligned}\|P_k\phi_i\restriction_{s=0}\|_{F_k(T)}&\le b_k(\sigma)2^{-\sigma k}, && \sigma\in\left[0,\tfrac{5}{4}\right],\\ \|P_k\phi_t\restriction_{s=0}\|_{L^4_{t,x}}&\le b_k(\sigma)2^{-(\sigma-1)k}, && \sigma\in\left[0,\tfrac{5}{4}\right].\end{aligned}$$

Then, if $\varepsilon>0$ is sufficiently small, for $\sigma\in[0,\frac{5}{4}]$ one has

$$\begin{aligned}\|P_kA_i(s)\|_{F_k(T)\cap S_k^{1/2}}&\lesssim h_{k,s}^{(1)}(\sigma)2^{-\sigma k}(1+s2^{2k})^{-4},\\ \|P_k\phi_i(s)\|_{F_k(T)}&\lesssim b_k^{(1)}(\sigma)2^{-\sigma k}(1+s2^{2k})^{-4},\\ \|P_kA_i\restriction_{s=0}\|_{L^4_{t,x}}&\lesssim b_k^{(1)}(\sigma)2^{-\sigma k},\quad i=1,2,\\ \|P_k\phi_t(s)\|_{L^4_{t,x}}&\lesssim b_k^{(1)}(\sigma)2^{-(\sigma-1)k}(1+2^{2k}s)^{-2},\\ \|P_kA_t\restriction_{s=0}\|_{L^2_{t,x}}&\lesssim \varepsilon b_k^{(1)}(\sigma)2^{-\sigma k}\quad \text{if } \sigma\in\left[\tfrac{1}{100},\tfrac{5}{4}\right],\\ \|P_kA_t\restriction_{s=0}\|_{L^2_{t,x}}&\lesssim\varepsilon^2.\end{aligned} \tag{1.30}$$

Remark 1.6. In Lemma 1.5, the $\{b_k^{(j)}\}$, $\{h_k^{(j)}\}$,$\{b_{k,s}^{(j)}\}$, $\{h_{k,s}^{(j)}\}$ are defined in Definition 6.2. The key of Lemma 1.5 is (1.30), and its proof is given in Lemma 6.2.

Step 8. G_k **bounds along the SMF direction for** $\sigma \in [0, \frac{5}{4}]$**.**

Lemma 1.6. *Assume that* $\sigma \in (\frac{99}{100}, \frac{5}{4}]$*. Let* $\epsilon_0 > 0$ *be a sufficiently small constant. Given any* $\mathcal{L} \in \mathbb{Z}_+$*, assume that* $T \in (0, 2^{2\mathcal{L}}]$*. Let* $\{c_k(\sigma)\}$ *be a frequency envelope of order* $\frac{1}{2}\delta$*, and let* $\{c_k\}$ *be an* ϵ_0*-frequency envelope of order* $\frac{1}{2}\delta$*. Let* $u \in \mathcal{H}_Q(T)$ *be the solution to SMF with initial data* u_0 *which satisfies*

$$\|P_k \nabla u_0\|_{L^2_x} \le c_k, \quad \|P_k \nabla u_0\|_{L^2_x} \le c_k(\sigma) 2^{-\sigma k}.$$

Denote by $\{\phi_i\}$ *the corresponding differential fields of the heat flow initiated from* u*. Then, when* $s = 0$*, given* $\sigma \in (\frac{99}{100}, \frac{5}{4}]$ *one has*

$$\|P_k \phi_i \restriction_{s=0}\|_{G_k(T)} \lesssim (c_k(\sigma) + c_k(\sigma - 3/8) c_k(3/8)) 2^{-\sigma k}.$$

Step 9. F_k **bounds of** $P_k \widetilde{\mathcal{G}}^{(1)}$ **and** G_k **bounds of** ϕ_x **along the SMF direction for** $\sigma \in [0, 1 + j/4]$**.**

Lemma 1.7. *Given* $j \ge 2$*, assume that* $\sigma \in [0, 1 + j/4]$*. Let* $Q \in \mathcal{N}$ *be a fixed point and* ϵ_0 *be a sufficiently small constant depending on* j*. Given any* $\mathcal{L} \in \mathbb{Z}_+$*, assume that* $T \in (0, 2^{2\mathcal{L}}]$*. Let* $u \in \mathcal{H}_Q(T)$ *be the solution to SMF with initial data* u_0*. Let* $\{c_k^{(j)}(\sigma)\}$ *be the frequency envelopes defined in Definition* 6.1*, and assume that* $\{c_k^{(j)}(0)\}$ *is an* ϵ_0*-frequency envelope with* $0 < \epsilon_0 \ll 1$*.*

- *For* $\sigma \in [0, 1 + \frac{j-1}{4}]$*, we have*

$$2^{k/2} \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t} \lesssim c_k^{(j)}(\sigma) 2^{-\sigma k} [(1 + 2^{2k+2k_0})^{-20} 1_{k+k_0 \ge 0} + 1_{k+k_0 \le 0} 2^{\delta |k+k_0|}],$$
$$\|P_k \widetilde{\mathcal{G}}\|_{F_k(T)} \lesssim c_k^{(j)}(\sigma) 2^{-\sigma k} [(1 + 2^{k+k_0})^{-7} 1_{k+k_0 \ge 0} 2^{k_0} + 1_{k+k_0 \le 0} 2^{-k}],$$

  for any $s \in [2^{2k_0 - 1}, 2^{2k_0+1})$ *and* $k_0, k \in \mathbb{Z}$*.*
- *Denote by* $\{\phi_i\}$ *the corresponding differential fields of the heat flow initiated from* u*. Then, for* $\sigma \in [0, 1 + j/4]$*,*

$$\|P_k \phi_i \restriction_{s=0}\|_{G_k(T)} \lesssim c_k^{(j)}(\sigma) 2^{-\sigma k}.$$

Step 10. Global regularity, global well-posedness and asymptotic behaviors. As in Step 9, performing the bootstrap-iteration scheme K times gives bounds of $2^{\sigma k} \|P_k \phi_j\|_{G_k}$ for $\sigma \in [0, K/4 + 1]$. Then, transforming the bounds of $\{\phi_j\} \restriction_{s=0}$ back to the solution u gives

$$\|u\|_{L^\infty_t \dot{H}^{\sigma+1}_x(T)} \lesssim \|u_0\|_{\dot{H}^1_x \cap \dot{H}^{\sigma+1}_x}.$$

Noticing that an $\dot{H}^1 \cap \dot{H}^{2+}$ uniform bound will rule out blow-up for SMF in $\mathbb{R}^2$, one step iteration suffices to show u is global. And the ϵ_0 only depends on the dimension and the target manifold $\mathcal{N}$. Moreover, we proceed the bootstrap-iteration scheme for K times and obtain uniform bounds for higher Sobolev norms.

The asymptotic behaviors stated in (1.5) will be proved following our recent work [22] on SMF on hyperbolic planes. The proof of Step 10 is presented in Section 7.

1.5. Main ideas

Let us explain the main ideas. To control the curvature terms, which is the non-self-contained part, we use dynamical separation and a bootstrap-iteration scheme to obtain an approximate constant sectional curvature nonlinearity with controllable remainder in finite steps of iteration. The essential advantage of this scheme is that it reduces estimates in frequency localized spaces such as F_k, G_k to decay estimates in Lebesgue spaces along the heat direction.

Iteration scheme for Step 1. We describe the iteration scheme for heat flows. The starting point of the heat flow iteration is bounds for $\partial_s v$.

1. First time iteration. Suppose that we have obtained parabolic decay estimates of $\|P_k \partial_s v\|_{L_t^\infty L_x^2}$ such as

$$\|P_k \partial_s v\|_{L_t^\infty L_x^2} \lesssim (1 + 2^{2k} s)^{-M_1} \gamma_{k,s}(\sigma) 2^{-\sigma k}, \quad \sigma \in \left[0, \tfrac{99}{100}\right].$$

By applying dynamical separation

$$S_{ij}^l(v(s)) = S_{ij}^l(Q) - \int_s^\infty (DS_{ij}^l)(v(s')) \cdot \partial_s v \, ds',$$

bounds for $\|P_k \partial_s v\|_{L_t^\infty L_x^2}$ yield improved frequency localized bounds for the second fundamental form term, i.e. $\|P_k S_{ij}^l(v(s))\|_{L_t^\infty L_x^2}$. The only potential trouble is the High $\times$ Low interaction of $(DS_{ij}^l)(v(s')) \cdot \partial_s v$. But we will see that this interaction can be handled by additionally proving the decay estimates

$$\|\partial_x^{L+1} DS_{ij}^l(v(s))\|_{L_t^\infty L^2} \lesssim_L \epsilon_1 s^{-L/2}, \quad \forall L \in \mathbb{N}.$$

Then back to the extrinsic map v, using the heat flow equation will give an improved bound for $\|P_k \partial_s v\|_{L_t^\infty L_x^2}$ for $\sigma \in [1, \frac{5}{4}]$.

2. m-th time iteration. Using dynamical separation of the schematic form

$$D^{m-1} S(v(s)) = D^{m-1} S(Q) - \int_s^\infty D^m S(v(s')) \partial_s v \, ds'$$

and the decay estimates

$$\|\partial_x^{L+1} D^{m-1} S_{ij}^l(v(s))\|_{L_t^\infty L^2} \lesssim_{L,m} \epsilon_1 s^{-L/2}, \quad \forall L \in \mathbb{N},$$

one gets frequency localized bounds of the extrinsic map v for $\sigma \in [1, 1 + m/4]$:

$$2^{\sigma k} \|P_k v(s)\|_{L_t^\infty L_x^2} \lesssim \epsilon_1 (1 + 2^{2k} s)^{-M_1 + m} \gamma_k^{(m)}(\sigma).$$

The motivation of decomposition in Step 2. To bound the curvature terms, by a kind of dynamical separation we have the decomposition of curvature terms denoted by $\mathcal{G}$:

$$\mathcal{G} = \text{constant} + \text{first order terms} + \text{quadratic terms}$$

(see Step 2). We observe that to control $\mathcal{G}$ in the F_k space, it suffices to prove parabolic decay estimates of $\mathcal{G}^{(j)}$. The same idea will be applied to bound the frames in frequency localized spaces. We remark that dynamical separation was previously used in [19, 38] to reveal the implicit null structures. Here, we apply dynamical separation to do iterations. Besides using dynamical separation, in order to give a bound for connection coefficients which is the key for bootstrap, we further decompose the curvature term into differential fields ϕ_i dominated terms and relatively smaller quadratic terms. By an appropriate bootstrap argument, bounding connection coefficients $\{A_j\}_{j=1}^2$ in the $F_k \cap S_k^{1/2}$ space reduces to deriving parabolic decay estimates of covariant derivatives of the curvatures $\mathcal{G}^{(j)}$ in the simpler $L_t^\infty L_x^2 \cap L^4$ spaces.

The motivation of adding (1.27) in Step 4.1. The key difficulty in bounding curvature involved terms is the High × Low → High interaction of curvatures and differential fields or heat tension fields, i.e., the frequency of curvatures occupies the dominating position compared with differential fields or heat tension fields.

- First of all, we observe that it suffices to control the F_k norms of curvatures $\mathcal{G}$. Then we further clarify that among the four blocks of the F_k space only the three blocks $L_t^\infty L_x^2$, L^4, $L_x^4 L_t^\infty$ need to be estimated for curvatures.
- Second, we find that using dynamic separation in the heat direction

$$\mathcal{G} := \mathbf{R}(e_{i_0}, e_{i_1}, e_{i_2}, e_{i_3}) = \Gamma^\infty - \int_s^\infty \psi_s^l (\widetilde{\nabla}\mathbf{R})(e_l; , e_{i_0}, e_{i_1}, e_{i_2}, e_{i_3})\, ds'$$
$$= \Gamma^\infty - \Gamma_l^{\infty,(1)} \int_s^\infty \psi_s^l \, ds' - \int_s^\infty \psi_s^l \widetilde{\mathcal{G}}_l^{(1)}\, ds'$$

  and the heat flow iteration scheme, the $L_t^\infty L_x^2$ and L^4 norms of curvatures can be controlled by corresponding norms of differential fields.
- The troublesome block of F_k is the $L_x^4 L_t^\infty$ norm. This norm of curvatures cannot be obtained by dynamical separation in the heat direction and the heat flow iteration scheme as before. The problem is that the High × Low → High interaction of $\widetilde{\mathcal{G}}_l^{(1)} \psi_s^l$ in $L_x^4 L_t^\infty$ fails if one only previously has bounds for $\widetilde{\mathcal{G}}^{(1)}$ in $L_t^\infty L_x^2 \cap L^4$. (Estimates of $\widetilde{\mathcal{G}}^{(1)}$ in $L_t^\infty L_x^2 \cap L^4$ are obtained in Step 3.) So we add the bootstrap assumption (1.27) to bound $L_x^4 L_t^\infty$ of $\widetilde{\mathcal{G}}^{(1)}$. The key is that one can indeed improve the assumption (1.27) and then close the bootstrap.

How to drop (1.27) in Step 4.2. Let us explain how to improve (1.27) and thus close the bootstrap of Step 4.

(I) With the assumption (1.27) we can prove bounds of A_t and ϕ_t in L^4 by envelopes of differential fields ϕ_x, loosely speaking say

$$\|P_k(A_t)\|_{L^4} \lesssim 2^{-\sigma k + k} h_k(\sigma)[(1 + 2^{2k+2k_0})^{-1} 1_{k+k_0 \ge 0} + 1_{k+k_0 \le 0} 2^{\delta |k+k_0|}],$$
$$\|P_k(\phi_t)\|_{L^4} \lesssim 2^k b_k (1 + 2^{2k} s)^{-2},$$

where $k, k_0 \in \mathbb{Z}$, $s \in [2^{2k_0-1}, 2^{2k_0+1})$, $\{h_k(\sigma)\}$ is the frequency envelope associated with the differential fields $\{\phi_i\}_{i=1}^2$ defined in (1.26), and $\{b_k\}$ is some frequency envelope with $\|b_k\|_{\ell^2}$ sufficiently small. This will give bounds for $\|P_k \partial_t \widetilde{\mathcal{G}}^{(1)}\|_{L^4}$.

(**II**) We improve the assumption (1.27) by interpolation

$$\|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t} \le \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4}^{3/4} \|P_k \partial_t \widetilde{\mathcal{G}}^{(1)}\|_{L^4}^{1/4}, \tag{1.31}$$

and the L^4 estimates obtained from Step 3 (Prop. 1.4 for heat flow iteration)

$$2^k \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4} \le h_k (1 + 2^{2k} s)^{-M}.$$

In fact, we can prove

$$2^{k/2} \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t} \le h_k 2^{-\sigma k} [(1 + 2^{2k+2k_0})^{-\frac{3}{4}M} 1_{k+k_0 \ge 0} + 1_{k+k_0 \le 0} 2^{\delta |k+k_0|}].$$

Then choosing sufficiently large M, one obtains better bounds of $\widetilde{\mathcal{G}}^{(1)}$ than in (1.27). This gives the way to bound curvatures in the block space $L^4_x L^\infty_t$ of F_k.

In (**I**), the key step is to obtain bounds of the connection coefficients; see Lemma 4.1. To prove Lemma 4.1, as mentioned before, we decompose the curvature term into differential fields ϕ_i dominated terms and the remaining quadratic terms; see Step 2 of Lemma 4.1. The additional smallness gained by the remaining quadratic terms gives us the chance to use a bootstrap argument to control the connection coefficients.

Iteration in Step 6 to Step 9. With these new ideas and [5]'s framework, the range of $\sigma \in [0, \frac{99}{100}]$ can be reached before performing iteration for the SMF evolution. In order to reach larger σ, one combines the heat flow iteration with an SMF iteration. For the SMF iteration scheme, the key is to improve the estimate $\|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t}$ step by step to reach larger σ.

1.6. Idea for higher dimensions

Let us give a prevue of the higher-dimensional case. As $d = 2$, for higher dimensions in order to track the curvature terms $\mathcal{G}$ in the F_k space along the heat flow direction, it suffices to control the first order covariant derivative of the curvature term $\widetilde{\mathcal{G}}^{(1)}$ in the simpler Lebesgue spaces in the heat direction. Thus the parabolic decay estimates of moving frame dependent quantities for $d \ge 3$ should be established. The difficulty is to bound all geometric quantities in fractional Sobolev spaces when d is odd. We solve this problem by using geodesic parallel transport and a difference characterization of Besov spaces. The idea is that the difference characterization reduces bounding fractional Sobolev norms to bounding differences of all these geometric quantities and their covariant derivatives in Lebesgue spaces. And geodesic parallel transport gives us the difference of the geometric quantities at different points of the base manifold.

We divide the whole theorem for $d \ge 2$ into two papers to make the main idea clear and avoid the paper being too long.

2. Preliminaries

2.1. Linear estimates

The following are the main linear estimates established in [5].

Proposition 2.1 ([5]). *Given $\mathcal{L} \in \mathbb{Z}_+$, assume that $T \in (0, 2^{2\mathcal{L}}]$. Then for every $u_0 \in L^2_x$ with frequency localized in I_k and every $F \in N_k(T)$, we have the following inhomogeneous estimate: If u solves*

$$\begin{cases} i\partial_t u + \Delta u = F, \\ u(0,x) = u_0(x), \end{cases} \tag{2.1}$$

then

$$\|u\|_{G_k(T)} \lesssim \|u_0\|_{L^2_x} + \|F\|_{N_k(T)}. \tag{2.2}$$

The following lemma will be used widely.

Lemma 2.1 ([5]). *For $f \in L^2_k(T)$,*

$$\|P_k f\|_{L^4} \le \|f\|_{F_k(T)}, \tag{2.3}$$

$$\|P_k f\|_{F_k(T)} \lesssim \|f\|_{L^2_x L^\infty_t} + \|f\|_{L^4}, \tag{2.4}$$

$$\|P_k f\|_{L^2_x L^\infty_t} \le \|f\|_{S^{1/2}_k}, \tag{2.5}$$

and

$$\|e^{s\Delta} g\|_{F_k(T)} \lesssim (1 + s2^{2k})^{-20} \|g\|_{F_k(T)}, \quad \forall s \ge 0, \tag{2.6}$$

provided that the RHS is finite.

2.2. Frequency envelopes

We recall the definition of envelopes introduced by Tao.

Definition 2.1. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a positive sequence. We call it a *frequency envelope* if

$$\sum_{k\in\mathbb{Z}} a_k^2 < \infty, \quad \text{and} \quad a_j \le 2^{\delta|l-j|} a_l, \quad \forall j, l \in \mathbb{Z}. \tag{2.7}$$

We call the frequency envelope $\{a_k\}$ an ϵ-*envelope* if it additionally satisfies

$$\sum_{k\in\mathbb{Z}} a_k^2 \le \epsilon^2.$$

For any nonnegative sequence $\{a_j\} \in \ell^2$, we define its frequency envelope by

$$\widetilde{a}_j := \sup_{j'\in\mathbb{Z}} a_{j'} 2^{-\delta|j-j'|}.$$

It satisfies

$$|a_j| \le \widetilde{a}_j, \quad \forall j \in \mathbb{Z}; \quad \sum_{j\in\mathbb{Z}} \widetilde{a}_j^2 \lesssim \sum_{j\in\mathbb{Z}} a_j^2.$$

Generally the δ in Definition 2.1 is not important if it has been fixed throughout the paper. But due to our iteration argument we shall introduce different δ in different steps of iterations. So we call $\{a_k\}$ satisfying (2.7) a frequency envelope *of order* δ.

In this paper, each time we mention a frequency envelope, we will clearly state its order.

We recall the following two facts on envelopes: (a) If $d_k \le b_k$ for all $k \in \mathbb{Z}$ and $\{b_k\}$ is a frequency envelope of order $\delta > 0$ then $\widetilde{d}_k \le b_k$ for all $k \in \mathbb{Z}$ as well, where $\{\widetilde{d}_k\}$ denotes the envelope of $\{d_k\}$ of the same order $\delta > 0$:

$$\widetilde{d}_k := \sup_{j\in\mathbb{Z}} d_j 2^{-\delta|k-j|}.$$

(b) If $\{d_k\}$ is already an envelope of order $\delta > 0$ then $d_k = \widetilde{d}_k$ for all $k \in \mathbb{Z}$.

We recall the classical result obtained in [31,36].

Lemma 2.2 ([31,36]). *Assume that* $u \in \mathcal{H}_Q(T)$ *satisfies*

$$\|\partial_x u\|_{L_t^\infty L_x^2} = \epsilon_1 \ll 1, \tag{2.8}$$

and let $v(s,t,x)$ *be the solution of the heat flow* (1.13) *with initial data* $u(t,x)$. *Then*

$$\|\partial_x^{j+1} v\|_{L_t^\infty L_x^2} \lesssim s^{-j/2}\epsilon_1, \tag{2.9}$$

and the corresponding differential fields and connection coefficients satisfy

$$s^{j/2}\|\partial_x^j \phi_i\|_{L_t^\infty L_x^2} \lesssim \epsilon_1, \tag{2.10}$$

$$s^{j/2}\|\partial_x^j A_i\|_{L_t^\infty L_x^2} \lesssim \epsilon_1, \tag{2.11}$$

$$s^{(j+1)/2}\|\partial_x^j \phi_i\|_{L_t^\infty L_x^\infty} \lesssim \epsilon_1, \tag{2.12}$$

$$s^{(j+1)/2}\|\partial_x^j A_i\|_{L_t^\infty L_x^\infty} \lesssim \epsilon_1, \tag{2.13}$$

for all $s \in [0,\infty)$, $i = 1,2$ *and any nonnegative integer* j.

3. Iteration for heat flows

3.1. Main results on the extrinsic map v solving HF

For $u \in \mathcal{H}_Q(T)$, define

$$\gamma_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'+k'} \|P_{k'} u\|_{L_t^\infty L_x^2}, \quad \sigma \ge 0,\ \delta = \tfrac{1}{800}. \tag{3.1}$$

Denote by $\{\gamma_k\}$ the frequency envelope for the energy norm, i.e.,

$$\gamma_k := \gamma_k(0).$$

Thus

$$2^k \|P_k u\|_{L_t^\infty L_x^2} \le 2^{-\sigma k}\gamma_k(\sigma), \quad \forall \sigma \ge 0.$$

Before going ahead, we recall the extrinsic formulations of heat flows. Assume that the target manifold $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^N$. Then the heat flow equation can be formulated as

$$\partial_s v^l - \Delta v^l = \sum_{a=1}^{2} \sum_{i,j=1}^{N} S^l_{ij} \partial_a v^i \partial_a v^j, \quad l = 1, \dots, N, \tag{3.2}$$

where $S = \{S^l_{ij}\}$ denotes the second fundamental form of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^N$.

Recall that $\{\gamma_k(\sigma)\}$ are defined to be the frequency envelopes of $u \in \mathcal{H}_Q(T)$. (See (3.1) and note that u **need not** solve SMF.) The frequency localized estimates of the extrinsic map v solving (3.2) with initial data u are given below.

Proposition 3.1. *Assume that $u \in \mathcal{H}_Q(T)$ satisfies*

$$\|\partial_x u\|_{L^\infty_t L^2_x} = \epsilon_1 \ll 1, \tag{3.3}$$

and let $v(s,t,x)$ be the solution of the heat flow (3.2) with initial data $u(t,x)$. Then v satisfies

$$\sup_{s \geq 0} (1 + s2^{2k})^{31} 2^k \|P_k v\|_{L^\infty_t L^2_x} \lesssim 2^{-\sigma k} \gamma_k(\sigma)$$

for all $\sigma \in [0, \frac{99}{100}]$ and $k \in \mathbb{Z}$. Moreover, for any $\sigma \in [\frac{99}{100}, \frac{5}{4}]$ and $k \in \mathbb{Z}$, we have

$$\sup_{s \geq 0} (1 + s2^{2k})^{30} 2^{\sigma k + k} \|P_k v\|_{L^\infty_t L^2_x} \lesssim \gamma_k(\sigma) + \gamma_k(\sigma - 3/8)\gamma_k(3/8).$$

Remark 3.1. The power of $1 + s2^{2k}$ in Proposition 3.1 can be chosen to be any $M \in \mathbb{Z}_+$ if we additionally assume that ϵ_1 is sufficiently small depending on M; see Proposition 3.4 below.

3.2. Before iteration

Given an initial data $v_0 \in \mathcal{H}_Q$ with energy sufficiently small, by Lemma 2.2 the corresponding heat flow is global with (2.9) holding. Combining this with the local Cauchy theory in Sobolev spaces for heat flows yields the following lemma.

Lemma 3.1. *Assume that $v_0 \in \mathcal{H}_Q$ has sufficiently small energy, and let $v(s,x)$ be the solution of the heat flow (3.2) with initial data v_0. Given an arbitrary $L \in \mathbb{Z}_+$, there exist constants $C_L, C_s > 0$ such that for any $s \geq 0$ and $0 \leq j \leq L$,*

$$\|\partial_x^{j+1} v(s)\|_{H^L_x} \leq C_L (1+s)^{-j/2}, \quad \|v(s) - Q\|_{L^2_x} \leq C_s.$$

Proposition 3.2. *Assume that $u \in \mathcal{H}_Q(T)$ satisfies*

$$\sum_{k \in \mathbb{Z}} 2^{2k} \|P_k u\|^2_{L^\infty_t L^2_x} = \epsilon_1^2 \ll 1, \tag{3.4}$$

and let $v(s,t,x)$ be the solution of the heat flow (3.2) with initial data $u(t,x)$. Then for $\sigma \in [0, \frac{99}{100}]$ and all $k \in \mathbb{Z}$, v satisfies

$$\sup_{s\in[0,\infty)} (1+s2^{2k})^{31} 2^{k+\sigma k} \|P_k v\|_{L_t^\infty L_x^2} \lesssim \gamma_k(\sigma). \tag{3.5}$$

Proof. Since v converges to a fixed point $Q \in \mathcal{N}$ as $s \to \infty$, we put

$$S_{ij}^l(v) := S_{ij}^l(Q) + (S_{ij}^l(v) - S_{ij}^l(Q)).$$

The $\{S_{ij}^l(Q)\}$ part is constant and makes an acceptable contribution to the final estimates by [5, Lemma 8.3]. Moreover, by Lemma 2.2, for all nonnegative integers L the remaining part satisfies

$$\|\partial_x^{L+1}(S_{ij}^l(v) - S_{ij}^l(Q))\|_{L_t^\infty L_x^2} \lesssim_L s^{-L/2} \|\nabla u\|_{L_t^\infty L_x^2}.$$

Thus for all $j \in \mathbb{Z}_+$ we have the bound

$$(1+2^{2k}s)^j \|\partial_x [P_k(S_{ij}^l(v) - S_{ij}^l(Q))]\|_{L_t^\infty L_x^2} \lesssim_j \epsilon_1. \tag{3.6}$$

Now, let us use [5, Lemma 8.3]'s arguments. Given $\sigma \in [0, \frac{99}{100}]$, define

$$B_{1,\sigma}(S) := \sup_{k\in\mathbb{Z},\, s\in[0,S)} \gamma_k^{-1}(\sigma)(1+s2^{2k})^{31} 2^{\sigma k} 2^k \|P_k v\|_{L_t^\infty L_x^2}.$$

By Lemma 3.1 and the fact $\{\gamma_k(\sigma)\}$ is a frequency envelope, $B_{1,\sigma}(S)$ is well-defined for $S \geq 0$ and continuous in S with

$$\lim_{S\to 0} B_{1,\sigma}(S) = 1.$$

Then by trilinear Littlewood–Paley decomposition (see (8.2) in Lemma 8.1), we have

$$\begin{aligned} 2^k \|P_k S_{ij}^l(f)\partial_a f^i \partial_a f^j\|_{L_t^\infty L_x^2} \lesssim 2^k \sum_{k_1\leq k} \mu_{k_1} 2^{k_1} \mu_k \\ + \sum_{k_2 \geq k} 2^{2k} \mu_{k_2}^2 + a_k \Big(\sum_{k_1\leq k} 2^{k_1}\mu_{k_1}\Big)^2 \\ + \sum_{k_2\geq k} 2^{2k} 2^{-k_2} a_{k_2} \mu_{k_2} \sum_{k_1\leq k_2} 2^{k_1}\mu_{k_1}, \end{aligned}$$

where

$$\begin{aligned} a_k &:= \sum_{|k-k'|\leq 20} \sum_{l,i,j=1}^N \|\partial_x P_{k'}(S_{ij}^l(v))\|_{L_t^\infty L_x^2}; \\ \mu_k &:= \sum_{l=1}^N \sum_{|k'-k|\leq 20} 2^{k'} \|P_{k'} v^l\|_{L_t^\infty L_x^2}. \end{aligned} \tag{3.7}$$

Then by definition of $B_1(S)$ and slow variation of envelopes, for $s \in [0,S], \sigma \in [0, \frac{99}{100}]$, we get

$$
\begin{aligned}
&2^k\|P_kS^l_{ij}(f)\partial_af^i\partial_af^j\|_{L^\infty_tL^2_x}\\
&\lesssim B_{1,\sigma}B_{1,0}(1+s2^{2k})^{-31}\gamma_k\sum_{k_1\le k}2^{k_1-\sigma k_1+k}\gamma_{k_1}(\sigma)\\
&\quad+B_{1,\sigma}B_{1,0}\sum_{k_2\ge k}2^{2k-\sigma k_2}(1+s2^{2k_2})^{-62}\gamma_{k_2}\gamma_{k_2}(\sigma)\\
&\quad+B_{1,\sigma}B_{1,0}2^ka_k\Big(\sum_{k_1\le k}2^{k_1}(1+s2^{2k_1})^{-31}\gamma_{k_1}\Big)\Big(\sum_{k_1\le k}2^{k_1-\sigma k_1}(1+s2^{2k_1}\Big)^{-31}\gamma_{k_1}(\sigma))\\
&\quad+B_{1,\sigma}B_{1,0}\sum_{k_2\ge k}2^{2k}(1+s2^{2k_2})^{-31}2^{-\sigma k_2}a_{k_2}\gamma_{k_2}(\sigma)\gamma_{k_2}\\
&\lesssim B_{1,\sigma}B_{1,0}(1+s2^{2k})^{-62}2^{2k-\sigma k}\gamma_k\gamma_k(\sigma)\\
&\quad+B_{1,\sigma}B_{1,0}\sum_{k_2\ge k}2^{2k-\sigma k_2}(1+s2^{2k_2})^{-31}\gamma_{k_2}\gamma_{k_2}(\sigma)\\
&+B_{1,\sigma}B_{1,0}\sum_{k_2\ge k}2^{-\sigma k_2}2^{2k}(1+s2^{2k_2})^{-31}a_{k_2}\gamma_{k_2}\gamma_{k_2}(\sigma)+a_k2^{-\sigma k}B_{1,\sigma}B_{1,0}2^{2k}\gamma_k\gamma_k(\sigma).
\end{aligned}
\tag{3.8}
$$

By applying (3.6) to $\{a_k\}$, we further bound

$$
\mathrm{RHS}(3.8)\lesssim 2^{-\sigma k}B_{1,\sigma}B_{1,0}2^{2k}\sum_{k_2\ge k}(1+s2^{2k_2})^{-31}\gamma_k\gamma_{k_2}(\sigma).
$$

Therefore, for $s\ge 0$, we conclude that the LHS of (3.8) satisfies

$$
2^k\|P_kS^l_{ij}(v)\partial_av^i\partial_av^j\|_{L^\infty_tL^2_x}\lesssim 2^{-\sigma k}2^{2k}B_{1,\sigma}B_{1,0}\sum_{k_2\ge k}(1+s2^{2k_2})^{-31}\gamma_{k_2}\gamma_{k_2}(\sigma).
$$

Hence by the Duhamel principle

$$
\begin{aligned}
(1+s2^{2k})^{31}2^{k+\sigma k}\|P_kv\|_{L^\infty_tL^2_x}&\lesssim(1+s2^{2k})^{31}e^{-s2^{2k}}2^{k+\sigma k}\|P_ku\|_{L^\infty_tL^2_x}\\
&\quad+B_{1,\sigma}B_{1,0}(1+s2^{2k})^{31}\int_0^se^{-(s-\tau)2^{2k}}2^{k+\sigma k}\|P_kS^l_{ij}(v)\partial_av^i\partial_av^j\|_{L^\infty_tL^2_x}\,d\tau
\end{aligned}
$$

and the inequality

$$
\int_0^se^{-(s-\tau)\lambda}(1+\tau\lambda_1)^{-31}\,d\tau\lesssim s(1+\lambda s)^{-31}(1+\lambda_1s)^{-1},\tag{3.9}
$$

we get

$$
\begin{aligned}
&(1+s2^{2k})^{31}2^{k+\sigma k}\|P_kv\|_{L^\infty_tL^2_x}\\
&\qquad\lesssim\gamma_k(\sigma)+B_{1,\sigma}B_{1,0}(S)2^{2k}s\sum_{k_2\ge k}\gamma_{k_2}\gamma_{k_2}(\sigma)(1+2^{2k_2}s)^{-1}\\
&\qquad\lesssim\gamma_k(\sigma)+B_{1,\sigma}(S)B_{1,0}(S)\epsilon_1\gamma_k(\sigma).
\end{aligned}
$$

Then $B_{1,0} \lesssim 1 + \epsilon_1 B_{1,0}^2$. Since $B_{1,0}(0) \le 1$ and ϵ_1 is sufficiently small, we have $B_{1,0}(S) \lesssim 1$ for all $S \ge 0$. Then using $B_{1,\sigma} \lesssim 1 + \epsilon_1 B_{1,0} B_{1,\sigma}$ and $B_{1,\sigma}(0) \le 1$, we get $B_{1,\sigma}(S) \lesssim 1$ for any $\sigma \in [0, \frac{99}{100}]$ and any $S \ge 0$ provided that ϵ_1 is sufficiently small. Thus (3.5) has been proved. ∎

Remark 3.2. The power of $1 + s2^{2k}$ in Proposition 3.2 can be chosen to be any $M \in \mathbb{Z}_+$ if we additionally assume that ϵ_1 is sufficiently small depending on M; see Proposition 3.4 below.

3.3. First time iteration

We state the first time iteration in the the following proposition.

