J. Eur. Math. Soc. 24, 2133–2167 (2022) DOI 10.4171/JEMS/1130

Michał Kowalczyk · Yvan Martel · Claudio Muñoz

Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes

Received April 5, 2019

Abstract. In this paper, we consider the dynamics of even solutions of the one-dimensional nonlinear Klein–Gordon equation $\partial_t^2\phi - \partial_x^2\phi + \phi - |\phi|^{2\alpha}\phi = 0$ for $\alpha > 1$, in the vicinity of the unstable soliton Q. Our main result is that stability in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ implies asymptotics stability in a local energy norm. In particular, there exists a Lipschitz graph of initial data leading to stable and asymptotically stable trajectories. The condition $\alpha > 1$ corresponds to cases where the linearized operator around Q has no resonance and no internal mode. Recall that the case $\alpha > 2$ is treated by Krieger, Nakanishi and Schlag [Math. Z. 272 (2012)] using Strichartz and other local dispersive estimates. Since these tools are not available for low power nonlinearities, our approach is based on virial type estimates and the particular structure of the linearized operator observed by Chang, Gustafson, Nakanishi and Tsai [SIAM J. Math. Anal. 39 (2007/08)].

Keywords. Nonlinear Klein–Gordon equation, soliton, asymptotic stability

1. Introduction

1.1. Main results

Consider the one-dimensional focusing nonlinear Klein–Gordon equation

$$\partial_t^2\phi - \partial_x^2\phi + \phi - f(\phi) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \qquad f(\phi) = |\phi|^{2\alpha}\phi, \tag{1.1}$$

where $\alpha > 0$. This equation also rewrites as a first-order system in time for the function $\boldsymbol{\phi} = (\phi, \partial_t\phi) = (\phi_1, \phi_2)$,

$$\begin{cases} \dot\phi_1 = \phi_2, \\ \dot\phi_2 = \partial_x^2\phi_1 - \phi_1 + f(\phi_1). \end{cases}$$

Michał Kowalczyk: Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile; kowalczy@dim.uchile.cl

Yvan Martel (corresponding author): CMLS, École polytechnique, CNRS, Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France; yvan.martel@polytechnique.edu

Claudio Muñoz: CNRS and Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile; cmunoz@dim.uchile.cl

Mathematics Subject Classification (2020): Primary 35L71; Secondary 35B40, 37K40

Let

$$F(\phi) = \int_0^{\phi} f(s)\,ds = \frac{1}{2\alpha+2}|\phi|^{2\alpha+2}.$$

Note that (1.1) is Hamiltonian. The conservation of energy of a solution $(\phi, \partial_t \phi)$ of (1.1) writes

$$E(\phi, \partial_t\phi) = \frac{1}{2}\int \left\{(\partial_t\phi)^2 + (\partial_x\phi)^2 + \phi^2 - 2F(\phi)\right\} = E(\phi(0), \partial_t\phi(0)). \tag{1.2}$$

For initial data in the energy space $H^1 \times L^2$, local well-posedness, as well as global well-posedness for small solutions, is well known (see for example [5, Theorem 6.2.2 and Proposition 6.3.3]).

Denote by Q the standing wave solution of (1.1), also called *soliton*, explicitly given by

$$Q(x) = \frac{(\alpha+1)^{\frac{1}{2\alpha}}}{\cosh^{\frac{1}{\alpha}}(\alpha x)}, \qquad Q'' - Q + Q^{2\alpha+1} = 0 \quad \text{on } \mathbb{R}.$$

The linearized operator L around Q writes

$$L = -\partial_x^2 + 1 - (2\alpha+1)Q^{2\alpha} = -\partial_x^2 + 1 - \frac{(2\alpha+1)(\alpha+1)}{\cosh^2(\alpha x)}. \tag{1.3}$$

For any $\alpha > 0$, the first eigenvalue of L is

$$\lambda_0 = -\alpha(\alpha+2) = -\nu_0^2 \quad (\nu_0 > 0)$$

with corresponding normalized eigenfunction

$$Y_0(x) = c_0(\cosh(\alpha x))^{-(1+\frac{1}{\alpha})}, \quad \langle Y_0, Y_0\rangle = 1, \quad LY_0 = -\nu_0^2 Y_0 \tag{1.4}$$

(we denote $\langle A, B\rangle = \int A \cdot B$). The second eigenvalue of the operator L is 0 with eigenfunction $Y_1 = c_1 Q'$. In the case $\alpha > 1$, there is no other eigenvalue in $[0, 1)$, which means that there is no *internal mode* for the model (see Section 1.3).

Let

$$\boldsymbol{Y}_\pm = \begin{pmatrix} Y_0 \\ \pm\nu_0 Y_0 \end{pmatrix}, \quad \boldsymbol{Z}_\pm = \begin{pmatrix} Y_0 \\ \pm\nu_0^{-1} Y_0 \end{pmatrix}.$$

The functions $\boldsymbol{u}_\pm(t,x) = e^{\pm\nu_0 t}\boldsymbol{Y}_\pm(x)$ are solutions of the linearized problem

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = -Lu_2 \end{cases} \tag{1.5}$$

illustrating the presence of exponentially stable and unstable modes both relevant in the dynamics of solutions in the vicinity of a soliton.

In this paper, by *global solution of* (1.1), we mean a function $\boldsymbol{\phi} \in \mathcal{C}([0,\infty), H^1 \times L^2)$ satisfying (1.1) for all $t \geq 0$. We only consider solutions with even symmetry.

Our main result is the following *conditional asymptotic stability theorem*.

Theorem 1. *Let $\alpha > 1$. There exists a constant $\delta > 0$ such that if a global even solution $\boldsymbol{\phi} = (\phi, \partial_t \phi)$ of (1.1) satisfies*

$$\|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \delta \quad \textit{for all } t \geq 0, \tag{1.6}$$

then, for any bounded interval I of $\mathbb{R}$,

$$\lim_{t\to+\infty} \|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1(I)\times L^2(I)} = 0. \tag{1.7}$$

For the sake of completeness, we provide a description of the set of initial data leading to global solutions satisfying the stability assumption (1.6) (see also Theorem 4.1 in [2]).

For $\delta_0 > 0$, let

$$\mathcal{A}_0 = \left\{\boldsymbol{\varepsilon} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) : \boldsymbol{\varepsilon} \text{ is even, } \|\boldsymbol{\varepsilon}\|_{H^1\times L^2} < \delta_0 \text{ and } \langle \boldsymbol{\varepsilon}, \boldsymbol{Z}_+\rangle = 0\right\}. \tag{1.8}$$

Theorem 2. *Let $\alpha > 1$. There exist $C, \delta_0 > 0$ and a Lipschitz function $h \colon \mathcal{A}_0 \to \mathbb{R}$ with $h(0) = 0$ and $|h(\boldsymbol{\varepsilon})| \leq C\|\boldsymbol{\varepsilon}\|^{3/2}_{H^1\times L^2}$ such that denoting*

$$\mathcal{M} = \{(Q, 0) + \boldsymbol{\varepsilon} + h(\boldsymbol{\varepsilon})\boldsymbol{Y}_+ : \boldsymbol{\varepsilon} \in \mathcal{A}_0\}$$

the following holds:

(1) *If $\boldsymbol{\phi}_0 \in \mathcal{M}$, then the solution $\boldsymbol{\phi}$ of (1.1) with initial data $\boldsymbol{\phi}_0$ is global and satisfies, for all $t \geq 0$,*

$$\|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leq C\|\boldsymbol{\phi}_0 - (Q, 0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}. \tag{1.9}$$

(2) *If a global even solution $\boldsymbol{\phi}$ of (1.1) satisfies, for all $t \geq 0$,*

$$\|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \frac{1}{2}\delta_0,$$

then for all $t \geq 0$, $\boldsymbol{\phi}(t) \in \mathcal{M}$.

1.2. Related results and comments on the proof

First, we comment on two articles devoted to soliton dynamics for the one-dimensional nonlinear Klein–Gordon equation (1.1).

Using techniques based on Strichartz and other local dispersive estimates, Krieger, Nakanishi and Schlag [21] have completely treated the case $\alpha > 2$ in the case of even data. Indeed, they classify all solutions whose energy does not exceed too much that of the ground state Q. This includes the construction, by the fixed point argument, of a $\mathcal{C}^1$ center-stable manifold around the soliton and the proof of asymptotic stability and scattering (linear behavior) around the ground state for solutions on the manifold. The method seems limited to $\alpha \geq 2$ because of the use of Strichartz estimates to control the nonlinear term, see comment in [21, Section 3.4].

By formal and numerical methods, Bizoń, Chmaj and Szpak [4] have shown that for even solutions trapped by the soliton, the convergence rate to Q heavily depends on the power α of the nonlinearity. In the L^∞ sense, they conjecture the following trichotomy:

(a) fast dispersive decay for $\alpha > 1$,

(b) slow decay for $\alpha = 1$,

(c) very slow decay for $0 < \alpha < 1$.

The threshold value $\alpha = 1$ corresponds to the emergence of a resonance at the linear level, while $\alpha < 1$ leads to one or several internal modes (see Section 1.3). Following these observations, unifying the case $\alpha > 1$ was the main motivation of the present work.

Our method does not give an explicit decay rate as $t \to +\infty$, but we notice as a by-product of the proof of Theorem 1 that, for any bounded interval I of $\mathbb{R}$, it holds

$$\int_0^{+\infty} \|\boldsymbol{\phi}(t) - (Q, 0)\|^2_{H^1(I)\times L^2(I)}\, dt < \infty. \tag{1.10}$$

This is to be compared with the results obtained in [18] on the (local) asymptotic stability of the kink for the ϕ^4 model under small odd perturbations. Indeed, in the latter case, the presence of an internal mode leads to a lower convergence rate since the component $z(t)$ of the solution along the internal mode only satisfies the weaker estimate

$$\int_0^{+\infty} |z(t)|^4\, dt < \infty$$

(see [18, Theorem 1.2]). Although we do not claim optimality of such results, in the case of (1.1) with $0 < \alpha \leq 1$, we do not expect estimates such as in (1.10) to hold.

The proof of Theorem 1 is mainly based on localized virial type arguments similar to that used in [18, 25, 28], for example. Unlike in these works, we avoid numerical computations of certain constants related to the coercivity of the virial functional by using factorization properties of the linearized operator described in [6] (see also references [29,37], cited in [6]). A formal presentation of this approach is given in Section 4.1. We point out that the same structure was crucially used in the construction of blow-up solutions for the wave maps, Yang–Mills and $O(3)$ σ-models in [32, 33]. Note that in the present paper, we compensate the loss of two derivatives due to the change of variables to still work in the energy space.

We refer to [1, 16, 17, 19, 20, 23, 35, 36] for various results of asymptotic stability for the nonlinear Klein–Gordon equation and ϕ^4 equation or variants of these models.

Several other conditional asymptotic stability results or classifications in a neighborhood of the ground state for the nonlinear Klein–Gordon in higher dimensions and for the nonlinear Schrödinger equation were also obtained in [10,11,30,34], for example. We also mention [22] where for the mass supercritical Schrödinger equation in one dimension, a finite co-dimensional manifold of initial data trapped by the soliton was constructed.

Concerning the generalized Korteweg-de Vries equation and related models, studies of the dynamics of the solutions close to the soliton are presented in [9, 14, 15, 24, 26–28, 31], in blow-up contexts or for bounded solutions. Note that the method introduced in [24,26], using the special structure of a transformed linearized problem, also has some analogy with our proof.

For global existence results in the case of semilinear and quasilinear wave equations, we refer to [12, 13].

Finally, we refer to [2, 3] and references therein for refined descriptions of dynamics of solutions in various settings.