Proposition 3.3. *Assume that $u \in \mathcal{H}_Q(T)$ satisfies (3.4), and let $v(s,t,x)$ be the solution of the heat flow (3.2) with initial data $u(t,x)$. Then for $\sigma \in (\frac{99}{100}, \frac{5}{4}]$ and any $k \in \mathbb{Z}$, v satisfies*

$$\sup_{s\in[0,\infty)} (1 + s2^{2k})^{30} 2^{k+\sigma k} \| P_k v \|_{L_t^\infty L_x^2} \lesssim \gamma_k(\sigma) + \gamma_k(\sigma - 3/8)\gamma_k(3/8). \tag{3.10}$$

Proof. The key point is to improve the bounds of $\{a_k\}$ defined by (3.7). For this, we use dynamic separation again. One has

$$S_{ij}^l(v)(s) = S_{ij}^l(Q) - \int_s^\infty (DS_{ij}^l)(v) \cdot \partial_s v \, ds'. \tag{3.11}$$

By Proposition 3.2, for $\sigma \in [0, \frac{99}{100}]$ and any $k \in \mathbb{Z}$, we get

$$2^{k+\sigma k} \| P_k \Delta v \|_{L_t^\infty L_x^2} \lesssim (2^{2k}s + 1)^{-31} 2^{2k} \gamma_k(\sigma),$$

and repeating the proof of Proposition 3.2 gives

$$\sum_{a=1,2} 2^{k+\sigma k} \| S_{ij}^l(\partial_a v^i, \partial_a v^j) \|_{L_t^\infty L_x^2} \lesssim 2^{2k} \sum_{k_1 \ge k} (2^{2k_1}s + 1)^{-31} \gamma_{k_1} \gamma_{k_1}(\sigma).$$

Thus, given $s \in [2^{2k_0-1}, 2^{2k_0+1})$, by the heat flow equation we get, for all $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$,

$$\begin{aligned} 2^{k+\sigma k} \| P_k \partial_s v \|_{L_t^\infty L_x^2} &\lesssim (2^{2k}s + 1)^{-31} 2^{2k} \gamma_k(\sigma) + \sum_{k_1 \ge k} (2^{2k}s + 1)^{-31} 2^{2k} \gamma_{k_1} \gamma_{k_1}(\sigma) \\ &\lesssim (2^{2k}s + 1)^{-31} 2^{2k} \gamma_k(\sigma) + 1_{k+k_0 \ge 0} (2^{2k}s + 1)^{-31} 2^{2k} \gamma_k \gamma_k(\sigma) \\ &\quad + 2^{2k} 1_{k+k_0 \le 0} \sum_{k \le l \le -k_0} \gamma_l \gamma_l(\sigma). \end{aligned} \tag{3.12}$$

Recall the bound

$$2^k \| P_k [(DS)(v)] \|_{L_t^\infty L_x^2} \lesssim \epsilon_1 (2^{2k}s + 1)^{-j} \tag{3.13}$$

for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then for $s \in [2^{2k_0-1}, 2^{2k_0+1})$, repeating bilinear arguments, (3.11) shows that if $k + k_0 \geq 0$ then

$$\begin{aligned}
&\|P_k[S^l_{ij}(v)(s)]\|_{L^\infty_t L^2_x}\\
&\quad\lesssim \int_s^\infty \sum_{|k_1-k|\leq 4} \|P_{\leq k-4}(DS(v))\|_{L^\infty_{t,x}} \|P_{k_1}\partial_s v\|_{L^\infty_t L^2_x}\, ds'\\
&\qquad + \int_s^\infty 2^k \sum_{|k_1-k_2|\leq 8,\, k_1,k_2\geq k-4} \|P_{k_2}(DS(v))\|_{L^\infty_t L^2_x} \|P_{k_1}\partial_s v\|_{L^\infty_t L^2_x}\, ds'\\
&\qquad + \int_s^\infty \sum_{|k_2-k|\leq 4,\, k_1\leq k-4} 2^{k_1} \|P_{k_2}(DS(v))\|_{L^\infty_t L^2_x} \|P_{k_1}\partial_s v\|_{L^\infty_t L^2_x}\, ds'\\
&\quad\lesssim 2^{-\sigma k - k}(2^{2k+2k_0}+1)^{-31} 2^{2k+2k_0}\gamma_k(\sigma)\gamma_k
\end{aligned} \tag{3.14}$$

provided $\sigma \in [0, \frac{99}{100}]$, where we applied (3.12) and (3.13) in the last line. Moreover, for any $\sigma \in [0, \frac{99}{100}]$, $k_0 \in \mathbb{Z}$, and $s \in [2^{2k_0-1}, 2^{2k_0+1})$, in the case $k + k_0 \leq 0$ one has

$$\|P_k[S^l_{ij}(v)(s)]\|_{L^\infty_t L^2_x} \lesssim \sum_{k_0\leq j\leq -k} 2^{-\sigma k} 2^{j+2\delta|k+j|}\gamma_k(\sigma)\gamma_k \lesssim 2^{-\sigma k-k}\gamma_k(\sigma)\gamma_k. \tag{3.15}$$

Thus (3.14) and (3.15) yield the following bounds for $\{a_k\}$:

$$2^{\sigma k} a_k \lesssim (1 + 2^{2k}s)^{-30}\gamma_k(\sigma)\gamma_k \tag{3.16}$$

provided that $\sigma \in [0, \frac{99}{100}]$. Now for a given $\sigma \in (\frac{99}{100}, \frac{5}{4}]$ define

$$B_{2,\sigma}(S) := \sup_{k\in\mathbb{Z},\, s\in[0,S)} (\gamma^{(1)}_k(\sigma))^{-1} 2^{\sigma k}(1 + s2^{2k})^{30} 2^k \|P_k v\|_{L^\infty_t L^2_x},$$

where

$$\gamma^{(1)}_k(\sigma) := \begin{cases} \gamma_k(\sigma), & \sigma \in \big[0, \frac{99}{100}\big],\\ \gamma_k(\sigma) + \gamma_k(\sigma - 3/8)\gamma_k(3/8), & \sigma \in \big(\frac{99}{100}, \frac{5}{4}\big]. \end{cases}$$

Moreover, by Lemma 3.1 and the fact that $\{\gamma^{(1)}_k(\sigma)\}$ is a frequency envelope of order 2δ, it is clear that $B_{2,\sigma} : [0,\infty) \to \mathbb{R}^+$ is well-defined and continuous with $\lim_{S\to 0} B_{2,\sigma}(S) = 1$. Then by trilinear Littlewood–Paley decomposition (see (8.2)), the definition of $B_{2,\sigma}$ and slow variation of envelopes, for $s \in [0, S]$ and $\sigma \in (\frac{99}{100}, \frac{5}{4}]$ we get

$$\begin{aligned}
&2^k \|P_k[S^l_{ij}(v)\partial_a v^i \partial_a v^j]\|_{L^\infty_t L^2_x}\\
&\lesssim B_{2,\sigma} B_{1,0} (1 + s2^{2k})^{-30} 2^{-\sigma k}\gamma^{(1)}_k(\sigma) \sum_{k_1\leq k} 2^{k_1+k}\gamma_{k_1}\\
&\quad + B_{2,\sigma} B_{1,0} \sum_{k_2\geq k} 2^{2k-\sigma k_2}(1 + s2^{2k_2})^{-60}\gamma_{k_2}\gamma^{(1)}_{k_2}(\sigma)\\
&\quad + B_{1,0} B_{1,3/8} a_k \Big(\sum_{k_1\leq k} 2^{k_1}(1+s2^{2k_1})^{-30}\gamma_{k_1}\Big)\Big(\sum_{k_1\leq k} 2^{k_1-\frac{3}{8}\sigma k_1}(1+s2^{2k_1})^{-30}\gamma_{k_1}(3/8)\Big)\\
&\quad + B_{2,\sigma} B_{1,0} \sum_{k_2\geq k} 2^{2k}(1 + s2^{2k_2})^{-30} 2^{-\sigma k_2} a_{k_2}\gamma^{(1)}_{k_2}(\sigma)\gamma_{k_2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim B_{2,\sigma} B_{1,0} \sum_{k_2\geq k} 2^{2k-\sigma k_2}(1+s2^{2k_2})^{-30}\gamma_{k_2}\gamma^{(1)}_{k_2}(\sigma)\\
&\quad + B_{2,\sigma} B_{1,0} \sum_{k_2\geq k} 2^{-\sigma k_2} 2^{2k}(1+s2^{2k_2})^{-30} a_{k_2}\gamma_{k_2}\gamma^{(1)}_{k_2}(\sigma)\\
&\quad + a_k 2^{-\frac{3}{8}\sigma k} B_{1,0} B_{1,3/8} 2^{2k}\gamma_k\gamma_k(3/8). \qquad (3.17)
\end{aligned}
$$

Then applying the trivial bound (3.6) to the RHS of (3.17) except the last term and applying (3.16) to $\{a_k\}$ in the last term, for all $\sigma \in (\frac{99}{100}, \frac{5}{4}]$ we get

$$
\begin{aligned}
2^{k+\sigma k}\|P_k[S^l_{ij}(v)\partial_a v^i\partial_a v^j]\|_{L^\infty_t L^2_x} &\lesssim B_{1,0}B_{2,\sigma}2^{2k}\sum_{k_2\geq k}(1+s2^{2k_2})^{-30}\gamma_{k_2}\gamma^{(1)}_{k_2}(\sigma)\\
&\quad + B_{1,0}B_{1,3/8}2^{2k}2^{-\sigma k}(1+s2^{2k})^{-30}\gamma_k(\sigma-3/8)\gamma_k(3/8)\gamma_k 1_{k+k_0\geq 0}\\
&\quad + B_{1,0}B_{1,3/8}2^{2k}2^{-\sigma k}2^{2\delta|k+k_0|}\gamma_k(\sigma-3/8)\gamma_k(3/8)\gamma_k 1_{k+k_0\leq 0}
\end{aligned}
$$

if $s \in [2^{2k_0-1}, 2^{2k_0+1})$. Then using the Duhamel principle, (3.9) and the inequality

$$
(1+2^{2k}s)^{30}e^{-2^{2k}s}\int_0^s e^{s'2^{2k}}(s'2^{2k})^{-\delta}1_{s'\leq 2^{-2k}}\,ds' \lesssim 2^{-2k},
$$

we obtain

$$
2^{k+\sigma k}(1+2^{2k}s)^{30}\|P_k v\|_{L^\infty_t L^2_x} \lesssim (1+\epsilon_1 B_{1,0}B_{1,3/8}+\epsilon_1 B_{2,\sigma}B_{1,0})\gamma^{(1)}_k(\sigma).
$$

Since $B_{1,\widetilde{\sigma}} \lesssim 1$ for $\widetilde{\sigma} \in [0, \frac{99}{100}]$ has been proved in Proposition 3.2, we arrive at

$$
B_{2,\sigma} \lesssim 1+\epsilon_1 B_{2,\sigma}, \quad \forall \sigma \in \left(\tfrac{99}{100}, \tfrac{5}{4}\right],
$$

which shows $B_{2,\sigma} \lesssim 1$, thus finishing the proof. ∎

We define the frequency envelope $\gamma^{(j)}_k(\sigma)$, $j=0,1$, by

$$
\gamma^{(0)}_k(\sigma) := \gamma_k(\sigma), \quad 0\leq\sigma<\tfrac{99}{100}, \qquad (3.18)
$$

$$
\gamma^{(1)}_k(\sigma) := \begin{cases} \gamma^{(0)}_k(\sigma), & 0\leq\sigma\leq\frac{99}{100},\\ \gamma_k(\sigma)+\gamma^{(0)}_k(\sigma-3/8)\gamma_k(3/8), & \frac{99}{100}<\sigma\leq\frac{5}{4}, \end{cases} \qquad (3.19)
$$

and the frequency envelopes $\gamma^{(j)}_k(\sigma)$, $j\geq 2$, are defined by induction:

$$
\gamma^{(j)}_k(\sigma) := \begin{cases} {\gamma_k}^{(j-1)}(\sigma), & 0\leq\sigma\leq (j+3)/4,\\ \gamma_k(\sigma)+\gamma^{(j-1)}_k(\sigma-3/8)\gamma_k(3/8), & (j+3)/4<\sigma\leq (j+4)/4. \end{cases} \qquad (3.20)
$$

For $j\in\mathbb{N}$ define the sequence $\{\gamma^{(j)}_{k,s}(\sigma)\}_{k\in\mathbb{Z}}$ by

$$
\gamma^{(j)}_{k,s}(\sigma) := \begin{cases} 2^{k+k_0}\gamma^{(j)}_k(\sigma)\gamma^{(j)}_{-k_0}(0), & k+k_0\geq 0,\\ \sum_{l=k}^{-k_0}\gamma^{(j)}_l(\sigma)\gamma^{(j)}_l(0), & k+k_0\leq 0, \end{cases} \qquad (3.21)
$$

for $s\in[2^{2k_0-1}, 2^{2k_0+1})$ and $k, k_0 \in \mathbb{Z}$.

We state the j-th time iteration in the following proposition.

Proposition 3.4 (j-th iteration). *Let $j \in \mathbb{N}$ and $M \in \mathbb{Z}_+$. Assume that $u \in \mathcal{H}_Q(T)$ satisfies (3.4) with ϵ_1 sufficiently small depending on $j + M$. Let $v(s,t,x)$ be the solution of the heat flow (3.2) with initial data $u(t,x)$. Then for $\sigma \in [0, 1 + j/4]$ and all $s \geq 0$,*

$$\sup_{s\in[0,\infty)} (1 + s2^{2k})^M 2^{k+\sigma k} \|P_k v\|_{L_t^\infty L_x^2} \lesssim \gamma_k^{(j)}(\sigma). \tag{3.22}$$

Proof. Define intervals $\{\mathbb{I}_l\}_{l=0}^\infty$ by

$$\mathbb{I}_0 := \big[0, \tfrac{99}{100}\big], \quad \mathbb{I}_1 = \big(\tfrac{99}{100}, \tfrac{5}{4}\big], \quad \mathbb{I}_l = \big(\tfrac{3+l}{4}, \tfrac{4+l}{4}\big], \quad l \geq 2.$$

Given $K \in \mathbb{Z}_+$ and $\sigma \in \mathbb{I}_l$, $l \in \mathbb{N}$, we denote

$$B_{l+1,\sigma,K}(S) := \sup_{s\in[0,S),\, k\in\mathbb{Z}} \frac{1}{\gamma_k^{(l)}(\sigma)} (1 + s2^{2k})^K 2^{k+\sigma k} \|P_k v\|_{L_t^\infty L_x^2},$$

and let

$$\mathbb{B}_{l+1,K}(S) := \sup_{\sigma\in\bigcup_{\ell\leq l}\mathbb{I}_\ell} B_{l+1,\sigma,K}(S).$$

(In this notation, Propositions 3.2 and 3.3 yield $\mathbb{B}_{2,30}(S) \lesssim 1$.)

Moreover, the argument of Proposition 3.3 indeed shows:

(i) For all $K_0 \geq 2$, $j \in \mathbb{N}$ and $0 \leq a \leq j + 1$,

$$2^k \|P_k[(D^a S)(v)]\|_{L_t^\infty L_x^2} \lesssim C_{K_0,j}\epsilon(1 + 2^{2k}s)^{-K_0-(j+1)}.$$

(ii) For all $K_0 \geq 2$ and $j \in \mathbb{N}$, if

$$\begin{cases} 2^k \|P_k[(D^{j+1}S)(v)]\|_{L_t^\infty L_x^2} \lesssim \epsilon(1 + 2^{2k}s)^{-K_0-j-1}, \\ 2^k \|P_k v\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\gamma_k^{(0)}(\sigma)(1 + 2^{2k}s)^{-K_0-j-1}, \end{cases}$$

then

$$2^k \|P_k[(D^j S)(v)]\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\gamma_k^{(0)}(\sigma)(1 + 2^{2k}s)^{-K_0-j},$$

where the implicit constant in the conclusion is of the form $C(1 + C_1^2 + C_2^2)$ if we denote by C_1, C_2 the implicit constants in the conditions of (ii). Here, C is universal and C_1, C_2 may depend on j, K_0.

(iii) For all $K_0 \geq 2$ and $j \in \mathbb{N}$, $0 \leq a \leq j + 1$, if

$$\begin{cases} 2^k \|P_k[(D^{a+1}S)(v)]\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\gamma_k^{(j-(a+1))}(\sigma)(1 + 2^{2k}s)^{-K_0-(a+1)}, \\ 2^k \|P_k v\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\gamma_k^{(j-a)}(\sigma)(1 + 2^{2k}s)^{-K_0-(a+1)}, \end{cases}$$

then

$$2^k \|P_k[(D^a S)(v)]\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\gamma_k^{(j-a)}(\sigma)(1 + 2^{2k}s)^{-K_0-a},$$

where the implicit constant in the conclusion is of the form $C(1 + C_1^2 + C_2^2)$ if we denote by C_1, C_2 the implicit constants in the conditions of (iii). Here, C is universal and C_1, C_2 may depend on j, K_0.

(iv) For any $K \geq 2$, $j \geq 1$, $1 \leq a \leq j+1$ and $\sigma \in \mathbb{I}_a$, if

$$\begin{cases} 2^k \|P_k[S(v)]\|_{L_t^\infty L_x^2} \leq C_K 2^{-\sigma k}\gamma_k^{(a-1)}(\sigma)(1+2^{2k}s)^{-K}, \\ 2^k \|P_k v\|_{L_t^\infty L_x^2} \leq B_{a+1,\sigma,K} 2^{-\sigma k}\gamma_k^{(a)}(\sigma)(1+2^{2k}s)^{-K}, \end{cases}$$

then for all $S \in [0,\infty)$,

$$B_{a+1,\sigma,K}(S) \leq C_*\Big(1 + \epsilon_1 \mathbb{B}_{a,K} B_{a+1,\sigma,K}(S) + C_K \sum_{l=1}^{a} \sup_{S\in[0,\infty)} \mathbb{B}^2_{l,K+l+1}(S)\Big),$$

where C_* depends only on d and emerges from trilinear Littlewood–Paley decomposition. Then our proposition follows by iteration. To be concrete, we make several remarks. First, in order to get the M-power decay in (3.22), it suffices to set $K_0 = M+4$ and the top derivative order involved is $D^{j+1}S$. Second, let us describe the iteration in a clearer way: In the first step, one verifies

$$\sup_{S\in[0,\infty)} B_{1,K_0+j+1}(S) \leq C_{K_0,j}, \tag{3.23}$$

i.e. the second conditions in (ii). This was presented in Proposition 3.2. (We emphasize that in this step, ϵ_1 shall be sufficiently small depending on K_0+j.) In the second step, one verifies $\sup_{S\in[0,\infty)} B_{2,K_0+j}(S) \leq C_{K_0,j}$, and in the a-th step one verifies $\sup_{S\in[0,\infty)} B_{a,K_0+j+2-a}(S) \leq C_{K_0,j}$. This is presented as (iii) and (iv). Thus in the j-th step, we get (3.22). ∎

3.4. Rough dynamical separation

Recall the notations $\psi_i^\alpha = \langle \partial_i v, e_\alpha\rangle$, $\psi_i^{\bar\alpha} = \langle \partial_i v, Je_\alpha\rangle$, $\alpha = 1,\ldots,n$, $i = 0,1,2,3$, and $\phi_i^\alpha = \psi_i^\alpha + \sqrt{-1}\,\psi_i^{\bar\alpha}$. Here, $i=0$ refers to the t variable and $i=3$ refers to the s variable.

We aim to bound the connection coefficients in the localized frequency spaces. As a preparation, we first derive a suitable form of the connection coefficients. By definitions, we see

$$\begin{aligned} \mathbf{R}(\mathbf{E}\phi_i, \mathbf{E}\phi_s) &= \mathbf{R}\big((\Re\phi_i^\alpha)e_\alpha + (\Im\phi_i^\alpha)e_{\bar\alpha}, (\Re\phi_s^\alpha)e_\beta + (\Im\phi_s^\alpha)e_{\bar\beta}\big) \\ &= (\phi_i^\alpha \wedge \phi_s^\beta)\mathbf{R}(e_\alpha, e_{\bar\beta}) + (\phi_i^\alpha \cdot \phi_s^\beta)\mathbf{R}(e_\alpha, e_\beta), \end{aligned}$$

where we denote $z_1 \wedge z_2 = -\Im(z_1\overline{z_2})$, $z_1\cdot z_2 = \Re z_1 \Re z_2 + \Im z_1 \Im z_2$ for complex numbers z_1, z_2. Thus schematically under the frame $\mathbf{E} = \{e_\alpha, e_{\bar\alpha}\}_{\alpha=1}^n$ we can write

$$\begin{cases} [A_i]_\theta^\gamma = \sum \int_s^\infty (\phi_i^\alpha \diamond \phi_s^\beta)\langle \mathbf{R}(e_\alpha, e_{\beta,\bar\beta})e_\gamma, e_\theta\rangle\, ds', \\ [A_i]_{\bar\theta}^\gamma = \sum \int_s^\infty (\phi_i^\alpha \diamond \phi_s^\beta)\langle \mathbf{R}(e_\alpha, e_{\beta,\bar\beta})e_\gamma, e_{\bar\theta}\rangle\, ds', \end{cases} \tag{3.24}$$

where $\diamond =$ "$\wedge$" when $e_{\beta,\bar\beta} = e_{\bar\beta}$, and $\diamond =$ "$\cdot$" when $e_{\beta,\bar\beta} = e_\beta$. For simplicity, with abuse of notation we schematically write

$$A_i(s) = \sum_{j_0,j_1,j_2,j_3} \int_s^\infty (\phi_i \diamond \phi_s)\langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle\, ds',$$

where $\{j_c\}_{c=0}^3$ run in $\{1,\ldots,2n\}$, and i runs in $\{0,1,2\}$. Recall also that $\phi_s = \sum_{l=1}^2 D_l\phi_l$.

With abuse of notation, set

$$\mathscr{G}(s) := \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle(s) \tag{3.25}$$

for any given $j_0, \dots, j_3 \in \{1, 2, \dots, 2n\}$. We expand $\mathscr{G}$ as

$$\begin{aligned}\langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle(s) &= \lim_{s\to\infty} \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle - \int_s^\infty \partial_s \langle \mathbf{R}(e_{j_0}, e_{j_1})e_{j_2}, e_{j_3}\rangle \, ds' \\ &= \Gamma^\infty - \int_s^\infty \psi_s^l (\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \dots, e_{j_3}) \, ds',\end{aligned}$$

where Γ^∞ denotes the limit part which is constant, and we have used the identity $\nabla_s e_p = 0$ for all $p = 1, \dots, 2n$ in the last line. Here, we view $\mathbf{R}$ as a type $(0, 4)$ tensor.

With the above notations, we write

$$A_i(s) = \sum_{j_0, j_1, j_2, j_3} \int_s^\infty (\phi_i \diamond \phi_s)\mathscr{G} \, ds', \tag{3.26}$$

and $\mathscr{G}$ is decomposed as

$$\mathscr{G} = \Gamma^\infty - \int_s^\infty \psi_s^l (\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \dots, e_{j_3}) \, ds'.$$

Of course, one can perform this separation for any time desired. Denote

$$\mathscr{G}^{(j)} := (\widetilde{\nabla}^j \mathbf{R})(\underbrace{e, \dots, e}_{j}; e_{j_0}, \dots, e_{j_3}), \quad \Gamma^{\infty,(j)} := \lim_{s\to\infty} \mathscr{G}^{(j)}(s).$$

Then we can schematically write

$$\mathscr{G} = \Gamma^\infty - \int_s^\infty \psi_s(s_1) \, ds_1 \Big(\Gamma^{\infty,(1)} - \int_{s_1}^\infty \psi_s(s_2) \, ds_2 \, (\Gamma^{\infty,(2)} + \cdots) \Big).$$

For simplicity we also denote

$$\widetilde{\mathscr{G}} := \mathscr{G} - \Gamma^\infty, \quad \widetilde{\mathscr{G}}^{(j)} := \mathscr{G}^{(j)} - \Gamma^{\infty,(j)}.$$

3.5. Intrinsic vs. extrinsic formulations in localized frequency pieces

Proposition 3.5. *Let $u \in \mathscr{H}_Q(T)$ satisfy*

$$\|\partial_x u\|_{L_t^\infty L_x^2} = \epsilon_1 \ll 1. \tag{3.27}$$

Here, we do not require u to solve SMF. Denote by $v(s, t, x)$ the solution to the heat flow with data $u(t, x)$, and by $\{\phi_i\}$ the corresponding differential fields under the caloric gauge. Assume that $\{\eta_k(\sigma)\}$ is a frequency envelope of order δ such that for all $i = 1, 2$ and $k \in \mathbb{Z}$,

$$2^{\sigma k} \| P_k \phi_i \restriction_{s=0} \|_{L_t^\infty L_x^2} \le \eta_k(\sigma). \tag{3.28}$$

Then

$$\gamma_k(\sigma) \lesssim \eta_k(\sigma), \tag{3.29}$$

$$(1+s2^{2k})^{30}2^{\sigma k}\|P_k A_i\|_{L_t^\infty L_x^2} \lesssim \eta_{k,s}^{(0)}(\sigma), \tag{3.30}$$

for any $\sigma \in [0, \frac{99}{100}]$ *and* $k \in \mathbb{Z}$. *Furthermore, assume that for* $\sigma \in [0, \frac{5}{4}]$, $\{\eta_k(\sigma)\}$ *is a frequency envelope of order* $\frac{1}{2}\delta$ *such that for all* $i = 1, 2$ *and* $k \in \mathbb{Z}$, (3.28) *holds. Then for any* $\sigma \in [0, \frac{5}{4}]$ *and* $k \in \mathbb{Z}$,

$$\gamma_k^{(1)}(\sigma) \lesssim \eta_k^{(1)}(\sigma), \tag{3.31}$$

$$(1+s2^{2k})^{29}2^{\sigma k}\|P_k A_i\|_{L_t^\infty L_x^2} \lesssim \eta_{k,s}^{(1)}(\sigma). \tag{3.32}$$

Proof. **Step 1.1:** $\sigma \in [0, \frac{99}{100}]$**.** Let $\mathcal{P}: \mathcal{N} \to \mathbb{R}^N$ be the isometric embedding. By definition, we see

$$\partial_i v = \sum_{l=1}^{2n} \psi^l d\mathcal{P}(e_l) = \sum_{l=1}^{2n} \psi_i^l \chi_l^\infty + \sum_{l=1}^{2n} \psi_i^l (d\mathcal{P}(e_l) - \chi_l^\infty), \tag{3.33}$$

where $\{\chi_l^\infty\}$ are the corresponding limits of $d\mathcal{P}(e_l)$ as $s \to \infty$, which are constant vectors belonging to $\mathbb{R}^N$. Denote

$$\omega_k(s) := \sum_{|k-k'|\le 20} \|P_{k'}\psi_i(s)\|_{L_t^\infty L_x^2},\ \nu_k(s) = \sum_{|k-k'|\le 20} 2^{k'}\|P_{k'}(d\mathcal{P}(e_l) - \chi_l^\infty)\|_{L_t^\infty L_x^2}. \tag{3.34}$$

Then we see by Lemma 2.2 that

$$\|\{\nu_k\}\|_{\ell^2} \lesssim \|\partial_i(d\mathcal{P}(e_l) - \chi_l^\infty)\|_{L_x^2} \lesssim \|\partial_i v\|_{L_x^2} + \|A_i\|_{L_x^2} \lesssim \epsilon_1.$$

Moreover, direct calculations give the inequality

$$\|\partial_x^L(d\mathcal{P}(e_l) - \chi_l^\infty)\|_{L_t^\infty L_x^2} \lesssim \sum_{0\le p,q\le L} \sum_{\mathcal{A}} |\partial_x^{\alpha_1}\phi_x|^{l_1}\dots|\partial_x^{\alpha_p}\phi_x|^{l_p}|\partial_x^{\beta_1}A|^{n_1}\dots|\partial_x^{\beta_q}A|^{n_q}, \tag{3.35}$$

where $\mathcal{A}$ is the set of nonnegative indices $l_1, \dots, n_q \in \mathbb{Z}$ and $(\alpha_1, \dots, \beta_q) \in \mathbb{Z}^2 \times \dots \times \mathbb{Z}^2$ which satisfy

$$l_1(|\alpha_1|+1) + \dots + l_p(|\alpha_p|+1) + n_1(|\beta_1|+1) + \dots + n_q(|\beta_q|+1) = L.$$

Suppose $l_1 \ge 1$. By Hölder and Lemma 2.2, we get

$$\begin{aligned}
&\|\partial_x^L(d\mathcal{P}(e_l) - \chi_l^\infty)\|_{L_x^2} \\
&\lesssim \epsilon_1 \sum s^{-\alpha_1/2} s^{-(l_1-1)\alpha_1+1/2} s^{-l_2(\alpha_2+1)/2-\dots-l_p(\alpha_p+1)/2} s^{-(|\beta_1|+1)n_1/2-\dots-(|\beta_q|+1)n_q/2} \\
&\lesssim \epsilon_1 s^{-(L-1)/2}.
\end{aligned}$$

Suppose that $n_1 \geq 1$. Then we also obtain the same bound as above. Thus we arrive at

$$\|\{(1+s2^{2k})^M \nu_k(s)\}\|_{\ell^2} \lesssim_M \epsilon_1 \tag{3.36}$$

for $M \in \mathbb{Z}_+$. Meanwhile, we see $\|d\mathcal{P}(e_l) - \chi_l^\infty\|_{L^\infty} \lesssim 1$. Thus by [5, (8.4)], we obtain

$$2^k \|P_k(\partial_i v)\|_{L_t^\infty L_x^2} \lesssim 2^k \omega_k + \nu_k \sum_{k_1 \leq k} \omega_{k_1} 2^{k_1} + \sum_{k_1 \geq k} 2^{-2|k_1-k|} \omega_{k_1} 2^{k_1} \nu_{k_1}. \tag{3.37}$$

Since $\omega_k(0) \leq 2^{-\sigma k}\eta_k(\sigma)$, by slow variation of envelopes one deduces

$$\begin{aligned}2^k \|P_k(\partial_i v)\|_{L_t^\infty L_x^2} &\lesssim 2^k 2^{-\sigma k}\eta_k(\sigma) + \nu_k \sum_{k_1 \leq k} 2^{k_1+\delta|k-k_1|} 2^{-\sigma k_1}\eta_{k_1}(\sigma)\\ &\quad + \sum_{k_1 \geq k} 2^{-2(k_1-k)} 2^{\delta|k_1-k|} 2^{k_1} \nu_{k_1} 2^{-\sigma k_1}\eta_{k_1}(\sigma)\\ &\lesssim 2^k 2^{-\sigma k}\eta_k(\sigma)(1+\epsilon_1)\end{aligned}$$

for $\sigma \in [0, \frac{99}{100}]$ and $s = 0$. Thus since $\{\eta_k(\sigma)\}$ is an envelope, by the definition of $\{\gamma_k(\sigma)\}$ we obtain

$$\gamma_k(\sigma) \lesssim \eta_k(\sigma). \tag{3.38}$$

Hence, (3.29) has been proved for $\sigma \in [0, \frac{99}{100}]$.

Step 1.2: $\sigma \in (\frac{99}{100}, \frac{5}{4}]$. Recall $\mathcal{P} : \mathcal{N} \hookrightarrow \mathbb{R}^N$ is the given isometric embedding. Viewing $d\mathcal{P}$ as a section of $T^*\mathcal{N} \otimes T\mathbb{R}^N$, the connection on $\mathcal{N}$ induces a covariant derivative $\mathbf{D}$ on the bundle $T^*\mathcal{N} \otimes T\mathbb{R}^N$. We have the identity

$$d\mathcal{P}(e_l) - \chi_l^\infty = -\int_s^\infty \psi_s^j \mathbf{D} d\mathcal{P}(e_j; e_l)\, ds'. \tag{3.39}$$

where we use the caloric condition $\nabla_s e_l = 0$ for all $l = 1, \ldots, 2n$. Similar to (3.36), direct calculations give

$$\|P_k(\mathbf{D} d\mathcal{P}(e_j; e_l))\|_{L_t^\infty L_x^2} \lesssim_M \epsilon_1 2^{-k}(1+s2^{2k})^{-M} \tag{3.40}$$

for any $M \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$.

By (3.12), we have the bound for $\partial_s v$:

$$2^{\sigma k+k}\|P_k(\partial_s v)\|_{L_t^\infty L_x^2} \lesssim 2^{2k}\Big[(1+2^{2k}s)^{-31} 1_{k+k_0 \geq 0}\gamma_k(\sigma) + \sum_{k \leq l \leq -k_0} \gamma_l(\sigma)\gamma_l\Big] \tag{3.41}$$

if $s \in [2^{2k_0-1}, 2^{2k_0+1})$ with $k_0 \in \mathbb{Z}$. And using the identity $\psi_s^l = (d\mathcal{P}e_l)\cdot \partial_s v$, (3.36) and (3.41) instead yield

$$\|P_k \psi_s\|_{L_t^\infty L_x^2} \lesssim 2^{k-\sigma k}\Big(1_{k+k_0 \geq 0}(1+s2^{2k})^{-31}\gamma_k(\sigma) + 1_{k+k_0 \leq 0} \sum_{k \leq l \leq -k_0} \gamma_l(\sigma)\gamma_l\Big) \tag{3.42}$$

for all $k \in \mathbb{Z}$, $\sigma \in [0, \frac{99}{100}]$, $s \in [2^{2k_0-1}, 2^{2k_0+1})$ and $k_0 \in \mathbb{Z}$.

Then applying bilinear Littlewood–Paley decomposition to (3.39), (3.42) and (3.40) shows that for any $\sigma \in [0, \frac{99}{100}]$,

$$(1 + s2^{2k})^{30} 2^k \| P_k (d\mathcal{P}(e_l) - \chi_l^\infty) \|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} \gamma_k(\sigma) \lesssim 2^{-\sigma k} \eta_k(\sigma), \tag{3.43}$$

where we have applied (3.38) in the last inequality. Then by (3.33), (3.43) and bilinear Littlewood–Paley decomposition, when $s = 0$ one has

$$\begin{aligned} \| P_k \partial_i v \|_{L_t^\infty L_x^2} &\lesssim \| P_k \psi_i \|_{L_t^\infty L_x^2} \| P_{\le k-4} [d\mathcal{P}(e_l) - \chi_l^\infty] \|_{L^\infty} \\ &\quad + \| P_k [d\mathcal{P}(e_l) - \chi_l^\infty] \|_{L_t^\infty L_x^2} \sum_{k_1 \le k-4} 2^{k_1} \| P_{k_1} \psi_i \|_{L_t^\infty L_x^2} \\ &\quad + 2^k \sum_{|k_1 - k_2| \le 8, k_1, k_2 \ge k-4} \| P_{k_1} \psi_i \|_{L_t^\infty L_x^2} \| P_k [d\mathcal{P}(e_l) - \chi_l^\infty] \|_{L_t^\infty L_x^2} \\ &\lesssim 2^{-\sigma k} \eta_k(\sigma) + 2^{-\sigma k} \eta_k(3/8) \eta_k(\sigma - 3/8). \end{aligned} \tag{3.44}$$

Thus (3.44) gives $\gamma_k^{(1)}(\sigma) \lesssim \eta_k^{(1)}(\sigma)$ for $\sigma \in (\frac{99}{100}, \frac{5}{4}]$. Combining Steps 1.1 and 1.2, we have proved (3.31) and (3.29).

Step 2.1: Bounds of connections for $\sigma \in [0, \frac{99}{100}]$**.** Applying Proposition 3.1 gives

$$(1 + 2^{2k} s)^{31} 2^{\sigma k + k} \| P_k v \|_{L_t^\infty L_x^2} \lesssim \gamma_k(\sigma) \tag{3.45}$$

for all $\sigma \in [0, \frac{99}{100}]$ and $s \ge 0$. Then using the identity $\psi_i^l = (d\mathcal{P} e_l) \cdot \partial_i v$ and the bounds (3.36), (3.29), (3.45), we infer from bilinear Littlewood–Paley decomposition that

$$2^{\sigma k} \| P_k \phi_i \|_{L_t^\infty L_x^2} \lesssim (1 + s2^{2k})^{-31} \eta_k(\sigma)$$

for all $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$. Then, by (3.41), applying bilinear Littlewood–Paley decomposition again gives

$$\| P_k (\phi_s \diamond \phi_i) \|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} (1 + 2^{2j+2k})^{-31} \big(2^{-j+k} \eta_{-j} \eta_k(\sigma) + 2^{-2j} \eta_{-j} \eta_{-j}(\sigma) \big) \tag{3.46}$$

for $j + k \ge 0$, $s \in [2^{2j-1}, 2^{2j+1})$ and $\sigma \in [0, \frac{99}{100}]$.

Recall that Section 3.4 shows A_i can be written in the form

$$A_i(s) = \int_s^\infty (\phi_s \diamond \phi_i) \mathcal{G} \, ds'. \tag{3.47}$$

Direct calculations and Lemma 2.2 imply that $\mathcal{G}$ modulo the constant part Γ^∞ satisfies

$$2^k \| P_k \widetilde{\mathcal{G}} \|_{L_t^\infty L_x^2} \lesssim_{M_1} (1 + s2^{2k})^{-M_1} \epsilon_1 \tag{3.48}$$

for all $M_1 \in \mathbb{Z}_+$, $k \in \mathbb{Z}$ and $s \ge 0$. Then applying bilinear Littlewood–Paley decomposition and (3.46) leads to

$$\int_s^\infty \| P_k ((\phi_i \diamond \phi_s) \mathcal{G}) \|_{L_t^\infty L_x^2} \, ds' \lesssim 2^{-\sigma k} (1 + s2^{2k})^{-30} \eta_k(\sigma) \eta_{-j} \tag{3.49}$$

for all $j \in \mathbb{Z}$, $s \in [2^{2j-1}, 2^{2j+1})$ and $k + j \geq 0$. Moreover, similarly one has

$$\int_s^\infty \|P_k((\phi_i \diamond \phi_s)\mathcal{G})\|_{L_t^\infty L_x^2}\, ds' \lesssim 2^{-\sigma k} \sum_{k \leq l \leq -j} \eta_l(\sigma)\eta_l$$

for any $j \in \mathbb{Z}$, $s \in [2^{2j-1}, 2^{2j+1})$ and $k + j \leq 0$.

Hence, (3.30) is proved.

Step 2.2: $\sigma \in (\frac{99}{100}, \frac{5}{4}]$. Recall that (3.29) and (3.31) have given

$$\gamma_k(\sigma) \lesssim \eta_k(\sigma), \quad \gamma_k^{(1)}(\sigma) \lesssim \eta_k^{(1)}(\sigma).$$

Now, we are ready to estimate A_x for $\sigma \in (\frac{99}{100}, \frac{5}{4}]$. By the identity $\psi_i^l = (d\mathcal{P}e_l) \cdot \partial_i v$, the bound (3.43) and

$$\|P_k \partial_i v\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}(1 + s2^{2k})^{-30}\eta_k^{(1)}(\sigma),$$

one obtains, by bilinear Littlewood–Paley decomposition,

$$\|P_k \phi_i\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}(1 + s2^{2k})^{-30}\eta_k^{(1)}(\sigma) \tag{3.50}$$

for any $\sigma \in (\frac{99}{100}, \frac{5}{4}]$, $k \in \mathbb{Z}$ and $s \geq 0$. For any $\sigma \in (\frac{99}{100}, \frac{5}{4}]$, the proof of Proposition 3.3 yields the bound

$$\|P_k \partial_s v\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k + k}\Big[1_{k+j\geq 0}(1 + s2^{2k})^{-30}\eta_k^{(1)}(\sigma) + 1_{k+j\leq 0} \sum_{k\leq l\leq -j} \eta_l^{(1)}(\sigma)\eta_l\Big],$$

which combined with (3.43) gives

$$\|P_k \phi_s\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k + k}\Big[1_{k+j\geq 0}(1 + s2^{2k})^{-30}\eta_k^{(1)}(\sigma) + 1_{k+j\leq 0} \sum_{k\leq l\leq -j} \eta_l^{(1)}(\sigma)\eta_l\Big] \tag{3.51}$$

for any $k \in \mathbb{Z}$, $s \in [2^{2j-1}, 2^{2j+1})$, $j \in \mathbb{Z}$ and $\sigma \in (\frac{99}{100}, \frac{5}{4}]$.

In order to apply (3.47), we also need to improve the bound of $\widetilde{\mathcal{G}}$ stated in (3.48). Recall the formula

$$\mathcal{G} := \langle \mathbf{R}(e_{j_0}, e_{j_1})(e_{j_2}), e_{j_3}\rangle = \Gamma^\infty - \int_s^\infty \psi_s^p (\widetilde{\nabla}\mathbf{R})(e_p; e_{j_0}, \ldots, e_{j_3})\, ds'.$$

By Lemma 2.2 and the direct calculations (see Step 1.1 for instance) we have the bounds:

$$2^k \|P_k((\widetilde{\nabla}\mathbf{R})(e_l; e_{j_0}, \ldots, e_{j_3}) - \Gamma_l^{\infty,(1)})\|_{L_t^\infty L_x^2} \lesssim_M (1 + s2^{2k})^{-M} \tag{3.52}$$

for all $M \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. Hence by (3.42) and bilinear Littlewood–Paley decomposition,

$$2^k \|P_k(\mathcal{G} - \Gamma^\infty)\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}(1 + s2^{2k})^{-30}\gamma_k(\sigma) \tag{3.53}$$

for any $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$.

Then by (3.53), (3.51), (3.50), using trilinear Littlewood–Paley decomposition as in Step 2.1 with additional modifications in the Low × High interaction of $P_k((\phi_s \diamond \phi_i)\widetilde{\mathcal{G}})$ (see Proposition 3.2 for instance), we conclude that

$$\int_s^\infty \|P_k((\phi_s \diamond \phi_i)\mathcal{G})\|_{L_t^\infty L_x^2}\, ds' \lesssim 2^{-\sigma k}(1+s2^{2k})^{-29}\eta_{k,s}^{(1)}(\sigma)$$

for any $k \in \mathbb{Z}$ and $\sigma \in (\frac{99}{100}, \frac{5}{4}]$. Thus, (3.32) is proved. ■

Lemma 3.2. *Let $j, M \in \mathbb{Z}_+$ and $u \in \mathcal{H}_Q(T)$, and let $v(s,t,x)$ be the solution of the heat flow with data $u(t,x)$. Denote by $\{\phi_i\}$ the differential fields of v under the caloric gauge. Assume that $\sigma \in [0, 1 + j/4]$, and $\{\eta_k(\sigma)\}$ are frequency envelopes of order $\frac{1}{2^j}\delta$ such that for all $i = 1, 2$ and $k \in \mathbb{Z}$,*

$$2^{\sigma k}\|P_k\phi_i\restriction_{s=0}\|_{L_t^\infty L_x^2} \le \eta_k(\sigma). \tag{3.54}$$

Then, given $j, M \in \mathbb{Z}_+$, there exists a sufficiently small constant ϵ_j depending only on M, j such that if $\|\nabla u\|_{L_t^\infty L_x^2} \le \epsilon_j$, then

$$\gamma_k^{(j)}(\sigma) \lesssim \eta_k^{(j)}(\sigma) \tag{3.55}$$

for $\sigma \in [0, 1 + j/4]$ and $k \in \mathbb{Z}$. Moreover, for $l = 0, \ldots, j$, we have

$$(1+s2^{2k})^M\|P_k\phi_i\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\eta_k^{(j)}(\sigma), \qquad i = 1, 2,\ \sigma \in [0, 1 + j/4), \tag{3.56}$$

$$(1+s2^{2k})^M 2^k\|P_k[\widetilde{d\mathcal{P}}]^{(l)}\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\eta_k^{(j-l)}(\sigma), \quad \sigma \in [0, 1 + (j-l)/4), \tag{3.57}$$

$$(1+s2^{2k})^M 2^k\|P_k\widetilde{\mathcal{G}}^{(l)}\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\eta_k^{(j-l)}(\sigma), \quad \sigma \in [0, 1 + (j-l)/4), \tag{3.58}$$

$$(1+s2^{2k})^M\|P_kA_i\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\eta_{k,s}^{(j)}(\sigma), \qquad \sigma \in [0, 1 + j/4), \tag{3.59}$$

where we denote $[d\mathcal{P}]^{(l)} = (\mathbf{D}^l d\mathcal{P})(\underbrace{e, \ldots, e}_{l}; e)$, and $[\widetilde{d\mathcal{P}}]^{(l)} = [d\mathcal{P}]^{(l)} - \lim_{s\to\infty}[d\mathcal{P}]^{(l)}$.

Proof. The case $\sigma \in [0, \frac{5}{4}]$ has been handled in Proposition 3.5. Let $\sigma \in [1 + j/4, 1 + (j+1)/4]$. The general case of (3.55) follows by iteration. The highest covariant derivative order of $\mathcal{G}$ and $d\mathcal{P}(e)$ one needs for the j-th iteration is $j+1$, and it suffices to take the decay power $M + 2 + j$, i.e.,

$$\|\partial_x^{L+1}\mathcal{G}^{(j+1)}\|_{L_t^\infty L_x^2} \lesssim_{L,j} \epsilon s^{-L/2}, \quad \forall L \in [0, M+2+j],$$
$$\|\partial_x^{L+1}[d\mathcal{P}]^{(j+1)}\|_{L_t^\infty L_x^2} \lesssim_{L,j} \epsilon s^{-L/2}, \quad \forall L \in [0, M+2+j],$$

where we denote

$$\mathcal{G}^{(k)} := (\widetilde{\nabla}^k\mathbf{R})(\underbrace{e, \ldots, e}_{k}; \underbrace{e, \ldots, e}_{4}) \quad \text{and} \quad [d\mathcal{P}]^{(k)} := (\mathbf{D}^k d\mathcal{P})(\underbrace{e, \ldots, e}_{k}; e).$$

These decay estimates are easy to check by using Lemma 2.2. If these decay estimates along the heat direction are verified, then (3.55) follows by repeating (j times) the arguments of Step 2 in Proposition 3.5. Moreover, (3.56)–(3.59) follow along the same lines by applying dynamical separation and j-fold iteration. ∎

Similar to Proposition 3.5, one also has

Corollary 3.1. *Let $v_0 \in \mathcal{H}_Q$ satisfy*

$$\|\partial_x v_0\|_{L^2_x} = \epsilon_1 \ll 1.$$

- *Let $\{d_k(\sigma)\}$ with $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{5}{4}]$ be frequency envelopes of order $\frac{1}{2}\delta$ satisfying*

$$2^{\sigma k+k}\|P_k v_0\|_{L^2_x} \le d_k(\sigma). \tag{3.60}$$

Denote by $v(s,x)$ the solution to the heat flow with data v_0, and denote by $\{\phi_i\}$ the corresponding differential fields under the caloric gauge. Then

$$\|P_k\phi_i\restriction_{s=0}\|_{L^2_x} \le 2^{-\sigma k} d^{(1)}_k(\sigma). \tag{3.61}$$

- *Let $j \in \mathbb{Z}_+$. Assume that $\{d_k(\sigma)\}$ with $k \in \mathbb{Z}$ and $\sigma \in [0, 1 + j/4]$ are frequency envelopes of order $\frac{1}{2^j}\delta$. Then for ϵ_1 sufficiently small depending only on j, similar results hold with $d^{(1)}_k(\sigma)$ replaced by $d^{(j)}_k(\sigma)$.*

Proof. By (3.60), Propositions 3.2 and 3.3 show that for $\sigma \in [0, \frac{5}{4}]$,

$$(1+s2^{2k})^{30} 2^{k+\sigma k}\|P_k v\|_{L^2_x} \lesssim d^{(1)}_k(\sigma). \tag{3.62}$$

Let us first consider $\sigma \in [0, \frac{99}{100}]$. Recall

$$2^k\|P_k(d\mathcal{P}(e_l) - \chi^\infty_l)\|_{L^2_x} \lesssim \epsilon_1(1+s2^{2k})^{-29}. \tag{3.63}$$

Then by the identity $\psi^l_i = d\mathcal{P}(e_l)\cdot\partial_i v$, (3.62) and (3.63), from the bilinear Litttlewood–Paley decomposition

$$\begin{aligned}\|P_k(d\mathcal{P}(e_l)\cdot\partial_i v)\|_{L^2_x} &\lesssim 2^{-\sigma k} d_k(\sigma)\|P_{\le k-4} d\mathcal{P}(e_l)\|_{L^\infty} \\ &\quad + 2^k \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} 2^{-\sigma k_1} d_{k_1}(\sigma)\|P_{k_2} d\mathcal{P}(e_l)\|_{L^\infty_t L^2_x} \\ &\quad + \sum_{|k-k_2|\le 4}\|P_{k_2}(d\mathcal{P}(e_l))\|_{L^2_x}\sum_{k_1\le k-4} 2^{k_1-\sigma k_1} d_{k_1}(\sigma)\end{aligned}$$

we deduce that for any $\sigma \in [0, \frac{99}{100}]$ and $k \in \mathbb{Z}$,

$$\|P_k\psi_i\restriction_{s=0}\|_{L^2_x} \lesssim 2^{-\sigma k} d_k(\sigma).$$

Using this bound and similar arguments to those before one can improve (3.63) to

$$2^k\|P_k(d\mathcal{P}(e_l) - \chi^\infty_l)\|_{L^2_x} \lesssim 2^{-\sigma k} d_k(\sigma)(1+s2^{2k})^{-29}, \quad \sigma \in \big[0, \tfrac{99}{100}\big],$$

giving (3.61). The second item of the corollary follows by similar arguments and Lemma 3.2. ∎

Lemma 3.3. *Let $u \in \mathcal{H}_Q(T)$ solve SMF, and let $v(s,t,x)$ be the solution of the heat flow (3.2) with initial data $u(t,x)$. Then given $L \in \mathbb{Z}_+$, $L \geq 200$, for any $0 \leq \sigma \leq 2L$, there exist constants $\epsilon_L, C_L, C_{L,T} > 0$ such that if $\|\partial_x u\|_{L_t^\infty L_x^2} \leq \epsilon_L \ll 1$, then for any $s \geq 0$, $i = 1,2$, $\rho = 0,1$ and $m = 0,1,\ldots,L$,*

$$\|\partial_t^\rho \partial_x^m (v - Q)\|_{L_t^\infty H_x^L} \leq C_{L,T}(s+1)^{-m/2}, \tag{3.64}$$

$$(2^{-k/2}1_{k\leq 0} + 2^{\sigma k})\|P_k\phi_i\|_{L_t^\infty L_x^2} \leq C(\|u-Q\|_{L_t^\infty H_x^{3L}})(2^{2k}s+1)^{-30}, \tag{3.65}$$

$$(2^{-k/2}1_{k\leq 0} + 2^{\sigma k})\|P_k A_i\|_{L_t^\infty L_x^2} \leq C(\|u-Q\|_{L_t^\infty H_x^{3L}})(2^{2k}s+1)^{-28}, \tag{3.66}$$

$$2^{mk}\|P_k\partial_t\phi_i\|_{L_t^\infty L_x^2} \leq C_{L,T}(2^{2k}s+1)^{-25}, \tag{3.67}$$

$$2^{mk}\|P_k\partial_t A_i\|_{L_t^\infty L_x^2} \leq C_{L.T}(2^{2k}s+1)^{-25}. \tag{3.68}$$

Proof. Fix arbitrary $L \in \mathbb{N}$, $L \geq 200$. Let $\lambda_k(\sigma)$ be the frequency envelope

$$\lambda_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\frac{1}{2^j}\delta|k-k'|}2^{\sigma k'}\|P_{k'}(u-Q)\|_{L_t^\infty L_x^2}$$

for $\sigma \in \mathbb{I}_j \cap [0,2L]$, and define

$$\widetilde{B}_{j,\sigma,K}(S) := \sup_{k\in\mathbb{Z}, s\in[0,S)} [\lambda_k^{(j)}(\sigma)]^{-1}(1+s2^{2k})^K 2^{\sigma k}\|P_k v\|_{L_t^\infty L_x^2}$$

for $\sigma \in \mathbb{I}_j \cap [0,2L]$, $j \in \mathbb{N}$ and $K \in \mathbb{Z}_+$. By Lemma 3.1 and the fact that $\{\lambda_k^{(j)}(\sigma)\}$ are frequency envelopes, $\widetilde{B}_{j,\sigma,K}(S)$ is well-defined for $S \geq 0$ and continuous in S with $\lim_{S\to 0}\widetilde{B}_{j,\sigma,K}(S) = 1$. Then applying Propositions 3.2–3.4 and their proofs, we get $\widetilde{B}_{j,\sigma,30}(S) \lesssim 1$, that is,

$$\|P_k v\|_{L_t^\infty L_x^2} \lesssim (1+s2^{2k})^{-30}2^{-\sigma k}\lambda_k^{(j)}(\sigma)$$

for $\sigma \in \mathbb{I}_j \cap [0,2L]$.

Recall the definition of $\gamma_k^{(j)}(\sigma)$ in Section 3.3. Then Corollary 3.1 together with Lemma 3.2 shows that for $\sigma \in [0, 1+j/4]$,

$$2^{\sigma k}\|P_k\phi_i\|_{L_t^\infty L_x^2} \lesssim \gamma^{(j)}(\sigma)(2^{2k}s+1)^{-30},$$
$$2^{\sigma k}\|P_k A_i\|_{L_t^\infty L_x^2} \lesssim \gamma_{k,s}^{(j)}(\sigma)(2^{2k}s+1)^{-28},$$

which verifies half of (3.65)–(3.66).

Moreover, by Lemma 3.2 one has

$$2^{\sigma k}\|P_k[d\mathcal{P}(e)]\|_{L_t^\infty L_x^2} \lesssim 2^{-k}(1+s2^{2k})^{-30} \tag{3.69}$$

for $\sigma \in [0,2L]$. Let us check the left half of (3.65)–(3.66). Recall the bounds

$$\|P_k v\|_{L_x^2} \lesssim 2^{-\sigma k}\lambda_k^{(0)}(\sigma)(1+s2^{2k})^{-30}$$

for $\sigma \in [0, \frac{99}{100}]$. Then (3.69) and bilinear Littlewood–Paley decomposition show that

$$
\begin{aligned}
\|P_k(\phi_i)\|_{L_t^\infty L_x^2} &\le \sum_{l}^{2n} \|P_k(d\mathcal{P}(e_l)\cdot \partial_i v)\|_{L_t^\infty L_x^2}\\
&\lesssim 2^k \lambda_k^{(0)}(0)(1+s2^{2k})^{-30}(1+\|P_{\le k-4} d\mathcal{P}(e_l)\|_{L^\infty})\\
&\quad + 2^k \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} 2^{k_1/2}\lambda_{k_1}^{(0)}(1/2)\|P_{k_2} d\mathcal{P}(e_l)\|_{L_t^\infty L_x^2}\\
&\quad + \sum_{|k-k_2|\le 4} \|P_{k_2}(d\mathcal{P}(e_l))\|_{L_x^2} \sum_{k_1\le k-4} 2^{2k_1}\lambda_{k_1}^{(0)}(0)\\
&\lesssim C(\|u-Q\|_{L_t^\infty H_x^1}) 2^{k/2}(1+s2^{2k})^{-30}
\end{aligned}
$$

for any $i = 1, 2$ and $k \le 0$. Similar arguments give

$$
\|P_k(A_i)\|_{L_t^\infty L_x^2} \lesssim C(\|u-Q\|_{L_t^\infty H_x^1}) 2^{k/2}(1+s2^{2k})^{-28} \tag{3.70}
$$

for any $i = 1, 2$ and $k \le 0$. Hence, (3.65) and (3.65) have been proved.

Since $\partial_t v = J(v)(\sum_{i=1,2} D_i\phi)$ at $s = 0$, we observe from $u \in \mathcal{H}_Q(T)$ that

$$
\|\partial_t v\restriction_{s=0}\|_{L_t^\infty H_x^l} \le C_{l,T}, \quad \forall l \in \mathbb{N}.
$$

Thus using the smoothing estimates of heat semigroups and applying ∂_t to (3.2), one obtains (3.64). From (3.64), (3.63), (3.69) and the identity

$$
\partial_x^l \psi_t^a = \sum_{l_1=0}^{l} \partial_x^{l_1}(d\mathcal{P}(e_a)) \cdot \partial_x^{l-l_1}(\partial_t v)
$$

we get

$$
2^{mk}\|P_k\phi_t\|_{L_t^\infty L_x^2} \lesssim (1+s2^{2k})^{-28}, \quad \forall 0 \le m \le L,
$$

which further gives bounds of $\|P_k A_t\|_{L_t^\infty L_x^2}$. Then applying similar bounds of $A_i\phi_i, A_t\phi_i$ and the identity $\partial_t\phi_i = -A_t\phi_i + D_i\phi_t$, we obtain (3.67). For (3.68), we use $\phi_s = D_i\phi_i$ and

$$
|\partial_x^l \partial_t A_i| \le \sum_{l_1=0}^{l} \int_s^\infty |D_x^{l-l_1} D_t\phi_i||D_x^{l_1}\phi_s|\,ds' + \int_s^\infty |D_x^{l-l_1}\phi_i||D_x^{l_1} D_t\phi_s|\,ds'. \quad \blacksquare
$$

3.6. Additional decay estimates for dynamical caloric gauge

Proposition 3.6. *Let $u \in \mathcal{H}_Q(T)$ be a solution of SMF. Denote by $v(s,t,x)$ the solution to the heat flow with data $u(t,x)$, and denote by $\{\phi_i\}_{i=0}^2$ the corresponding differential fields under the caloric gauge. Assume that $\{\beta_k(\sigma)\}$ is a frequency envelope of order δ such that for all $i = 1, 2$ and $k \in \mathbb{Z}$,*

$$
2^{\sigma k}\|P_k\phi_i\restriction_{s=0}\|_{L_t^\infty L_x^2 \cap L_{t,x}^4} \le \beta_k(\sigma). \tag{3.71}
$$

- *There exists a sufficiently small constant $\epsilon > 0$ such that if*

$$
\sum_{k\in\mathbb{Z}} |\beta_k(0)|^2 < \epsilon, \tag{3.72}
$$

then for any $l \in \mathbb{N}$,

$$\|P_k\widetilde{\mathcal{G}}^{(l)}\|_{L^4\cap L_t^\infty L_x^2} \lesssim_l 2^{-\sigma k-k}\beta_k(\sigma)(1+s2^{2k})^{-30}, \tag{3.73}$$

$$\|P_k\phi_s\|_{L^4\cap L_t^\infty L_x^2} \lesssim 2^{-\sigma k+k}\Big[1_{k+j\geq 0}(1+s2^{2k})^{-30}\beta_k(\sigma)+1_{k+j\leq 0}\sum_{k\leq l\leq -j}\beta_l(\sigma)\beta_l\Big], \tag{3.74}$$

$$(1+s2^{2k})^{29}2^{\sigma k+k}\|P_k(d\mathcal{P}(e))\|_{L^4} \lesssim \beta_k(\sigma), \tag{3.75}$$

for any $\sigma \in [0, \frac{99}{100}]$, $s \in [2^{2j-1}, 2^{2j+1})$ and $j, k \in \mathbb{Z}$. Furthermore, assume that for $\sigma \in [0, \frac{5}{4}]$, $\{\beta_k(\sigma)\}$ is a frequency envelope of order $\frac{1}{2}\delta$ such that for all $i = 1, 2$ and $k \in \mathbb{Z}$, (3.71) and (3.72) hold. Then for any $\sigma \in [0, \frac{5}{4}]$ and $k \in \mathbb{Z}$,

$$(1+s2^{2k})^{27}2^{\sigma k}\|P_kA_i\|_{L_t^\infty L_x^2} \lesssim \beta_{k,s}^{(1)}(\sigma). \tag{3.76}$$

- *If $\{\beta_k(\sigma)\}$ is a frequency envelope of order $\frac{1}{2^j}\delta$, then similar results hold for $\sigma \in [0, 1+j/4]$ and ϵ sufficiently small depending only on $j \in \mathbb{Z}_+$ (see Prop. 1.4 for instance).*

Proof. By Proposition 3.5 and its proof, we have

$$(1+s2^{2k})^{31}2^{\sigma k}2^k\|P_kv\|_{L_t^\infty L_x^2} \lesssim \beta_k(\sigma), \tag{3.77}$$

$$(1+s2^{2k})^{30}2^{\sigma k}2^k\|P_kS(v)\|_{L_t^\infty L_x^2} \lesssim \beta_k(\sigma), \tag{3.78}$$

$$\|P_k\phi_s\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k+k}\Big[1_{k+j\geq 0}(1+s2^{2k})^{-30}\beta_k(\sigma)+1_{k+j\leq 0}\sum_{k\leq l\leq -j}\beta_l(\sigma)\beta_l\Big], \tag{3.79}$$

$$(1+s2^{2k})^{30}2^k\|P_k(d\mathcal{P}(e_l)-\chi_l^\infty)\|_{L_x^2} \lesssim 2^{-\sigma k}\beta_k(\sigma), \tag{3.80}$$

$$\sum_{i=1,2}(1+s2^{2k})^{29}2^{\sigma k}\|P_kA_i\|_{L_t^\infty L_x^2} \lesssim \beta_{k,s}^{(0)}(\sigma), \tag{3.81}$$

for any $\sigma \in [0, \frac{99}{100}]$, $j, k \in \mathbb{Z}$ and $s \in [2^{2j-1}, 2^{2j+1})$ if $\{\beta_k(\sigma)\}$ is a frequency envelope of order δ. And Proposition 3.5 and its proof give similar results for $\sigma \in [0, \frac{5}{4}]$ if $\{\beta_k(\sigma)\}$ is a frequency envelope of order $\frac{1}{2}\delta$.

Step 1. When $s = 0$, using $\partial_i v = \sum_l \mathcal{P}(e_l)\psi_i^l$, from the bilinear Littlewood–Paley decomposition

$$\begin{aligned}\|P_k(fg)\|_{L^4} \lesssim &\sum_{|k-k_2|\leq 4}\|P_{\leq k-4}f\|_{L_{t,x}^\infty}\|P_{k_2}g\|_{L^4}\\ &+2^k\sum_{k_1,k_2\geq k-4,\,|k_1-k_2|\leq 8}\|P_{k_1}f\|_{L_t^\infty L_x^2}\|P_{k_2}g\|_{L^4}\\ &+\sum_{k_2\leq k-4,\,|k_1-k|\leq 4}2^{k_1/2}\|P_{k_1}f\|_{L_t^\infty L_x^2}2^{\frac{1}{2}k_2}\|P_{k_2}g\|_{L^4}\end{aligned} \tag{3.82}$$

and (3.71), (3.80) we get

$$\|P_k(\partial_i v)\|_{L^4} \lesssim 2^{-\sigma k}\beta_k(\sigma) \tag{3.83}$$

for $s = 0$ and any $\sigma \in [0, \frac{99}{100}]$ and $k \in \mathbb{Z}$. Then we turn to the heat flow equation (1.13) to obtain

$$(1 + s2^{2k})^{30}\|P_k(\partial_i v)\|_{L^4} \lesssim 2^{-\sigma k}\beta_k(\sigma) \tag{3.84}$$

for any $s \geq 0$, $\sigma \in [0, \frac{99}{100}]$ and $k \in \mathbb{Z}$. In fact, set

$$Z_1(S) := \sup_{s\in[0,S),\, k\in\mathbb{Z}} \frac{1}{\beta_k(\sigma)} 2^{\sigma k}(1 + s2^{2k})^{30}\|P_k(\partial_i v)\|_{L^4}.$$

Then $Z_1(S)$ is well-defined, continuous and tends to 1 as $S \to 0$ by (3.83). Using the trilinear Littlewood–Paley decomposition

$$\begin{aligned} 2^k\|P_k S(v)(\partial_x v, \partial_x v)\|_{L^4_{t,x}} &\lesssim 2^k\widetilde{\beta}_k \sum_{k_1\leq k}\widetilde{\beta}_{k_1}2^{k_1} + \sum_{k_2\geq k} 2^{-2|k-k_2|}2^{2k_2}\widetilde{\beta}^2_{k_2} \\ &\quad + 2^{k/2}\widetilde{\alpha}_k\Big(\sum_{k_1\leq k}2^{k_1}\widetilde{\beta}_{k_1}\Big)^2 + \sum_{k_2\geq k}2^{2k-k_2}\widetilde{\alpha}_{k_2}\widetilde{\beta}_{k_2}\sum_{k_1\leq k_2}2^{k_1}\widetilde{\beta}_{k_1}, \end{aligned} \tag{3.85}$$

where we denote

$$\widetilde{\beta}_k := \sum_{|k'-k|\leq 30} 2^{k'}\|P_{k'}v\|_{L^\infty_t L^2_x \cap L^4_{t,x}}, \quad \widetilde{\alpha}_k := \sum_{|k'-k|\leq 30} 2^{k'}\|P_{k'}(S(v))\|_{L^\infty_t L^2_x},$$

and using similar arguments to those for Proposition 3.2, we deduce from (3.77) and (3.78) that

$$Z_1(S) \lesssim 1 + \epsilon Z_1^2(S)$$

for any $S \geq 0$. Then $Z_1(S) \lesssim 1$ since $\lim_{S\to 0} Z_1(S) = 1$. Thus (3.84) follows.

Step 2. With (3.84) in hand, using the heat flow equation one obtains bounds for $\|P_k\partial_s v\|_{L^4}$. Then the bound of $\|P_k\phi_s\|_{L^4}$ follows by the bilinear Littlewood–Paley decomposition (3.82) and (3.80). By performing dynamical separation for $d\mathcal{P}(e)$, we get bounds of $\|P_k(d\mathcal{P}(e))\|_{L^4}$ from $\|P_k\phi_s\|_{L^4}$ and $\|P_k(\mathbf{D}d\mathcal{P}(e;e))\|_{L^\infty_t L^2_x}$. Then one obtains bounds of $\|P_k(\mathcal{G})\|_{L^4}$, $\|P_k\phi_i\|_{L^4}$ for all $s \geq 0$, which further yields bounds of $\|A_i\|_{L^4}$ for any $s \geq 0$. See Proposition 3.5 for the details. ∎

Outline of proof before iteration

One of the ingredients of the proof before iteration is the framework of [5]. The other main ingredient is the decomposition of curvatures mentioned in Step 2 of Section 1.4. And at the technical level we need a bootstrap assumption on $\|P_k\widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t(T)}$ and some ideas to improve this bound; see Steps 4.1 and 4.2 of Section 1.4 for instance.

We outline the framework of [5] for the reader's convenience. Since the gauged equation is now not self-contained due to the curvature terms, several key new ideas as mentioned above will be used. But to give the reader a whole picture, we just sketch the framework of [5] rather than presenting all technically complex issues.

The proof is a bootstrap argument. Let $\sigma \in [0, \frac{99}{100})$ be given. Given $\mathcal{L} \in \mathbb{Z}_+$ and $Q \in \mathcal{N}$, let $T \in (0, 2^{2\mathcal{L}}]$. Assume that $\{c_k\}$ is an ϵ_0-frequency envelope of order δ and $\{c_k(\sigma)\}$ is another frequency envelope of order δ. Let u_0 be the initial data of SMF which satisfies

$$\|P_k \nabla u_0\|_{L^2_x} \le c_k(\widetilde{\sigma}) 2^{-\widetilde{\sigma} k}, \quad \widetilde{\sigma} \in \left[0, \tfrac{99}{100}\right]. \tag{3.86}$$

Denote by u the solution to SMF with initial data u_0. Assume that u satisfies

Bootstrap I. $\|P_k \nabla u\|_{L^\infty_t L^2_x} \le \epsilon_0^{-1/2} c_k$.

Denote by $v(s,t,x)$ the solution of the heat flow with initial data $u(t,x)$, and A_i, A_t, A_s the corresponding connection coefficients. And denote the heat tension field by ϕ_s and the differential fields by $\{\phi_i\}$, ϕ_t respectively. Suppose that $\{\phi_i\}_{i=1}^2$ satisfy the following condition at $s = 0$:

Bootstrap II. $\|P_k \phi_i \restriction_{s=0}\|_{G_k(T)} \le \epsilon_0^{-1/2} c_k$.

In Step 1, by studying the heat equations (1.15), (1.16), we prove that Bootstraps I–II in fact give parabolic estimates for A_i, A_t and $\phi_{i,t}$ along the heat flow direction:

$$\begin{aligned}
\|P_k \phi_i(s)\|_{F_k(T)} &\le c_k(\sigma) 2^{-\sigma k}(1 + s2^{2k})^{-4}, && \sigma \in \left[0, \tfrac{99}{100}\right],\\
\|P_k \phi_t(s)\|_{L^4_{t,x}} &\le c_k(\sigma) 2^{-\sigma k + k}(1 + s2^{2k})^{-2}, && \sigma \in \left[0, \tfrac{99}{100}\right],\\
\|P_k A_i \restriction_{s=0}\|_{L^4_{t,x}} &\le c_k(\sigma) 2^{-\sigma k}, && \sigma \in \left[0, \tfrac{99}{100}\right],\\
\|P_k A_t \restriction_{s=0}\|_{L^2_{t,x}} &\lesssim \epsilon_0.
\end{aligned}$$

In Step 2, by studying the Schrödinger equations (1.17), we prove that Bootstraps I–II indeed yield improved estimates for ϕ_i along the Schrödinger flow direction:

$$\|P_k \phi_i \restriction_{s=0}\|_{G_k(T)} \lesssim c_k(\sigma) 2^{-\sigma k}, \quad \sigma \in \left[0, \tfrac{99}{100}\right].$$

In Step 3, we prove

$$\|P_k \phi_i \restriction_{s=0}\|_{G_k(T)} \lesssim c_k \tag{3.87}$$

with Bootstraps I–II dropped.

4. Evolution of SMF solutions along the heat direction

4.1. Parabolic estimates for differential fields

The main result of this section is the following.

Proposition 4.1. *Let $\{b_k\}$ be an ε-frequency envelope. Assume that for $i = 1, 2$,*

$$\|P_k \phi_i \restriction_{s=0}\|_{F_k(T)} \le b_k(\sigma) 2^{-\sigma k}, \quad \sigma \in \left[0, \tfrac{99}{100}\right], \tag{4.1}$$

$$\|P_k \phi_t \restriction_{s=0}\|_{L^4_{t,x}} \lesssim b_k(\sigma) 2^{-(\sigma-1)k}, \quad \sigma \in \left[0, \tfrac{99}{100}\right], \tag{4.2}$$

and

$$\|P_k\phi_i(s)\|_{F_k(T)} \le \varepsilon^{-1/2} b_k (1+s2^{2k})^{-4}. \tag{4.3}$$

Then if $\varepsilon > 0$ *is sufficiently small, for* $\sigma \in [0, \frac{99}{100}]$ *one has*

$$\|P_k\phi_i(s)\|_{F_k(T)} \lesssim b_k(\sigma) 2^{-\sigma k}(1+s2^{2k})^{-4}, \tag{4.4}$$

$$\|P_k A_i\restriction_{s=0}\|_{L^4_{t,x}} \lesssim b_k(\sigma)2^{-\sigma k}, \quad i=1,2, \tag{4.5}$$

$$\|P_k\phi_t(s)\|_{L^4_{t,x}} \lesssim b_k(\sigma)2^{-(\sigma-1)k}(1+2^{2k}s)^{-2},$$

$$\|P_k A_t\restriction_{s=0}\|_{L^2_{t,x}} \lesssim \varepsilon b_k(\sigma)2^{-\sigma k} \quad \text{if } \sigma \in \left[\tfrac{1}{100}, \tfrac{99}{100}\right], \tag{4.6}$$

$$\|P_k A_t\restriction_{s=0}\|_{L^2_{t,x}} \lesssim \varepsilon^2.$$

Remark. Assumption (4.3) can be dropped. It suffices to apply Sobolev embeddings, Lemma 3.3 and [5, p. 1463]'s argument.

4.2. *Proof of Proposition* 4.1

Now we turn to prove the parabolic estimates in Proposition 4.1.

Denote

$$h(k) := \sup_{s\ge 0} (1+s2^{2k})^4 \sum_{i=1}^{2} \|P_k\phi_i(s)\|_{F_k(T)}. \tag{4.7}$$

Define the corresponding envelope by

$$h_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{\sigma k'} 2^{-\delta|k'-k|} h(k'). \tag{4.8}$$

Assume that

$$2^{k/2}\|P_k(\widetilde{\mathcal{G}}^{(1)})\|_{L^4_x L^\infty_t(T)} \le \varepsilon^{-1/4} h_k[(1+2^{2k}s)^{-20}1_{j+k\ge 0} + 1_{j+k\le 0}2^{\delta|k+j|}] \tag{4.9}$$

for any $s \in [2^{2j-1}, 2^{2j+1})$ and $k, j \in \mathbb{Z}$.

Lemma 4.1. *Under the assumptions of Proposition* 4.1 *and* (4.9)*, for any* $k \in \mathbb{Z}$*,* $s \ge 0$ *and* $i = 1, 2$*, we have*

$$\|P_k(A_i(s))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1+s2^{2k})^{-4} h_{k,s}(\sigma), \tag{4.10}$$

where the sequences $\{h_{k,s}\}$ *with* $2^{2k_0-1} \le s < 2^{2k_0+1}$ *and* $k_0 \in \mathbb{Z}$ *are defined by*

$$h_{k,s}(\sigma) := \begin{cases} 2^{k+k_0} h_{-k_0} h_k(\sigma) & \text{if } k+k_0 \ge 0, \\ \sum_{l=k}^{-k_0} h_l h_l(\sigma) & \text{if } k+k_0 \le 0. \end{cases} \tag{4.11}$$

Proof. By assumption (4.3) of Proposition 4.1 and noticing $\{b_k\}$ is an ε-envelope, we have

$$\|\{h_k\}\|^2_{\ell^2} \le \varepsilon. \tag{4.12}$$

In order to prove (4.10), let B_1 denote the smallest number in $[1,\infty)$ such that for all $\sigma\in[0,\frac{99}{100}]$, $s\geq 0$, $k\in\mathbb{Z}$ and $i=1,2$,

$$\|P_k(A_i(s))\|_{F_k(T)\cap S_k^{1/2}(T)}\leq B_1 2^{-\sigma k}(1+s2^{2k})^{-4}h_{k,s}(\sigma). \tag{4.13}$$

Step 1. Recall that Section 3.4 shows A_i is schematically written as

$$A_i(s)=\sum_{j_0,j_1,j_2,j_3}\int_s^\infty(\phi_i\diamond\phi_s)\langle\mathbf{R}(e_{j_0},e_{j_1})(e_{j_2}),e_{j_3}\rangle\,ds', \tag{4.14}$$

where $\{j_c\}_{c=0}^3$ run in $\{1,\dots,2n\}$, and i runs in $\{1,2\}$. Recall also that $\phi_s=\sum_{l=1}^2 D_l\phi_l$. Applying P_k to (4.14) we have

$$\begin{aligned}
&\|P_k(A_i(s))\|_{F_k(T)\cap S_k^{1/2}(T)}\leq\\
&\sum_{|k_1-k_2|\leq 8,\,k_1,k_2\geq k-4}\int_s^\infty\|P_k[P_{k_1}(\phi_i\diamond\phi_s)P_{k_2}\langle\mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle]\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds'\\
&+\sum_{|k_1-k|\leq 4}\int_s^\infty\|P_k[P_{k_1}(\phi_i\diamond\phi_s)P_{\leq k-4}\langle\mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle]\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds'\\
&+\sum_{|k_2-k|\leq 4,\,k_1\leq k-4}\int_s^\infty\|P_k[P_{k_1}(\phi_i\diamond\phi_s)P_{k_2}\langle\mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle]\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds'.
\end{aligned} \tag{4.15}$$

The above three subcases according to their order are usually called (a) High × High → Low, (b) High × Low → High, (c) Low × High → High.