1.3. Resonances and internal modes

As mentioned before, the absence of any other eigenvalue in $[0, 1)$ for the operator L when $\alpha > 1$ is important in our proof. For $0 < \alpha \leq 1$, we continue the description of the spectrum of L. For $\alpha = 1$, there is an even resonance at 1. For any $0 < \alpha < 1$, there is a third eigenvalue associated to an even eigenfunction

$$Y_2(x) = c_2 Y_0(x)\left(1 - \frac{2}{\alpha}\sinh^2(\alpha x)\right), \quad \lambda_2 = \alpha(2-\alpha), \quad \nu_2 = \lambda_2^{\frac{1}{2}}.$$

In particular, for any $0 < \alpha < 1$, the function

$$\boldsymbol{u}(t) = (\cos(\nu_2 t)Y_2, -\nu_2 \sin(\gamma_2 t)Y_2)$$

is solution of (1.5). This solution is typical of the notion of *internal modes* and shows that asymptotic stability (even up to the exponential instable mode) cannot be true at the linear level for such value of α. An important issue is the nature of the interaction of such internal mode with the nonlinearity. We recall that such an internal mode was treated in the context of the ϕ^4 equation in [18]. Pioneering results on internal modes were obtained in [35]. See other references in [18].

For $\alpha \in (\frac{1}{2}, 1)$, there are no other eigenvalue on $[0, 1)$. For $\alpha = \frac{1}{2}$, there is an odd resonance at 1. For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, there is a fourth eigenvalue, associated to an odd eigenfunction. For $\alpha \in (\frac{1}{4}, \frac{1}{3})$, there are five eigenvalues, three of them being associated to even eigenfunctions. In particular, there are two even internal modes. This procedure can be continued for all $\alpha > 0$, showing the emergence of arbitrarily many internal modes (and sometimes resonances) as $\alpha \to 0^+$.

The above information is taken from [6, Section 3].

2. Preliminaries

2.1. Decomposition of a solution in a vicinity of the soliton

Let $\boldsymbol{\phi} = (\phi, \partial_t \phi)$ be a solution of (1.1) satisfying (1.6) for some small $\delta > 0$. We decompose $(\phi, \partial_t \phi)$ as follows:

$$\begin{cases} \phi(t,x) = Q(x) + a_1(t)Y_0(x) + u_1(t,x), \\ \partial_t \phi(t,x) = a_2(t)\nu_0 Y_0(x) + u_2(t,x), \end{cases} \tag{2.1}$$

where

$$a_1(t) = \langle \phi(t) - Q, Y_0 \rangle, \quad a_2(t) = \frac{1}{\nu_0}\langle \partial_t \phi(t), Y_0 \rangle,$$

so that

$$\langle u_1(t), Y_0\rangle = \langle u_2(t), Y_0\rangle = 0. \tag{2.2}$$

Setting

$$b_+ = \frac{1}{2}(a_1 + a_2), \qquad b_- = \frac{1}{2}(a_1 - a_2), \tag{2.3}$$

we observe that $\boldsymbol{\phi}$ also writes as

$$\boldsymbol{\phi} = (Q, 0) + \boldsymbol{u} + b_-\boldsymbol{Y}_- + b_+\boldsymbol{Y}_+, \quad \boldsymbol{u} = (u_1, u_2). \tag{2.4}$$

From (1.6), for all $t \in [0, \infty)$, it holds

$$\|u_1(t)\|_{H^1} + \|u_2(t)\|_{L^2} + |a_1(t)| + |a_2(t)| + |b_+(t)| + |b_-(t)| \le C_0\delta. \tag{2.5}$$

Moreover, using $Q'' - Q + f(Q) = 0$, $LY_0 = -\nu_0^2 Y_0$ and (2.2), the systems of equations of (a_1, a_2) and (u_1, u_2) write

$$\begin{cases} \dot a_1 = \nu_0 a_2, \\ \dot a_2 = \nu_0 a_1 + \dfrac{N_0}{\nu_0}, \end{cases} \quad \text{equivalently} \quad \begin{cases} \dot b_+ = \nu_0 b_+ + \dfrac{N_0}{2\nu_0}, \\ \dot b_- = -\nu_0 b_- - \dfrac{N_0}{2\nu_0}, \end{cases} \tag{2.6}$$

and

$$\begin{cases} \dot u_1 = u_2, \\ \dot u_2 = -Lu_1 + N^\perp, \end{cases} \tag{2.7}$$

where

$$\begin{aligned} N &= f(Q + a_1Y_0 + u_1) - f(Q) - f'(Q)a_1Y_0 - f'(Q)u_1, \\ N_0 &= \langle N, Y_0\rangle, \quad N^\perp = N - N_0Y_0. \end{aligned} \tag{2.8}$$

2.2. *Notation for virial arguments*

Let ρ be the following weight function:

$$\rho(x) = \operatorname{sech}\left(\frac{x}{10}\right). \tag{2.9}$$

For any function $w \in H^1$, consider the norm

$$\|w\|_\rho = \left[\int \left((\partial_x w)^2 + \rho w^2\right)\right]^{\frac{1}{2}}. \tag{2.10}$$

We consider a smooth even function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \chi = 1 & \text{on } [-1, 1], \\ \chi = 0 & \text{on } (-\infty, -2] \cup [2, +\infty), \\ \chi' \le 0 & \text{on } [0, +\infty). \end{cases} \tag{2.11}$$

For $A > 0$, we define the functions ζ_A and φ_A as follows:

$$\zeta_A(x) = \exp\left(-\frac{1}{A}(1 - \chi(x))|x|\right), \quad \varphi_A(x) = \int_0^x \zeta_A^2(y)\, dy, \quad x \in \mathbb{R}.$$

For $B > 0$, we also define

$$\zeta_B(x) = \exp\left(-\frac{1}{B}(1-\chi(x))|x|\right), \quad \varphi_B(x) = \int_0^x \zeta_B^2(y)\,dy, \quad x \in \mathbb{R}, \tag{2.12}$$

and we consider the function ψ defined as

$$\psi_B(x) = \chi_B^2(x)\varphi_B(x), \quad \text{where } \chi_B(x) = \chi\left(\frac{x}{B^2}\right), \quad x \in \mathbb{R}. \tag{2.13}$$

The notation $X \lesssim Y$ means $X \le CY$ for a constant independent of A and B.

These functions ζ_A, φ_A, ζ_B, φ_B and ψ_B will be used in two distinct virial arguments with different scales

$$A \gg B^2 \gg B \gg 1. \tag{2.14}$$

3. Virial argument in u

Set

$$\mathcal{I} = \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) u_2 \tag{3.1}$$

and

$$w = \zeta_A u_1. \tag{3.2}$$

We refer to [18] for the use of such virial argument in a similar context. Here, w represents a localized version of u_1, in the scale A (see (2.14)). We shall prove the following result.

Proposition 1. *There exist $C_1 > 0$ and $\delta_1 > 0$ such that for any $0 < \delta \le \delta_1$, the following holds. Fix $A = \delta^{-1}$. Assume that for all $t \ge 0$, (2.5) holds. Then, for all $t \ge 0$,*

$$\dot{\mathcal{I}} \le -\frac{1}{2}\int (\partial_x w)^2 + C_1 \int \operatorname{sech}\left(\frac{x}{2}\right) w^2 + C_1 |a_1|^4. \tag{3.3}$$

Remark 1. Note that estimate (3.3) does not involve any type of spectral analysis. Its purpose is to give a simple control of $\int (\partial_x w)^2$ in terms of $\int \operatorname{sech}(\frac{x}{2}) w^2$ and $|a_1|^4$.

The rest of this section is devoted to the proof of Proposition 1. We compute from (3.1)

$$\dot{\mathcal{I}} = \int \left(\varphi_A \partial_x \dot{u}_1 + \frac{1}{2}\varphi_A' \dot{u}_1\right) u_2 + \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right) \dot{u}_2.$$

Replacing $\dot{u}_1$ by u_2 and integrating by parts, the first integral in the right-hand side vanishes. The expression of $\dot{u}_2$ in (2.7) rewrites

$$\dot{u}_2 = \partial_x^2 u_1 - u_1 + f(Q + a_1 Y_0 + u_1) - f(Q) - f'(Q) a_1 Y_0 - N_0 Y_0,$$

and so

$$\begin{aligned} \dot{\mathcal{I}} = &\int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)(\partial_x^2 u_1 - u_1) \\ &+ \int \left(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\right)\left[f(Q + a_1 Y_0 + u_1) - f(Q) - f'(Q) a_1 Y_0 - N_0 Y_0\right]. \end{aligned}$$

To treat the first line in the expression of $\dot{\mathcal{I}}$, we claim the following.

Lemma 1. *It holds*

$$\int \Big(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\Big)(\partial_x^2 u_1 - u_1) = -\int (\partial_x w)^2 - \frac{1}{2}\int \Big(\frac{\zeta_A''}{\zeta_A} - \frac{(\zeta_A')^2}{\zeta_A^2}\Big) w^2. \quad (3.4)$$

Moreover,

$$\frac{\zeta_A''}{\zeta_A} - \frac{(\zeta_A')^2}{\zeta_A^2} = \frac{1}{A}\big[\chi''(x)|x| + 2\chi'(x)\,\mathrm{sgn}(x)\big] \quad (3.5)$$

and

$$\left|\frac{\zeta_A''}{\zeta_A} - \frac{(\zeta_A')^2}{\zeta_A^2}\right| \lesssim \frac{\mathbf{1}_{1\le|x|\le 2}(x)}{A} \lesssim \frac{\mathrm{sech}(x)}{A}. \quad (3.6)$$

Proof. Proof of (3.4). By integration by parts

$$\int \Big(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\Big)(\partial_x^2 u_1 - u_1) = -\int \varphi_A'(\partial_x u_1)^2 + \frac{1}{4}\int \varphi_A''' u_1^2.$$

We rewrite the above expression using the auxiliary function w. Indeed,

$$\int (\partial_x w)^2 = \int (\zeta_A \partial_x u_1 + \zeta_A' u_1)^2 = \int \zeta_A^2 (\partial_x u_1)^2 + 2\int \zeta_A \zeta_A' u_1 \partial_x u_1 + \int (\zeta_A')^2 u_1^2$$
$$= \int \varphi_A'(\partial_x u_1)^2 - \int \zeta_A \zeta_A'' u_1^2 = \int \varphi_A'(\partial_x u_1)^2 - \int \frac{\zeta_A''}{\zeta_A} w^2$$

and so

$$\int \varphi_A'(\partial_x u_1)^2 = \int (\partial_x w)^2 + \int \frac{\zeta_A''}{\zeta_A} w^2.$$

Next,

$$\int \varphi_A''' u_1^2 = \int \frac{(\zeta_A^2)''}{\zeta_A^2} w^2 = 2\int \Big(\frac{\zeta_A''}{\zeta_A} + \frac{(\zeta_A')^2}{\zeta_A^2}\Big) w^2. \quad (3.7)$$

Identity (3.4) follows.

Proof of (3.5)–(3.6). By elementary computations, we have

$$\frac{\zeta_A'}{\zeta_A} = -\frac{1}{A}\big[-\chi'(x)|x| + (1-\chi(x))\,\mathrm{sgn}(x)\big],$$
$$\frac{\zeta_A''}{\zeta_A} = \frac{1}{A^2}\big[-\chi'(x)|x| + (1-\chi(x))\,\mathrm{sgn}(x)\big]^2 + \frac{1}{A}\big[\chi''(x)|x| + 2\chi'(x)\,\mathrm{sgn}(x)\big],$$

which proves (3.5). Estimate (3.6) then follows from the definition of χ. ■

To treat the second line in the expression of $\dot{\mathcal{J}}$, we claim the following.