Case (b): High × Low → High. In [5, Lemma 5.2, p. 1470], the authors have proved

$$\sum_{i=1}^2\int_s^\infty\|P_k(\phi_i\diamond\phi_s)\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds'\lesssim\varepsilon B_1 2^{-\sigma k}(1+s2^{2k})^{-4}h_{k,s}(\sigma), \tag{4.16}$$

with slightly different notations. Thus in case (b), by (8.5) and applying the trivial bound

$$\|\langle\mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle\|_{L^\infty_{t,s,x}}\lesssim K(\mathcal{N}) \tag{4.17}$$

to the $P_{\leq k-4}$ part and (4.16) to the P_{k_1} part, we obtain

$$\begin{aligned}
&\sum_{|k_1-k|\leq 4}\int_s^\infty\|P_k(P_{k_1}(\phi_i\diamond\phi_s)P_{\leq k-4}\langle\mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle)\|_{F_k(T)\cap S_k^{1/2}(T)}\\
&\qquad\lesssim\varepsilon B_1 2^{-\sigma k}(1+s2^{2k})^{-4}h_{k,s}(\sigma).
\end{aligned} \tag{4.18}$$

Step 2. Refined dynamic separation. For the Low × High and High × High part, we need to further decompose the curvature term. The dynamic separation performed in Section 3.4 also needs to be refined. Recall the notation

$$\mathcal{G}(s)=\langle\mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle(s),$$

for any given $j_0,\dots,j_3\in\{1,\dots,2n\}$, and the decomposition of $\mathcal{G}$ in Section 3.4. Thus using $\psi_s=\sum_{i=1,2}(\partial_i+A_i)\psi_i$, after the second time-dynamic separation, $\mathcal{G}$ can be decomposed into

$$\begin{aligned}\mathcal{G}(s) &= \langle \mathbf{R}(e_{j_0},e_{j_1})e_{j_2},e_{j_3}\rangle(s)\\ &= \Gamma^{\infty}-\Gamma_l^{\infty,(1)}\int_s^{\infty}\psi_s^l(\widetilde{s})\,d\widetilde{s}\\ &\quad-\int_s^{\infty}\psi_s^l(\widetilde{s})\Big(\int_{\widetilde{s}}^{\infty}\psi_s^p(s')(\widetilde{\nabla}^2\mathbf{R})(e_l,e_p;e_{j_0},\dots,e_{j_3})\,ds'\Big)\,d\widetilde{s}\\ &= \Gamma^{\infty}+\mathcal{U}_{00}+\mathcal{U}_{01}+\mathcal{U}_I+\mathcal{U}_{II},\end{aligned}\tag{4.19}$$

where

$$\begin{aligned}\mathcal{U}_{00} &:= -\Gamma_l^{\infty,(1)}\int_s^{\infty}\sum_{i=1}^2(\partial_i\psi_i)\,ds'\\ \mathcal{U}_{01} &:= -\int_s^{\infty}\sum_{i=1}^2(\partial_i\psi_i)^l\big((\widetilde{\nabla}\mathbf{R})(e_l;e_{j_0},\dots,e_{j_3})-\Gamma_l^{\infty,(1)}\big)\,ds'\\ \mathcal{U}_{I} &:= -\Gamma_l^{\infty,(1)}\int_s^{\infty}\sum_{i=1}^2(A_i\psi_i)^l\,ds'\\ \mathcal{U}_{II} &:= -\int_s^{\infty}\sum_{i=1}^2(A_i\psi_i)^l(\widetilde{s})\Big(\int_{\widetilde{s}}^{\infty}\psi_s^p(s')(\widetilde{\nabla}^2\mathbf{R})(e_l,e_p;e_{j_0},\dots,e_{j_3})\,ds'\Big)\,d\widetilde{s}\\ &= -\int_s^{\infty}\sum_{i=1}^2(A_i\psi_i)^l(\widetilde{s})\big((\widetilde{\nabla}\mathbf{R})(e_l;e_{j_0},\dots,e_{j_3})-\Gamma_l^{\infty,(1)}\big)\,d\widetilde{s}.\end{aligned}$$

It is easy to prove that

$$\begin{aligned}&\Big\|\int_{2^{2k_0-1}}^{\infty}\sum_{i=1}^2(\partial_i\psi_i)\,ds'\Big\|_{F_k(T)}\\ &\quad\lesssim 2^{-\sigma k}h_k(\sigma)(1_{k_0+k\le 0}2^{-k}+1_{k_0+k\ge 0}2^{2k_0+k})(1+2^{2k_0+2k})^{-4}.\end{aligned}\tag{4.20}$$

And recall that [5, Lemma 5.2] shows that for $s\in[2^{2j-1},2^{2j+2})$,

$$\begin{aligned}&\|P_k(\phi_i\diamond\phi_s)(s)\|_{F_k(T)\cap S_k^{1/2}(T)}\\ &\quad\lesssim\begin{cases}2^{-\sigma k}(1+2^{2k}s)^{-4}\big(\widetilde{h}_{k,s}(\sigma)+B_1\varepsilon 2^{-2j}h_{k,s}(\sigma)\big) & \text{if } k+j\ge 0,\\ 2^{-\sigma k}\big(\widetilde{h}_{k,s}(\sigma)+B_1\varepsilon 2^{-2j}h_{-j}h_{-j}(\sigma)\big) & \text{if } k+j\le 0,\end{cases}\end{aligned}\tag{4.21}$$

where

$$\widetilde{h}_{k,s}(\sigma):=2^{-j}h_{-j}(2^kh_k(\sigma)+2^{-j}h_{-j}(\sigma)).$$

Then repeating the bilinear estimates of [5, Lemma 5.1], we have

$$\int_s^\infty \|P_k(\mathcal{U}_{00}(\phi_s \diamond \phi_i))\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds' \lesssim (1+\varepsilon B_1)2^{-\sigma k}h_{k,s}(\sigma)(1+2^{2k_0+2k})^{-4}. \tag{4.22}$$

The Γ^∞ part

$$\int_s^\infty \|P_k(\Gamma^\infty(\phi_s \diamond \phi_i))\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds' \lesssim (1+\varepsilon B_1)2^{-\sigma k}h_{k,s}(\sigma)(1+2^{2k_0+2k})^{-4}$$

follows by directly applying [5, Lemma 5.2], since Γ^∞ is just a constant.

Recall the notation $\widetilde{\mathcal{G}}^{(1)} = (\widetilde{\nabla}\mathbf{R})(e; e_{j_0}, e_{j_1}, e_{j_2}, e_{j_3}) - \Gamma^{\infty,(1)}$. For $\mathcal{U}_{01}$, applying (3.73) which says

$$2^k\|P_k\widetilde{\mathcal{G}}^{(1)}\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}h_k(\sigma)(1+2^{2k}s)^{-30}, \tag{4.23}$$

and (4.9) which says, for $s \in [2^{2j-1}, 2^{2j+1})$,

$$2^{k/2}\|P_k\widetilde{\mathcal{G}}^{(1)}\|_{L_x^4 L_t^\infty} \lesssim 2^{-\sigma k}h_k(\sigma)[(1+2^{2k}s)^{-20}1_{j+k\ge 0} + 2^{\delta|j+k|}1_{k+j\le 0}], \tag{4.24}$$

we find by Lemma 4.2 that

$$\begin{aligned}
&\|P_k((\partial_i\psi_i)\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)} \\
&\quad\lesssim \sum_{|k_1-k|\le 4}\|P_{k_1}\partial_i\phi_i\|_{F_{k_1}(T)}\|P_{\le k-4}\widetilde{\mathcal{G}}^{(1)}\|_{L^\infty} \\
&\qquad+ \sum_{|k_2-k|\le 4,\,k_1\le k-4} 2^{k_1/2}\|P_{k_1}\partial_i\phi_i\|_{F_{k_1}(T)}\|P_{k_2}\widetilde{\mathcal{G}}^{(1)}\|_{L_x^4L_t^\infty} \\
&\qquad+ \sum_{|k_2-k|\le 4,\,k_1\le k-4} 2^{k_1}\|P_{k_1}\partial_i\phi_i\|_{F_{k_1}(T)}\|P_{k_2}\widetilde{\mathcal{G}}^{(1)}\|_{L^4} \\
&\qquad+ \sum_{|k_2-k_1|\le 8,\,k_1,k_2\ge k-4}\|P_{k_1}\partial_i\phi_i\|_{F_{k_1}(T)}(\|P_{k_2}\widetilde{\mathcal{G}}^{(1)}\|_{L^\infty} + 2^{k_1/2}\|P_{k_2}\widetilde{\mathcal{G}}^{(1)}\|_{L_x^4L_t^\infty}).
\end{aligned}$$

Thus by the slow variation of envelopes we further have

$$\|P_k((\partial_i\psi_i)\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)} \lesssim 2^{-\sigma k}h_k(\sigma)\big(1_{k+j\ge 0}2^k(1+2^{2k+2j})^{-4} + 1_{k+j\le 0}2^{-j}2^{\delta|j+k|}\big)$$

for $s \in [2^{2j-1}, 2^{2j+1})$ and $k, j \in \mathbb{Z}$. Notice that the large constant $\varepsilon^{-1/4}$ is absorbed by $\|\{h_k\}\|_{\ell^\infty} \lesssim \varepsilon^{1/2}$. Also notice that in the Low $\times$ High interaction of $(\partial_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$ it is possible to deduce $2^{-\sigma k}$ for $\sigma \in [0, \frac{99}{100}]$ from $\partial_i\psi_i$ due to the fact that the series $\sum_{k_1\le k-4} 2^{2k-\sigma k}h_k(\sigma)$ is summable for $\sigma < 2$. Thus for $s \in [2^{2k_0-1}, 2^{2k_0+1})$, summing the above formula over $j \ge k_0$ yields

$$\begin{aligned}
&\int_s^\infty \|P_k((\partial_i\psi_i)\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\,ds' \\
&\qquad\lesssim 2^{-\sigma k}h_k(\sigma)\big(1_{k+k_0\ge 0}2^{k+2k_0}(1+2^{2k+2k_0})^{-4} + 1_{k+k_0\le 0}2^{-k}\big),
\end{aligned}$$

which is the same as (4.20). Thus (4.21) and bilinear estimates give

$$\int_s^\infty \|P_k(\mathcal{U}_{01}(\phi_s \diamond \phi_i))\|_{F_k(T)\cap S_k^{1/2}(T)}\, ds' \lesssim (1+\varepsilon B_1)2^{-\sigma k} h_{k,s}(\sigma)(1+2^{2k_0+2k})^{-4}.$$

Therefore, it remains to estimate $\mathcal{U}_I$ and $\mathcal{U}_{II}$.

Step 3. Proof of our lemma with a bootstrap condition. We first prove our lemma with an additional bootstrap condition. In the final step we will drop the bootstrap condition and finish the whole proof.

Bootstrap Assumption A. Assume that for all $k, j \in \mathbb{Z}$ and $s \in [2^{2j-1}, 2^{2j+1})$,

$$\|P_k \mathcal{U}_I\|_{F_k(T)\cap S_k^{1/2}(T)} \le \varepsilon^{-1/2}(1+2^{k+j})^{-7} T_{k,j} h_k c_0^*, \tag{4.25}$$

$$\|P_k \mathcal{U}_{II}\|_{F_k(T)} \le \varepsilon^{-1/2}(1+2^{k+j})^{-7} T_{k,j} h_k c_0^*, \tag{4.26}$$

where $c_0^* := \|\{h_k\}\|_{\ell^2}$ and we denote

$$T_{k,j} = 1_{k+j\le 0} 2^{-k} + 1_{k+j\ge 0} 2^{j}. \tag{4.27}$$

Our aim for this step is to prove that B_1 defined by (4.15) satisfies

$$B_1 \lesssim 1 + \varepsilon B_1,$$

assuming Bootstrap Assumption A.

For $s \in [2^{2j-1}, 2^{2j+1})$ and $j \in \mathbb{Z}$, (4.25) and (4.26) show $\mathcal{U} := \mathcal{U}_I + \mathcal{U}_{II}$ satisfies

$$2^k(1+2^{k+j})^6 \|P_k \mathcal{U}\|_{F_k(T)} \lesssim 1. \tag{4.28}$$

Case (a): High × High → Low. The bound (4.28) suffices to control the High × High interaction. For $k + k_0 \ge 0$, applying the bounds (4.28), (4.21) and (8.4) of Lemma 8.2 with $\omega = 1/2$, one finds that in the High × High case,

$$\begin{aligned}
&\sum_{j\ge k_0} \int_{2^{2j-1}}^{2^{2j+1}} \sum_{|k_1-k_2|\le 8,\, k_1,k_2\ge k-4} \|P_k(P_{k_1}(\phi_i \diamond \phi_s) P_{k_2}\mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)}\, ds' \\
&\quad\lesssim 2^{-\sigma k} \sum_{j\ge k_0} \sum_{k_1\ge k-4} 2^{\frac{k-k_1}{2}} (1+2^{2k_1+2j})^{-6} h_{-j}\big(2^{k_1+j} h_{k_1}(\sigma) + h_{-j}(\sigma)\big) \\
&\qquad + 2^{-\sigma k} \sum_{j\ge k_0} \sum_{k_1\ge k-4} 2^{\frac{k-k_1}{2}} B_1 \varepsilon (1+2^{2k_1+2j})^{-6} h_{k_1,2^{2j}}(\sigma),
\end{aligned} \tag{4.29}$$

which by slow variation of envelopes is further bounded by

$$\begin{aligned}
&2^{-\sigma k} \sum_{j\ge k_0} \sum_{k_1\ge k-4} 2^{\frac{k-k_1}{2}} (1+2^{k_1+j})^{-10} 2^{\delta|j-k_0|} h_{-k_0} h_k(\sigma)(2^{k_1+j+\delta|k_1-k|} + 2^{\delta|k+j|}) \\
&\quad + 2^{-\sigma k} \sum_{j\ge k_0} \sum_{k_1\ge k-4} 2^{\frac{k-k_1}{2}} B_1 \varepsilon (1+2^{k_1+j})^{-10} 2^{\delta|j-k_0|} 2^{\delta|k_1-k|} h_{-k_0} h_k(\sigma).
\end{aligned}$$

Since $k_1 + j \gtrsim k + j \geq k + k_0 \geq 0$, it is easy to see the above formula is acceptable because

$$\sum_{j\geq k_0} 2^{-\sigma k}(1+\varepsilon B_1)h_{-k_0}h_k(\sigma)2^{\delta|j-k_0|}2^{\delta|k+j|}(1+2^{k+j})^{-8} \lesssim (1+\varepsilon B_1)2^{-\sigma k}h_{-k_0}h_k(\sigma)2^{k_0+k}(1+2^{k+k_0})^{-8}.$$

Therefore, the $k + k_0 \geq 0$ case for the High $\times$ High interaction has been settled.

Assume $k + k_0 \leq 0$. Applying the bounds (4.21), (4.28) with $\sigma = 0$ and (8.4) with $\omega = \frac{1}{2}$, for $j + k \leq 0$, by slow variation of envelopes we have

$$\begin{aligned}
&\sum_{k_0\leq j\leq -k} \int_{2^{2j-1}}^{2^{2j+1}} \sum_{|k_1-k_2|\leq 8,\, k_1,k_2\geq k-4} \|P_k(P_{k_1}(\phi_i \diamond \phi_s)P_{k_2}\mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)}\, ds' \\
&\lesssim \sum_{k_0\leq j\leq -k} \sum_{k_1\geq k-4} 2^{\frac{k-k_1}{2}} 2^{-\sigma k_1}(1+2^{2k_1+2j})^{-6}h_{-j}(2^{k_1+j}h_{k_1}(\sigma)+h_{-j}(\sigma)) \\
&\quad + B_1\varepsilon 2^{-\sigma k} \sum_{k_0\leq j\leq -k}\Big(\sum_{k-4\leq k_1\leq -j} 2^{\frac{k-k_1}{2}} h_{-j}h_{-j}(\sigma) \\
&\qquad + \sum_{k_1\geq -j} 2^{\frac{k-k_1}{2}}(1+2^{2j+2k_1})^{-4}h_{k_1,2^{2j}}(\sigma)\Big) \\
&\lesssim \sum_{k_0\leq j\leq -k} 2^{-\sigma k}(1+\varepsilon B_1)h_{-j}h_{-j}(\sigma).
\end{aligned}$$

Therefore, for $k_0 + k \leq 0$, the High $\times$ High part is bounded by

$$\begin{aligned}
&\sum_{j\geq k_0} \int_{2j-1}^{2j+1} \sum_{|k_1-k_2|\leq 8,\, k_1,k_2\geq k-4} \|P_k(P_{k_1}(\phi_i \diamond \phi_s)P_{k_2}\mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)}\, d\tau \\
&\quad\lesssim \sum_{k_0\leq j\leq -k} C(1+2^{k+j})^{-12}2^{-\sigma k}h_{-j}h_{-j}(\sigma)2^{(1\pm\delta)(k+j)} + \sum_{j\geq -k}(\dots) \\
&\quad\lesssim (1+B_1\varepsilon) \sum_{k_0\leq j\leq -k} 2^{-\sigma k}h_{-j}h_{-j}(\sigma) + (1+B_1\varepsilon)2^{-\sigma k}h_k h_k(\sigma) \\
&\quad\lesssim \sum_{k_0\leq j\leq -k} (1+B_1\varepsilon)2^{-\sigma k}h_{-j}h_{-j}(\sigma).
\end{aligned} \tag{4.30}$$

where the $\sum_{j\geq -k}(\dots)$ part in (4.30) is bounded by $(1 + B_1\varepsilon)2^{-\sigma k}h_k h_k(\sigma)$ by directly using results of the $k + k_0 \geq 0$ case. Therefore, we conclude that

$$\sum_{j\geq k_0} \int_{2j-1}^{2j+1} \sum_{|k_1-k_2|\leq 8,\, k_1,k_2\geq k-4} \|P_k(P_{k_1}(\phi_i \diamond \phi_s)P_{k_2}\mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)}\, d\tau \lesssim (1+\varepsilon B_1)2^{-\sigma k}(1+2^{j+k})^{-8}h_{k,s}(\sigma).$$

Case (c): Low $\times$ High $\to$ High. In case (c), we assume $|k-k_2| \le 4$ and $k \ge k_1+4$, i.e. Low $\times$ High $\to$ High. We apply the bound

$$\|P_k(f_{k_1} g_{k_2})\|_{F_k \cap S_k^{1/2}} \lesssim 2^{k_1} \|P_{k_1} f\|_{F_{k_1} \cap S_{k_1}^{1/2}} \|P_{k_2} g\|_{F_{k_2}(T)}$$

provided that $|k-k_2| \le 20$. To avoid a long formula, we recall the notation

$$T_{k,j} = 1_{k+j\ge 0} 2^j + 1_{k+j\le 0} 2^{-k}.$$

Thus by (4.21), (4.25) and (4.26), for $j \ge k_0$ and $\sigma \in [0, \frac{99}{100}]$, we find that the Low $\times$ High $\to$ High part is bounded by

$$\begin{aligned}
&\sum_{j\ge k_0} \int_{2^{2j-1}}^{2^{2j+1}} \sum_{|k_2-k|\le 4,\, k_1\le k-4} \|P_k(P_{k_1}(\phi_s \diamond \phi_i) P_{k_2} \mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)} \\
&\lesssim \sum_{j\ge k_0} \sum_{|k-k_2|\le 4} \sum_{k_1\le k-4} \int_{2^{2j-1}}^{2^{2j+1}} 2^{k_1} \|P_{k_1}(\phi_s \diamond \phi_i)\|_{F_{k_1}\cap S_{k_1}^{1/2}(T)} \|P_{k_2}(\mathcal{U})\|_{F_{k_2}(T)} \\
&\lesssim \sum_{j\ge k_0} h_k T_{k,j} (1+2^{j+k})^{-7} \sum_{k_1\le k} 2^{k_1-\sigma k_1} h_{-j} (2^{k_1+j} h_{k_1}(\sigma) + h_{-j}(\sigma))(1+2^{k_1+j})^{-7} \\
&\quad + \sum_{j\ge k_0} h_k T_{k,j} (1+2^{j+k})^{-7} B_1 \varepsilon 1_{k+j\ge 0} \sum_{k_1\le -j} 2^{k_1-\sigma k_1} h_{-j} h_{-j}(\sigma) \\
&+ \sum_{j\ge k_0} h_k T_{k,j} (1+2^{j+k})^{-7} B_1 \varepsilon 1_{k+j\ge 0} \sum_{-j\le k_1\le k} 2^{k_1-\sigma k_1} 2^{k_1+j} h_{-j} h_{k_1}(\sigma)(1+2^{k_1+j})^{-7} \\
&\quad + \sum_{j\ge k_0} h_k T_{k,j} (1+2^{j+k})^{-7} B_1 \varepsilon 1_{k+j\le 0} \sum_{k_1\le k} 2^{k_1-\sigma k_1} h_{-j} h_{-j}(\sigma).
\end{aligned}$$

Therefore, for $k+k_0 \ge 0$ we conclude that

$$\begin{aligned}
&\sum_{j\ge k_0} \int_{2^{2j-1}}^{2^{2j+1}} \sum_{|k_2-k|\le 4,\, k_1\le k-4} \|P_k(P_{k_1}(\phi_s \diamond \phi_i) P_{k_2} \mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)} \\
&\qquad \lesssim 2^{-\sigma k} \sum_{j\ge k_0} (1+B_1\varepsilon)(1+2^{k+j})^{-7} 2^{\delta|j-k_0|} h_{-k_0} h_k(\sigma) \\
&\qquad \lesssim 2^{-\sigma k} (1+B_1\varepsilon)(1+2^{2k+2j})^{-4} h_{k,2^{2k_0}}(\sigma),
\end{aligned}$$

and for $k+k_0 \le 0$, we also have

$$\begin{aligned}
&\sum_{j\ge k_0} \sum_{|k_2-k|\le 8,\, k_1\le k} \int_{2^{2j-1}}^{2^{2j+1}} \|P_k(P_{k_1}(\phi_s \diamond \phi_i) P_{k_2} \mathcal{U})\|_{F_k(T)\cap S_k^{1/2}(T)} \\
&\qquad \lesssim 2^{-\sigma k} (1+B_1\varepsilon^{1/2})(1+2^{2k+2j})^{-4} h_{k,2^{2k_0}}(\sigma).
\end{aligned}$$

Thus the Low $\times$ High part has been handled for $\mathcal{U}$ as well.

Therefore, combining the three cases, we summarize

$$\|P_k(A_i(s))\|_{F_k\cap S_k^{1/2}} \lesssim (\varepsilon^{1/2}B_1+1)2^{-\sigma k}(1+2^{k+k_0})^{-8}h_{k,2^{2k_0}}(\sigma). \tag{4.31}$$

This shows

$$B_1 \lesssim \varepsilon^{1/2}B_1+1. \tag{4.32}$$

Hence $B_1 \lesssim 1$. So, we have obtained our lemma for $\sigma \in [0, \frac{99}{100}]$ assuming Bootstrap Assumption A.

Step 4. In this step, we prove that our lemma remains valid if we drop the Bootstrap Assumption A in (4.25) and (4.26). First, we prove a claim.

Claim A. *If (4.25)–(4.26) hold, then for all $k, j \in \mathbb{Z}$, $\sigma \in [0, \frac{99}{100}]$ and $s \in [2^{2j-1}, 2^{2j+1})$,*

$$\|P_k\mathcal{U}_I\|_{F_{k_2}(T)\cap S_k^{1/2}(T)} \le c_0^* 2^{-\sigma k}(1+2^{k+j})^{-7}T_{k,j}h_k(\sigma), \tag{4.33}$$

$$\|P_k\mathcal{U}_{II}\|_{F_{k_2}(T)} \le c_0^* 2^{-\sigma k}(1+2^{k+j})^{-7}T_{k,j}h_k(\sigma). \tag{4.34}$$

Recall the definition of $\mathcal{U}_I$:

$$\mathcal{U}_I = -\Gamma_l^{\infty,(1)}\int_s^\infty \sum_{i=1,2}(A_i\psi_i)^l(s')\,ds'.$$

For $\mathcal{U}_{II}$, it is better to use

$$\mathcal{U}_{II} = -\int_s^\infty \sum_{i=1,2}(A_i\psi_i)^p(s')[(\widetilde{\nabla}\mathbf{R})(e_p;e_{j_0},e_{j_1},e_{j_2},e_{j_3})-\Gamma_p^{\infty,(1)}]\,ds'.$$

Recall the notation

$$\widetilde{\mathcal{G}}^{(1)} = (\widetilde{\nabla}\mathbf{R})(e;e_{j_0},\dots,e_{j_3})-\Gamma^{\infty,(1)}.$$

Moreover, by (3.73) and since $c_0^* := \|\{h_k\}\|_{\ell^2}$ we have

$$\|P_k(\widetilde{\mathcal{G}}^{(1)})(s)\|_{L^\infty_{t,x}} \lesssim 2^{-\sigma k}h_k(\sigma)(1+s2^{2k})^{-20}, \quad \sigma\in\big[0,\tfrac{99}{100}\big]. \tag{4.35}$$

Thus in order to prove Claim A for $\mathcal{U}_{II}$, it suffices to prove

$$\int_s^\infty \|(A_i\psi_i)\widetilde{\mathcal{G}}^{(1)}\|_{F_k(T)}\,ds' \lesssim (1+2^k s^{1/2})^{-7}c_0^*T_{k,j}. \tag{4.36}$$

The bound claimed for $\mathcal{U}_I$ is easier to verify. Since now $B_1 \lesssim 1$, applying bilinear Lemma 8.2 to $A_i\psi_i$, one has for $j+k\ge 0$ and $s\in[2^{2j-1},2^{2j+1})$,

$$\begin{aligned}&\|(A_i\psi_i)\|_{F_k(T)\cap S_{k_1}^{1/2}(T)}\\ &\quad\lesssim (1_{k+j\ge 0}2^{-j}+1_{k+j\le 0}2^{\frac{k-j}{2}})(1+2^{2k+2j})^{-4}2^{-\sigma k}c_0^*h_{k,s}(\sigma).\end{aligned} \tag{4.37}$$

Summing the above formula over $j\ge k_0$, we get

$$\int_s^\infty \|A_i\psi_i\|_{F_k(T)\cap S_k^{1/2}(T)}\,ds' \lesssim c_0^*2^{-\sigma k}T_{k,k_0}(1+2^{k_0+k})^{-7}h_k(\sigma). \tag{4.38}$$

Thus Claim A has been verified for $\mathcal{U}_I$.

For $\mathcal{U}_{II}$, we will use the inequality (see (4.45))

$$\|P_k(P_{k_1} f P_{k_2} g)\|_{F_k(T)} \lesssim \|P_{k_2} g\|_{L_x^\infty} \|P_{k_1} f\|_{F_{k_1}(T)\cap S_{k_1}^{1/2}(T)}.$$

Then by Littlewood–Paley bilinear decomposition,

$$\begin{aligned}\|P_k((A_i\psi_i)\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)} \lesssim &\sum_{|k_1-k|\le 4} \|P_{k_1}(A_i\psi_i)\|_{F_{k_1}(T)\cap S_{k_1}^{1/2}(T)} \|P_{\le k-4}\widetilde{\mathcal{G}}^{(1)}\|_{L^\infty}\\ &+ \sum_{|k_1-k_2|\le 8,\, k_1,k_2\ge k-4} \|P_{k_1}(A_i\psi_i)\|_{F_{k_1}(T)\cap S_{k_1}^{1/2}(T)} \|P_{k_2}\widetilde{\mathcal{G}}^{(1)}\|_{L^\infty}\\ &+ \sum_{|k_2-k|\le 4,\, k_1\le k-4} \|P_{k_1}(A_i\psi_i)\|_{F_{k_1}(T)\cap S_{k_1}^{1/2}(T)} \|P_{k_2}\widetilde{\mathcal{G}}^{(1)}\|_{L^\infty}.\end{aligned}$$

Thus by (4.37), the High × Low part of $(A_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$ is dominated by

$$\begin{aligned}&\sum_{|k-k_1|\le 4} \|P_k(P_{k_1}(A_i\psi_i)P_{\le k-4}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\\ &\quad\lesssim c_0^* 2^{-\sigma k}\big(1_{k+j\le 0} 2^{\frac{k-j}{2}+\delta|k+j|} h_k(\sigma) + 1_{k+j\ge 0} 2^{-j}(1+2^{k+j})^{-7} h_k(\sigma) h_{-j}\big).\end{aligned}$$

Summing over $j \ge k_0$ yields

$$\begin{aligned}&\sum_{j\ge k_0} 2^{2j} \sum_{|k-k_1|\le 4} \|P_k(P_{k_1}(A_i\psi_i)P_{\le k}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\\ &\quad\lesssim c_0^* 2^{-\sigma k} h_k(\sigma)\big(1_{k+k_0\ge 0} 2^{k_0}(1+2^{k+k_0})^{-7} + 1_{k+k_0\le 0} 2^{-k}\big).\end{aligned}$$

Using (4.35) and (4.37), the High × High part of $(A_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$ is dominated by

$$\begin{aligned}&\sum_{|k_2-k_1|\le 8,\, k_1,k_2\ge k-4} \|P_k(P_{k_1}(A_i\psi_i)P_{k_2}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\\ &\lesssim c_0^* 1_{k+j\ge 0} \sum_{k_1\ge k-4} 2^{-\sigma k_1}(1+2^{2j+2k})^{-7} 2^{k_1} h_{k_1}(\sigma)\\ &\quad + c_0^* 1_{k+j\le 0}\Big[\sum_{k-4\le k_1\le -j} 2^{\frac{k_1-j}{2}} 2^{\delta|k_1+j|} 2^{-\sigma k_1} h_{k_1}(\sigma)\\ &\qquad\qquad + \sum_{k_1\ge -j} 2^{-\sigma k_1}(1+2^{2j+2k})^{-7} 2^{k_1} h_{k_1}(\sigma)\Big]\\ &\lesssim c_0^* 1_{k+j\ge 0} 2^{-j}(1+2^{j+k})^{-10} 2^{-\sigma k} h_k(\sigma) + c_0^* 1_{k+j\le 0} 2^{-j} 2^{\delta|k+j|} 2^{-\sigma k} h_k(\sigma).\end{aligned}$$

Summing over $j \ge k_0$ also gives

$$\begin{aligned}&\sum_{j\ge k_0} 2^{2j} \sum_{|k_2-k_1|\le 8,\, k_1,k_2\ge k-4} \|P_k(P_{k_1}(A_i\psi_i)P_{k_2}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\\ &\quad\lesssim c_0^* 2^{-\sigma k} 1_{k+k_0\ge 0} 2^{k_0}(1+2^{k+k_0})^{-7} h_k(\sigma) + c_0^* 1_{k+k_0\le 0} 2^{-k} 2^{-\sigma k} h_k(\sigma).\end{aligned}$$

Then using (4.35), the Low $\times$ High part of $(A_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$ is bounded as

$$\begin{aligned}
&\sum_{|k-k_2|\le 4,\,k_1\le k-4} \|P_k(P_{k_1}(A_i\psi_i)P_{k_2}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\\
&\lesssim h_k(\sigma)2^{-\sigma k}1_{k+j\le 0}\sum_{k_1\le k} h_{k_1}2^{\frac{1}{2}(k_1-j)}2^{\delta|k_1+j|}\\
&\quad + h_k(\sigma)2^{-\sigma k}(1+2^{2j+2k})^{-20}1_{k+j\ge 0}\Big[\sum_{-j\le k_1\le k} h_{k_1}2^{k_1}(1+2^{2j+2k_1})^{-4}\Big]\\
&\quad + h_k(\sigma)2^{-\sigma k}(1+2^{2j+2k})^{-20}1_{k+j\ge 0}\Big[\sum_{k_1\le -j} h_{k_1}2^{\frac{k_1-j}{2}}2^{\delta|k_1+j|}\Big]\\
&\lesssim c_0^*2^{-\sigma k}1_{k+j\ge 0}2^{-j}(1+2^{j+k})^{-7}2^{-\sigma k}h_k(\sigma)+c_0^*2^{-\sigma k}1_{k+j\le 0}2^{\frac{k-j}{2}}2^{\delta|k+j|}2^{-\sigma k}h_k(\sigma).
\end{aligned}$$

Summing over $j\ge k_0$ as well yields

$$\begin{aligned}
&\sum_{j\ge k_0}2^{2j}\sum_{|k-k_2|\le 4,\,k_1\le k+4}\|P_k(P_{k_1}(A_i\psi_i)P_{k_2}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}\\
&\qquad\lesssim c_0^*2^{-\sigma k}1_{k+k_0\ge 0}2^{k_0}(1+2^{k+k_0})^{-7}h_k(\sigma)+c_0^*1_{k+k_0\le 0}2^{-k}2^{-\sigma k}h_k(\sigma).
\end{aligned}$$

Thus back to the LHS of (4.36), we conclude that if (4.26) holds, then

$$\|P_k\mathcal{U}_{II}\|_{F_k(T)}\lesssim c_0^*2^{-\sigma k}1_{k+k_0\ge 0}2^{k_0}(1+2^{k+k_0})^{-7}h_k(\sigma)+c_0^*1_{k+k_0\le 0}2^{-k}2^{-\sigma k}h_k(\sigma).$$

In particular, (4.26) holds, thus proving Claim A.

Now we are ready to prove our lemma with (4.25) and (4.26) being dropped. Define a function of $T'\in[0,T]$ by

$$\begin{aligned}
\Phi(T') = \sum_{\{j_c\}\subset\{1,\dots,2n\}}\sup_{k,j\in\mathbb{Z}}\sup_{s\in[2^{2j-1},2^{2j+1}]}&(c_0^*)^{-1}(1+2^k2^{s/2})^7T_{k,j}^{-1}h_k^{-1}\\
&\times(\|P_k\mathcal{U}_I\|_{F_k(T')}+\|P_k\mathcal{U}_{II}\|_{F_k(T')}).
\end{aligned}$$

Using Lemma 3.3 and Sobolev embeddings, we find that Φ is a continuous function on $[0,T]$. In order to proving our lemma, it suffices to prove $\Phi\lesssim 1$. It is easy to see that Φ is also an increasing continuous function on $[0,T]$. And Claim A shows

$$\Phi(T')\le\varepsilon^{-1/2}\implies\Phi(T')\lesssim 1.$$

Hence it suffices to verify

$$\lim_{T'\to 0}\Phi(T')\lesssim 1.$$

This reduces to proving that for all $j,k\in\mathbb{Z}$ and $s\in[2^{2j-1},2^{2j+1})$,

$$\begin{aligned}
\sum_{\{j_c\}\subset\{1,\dots,2n\}}&\Big\|P_k\int_s^\infty(A_i\psi_i)^q[(\widetilde{\nabla}\mathbf{R})(e_q;e_{j_0},\dots,e_{j_3})-\Gamma_q^{\infty,(1)}]\Big\|_{L_x^2}\\
&\qquad+\int_s^\infty\|P_k(A_i\psi_i)\|_{L_x^2}\,ds'\lesssim c_0^*(1+2^k2^{s/2})^{-7}T_{k,j}h_k,
\end{aligned}$$

where all the fields ψ_i and the matrices A_i are associated with the heat flow with initial data u_0. This can be proved by applying the results of Section 3. In fact, by the definition of $h_k(\sigma)$ one has

$$2^{\sigma k}\|\phi_i\restriction_{s=0}\|_{L_t^\infty L_x^2} \le h_k(\sigma)$$

if $\sigma \in [0, \frac{99}{100}]$. Then by Proposition 3.5 with $\eta_k(\sigma) = h_k(\sigma)$ we get

$$(1+2^{2k}s)^{29}\|P_k A_i(s)\|_{L_t^\infty L_x^2} \le 2^{-\sigma k} h_{k,s}(\sigma). \tag{4.39}$$

The proof of Proposition 3.5 shows

$$(1+2^{2k}s)^{30}\|P_k \psi_i(s)\|_{L_t^\infty L_x^2} \le 2^{-\sigma k} h_k(\sigma) \tag{4.40}$$

if $\sigma \in [0, \frac{99}{100}]$. Then by (4.40), (4.39) and bilinear Littlewood–Paley decomposition, one obtains

$$(1+2^{2k}s)^{28}\int_s^\infty \|P_k(A_i\psi_i)\|_{L_t^\infty L_x^2}\,ds' \lesssim \|\{h_k\}\|_{\ell^2} T_{k,j} h_k(\sigma) 2^{-\sigma k}$$

for $s \in [2^{2j-1}, 2^{2j+1})$ and $k, j \in \mathbb{Z}$. It remains to prove

$$(1+2^{2k}s)^{28}\int_s^\infty \|P_k[A_i\psi_i\widetilde{\mathcal{G}}^{(1)}]\|_{L_t^\infty L_x^2}\,ds' \lesssim \|\{h_k\}\|_{\ell^2} T_{k,j} h_k(\sigma) 2^{-\sigma k}, \quad \sigma \in \big[0, \tfrac{99}{100}\big]. \tag{4.41}$$

This follows by (4.39), (4.40), (3.52) and bilinear Littlewood–Paley decomposition as well. ∎

Remark 4.1. Checking the proof of Lemma 4.1, we see the range of $\sigma \in [0, \frac{99}{100}]$ was only used in the Low × High interaction of $(\partial_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$, $(A_i\psi_i)(\mathcal{U}_I + \mathcal{U}_{II})$ of Step 2 and Step 3 respectively.