Lemma 2. *One has*

$$\left|\int \Big(\varphi_A \partial_x u_1 + \frac{1}{2}\varphi_A' u_1\Big)\big[f(Q + a_1 Y_0 + u_1) - f(Q) - a_1 f'(Q) Y_0 - N_0 Y_0\big]\right|$$
$$\lesssim |a_1|^4 + \int \mathrm{sech}\Big(\frac{x}{2}\Big) w_1^2 + A^2 \|u_1\|_{L^\infty}^{2\alpha} \int |\partial_x w|^2. \quad (3.8)$$

Proof. First, we treat the term $-\int(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A' u_1)N_0Y_0$. By Taylor's expansion, one has

$$|N|\lesssim a_1^2Q^{2\alpha-1}Y_0^2+Q^{2\alpha-1}u_1^2+|a_1|^{2\alpha+1}Y_0^{2\alpha+1}+|u_1|^{2\alpha+1}, \tag{3.9}$$

and thus, by decay estimates on Q and Y_0, and by (2.5), $|a_1|\lesssim 1$, $\|u_1\|_{L^\infty}\lesssim\|u_1\|_{H^1}\lesssim 1$, $A\geq 4$, it holds

$$|N_0|\lesssim a_1^2+\int \operatorname{sech}(x)u_1^2\lesssim a_1^2+\int\operatorname{sech}\left(\frac{x}{2}\right)w^2. \tag{3.10}$$

Using integration by parts,

$$-\int\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A' u_1\right)Y_0=\int u_1\left(\varphi_A\partial_x Y_0+\frac{1}{2}\varphi_A' Y_0\right).$$

Note that for all $x\in\mathbb{R}$, $|\varphi_A'(x)|\leq 1$ and $|\varphi_A(x)|\leq|x|$, and so

$$|\varphi_A(x)\operatorname{sech}(x)|+|\varphi_A'(x)\operatorname{sech}(x)|\leq(|x|+1)\operatorname{sech}(x)\lesssim\operatorname{sech}\left(\frac{3}{4}x\right) \tag{3.11}$$

for an implicit constant independent of A. Thus, by the Cauchy–Schwarz inequality,

$$\left|N_0\int\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A' u_1\right)Y_0\right|\lesssim a_1^4+\int\operatorname{sech}\left(\frac{x}{2}\right)w_1^2.$$

Second, we decompose

$$\begin{aligned}
&\int\left(\varphi_A\partial_x u_1+\frac{1}{2}\varphi_A' u_1\right)\left[f(Q+a_1Y_0+u_1)-f(Q)-f'(Q)a_1Y_0\right]\\
&=\int\varphi_A\partial_x\left[F(Q+a_1Y_0+u_1)-F(Q+a_1Y_0)-(f(Q)+f'(Q)a_1Y_0)u_1\right]\\
&\quad-\int\varphi_A Q'\left[f(Q+a_1Y_0+u_1)-f(Q+a_1Y_0)-(f'(Q)+f''(Q)a_1Y_0)u_1\right]\\
&\quad-a_1\int\varphi_A Y_0'\left[f(Q+a_1Y_0+u_1)-f(Q+a_1Y_0)-f'(Q)u_1\right]\\
&\quad+\frac{1}{2}\int\varphi_A' u_1\left[f(Q+a_1Y_0+u_1)-f(Q)-f'(Q)a_1Y_0\right]\\
&=I_1+I_2+I_3+I_4.
\end{aligned}$$

We rewrite I_1, I_2, I_3 and I_4 as follows:

$$\begin{aligned}
I_1&=-\int\varphi_A'\left[F(Q+a_1Y_0+u_1)-F(Q+a_1Y_0)-F'(Q+a_1Y_0)u_1-F(u_1)\right]\\
&\quad-\int\varphi_A'\left[f(Q+a_1Y_0)-f(Q)-f'(Q)a_1Y_0\right]u_1-\int\varphi_A' F(u_1),\\
I_2&=-\int\varphi_A Q'\left[f(Q+a_1Y_0+u_1)-f(Q+a_1Y_0)-f'(Q+a_1Y_0)u_1\right]\\
&\quad-\int\varphi_A Q'\left[f'(Q+a_1Y_0)-f'(Q)-f''(Q)a_1Y_0\right]u_1,
\end{aligned}$$

$$I_3 = -a_1 \int \varphi_A Y_0' \big[f(Q + a_1 Y_0 + u_1) - f(Q + a_1 Y_0) - f'(Q + a_1 Y_0) u_1 \big] \\ - a_1 \int \varphi_A Y_0' \big[f'(Q + a_1 Y_0) - f'(Q) \big] u_1,$$

$$I_4 = \frac{1}{2} \int \varphi_A' u_1 \big[f(Q + a_1 Y_0 + u_1) - f(Q + a_1 Y_0) - f(u_1) \big] \\ + \frac{1}{2} \int \varphi_A' u_1 [f(Q + a_1 Y_0) - f(Q) - f'(Q) a_1 Y_0] + \frac{1}{2} \int \varphi_A' u_1 f(u_1).$$

To control the two terms that are purely nonlinear in u_1, we need the following claim.

Claim 1. *It holds*

$$\int \zeta_A^2 |u_1|^{2\alpha+2} = \int \zeta_A^{-2\alpha} |w|^{2\alpha+2} \lesssim A^2 \|u_1\|_{L^\infty}^{2\alpha} \int |\partial_x w|^2. \tag{3.12}$$

Proof of Claim 1. The first equality in (3.12) corresponds to the definition of w in (3.2). Next, by integration by parts and standard estimates, we have

$$\begin{aligned} &\int_0^{+\infty} \exp\Big(\frac{2\alpha}{A} x\Big) |w|^{2\alpha+2} \, dx \\ &= -\frac{A}{2\alpha} |w(0)|^{2\alpha+2} - \frac{A}{2\alpha} \int_0^{+\infty} \exp\Big(\frac{2\alpha}{A} x\Big) \partial_x (|w|^{2\alpha+2}) \, dx \\ &\le -\frac{\alpha+1}{\alpha} A \int_0^{+\infty} \exp\Big(\frac{2\alpha}{A} x\Big) (\partial_x w) w |w|^{2\alpha} \, dx \\ &\le \frac{\alpha+1}{\alpha} A \|u_1\|_{L^\infty}^{\alpha} \int_0^{+\infty} \exp\Big(\frac{\alpha}{A} x\Big) |\partial_x w| |w|^{\alpha+1} \, dx \\ &\le \Big(\frac{\alpha+1}{\alpha}\Big)^2 A^2 \|u_1\|_{L^\infty}^{2\alpha} \int_0^{+\infty} |\partial_x w|^2 \, dx + \frac{1}{4} \int_0^{+\infty} \exp\Big(\frac{2\alpha}{A} x\Big) |w|^{2\alpha+2} \, dx. \end{aligned}$$

Thus,

$$\int_0^{+\infty} \exp\Big(\frac{2\alpha}{A} x\Big) |w|^{2\alpha+2} \, dx \le \frac{4}{3} \Big(\frac{\alpha+1}{\alpha}\Big)^2 A^2 \|u_1\|_{L^\infty}^{2\alpha} \int_0^{+\infty} |\partial_x w|^2 \, dx,$$

which implies (3.12). ∎

In particular, (3.12) implies that

$$\int \varphi_A' F(u_1) + \int \varphi_A' u_1 f(u_1) \lesssim \int \zeta_A^2 |u_1|^{2\alpha+2} \lesssim A^2 \|u_1\|_{L^\infty}^{2\alpha} \int |\partial_x w|^2,$$

which takes care of the last terms in I_1 and I_4.

By Taylor expansion, $\alpha \ge 1$, $|a_1| \lesssim 1$ and $\|u_1\|_{L^\infty} \lesssim 1$, we have

$$\big| F(Q + a_1 Y_0 + u_1) - F(Q + a_1 Y_0) - F'(Q + a_1 Y_0) u_1 - F(u_1) \big| \\ \lesssim |Q + a_1 Y_0|^{2\alpha} u_1^2 + |Q + a_1 Y_0| |u_1|^{2\alpha+1} \lesssim \operatorname{sech}(x) u_1^2 \lesssim \operatorname{sech}\Big(\frac{x}{2}\Big) w_1^2.$$

Similarly, using also (3.11) and $A \geq 4$, we find the following estimates:

$$\left|\varphi_A Q'\left[f(Q + a_1 Y_0 + u_1) - f(Q + a_1 Y_0) - f'(Q + a_1 Y_0)u_1\right]\right| \lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2,$$

$$\left|a_1 \varphi_A Y_0'\left[f(Q + a_1 Y_0 + u_1) - f(Q + a_1 Y_0) - f'(Q + a_1 Y_0)u_1\right]\right| \lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2,$$

$$\left|\varphi_A' u_1\left[f(Q + a_1 Y_0 + u_1) - f(Q + a_1 Y_0) - f(u_1)\right]\right| \lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2.$$

Moreover, again by Taylor expansion and (3.11) (with $A > 8$), we have

$$\begin{aligned}
&\left|\varphi_A'\left[f(Q + a_1 Y_0 + u_1) - f(Q) - f'(Q)a_1 Y_0\right]u_1\right| \\
&\quad + \left|\varphi_A Q'\left[f'(Q + a_1 Y_0) - f'(Q) - f''(Q)a_1 Y_0\right]u_1\right| \\
&\quad + \left|a_1 \varphi_A Y_0'\left[f'(Q + a_1 Y_0) - f'(Q)\right]u_1\right| \\
&\quad + \left|\varphi_A' u_1[f(Q + a_1 Y_0) - f(Q) - f'(Q)a_1 Y_0]\right| \\
&\lesssim \operatorname{sech}\left(\frac{x}{2}\right)|a_1|^2|u_1| \lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2 + \operatorname{sech}\left(\frac{x}{4}\right)|a_1|^4.
\end{aligned}$$

Collecting these estimates, (3.8) is proved.

Taking $\|u_1\|_{L^\infty} \leq \delta_A$, for δ_A small enough, we have proved

$$\dot{\mathcal{J}} \leq -\int (\partial_x w)^2 + C \int w^2 \operatorname{sech}\left(\frac{x}{2}\right) + C a_1^4 + A^2 \|u_1\|_{L^\infty}^{2\alpha} \int (\partial_x w)^2.$$

Using $A = \delta^{-1}$ and $\|u_1\|_{L^\infty}^{2\alpha} \lesssim \delta^{2\alpha}$ (from (2.5)), for δ_1 small enough, we obtain (3.3). ■

4. Virial argument for the transformed problem

4.1. Heuristic

We recall results from [6, pp. 1086–1087]. Let

$$L = -\partial_x^2 + 1 - (2\alpha + 1)Q^{2\alpha}, \quad L_- = -\partial_x^2 + 1 - Q^{2\alpha},$$

and

$$U = Y_0 \cdot \partial_x \cdot Y_0^{-1}, \quad U^\star = -Y_0^{-1} \cdot \partial_x \cdot Y_0.$$

(The above notation means $Uf = Y_0(Y_0^{-1} f)'$.) Then the operators L and L_- rewrite as $L = U^\star U + \lambda_0$, $L_- = U U^\star + \lambda_0$ and it follows that

$$UL = L_- U.$$

Now, let

$$L_0 = -\partial_x^2 + 1 + \frac{\alpha - 1}{\alpha + 1} Q^{2\alpha}, \tag{4.1}$$

and

$$S = Q \cdot \partial_x \cdot Q^{-1}, \quad S^\star = -Q^{-1} \cdot \partial_x \cdot Q.$$

A similar structure $L_- = S^\star S$, $L_0 = SS^\star$, leads to

$$SL_- = L_0 S \quad \text{and thus} \quad SUL = L_0 SU.$$

In particular, let (u_1, u_2) be a solution of (1.5), and set $\tilde{u}_1 = Uu_1$, $\tilde{u}_2 = Uu_2$. Then

$$\begin{cases} \dot{\tilde{u}}_1 = \tilde{u}_2, \\ \dot{\tilde{u}}_2 = -L_- \tilde{u}_1. \end{cases}$$

Next, set

$$v_1 = S\tilde{u}_1 = SUu_1 \quad \text{and} \quad v_2 = S\tilde{u}_2 = SUu_2.$$

Then, (v_1, v_2) satisfies the following transformed problem:

$$\begin{cases} \dot{v}_1 = v_2, \\ \dot{v}_2 = -L_0 v_1. \end{cases}$$

The key point for our analysis is that for $\alpha > 1$, the potential in L_0 is positive. This property happens to be the only spectral information needed for the proof of Theorem 1.

Observe that $UY_0 = 0$, $UQ' = -\alpha Q$ and $SQ = 0$, which means that the prior decomposition of the solution $(\phi, \partial_t \phi)$ as in Section 2.1 and a coercivity argument as in Section 5 are necessary to avoid loosing information through the transformation. (Here, we work with even functions and so only the direction Y_0 is relevant.)

4.2. Transformed problem

With respect to the above heuristic, we need to localize and regularize the functions involved. For $\gamma > 0$ small to be defined later, set

$$\begin{cases} v_1 = (1 - \gamma \partial_x^2)^{-1} SU(\chi_B u_1), \\ v_2 = (1 - \gamma \partial_x^2)^{-1} SU(\chi_B u_2), \end{cases} \tag{4.2}$$

where χ_B is defined in (2.13). We refer to Section 5 for coercivity results relating u_1 and v_1. The introduction of the operator $(1 - \gamma \partial_x^2)^{-1}$ with a small constant γ is needed to compensate the loss of two derivatives due to the operator SU, without destroying the special algebra described heuristically. Now, we explain the role of the localization term χ_B in the definitions of v_1 and v_2. Note that Proposition 1 provides an estimate on the function w, which is a localized version of u (see (3.2)). To use this information, the functions v_1 and v_2 also need to contain a certain localization.

We deduce the following system for (v_1, v_2) from the one for (u_1, u_2) in (2.7):

$$\begin{cases} \dot{v}_1 = v_2, \\ \dot{v}_2 = -(1 - \gamma \partial_x^2)^{-1} SU(\chi_B L u_1) + (1 - \gamma \partial_x^2)^{-1} SU(\chi_B N^\perp). \end{cases}$$

First, we note that

$$\chi_B L u_1 = L(\chi_B u_1) + 2\chi_B' \partial_x u_1 + \chi_B'' u_1.$$

Moreover, since $SUL = L_0 SU$, it holds

$$\begin{aligned}-(1-\gamma\partial_x^2)^{-1}SUL(\chi_B u_1) &= -(1-\gamma\partial_x^2)^{-1}L_0 SU(\chi_B u_1)\\ &= -(1-\gamma\partial_x^2)^{-1}L_0[(1-\gamma\partial_x^2)v_1]\\ &= \partial_x^2 v_1 - v_1 - \frac{\alpha-1}{\alpha+1}(1-\gamma\partial_x^2)^{-1}\big[Q^{2\alpha}(1-\gamma\partial_x^2)v_1\big].\end{aligned}$$

Since

$$(1-\gamma\partial_x^2)[Q^{2\alpha}v_1] = Q^{2\alpha}(1-\gamma\partial_x^2)v_1 - 2\gamma(Q^{2\alpha})'\partial_x v_1 - \gamma(Q^{2\alpha})''v_1,$$

we obtain

$$\begin{aligned}&-(1-\gamma\partial_x^2)^{-1}SUL(\chi_B u_1)\\ &\quad= -L_0 v_1 - \frac{\alpha-1}{\alpha+1}\gamma(1-\gamma\partial_x^2)^{-1}\big[2(Q^{2\alpha})'\partial_x v_1 + (Q^{2\alpha})''v_1\big].\end{aligned}$$

Therefore, we have obtained the following system for (v_1, v_2):

$$\begin{cases}\dot v_1 = v_2,\\ \dot v_2 = -L_0 v_1 - \dfrac{\alpha-1}{\alpha+1}\gamma(1-\gamma\partial_x^2)^{-1}\big[2(Q^{2\alpha})'\partial_x v_1 + (Q^{2\alpha})''v_1\big],\\ \qquad - (1-\gamma\partial_x^2)^{-1}SU\big[2\chi_B'\partial_x u_1 + \chi_B'' u_1\big] + (1-\gamma\partial_x^2)^{-1}SU[\chi_B N^\perp].\end{cases} \tag{4.3}$$

For this transformed system we construct a second virial functional, where the spectral analysis reduces to the fact that the potential in L_0 is positive.

4.3. Virial functional for the transformed problem

We set

$$\mathcal{J} = \int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big) v_2$$

and (see (2.12) and (2.13))

$$z = \chi_B \zeta_B v_1. \tag{4.4}$$

Here, z represents a localized version of the function v_1. The scale of localization B is intermediate between the one involved in the definition of w from u_1 (see (2.14) and (3.2)) and the weight function ρ defined in (2.9) (similar to a localization at the soliton scale).

Proposition 2. *There exist $C_2 > 0$ and $\delta_2 > 0$ such that for γ small enough and for any $0 < \delta \le \delta_2$, the following holds. Fix $B = \delta^{-\frac14}$. Assume that for all $t \ge 0$, (2.5) holds. Then, for all $t \ge 0$,*

$$\dot{\mathcal{J}} \le -C_2\|z\|_\rho^2 + \delta^{\frac18}\|w\|_\rho^2 + |a_1|^3. \tag{4.5}$$

Remark 2. The objective of estimate (4.5) is to control the local norm $\|z\|_\rho^2$ up to small error in terms of $\|w\|_\rho^2$ and $|a_1|^3$.

The rest of this section is devoted to the proof of Proposition 2. As in the computation of $\dot{\mathcal{J}}$ in the proof of Proposition 1, we have from (4.3),

$$\begin{aligned}\dot{\mathcal{J}} &= \int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big)\dot{v}_2 \\ &= -\int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big) L_0 v_1 \\ &\quad - \frac{\alpha-1}{\alpha+1}\gamma \int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big)(1-\gamma\partial_x^2)^{-1}\big[2(Q^{2\alpha})'\partial_x v_1 + (Q^{2\alpha})'' v_1\big] \\ &\quad - \int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big)(1-\gamma\partial_x^2)^{-1} SU\big[2\chi_B'\partial_x u_1 + \chi_B'' u_1\big] \\ &\quad + \int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big)(1-\gamma\partial_x^2)^{-1} SU[\chi_B N^{\perp}] = J_1 + J_2 + J_3 + J_4.\end{aligned}$$

First, using the definition of L_0 in (4.1) and integrating by parts, we have

$$J_1 = -\int \psi_B'(\partial_x v_1)^2 + \frac{1}{4}\int \psi_B''' v_1^2 - \frac{\alpha-1}{\alpha+1}\int \Big(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big) Q^{2\alpha} v_1.$$

From (2.13), we note that $\psi_B' = \chi_B^2\zeta_B^2 + (\chi_B^2)'\varphi_B$ and

$$\psi_B''' = \chi_B^2(\zeta_B^2)'' + 3(\chi_B^2)'(\zeta_B^2)' + 3(\chi_B^2)''\zeta_B^2 + (\chi_B^2)'''\varphi_B.$$

Thus,

$$\begin{aligned}\int \psi_B'(\partial_x v_1)^2 - \frac{1}{4}\int \psi_B''' v_1^2 &= \int \chi_B^2\zeta_B^2(\partial_x v_1)^2 - \frac{1}{4}\int \chi_B^2(\zeta_B^2)'' v_1^2 \\ &\quad - \frac{3}{4}\int (\chi_B^2)'(\zeta_B^2)' v_1^2 - \frac{3}{4}\int (\chi_B^2)''\zeta_B^2 v_1^2 \\ &\quad + \int (\chi_B^2)'\varphi_B(\partial_x v_1)^2 - \frac{1}{4}\int (\chi_B^2)'''\varphi_B v_1^2.\end{aligned}$$

By the definition of z in (4.4), proceeding as in the proof of (3.7) in Lemma 1, we have

$$\begin{aligned}\int \chi_B^2\zeta_B^2(\partial_x v_1)^2 &= \int (\partial_x z)^2 + \int (\chi_B\zeta_B)''\chi_B\zeta_B v_1^2 \\ &= \int (\partial_x z)^2 + \int \frac{\zeta_B''}{\zeta_B} z^2 + \int \chi_B''\chi_B\zeta_B^2 v_1^2 + \frac{1}{2}\int (\chi_B^2)'(\zeta_B^2)' v_1^2\end{aligned}$$

and

$$\frac{1}{4}\int \chi_B^2(\zeta_B^2)'' v_1^2 = \frac{1}{2}\int \Big(\frac{\zeta_B''}{\zeta_B} + \frac{\zeta_B'^2}{\zeta_B^2}\Big) z^2.$$

Thus,

$$-\int \psi_B'(\partial_x v_1)^2 + \frac{1}{4}\int \psi_B''' v_1^2 = -\Big\{\int (\partial_x z)^2 + \frac{1}{2}\int \Big(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\Big) z^2\Big\} + \widetilde{J}_1,$$

where we have set

$$\widetilde{J}_1 = \frac{1}{4}\int (\chi_B^2)'(\zeta_B^2)'v_1^2 + \frac{1}{2}\int \big[3(\chi_B')^2 + \chi_B''\chi_B\big]\zeta_B^2 v_1^2 - \int (\chi_B^2)'\varphi_B(\partial_x v_1)^2 + \frac{1}{4}\int (\chi_B^2)'''\varphi_B v_1^2.$$

Recalling (4.4), (2.13), (2.12) and integrating by parts,

$$\int \Big(\psi_B\partial_x v_1 + \frac{1}{2}\psi_B' v_1\Big)Q^{2\alpha}v_1 = \frac{1}{2}\int Q^{2\alpha}\partial_x(\psi_B v_1^2) = -\alpha\int \frac{\varphi_B}{\zeta_B^2}Q^{2\alpha-1}Q'z^2.$$

Therefore, setting

$$V = \frac{1}{2}\Big(\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\Big) - \alpha\frac{\alpha-1}{\alpha+1}\frac{\varphi_B}{\zeta_B^2}Q^{2\alpha-1}Q',$$

we have obtained

$$J_1 = -\int \big[(\partial_x z)^2 + Vz^2\big] + \widetilde{J}_1.$$

Lemma 3. *There exists $B_0 > 0$ such that for all $B \geq B_0$, $V \geq 0$ on $\mathbb{R}$. More precisely,*

$$V \geq V_0, \quad \textit{where } V_0 = \frac{\alpha}{2}\frac{\alpha-1}{\alpha+1}|xQ'|Q^{2\alpha-1} \geq 0. \tag{4.6}$$

Proof. First, from (3.6) (with A replaced by B), it holds

$$\left|\frac{\zeta_B''}{\zeta_B} - \frac{(\zeta_B')^2}{\zeta_B^2}\right| \lesssim \frac{\mathbf{1}_{1\leq|x|\leq 2}(x)}{B}.$$

Second, since for $x \in [0,+\infty) \mapsto \zeta_B(x)$ is non-increasing, we have for $x \geq 0$,

$$\frac{\varphi_B}{\zeta_B^2} = \frac{\int_0^x \zeta_B^2}{\zeta_B^2} \geq x.$$

Since $Q'(x) \leq 0$ for $x \geq 0$, we obtain, for a constant $C > 0$,

$$V(x) \geq -\frac{C}{B}\mathbf{1}_{1\leq|x|\leq 2}(x) + \alpha\frac{\alpha-1}{\alpha+1}|xQ'(x)|Q^{2\alpha-1}(x) \geq \frac{\alpha}{2}\frac{\alpha-1}{\alpha+1}|xQ'(x)|Q^{2\alpha-1}(x),$$

choosing B_0 large enough. By parity, this estimate holds for any $x \in \mathbb{R}$. ∎

Using this lemma, and the above computations for J_1, we conclude

$$\dot{\mathcal{J}} \leq -\int \big[(\partial_x z)^2 + V_0 z^2\big] + \widetilde{J}_1 + J_2 + J_3 + J_4. \tag{4.7}$$

To control the terms $\widetilde{J}_1$, J_2, J_3 and J_4, we need some technical estimates.

4.4. Technical estimates

Lemma 4. *We have the following estimates:*

(1) *on w:*

$$\int_{|x|\leq 2B^2} w^2 \lesssim B^4 \int (\partial_x w)^2 + B^2 \int w^2 \operatorname{sech}\Big(\frac{x}{2}\Big), \tag{4.8}$$

$$\|w\|_\rho^2 \lesssim \int (\partial_x w)^2 + \int_{|x|<1} w^2 \lesssim \int (\partial_x w)^2 + \int w^2 \operatorname{sech}\Big(\frac{x}{2}\Big). \tag{4.9}$$

(2) *on z:*

$$\|z\|_\rho^2 \lesssim \int (\partial_x z)^2 + \int V_0 z^2 \lesssim \|z\|_\rho^2, \tag{4.10}$$

$$\int z^2 \zeta_B \lesssim B^2 \int (\partial_x z)^2 + B \int V_0 z^2 \lesssim B^2 \|z\|_\rho^2. \tag{4.11}$$

(3) *on v_1:*

$$\|v_1\|_{L^2} \lesssim \gamma^{-1} B^2 \|w\|_\rho, \tag{4.12}$$

$$\|\partial_x v_1\|_{L^2} \lesssim \gamma^{-1} \|w\|_\rho. \tag{4.13}$$

Proof. Proof of (4.8) *and* (4.9). For any $x, y \in \mathbb{R}$, using $w(x) = w(y) + \int_y^x \partial_x w$ and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned} w^2(x) &\leq 2w^2(y) + 2\Big(\int_y^x \partial_x w\Big)^2 \leq 2w^2(y) + 2|x-y| \int (\partial_x w)^2 \\ &\leq 2w^2(y) + 2(|x|+|y|) \int (\partial_x w)^2. \end{aligned} \tag{4.14}$$

Integrating (4.14) in $x \in [-2B^2, 2B^2]$ and $y \in [-1,1]$, we find (4.8). Multiplying (4.14) by $\operatorname{sech}(\frac{x}{10})$ and integrating in $x \in \mathbb{R}$ and $y \in [-1,1]$, we find (4.9).