Lemma 4.2. *If* $|k_1 - k| \le 4$*, then*

$$\|P_k(P_{k_1} f P_{\le k-4} g)\|_{F_k(T)} \lesssim \|P_{k_1} g\|_{F_{k_1}(T)} \|P_{\le k-4} g\|_{L^\infty}. \tag{4.42}$$

If $|k_2 - k_1| \le 8$ *and* $k_1, k_2 \ge k-4$*, then*

$$\|P_k(P_{k_1} f P_{k_2} g)\|_{F_k(T)} \lesssim \|P_{k_1} f\|_{F_{k_1}(T)} (\|P_{k_2} g\|_{L^\infty} + 2^{k_2/2}\|P_{k_2} g\|_{L_x^4 L_t^\infty}). \tag{4.43}$$

If $|k_2 - k| \le 4$ *and* $k_1 \le k-4$*, then*

$$\begin{aligned}\|P_k(P_{k_1} f P_{k_2} g)\|_{F_k(T)} \\ \lesssim 2^{k_1/2}\|P_{k_1} f\|_{F_{k_1}(T)} \|P_{k_2} g\|_{L_x^4 L_t^\infty} + 2^{k_1}\|P_{k_1} f\|_{F_{k_1}(T)} \|P_{k_2} g\|_{L^4}.\end{aligned} \tag{4.44}$$

For any $k_1, k_2, k \in \mathbb{Z}$*, one has*

$$\|P_k(P_{k_1} f P_{k_2} g)\|_{F_k(T)} \lesssim \|P_{k_1} f\|_{F_{k_1}(T)\cap S_{k_1}^{1/2}(T)} \|P_{k_2} g\|_{L^\infty}. \tag{4.45}$$

Proof. (4.42) has been given in [5]. And (4.43) follows by Hölder and [5, (3.17)] which says

$$\|P_k f\|_{F_k(T)} \lesssim \|P_k f\|_{L_x^2 L_t^\infty} + \|P_k f\|_{L_{t,x}^4}.$$

(4.44) follows for the same reason with additionally using the Bernstein inequality. Moreover, by definition,

$$\|P_k f\|_{L_{t,x}^4} \le \|f\|_{F_k(T)}, \quad \|P_k f\|_{L_x^2 L_t^\infty} \le \|f\|_{S_k^{1/2}(T)}.$$

Thus one obtains

$$\begin{aligned}\|P_k(P_{k_1} f P_{k_2} g)\|_{F_k(T)} &\lesssim \|P_{k_1} f P_{k_2} g\|_{L_x^2 L_t^\infty} + \|P_{k_1} f P_{k_2} g\|_{L_{t,x}^4}\\ &\lesssim (\|P_{k_1} f\|_{L_x^2 L_t^\infty} + \|P_{k_1} f\|_{L_{t,x}^4})\|P_{k_2} g\|_{L^\infty}\\ &\lesssim \|P_{k_2} g\|_{L^\infty}\|P_{k_1} f\|_{F_{k_1}(T)\cap S_{k_1}^{1/2}(T)}.\end{aligned}$$

■

The proof of Lemma 4.1 yields

Corollary 4.1. *Under the assumptions of Proposition 4.1 and (4.9), for $s\in[2^{2j-1}, 2^{2j+1})$, $\sigma \in [0, \frac{99}{100}]$ and $j, k \in \mathbb{Z}$,*

$$\|P_k(\widetilde{\mathcal{G}})\|_{F_k(T)} \lesssim 2^{-\sigma k} h_k(\sigma) T_{k,j}(1+2^{j+k})^{-7},$$

where $T_{k,j}$ is defined by (4.27). When $s = 0$, we have

$$\|P_k(\widetilde{\mathcal{G}})\upharpoonright_{s=0}\|_{F_k(T)} \lesssim 2^{-\sigma k} h_k(\sigma) 2^{-k}.$$

Proof. Lemma 4.1 gives

$$\begin{aligned}\|P_k(\mathcal{U}_{00})\|_{F_k(T)} + \|P_k(\mathcal{U}_{01})\|_{F_k(T)} &\lesssim 2^{-\sigma k} T_{k,j} h_k(\sigma)(1+2^{j+k})^{-7},\\ \|P_k \mathcal{U}_I\|_{F_k(T)\cap S_k^{1/2}(T)} &\lesssim 2^{-\sigma k} T_{k,j} h_k(\sigma)(1+2^{j+k})^{-7},\\ \|P_k \mathcal{U}_{II}\|_{F_k(T)} &\lesssim 2^{-\sigma k} T_{k,j} h_k(\sigma)(1+2^{j+k})^{-7}.\end{aligned}$$

Then the corollary follows by the decomposition

$$\widetilde{\mathcal{G}} = \mathcal{G} - \Gamma^\infty = \mathcal{U}_{00} + \mathcal{U}_{01} + \mathcal{U}_I + \mathcal{U}_{II},$$

and the inequality $(1+2^{j+k})^{-1} T_{k,j} \le 2^{-k}$ for all $j, k \in \mathbb{Z}$. ■

4.3. Evolution of differential fields along the heat flow

Recall the evolution equation for ϕ_i along the heat flow:

$$\begin{aligned}&(\partial_s - \Delta)\phi_i = K_i,\\ &K_i := 2\sum_{j=1}^2 \partial_j(A_j\phi_i) + \sum_{j=1}^2 (A_j^2 - \partial_j A_j)\phi_i + \sum_{j=1}^2 \phi_j \diamond \phi_i \diamond \phi_j \mathcal{G}. \qquad (4.46)\end{aligned}$$

Now we control the nonlinearities in the above equations.

Lemma 4.3. *Under the assumptions of Proposition 4.1 and (4.9), for all* $s \in [0, \infty)$*,* $i = 1, 2$ *and* $\sigma \in [0, \frac{99}{100}]$*, we have*

$$\left\| \int_0^s e^{(s-\tau)\Delta} P_k K_i(\tau)\, d\tau \right\|_{F_k(T)} \lesssim \varepsilon (1 + s2^{2k})^{-4} 2^{-\sigma k} h_k(\sigma). \tag{4.47}$$

Proof. First, we consider the quartic term $\mathcal{G}(\phi_i \diamond \phi_j) \diamond \phi_j$ in K_i. In [5, (5.25)] it is proved that for $\tau \in [2^{2j-1}, 2^{2j+1})$,

$$\begin{aligned} &\|P_k(\phi_i \diamond \phi_p \diamond \phi_l)(\tau)\|_{F_k(T) \cap S_k^{1/2}(T)} \\ &\qquad \lesssim \varepsilon 2^{-\sigma k} 2^{2k} (1 + 2^{2k+2j})^{-4} [h_k(\sigma) + 2^{-\frac{3}{2}(k+j)} h_{-j}(\sigma)]. \end{aligned} \tag{4.48}$$

Recall that $\mathcal{G} = \widetilde{\mathcal{G}} + \Gamma^\infty$. The constant part follows by (4.48). By bilinear Littlewood–Paley decomposition we have

$$\begin{aligned} &\|P_k(\phi_i \diamond \phi_p \diamond \phi_l \widetilde{\mathcal{G}})\|_{F_k(T)} \\ &\lesssim \sum_{\substack{k_1 \geq k-4 \\ |k_1 - k_2| \leq 8}} \|P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2} \widetilde{\mathcal{G}}\|_{F_k(T)} + \sum_{\substack{k_1 \leq k-4 \\ |k_2 - k| \leq 4}} \|P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2} \widetilde{\mathcal{G}}\|_{F_k(T)} \\ &\quad + \sum_{|k_1 - k| \leq 4} \|P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{\leq k-4} \widetilde{\mathcal{G}}\|_{F_k(T)}. \end{aligned}$$

For the High $\times$ Low term, directly applying $\|\widetilde{\mathcal{G}}\|_{L^\infty_{t,x}} \leq K(\mathcal{N})$ gives

$$\begin{aligned} \|P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{\leq k-4} \widetilde{\mathcal{G}}\|_{F_k(T)} &\lesssim \|P_k(\phi_i \diamond \phi_p \diamond \phi_l)\|_{F_k(T) \cap S_k^{1/2}(T)} \\ &\lesssim \varepsilon 2^{-\sigma k} 2^{2k} (1 + 2^{2k+2j})^{-4} [h_k(\sigma) + 2^{-\frac{3}{2}(k+j)} h_{-j}(\sigma)]. \end{aligned}$$

For the High $\times$ High term, denoting $\mathcal{V} := \phi_i \diamond \phi_p \diamond \phi_l$, Corollary 4.1 and (8.4) show

$$\begin{aligned} &\sum_{|k_1 - k_2| \leq 8,\, k_1, k_2 \geq k-4} \|P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2} \mathcal{G}\|_{F_k(T)} \\ &\lesssim \sum_{|k_1 - k_2| \leq 8,\, k_1, k_2 \geq k-4} 2^{\frac{k_1 + k}{2}} \|P_{k_1} \mathcal{V}\|_{F_{k_1}(T) \cap S_{k_1}^{1/2}(T)} \|P_{k_2} \mathcal{G}\|_{F_{k_2}(T)} \\ &\lesssim \sum_{k_1, k_2 \geq k-4,\, |k_1 - k_2| \leq 8} 2^{\frac{k_1 + k}{2}} 2^{-\sigma k_1 + 2k_1} (1 + 2^{k_1 + j})^{-15} \\ &\qquad \times [h_{k_1}(\sigma) + 2^{-\frac{3}{2}(k_1 + j)} h_{-j}(\sigma)] T_{k_2, j} h_{k_2}. \end{aligned} \tag{4.49}$$

If $k + j \geq 0$, then by slow variation of envelopes, (4.49) is bounded by

$$2^{3k+j} 2^{-\sigma k} (1 + 2^{k+j})^{-14} h_k(\sigma) h_k.$$

If $k + j \leq 0$, using $(1 + 2^{k+j})^{-1} T_{k,j} \leq 2^{-k}$ for all $k \in \mathbb{Z}$, (4.49) is dominated by

$$2^{k/2 - 3j/2} 2^{-\sigma k} h_k(\sigma) h_k 2^{\delta |j+k|}.$$

Therefore, for the High × High interaction, if $s \in [2^{2j-1}, 2^{2j+1})$ then

$$\sum_{|k_1-k_2|\le 8,\, k_1,k_2\ge k-4} \| P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2} \widetilde{\mathcal{G}} \|_{F_k(T)} \lesssim 2^{-\sigma k - \frac{3}{2}j} 2^{k/2} 2^{\delta|j+k|} h_k h_k(\sigma)(1 + 2^{2k+2j})^{-5}. \quad (4.50)$$

To finish estimates for the High × High interaction, we verify the corresponding part in (4.47). We use (4.50) to verify (4.47). Let $s \in [2^{2k_0-1}, 2^{2k_0+1})$ with $k_0 \in \mathbb{Z}$ fixed. For $k + k_0 \le 0$, by (2.6) one has

$$\begin{aligned}
&\left\| \int_0^s e^{(s-\tau)\Delta} \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} P_k(P_{k_1} \mathcal{V} P_{k_2} \widetilde{\mathcal{G}})\, d\tau \right\|_{F_k(T)} \\
&\lesssim \sum_{j\le k_0} \int_{2^{2j-1}}^{2^{2j+1}} \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} \| P_k(P_{k_1} \mathcal{V} p_{k_2} \widetilde{\mathcal{G}}) \|_{F_k(T)}\, d\tau \\
&\lesssim \varepsilon \sum_{j\le k_0} 2^{-\sigma k} h_k(\sigma) \big(2^{\frac{1}{2}(j+k)} + 2^{(\frac{1}{2}\pm\delta)(k+j)} \big) \\
&\lesssim \varepsilon 2^{-\sigma k} h_k(\sigma).
\end{aligned}$$

For $k + k_0 \ge 0$, by (2.6) and (4.50), one has

$$\begin{aligned}
&\int_0^s \left\| e^{(s-\tau)\Delta} \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} P_k(P_{k_1} \mathcal{V} P_{k_2} \widetilde{\mathcal{G}}) \right\|_{F_k(T)} d\tau \\
&\lesssim \int_0^{s/2} \ldots\, d\tau + \int_{s/2}^s \ldots\, d\tau \\
&\lesssim \sum_{j\le -k_0-1} \int_{2^{2j-1}}^{2^{2j+1}} 2^{-20(k+k_0)} \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} \| P_k(P_{k_1} \mathcal{V} P_{k_2} \widetilde{\mathcal{G}}) \|_{F_k(T)}\, d\tau \\
&\quad + 2^{2k_0} \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} \sup_{\tau\in[2^{2k_0-2}, 2^{2k_0+1}]} \| P_k(P_{k_1} \mathcal{V} P_{k_2} \widetilde{\mathcal{G}}) \|_{F_{k(T)}} \\
&\lesssim \varepsilon 2^{-20(k+k_0)} \sum_{j\le -k_0-1} 2^{-\sigma k} h_k(\sigma) \big(2^{\frac{1}{2}(j+k)} + 2^{(\frac{1}{2}\pm\delta)(j+k)} \big) \\
&\quad + \varepsilon 2^{-\sigma k} h_k(\sigma)(1 + 2^{k+k_0})^{-10} \big[2^{\frac{1}{2}(k_0+k)} + 2^{(\frac{1}{2}\pm\delta)(k_0+k)} \big] \\
&\lesssim \varepsilon (1 + 2^{k+k_0})^{-8} 2^{-\sigma k} h_k(\sigma).
\end{aligned}$$

Thus we conclude

$$\int_0^s \left\| e^{(s-\tau)\Delta} \sum_{k_1\ge k-4,\, |k_1-k_2|\le 8} P_k(P_{k_1} \mathcal{V} P_{k_2} \widetilde{\mathcal{G}}) \right\|_{F_k(T)} d\tau \lesssim \varepsilon 2^{-\sigma k} h_k(\sigma)(1 + 2^{2k} s)^{-4}.$$

For the Low × High term, using

$$\| P_{k_1}(\mathcal{V}) \|_{L^\infty_{t,x}} \lesssim 2^{k_1} \varepsilon 2^{-\sigma k_1 + 2k_1} (1 + 2^{k_1+j})^{-8} [h_{k_1}(\sigma) + 2^{-\frac{3}{2}(k_1+j)} h_{-j}(\sigma)]$$

we deduce by Corollary 4.1 and Lemma 4.2 that

$$\sum_{k_1\le k-4,\,|k-k_2|\le 4} \big\| P_k\big(P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2}\widetilde{\mathcal{G}}\big)\big\|_{F_k(T)} \lesssim 2^{-\sigma k} T_{k,j}(1+2^{j+k})^{-7} h_k(\sigma)\varepsilon \sum_{k_1\le k-4} 2^{3k_1}(1+2^{k_1+j})^{-8}(1+2^{-\frac{3}{2}(k_1+j)}).$$

For $k+j\ge 0$, we have

$$\sum_{k_1\le k-4,\,|k-k_2|\le 4} \big\| P_k\big(P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2}\widetilde{\mathcal{G}}\big)\big\|_{F_k(T)} \lesssim \varepsilon 2^{-\sigma k} 2^{-2j}(1+2^{k+j})^{-7} h_k(\sigma). \tag{4.51}$$

For $k+j\le 0$, we have

$$\sum_{k_1\le k-4,\,|k-k_2|\le 4} \big\| P_k\big(P_{k_1}(\phi_i \diamond \phi_p \diamond \phi_l) P_{k_2}\widetilde{\mathcal{G}}\big)\big\|_{F_k(T)} \lesssim \varepsilon 2^{-\sigma k} 2^{\frac{1}{2}k-\frac{3}{2}j}(1+2^{k+j})^{-7} h_k(\sigma). \tag{4.52}$$

As a summary, we use (4.51) and (4.52) to verify (4.47). Let $s\in[2^{2k_0-1},2^{2k_0+1})$ with k_0 fixed. For $k+k_0\le 0$, by (2.6) one sees for the Low × High interaction that

$$\begin{aligned}\int_0^s \Big\| e^{(s-\tau)\Delta} \sum_{k_1\le k-4,\,|k-k_2|\le 4} P_k(P_{k_1}\mathcal{V} P_{k_2}\widetilde{\mathcal{G}})\Big\|_{F_k(T)} d\tau &\lesssim \sum_{j\le k_0}\int_{2^{2j-1}}^{2^{2j+1}} \dots\, d\tau \\ &\lesssim \varepsilon \sum_{j\le k_0} 2^{j/2+k/2} 2^{-\sigma k} h_k(\sigma) \lesssim 2^{-\sigma k}\varepsilon h_k(\sigma).\end{aligned}$$

For $k+k_0\ge 0$, similarly we have

$$\begin{aligned}&\int_0^s \Big\| e^{(s-\tau)\Delta} \sum_{k_1\le k-4,\,|k-k_2|\le 4} P_k(P_{k_1}\mathcal{V} P_{k_2}\widetilde{\mathcal{G}})\Big\|_{F_k(T)} d\tau \\ &\quad\lesssim \varepsilon 2^{-20(k+k_0)} 2^{-\sigma k} h_k(\sigma) \sum_{j\le -k_0-1} 2^{\frac{1}{2}(j+k)} + \varepsilon 2^{-\sigma k} h_k(\sigma)(1+2^{k+k_0})^{-7} 2^{-2k_0-2k} \\ &\quad\lesssim \varepsilon(1+2^{k+k_0})^{-8} 2^{-\sigma k} h_k(\sigma).\end{aligned}$$

Thus we conclude

$$\int_0^s \Big\| e^{(s-\tau)\Delta} \sum_{k_1\le k-4,\,|k-k_2|\le 4} P_k(P_{k_1}\mathcal{V} P_{k_2}\widetilde{\mathcal{G}})\Big\|_{F_k(T)} d\tau \lesssim \varepsilon(1+2^{k+k_0})^{-8} 2^{-\sigma k} h_k(\sigma).$$

And the High × Low case is easy by repeating the same argument or directly applying the result of [5, Lemma 5.3]. Therefore, the curvature term has been handled:

$$\Big\| \int_0^s e^{(s-\tau)\Delta} P_k(\phi_i \diamond \phi_p \diamond \phi_l \widetilde{\mathcal{G}})\, d\tau \Big\|_{F_k(T)} \lesssim \varepsilon(1+s2^{2k})^{-4} 2^{-\sigma k} h_k(\sigma).$$

Step 2. Connection coefficient terms. In this step, we turn to estimating the terms $\partial_l(A_l\psi_i)$, $\partial_l A_l\phi_i$ and $A_l^2\phi_i$. With Lemma 4.1 in hand, all these terms follow directly by repeating arguments of [5, Lemma 5.3]. ■

Lemma 4.4. *Under the assumptions of Proposition 4.1 and (4.9), for all $k\in\mathbb{Z}$, $s\geq 0$ and $i=1,2$, we have*

$$\|P_k\phi_i(s)\|_{F_k(T)}\lesssim b_k(\sigma)2^{-\sigma k}(1+s2^{2k})^{-4},\quad \sigma\in\big[0,\tfrac{99}{100}\big]. \tag{4.53}$$

Proof. By the Duhamel principle and (4.47), we get

$$\sup_{s\geq 0}(1+s2^{2k})^4 2^{\sigma k}\|P_k\phi_i(s)\|_{F_k(T)}\lesssim b_k(\sigma)+\varepsilon h_k(\sigma). \tag{4.54}$$

Since the RHS of (4.54) is a frequency envelope of order δ, by the definition of $\{h_k(\sigma)\}$ we get

$$h_k(\sigma)\lesssim b_k(\sigma)+\varepsilon h_k(\sigma), \tag{4.55}$$

which by letting ε be sufficiently small yields

$$h_k(\sigma)\lesssim b_k(\sigma). \tag{4.56}$$

■

Lemma 4.5. *Under the assumptions of Proposition 4.1 and (4.9), for all $k\in\mathbb{Z}$, $s\geq 0$ and $i=1,2$ we have*

$$\|P_kA_i\restriction_{s=0}\|_{L^4_{t,x}}\lesssim b_k(\sigma)2^{-\sigma k}. \tag{4.57}$$

Proof. In the proof of Lemma 4.4 we have shown (4.56). Then the previous bounds in Lemma 4.1 now hold with $h_k(\sigma)$ replaced by $b_k(\sigma)$. Recall that in [5, pp. 1473–1474] it is proved that

$$\|P_k\phi_s\|_{L^4_{t,x}}\lesssim 2^k2^{-\sigma k}b_k(\sigma)(2^{2k}s)^{-3/8}(1+s2^{2k})^{-3}. \tag{4.58}$$

We also recall the bilinear estimate of [5, Lemma 5.4] in our Appendix A, Lemma 8.4. Then (4.58) and (4.53) show

$$\begin{aligned}\|P_k(\phi_i\diamond\phi_s)\|_{L^4_{t,x}}&\lesssim 2^{-\sigma k}\sum_{l\leq k}b_k(\sigma)b_l2^{l+k}(s2^{2k})^{-3/8}(1+s2^{2k})^{-3}\\&\quad+2^{-\sigma k}\sum_{l\leq k}b_k(\sigma)b_l2^{2l}2^{\frac12(k-l)}(2^{2k}s)^{-3/8}(1+s2^{2k})^{-4}\\&\quad+\sum_{l\geq k}2^{-\sigma l}b_l(\sigma)b_l2^{k+l}(2^{2l}s)^{-3/8}(1+s2^{2l})^{-7}.\end{aligned}$$

Thus given $s\in[2^{2j-1},2^{2j+2})$ with $j\in\mathbb{Z}$, we conclude that for $k+j\geq 0$,

$$\|P_k(\phi_i\diamond\phi_s)\|_{L^4_{t,x}}\lesssim b_k(\sigma)b_k2^{2k-\sigma k}(s2^{2k})^{-3/8}(1+s2^{2k})^{-3}, \tag{4.59}$$

and for $k+j\leq 0$,

$$\|P_k(\phi_i\diamond\phi_s)\|_{L^4_{t,x}}\lesssim b_k(\sigma)b_k2^{2\delta|k+j|}2^{-\sigma k}2^k2^{-j}. \tag{4.60}$$

Recalling that for $i = 1, 2$,

$$A_i(0) = \int_0^\infty (\phi_i \diamond \phi_s)\mathcal{G}\, ds, \tag{4.61}$$

we see that it remains to deal with the interaction of $\phi_s \diamond \phi_i$ with $\mathcal{G}$. Recall that $\mathcal{G} = \Gamma^\infty + \widetilde{\mathcal{G}}$ with

$$\|P_k(\widetilde{\mathcal{G}})\|_{F_k(T)} \lesssim 2^{-\sigma k} T_{k,j} (1 + 2^{k+j})^{-7} h_k(\sigma) \tag{4.62}$$

for $s \in [2^{2j-1}, 2^{2j+1})$ with $j \in \mathbb{Z}$. (4.59) and (4.60) show that the constant part Γ^∞ contributes $b_k(\sigma)2^{-\sigma k}$ to $\|A_i(0)\|_{L^4_{t,x}}$. Thus it suffices to control $(\phi_i \diamond \phi_s)\widetilde{\mathcal{G}}$.

As before, we consider three cases according to Littlewood–Paley decomposition. Using the trivial bound $\|\widetilde{\mathcal{G}}\|_{L^\infty_{t,x}} \le K(\mathcal{N})$ in the High $\times$ Low part gives

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \int_{2^{2j-1}}^{2^{2j+1}} \sum_{|k_1-k|\le 4} \|P_k[P_{k_1}(\phi_s \diamond \phi_i) P_{\le k-4}\widetilde{\mathcal{G}}]\|_{L^4_{t,x}}\, ds \\
&\quad \lesssim \sum_{j\ge -k} 1_{k+j\ge 0} b_k(\sigma) b_k 2^{2k+2j-\sigma k} (2^{2k+2j})^{-3/8} (1 + 2^{2k+2j})^{-3} \\
&\qquad + \sum_{j\le -k} 1_{k+j\le 0} b_k(\sigma) b_k 2^{2\delta|k+j|} 2^{-\sigma k} 2^k 2^j \\
&\quad \lesssim b_k(\sigma) b_k 2^{-\sigma k}.
\end{aligned}$$

Notice that Lemma 8.4 shows

$$\sum_{|k_1-k_2|\le 8,\, k_1,k_2\ge k-4} \|P_k(P_{k_1} f P_{k_2} g)\|_{L^4_{t,x}} \lesssim \sum_{k_1\ge k} 2^{k(1+\omega)} 2^{-\omega k_1} \mu_{k_1} \nu_{k_1},$$

where $\mu_k = \sum_{|k-k'|\le 20} \|P_{k'} f\|_{S^\omega_{k'}}$ and $\nu_k = \sum_{|k-k'|\le 20} \|P_{k'} g\|_{L^4_{t,x}}$. Thus using (4.62), we have by choosing $\omega = 0$ that

$$\begin{aligned}
&\int_0^\infty \sum_{|k_1-k_2|\le 8,\, k_1,k_2\ge k-4} \|P_k[P_{k_1}(\phi_s \diamond \phi_i) P_{k_2}\widetilde{\mathcal{G}}]\|_{L^4_{t,x}}\, ds \\
&\quad \lesssim \sum_{j\ge -k} 2^{2j} \varepsilon 2^k \sum_{k_1\ge k-4} b_{k_1} b_{k_1}(\sigma) 2^{k_1} 2^{-\sigma k_1} (1 + 2^{2j+2k_1})^{-5} \\
&\qquad + \sum_{j\le -k} 2^{-\sigma k} 2^k \sum_{k-4\le k_1\le -j} 2^j b_{k_1}(\sigma) b_{k_1} 2^{2\delta|k_1+j|} \\
&\qquad + 2^{-\sigma k} \sum_{j\le -k} 2^{2j} 2^k \sum_{k_1\ge -j} b_{k_1}(\sigma) b_{k_1} 2^{2\delta|k_1+j|} 2^{k_1} (2^{2j+2k_1})^{-3/8} (1 + 2^{2j+2k_1})^{-5} \\
&\quad \lesssim \sum_{j\ge -k} 2^{-\sigma k} 2^{2j+2k} (1 + 2^{2k+2j})^{-5} b_k b_k(\sigma) + \sum_{j\le -k} 2^{-\sigma k} 2^{(k+j)+\delta|k+j|} b_k b_k(\sigma) \\
&\quad \lesssim b_k b_k(\sigma) 2^{-\sigma k},
\end{aligned}$$

where we have used $(1 + 2^{k+j})^{-1} T_{k,j} \le 2^{-k}$ for all $k \in \mathbb{Z}$. In the Low $\times$ High part, Lemma 8.4 shows

$$\sum_{|k_2-k|\le 4,\, k_1\le k-4} \|P_k(P_{k_1} f P_{k_2} g)\|_{L^4_{t,x}} \lesssim \sum_{l\le k} 2^l \mu_l \nu_k,$$

where $\mu_k = \sum_{|k'-k|\le 20} \|P_{k'} f\|_{S^\omega_k}$ and $\nu_k = \sum_{|k'-k|\le 20} \|P_{k'} g\|_{L^4_{t,x}}$. Then by (4.62) we have

$$\begin{aligned}
&\sum_{|k_2-k|\le 4,\, k_1\le k-4} \|P_k[P_{k_1}(\phi_s \diamond \phi_i) P_{k_2}\widetilde{\mathcal{G}}]\|_{L^4_{t,x}} \\
&\quad\lesssim 2^{-\sigma k} b_k(\sigma)\varepsilon T_{k,j}(1+2^{j+k})^{-7} \\
&\qquad\times \sum_{l\le k} 2^l b_l b_l \big(1_{l+j\le 0} 2^{2\delta|l+j|} 2^{l-j} + 1_{j+l\ge 0} 2^{2l} 2^{-\frac{3}{4}(j+l)}(1+2^{2j+2l})^{-3}\big) \\
&\quad\lesssim 2^{-\sigma k} b_k(\sigma) b_k \big(1_{k+j\ge 0} 2^{-2j}(1+2^{k+j})^{-4} + 1_{k+j\le 0} 2^{k-j} 2^{\delta|k+j|}\big).
\end{aligned}$$

Hence for the Low $\times$ High part we conclude that

$$\begin{aligned}
&\int_0^\infty \sum_{|k_2-k|\le 4,\, k_1\le k-4} \|P_k[P_{k_1}(\phi_s \diamond \phi_i) P_{k_2}\widetilde{\mathcal{G}}]\|_{L^4_{t,x}}\, ds \\
&\quad\lesssim 2^{-\sigma k} \sum_{j\in\mathbb{Z}} b_k b_k(\sigma) 1_{k+j\ge 0}(1+2^{k+j})^{-4} + 2^{-\sigma k} \sum_{j\in\mathbb{Z}} b_k b_k(\sigma) 1_{k+j\le 0} 2^{k+j} 2^{\delta|k+j|} \\
&\quad\lesssim b_k b_k(\sigma) 2^{-\sigma k}.
\end{aligned}$$

Therefore, we get

$$\|P_k(A_i(0))\|_{L^4_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma). \qquad \blacksquare$$

Now we turn to the bounds for ϕ_t stated in Proposition 4.1.

Lemma 4.6. *Assume that the assumptions of Proposition 4.1 and (4.9) hold. Then for $\sigma \in [0, \frac{99}{100}]$, one has*

$$\|P_k \phi_t(s)\|_{L^4_{t,x}} \lesssim b_k(\sigma) 2^{-(\sigma-1)k}(1+2^{2k}s)^{-2}. \tag{4.63}$$

Proof. Recall that ϕ_t satisfies

$$\begin{aligned}
\partial_s \phi_t - \Delta \phi_t &= L(\phi_t), \\
L(\phi_t) &:= L_1(\phi_t) + L_2(\phi_t), \\
L_1(\phi_t) &:= \sum_{i=1}^2 2\partial_i(A_i \phi_t) + \Big(\sum_{l=1}^2 A_l^2 - \partial_l A_l\Big)\phi_t, \\
L_2(\phi_t) &:= \sum_{i=1}^2 (\phi_t \diamond \phi_i) \diamond \phi_i \mathcal{G}.
\end{aligned}$$

By the Duhamel principle, ϕ_t can be written as

$$\phi_t = e^{s\Delta}\phi_t\restriction_{s=0} + \int_0^s e^{(s-\tau)\Delta} L(\phi_t(\tau))\, d\tau. \tag{4.64}$$

By a uniqueness argument as in [5, Lemma 5.6], in order to prove (4.63), it suffices to show that

$$\|P_k \phi_t(s)\|_{L^4_{t,x}} \lesssim b_k(\sigma) 2^{-(\sigma-1)k}(1+2^{2k}s)^{-2} \tag{4.65}$$

implies

$$\int_0^s \|e^{(s-\tau)\Delta}L(\phi_t(\tau))\|_{L^4_{t,x}}\,d\tau \lesssim \varepsilon^2 b_k(\sigma)2^{-(\sigma-1)k}(1+2^{2k}s)^{-2}. \tag{4.66}$$

The $L_1(\phi_t)$ part of (4.66) has been shown in [5, Lemma 5.6]. It suffices to prove (4.66) for $L_2(\phi_t)$ under the assumption of (4.65). Recall also that $\mathcal{G}=\Gamma^\infty+\widetilde{\mathcal{G}}$ satisfies

$$\|P_k(\mathcal{G}-\Gamma^\infty)\|_{F_k(T)} \lesssim 2^{-\sigma k}(1+2^{j+k})^{-7}T_{k,j}. \tag{4.67}$$

By the proof of [5, Lemma 5.6],

$$\|P_k(\phi_t(s)\diamond\phi_i\diamond\phi_l)\|_{L^4_{t,x}} \lesssim b_k^2 2^{-(\sigma-3)k}(1+2^{2k}s)^{-2}(s2^{2k})^{-7/8}b_k(\sigma). \tag{4.68}$$

Then the Γ^∞ part of $L_2(\phi_t)$ follows directly from that proof.

Denote $\mathbf{P}=\phi_t(s)\diamond\phi_i\diamond\phi_l$. In order to control $\mathbf{P}(\mathcal{G}-\Gamma^\infty)$, we first control $\|(\phi_i\diamond\phi_l)\widetilde{\mathcal{G}}\|_{S_k^{1/2}(T)}$. We have seen

$$\|P_k(\phi_i\diamond\phi_l)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1+2^{2k+2j})^{-4}2^{-j}b_{-j}b_{\max(k,-j)}(\sigma). \tag{4.69}$$

Thus applying bilinear Littlewood–Paley decomposition, we find by (4.67) that

$$\begin{aligned}
\|P_k(\phi_i\diamond\phi_l\widetilde{\mathcal{G}})\|_{F_k(T)\cap S_k^{1/2}(T)} &\lesssim \sum_{|k_1-k|\le 4}\|P_{k_1}(\phi_i\diamond\phi_l)\|_{F_k(T)\cap S_k^{1/2}(T)}\|P_{\le k-4}\widetilde{\mathcal{G}}\|_{L^\infty}\\
&\quad+\sum_{|k_1-k_2|\le 8,\,k_1,k_2\ge k-4} 2^{\frac{k+k_1}{2}}\|P_{k_1}(\phi_i\diamond\phi_l)\|_{F_k(T)\cap S_k^{1/2}(T)}\|P_{k_2}\widetilde{\mathcal{G}}\|_{F_{k_2}(T)}\\
&\quad+\sum_{|k_2-k|\le 4,\,k_1\le k-4} 2^{k_1}\|P_{k_1}(\phi_i\diamond\phi_l)\|_{F_k(T)\cap S_k^{1/2}(T)}\|P_{k_2}\widetilde{\mathcal{G}}\|_{F_{k_2}(T)}\\
&\lesssim 2^{-\sigma k}b_k(\sigma)b_k 2^{\delta|k+j|}\big(1_{k+j\le 0}2^{-j}+2^k 1_{k+j\ge 0}(1+2^{k+j})^{-7}\big).
\end{aligned}$$

Then using Lemma 8.4 with $\omega=\frac{1}{2}$ and (4.67), $\mathbf{P}\widetilde{\mathcal{G}}$ is dominated by

$$\begin{aligned}
&\|P_k(P_{k_1}\phi_t P_{k_2}(\phi_i\diamond\phi_l\widetilde{\mathcal{G}}))\|_{L^4_{t,x}}\\
&\quad\lesssim \sum_{|k_1-k|\le 4,\,k_2\le k-4} 2^{k_2}\|P_{k_1}\phi_t\|_{L^4_{t,x}}\|P_{k_2}(\phi_i\diamond\phi_l\widetilde{\mathcal{G}})\|_{S_k^{1/2}(T)}\\
&\quad\quad+2^k\sum_{|k_1-k_2|\le 8,\,k_2,k_1\ge k-4} 2^{-\frac{1}{2}(k_1-k)}\|P_{k_1}\phi_t\|_{L^4_{t,x}}\|P_{k_2}(\phi_i\diamond\phi_l\widetilde{\mathcal{G}})\|_{S_k^{1/2}(T)}\\
&\quad\quad+\sum_{|k_2-k|\le 4,\,k_1\le k-4} 2^{\frac{k+k_1}{2}}\|P_{k_1}\phi_t\|_{L^4_{t,x}}\|P_{k_2}(\phi_i\diamond\phi_l\widetilde{\mathcal{G}})\|_{S_k^{1/2}(T)}\\
&\quad\lesssim 2^{-\sigma k}b_k(\sigma)b_k 2^{\delta|k+j|}1_{k+j\le 0}(2^{\frac{3}{2}k-\frac{3}{2}j}+2^{2k-j})\\
&\quad\quad+2^{-\sigma k}b_k(\sigma)b_k 1_{k+j\ge 0}\\
&\quad\quad\times\big(2^{\delta|k+j|}2^{3k}(1+2^{k+j})^{-10}+2^{\frac{3}{2}(k-j)}(1+2^{k+j})^{-7}+2^{k-2j}(1+2^{k+j})^{-4}\big)
\end{aligned}$$

for $s \in [2^{2j-1}, 2^{2j+1})$ and $j, k \in \mathbb{Z}$. As a summary, inserting this bound to the heat estimates

$$\int_0^{\widetilde{s}} \|e^{(\widetilde{s}-s)\Delta} P_k[\phi_t \diamond \phi_i \diamond \phi_l \widetilde{\mathcal{G}}]\|_{L^4_{t,x}}\, ds \lesssim \int_0^{\widetilde{s}} (1 + |\widetilde{s} - s|)^{-N} (\dots)\, ds,$$

we conclude that

$$\left\| P_k \int_0^s e^{(s-\tau)\Delta} L_2(\phi_t(\tau))\, d\tau \right\|_{L^4_{t,x}} \lesssim \varepsilon(1 + 2^{2k}s)^{-2} 2^{-\sigma k + k} b_k(\sigma).$$

Since L_1 has been handled before, we have finished the proof. ■

Lemma 4.7. *With the assumptions of Proposition 4.1, the bootstrap assumption (4.9) can be improved to*

$$2^{k/2} \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t} \lesssim h_k (1_{k+j\geq 0}(1 + s2^{2k})^{-20} + 2^{\delta|k+j|} 1_{k+j\leq 0})$$

for any $k, j \in \mathbb{Z}$ and $s \in [2^{2j-1}, 2^{2j+1})$.