Proof of (4.10) *and* (4.11). The proof is similar. For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we have

$$z^2(x) \leq 2z^2(y) + 2(|x|+|y|) \int (\partial_x z)^2.$$

We multiply by $\operatorname{sech}(\frac{x}{10})$ and $V_0(y) \geq 0$ and integrate in $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Since $\int V_0 > 0$ and $\int |y| V_0(y)\, dy < \infty$ from (4.6), we obtain (4.10).

We multiply by $\zeta_B(x)$ and $V_0(y)$ and integrate in $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Since

$$\int \zeta_B \lesssim B, \quad \int |x| \zeta_B \lesssim B^2 \quad \text{and} \quad \int |y| V_0 \lesssim 1,$$

we obtain (4.11).

Proof of (4.12) *and* (4.13). Note by direct computations that

$$\begin{aligned} SUf &= f'' - \Big[\frac{Q'}{Q} + \frac{Y_0'}{Y_0}\Big] f' + \Big[-\Big(\frac{Y_0'}{Y_0}\Big)' + \frac{Q'}{Q}\frac{Y_0'}{Y_0}\Big] f \\ &= f'' + (\alpha+2)\tanh(\alpha x) f' + (\alpha+1)\Big(1 + \frac{\alpha-1}{\cosh^2(\alpha x)}\Big) f. \end{aligned}$$

Thus,

$$\|SUf\|_{L^2} \lesssim \|f\|_{H^2}.$$

Moreover,

$$\|(1-\gamma\partial_x^2)^{-1} f\|_{H^2} \lesssim \gamma^{-1}\|f\|_{L^2}.$$

As a consequence, it holds

$$\|(1-\gamma\partial_x^2)^{-1} SUf\|_{L^2} \lesssim \gamma^{-1}\|f\|_{L^2}. \tag{4.15}$$

Using (4.15), the definition of v_1 in (4.2), the definition of w in (3.2) and $A \gg B^2$, we obtain

$$\|v_1\|_{L^2} \lesssim \gamma^{-1}\|\chi_B u_1\|_{L^2} \lesssim \gamma^{-1}\|u_1\|_{L^2(|x|\le 2B^2)} \lesssim \gamma^{-1}\|w\|_{L^2(|x|\le 2B^2)},$$

and then (4.8) implies (4.12).

Moreover, by direct computation

$$\partial_x(SUf) = SUf' + (\alpha+2)\alpha\operatorname{sech}^2(\alpha x) f' + \alpha(\alpha^2-1)\operatorname{sech}^2(\alpha x)\tanh(\alpha x) f.$$

Thus, similarly,

$$\|\partial_x(1-\gamma\partial_x^2)^{-1} SUf\|_{L^2} \lesssim \gamma^{-1}\|f'\|_{L^2} + \|f\operatorname{sech}(x)\|_{L^2}. \tag{4.16}$$

Using (4.16), we obtain

$$\|\partial_x v_1\|_{L^2} \lesssim \gamma^{-1}\|\partial_x(\chi_B u_1)\|_{L^2} + \|\chi_B u_1 \operatorname{sech}(x)\|_{L^2}.$$

By the definition of w, $A \gg B^2$ and the definition of χ_B and ζ_A, we have

$$\begin{aligned}|\partial_x(\chi_B u_1)|^2 = \left|\partial_x\left(\frac{\chi_B}{\zeta_A} w\right)\right|^2 &\lesssim \left|\frac{\chi_B}{\zeta_A}\right|^2 |\partial_x w|^2 + \left|\left(\frac{\chi_B}{\zeta_A}\right)'\right|^2 w^2 \\ &\lesssim |\partial_x w|^2 + B^{-4} w^2 \mathbf{1}_{|x|\le 2B^2},\end{aligned}$$

and $\|\chi_B u_1 \operatorname{sech}(x)\|_{L^2} \lesssim \|w\operatorname{sech}(x)\|_{L^2}$. Thus, estimate (4.8) imply (4.13). ■

Lemma 5. *For any* $0 < K \le 1$ *and* $\gamma > 0$ *small enough, for any* $f \in L^2$,

$$\|\operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1} f\|_{L^2} \lesssim \|\operatorname{sech}(Kx) f\|_{L^2}. \tag{4.17}$$

where the implicit constant is independent of γ *and* K.

Proof. We set $g = \operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1} f$ and $k = \operatorname{sech}(Kx) f$. We have

$$\begin{aligned}\cosh(Kx)k &= (1-\gamma\partial_x^2)[\cosh(Kx)g] \\ &= \cosh(Kx)g - \gamma K^2\cosh(Kx)g - 2\gamma K\sinh(Kx)g' - \gamma\cosh(Kx)g''.\end{aligned}$$

Thus,

$$k = \left[(1-\gamma K^2) - \gamma\partial_x^2\right]g - 2\gamma K\tanh(Kx)g'.$$

For $0 < K \le 1$ and $\gamma \le \frac{1}{2}$, we apply the operator $[(1-\gamma K^2) - \gamma\partial_x^2]^{-1}$, to obtain

$$g = \left[(1-\gamma K^2) - \gamma\partial_x^2\right]^{-1} k + 2\gamma K\left[(1-\gamma K^2) - \gamma\partial_x^2\right]^{-1}[\tanh(Kx)g'].$$

For $0 < K \le 1$ and $\gamma \le \frac{1}{2}$, one has

$$\left\| \left[(1-\gamma K^2) - \gamma \partial_x^2 \right]^{-1} \right\|_{\mathcal{L}(L^2,L^2)} \lesssim 1,$$

$$\left\| \left[(1-\gamma K^2) - \gamma \partial_x^2 \right]^{-1} \partial_x \right\|_{\mathcal{L}(L^2,L^2)} \lesssim \gamma^{-\frac{1}{2}}.$$

Thus, $\|[(1-\gamma K^2)-\gamma\partial_x^2]^{-1}k\|_{L^2} \lesssim \|k\|_{L^2}$, and

$$\begin{aligned}
&\left\| \left[(1-\gamma K^2) - \gamma \partial_x^2 \right]^{-1} [\tanh(Kx) g'] \right\|_{L^2} \\
&\lesssim \left\| \left[(1-\gamma K^2) - \gamma \partial_x^2 \right]^{-1} \partial_x [\tanh(Kx) g] \right\|_{L^2} \\
&\qquad + \left\| \left[(1-\gamma K^2) - \gamma \partial_x^2 \right]^{-1} [\operatorname{sech}^2(Kx) g] \right\|_{L^2} \\
&\lesssim \gamma^{-\frac{1}{2}} \|g\|_{L^2}.
\end{aligned}$$

We deduce, for a constant C independent of γ,

$$\|g\|_{L^2} \le C\|k\|_{L^2} + C\gamma^{\frac{1}{2}}\|g\|_{L^2},$$

which implies (4.17) for γ small enough. ∎

4.5. *Control of error terms*

Now, we are in a position to control the error terms in (4.7).

Control of $\widetilde{J}_1$. By the definition of ζ_B, it holds

$$\zeta_B(x) \lesssim e^{-\frac{|x|}{B}}, \quad |\zeta_B'(x)| \lesssim \frac{1}{B} e^{-\frac{|x|}{B}}.$$

Thus, using the properties of χ in (2.11), we have

$$\int \left[|\chi_B''|\chi_B\zeta_B^2 + (\chi_B')^2\zeta_B^2 + |\chi_B'\zeta_B'|\chi_B\zeta_B \right] v_1^2 \lesssim \int_{B^2 \le |x| \le 2B^2} e^{-\frac{2|x|}{B}} v_1^2 \lesssim e^{-2B}\|v_1\|_{L^2}^2.$$

Next, since $|\varphi_B| \lesssim B$ and $|(\chi_B^2)'| \lesssim B^{-2}$, $|(\chi_B^2)'''| \lesssim B^{-6}$, we have

$$\int |(\chi_B^2)'\varphi_B|(\partial_x v_1)^2 \lesssim B^{-1}\|\partial_x v_1\|_{L^2}^2 \quad \text{and} \quad \int |(\chi_B^2)'''\varphi_B| v_1^2 \lesssim B^{-5}\|v_1\|_{L^2}^2.$$

Using (4.12)–(4.13), we conclude for this term

$$|\widetilde{J}_1| \lesssim \gamma^{-2} B^{-1} \|w\|_\rho^2. \tag{4.18}$$

Control of J_2. By the Cauchy–Schwarz inequality,

$$|J_2| \lesssim \gamma \left\| Q^\alpha (1-\gamma\partial_x^2)^{-1} \left(\psi_B \partial_x v_1 + \frac{1}{2}\psi_B' v_1 \right) \right\|_{L^2} \left(\|Q^\alpha v_1\|_{L^2} + \|Q^\alpha \partial_x v_1\|_{L^2} \right).$$

First, we estimate using (4.17)

$$\left\| Q(1-\gamma\partial_x^2)^{-1}(\psi_B\partial_x v_1) \right\|_{L^2} \lesssim \|Q\psi_B\partial_x v_1\|_{L^2}.$$

From the definition of z in (4.4), we have

$$\partial_x z = \zeta_B\chi_B\partial_x v_1 + (\zeta_B\chi_B)' v_1,$$

and so

$$\zeta_B^2\chi_B^2|\partial_x v_1|^2 \lesssim |\partial_x z|^2 + |(\zeta_B\chi_B)'v_1|^2.$$

Using $|\chi'| \lesssim 1$, the definitions of χ_B and ζ_B and again the definition of z,

$$|(\zeta_B\chi_B)'v_1|^2\chi_B^2 \lesssim B^{-2}\zeta_B^2\chi_B^2 v_1^2 \lesssim B^{-2}z^2,$$

and so

$$\zeta_B^2\chi_B^4|\partial_x v_1|^2 \lesssim |\partial_x z|^2\chi_B^2 + B^{-2}z^2 \lesssim |\partial_x z|^2 + z^2. \tag{4.19}$$

Thus, using $|\psi_B| \lesssim |x|\chi_B^2$,

$$|Q\psi_B\partial_x v_1|^2 \lesssim |x|^2Q^2\chi_B^4|\partial_x v_1|^2 \lesssim Q\zeta_B^2\chi_B^4|\partial_x v_1|^2 \lesssim |\partial_x z|^2 + Qz^2.$$

It follows that

$$\|Q\psi_B\partial_x v_1\|_{L^2} \lesssim \|z\|_\rho.$$

Second, we also estimate using (4.17)

$$\left\|Q(1-\gamma\partial_x^2)^{-1}(\psi_B' v_1)\right\|_{L^2} \lesssim \|Q\psi_B' v_1\|_{L^2}$$

We claim

$$(\psi_B')^2 \lesssim \chi_B^2. \tag{4.20}$$

Indeed, using $|\chi_B'| \lesssim B^{-2}$, $|\varphi_B| \lesssim |x|$, $\chi_B = 0$ for $|x| \geq 2B^2$ and $\zeta_B \leq 1$,

$$(\psi_B')^2 \lesssim [\chi_B'\chi_B]^2\varphi_B^2 + \zeta_B^4\chi_B^4 \lesssim \chi_B^2.$$

Using (4.20), we infer that $|(\psi_B')^2v_1^2| \lesssim \chi_B^2 v_1^2$, thus $|Q(\psi_B')^2v_1^2| \lesssim z^2$, and so

$$\|Q\psi_B' v_1\|_{L^2} \lesssim \|Q^{\frac{1}{2}}z\|_{L^2} \lesssim \|z\|_\rho.$$

Now, we estimate $\|Q^\alpha v_1\|_{L^2}$ and $\|Q^\alpha\partial_x v_1\|_{L^2}$. From the definition of z in (4.4), we have $e^{-|x|}v_1^2\chi_B^2 \lesssim z^2$. Thus, from the definition of χ_B,

$$e^{-2|x|}v_1^2 \lesssim e^{-2|x|}v_1^2\chi_B^2 + e^{-2B^2}v_1^2 \lesssim e^{-|x|}z^2 + e^{-2B^2}v_1^2.$$

It follows using also (4.12) that

$$\|e^{-|x|}v_1\|_{L^2} \lesssim \|z\|_\rho + e^{-\frac{1}{2}B^2}\gamma^{-1}\|w\|_\rho.$$

Differentiating $z = \chi_B\zeta_B v_1$, we have

$$\chi_B\zeta_B\partial_x v_1 = \partial_x z - \frac{\zeta_B'}{\zeta_B}z - \chi_B'\zeta_B v_1.$$

Thus, as before,

$$e^{-2|x|}(\partial_x v_1)^2 \lesssim e^{-|x|}\left[(\partial_x z)^2 + z^2\right] + e^{-2B^2}\left[(\partial_x v_1)^2 + v_1^2\right].$$

It follows using (4.12) and (4.13) that

$$\|e^{-|x|}\partial_x v_1\|_{L^2} \lesssim \|z\|_\rho + e^{-\frac{1}{2}B^2}\gamma^{-1}\|w\|_\rho.$$

Collecting these estimates, we conclude

$$|J_2| \lesssim \gamma\|z\|_\rho^2 + e^{-B}\|w\|_\rho\|z\|_\rho. \tag{4.21}$$

Control of J_3. Using Cauchy–Schwarz inequality and (4.15), we have

$$|J_3| \lesssim \gamma^{-1}\big(\|\psi_B \partial_x v_1\|_{L^2} + \|\psi_B' v_1\|_{L^2}\big)\big(\|\chi_B' \partial_x u_1\|_{L^2} + \|\chi_B'' u_1\|_{L^2}\big).$$

First, using $|\psi_B| \lesssim B$ (from its definition and $|\varphi_B| \lesssim B$) and (4.13),

$$\|\psi_B \partial_x v_1\|_{L^2} \lesssim B\|\partial_x v_1\|_{L^2} \lesssim \gamma^{-1} B\|w\|_\rho.$$

Then, since $|\varphi_B| \lesssim B$ and $\varphi_B' = \zeta_B^2$,

$$|\psi_B'| = \big|2\chi_B'\chi_B\varphi_B + \zeta_B^2\chi_B^2\big| \lesssim B^{-1} + \zeta_B^2\chi_B^2.$$

Thus, using the definition (4.4), $z = \chi_B\zeta_B v_1$ and then (4.12),

$$\|\psi_B' v_1\|_{L^2}^2 \lesssim B^{-2}\|v_1\|_{L^2}^2 + \int \zeta_B^2 z^2 \lesssim \gamma^{-2}B^2\|w\|_\rho^2 + B^2\|z\|_\rho^2.$$

In conclusion,

$$\|\psi_B \partial_x v_1\|_{L^2} + \|\psi_B' v_1\|_{L^2} \lesssim \gamma^{-1}B\|w\|_\rho + B\|z\|_\rho. \tag{4.22}$$

Second, differentiating $w = \zeta_A u_1$, we have

$$\partial_x w = \zeta_A' u_1 + \zeta_A \partial_x u_1,$$

so that (using also the assumption $A \gg B^2$)

$$|\partial_x u_1|^2 \lesssim A^{-2}|u_1|^2 + |\partial_x w|^2 \lesssim B^{-4}|w|^2 + |\partial_x w|^2 \quad \text{for } |x| < A.$$

Thus, using also (4.8),

$$\begin{aligned}\|\chi_B' \partial_x u_1\|_{L^2}^2 &\lesssim B^{-4}\int_{B^2<|x|<2B^2} |\partial_x u_1|^2\\ &\lesssim B^{-4}\left[\int |\partial_x w|^2 + B^{-4}\int_{|x|<2B^2} |w|^2\right]\\ &\lesssim B^{-4}\|w\|_\rho^2.\end{aligned}$$

Next, by the definition of χ_B and (4.8),

$$\begin{aligned}\|\chi_B'' u_1\|_{L^2}^2 &\lesssim B^{-8}\int_{B^2<|x|<2B^2} |u_1|^2\\ &\lesssim B^{-8}\int_{|x|<2B^2} |w|^2\\ &\lesssim B^{-4}\|w\|_\rho^2.\end{aligned}$$

In conclusion,

$$\|\chi_B' \partial_x u_1\|_{L^2} + \|\chi_B'' u_1\|_{L^2} \lesssim B^{-2}\|w\|_\rho. \tag{4.23}$$

Collecting (4.22) and (4.23), we obtain

$$|J_3| \lesssim \gamma^{-2}B^{-1}\|w\|_\rho^2 + \gamma^{-1}B^{-1}\|w\|_\rho\|z\|_\rho. \tag{4.24}$$

Control of J_4. Using the Cauchy–Schwarz inequality, (4.15) and then $N^\perp = N - N_0 Y_0$, we have

$$|J_4| \lesssim \gamma^{-1}\big(\|\psi_B \partial_x v_1\|_{L^2} + \|\psi_B' v_1\|_{L^2}\big)\|\chi_B N^\perp\|_{L^2}$$
$$\lesssim \gamma^{-1}\big(\|\psi_B \partial_x v_1\|_{L^2} + \|\psi_B' v_1\|_{L^2}\big)\big(\|\chi_B N\|_{L^2} + |N_0|\big).$$

By (3.9), $|a_1| \lesssim 1$, $\|u_1\|_{L^\infty} \lesssim 1$, and decay properties of Y_0 and Q, we have

$$\|\chi_B N\|_{L^2} \lesssim a_1^2 + \|u_1\|_{L^\infty}\|Q\chi_B u_1\|_{L^2} + |a_1|^{2\alpha+1} + \|u_1\|_{L^\infty}^{2\alpha}\|\chi_B u_1\|_{L^2}$$
$$\lesssim a_1^2 + \|u_1\|_{L^\infty}\|\chi_B u_1\|_{L^2}.$$

Using $\chi_B \lesssim \zeta_A$ (since $A \gg B^2$ in (2.14)) and (4.8), it holds

$$\|\chi_B u_1\|_{L^2}^2 \lesssim \int_{|x|\le 2B^2} w^2 \lesssim B^4\|w\|_\rho^2.$$

Moreover, from (3.10),

$$|N_0| \lesssim a_1^2 + \|u_1\|_{L^\infty}\|w\|_\rho.$$

Therefore, using again (4.22), we obtain

$$|J_4| \lesssim \gamma^{-2}B\big(\|w\|_\rho + \|z\|_\rho\big)\big(a_1^2 + B^2\|u_1\|_{L^\infty}\|w\|_\rho\big). \tag{4.25}$$

4.6. *End of proof of Proposition* 2

From (4.7), (4.10), (4.18), (4.21), (4.24) and (4.25), it follows that there exist $C_2 > 0$ and $C > 0$ such that

$$\dot{\mathcal{J}} \le -4C_2\|z\|_\rho^2 + C\gamma^{-2}B^{-1}\|w\|_\rho^2 + C\gamma\|z\|_\rho^2 + Ce^{-B}\|w\|_\rho\|z\|_\rho$$
$$+ C\gamma^{-1}B^{-1}\|w\|_\rho\|z\|_\rho + C\gamma^{-2}B\big(\|w\|_\rho + \|z\|_\rho\big)\big(a_1^2 + B^2\|u_1\|_{L^\infty}\|w\|_\rho\big).$$

We fix $\gamma > 0$ such that $C\gamma \le 2C_2$ and also small enough to satisfy Lemma 5.

The value of γ being now fixed, we do not mention anymore dependency in γ. Using standard inequalities and B large enough, we obtain, for a possibly large constant $C > 0$,

$$\dot{\mathcal{J}} \le -C_2\|z\|_\rho^2 + CB^{-1}\|w\|_\rho^2 + CB^3\big(a_1^2 + B^2\|u_1\|_{L^\infty}\|w\|_\rho\big)^2.$$

Choosing (as specified in the statement of Proposition 2)

$$B = \delta^{-\frac{1}{4}},$$

and next using the assumption (2.5), we have

$$B^3(B^2\|u_1\|_{L^\infty}\|w\|_\rho)^2 \lesssim \delta^{-\frac{7}{4}}\|u_1\|_{L^\infty}^2\|w\|_\rho^2 \lesssim \delta^{\frac{1}{4}}\|w\|_\rho^2.$$

Therefore, using again (2.5), for δ small enough (to absorb some constants), we obtain

$$\dot{\mathcal{J}} \le -C_2\|z\|_\rho^2 + C\delta^{\frac{1}{4}}\|w\|_\rho^2 + B^3a_1^4 \le -C_2\|z\|_\rho^2 + \delta^{\frac{1}{8}}\|w\|_\rho^2 + |a_1|^3.$$

This estimate completes the proof of Proposition 2.

5. Coercivity and proof of Theorem 1

In this section, the constant γ is fixed as in Proposition 2.

5.1. Coercivity results

Lemma 6. *Let $B > 2$. Let u and v be Schwartz functions related by*

$$v = (1 - \gamma\partial_x^2)^{-1} SU(\chi_B u). \tag{5.1}$$

Assume

$$\langle u, Y_0\rangle = \langle u, Q'\rangle = 0. \tag{5.2}$$

It holds

$$\int (\chi_B u)^2 \operatorname{sech}\left(\frac{x}{2}\right) \lesssim \int \left[(\partial_x v)^2 + v^2\right]\rho^2 + e^{-B} \int u^2 \operatorname{sech}\left(\frac{x}{2}\right). \tag{5.3}$$

Proof. Using the expression of S and U, we rewrite (5.1) as

$$v - \gamma\partial_x^2 v = Q\partial_x\left(\frac{Y_0}{Q}\partial_x\left(\frac{\chi_B u}{Y_0}\right)\right),$$

and thus

$$\partial_x\left(\frac{Y_0}{Q}\partial_x\left(\frac{\chi_B u}{Y_0}\right) + \gamma\frac{\partial_x v}{Q}\right) = \frac{1}{Q}\left(v - \gamma\frac{Q'}{Q}\partial_x v\right).$$

Integrating between 0 and $x > 0$, this yields, for some constant a,

$$\frac{Y_0}{Q}\partial_x\left(\frac{\chi_B u}{Y_0}\right) + \gamma\frac{\partial_x v}{Q} = a + \int_0^x \left[\frac{1}{Q}\left(v - \gamma\frac{Q'}{Q}\partial_x v\right)\right],$$

which rewrites as

$$\partial_x\left(\frac{\chi_B u}{Y_0}\right) = a\frac{Q}{Y_0} - \gamma\frac{\partial_x v}{Y_0} + \frac{Q}{Y_0}\int_0^x \left[\frac{1}{Q}\left(v - \gamma\frac{Q'}{Q}\partial_x v\right)\right].$$

Integrating on $[0, x]$, $x > 0$, and multiplying by Y_0, it holds, for some constant b,

$$\chi_B u = bY_0 + aY_0\int_0^x \frac{Q}{Y_0} + \tilde{u}, \tag{5.4}$$

where

$$\tilde{u} = Y_0\int_0^x \left\{-\gamma\frac{\partial_x v}{Y_0} + \frac{Q}{Y_0}\int_0^y \left[\frac{1}{Q}\left(v - \gamma\frac{Q'}{Q}\partial_x v\right)\right]\right\}.$$

Let us now estimate $\int \tilde{u}^2 \operatorname{sech}(\frac{x}{2})$. First, by the Cauchy–Schwarz inequality,

$$Y_0\int_0^x \frac{|\partial_x v|}{Y_0} \lesssim Y_0\left(\int (\partial_x v)^2\rho^2\right)^{\frac{1}{2}}\left(\int_0^x (\rho Y_0)^{-2}\right)^{\frac{1}{2}} \lesssim \rho^{-1}\left(\int (\partial_x v)^2\rho^2\right)^{\frac{1}{2}}.$$