Proof. The proof of Lemma 4.1 has shown that for any $\sigma \in [0, \frac{99}{100}]$, $k, j \in \mathbb{Z}$ and $s \in [2^{2j-1}, 2^{2j+1})$,

$$\|P_k(\mathcal{G} D_i \phi_i)\|_{L^4 \cap L^\infty_t L^2_x} \lesssim h_k(\sigma) 2^{-\sigma k + k} (1 + s2^{2k})^{-3} 1_{j+k\geq 0} + 2^{-j} 2^{\delta|k+j|} 1_{k+j\leq 0}. \tag{4.70}$$

Meanwhile, Lemma 4.6 yields

$$\|P_k \phi_t\|_{L^4} \lesssim b_k 2^k (1 + s2^{2k})^{-2}. \tag{4.71}$$

Recall that $b_k \leq \varepsilon^{1/2}$ for any $k \in \mathbb{Z}$. Then bilinear Littlewood–Paley decomposition shows

$$\begin{aligned}&\|P_k(\phi_t(D_i \phi_i \mathcal{G}))\|_{L^4} \\ &\quad \lesssim h_k(\sigma) 2^{-\sigma k + 3k} (1 + s2^{2k})^{-2} 1_{k+j\geq 0} + 2^{-2j} 2^{k-\sigma k} 2^{2\delta|k+j|} h_k(\sigma) h_k 1_{k+j\leq 0}\end{aligned}$$

for $i = 1, 2$, $\sigma \in [0, \frac{99}{100}]$, $k, j \in \mathbb{Z}$ and $s \in [2^{2j-1}, 2^{2j+1})$. Here, in the High × Low interaction of $\phi_t(D_i \phi_i \mathcal{G})$ we use

$$\begin{aligned}&\sum_{|k_1-k|\leq 4,\, k_2\leq k-4} \|P_k(P_{k_1}\phi_t P_{k_2}(D_i \phi_i \mathcal{G}))\|_{L^4} \\ &\quad \lesssim b_k 2^k \sum_{k_2\leq k-4} h_{k_2}(\sigma) 2^{-\sigma k_2 + 2k_2} \big((1 + s2^{2k_2})^{-3} 1_{j+k_2\geq 0} + 2^{-j} 2^{\delta|k_2+j|} 1_{k_2+j\leq 0}\big).\end{aligned}$$

The other two frequency interactions are standard. Thus

$$\begin{aligned}\int_s^\infty \|P_k(\phi_t D_i \phi_i)\mathcal{G}\|_{L^4}\, ds' &\lesssim h_k(\sigma) 2^{-\sigma k + k} (1 + 2^{2k+2k_0})^{-1} 1_{k+k_0\geq 0} \\ &\quad + 2^{k-\sigma k} h_k(\sigma)(1 + 2^{2\delta|k+k_0|} h_k) 1_{k+k_0\leq 0}\end{aligned} \tag{4.72}$$

for any $\sigma \in [0, \frac{99}{100}]$, $k, k_0 \in \mathbb{Z}$ and $s \in [2^{2k_0-1}, 2^{2k_0+1})$. Recalling that

$$A_t = \int_s^\infty (\phi_t \diamond \phi_s)\mathcal{G}\, ds', \quad \phi_s = \sum_{i=1,2} D_i\phi_i,$$

we see that $\|P_k A_t\|_{L^4}$ is bounded by the RHS of (4.72). By the schematic formula

$$\partial_t(\widetilde{\mathcal{G}}^{(1)}) = \phi_t \mathcal{G}^{(2)} + A_t \mathcal{G}^{(1)},$$

and the bounds

$$\|P_k(\widetilde{\mathcal{G}}^{(l)})\|_{L^4 \cap L_t^\infty L_x^2} \lesssim (1+s2^{2k})^{-30} 2^{-\sigma k - k} h_k(\sigma), \quad \forall l = 1, 2,$$

we deduce from bilinear Littlewood–Paley decomposition that

$$\|P_k \partial_t(\widetilde{\mathcal{G}}^{(1)})\|_{L^4} \lesssim h_k 2^k (1 + 2^{2\delta|k+k_0|} h_k \mathbf{1}_{k+k_0 \le 0}).$$

Then by the Gagliardo–Nirenberg inequality we get

$$\begin{aligned} 2^{k/2} \|P_k(\widetilde{\mathcal{G}}^{(1)})\|_{L_x^4 L_t^\infty} &\lesssim \|P_k(\widetilde{\mathcal{G}}^{(1)})\|_{L^4}^{3/4} \|\partial_t P_k(\widetilde{\mathcal{G}}^{(1)})\|_{L^4}^{1/4} \\ &\lesssim h_k(1+s2^{2k})^{-20} \mathbf{1}_{k+k_0 \ge 0} + h_k 2^{\delta|k+k_0|} \mathbf{1}_{k+k_0 \le 0} \end{aligned}$$

for $k_0, k \in \mathbb{Z}$ and $s \in [2^{2k_0-1}, 2^{2k_0+1})$. ∎

4.4. *End of proof of Proposition 4.1*

By Lemma 4.7, the assumption (4.9) in Lemmas 4.4–4.6 can be dropped. In fact, let

$$\widetilde{\Phi}(T') := \sup_{k,j \in \mathbb{Z}} \sup_{s \in [2^{2j-1}, 2^{2j+1})} h_k^{-1} \big(\mathbf{1}_{k+j \ge 0}(1+s2^{2k})^{-20} + 2^{\delta|k+j|}\mathbf{1}_{k+j \le 0}\big)^{-1} 2^{k/2} \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L_x^4 L_t^\infty(T')}.$$

Lemma 3.3 and Sobolev embeddings imply $\widetilde{\Phi}$ is an increasing continuous function on $T' \in [0, T]$. Lemma 4.7 shows $\widetilde{\Phi}(T') \le \varepsilon^{-1/4} \Rightarrow \widetilde{\Phi}(T') \lesssim 1$. Then by the Bernstein inequality and letting $T' \to 0$, it remains to prove

$$2^k \|P_k \widetilde{\mathcal{G}}^{(1)}\|_{L_x^2} \lesssim h_k \big(\mathbf{1}_{k+j \ge 0}(1+s2^{2k})^{-20} + 2^{\delta|k+j|}\mathbf{1}_{k+j \le 0}\big)$$

along the heat flow initiated from u_0 for any $k, j \in \mathbb{Z}$ and $s \in [2^{2j-1}, 2^{2j+1})$. This follows by (3.73).

By Lemma 3.3 and similar arguments, assumption (4.3) can also be dropped. Thus Lemmas 4.4–4.6 all hold only assuming (4.1) and (4.2) of Proposition 4.1. To complete the proof of Proposition 4.1, it remains to prove the $L_{t,x}^2$ bound for A_t.

Lemma 4.8. *With the assumptions (4.1), (4.2) of Proposition 4.1, for all $k \in \mathbb{Z}$, one has*

$$\|P_k A_t \restriction_{s=0}\|_{L_{t,x}^2} \lesssim \varepsilon b_k(\sigma) 2^{-\sigma k}, \quad \sigma \in \big[\tfrac{1}{100}, \tfrac{99}{100}\big], \tag{4.73}$$

$$\|A_t \restriction_{s=0}\|_{L_{t,x}^2} \lesssim \varepsilon^2, \quad \sigma \in \big[0, \tfrac{99}{100}\big]. \tag{4.74}$$

Proof. Recall that in [5, Lemma 5.7] it is proved that

$$\begin{aligned}\|P_k(\phi_t\diamond\phi_s)\|_{L^2_{t,x}}\lesssim&\sum_{l\le k}2^{-\sigma l}2^{l+k}b_k(\sigma)b_l(s2^{2l})^{-3/8}(1+s2^{2k})^{-2}\\&+\sum_{l\ge k}2^{-\sigma l}2^{2l}b_l(\sigma)b_l(s2^{2l})^{-3/8}(1+s2^{2l})^{-4}.\end{aligned}\tag{4.75}$$

Denote the RHS of (4.75) by $\mathbf{a}_k(\sigma)$ for simplicity.

Since $A_t(0)=\int_0^\infty(\phi_t\diamond\phi_s)\mathcal{G}\,ds$, (4.74) follows by directly applying [5, Lemma 5.7] and $\|\mathcal{G}\|_{L^\infty}\lesssim K(\mathcal{N})$. For (4.73), we need to clarify the frequency interaction between $\phi_t\diamond\phi_s$ and $\mathcal{G}$ as before. The constant part of $\mathcal{G}$ follows by (4.75). It remains to deal with the $\widetilde{\mathcal{G}}$ part. In the High $\times$ Low part of $P_k[(\phi_t\diamond\phi_s)\widetilde{\mathcal{G}}]$, we have

$$\sum_{|k_1-k|\le 4}\|P_{k_1}(\phi_t\diamond\phi_s)P_{\le k-4}\widetilde{\mathcal{G}}\|_{L^2_{t,x}}\lesssim\sum_{|k_1-k|\le 4}\|P_{k_1}(\phi_t\diamond\phi_s)\|_{L^2_{t,x}}\|\widetilde{\mathcal{G}}\|_{L^\infty}\lesssim\mathbf{a}_k(\sigma).$$

Thus the High $\times$ Low part is acceptable by directly repeating [5, Lemma 5.7].

From now on until the end of this proof, we assume $\sigma\in[\frac{1}{100},\frac{99}{100}]$. In the High $\times$ High part of $P_k[(\phi_t\diamond\phi_s)\widetilde{\mathcal{G}}]$, using $\|P_kf\|_{L^\infty}\lesssim 2^k\|f\|_{F_k}$ and (4.62), we have

$$\begin{aligned}&\sum_{|k_1-k_2|\le 8,\,k_1,k_2\ge k-4}\|P_{k_1}(\phi_t\diamond\phi_s)P_{k_2}\widetilde{\mathcal{G}}\|_{L^2_{t,x}}\\
&\lesssim\sum_{|k_1-k|\le 8,\,k_1,k_2\ge k-4}\|P_{k_1}(\phi_t\diamond\phi_s)\|_{L^2_{t,x}}2^{k_2}\|\widetilde{\mathcal{G}}\|_{F_{k_2}}\\
&\lesssim\sum_{k_1\ge k-4}\mathbf{a}_{k_1}(0)(1+s2^{2k_1})^{-3}2^{-\sigma k_1}b_{k_1}(\sigma)\\
&\lesssim\sum_{k_1\ge k-4}2^{-\sigma k_1}b_{k_1}(\sigma)(1+s2^{2k_1})^{-3}\sum_{l\le k_1}2^{l+k_1}b_{k_1}b_l(s2^{2l})^{-3/8}(1+s2^{2k_1})^{-2}\\
&\quad+\sum_{k_1\ge k-4}2^{-\sigma k_1}b_{k_1}(\sigma)(1+s2^{2k_1})^{-3}\sum_{l\ge k_1}2^{2l}b_lb_l(s2^{2l})^{-3/8}(1+s2^{2l})^{-4}.\end{aligned}\tag{4.76}$$

Thus for $j\in\mathbb{Z}$ and $s\in[2^{2j-1},2^{2j+1})$, when $k+j\ge 0$, the above quantity is bounded by

$$(1+2^{2j+2k})^{-2}2^{2k-\sigma k}b_kb_k(\sigma)(2^{2k+2j})^{-3/8}.$$

When $k+j\le 0$, by (4.76) the High $\times$ High part is dominated by

$$\begin{aligned}&\Big(\sum_{k_1\ge -j}+\sum_{k-4\le k_1\le -j}\Big)(1+2^{2k_1+2j})^{-3}2^{2k_1-\sigma k_1}b^2_{k_1}b_{k_1}(\sigma)2^{-\frac34(k_1+j)}\\
&+\sum_{k_1\ge k-4}1_{k_1+j\ge 0}(1+2^{2k_1+2j})^{-3}2^{-\sigma k_1}b_{k_1}(\sigma)\Big[\sum_{l\ge k_1}2^{2l}b_lb_l2^{-\frac34(j+l)}(1+2^{2l+2j})^{-4}\Big]\\
&+\sum_{k_1\ge k-4}1_{k_1+j\le 0}(1+2^{2k_1+2j})^{-3}2^{-\sigma k_1}b_{k_1}(\sigma)\\
&\qquad\times\Big[\Big(\sum_{l\ge -j}+\sum_{k_1\le l\le -j}\Big)2^{2l}b_lb_l2^{-\frac34(j+l)}(1+2^{2l+2j})^{-4}\Big],\end{aligned}$$

which is further bounded by

$$\begin{aligned}
&\sum_{k_1\geq -j} b_{k_1}(\sigma)b_{-j}^2 2^{2\delta|k_1+j|}2^{2k_1-\sigma k_1}2^{-\frac{3}{4}(k_1+j)}2^{-6(k_1+j)}\\
&\quad+\sum_{k\leq k_1\leq -j} b_{k_1}(\sigma)b_{-j}^2 2^{-\sigma k_1}2^{2\delta|k_1+j|}2^{2k_1}2^{-\frac{3}{4}(k_1+j)}\\
&\quad+b_{-j}^2 b_{-j}(\sigma)\sum_{k_1\geq k-4} 1_{k_1+j\geq 0}(1+2^{2k_1+2j})^{-7}2^{2k_1-\sigma k_1}2^{\delta|k_1+j|}2^{-\frac{3}{4}(j+k_1)}\\
&\quad+\sum_{k_1\geq k-4} 1_{k_1+j\leq 0} b_{-j}^2 b_{k_1}(\sigma)\Big[2^{-\sigma k_1}2^{-2j}+2^{-\sigma k_1}\sum_{k_1\leq l\leq -j} 2^{2l}2^{\delta|l+j|}2^{-\frac{3}{4}(j+l)}\Big]\\
&\lesssim b_{-j}^2 b_{-j}(\sigma)2^{\sigma j}2^{-2j}+\sum_{k_1\geq k-4} 1_{k_1+j\leq 0} b_{-j}^2 b_{k_1}(\sigma)2^{-\sigma k_1}2^{-2j}\\
&\lesssim b_{-j}^2 b_{-j}(\sigma)2^{\sigma j}2^{-2j}+2^{-\sigma k}b_k(\sigma)b_{-j}^2 2^{-2j},
\end{aligned}$$

where in the last line we have used $\sigma\geq\frac{1}{100}$. Summing over $j\geq k_0$ we see the High $\times$ High part satisfies

$$\begin{aligned}
&\int_0^\infty \sum_{|k_1-k_2|\leq 8,\, k_1,k_2\geq k-4} \|P_k[P_{k_1}(\phi_t\diamond\phi_s)P_{k_2}\widetilde{\mathcal{G}}]\|_{L^2_{t,x}}\,ds'\\
&\qquad\lesssim \sum_{j\leq -k} 2^{\sigma j}b_{-j}(\sigma)b_{-j}^2+\sum_{j\leq -k} b_{-j}^2 2^{-\sigma k}b_k(\sigma)\\
&\qquad\quad+\sum_{j\geq -k}(1+2^{2j+2k})^{-2}2^{2k+2j-\sigma k}b_k b_k(\sigma)(2^{2k+2j})^{-3/8}\\
&\qquad\lesssim \varepsilon^2 2^{-\sigma k}b_k(\sigma),
\end{aligned}$$

where we have applied $\sigma\geq\frac{1}{100}$ in the last line again. Now let us consider the Low $\times$ High part of $P_k[(\phi_t\diamond\phi_s)\widetilde{\mathcal{G}}]$. For the same reason as High $\times$ High, the Low $\times$ High part is dominated by

$$\begin{aligned}
&\int_0^\infty \sum_{|k_2-k|\leq 4} \|P_{\leq k-4}(\phi_t\diamond\phi_s)P_{k_2}\widetilde{\mathcal{G}}\|_{L^2_{t,x}}\,ds'\\
&\qquad\lesssim \int_0^\infty \sum_{|k_2-k|\leq 4} \|P_{\leq k-4}(\phi_t\diamond\phi_s)\|_{L^2_{t,x}}2^{k_2}\|P_{k_2}\widetilde{\mathcal{G}}\|_{L^\infty_t L^2_x}\,ds'\\
&\qquad\lesssim b_k(\sigma)2^{-\sigma k}\int_0^\infty \|(\phi_t\diamond\phi_s)\|_{L^2_{t,x}}\,ds'\lesssim b_k(\sigma)2^{-\sigma k}\varepsilon^2,
\end{aligned}$$

where we have applied (4.62) and $2^k T_{k,j}(1+2^{j+k})^{-1}\lesssim 1$ in the third inequality. ∎

5. Evolution along the Schrödinger map flow direction

In this section, we prove the following proposition, which is the key to closing the bootstrap for solutions in the Schrödinger evolution direction.

Proposition 5.1. *Assume that $\sigma \in [0, \frac{99}{100}]$. Let $Q \in \mathcal{N}$ be a fixed point and ϵ_0 be a sufficiently small constant. Given any $\mathcal{L} \in \mathbb{Z}_+$, assume that $T \in (0, 2^{2\mathcal{L}}]$. Let $\{c_k\}$ be an ϵ_0-frequency envelope of order δ, and let $\{c_k(\sigma)\}$ be another frequency envelope of order δ. Let $u \in \mathcal{H}_Q(T)$ be the solution to SMF with initial data u_0 which satisfies*

$$\|P_k \nabla u_0\|_{L^2_x} \le c_k, \tag{5.1}$$

$$\|P_k \nabla u_0\|_{L^2_x} \le c_k(\sigma) 2^{-\sigma k}, \tag{5.2}$$

Denote by $\{\phi_i\}$ the corresponding differential fields of the heat flow initiated from u. Suppose also that at the heat initial time $s = 0$,

$$\|P_k \phi_i\|_{G_k(T)} \le \epsilon_0^{-1/2} c_k. \tag{5.3}$$

Then when $s = 0$, for all $i = 1, 2$ and $k \in \mathbb{Z}$ we have

$$\|P_k \phi_i\|_{G_k(T)} \lesssim c_k, \tag{5.4}$$

$$\|P_k \phi_i\|_{G_k(T)} \lesssim c_k(\sigma) 2^{-\sigma k}. \tag{5.5}$$

The proof of Proposition 5.1 will be divided into several lemmas. First of all, Corollary 3.1 shows

$$\sum_{i=1}^{2} \|P_k \phi_i \upharpoonright_{s=0,t=0}\|_{L^2_x} \lesssim 2^{-\sigma k} c_k(\sigma) \tag{5.6}$$

for any $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$.

Second, we reduce the proof to frequency envelope bounds. Let

$$b(k) := \sum_{i=1}^{2} \|P_k \phi_i \upharpoonright_{s=0}\|_{G_k(T)}. \tag{5.7}$$

For $\sigma \in [0, \frac{99}{100}]$, define the frequency envelopes:

$$b_k(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{\sigma k'} 2^{-\delta|k-k'|} b(k'). \tag{5.8}$$

By Proposition 3.1 and Sobolev embeddings, they are finite and ℓ^2 summable. And

$$\|P_k \phi_i \upharpoonright_{s=0}\|_{G_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma). \tag{5.9}$$

To prove (5.4) and (5.5), it suffices to show

$$b_k(\sigma) \lesssim c_k(\sigma). \tag{5.10}$$

By (5.3), we have $b_k \le \varepsilon_0^{-1/2} c_k$, and in particular

$$\sum_{k \in \mathbb{Z}} b_k^2 \le \epsilon_0. \tag{5.11}$$

The assumption (4.1) of Proposition 4.1 follows from the inclusion $G_k \subset F_k$. The following lemma will show that the assumption (4.2) holds as a corollary of (5.9) if u solves SMF.

Lemma 5.1. *If* $\{b_k(\sigma)\}$ *are defined as above, then the field* ϕ_t *at the heat initial time* $s=0$ *satisfies*

$$\|P_k\phi_t\upharpoonright_{s=0}\|_{L^4_{t,x}} \lesssim b_k(\sigma)2^{-(\sigma-1)k}. \tag{5.12}$$

Proof. When $s=0$, $\phi_t(0)=\sqrt{-1}\sum_{i=1}^d \partial_i\phi_i(0)+A_i(0)\phi_i(0)$. The terms $\psi_i(0)$, $A_i(0)$ have been estimated before in Section 4. Thus copying the proof of [5, Lemma 6.1] gives (5.12). ∎

Thus both the assumption (4.1) and the assumption (4.2) of Proposition 4.1 are verified. Now one can apply Proposition 4.1, since (4.3) can be dropped. We summarize the results in the following:

$$\begin{cases} \|P_k(\phi_i(s))\|_{F_k(T)} \lesssim 2^{-\sigma k}b_k(\sigma)(1+2^{2k}s)^{-4}, \\ \|P_k(D_i\phi_i(s))\|_{F_k(T)} \lesssim 2^k 2^{-\sigma k}b_k(\sigma)(s2^{2k})^{-3/8}(1+2^{2k}s)^{-2}, \end{cases} \tag{5.13}$$

and for $F\in\{\psi_i\diamond\psi_j, A_l^2\}_{l,i,j=1}^2$

$$\|P_kF\upharpoonright_{s=0}\|_{L^2_{t,x}} \lesssim 2^{-\sigma k}b^2_{>k}(\sigma), \quad \|F\upharpoonright_{s=0}\|_{L^2_{t,x}} \lesssim \epsilon_0. \tag{5.14}$$

Then at $s=0$, A_t satisfies

$$\begin{aligned} \|A_t(0)\|_{L^2_{t,x}} &\lesssim \epsilon_0, & \sigma &\in \left[0,\tfrac{99}{100}\right], \\ \|P_kA_t(0)\|_{L^2_{t,x}} &\lesssim 2^{-\sigma k}b_k(\sigma), & \sigma &\in \left[\tfrac{1}{100},\tfrac{99}{100}\right]. \end{aligned}$$

Recall that when $s=0$, the evolution equation of ϕ_i along the Schrödinger map flow direction (see Lemma 1.1) is

$$-\sqrt{-1}\,D_t\phi_i=\sum_{j=1}^2 D_jD_j\phi_i+\sum_{j=1}^2\mathcal{R}(\phi_i,\phi_j)\phi_j. \tag{5.15}$$

5.1. *Control of nonlinearities*

Now let us deal with the nonlinearities in (5.15). **In this section we always assume $s=0$.** Denote

$$L'_j := A_t\phi_j+\sum_{i=1}^2 A_i^2\phi_j+2\sum_{i=1}^2\partial_i(A_i\phi_j)-\sum_{i=1}^2(\partial_iA_i)\phi_j. \tag{5.16}$$

Proposition 5.2 ([5]). *For all* $j\in\{1,2\}$ *and* $\sigma\in[0,\frac{99}{100}]$ *we have*

$$\|P_k(L'_j)\upharpoonright_{s=0}\|_{N_k(T)} \lesssim \epsilon_0 2^{-\sigma k}b_k(\sigma), \tag{5.17}$$

$$\sum_{j_0,j_1,j_3=1}^2 \|P_k(\phi_{j_0}\diamond\phi_{j_1}\diamond\phi_{j_3})\upharpoonright_{s=0}\|_{N_k(T)} \lesssim \epsilon_0 2^{-\sigma k}b_k(\sigma). \tag{5.18}$$

Proof. (5.17) and (5.18) have been proved in [5, Proposition 6.2]. We emphasize that to bound $\|A_t\phi_i\|_{N_k}$, [5, Proposition 6.2] used $\|A_t\|_{L^2_{t,x}}\le\varepsilon^2$ when $\sigma\in[0,\frac{1}{12}]$ and

$\|P_k A_t\|_{L^2_{t,x}} \leq 2^{-\sigma k} b_k(\sigma)$ when $\sigma \geq \frac{1}{12}$. Thus our bounds (4.73) and (4.74) suffice to bound $\|A_t \phi_i\|_{N_k}$ as well, although (4.73)–(4.74) themselves differ from the bounds stated in [5, Lemma 5.7]. ∎

Now we turn to the remaining curvature term in (5.15).

Proposition 5.3. *For all $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$ we have*

$$\sum_{j_0,j_1,j_3=1}^{2} \|P_k((\phi_{j_0} \diamond \phi_{j_1} \diamond \phi_{j_3})\mathcal{G})\|_{N_k(T)} \lesssim 2^{-\sigma k} \epsilon_0 b_k(\sigma). \tag{5.19}$$

Proof. Recall $\mathcal{G} = \Gamma^\infty + \widetilde{\mathcal{G}}$. The constant part Γ^∞ satisfies (5.19) by directly applying (5.18). It suffices to control the $\widetilde{\mathcal{G}}$ part.

As a preparation, we first prove the following estimate:

$$\sum_{i=1}^{2} \|P_k(\widetilde{\mathcal{G}}\phi_i)\|_{F_k(T)} \lesssim \begin{cases} 2^{-\sigma k} b_k(\sigma), & \frac{1}{100} < \sigma \leq \frac{99}{100}, \\ 2^{-\sigma k} \sum_{j \geq k} b_j b_j(\sigma), & 0 \leq \sigma \leq \frac{1}{100}. \end{cases} \tag{5.20}$$

This follows directly by applying Corollary 4.1 and Lemma 8.2: If $\sigma > \frac{1}{100}$, then

$$\begin{aligned} \|P_k(\widetilde{\mathcal{G}}\phi_i)\|_{F_k(T)} &\lesssim 2^{-\sigma k} b_k(\sigma) + 2^{-k-\sigma k} b_k(\sigma) \sum_{l \leq k} 2^{\delta|k-l|} 2^l b_l + b_k(\sigma) \sum_{j \geq k} 2^{-\sigma j} 2^{2\delta|k-j|} \\ &\lesssim 2^{-\sigma k} b_k(\sigma). \end{aligned}$$

If $\sigma \in [0, \frac{1}{100}]$, for the High × High interaction we directly use

$$\begin{aligned} &\sum_{|k_1-k_2| \leq 8,\, k_1,k_2 \geq k-4} \|P_k(P_{k_1}\widetilde{\mathcal{G}} P_{k_2}\phi_i)\|_{F_k(T)} \\ &\lesssim \sum_{j \geq k-4} 2^j \Big(\sum_{|k_1-j| \leq 28} \|P_{k_1}\widetilde{\mathcal{G}}\|_{F_{k_1}(T)} \Big) \Big(\sum_{|k_2-j| \leq 28} \|P_{k_2}\phi_i\|_{F_{k_2}(T)} \Big) \\ &\lesssim 2^{-\sigma k} \sum_{j \geq k} b_j b_j(\sigma). \end{aligned}$$

The other two interactions are all the same as for $\sigma \geq \frac{1}{100}$. Thus (5.20) follows.

As before, denoting $\mathbf{F} = \phi_{j_0} \diamond \phi_{j_1}$, by bilinear Littlewood–Paley decomposition, we have

$$\begin{aligned} &\|P_k(\mathbf{F} \diamond (\phi_{j_3}\widetilde{\mathcal{G}}))\|_{N_k(T)} \\ &= \sum_{|l-k| \leq 4} \|P_k(P_{<k-100}\mathbf{F} P_l(\widetilde{\mathcal{G}}\phi_{j_3}))\|_{N_k(T)} + \sum_{|k_1-k| \leq 4} \|P_k(P_{k_1}\mathbf{F} P_{<k-100}(\widetilde{\mathcal{G}}\phi_{j_3}))\|_{N_k(T)} \\ &\quad + \sum_{\substack{|k_1-k_2| \leq 120 \\ k_1,k_2 \geq k-100}} \|P_k(P_{k_1}\mathbf{F} P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3}))\|_{N_k(T)}. \end{aligned} \tag{5.21}$$

For the first RHS term of (5.21), applying (8.12) and the trivial bounds

$$\|\widetilde{\mathcal{G}}\|_{L^\infty_{t,x}} \lesssim 1, \tag{5.22}$$

$$\|\phi_x\|^2_{L^4_{t,x}} \lesssim \epsilon_0, \tag{5.23}$$

and (5.20), for $\sigma \in [\frac{1}{100}, \frac{99}{100}]$ we get

$$\begin{aligned}\sum_{|k_0-k|\le 4} \|P_k(P_{<k-100}\mathbf{F}P_{k_0}(\widetilde{\mathcal{G}}\phi_{j_3}))\|_{N_k(T)} &\lesssim \|\phi_{j_0}\phi_{j_1}\|_{L^2_{t,x}}\|P_k(\widetilde{\mathcal{G}}\phi_{j_3})\|_{F_k(T)}\\ &\lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma).\end{aligned}$$

For the first RHS term of (5.21), when $\sigma \in [0, \frac{1}{100}]$, one further decomposes $P_{[k-4,k+4]}(\widetilde{\mathcal{G}}\phi_{j_3})$ into High $\times$ High, Low $\times$ High, High $\times$ Low. We schematically write

$$\begin{aligned}&\sum_{|k_0-k|\le 4} \|P_k(P_{<k-100}\mathbf{F}P_{k_0}(\widetilde{\mathcal{G}}\phi_{j_3}))\|_{N_k(T)}\\ &\quad\lesssim \sum_{|l-k|\le 8} \|P_k((P_{<k-100}\mathbf{F})P_l\phi_{j_3}(P_{\le k-8}\widetilde{\mathcal{G}}))\|_{N_k(T)} &(5.24)\\ &\qquad + \sum_{|l-k|\le 8} \|P_k((P_{<k-100}\mathbf{F})(P_{\le k-8}\phi_{j_3}P_l\widetilde{\mathcal{G}}))\|_{N_k} &(5.25)\\ &\qquad + \sum_{|k_1-k_2|\le 16,\, k_1,k_2\ge k-8} \|P_k((P_{<k-100}\mathbf{F})P_{k_1}\phi_{j_3}(P_{k_2}\widetilde{\mathcal{G}}))\|_{N_k(T)}. &(5.26)\end{aligned}$$

Since for all $\sigma \in [0, \frac{99}{100}]$, the Low $\times$ High (denoted by P_k^{lh} for short) and High $\times$ Low (denoted by P_k^{hl} for short) interactions lead to $\|(P_k^{\mathrm{lh}} + P_k^{\mathrm{hl}})(\widetilde{\mathcal{G}}\phi_{j_3})\|_{F_k} \lesssim 2^{-\sigma k} b_k(\sigma)$, we conclude that

$$\begin{aligned}(5.24) + (5.25) &\lesssim \|\phi_{j_0}\phi_{j_1}\|_{L^2_{t,x}}\big(\|P_k^{\mathrm{lh}}(\mathcal{G}\phi_{j_3})\|_{F_k(T)} + \|P_k^{\mathrm{hl}}(\mathcal{G}\phi_{j_3})\|_{F_k(T)}\big)\\ &\lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma).\end{aligned}$$

For the (5.26) term, applying (8.14) yields

$$\begin{aligned}(5.26) &\lesssim \sum_{k_2\ge k-8}\sum_{|k_1-k_2|\le 16} \|P_k[((P_{<k-100}\mathbf{F})P_{k_2}\widetilde{\mathcal{G}})P_{k_1}\phi_{j_3}]\|_{N_k}\\ &\lesssim \sum_{k_2\ge k-8,\,|k_1-k_2|\le 16} \|(P_{<k-100}\mathbf{F})P_{k_2}\widetilde{\mathcal{G}}\|_{L^2_{t,x}} 2^{\frac{k-k_1}{6}}\|P_{k_1}\phi_{j_3}\|_{G_{k_1}}\\ &\lesssim \sum_{k_1\ge k-12} \|\mathbf{F}\|_{L^2_{t,x}} 2^{\frac{k-k_1}{6}} 2^{-\sigma k_1} b_{k_1}(\sigma)\\ &\lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma).\end{aligned}$$

Thus the first RHS term of (5.21) has been handled.

For the second RHS term of (5.21), we further divide $\mathbf{F}$ into

$$\sum_{|k_1-k|\le 4} \|P_k(P_{k_1}\mathbf{F}P_{<k-100}(\mathcal{G}\phi_{j_3}))\|_{N_k(T)}$$

$$\lesssim \sum_{|l-k|\le 8} \|P_k[P_l\phi_{j_0})(P_{\le k-8}\phi_{j_1})P_{\le k-100}(\mathcal{G}\phi_{j_3})]\|_{N_k(T)} \tag{5.27}$$

$$+ \sum_{|l-k|\le 8} \|P_k[(P_l\phi_{j_1})(P_{\le k-8}\phi_{j_0})P_{\le k-100}(\mathcal{G}\phi_{j_3})]\|_{N_k(T)} \tag{5.28}$$

$$+ \sum_{|l_1-l_2|\le 16,\, l_1,l_2\ge k-8} \|P_k[(P_{l_1}\phi_{j_0})(P_{l_2}\phi_{j_1})P_{\le k-100}(\mathcal{G}\phi_{j_3})]\|_{N_k(T)}. \tag{5.29}$$

Using again (8.12) and the bounds (5.22), (5.23), we obtain

$$(5.28) + (5.27) \lesssim \|P_k(\phi_x)\|_{F_k(T)}\|\phi_x\|^2_{L^4_{t,x}} \lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma),$$

and using (8.14) and the bounds (5.22), (5.23), we have

$$(5.29) \lesssim \sum_{|l_1-l_2|\le 16,\, l_1,l_2\ge k-8} 2^{\frac{k-l_2}{6}} \|(P_{\le k-100}(\widetilde{\mathcal{G}}\phi_{j_3}))P_{l_1}\phi_{j_1}\|_{L^2_{t,x}} \|P_{l_2}\phi_{j_0}\|_{G_l(T)}$$

$$\lesssim \|\phi_x\|^2_{L^4_{t,x}} \sum_{l_2\ge k-8} 2^{\frac{k-l_2}{6}} 2^{-\sigma l_2} b_{l_2}(\sigma) \lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma),$$

where we have used the embedding $L^4_k(T) \hookrightarrow F_k(T) \hookrightarrow G_k(T)$ and the fact $\|P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})\|_{L^4} \lesssim \|\phi_x\|_{L^4}$ in the second inequality. Thus the first two RHS terms of (5.21) are handled.

For the third term of (5.21), applying Littlewood–Paley decomposition to $\mathbf{F}$ shows

$$\sum_{\substack{k_1,k_2\ge k-100\\ |k_1-k_2|\le 120}} \|P_k(P_{k_1}\mathbf{F}P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3}))\|_{N_k}$$

$$\lesssim \sum_{\substack{k_1,k_2\ge k-100\\ |k_1-k_2|\le 120}} \sum_{|l-k_1|\le 4} \|P_k[P_l\phi_{j_0}P_{\le k_1-8}\phi_{j_1}P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})]\|_{N_k} \tag{5.30}$$

$$+ \sum_{\substack{k_1,k_2\ge k-100\\ |k_1-k_2|\le 120}} \sum_{|l-k_1|\le 4} \|P_k[P_l\phi_{j_1}P_{\le k_1-8}\phi_{j_0}P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})]\|_{N_k} \tag{5.31}$$

$$+ \sum_{\substack{k_1,k_2\ge k-100\\ |k_1-k_2|\le 120}} \sum_{\substack{l_1,l_2\ge k_1-8\\ |l_1-l_2|\le 16}} \|P_k[P_{l_1}\phi_{j_1}P_{l_2}\phi_{j_0}P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})]\|_{N_k}. \tag{5.32}$$

By Lemma 8.5 and (5.20), (5.23), we have

$$(5.30) + (5.31)$$
$$\lesssim \sum_{k_1\ge k-100} \sum_{|k_1-k_2|\le 120} \sum_{|l-k_1|\le 4} 2^{\frac{k-l}{6}} \|P_l\phi_x\|_{G_l(T)}\|\phi_x\|_{L^4_{t,x}}\|P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})\|_{L^4_{t,x}}$$
$$\lesssim \sum_{k_1\ge k-100} \varepsilon_0 2^{\frac{k-k_1}{6}} 2^{-\sigma k_1} b_{k_1}(\sigma) \lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma).$$

And using Lemma 8.5, especially (8.14) and (8.12), we see

$$
\begin{aligned}
(5.32) &\lesssim \sum_{\substack{k_1,k_2\geq k-100\\ |k_1-k_2|\leq 120}} \sum_{\substack{l_1,l_2\geq k_1-8\\ |l_1-l_2|\leq 16}} 2^{\frac{k-l_1}{6}} \|P_{l_1}\phi_{j_1}\|_{G_{l_1}(T)} \|(P_{l_2}\phi_{j_0})P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})\|_{L^2_{t,x}} \\
&\lesssim \sum_{\substack{k_1,k_2\geq k-100\\ |k_1-k_2|\leq 120}} \sum_{\substack{l_1,l_2\geq k_1-8\\ |l_1-l_2|\leq 16}} 2^{\frac{k-l_1}{6}} \|P_{l_1}\phi_{j_1}\|_{G_{l_1}(T)} \|P_{l_2}\phi_{j_0}\|_{L^4} \|P_{k_2}(\widetilde{\mathcal{G}}\phi_{j_3})\|_{L^4} \\
&\lesssim \epsilon_0 \sum_{k_1\geq k-100} \sum_{\substack{l_1\geq k_1-4\\ |l_1-l_2|\leq 16}} 2^{\frac{k-l_1}{6}} 2^{-\sigma l_1} b_{l_1}(\sigma) \lesssim \epsilon_0 2^{-\sigma k} b_k(\sigma).
\end{aligned}
$$

Thus the third RHS term of (5.21) has been handled. Hence, the proof is finished. ∎

Corollary 5.1 (Proof of Proposition 5.1). *Under the assumptions of Proposition 5.1, for all $i \in \{1,2\}$ and $\sigma \in [0, \frac{99}{100}]$ we have*

$$\|P_k\phi_i\|_{G_k(T)} \lesssim 2^{-\sigma k} c_k(\sigma). \tag{5.33}$$

Proof. (5.6) shows that for any $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$,

$$2^{\sigma k}\|P_k\phi_i(0,0,\cdot)\|_{L^2_x} \lesssim c_k(\sigma). \tag{5.34}$$

Then by Proposition 5.3, Proposition 5.2 and the linear estimates of Proposition 2.1, one has

$$b_k(\sigma) \lesssim c_k(\sigma) + \epsilon_0 b_k(\sigma) \tag{5.35}$$

for all $\sigma \in [0, \frac{99}{100}]$. Thus $b_k(\sigma) \lesssim c_k(\sigma)$, and our result follows by the definition of $\{b_k(\sigma)\}$ in Section 5. ∎

5.2. *Unform bounds for $\sigma \in [0, \frac{99}{100}]$*

We end the arguments for $\sigma \in [0, \frac{99}{100}]$ with the following proposition.

Proposition 5.4. *Assume that $\sigma \in [0, \frac{99}{100}]$. Let $Q \in \mathcal{N}$ be a fixed point and ϵ_0 be a sufficiently small constant. Given any $\mathcal{L} \in \mathbb{Z}_+$, assume that $T \in (0, 2^{2\mathcal{L}}]$. Let $\{c_k\}$ be an ϵ_0-frequency envelope of order δ, and let $\{c_k(\sigma)\}$ be another frequency envelope of order δ. Let $u \in \mathcal{H}_Q(T)$ be the solution to SMF with initial data u_0 which satisfies*

$$\|P_k\nabla u_0\|_{L^2_x} \leq c_k, \tag{5.36}$$

$$\|P_k\nabla u_0\|_{L^2_x} \leq c_k(\sigma)2^{-\sigma k}. \tag{5.37}$$

Denote by $\{\phi_i\}$ the corresponding differential fields of the heat flow initiated from u. Then for all $i = 1,2$, $k \in \mathbb{Z}$ and $\sigma \in [0, \frac{99}{100}]$ we have

$$\|P_k\phi_i\restriction_{s=0}\|_{G_k(T)} \lesssim c_k, \tag{5.38}$$

$$\|P_k\phi_i\restriction_{s=0}\|_{G_k(T)} \lesssim c_k(\sigma)2^{-\sigma k}, \tag{5.39}$$

$$\sup_{s\geq 0}(1+s2^{2k})^4\|P_k\phi_i(s)\|_{F_k(T)} \lesssim c_k(\sigma)2^{-\sigma k}. \tag{5.40}$$

Proof. Define the function $\Theta : [-T.T] \to \mathbb{R}^+$ by

$$\Theta(T') := \sup_{k\in\mathbb{Z}} c_k^{-1}(\|P_k\phi_i\restriction_{s=0}\|_{G_k(T')} + \|P_k\nabla u\|_{L_t^\infty L_x^2(T')}).$$

By Lemma 3.3, the function Θ is continuous on $[0, T]$. Then Proposition 5.1 implies

$$\Theta(T') \le \epsilon_0^{-1/2} \implies \sup_{k\in\mathbb{Z}} c_k^{-1}(\|P_k\phi_i\restriction_{s=0}\|_{G_k(T')}) \lesssim 1.$$

And by Proposition 3.5,

$$\sup_{k\in\mathbb{Z}} c_k^{-1}(\|P_k\phi_i\restriction_{s=0}\|_{G_k(T')}) \lesssim 1 \implies \sup_{k\in\mathbb{Z}} c_k^{-1}(\|P_k\nabla u\|_{L_t^\infty L_x^2(T')}) \lesssim 1.$$

Hence, we conclude

$$\Theta(T') \le \epsilon_0^{-1/2} \implies \Theta(T') \lesssim 1.$$

And it is easy to see $\Theta(T')$ is increasing. Moreover,

$$\lim_{T'\to 0} \Theta(T') \lesssim 1,$$

by the definition of $\Theta(T')$, $G_k(T')$ and Corollary 3.1. Therefore, from the continuity of Θ we conclude that (5.36) and (5.37) suffice to get

$$\Theta(T) \lesssim 1,$$

thus giving (5.38). Then Proposition 5.1 yields (5.39), and (5.40) follows by the inclusion $G_k \subset F_k$ and Proposition 4.1. ∎

6. Iteration scheme

From now on, the notations $a_k^{(j)}(\sigma)$ and $a_{k,s}^{(j)}(\sigma)$ differ from the ones defined in Section 3. They are defined as follows.

Definition 6.1. Assume that $u_0 \in \mathcal{H}_Q$. Given $j \in \mathbb{N}$, let

$$c_{k,(j)}(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\frac{1}{2^j}\delta|k-k'|}\|P_{k'}\nabla u_0\|_{L_x^2}, \quad k \in \mathbb{Z}.$$

- For $\sigma \in [0, \frac{99}{100}]$, define

$$c_k^{(0)}(\sigma) := c_{k,(0)}(\sigma).$$

- For $\sigma \in [0, \frac{5}{4}]$, define

$$c_k^{(1)}(\sigma) := \begin{cases} c_{k,(1)}(\sigma), & \sigma \in \left[0, \frac{99}{100}\right], \\ c_{k,(1)}(\sigma) + c_{k,(1)}(3/8)c_{k,(1)}(\sigma - 3/8), & \sigma \in \left(\frac{99}{100}, \frac{5}{4}\right]. \end{cases}$$

- Given an integer $j \geq 2$, for $\sigma \in [0, j/4+1]$, define $\{c_k^{(j)}(\sigma)\}$ by induction:

$$c_k^{(j)}(\sigma) := \begin{cases} c_{k,(j)}(\sigma), & \sigma \in \big[0, \frac{99}{100}\big], \\ c_{k,(j)}(\sigma) + c_{k,(j)}(3/8)c_{k,(j)}(\sigma - 3/8), & \sigma \in \big(\frac{99}{100}, \frac{5}{4}\big], \\ \dots \\ c_{k,(j)}(\sigma) + c_{k,(j)}(3/8)c_k^{(j)}(\sigma - 3/8), & \sigma \in \big(\frac{m+3}{4}, \frac{m}{4}+1\big], \\ \dots \\ c_{k,(j)}(\sigma) + c_{k,(j)}(3/8)c_k^{(j)}(\sigma - 3/8), & \sigma \in \big(\frac{j+3}{4}, \frac{j}{4}+1\big]. \end{cases}$$

Definition 6.2. • Assume that $\{a_k(\sigma)\}$ are frequency envelopes of order δ with $\sigma \in [0, \frac{99}{100}]$. Define

$$a_k^{(0)}(\sigma) := c_k^{(0)}(\sigma), \quad \forall \sigma \in \big[0, \tfrac{99}{100}\big].$$

- Assume that $\{a_k(\sigma)\}$ are frequency envelopes of order δ with $\sigma \in [0, \frac{99}{100}]$. Define

$$a_k^{(1)}(\sigma) := \begin{cases} c_k^{(1)}(\sigma), & \sigma \in \big[0, \frac{99}{100}\big], \\ a_k(\sigma) + c_k^{(1)}(3/8)c_k^{(1)}(\sigma - 3/8), & \sigma \in \big(\frac{99}{100}, \frac{5}{4}\big]. \end{cases}$$

- Given an integer $j \geq 2$, assume that $\{a_k(\sigma)\}$ are frequency envelopes of order δ with $\sigma \in [0, j/4+1]$. Define

$$a_k^{(j)}(\sigma) := \begin{cases} c_k^{(j)}(\sigma), & \sigma \in \big[0, \frac{j+3}{4}\big], \\ a_k(\sigma) + c_k^{(j)}(3/8)c_k^{(j)}(\sigma - 3/8), & \sigma \in \big(\frac{j+3}{4}, \frac{j}{4}+1\big]. \end{cases}$$

Given an integer $j \in \mathbb{N}$, assume that $\{a_k(\sigma)\}$ are frequency envelopes of order δ with $\sigma \in [0, j/4+1]$, and define

$$a_{k,s}^{(j)}(\sigma) := \begin{cases} 2^{k+k_0} a_{-k_0}(0) a_k^{(j)}(\sigma) & \text{if } k + k_0 \geq 0, \\ \sum_{l=k}^{-k_0} a_l(0) a_l^{(j)}(\sigma) & \text{if } k + k_0 \leq 0, \end{cases}$$

for $s \in [2^{2k_0-1}, 2^{2k_0+1})$ and $k, k_0 \in \mathbb{Z}$.

Remark 6.1. Given $j \geq 2$, we infer from Definition 6.1 that $\{c^{(j)}(\sigma)\}$ is of order $\frac{1}{2^m}\delta$ if $\sigma \in (\frac{m+3}{4}, \frac{m}{4}+1]$, $2 \leq m \leq j$. In particular, $\{c^{(j)}(\sigma)\}$ is of order δ for all $\sigma \in [0, j/4+1]$. One can also see from Definition 6.2 that $\{a^{(j)}(\sigma)\}$ are of order δ for all $\sigma \in [0, j/4+1]$.

Now we iterate the argument of previous sections to obtain uniform bounds for all $\sigma \in [0, \frac{5}{4}]$. We aim to prove the following proposition:

Proposition 6.1. *Assume that $\sigma \in [0, \frac{5}{4}]$. Let $Q \in \mathcal{N}$ be a fixed point and ϵ_0 be a sufficiently small constant. Given any $\mathcal{L} \in \mathbb{Z}_+$, assume that $T \in (0, 2^{2\mathcal{L}}]$. Let $u \in \mathcal{H}_Q(T)$ be the solution to SMF with initial data u_0. Let $\{c_k^{(1)}(\sigma)\}$ be frequency envelopes defined by Definition 6.1, and assume that $\{c_k^{(1)}(0)\}$ is an ϵ_0-frequency envelope. Denote by $\{\phi_i\}$ the* *corresponding differential fields of the heat flow initiated from u. Then for all* $i = 1, 2$, $k \in \mathbb{Z}$ *and* $\sigma \in [0, \frac{5}{4}]$, *we have*

$$2^{\sigma k} \| P_k \phi_i \restriction_{s=0} \|_{G_k(T)} \lesssim c_k^{(1)}(\sigma).$$

As before, this proposition will be divided into two propositions, one for the heat flow evolution and the other for the Schrödinger map flow evolution. In the statements of the following propositions or lemmas, the notation ✓ means that the relevant line can be dropped.

Proposition 6.2. *Let* $\sigma \in [0, \frac{5}{4}]$. *Let* $\{b_k(\sigma)\}$ *be frequency envelopes of order* δ *such that* $b_k(\sigma) \lesssim c_k^{(1)}(\sigma)$ *for* $\sigma \in [0, \frac{99}{100}]$. *Assume that* $\{c_k^{(1)}(0)\}$ *is an* ϵ_0*-frequency envelope.*

- *Assume that for* $i = 1, 2$,

$$\| P_k \phi_i \restriction_{s=0} \|_{F_k(T)} \le b_k(\sigma') 2^{-\sigma' k}, \quad \sigma' \in \left[0, \tfrac{5}{4}\right], \tag{6.1}$$

$$\checkmark \| P_k \phi_i(s) \|_{F_k(T)} \le \varepsilon^{-1} b_k^{(1)}(0)(1 + s 2^{2k})^{-4}. \tag{6.2}$$

Then for $\sigma \in [0, \frac{5}{4}]$ *and* $i = 1, 2$,

$$\| P_k \phi_i(s) \|_{F_k(T)} \lesssim 2^{-\sigma k}(1 + s 2^{2k})^{-4} b_k^{(1)}(\sigma), \tag{6.3}$$

$$\| P_k A_i \restriction_{s=0} \|_{L^4_{t,x}} \lesssim b_k^{(1)}(\sigma) 2^{-\sigma k}. \tag{6.4}$$

- *Assume further that*

$$\| P_k \phi_t \restriction_{s=0} \|_{L^4_{t,x}} \lesssim b_k(\sigma') 2^{-(\sigma'-1)k}, \quad \sigma' \in \left[0, \tfrac{5}{4}\right]. \tag{6.5}$$

Then for $\sigma \in [0, \frac{5}{4}]$, *one has*

$$\| A_t \restriction_{s=0} \|_{L^2_{t,x}} \lesssim \varepsilon^2, \tag{6.6}$$

$$\| P_k \phi_t(s) \|_{L^4_{t,x}} \lesssim b_k^{(1)}(\sigma) 2^{-(\sigma-1)k}(1 + 2^{2k} s)^{-2}, \tag{6.7}$$

$$\| P_k A_t \restriction_{s=0} \|_{L^2_{t,x}} \lesssim \varepsilon b_k^{(1)}(\sigma) 2^{-\sigma k}. \tag{6.8}$$

Proof. Recalling the definitions of $c_k^{(1)}(\sigma)$, $b_k^{(1)}(\sigma)$ in Definitions 6.1 and 6.2, by Propositions 4.1 and 5.4, we see (6.3), (6.4), (6.7) and (6.8) are already proved for $\sigma \in [0, \frac{99}{100}]$. Moreover, (6.6) and the assumption (6.2) hold naturally. It remains to prove (6.3), (6.4), (6.7) and (6.8) for $\sigma \in [\frac{99}{100}, \frac{5}{4}]$.

The key and starting point for the SMF iteration scheme is to improve $\| P_k \widetilde{\mathscr{G}}^{(1)} \|_{L^4_x L^\infty_t}$ step by step.

Lemma 6.1. *Let* $u \in \mathcal{H}_Q(T)$ *solve SMF with data* u_0. *Given any* $\sigma \in [0, \frac{99}{100}]$, *let* $\{c_k^{(1)}(\sigma)\}$ *be frequency envelopes defined in Definition 6.1. Assume also that* $\{c_k^{(1)}(0)\}$ *is an* ϵ_0*-frequency envelope. Then for* ϵ_0 *sufficiently small,*

$$2^{k/2} \| P_k \widetilde{\mathscr{G}}^{(1)} \|_{L^4_x L^\infty_t} \le c_k^{(1)}(\sigma) 2^{-\sigma k} [(1 + 2^{2k+2k_0})^{-20} 1_{k+k_0 \ge 0} + 1_{k+k_0 \le 0} 2^{\delta |k+k_0|}]$$

for any $\sigma \in [0, \frac{99}{100}]$, $k, k_0 \in \mathbb{Z}$ *and* $s \in [2^{2k_0-1}, 2^{2k_0+1})$.

Proof. By combining Propositions 5.1 and 4.1 we obtain

$$\begin{aligned}\|P_k\phi_t\|_{L^4} &\lesssim (1+s2^{2k})^{-2}2^{-\sigma k+k}c_k^{(1)}(\sigma), \quad \sigma\in\big[0,\tfrac{99}{100}\big],\\ \|P_k\phi_i\|_{L^4} &\lesssim (1+s2^{2k})^{-4}2^{-\sigma k}c_k^{(1)}(\sigma), \qquad \sigma\in\big[0,\tfrac{99}{100}\big].\end{aligned}$$

Then Proposition 3.6 yields

$$\begin{aligned}\|P_k\widetilde{\mathcal{G}}\|_{L^4_{t,x}\cap L^\infty_t L^2_x} &\lesssim (1+s2^{2k})^{-30}2^{-\sigma k-k}c_k^{(1)}(\sigma), \quad \sigma\in\big[0,\tfrac{99}{100}\big],\\ \|P_k\widetilde{\mathcal{G}}^{(m)}\|_{L^4_{t,x}\cap L^\infty_t L^2_x} &\lesssim (1+s2^{2k})^{-30}2^{-\sigma k-k}c_k^{(1)}(\sigma), \quad m=1,2,\ \sigma\in\big[0,\tfrac{99}{100}\big].\end{aligned}$$

So using the schematic formula

$$A_t=\int_s^\infty \phi_t\diamond(D_i\phi_i)\mathcal{G}\,ds'$$

and bilinear Littlewood–Paley decomposition (see the proof of Lemma 4.7), we get

$$\|P_kA_t\|_{L^4}\le c_k^{(1)}(\sigma)2^{-\sigma k+k}[(1+2^{2k+2j})^{-1}1_{k+j\ge 0}+1_{k+j\le 0}c_k^{(1)}2^{\delta|k+j|}]$$

for any $\sigma\in[0,\frac{99}{100}]$, $k,j\in\mathbb{Z}$ and $s\in[2^{2j-1},2^{2j+1})$. Thus using $\partial_t\widetilde{\mathcal{G}}^{(1)}=A_t\mathcal{G}^{(1)}+\mathcal{G}^{(2)}\phi_t$ and interpolation (see the proof of Lemma 4.7), one deduces that

$$2^{k/2}\|P_k\widetilde{\mathcal{G}}^{(1)}\|_{L^4_xL^\infty_t}\le c_k^{(1)}(\sigma)2^{-\sigma k}[(1+2^{2k+2k_0})^{-20}1_{k+k_0\ge 0}+1_{k+k_0\le 0}2^{\delta|k+k_0|}]$$

for any $\sigma\in[0,\frac{99}{100}]$, $k,k_0\in\mathbb{Z}$ and $s\in[2^{2k_0-1},2^{2k_0+1})$. ■

As before, we start with the bound for connection forms.

Lemma 6.2. *Let* $\sigma\in[\frac{99}{100},\frac{5}{4}]$. *Denote*

$$h(k):=\sup_{s\ge 0}(1+s2^{2k})^4\sum_{i=1}^2\|P_k\phi_i(s)\|_{F_k(T)}. \tag{6.9}$$

Define the corresponding envelope by

$$h_k(\sigma):=\sup_{k'\in\mathbb{Z}}2^{\sigma k'}2^{-\delta|k'-k|}h(k'). \tag{6.10}$$

Then under the assumptions of Proposition 6.2, for all $k\in\mathbb{Z}$, $s\ge 0$ *and* $i=1,2$ *we have*

$$\|P_k(A_i(s))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim 2^{-\sigma k}(1+s2^{2k})^{-4}h_{k,s}^{(1)}(\sigma), \tag{6.11}$$

where the sequence $\{h_{k,s}^{(1)}\}$ *when* $2^{2k_0-1}\le s<2^{2k_0+1}$, $k_0\in\mathbb{Z}$, *is defined by*

$$h_{k,s}^{(1)}(\sigma):=\begin{cases}2^{k+k_0}h_{-k_0}h_k^{(1)}(\sigma) & \text{if } k+k_0\ge 0,\\ \sum_{l=k}^{-k_0}h_lh_l^{(1)}(\sigma) & \text{if } k+k_0\le 0,\end{cases} \tag{6.12}$$

with

$$h_k^{(1)}(\sigma') := \begin{cases} c_k^{(1)}(\sigma'), & \sigma' \in \left[0, \frac{99}{100}\right], \\ h_k(\sigma') + c_k^{(1)}(3/8)c_k^{(1)}(\sigma' - 3/8), & \sigma' \in \left(\frac{99}{100}, \frac{5}{4}\right]. \end{cases} \tag{6.13}$$

Proof. The proof is almost the same as for Lemma 4.1. The difference is that more concern is needed for the High × Low interaction of $P_k[\widetilde{\mathcal{G}}^{(1)}\psi_s]$ in Step 4 of Lemma 4.1. First of all we point out that (5.40) of Proposition 5.4 shows that for all $\sigma' \in [0, \frac{99}{100}]$,

$$h_k(\sigma') \lesssim c_k^{(1)}(\sigma'). \tag{6.14}$$

Let $B_1^{(1)}$ be the smallest constant such that for all $\sigma \in [\frac{99}{100}, \frac{5}{4}]$, $s \geq 0$ and $k \in \mathbb{Z}$,

$$\|P_k(A_i(s))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim B_1^{(1)} 2^{-\sigma k}(1 + s2^{2k})^{-4} h_{k,s}^{(1)}(\sigma). \tag{6.15}$$

Recall the following decomposition of $\mathcal{G}$:

$$\mathcal{G} = \Gamma^{\infty} - \Gamma_p^{\infty,(1)} \int_s^{\infty} \psi_s^p \, ds' - \int_s^{\infty} \psi_s^p \widetilde{\mathcal{G}}^{(1)} \, ds'.$$

Since $\psi_s = \sum_{i=1}^{2} \partial_i \psi_i + A_i \psi_i$, we separate the ψ_i part away. And thus schematically one has

$$\begin{aligned} \mathcal{G} = \Gamma^{\infty} - \Gamma_l^{\infty,(1)} \int_s^{\infty} (\partial_i \psi_i)^l \, ds' - \int_s^{\infty} (\partial_i \psi_i)^l \widetilde{\mathcal{G}}_l^{(1)} \, ds' \\ - \Gamma_l^{\infty,(1)} \int_s^{\infty} (A_i \psi_i)^l \, ds' - \int_s^{\infty} (A_i \psi_i)^l \widetilde{\mathcal{G}}_l^{(1)} \, ds'. \end{aligned}$$

In order to prove our lemma, as before we first prove $B_1^{(1)} \lesssim 1$ under the **Bootstrap Assumption B**: For a fixed given $\sigma \in (\frac{99}{100}, \frac{5}{4}]$,

$$\int_s^{\infty} \|P_k(A_i \psi_i)\|_{F_k(T)} \, ds' \lesssim \varepsilon^{-1/2} 2^{-\sigma k} T_{k,j} (1 + s^{1/2} 2^k)^{-7} h_k^{(1)}(\sigma) c_0^*,$$

$$\int_s^{\infty} \|P_k[(A_i \psi_i)\widetilde{\mathcal{G}}^{(1)}]\|_{F_k(T)} \, ds' \lesssim \varepsilon^{-1/2} 2^{-\sigma k} T_{k,j} (1 + s^{1/2} 2^k)^{-7} h_k^{(1)}(\sigma) c_0^*,$$

where $c_0^* := \|\{h_k\}\|_{\ell^2}$, $s \in [2^{2j-1}, 2^{2j+1})$, and $T_{k,j}$ is defined in (4.27). This part is the same as Step 2 of Lemma 4.1 except controlling

$$\left\| P_k \left(\int_s^{\infty} (\partial_i \psi_i)(\widetilde{\mathcal{G}}^{(1)}) \, ds' \right) \right\|_{F_k(T)}, \tag{6.16}$$

which was labeled as $\mathcal{U}_{01}$ in Lemma 4.1. To estimate (6.16), recall the bounds in Lemma 6.1 and Proposition 3.6 for $\widetilde{\mathcal{G}}^{(1)}$:

$$\begin{aligned} &2^k \|P_k(\widetilde{\mathcal{G}}^{(1)})\|_{L_t^{\infty} L_x^2 \cap L^4} + 2^{k/2} \|P_k(\widetilde{\mathcal{G}}^{(1)})\|_{L_x^4 L_t^{\infty}} \\ &\quad \lesssim 2^{-\widetilde{\sigma} k} c_k(\widetilde{\sigma}) [1_{k+j\geq 0}(1 + 2^{2k}s)^{-20} + 1_{k+j\leq 0} 2^{\delta|k+j|}] \end{aligned} \tag{6.17}$$

for any $k, j \in \mathbb{Z}$, $s \in [2^{2j-1}, 2^{2j+1})$ and $\widetilde{\sigma} \in [0, \frac{99}{100}]$. By bilinear Littlewood–Paley decomposition and Lemma 4.2, we have

$$
\begin{aligned}
&\| P_k((\partial_i \psi_i)\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)} \\
&\lesssim \sum_{|k_1-k|\le 4} \| P_{k_1}(\partial_i \psi_i)\|_{F_{k_1}(T)} \| P_{\le k-4}\widetilde{\mathcal{G}}^{(1)}\|_{L^\infty} \\
&\quad + \sum_{|k_1-k_2|\le 8,\, k_1,k_2\ge k-4} \| P_{k_1}(\partial_i \psi_i)\|_{F_{k_1}(T)} (\| P_{k_2}(\widetilde{\mathcal{G}}^{(1)})\|_{L^\infty} + 2^{k_1/2}\| P_{k_2}(\widetilde{\mathcal{G}}^{(1)})\|_{L^4_x L^\infty_t}) \\
&\quad + \sum_{|k_2-k|\le 4,\, k_1\le k-4} 2^{k_1/2}\| P_{k_1}(\partial_i \psi_i)\|_{F_{k_1}(T)} \| P_{k_2}(\widetilde{\mathcal{G}}^{(1)})\|_{L^4_x L^\infty_t} \\
&\quad + 2^{k_1}\| P_{k_1}(\partial_i \psi_i)\|_{F_{k_1}(T)} \| P_{k_2}(\widetilde{\mathcal{G}}^{(1)})\|_{L^4} \\
&\lesssim 2^{-\sigma k} h_k^{(1)}(\sigma) + 2^{-\sigma k} h_k h_k^{(1)}(\sigma)[1_{k+j\ge 0} 2^k (1+2^{2k}s)^{-4} + 1_{k+j\le 0} 2^{\delta|k+j|} 2^{-j}] \\
&\quad + R_{j,k} 2^{-(\sigma-3/8)k} c_k^{(1)}(\sigma - 3/8) \\
&\quad \times \Big[2^{-k/2} \sum_{k_1\le k-4} 2^{\frac{3}{2}k_1 - \frac{3}{8}k_1} c_{k_1}^{(1)}(3/8) + 2^{-k} \sum_{k_1\le k-4} 2^{2k_1 - \frac{3}{8}k_1} c_{k_1}^{(1)}(3/8) \Big]
\end{aligned}
$$

where $R_{j,k} := 1_{k+j\ge 0}(1+2^{2k}s)^{-20} + 1_{k+j\le 0} 2^{\delta|k+j|}$ and we have used (6.14). Thus by slow variation of envelopes we get

$$
\| P_k((\partial_i \psi_i)\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)} \lesssim 2^{-\sigma k} h_k^{(1)}(\sigma)\big(1_{k+j\ge 0} 2^k (1+2^{2k_2}s)^{-4} + 1_{k+j\le 0} 2^{-j} 2^{\delta|k+j|}\big).
$$

for $s \in [2^{2j-1}, 2^{2j+1})$ and $j, k \in \mathbb{Z}$. This bound is the same as for $\mathcal{U}_{01}$ in Lemma 4.1 and acceptable.

In the third step, we prove the claim: If Bootstrap Assumption B holds, then

$$
\int_s^\infty \| P_k(A_i \psi_i)\|_{F_k(T)}\, ds' \lesssim 2^{-\sigma k} T_{k,j} (1 + s^{1/2} 2^k)^{-7} h_k^{(1)}(\sigma) c_0^*, \tag{6.18}
$$

$$
\int_s^\infty \| P_k(A_i \psi_i)\widetilde{\mathcal{G}}^{(1)}\|_{F_k(T)}\, ds' \lesssim 2^{-\sigma k} T_{k,j} (1 + s^{1/2} 2^k)^{-7} h_k^{(1)}(\sigma) c_0^*. \tag{6.19}
$$

The proof of (6.18) is the same as Step 4 of Lemma 4.1. For (6.19), the Low × High interaction of $P_k[(A_i \psi_i)\widetilde{\mathcal{G}}^{(1)}]$ is different due to the larger σ. The other two interactions are the same. We present the necessary modifications. Since under Bootstrap Assumption B one has $B^{(1)} \lesssim 1$, $P_k(A_i \psi_i)$ enjoys the same $F_k \cap S_k^{1/2}$ bound as in Lemma 4.1 with $h_k(\sigma)$ replaced by $h_k^{(1)}(\sigma)$:

$$
\begin{aligned}
\| P_k(A_i \psi_i)\|_{F_k(T)\cap S_k^{1/2}(T)} &\lesssim c_0^* 2^{-\sigma k} 1_{k+j\le 0} h_k^{(1)}(\sigma) 2^{1/2(k-j)} 2^{\delta|k+j|} \\
&\quad + c_0^* 2^{-\sigma k} 1_{k+j\ge 0} h_k^{(1)}(\sigma) 2^k (1 + 2^{j+k})^{-8}
\end{aligned}
$$

for all $\sigma \in [0, \frac{5}{4}]$, $s \in [2^{2j-1}, 2^{2j+1})$ and $j, k \in \mathbb{Z}$.

Then by (6.17), (6.14) and (4.45), the Low × High part of $(A_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$ is dominated by

$$\sum_{|k-k_2|\le 4,\, k_1\le k+4} \|P_k(P_{k_1}(A_i\psi_i)P_{k_2}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}$$

$$\lesssim c_0^* 2^{-(\sigma-3/8)k} c_k^{(1)}(\sigma-3/8)1_{k+j\le 0} \sum_{k_1\le k-4} c_{k_1}^{(1)}(3/8)2^{\frac{1}{2}(k_1-j)}2^{\delta|k_1+j|}2^{-\frac{3}{8}k_1}$$
$$+ c_0^* 2^{-(\sigma-3/8)k}(1+2^{2j+2k})^{-20} c_k^{(1)}(\sigma-3/8)1_{k+j\ge 0}$$
$$\times\Big[\sum_{-j\le k_1\le k} c_{k_1}^{(1)}(3/8)2^{k_1-\frac{3}{8}k_1}(1+2^{2j+2k_1})^{-4}\Big]$$
$$+ c_0^* 2^{-(\sigma-3/8)k}(1+2^{2j+2k})^{-20} c_k^{(1)}(\sigma-3/8)1_{k+j\ge 0}$$
$$\times\Big[\sum_{k_1\le -j} c_{k_1}^{(1)}(3/8)2^{k_1-j/2}2^{\delta|k_1+j|}2^{-\frac{3}{8}k_1}\Big]$$
$$\lesssim c_0^* 2^{-\sigma k} c_k^{(1)}(\sigma-3/8)c_k^{(1)}(3/8)\big(1_{k+j\ge 0}2^{-j}(1+2^{j+k})^{-7} + 1_{k+j\le 0}2^{k-j/2}2^{\delta|k+j|}\big).$$

Summing over $j\ge k_0$ as well yields

$$\sum_{j\ge k_0} 2^{2j} \sum_{|k-k_2|\le 4,\, k_1\le k+4} \|P_k(P_{k_1}\psi_s P_{k_2}\widetilde{\mathcal{G}}^{(1)})\|_{F_k(T)}$$
$$\lesssim c_0^* 2^{-\sigma k} h_k^{(1)}(\sigma)\big(1_{k+k_0\ge 0}2^{k_0}(1+2^{k+k_0})^{-7} + 2^{-k}1_{k+k_0\le 0}\big)$$

for $s\in[2^{2k_0-1}, 2^{2k_0+1})$ and $k_0,k\in\mathbb{Z}$. This bound is again the same as for $\mathcal{U}_{II}$ in Lemma 4.1 and acceptable.

Finally, we need to prove that (6.18), (6.19) of Bootstrap Assumption B hold when $T\to 0$. Let us verify it. Using (3.32) and (3.52) and putting $\frac{3}{8}$-th order derivatives on $A_i\psi_i$, when estimating the Low × High interaction of $(A_i\psi_i)\widetilde{\mathcal{G}}^{(1)}$, we also have

$$\int_s^\infty \|P_k[A_i\psi_i\widetilde{\mathcal{G}}^{(1)}]\|_{L_t^\infty L_x^2}\, ds' \lesssim \|\{h_k\}\|_{\ell^2} T_{k,j} h_k^{(1)}(\sigma)2^{-\sigma k}$$

for $\sigma\in[\frac{99}{100},\frac{5}{4}]$.

Therefore, combining the above four steps gives Lemma 6.2. ∎

The proof of Lemma 6.2 gives an F_k bound for $\widetilde{\mathcal{G}}$.

Lemma 6.3. *For all $\sigma\in(\frac{99}{100},\frac{5}{4}]$ and $k\in\mathbb{Z}$,*

$$\|P_k(\widetilde{\mathcal{G}})\|_{F_k(T)} \lesssim \begin{cases} 2^{-\sigma k}(1+s2^{2k})^{-4}2^j h_k^{(1)}(\sigma) & \text{if } j+k\ge 0,\\ 2^{-\sigma k}2^{-k}h_k^{(1)}(\sigma) & \text{if } j+k\le 0,\end{cases} \tag{6.20}$$

when $2^{2j-1}\le s<2^{2j+1}$, $j\in\mathbb{Z}$. Moreover, for $s=0$,

$$\|P_k\widetilde{\mathcal{G}}\restriction_{s=0}\|_{F_k(T)} \lesssim 2^{-k-\sigma k}h_k^{(1)}(\sigma). \tag{6.21}$$

Proof of Proposition 6.2. With this improved bound of $\widetilde{\mathcal{G}}$, running the program of Section 4 again gives

$$\sup_{s\geq 0} 2^{\sigma k}(1+s2^{2k})^4 \sum_{i=1}^{2} \|P_k\phi_i(s)\|_{F_k(T)} \lesssim b_k(\sigma) + \varepsilon h_k^{(1)}(\sigma).$$

Since the right side is a frequency envelope of order δ, we have

$$h_k(\sigma) \lesssim b_k(\sigma) + \varepsilon h_k^{(1)}(\sigma).$$

By the definition of $h_k^{(1)}(\sigma)$, we conclude that for $\sigma \in (\frac{99}{100}, \frac{5}{4}]$,

$$h_k(\sigma) \lesssim b_k(\sigma) + c_k^{(1)}(3/8)c_k^{(1)}(\sigma - 3/8),$$

thus proving (6.3). The remaining (6.4), (6.7) and (6.8) are the same. ∎

In the following proposition, we finish iteration of σ in the Schrödinger direction.

Proposition 6.3. *Given $\mathcal{L} \in \mathbb{Z}_+$, suppose that $T \in (0, 2^{2\mathcal{L}}]$ and $Q \in \mathcal{N}$. Assume that $\sigma \in [0, \frac{5}{4}]$. Let $u \in \mathcal{H}_Q(T)$ be a solution to SMF with initial data u_0, let $\{c_k^{(1)}(\sigma)\}_{k\in\mathbb{Z}}$ be frequency envelopes defined in Definition 6.1, and assume that $\{c_k^{(1)}(0)\}$ is an ϵ_0-frequency envelope with $0 < \epsilon_0 \ll 1$. Then for any $\sigma \in [0, \frac{5}{4}]$, $k \in \mathbb{Z}$, we have*

$$\|P_k\phi_i\restriction_{s=0}\|_{G_k(T)} \lesssim c_k^{(1)}(\sigma). \tag{6.22}$$

Proof. (6.22) has been proved for $\sigma \in [0, \frac{99}{100}]$ in Section 5. Thus, it suffices to consider $\sigma \in (\frac{99}{100}, \frac{5}{4}]$. Let

$$b(k) = \sum_{i=1}^{2} \|P_k\phi_i\restriction_{s=0}\|_{G_k(T)}.$$

For $\sigma \in [0, \frac{5}{4}]$, define the frequency envelopes

$$b_k(\sigma) = \sup_{k'\in\mathbb{Z}} 2^{\sigma k'} 2^{-\delta|k-k'|} b(k').$$

By Proposition 3.1, they are finite and ℓ^2 summable, and

$$\|P_k\phi_i\restriction_{s=0}\|_{G_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma).$$

The assumption (6.5) holds by repeating the argument of Lemma 4.1. Thus using Proposition 6.2, we see (6.3)–(6.8) hold. With Lemma 6.3, repeating the argument in Section 5, one finds that when $s = 0$,

$$\|P_k\phi_i\restriction_{s=0}\|_{G_k(T)} \lesssim c_k^{(1)}(\sigma) + \epsilon_0(b_k(\sigma) + c_k^{(1)}(3/8)c_k^{(1)}(\sigma - 3/8)), \quad \sigma \in \left(\tfrac{99}{100}, \tfrac{5}{4}\right].$$

Since the RHS is a frequency envelope of order δ, we conclude

$$b_k(\sigma) \lesssim c_k^{(1)}(\sigma).$$

This gives (6.22) and finishes the proof. ∎

7. Proofs of Theorems 1.1 and 1.2

7.1. Global regularity

In order to prove u is global, it suffices to verify (see Appendix B)

$$\|\nabla u\|_{L^\infty_{t,x}} \lesssim 1. \tag{7.1}$$

To prove (7.1), it suffices to give a uniform bound for $\|u(t)\|_{\dot H^1\cap \dot H^{2+}}$. Since energy is preserved, it remains to bound $\|u(t)\|_{\dot H^{2+}}$, which is related to frequency envelopes with $\sigma = 1+$. Thus we need to transform the intrinsic bound (6.22) to bounds for u.

The following lemma follows directly from Corollary 3.1.

Lemma 7.1. *Let $u \in \mathcal{H}_Q(T)$ solve SMF with data u_0 of small energy. For $\sigma \in [0, \frac{5}{4}]$, suppose that $\{c^{(1)}_k(\sigma)\}$ are frequency envelopes as in Definition* 6.1. *Assume that the differential fields $\{\phi_i\}$ associated with u under the caloric gauge satisfy*

$$\sum_{i=1,2} \|P_k\phi_i\restriction_{s=0}\|_{L^\infty_t L^2_x} \le 2^{-\sigma k} c^{(1)}_k(\sigma), \quad \forall k \in \mathbb{Z}. \tag{7.2}$$

Then

$$2^k \|P_k u\|_{L^\infty_t L^2_x} \le 2^{-\sigma k} c^{(1)}_k(\sigma), \quad \forall k \in \mathbb{Z}. \tag{7.3}$$

Proposition 6.3 shows the assumption (7.2) of Lemma 7.1 holds. And thus by applying Lemma 7.1, we conclude that

$$\|u\|_{\dot H^\rho\cap \dot H^1} \lesssim C(\|u_0\|_{\dot H^\rho\cap \dot H^1}) \tag{7.4}$$

for all $\rho \in [0, \frac{9}{4}]$. In particular, $\|\nabla u\|_{L^\infty_{t,x}} \lesssim 1$ by Sobolev embedding. Therefore, u is global by Appendix B, and global regularity follows by the local theory of [24].

The remaining part for Theorem 1.1 is (1.4) and (1.5). These will be proved in Sections 7.4 and 7.5 respectively.