Second,

$$\frac{Q}{Y_0}\int_0^y \frac{|v|}{Q} \lesssim \frac{Q}{Y_0}\left(\int v^2\rho^2\right)^{\frac{1}{2}}\left(\int_0^y (\rho Q)^{-2}\right)^{\frac{1}{2}} \lesssim (\rho Y_0)^{-1}\left(\int v^2\rho^2\right)^{\frac{1}{2}}.$$

Thus,

$$Y_0\int_0^x \frac{Q}{Y_0}\int_0^y \frac{|v|}{Q} \lesssim \left(\int v^2\rho^2\right)^{\frac{1}{2}} Y_0\int_0^x (\rho Y_0)^{-1} \lesssim \rho^{-1}\left(\int v^2\rho^2\right)^{\frac{1}{2}}.$$

Third, since $\frac{|Q'|}{Q} \lesssim 1$, we obtain similarly,

$$Y_0\int_0^x \frac{Q}{Y_0}\int_0^y \frac{|Q'\partial_x v|}{Q^2} \lesssim \rho^{-1}\left(\int (\partial_x v)^2\rho^2\right)^{\frac{1}{2}}.$$

Collecting these estimates, we obtain, for all $x \geq 0$,

$$\tilde{u}^2\rho^2 \lesssim \int [(\partial_x v)^2 + v^2]\rho^2.$$

The same holds for $x \leq 0$, and thus

$$\int \tilde{u}^2 \operatorname{sech}\left(\frac{x}{2}\right) \lesssim \int [(\partial_x v)^2 + v^2]\rho^2.$$

To complete the proof, we estimate the constants a and b in (5.4). Using (5.2) and parity property, projecting (5.4) on Y_0 yields

$$\langle \chi_B u, Y_0\rangle = \langle (\chi_B - 1)u, Y_0\rangle = b + \langle \tilde{u}, Y_0\rangle.$$

Thus,

$$\begin{aligned} b^2 &\lesssim \int \tilde{u}^2 \operatorname{sech}(x) + \int u^2 \operatorname{sech}(x)(1-\chi_B)^2 \\ &\lesssim \int \tilde{u}^2 \operatorname{sech}(x) + e^{-\frac{1}{2}B^2}\int u^2 \operatorname{sech}\left(\frac{x}{2}\right). \end{aligned}$$

Using (5.2), $Y_0\int_0^x \frac{Q}{Y_0} = -\alpha^{-1}Q'$ and projecting (5.4) on Q' yields similarly

$$a^2 \lesssim \int \tilde{u}^2 \operatorname{sech}(x) + e^{-\frac{1}{2}B^2}\int u^2 \operatorname{sech}\left(\frac{x}{2}\right).$$

We conclude the proof using again (5.4). ■

The next result is a consequence of the previous general lemma, in the framework of the time-dependent functions introduced in (2.2), (3.2), (4.2) and (4.4).

Lemma 7. *For B large enough, it holds*

$$\int w^2 \operatorname{sech}\left(\frac{x}{2}\right) \lesssim \|z\|_\rho^2 + e^{-B}\|\partial_x w\|_{L^2}^2, \tag{5.5}$$

$$\|w\|_\rho^2 \lesssim \|z\|_\rho^2 + \|\partial_x w\|_{L^2}^2. \tag{5.6}$$

Proof. Recall that the function u_1 is even so that it satisfies $\langle u_1, Q'\rangle = 0$ in addition to the orthogonality (2.2). Therefore, applying (5.3),

$$\int (\chi_B u_1)^2 \operatorname{sech}\left(\frac{x}{2}\right) \lesssim \int [(\partial_x v_1)^2 + v_1^2]\rho^2 + e^{-B} \int u_1^2 \operatorname{sech}\left(\frac{x}{2}\right),$$

which implies by (3.2) and (2.10)

$$\int (\chi_B w)^2 \operatorname{sech}\left(\frac{x}{2}\right) \lesssim \int [(\partial_x v_1)^2 + v_1^2]\rho^2 + e^{-B}\|w\|_\rho^2. \tag{5.7}$$

By (4.4) and (4.19), it holds

$$\rho|\partial_x v_1|^2 + \rho|v_1|^2 \lesssim |\partial_x z|^2 + z^2 \quad \text{for } |x| < B^2.$$

Thus, using (4.12)–(4.13),

$$\begin{aligned}\int [(\partial_x v_1)^2 + v_1^2]\rho^2 &\lesssim \int_{|x|<B^2} [(\partial_x v_1)^2 + v_1^2]\rho^2 + e^{-\frac{B^2}{5}}\|v_1\|_{H^1}^2 \\ &\lesssim \|z\|_\rho^2 + e^{-\frac{B^2}{5}}\|v_1\|_{H^1}^2 \lesssim \|z\|_\rho^2 + e^{-\frac{B^2}{10}}\|w\|_\rho^2.\end{aligned}$$

Using (4.9) and the definition of χ_B in (2.13), it holds

$$\|w\|_\rho^2 \lesssim \int (\partial_x w)^2 + \int_{|x|<1} w^2 \lesssim \int (\partial_x w)^2 + \int (\chi_B w)^2 \operatorname{sech}\left(\frac{x}{2}\right).$$

Inserting these estimates into (5.7), it follows for B large enough that

$$\int (\chi_B w)^2 \operatorname{sech}\left(\frac{x}{2}\right) \lesssim \|z\|_\rho^2 + e^{-B}\|\partial_x w\|_{L^2}^2.$$

The last two estimates imply (5.6).

Finally,

$$\begin{aligned}\int w^2 \operatorname{sech}\left(\frac{x}{2}\right) &\lesssim \int (\chi_B w)^2 \operatorname{sech}\left(\frac{x}{2}\right) + e^{-\frac{B^2}{4}} \int w^2 \rho \\ &\lesssim \int (\chi_B w)^2 \operatorname{sech}\left(\frac{x}{2}\right) + e^{-B}\|w\|_\rho^2,\end{aligned}$$

and (5.5) follows. ∎

5.2. *Proof of Theorem 1*

Recall that the constants $\gamma > 0$, $\delta_1, \delta_2 > 0$ were defined in Propositions 1 and 2.

Proposition 3. *There exist $C_3 > 0$ and $0 < \delta_3 \le \min(\delta_1, \delta_2)$ such that for any δ with $0 < \delta \le \delta_3$, the following holds. Fix $A = \delta^{-1}$ and $B = \delta^{-\frac{1}{4}}$. Assume that for all $t \ge 0$, (2.5) holds. Let*

$$\mathcal{H} = \mathcal{J} + 8\delta_3^{\frac{1}{10}}\mathcal{I}. \tag{5.8}$$

Then, for all $t \ge 0$,

$$\dot{\mathcal{H}} \le -C_3\|w\|_\rho^2 + 2|a_1|^3. \tag{5.9}$$

Proof. In the context of Propositions 1 and 2, observe that fixing $A = \delta^{-1}$ and $B = \delta^{-\frac{1}{4}}$, for $\delta > 0$ small is consistent with the requirement $A \gg B^2 \gg B \gg 1$ in (2.14).

Combining (4.5) with (5.6) and (3.3) with (5.5), for $\delta_3 > 0$ small enough and δ satisfying $0 < \delta \le \delta_3$, one obtains, for a constant $C > 0$,

$$\dot{\mathcal{J}} \le -\frac{C_2}{2}\|z\|_\rho^2 + \delta_3^{\frac{1}{10}}\|\partial_x w\|_{L^2}^2 + |a_1|^3,$$
$$\dot{\mathcal{I}} \le -\frac{1}{4}\|\partial_x w\|_{L^2}^2 + C\|z\|_\rho^2 + |a_1|^3.$$

Define $\mathcal{H}$ as in (5.8). It follows by combining the above estimates that

$$\dot{\mathcal{H}} \le -\frac{C_2}{2}\|z\|_\rho^2 - \delta_3^{\frac{1}{10}}\|\partial_x w\|_{L^2}^2 + 8C\delta_3^{\frac{1}{10}}\|z\|_\rho^2 + (1 + 8\delta_3^{\frac{1}{10}})|a_1|^3.$$

Possibly choosing a smaller δ_3, we obtain

$$\dot{\mathcal{H}} \le -\frac{C_2}{4}\|z\|_\rho^2 - \delta_3^{\frac{1}{10}}\|\partial_x w\|_{L^2}^2 + 2|a_1|^3.$$

This estimate, together with (5.6), implies (5.9) for some $C_3 > 0$ (depending on δ_3). ∎

We set

$$\mathcal{B} = b_+^2 - b_-^2.$$

Lemma 8. *There exist $C_4 > 0$ and $0 < \delta_4 \le \delta_3$ such that for any δ with $0 < \delta \le \delta_4$, the following holds. Fix $A = \delta^{-1}$. Assume that for all $t \ge 0$, (2.5) holds. Then, for all $t \ge 0$,*

$$|\dot{b}_+ - \nu_0 b_+| + |\dot{b}_- + \nu_0 b_-| \le C_4(b_+^2 + b_-^2 + \|w\|_\rho^2) \tag{5.10}$$

and

$$\left|\frac{d}{dt}(b_+^2) - 2\nu_0 b_+^2\right| + \left|\frac{d}{dt}(b_-^2) + 2\nu_0 b_-^2\right| \le C_4\big(b_+^2 + b_-^2 + \|w\|_\rho^2\big)^{\frac{3}{2}}. \tag{5.11}$$

In particular,

$$\dot{\mathcal{B}} \ge \nu_0(b_+^2 + b_-^2) - C_4\|w\|_\rho^2 = \frac{\nu_0}{2}(a_1^2 + a_2^2) - C_4\|w\|_\rho^2. \tag{5.12}$$

Proof. From (3.10) and (2.3), it holds

$$|N_0| \lesssim a_1^2 + \|w\|_\rho^2 \lesssim b_+^2 + b_-^2 + \|w\|_\rho^2.$$

Estimates (5.10) and (5.11) then follow from (2.6). Last, estimate (5.12) is a consequence of (5.11) taking $\delta_4 > 0$ small enough. ∎

Combining (5.9) and (5.12), it holds

$$\dot{\mathcal{B}} - 2\frac{C_4}{C_3}\dot{\mathcal{H}} \ge \frac{\nu_0}{2}(a_1^2 + a_2^2) + C_4\|w\|_\rho^2 - 4\frac{C_4}{C_3}|a_1|^3,$$

and thus, for possibly smaller $\delta > 0$,

$$\dot{\mathcal{B}} - 2\frac{C_4}{C_3}\dot{\mathcal{H}} \ge \frac{\nu_0}{4}(a_1^2 + a_2^2) + C_4\|w\|_\rho^2. \tag{5.13}$$

By the choice of $A = \delta^{-1}$, the bound $|\varphi_A| \lesssim A$, and (2.5), we have for all $t \geq 0$,

$$|\mathcal{I}| \lesssim A\|u_1\|_{H^1}\|u_2\|_{L^2} \lesssim \delta.$$

Similarly, using also (4.15), it holds

$$|\mathcal{J}| \lesssim B\|v_1\|_{H^1}\|v_2\|_{L^2} \lesssim \delta \quad \text{and thus} \quad |\mathcal{H}| \lesssim \delta.$$

Estimate $|\mathcal{B}| \lesssim \delta^2$ is also clear from (2.5).