7.2. Uniform Sobolev norm bounds of solutions to SMF

To get uniform Sobolev norm bounds for SMF up to $\sigma = 1 + K/4$, $K \in \mathbb{Z}_+$, in the heat flow iteration scheme it suffices to begin with the parabolic decay estimates

$$\|\partial_x^{L+1}\mathcal{G}^{(K+1)}\|_{L^\infty_t L^2_x} \lesssim \epsilon s^{-L/2}, \quad \forall L \in [0, 100+K],$$
$$\|\partial_x^{L+1}[d\mathcal{P}]^{(K+1)}\|_{L^\infty_t L^2_x} \lesssim \epsilon s^{-L/2}, \quad \forall L \in [0, 100+K].$$

And in the SMF iteration scheme, for the j-th iteration we always begin by proving

$$2^{k/2}\|P_k\widetilde{\mathcal{G}}^{(1)}\|_{L^4_x L^\infty_t} \le c^{(j)}_k(\sigma)2^{-\sigma k}[(1+2^{2k+2k_0})^{-20}1_{k+k_0\ge 0} + 1_{k+k_0\le 0}2^{\delta|k+k_0|}],$$
$$\sigma \in [0, 1+(j-1)/4],$$

for any $s \in [2^{2k_0-1}, 2^{2k_0+1})$ and $k_0, k \in \mathbb{Z}$. Then repeating (K times) the argument of the first time iteration we obtain

$$2^k \| P_k d\mathcal{P}(e)\restriction_{s=0} \|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} c_k^{(K)}(\sigma),$$
$$\| P_k \phi_x \restriction_{s=0} \|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} c_k^{(K)}(\sigma).$$

By bilinear estimates we then arrive at

$$\| P_k \partial_x u \|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k} c_k^{(K)}(\sigma), \tag{7.5}$$

from which the uniform Sobolev bounds follow. Each time iteration requires ϵ_* to be smaller in our arguments. We emphasize that the key to the succeeding SMF iterations is to improve $\| P_k \widetilde{\mathcal{G}}^{(1)} \|_{L_x^4 L_t^\infty}$ step by step (see e.g. Lemma 6.1).

Therefore, we have the following result:

Proposition 7.1. *For any $j \geq 1$, there exists a constant $\epsilon_j > 0$ such that if $u_0 \in \mathcal{H}_Q$ with $\|u_0\|_{\dot H^1} \leq \epsilon_j$, then $\|u(t)\|_{\dot H_x^j} \leq C(\|u_0\|_{\dot H^1 \cap \dot H^j})$ for all $t \in \mathbb{R}$.*

Since the mass of SMF solutions is not conserved, the $\|u - Q\|_{L_x^2}$ norm should be handled separately. This will be proved as a corollary of well-posedness; see the next section.

7.3. Well-posedness

In fact, the well-posedness stated in Theorem 1.2 follows closely by [41] and [5]'s original arguments. We sketch them for the reader's convenience.

Tataru [41, Prop. 3.13] proved that given $u_0^0, u_0^1 \in \mathcal{H}_Q$ with $\|u_0^h\|_{\dot H^1} \ll 1$ for $h = 0, 1$, there exists a smooth one-parameter family $\{u_0^h\}_{h\in[0,1]} \in C^\infty([0,1]; \mathcal{H}_Q)$ of initial data which satisfies

$$\|u_0^h\|_{\dot H^1} \ll 1, \quad h \in [0,1], \tag{7.6}$$
$$\int_0^1 \|P_k \partial_x u^h\|_{L_x^2}\, dh \approx \|u_0^0 - u_0^1\|_{L_x^2}. \tag{7.7}$$

Given $h \in [0,1]$, Theorem 1.1 yields a solution $u^h(t,x) \in C(\mathbb{R}; \mathcal{H}_Q)$ with initial data u_0^h. Then under the caloric gauge $\{e_\alpha, Je_\alpha\}$ for $u^h(t,x)$, define the differential field ϕ_h by

$$\phi_h^\alpha = \langle \partial_h u^h, e_\alpha \rangle + \sqrt{-1}\, \partial_h u^h, Je_\alpha \rangle, \quad \alpha = 1, \ldots, n, \tag{7.8}$$

and define $\{\phi_i\}_{i=0}^2$ as before. Since $-\sqrt{-1}\,\phi_t = \sum_{i=1,2} D_i \phi_i$ at $s = 0$ (because for all $h \in [0,1]$, $u^h(t,x)$ solves SMF), applying $D_h = \partial_h + A_h$ to both sides gives

$$-\sqrt{-1}\, D_t \phi_h = \sum_{i=1}^2 D_i D_i \phi_h + \sum_{i=1}^2 \mathcal{R}(u^h(t,x))(\phi_i, \phi_h)\phi_i \quad \text{when } s = 0,$$

which as before can be further schematically written as

$$-\sqrt{-1}\, D_t \phi_h = \sum_{i=1}^2 D_i D_i \phi_h + \sum (\phi_i \diamond \phi_h)\phi_i \mathcal{G} \quad \text{when } s = 0. \tag{7.9}$$

Given $\sigma \in [0, 1 + j/4)$ with $j \in \mathbb{Z}_+$, let

$$c_{k,(j),h}(\sigma) := \sup_{k' \in \mathbb{Z}} 2^{-\frac{1}{2^j}\delta|k'-k|} 2^{\sigma k' + k'} \| P_{k'} u_0^h \|_{L_x^2},$$

and define $\{c_{k,h}^{(j)}(\sigma)\}$ as in Definition 6.1. Then Section 7.2 gives

$$\sum_{i=1}^{2} 2^{\sigma k} \| P_k \phi_i (s = 0, h, \cdot, \cdot) \|_{G_k(T)} \lesssim c_{k,h}^{(j)}(\sigma), \tag{7.10}$$

and thus

$$2^{\sigma k + k} \| P_k \widetilde{\mathcal{G}}(s = 0, h, \cdot, \cdot) \|_{F_k(T)} \lesssim c_{k,h}^{(j)}(\sigma). \tag{7.11}$$

Using (7.10), (7.11) we infer by (7.9) that

$$\sum_{k \in \mathbb{Z}} \| P_k \phi_h (s = 0) \|_{G_k(T)}^2 \lesssim \| \phi_h (s = 0, t = 0) \|_{L_x^2}^2.$$

Transforming this bound to $\partial_h u^h$ yields

$$\| \partial_h u^h \|_{L_t^\infty L_x^2} \lesssim \| \partial_h u_0^h \|_{L_x^2}.$$

Then (7.7) leads to

$$\| u^1 - u^0 \|_{L_t^\infty L_x^2} \lesssim \| u_0^1 - u_0^0 \|_{L_x^2}. \tag{7.12}$$

With (7.12) in hand, the continuity of S_Q from $\mathfrak{B}_\epsilon^\sigma$ to $C(\mathbb{R}; H_Q^{\sigma+1})$ follows by the arguments of [5, pp. 1467–1468] if $\epsilon > 0$ is sufficiently small depending only on j thus σ.

Moreover, letting $u_0^1 = Q$, $u_0^0 = u_0$ in (7.12) one obtains

$$\| u - Q \|_{L_t^\infty L_x^2} \lesssim \| u_0 - Q \|_{L_x^2},$$

which combined with Proposition 7.1 gives (1.6).

7.4. Asymptotic behavior

Let us prove (1.4). First, we notice

$$|u(t, x) - Q| = \int_0^\infty |\partial_s v(s, t, x)| \, ds' \lesssim \int_0^\infty |\phi_s| \, ds'. \tag{7.13}$$

Step 1.1. Recall the definition of $\{c_k^{(j)}(\sigma)\}$ in Definition 6.1. Applying (3.74) with $\beta_k(\sigma) = c_k^{(0)}(\sigma)$, and its analogues in succeeding iterations, by the Bernstein inequality we get

$$\| \phi_s \|_{L_t^4 L_x^\infty} \lesssim s^{-1/4} \sum_{k \in \mathbb{Z}} c_k^{(1)}(1), \tag{7.14}$$

$$\| \phi_s \|_{L_t^4 L_x^\infty} \lesssim s^{-3/4} \sum_{k \in \mathbb{Z}} c_k^{(0)}(0). \tag{7.15}$$

We find by Young's inequality and the triangle inequality that

$$2^{\frac{1}{2^{j+4}}\delta|k|}c_k^{(j)} \lesssim \sup_{k'\in\mathbb{Z}} 2^{\frac{1}{2^{j+4}}\delta|k'|}\|P_{k'}\nabla u_0\|_{L_x^2},$$

and thus

$$\sum_{k\in\mathbb{Z}} c_k^{(j)} \lesssim \sup_{k'\in\mathbb{Z}} 2^{\frac{1}{2^{j+4}}\delta|k'|}\|P_{k'}\nabla u_0\|_{L_x^2} \lesssim 1, \tag{7.16}$$

since $u_0 \in \mathcal{H}_Q$. Then (7.14) and (7.15) show

$$\|\phi_s\|_{L_t^4 L_x^\infty} \lesssim \min(s^{-1/4}, s^{-3/4}). \tag{7.17}$$

We see (7.17) is not enough to put $\|\phi_s\|_{L_x^\infty}$ in L_s^1, but useful for Step 2 below.

Step 1.2. Applying (3.74) and (3.77) with $\beta_k(\sigma) = c_k^{(0)}(\sigma)$, $\sigma = 0$, and by interpolation, we see that for any $p \in (4,\infty)$ and $\widetilde{p} \in (2,4)$ satisfying $1/p + 1/\widetilde{p} = \frac{1}{2}$, we have

$$\|\phi_s\|_{L_t^p L_x^{\widetilde{p}}} \lesssim 2^k 1_{k+j\geq 0}(1+2^{2j+2k})^{-4}c_k^{(0)}(0) + 2^k 1_{k+j\leq 0}2^{\delta|k+j|}c_k^{(0)}(0)$$

for $s \in [2^{2j-1}, 2^{2j+1})$ and $k, j \in \mathbb{Z}$. Then by the Bernstein inequality,

$$\begin{aligned}\int_0^\infty \|\phi_s\|_{L_t^p L_x^\infty}\, ds' &\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\leq -k} 2^{2j+k}2^{2k/\widetilde{p}}c_k^{(0)}(0)2^{\delta|k+j|} \\ &\quad + \sum_{k\in\mathbb{Z}}\sum_{j\geq -k} 2^{2j+k}2^{2k/\widetilde{p}}(1+2^{k+j})^{-8}c_k^{(0)}(0) \\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{(2/\widetilde{p}-1)k}c_k^{(0)}(0). \end{aligned}\tag{7.18}$$

Taking $\widetilde{p} \in (2,4)$ such that $|2/\widetilde{p} - 1| \leq \frac{1}{8}\delta$, one finds that (7.18) is finite by (7.16). Hence, there exists a $p \in (4,\infty)$ such that

$$\int_0^\infty \|\phi_s\|_{L_t^p L_x^\infty}\, ds' \lesssim 1. \tag{7.19}$$

Step 1.3. We aim to prove

$$\lim_{t\to\infty}\int_0^\infty \|\phi_s(t)\|_{L_x^\infty}\, ds' = 0. \tag{7.20}$$

If (7.20) fails, then for some $\varrho > 0$, there exists a time sequence $\{t_\nu^1\}$ such that $\lim_{\nu\to\infty} t_\nu^1 = \infty$ and

$$\int_0^\infty \|\phi_s(t_\nu^1)\|_{L_x^\infty}\, ds' > \varrho, \quad \forall \nu \in \mathbb{Z}_+. \tag{7.21}$$

We can also assume $t_\nu^1 \leq t_{\nu+1}^1 - 4$ for any $\nu \in \mathbb{Z}_+$. Thus by (7.19) there must exist a sufficiently large constant N and a time sequence $\{t_\nu^2\}$ such that

$$t_\nu^1 - 1 \leq t_\nu^2 \leq t_\nu^1 + 1, \tag{7.22}$$

$$\int_0^\infty \|\phi_s(t_\nu^2)\|_{L_x^\infty}\, ds' \leq \tfrac{1}{8}\varrho, \quad \forall \nu \geq N. \tag{7.23}$$

Step 2. On the other hand, we have

$$\begin{aligned}\partial_t\phi_s &= D_t\phi_s - A_t\phi_s = D_s\phi_t - A_t\phi_s\\ &= \Delta\phi_t + \sum_{i=1,2}(2A_i\partial_i\phi_t + A_iA_i\phi_t + \phi_t\partial_i A_i + \mathcal{R}(\phi_i,\phi_t)\phi_i) - A_t\phi_s.\end{aligned}$$

Using Proposition 6.2 ((6.12), (6.13)) with $b_k(\sigma)$ replaced by $c_k^{(1)}(\sigma)$ and similar results for succeeding iterations, we see

$$\begin{aligned}\|\phi_t\|_{L_t^4L_x^\infty} &\lesssim \sum_{k\in\mathbb{Z}} c_k^{(2)}(3/2) \lesssim 1,\\ \|\partial_x\phi_t\|_{L_t^4L_x^\infty} &\lesssim \sum_{k\in\mathbb{Z}} c_k^{(6)}(5/2) \lesssim 1,\\ \|\partial_x^2\phi_t\|_{L_t^4L_x^\infty} &\lesssim \sum_{k\in\mathbb{Z}} c_k^{(10)}(7/2) \lesssim 1,\end{aligned}$$

since as before one has

$$\sum_{k\in\mathbb{Z}} 2^{\frac{1}{2j+4}\delta|k|} c_k^{(j)}(\sigma) \lesssim 1.$$

And for the same reason,

$$\begin{aligned}\|\partial_x^2\phi_t\|_{L_t^4L_x^\infty} &\lesssim s^{-5/4}\sum_{k\in\mathbb{Z}} c_k^{(1)}(1) \lesssim s^{-5/4},\\ \|\partial_x\phi_t\|_{L_t^4L_x^\infty} &\lesssim s^{-3/4}\sum_{k\in\mathbb{Z}} c_k^{(1)}(1) \lesssim s^{-3/4}.\end{aligned}$$

Meanwhile, Lemma 3.3 and (1.6) show

$$\begin{aligned}\|\phi_i\|_{L^\infty} &\lesssim (1+s)^{-3/4}, \qquad i = 1,2,\\ \|\partial_x^j A_i\|_{L^\infty} &\lesssim (1+s)^{-3/4-j/2}, \quad j = 0,1.\end{aligned}$$

Thus we arrive at

$$\int_0^\infty \|\Delta\phi_t\|_{L_t^4L_x^\infty} + \sum_{i=1,2}\|2A_i\partial_i\phi_t + A_iA_i\phi_t + \phi_t\partial_iA_i + \mathcal{R}(\phi_i,\phi_t)\phi_i\|_{L_t^4L_x^\infty}\, ds \lesssim 1.$$

For the rest $A_t\phi_s$, by the proof of Lemma 6.1 and its analogues in succeeding iterations, we see that

$$\|A_t\|_{L_t^4L_x^\infty} \lesssim s^{-1/4}\sum_{k\in\mathbb{Z}} c_k^{(1)}(1) \lesssim s^{-1/4}, \quad \|A_t\|_{L_t^4L_x^\infty} \lesssim s^{-3/4}\sum_{k\in\mathbb{Z}} c_k^{(0)}(0) \lesssim s^{-3/4}.$$

Hence, (7.17) implies

$$\int_0^\infty \|A_t\phi_s\|_{L_t^2L_x^\infty}\, ds' \lesssim 1.$$

Therefore, we conclude in this step that there exists a decomposition $\partial_t \phi_s = I_1 + I_2$ such that

$$\int_0^\infty \|I_1\|_{L_t^4 L_x^\infty} \, ds' \lesssim 1, \quad \int_0^\infty \|I_2\|_{L_t^2 L_x^\infty} \, ds' \lesssim 1. \tag{7.24}$$

Step 3. (7.22) and (7.24) show

$$\begin{aligned} &\int_0^\infty \|\phi_s(t_\nu^1) - \phi_s(t_\nu^2)\|_{L_x^\infty} \, ds' \\ &\quad \lesssim \int_0^\infty \big(\|I_1\|_{L_t^2 L_x^\infty([t_\nu^2-1, t_\nu^2+1]\times\mathbb{R}^2)} + \|I_2\|_{L_t^2 L_x^\infty([t_\nu^2-1, t_\nu^2+1]\times\mathbb{R}^2)}\big) \, ds'. \end{aligned} \tag{7.25}$$

Then as $\nu \to \infty$, (7.24) further implies the RHS of (7.25) goes to zero. Thus (7.23) yields

$$\int_0^\infty \|\phi_s(t_\nu^1)\|_{L_x^\infty} \, ds' \le \tfrac{1}{4}\varrho$$

for ν sufficiently large, which contradicts (7.21). So we have verified (7.20).

Similar to (7.20) we also have

$$\lim_{t\to-\infty} \int_0^\infty \|\phi_s(t)\|_{L_x^\infty} \, ds' = 0.$$

Then (1.4) follows by (7.13).

7.5. *Proof of* (1.5)

The proof of (1.5) can be reduced to the following lemma.

Lemma 7.2. *Given $s > 0$, there exists a function $f_s : \mathbb{R}^2 \to \mathbb{C}^n$ belonging to $\dot{H}^1$ such that*

$$\lim_{t\to\infty} \|\phi_s(t) - e^{it\Delta} f_s\|_{\dot{H}_x^1} = 0.$$

Moreover, f_s satisfies

$$\|f_s\|_{\dot{H}_x^1} \lesssim \mathbf{1}_{s\in[0,1]} + \mathbf{1}_{s\ge 1} s^{-3/2}.$$

Now, let us prove (1.5) by assuming Lemma 7.2. Recall that

$$\phi_i = -\int_s^\infty (\partial_i \phi_s + A_i \phi_s) \, ds'.$$

Since $\|A\|_{L_{s,t}^\infty L_x^2} \lesssim 1$, (7.20) shows

$$\lim_{t\to\infty} \left\| \int_s^\infty |A_i \phi_s| \, ds' \right\|_{L_x^2} = 0.$$

Then Lemma 7.2 yields

$$\lim_{t\to\infty} \|\phi(0, t, x) - \nabla e^{it\Delta} f_+\|_{L_x^2} = 0, \tag{7.26}$$

where $f_+ \in \dot{H}^1$ is defined by

$$f_+ = -\int_0^\infty f_s \, ds'.$$

Let $\mathcal{P}$ denote the isometric embedding of $\mathcal{N}$ into $\mathbb{R}^N$. Recall that $\{e_\alpha, Je_\alpha\}_{\alpha=1}^n$ denotes the caloric gauge. Then the caloric gauge condition shows

$$\sum_{l=1}^{2n} |d\mathcal{P}(e_l) - d\mathcal{P}(e_l^\infty)| \lesssim \int_0^\infty |\phi_s| \, ds',$$

which combined with (7.20) implies, for $s = 0$,

$$\lim_{t\to\infty} \|d\mathcal{P}(e_l) - d\mathcal{P}(e_l^\infty)\|_{L_x^\infty} = 0, \quad \forall l = 1, \ldots, 2n. \tag{7.27}$$

Thus, we deduce from

$$\partial_j u = \sum_{\alpha=1}^n (\Re(\phi_j^\alpha) e_\alpha + \Im(\phi_j^\alpha) Je_\alpha)$$

that for $s = 0$,

$$\begin{aligned}
&\Big\| d\mathcal{P}(\nabla u) - \sum_{\alpha=1}^n \Re(\nabla e^{it\Delta} f_+)^\alpha d\mathcal{P}(e_\alpha^\infty) - \sum_{\alpha=1}^n \Im(\nabla e^{it\Delta} f_+)^\alpha d\mathcal{P}(Je_\alpha^\infty) \Big\|_{L_x^2} \\
&\quad \lesssim \|\phi - \nabla e^{it\Delta} f_+\|_{L_x^2} + \big\| |e^{it\Delta} \nabla f_+| \, |d\mathcal{P}(e) - d\mathcal{P}(e^\infty)| \big\|_{L_x^2} \\
&\qquad + \big\| |\phi - \nabla e^{it\Delta} f_+| \, |d\mathcal{P}(e) - d\mathcal{P}(e^\infty)| \big\|_{L_x^2}.
\end{aligned}$$

Therefore, (7.27) and (7.26) give

$$\lim_{t\to\infty} \Big\| d\mathcal{P}(\nabla u) - \sum_{\alpha=1}^n \big(\Re(\nabla e^{it\Delta} f_+)^\alpha d\mathcal{P}(e_\alpha^\infty) - \Im(\nabla e^{it\Delta} f_+)^\alpha d\mathcal{P}(Je_\alpha^\infty) \big) \Big\|_{L_x^2} = 0.$$

Then, letting $\vec{v}_\alpha := d\mathcal{P}(e_\alpha^\infty)$ and $\vec{v}_{\alpha+n} := d\mathcal{P}(Je_\alpha^\infty)$, we get

$$\lim_{t\to\infty} \Big\| u(t) - \sum_{j=1}^n \Re(e^{it\Delta} f_+)^j \vec{v}_j - \sum_{j=1}^n \Im(e^{it\Delta} f_+)^j \vec{v}_{j+n} \Big\|_{\dot{H}_x^1} = 0. \tag{7.28}$$

Thus, (1.5) follows from (7.28) by setting

$$h_+^j := f_+^j \vec{v}_j, \quad g_+^j := f_+^j \vec{v}_{j+n}, \quad j = 1, \ldots, n.$$

Now, let us prove Lemma 7.2. The convenient way is to introduce the so-called Schrödinger map tension field $Z := \phi_s - i\phi_t$. Then the heat tension field ϕ_s satisfies, for any $s \geq 0$,

$$(i\partial_t + \Delta)\phi_s = \mathbf{N}, \tag{7.29}$$

$$\mathbf{N} := -\Big(\sum_{k=1}^2 \partial_k A_k\Big)\phi_s - \sum_{j=1}^2 (2A_j \partial_j \phi_s - A_j A_j \phi_s) + i\partial_s Z + \sum_{j=1}^2 \mathcal{R}(\phi_j, \phi_s)\phi_j. \tag{7.30}$$

And the Schrödinger map tension field Z satisfies the heat equation

$$\begin{cases} (\partial_s - \Delta)Z = (\sum_{k=1}^2 \partial_k A_k)Z + \sum_{j=1}^2 [2A_j \partial_j Z + A_j A_j Z] \\ \qquad\qquad + \sum_{j=1}^2 [\mathcal{R}(Z, \phi_j)\phi_j + i\mathcal{R}(\phi_j, \phi_s)\phi_j - \mathcal{R}(\phi_j, i\phi_s)\phi_j], \\ Z(0,t,x) = 0. \end{cases} \tag{7.31}$$

To prove Lemma 7.2, it suffices to verify

$$\|\{2^k \|P_k \mathbf{N}\|_{N_k}\}\|_{\ell^2} \lesssim (1+s)^{-3/2},$$

where $\mathbf{N}$ is given by (7.30). Except for the $\partial_s Z$ term in $\mathbf{N}$, the other terms have been handled before. It remains to dominate $\|P_k \partial_s Z\|_{N_k}$. In fact, one can prove a stronger result for Z:

$$\big\|\{(1+2^{2k})2^k \|P_k Z\|_{L^{4/3}}\}\big\|_{\ell^2} \lesssim (1+s)^{-3/2}. \tag{7.32}$$

We see that (7.32) follows by bootstrap and (7.31). Therefore, Lemma 7.2 follows.

Hence, we have finished the proofs of Theorems 1.1 and 1.2.

8. Appendix A. Bilinear estimates

Lemma 8.1. *Let $S : \mathbb{R}^N \to \mathbb{R}$ and $f : (-T,T) \times \mathbb{R}^2 \to \mathbb{R}^N$ be smooth. Let*

$$\mu_k := \sum_{|k_1 - k| \le 20} 2^{k_1} \|P_{k_1} f\|_{L^\infty_t L^2_x}. \tag{8.1}$$

Assume that $\|f\|_{L^\infty_x} \lesssim 1$ and $\sup_{k\in\mathbb{Z}} \mu_k \le 1$. Then

$$\begin{aligned} 2^k \|P_k S(f)(\partial_a f \partial_b f)\|_{L^\infty_t L^2_x} &\lesssim 2^k \sum_{k_1 \le k} \mu_{k_1} 2^{k_1} \mu_k + \sum_{k_2 \ge k} 2^{2k} \mu_{k_2}^2 \\ &+ a_k \Big(\sum_{k_1 \le k} 2^{k_1} \mu_{k_1}\Big)^2 + \sum_{k_2 \ge k} 2^{2k} 2^{-k_2} a_{k_2} \mu_{k_2} \sum_{k_1 \le k_2} 2^{k_1} \mu_{k_1}. \end{aligned} \tag{8.2}$$

where

$$a_k := \|\nabla P_k(S(f))\|_{L^\infty_t L^2_x}. \tag{8.3}$$

Proof. The proof of [5, Lemma 8.2] shows

$$\begin{aligned} 2^k \|P_k S(f)(\partial_a f \partial_b f)\|_{L^\infty_t L^2_x} &\lesssim 2^{2k} \sum_{k_1 \le k} \mu_{k_1} 2^k \mu_k + \sum_{k_2 \ge k} 2^{-2(k_2-k)} 2^{2k_2} \mu_{k_2}^2 \\ &+ a_k \Big(\sum_{k_1 \le k} 2^{k_1} \mu_{k_1}\Big)^2 + \sum_{k_2 \ge k} 2^{2k} 2^{-2k_2} 2^{k_2} a_{k_2} \mu_{k_2} \sum_{k_1 \le k_2} 2^{k_1} \mu_{k_1}. \end{aligned}$$

The only difference is that we use

$$\|P_k(S(f))\|_{L^2_x} \le 2^{-k} \|\nabla P_k(S(f))\|_{L^2_x}$$

when $S(f)$ lies in the high frequency with respect to $\partial_a f \partial_b f$, and the trivial bound

$$\|P_k(S(f))\|_{L_x^\infty} \lesssim 1$$

when $S(f)$ lies in the relatively low frequency. ■

Denote by $H^{\infty,\infty}(T)$ the set of functions $f:[-T,T]\times\mathbb{R}^2\to\mathbb{R}$ satisfying $\partial_t^{b_1}\partial_x^{b_2} f \in L^2([-T,T]\times\mathbb{R}^2)$ for any $b_1, b_2 \in \mathbb{N}$.

Lemma 8.2 ([5, Lemma 5.1]). *Given $\mathcal{L}\in\mathbb{Z}_+$, $\omega\in[0,\frac{1}{2}]$ and $T\in(0,2^{2\mathcal{L}}]$. Suppose that $f,g\in H^{\infty,\infty}(T)$, and let*

$$\alpha_k := \sum_{|k-k'|\le 20} \|f_{k'}\|_{S^\omega_{k'}(T)\cap F_{k'}(T)}, \quad \beta_k := \sum_{|k-k'|\le 20} \|g_{k'}\|_{S^0_{k'}(T)},$$

If $|k_1-k_2|\le 8$, then

$$\|P_k(P_{k_1} f P_{k_2} g)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^k 2^{(k_2-k)(1-\omega)} \alpha_{k_1}\beta_{k_2}. \tag{8.4}$$

If $|k-k_1|\le 4$, then

$$\|P_k(g P_{k_1} f)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \|g\|_{L^\infty}\alpha_{k_1}. \tag{8.5}$$

Lemma 8.3 ([5, Lemma 5.4]). *Given $\mathcal{L}\in\mathbb{Z}_+$, $\omega\in[0,\frac{1}{2}]$ and $T\in(0,2^{2\mathcal{L}}]$. Then for $f,g\in H^{\infty,\infty}(T)$,*

$$\|P_k(fg)\|_{L^4_{t,x}} \lesssim \sum_{l\le k} 2^l(\mathbf{a}_l\mathbf{b}_k + 2^{\frac{1}{2}(k-l)}\mathbf{a}_k\mathbf{b}_l) + 2^k\sum_{l\ge k} 2^{-\omega(l-k)}\mathbf{a}_l\mathbf{b}_l. \tag{8.6}$$

where

$$\mathbf{a}_k := \sum_{|l-k|\le 20} \|P_k f\|_{S_l^\omega(T)}, \quad \mathbf{b}_k := \sum_{|l-k|\le 20} \|P_k g\|_{L^4_{t,x}(T)}. \tag{8.7}$$

Lemma 8.4 ([5, Lemma 5.4]). *Given $\mathcal{L}\in\mathbb{Z}_+$, $\omega\in[0,\frac{1}{2}]$ and $T\in(0,2^{2\mathcal{L}}]$. Suppose that $f,g\in H^{\infty,\infty}(T)$, and $P_k f\in S_k^\omega(T)$, $P_k g\in L^4_{t,x}$ for all $k\in\mathbb{Z}$. Let*

$$\mu_k := \sum_{|l-k|\le 20} \|P_k f\|_{S_l^\omega}(T), \quad \nu_k := \sum_{|l-k|\le 20} \|P_k g\|_{L^4_{t,x}(T)}. \tag{8.8}$$

If $|k_2-k|\le 4$ and $k_1\le k-4$, then

$$\|P_k(f_{k_1} g_{k_2})\|_{L^4_{t,x}} \lesssim 2^{k_1}\mu_{k_2}\nu_k. \tag{8.9}$$

If $|k_1-k|\le 4$ and $k_2\le k-4$, then

$$\|P_k(f_{k_1} g_{k_2})\|_{L^4_{t,x}} \lesssim 2^{k_2} 2^{\frac{1}{2}(k-k_2)}\mu_k\nu_{k_2}. \tag{8.10}$$

If $|k_1-k_2|\le 8$ and $k_1,k_2\ge k-4$, then

$$\|P_k(f_{k_1} g_{k_2})\|_{L^4_{t,x}} \lesssim 2^{k(1+\omega)} 2^{-\omega k_2}\mu_{k_2}\nu_{k_2}. \tag{8.11}$$

Lemma 8.5 ([5, Lemma 6.3]). *If* $|l-k| \le 80$ *and* $f \in F_l(T)$, *then*

$$\|P_k(gf)\|_{N_k(T)} \lesssim \|g\|_{L_t^2 L_x^2} \|f\|_{F_l(T)}. \tag{8.12}$$

If $l \le k-80$ *and* $f \in F_l(T)$, *then*

$$\|P_k(gf)\|_{N_k(T)} \lesssim 2^{\frac{l-k}{2}} \|g\|_{L_t^2 L_x^2} \|f\|_{F_l(T)}. \tag{8.13}$$

If $k \le l-80$ *and* $f \in G_l(T)$, *then*

$$\|P_k(gf)\|_{N_k(T)} \lesssim 2^{\frac{k-l}{6}} \|g\|_{L_t^2 L_x^2} \|f\|_{G_l(T)}. \tag{8.14}$$

Lemma 8.6 ([5, Lemma 6.5]). *If* $k \le l$ *and* $f \in F_k(T)$, $g \in F_l(T)$ *then*

$$\|fg\|_{L^2_{t,x}} \lesssim \|f\|_{F_k(T)} \|g\|_{F_l(T)}. \tag{8.15}$$

If $k \le l$ *and* $f \in F_k(T)$, $g \in G_l(T)$ *then*

$$\|fg\|_{L^2_{t,x}} \lesssim 2^{\frac{k-l}{2}} \|f\|_{F_k(T)} \|g\|_{G_l(T)}. \tag{8.16}$$

9. Appendix B. Proof of remaining claims

It seems that the following blow-up criterion has not been explicitly written down in the literature. This result is well-known for energy critical heat flows. For completeness, we give a proof.

Proposition 9.1. *Suppose that* $u_0 \in H_Q^L$ *with* $L \ge 4$ *is the initial data to SMF. If in the time interval* $[-T,T]$, *the SMF solution* u *satisfies*

$$\|u(t)\|_{L^\infty_{t,x}(T)} \le B < \infty, \tag{9.1}$$

then u *has the bound*

$$\|u(t)\|_{L_t^\infty H_x^L} \le C(B,T,\|u_0\|_{H_x^L}) < \infty. \tag{9.2}$$

As a corollary, if (9.1) *holds then* u *can be extended beyond* $[-T,T]$ *to* $C([-T-\rho,T+\rho];H_Q^L)$ *for some* $\rho > 0$.

Proof. Recall the tension field $\tau(u) = \sum_{j=1}^2 \nabla_j \partial_j u$. By integration by parts,

$$\begin{aligned}\int_{\mathbb{R}^2} \langle \tau(u), \tau(u)\rangle \, dx &= \int_{\mathbb{R}^2} \sum_{j,k=1}^2 \langle \nabla_j \partial_j u, \nabla_k \partial_k u\rangle \, dx \\ &= \int_{\mathbb{R}^2} \langle \nabla_k \nabla_j \partial_k u, \nabla_k \nabla_j \partial_k u\rangle \, dx + \int_{\mathbb{R}^2} O(|du|^4)\, dx. \end{aligned}\tag{9.3}$$

Since u solves SMF, by integration by parts we get

$$\frac{d}{dt}\int_{\mathbb{R}^2}\langle\tau(u),\tau(u)\rangle\,dx = 2\sum_{j=1}^{2}\int_{\mathbb{R}^2}\langle\nabla_j\partial_j\partial_t u,\tau(u)\rangle\,dx + \int_{\mathbb{R}^2} O(|du|^2|\partial_t u|\,|\tau(u)|)\,dx$$
$$= 2\sum_{j=1}^{2}\int_{\mathbb{R}^2}\langle\nabla_j J\tau(u),\nabla_j\tau(u)\rangle\,dx + \int_{\mathbb{R}^2} O(|du|^2|\partial_t u|\,|\tau(u)|)\,dx.$$

Since J commutes with ∇_j, $\langle JX,X\rangle = 0$, we then arrive at

$$\frac{d}{dt}\|\tau(u)\|^2_{L^2_x} \lesssim \|du\|^2_{L^\infty_{t,x}}\|\tau(u)\|^2_{L^2_x}.$$

The Gronwall inequality and (9.1) show

$$\|\tau(u)\|_{L^2_x} \lesssim e^{Bt}\|\tau(u_0)\|_{L^2_x}.$$

Using the energy bound

$$\|\nabla u\|_{L^\infty_t L^2_x} \lesssim \|\nabla u_0\|_{L^2_x}$$

and (9.3) gives

$$\|u(t)\|_{\mathcal{W}^{2,2}} \lesssim B\|\nabla u_0\|_{L^2_x} + e^{Bt}\|\tau(u_0)\|_{L^2_x}. \tag{9.4}$$

By integration by parts,

$$\int_{\mathbb{R}^2}\langle\nabla_i\tau(u),\nabla_i\tau(u)\rangle\,dx = \int_{\mathbb{R}^2}\sum_{j,k=1}^{2}\langle\nabla_i\nabla_j\partial_j u,\nabla_i\nabla_k\partial_k u\rangle\,dx$$
$$=\int_{\mathbb{R}^2}\langle\nabla_i\nabla_j\partial_k u,\nabla_i\nabla_j\partial_k u\rangle\,dx+\int_{\mathbb{R}^2} O(|du|^3|\nabla^2 du|+|\nabla u|^2|\nabla du|^2+|\nabla du|\,|du|^2)\,dx.$$

Thus we have

$$\|\nabla^2 du(t)\|^2_{L^2_x} \lesssim \|\nabla\tau(u)\|^2_{L^2_x} + \|du\|^6_{L^6_x} + \|du\|^2_{L^\infty_x}\|\nabla du\|^2_{L^2_x} + \|du\|^2_{L^4_x}\|\nabla du\|_{L^2_x}$$
$$\lesssim \|\nabla\tau(u)\|^2_{L^2_x} + C(B,t,\|u_0\|_{\mathcal{W}^{2,2}}). \tag{9.5}$$

And applying integration by parts furthermore gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla\tau(u)\|^2_{L^2_x} = \int_{\mathbb{R}^2}\sum_{i,j}\langle\nabla_i\nabla_j\partial_j u,\nabla_t\nabla_i\nabla_j\partial_j u\rangle\,dx$$
$$= \int_{\mathbb{R}^2}\sum_{i,j}\langle\nabla_i\tau(u),\nabla_i\nabla_j\nabla_j\partial_t u\rangle\,dx + \int_{\mathbb{R}^2}|\nabla\tau(u)|\,|\nabla\partial_t u|\,|du|^2\,dx$$
$$+ \int_{\mathbb{R}^2}|\nabla^2 du|\,|\nabla du|\,|\partial_t u|\,|du|\,dx + \int_{\mathbb{R}^2}|du|^3|\partial_t u|\,|\nabla^2 du|\,dx$$
$$\lesssim \int_{\mathbb{R}^2} -\Big\langle\sum_i\nabla_i\nabla_i\tau(u), J\sum_j\nabla_j\nabla_j\tau(u)\Big\rangle\,dx + B^2\|\nabla\tau(u)\|^2_{L^2_x}$$
$$+ B\|\nabla\tau(u)\|_{L^2_x}\|\nabla du\|^2_{L^4_x} + B^3\|\nabla\tau(u)\|_{L^2_x}\|\tau(u)\|_{L^2_x}$$
$$\lesssim B^2\|\nabla\tau(u)\|^2_{L^2_x} + B\|\nabla\tau(u)\|_{L^2_x}\|\nabla^2 du\|_{L^2_x}\|\nabla du\|_{L^2_x} + B^3\|\nabla\tau(u)\|_{L^2_x}\|\tau(u)\|_{L^2_x}.$$

Hence, denoting $F(t) = \|\nabla\tau(u)\|_{L_x^2}$, (9.4) and (9.5) show

$$\frac{1}{2}\frac{d}{dt}F(t)^2 \lesssim C_1(B,T)F(t)[F(t) + C_2(B,T)],$$

where $C_1(B,T)$ and $C_2(B,T)$ are smooth functions of B, T. So the Sobolev norm of u has a uniform bound in $[-T, T]$ up to order 3. This together with the classical local existence theory (see [9] or [24]) implies u can be extended to $[-\rho - T, T + \rho]$ for some $\rho > 0$. And the bounds for the higher order Sobolev norms follow from [24, Theorem 3.3] or by induction. Then by Sobolev embedding, u is smooth in $[-\rho - T, T + \rho]$ if $u_0 \in \mathcal{H}_Q$. ∎

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