Therefore, integrating estimate (5.13) on $[0,t]$ and passing to the limit as $t \to +\infty$, it follows that

$$\int_0^\infty [a_1^2 + a_2^2 + \|w\|_\rho^2]\, dt \lesssim \delta.$$

Since $\int [(\partial_x u_1)^2 + u_1^2] \operatorname{sech}(x) \lesssim \|w\|_\rho^2$, this implies

$$\int_0^\infty \left\{ a_1^2 + a_2^2 + \int \left[(\partial_x u_1)^2 + u_1^2\right] \operatorname{sech}(x) \right\} dt \lesssim \delta. \tag{5.14}$$

Using (5.14), we conclude the proof of Theorem 1 as in [18, Section 5.2]. Let

$$\mathcal{K} = \int u_1 u_2 \operatorname{sech}(x) \quad \text{and} \quad \mathcal{G} = \frac{1}{2}\int \left[(\partial_x u_1)^2 + u_1^2 + u_2^2\right] \operatorname{sech}(x).$$

Using (2.7), we have

$$\begin{aligned}
\dot{\mathcal{K}} &= \int [\dot{u}_1 u_2 + u_1 \dot{u}_2] \operatorname{sech}(x) \\
&= \int \left[u_2^2 + u_1(-Lu_1 + N^\perp)\right] \operatorname{sech}(x) \\
&= \int \left[u_2^2 - (\partial_x u_1)^2 - u_1^2\right] \operatorname{sech}(x) + \frac{1}{2}\int u_1^2 \operatorname{sech}''(x) \\
&\quad + \int \left[f(Q + a_1 Y_0 + u_1) - f(Q) - a_1 f'(Q) Y_0 - N_0 Y_0\right] u_1 \operatorname{sech}(x).
\end{aligned}$$

We check that

$$\begin{aligned}
&\left| \int \left[f(Q + a_1 Y_0 + u_1) - f(Q) - a_1 f'(Q) Y_0 - N_0 Y_0\right] u_1 \operatorname{sech}(x) \right| \\
&\quad \lesssim a_1^2 + \int u_1^2 \operatorname{sech}(x).
\end{aligned}$$

(See (3.9)–(3.10) in the proof of Lemma 2.) In particular, it follows that

$$\int u_2^2 \operatorname{sech}(x) \leq \dot{\mathcal{K}} + C a_1^2 + C \int \left[(\partial_x u_1)^2 + u_1^2\right] \operatorname{sech}(x).$$

Using the bound $|\mathcal{K}| \lesssim \delta^2$ and (5.14), we deduce

$$\int_0^\infty \left[a_1^2 + a_2^2 + \mathcal{G}\right] dt \lesssim \delta. \tag{5.15}$$

Similarly, we check that

$$\begin{aligned}\dot{\mathcal{G}} &= \int \left[(\partial_x \dot{u}_1)(\partial_x u_1) + \dot{u}_1 u_1 + \dot{u}_2 u_2\right] \operatorname{sech}(x) \\ &= \int \left[(\partial_x u_2)(\partial_x u_1) + u_2 u_1 + (-L u_1 + N^{\perp}) u_2\right] \operatorname{sech}(x) \\ &= -\int (\partial_x u_1) u_2 \operatorname{sech}'(x) \\ &\quad + \int \left[f(Q + a_1 Y_0 + u_1) - f(Q) - a_1 f'(Q) Y_0 - N_0 Y_0\right] u_2 \operatorname{sech}(x),\end{aligned}$$

and so, as before

$$|\dot{\mathcal{G}}| \lesssim a_1^2 + \mathcal{G}. \tag{5.16}$$

By (5.15), there exists an increasing sequence $t_n \to +\infty$ such that

$$\lim_{n\to\infty} \left[a_1^2(t_n) + a_2^2(t_n) + \mathcal{G}(t_n)\right] = 0.$$

For $t \geq 0$, integrating (5.16) on $[t, t_n]$, and passing to the limit as $n \to \infty$, we obtain

$$\mathcal{G}(t) \lesssim \int_t^\infty [a_1^2 + \mathcal{G}]\, dt.$$

By (5.15), we deduce that

$$\lim_{t\to\infty} \mathcal{G}(t) = 0.$$

Finally, by (2.6) and (3.10), we have

$$\left|\frac{d}{dt}(a_1^2)\right| + \left|\frac{d}{dt}(a_2^2)\right| \lesssim a_1^2 + a_2^2 + \int u_1^2 \operatorname{sech}(x),$$

and so as before, by integration on $[t, t_n]$ and $n \to \infty$,

$$a_1^2(t) + a_2^2(t) \lesssim \int_t^\infty \left[a_1^2 + a_2^2 + \mathcal{G}\right] dt,$$

which proves

$$\lim_{t\to\infty} |a_1(t)| + |a_2(t)| = 0.$$

By the decomposition (2.1), this clearly implies (1.7). The proof of Theorem 1 is complete.

6. Proof of Theorem 2

6.1. Conservation of energy

Using (1.3) and (1.4) and performing a standard computation, we expand the conservation of energy (1.2) for a solution $(\phi, \partial_t \phi)$ written under the form (2.1) with the orthogonality conditions (2.2), to obtain

$$\begin{aligned}
&2\{E(\phi,\partial_t\phi)-E(Q,0)\}\\
&\quad=\int\big\{(\partial_t\phi)^2+(\partial_x\phi)^2+\phi^2-2F(\phi)\big\}-2E(Q,0)\\
&\quad=a_2^2\nu_0^2\langle Y_0,Y_0\rangle+a_1^2\langle LY_0,Y_0\rangle+\|u_2\|_{L^2}^2+\langle Lu_1,u_1\rangle\\
&\qquad\quad+O\big(|a_1|^3+|a_2|^3+\|u_1\|_{H^1}^3\big)\\
&\quad=\nu_0^2(a_2^2-a_1^2)+\|u_2\|_{L^2}^2+\langle Lu_1,u_1\rangle+O\big(|a_1|^3+|a_2|^3+\|u_1\|_{H^1}^3\big).
\end{aligned}$$

Using the notation (2.3), we have

$$\begin{aligned}
2\{E(\phi,\partial_t\phi)-E(Q,0)\}=-4\nu_0b_+b_-+\|u_2\|_{L^2}^2+\langle Lu_1,u_1\rangle\\
+O\big(|b_+|^3+|b_-|^3+\|u_1\|_{H^1}^3\big).
\end{aligned}\tag{6.1}$$

Let $\delta_0>0$ be defined by

$$\delta_0^2=b_+^2(0)+b_-^2(0)+\|u_1(0)\|_{H^1}^2+\|u_2(0)\|_{L^2}^2.$$

Then (6.1) applied at $t=0$ gives

$$|2\{E(\phi,\partial_t\phi)-E(Q,0)\}|\lesssim\delta_0^2.$$

Thus, by conservation of energy, estimate (6.1) at some $t>0$ gives

$$\big|-4\nu_0b_+b_-+\|u_2\|_{L^2}^2+\langle Lu_1,u_1\rangle+O\big(|b_+|^3+|b_-|^3+\|u_1\|_{H^1}^3\big)\big|\lesssim\delta_0^2.$$

Under the orthogonality conditions (2.2), the parity of u_1, from the spectral analysis recalled in the Introduction (see [6]), it follows that for some $\mu>0$,

$$\langle Lu_1,u_1\rangle\ge\mu\|u_1\|_{H^1}^2.\tag{6.2}$$

Thus, as long as $\|u_1\|_{H^1}+\|u_2\|_{L^2}+|b_+|+|b_-|\lesssim\delta_0^{\frac12}$, the following energy estimate holds:

$$\|u_1\|_{H^1}^2+\|u_2\|_{L^2}^2\lesssim|b_+|^2+|b_-|^2+\delta_0^2.\tag{6.3}$$

6.2. *Construction of the graph*

By the energy estimate (6.3), Lemma 8 and a standard contradiction argument, we construct initial data leading to global solutions close to the ground state Q.

Let $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2)\in\mathcal{A}_0$ (see (1.8)). Then the condition $\langle\boldsymbol{\varepsilon},\boldsymbol{Z}_+\rangle=0$ rewrites

$$\langle\varepsilon_1,Y_0\rangle+\langle\varepsilon_2,\nu_0^{-1}Y_0\rangle=0.$$

Define $b_-(0)$ and $(u_1(0),u_2(0))$ such that

$$b_-(0)=\langle\varepsilon_1,Y_0\rangle=-\langle\varepsilon_2,\nu_0^{-1}Y_0\rangle$$

and

$$\varepsilon_1=b_-(0)Y_0+u_1(0),\quad\varepsilon_2=-b_-(0)\nu_0Y_0+u_2(0).$$

Then it holds

$$\langle u_1(0), Y_0\rangle = \langle u_2(0), Y_0\rangle = 0.$$

This means that the initial data in the statement of Theorem 2 decomposes as (see (2.4))

$$\boldsymbol{\phi}_0 = \boldsymbol{\phi}(0) = (Q, 0) + (u_1, u_2)(0) + b_-(0)\boldsymbol{Y}_- + h(\boldsymbol{\varepsilon})\boldsymbol{Y}_+.$$

Now, we prove that there exists at least a choice of $h(\boldsymbol{\varepsilon}) = b_+(0)$ such that the corresponding solution $\boldsymbol{\phi}$ is global and satisfies (1.9).

Let $\delta_0 > 0$ small enough and $K > 1$ large enough to be chosen. We introduce the following bootstrap estimates:

$$\|u_1\|_{H^1} \le K^2\delta_0 \quad \text{and} \quad \|u_2\|_{L^2} \le K^2\delta_0, \tag{6.4}$$

$$|b_-| \le K\delta_0, \tag{6.5}$$

$$|b_+| \le K^5\delta_0^2. \tag{6.6}$$

Given any $(u_1(0), u_2(0))$ and $b_-(0)$ such that

$$\|u_1(0)\|_{H^1} \le \delta_0, \quad \|u_2(0)\|_{L^2} \le \delta_0, \quad |b_-(0)| \le \delta_0, \tag{6.7}$$

and $b_+(0)$ satisfying

$$|b_+(0)| \le K^5\delta_0^2,$$

we define

$$T = \sup\{t \ge 0 : (6.4)\text{–}(6.6) \text{ hold on } [0, t]\}.$$

Note that since $K > 1$, T is well defined in $[0, +\infty]$. We aim at proving that there exists at least one value of $b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2]$ such that $T = \infty$. We argue by contradiction, assuming that any $b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2]$ leads to $T < \infty$.

First, we strictly improve the estimate on (u_1, u_2) in (6.4). Indeed, by estimates (6.3) and (6.5)–(6.6), it holds

$$\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 \le C_5(K^{10}\delta_0^4 + K^2\delta_0^2 + \delta_0^2)$$

for some constant $C_5 > 0$. Thus, under the constraints

$$C_5K^{10}\delta_0^2 \le \frac{1}{4}K^4, \quad C_5K^2 \le \frac{1}{4}K^4, \quad C_5 \le \frac{1}{4}K^4, \tag{6.8}$$

it holds

$$\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 \le \frac{3}{4}K^4\delta_0^2,$$

which strictly improves (6.4).

Second, we use (5.11) to control b_-. By (6.4)–(6.6), since $\|w\|_\rho \lesssim \|u_1\|_{H^1}$, it holds

$$\left|\frac{d}{dt}\left(e^{2\nu_0 t}b_-^2\right)\right| \le C_6\left(K^{15}\delta_0^6 + K^6\delta_0^3\right)e^{2\nu_0 t}$$

for some constant $C_6 > 0$. Thus, by integration on $[0, t]$ and using (6.7), we obtain

$$b_-^2 \le \frac{C_6}{2\nu_0}\left(K^{15}\delta_0^6 + K^6\delta_0^3\right) + \delta_0^2.$$

Under the constraints

$$\frac{C_6}{2\nu_0}K^{15}\delta_0^4 \leq \frac{1}{4}K^2, \quad C_6 K^6 \delta_0 \leq \frac{1}{4}K^2, \quad 1 \leq \frac{1}{4}K^2, \tag{6.9}$$

it holds

$$b_-^2 \leq \frac{3}{4}K^2\delta_0^2$$

which strictly improves (6.5).

By the previous estimates (under the constraints (6.8)–(6.9)) and a continuity argument, we see that if $T < +\infty$, then $|b_+(T)| = K^5\delta_0^2$.

Third, we observe that if $t \in [0, T]$ is such that $|b_+(t)| = K^5\delta_0^2$, it follows from (5.10) that

$$\begin{aligned}\frac{d}{dt}(b_+^2) &\geq 2\nu_0 b_+^2 - 2C_4|b_+|(b_+^2 + b_-^2 + \|w\|_\rho^2)\\ &\geq 2\nu_0 K^{10}\delta_0^4 - C_7 K^5\delta_0^2\big(K^{10}\delta_0^4 + K^4\delta_0^2\big)\end{aligned}$$

for some constant $C_7 > 0$. Under the constraints

$$C_7 K^{15}\delta_0^2 \leq \frac{1}{2}\nu_0 K^{10}, \quad C_7 K^9 \leq \frac{1}{2}\nu_0 K^{10}, \tag{6.10}$$

the inequality

$$\frac{d}{dt}(b_+^2) \geq \nu_0 K^{10}\delta_0^4 > 0$$

holds. By standard arguments, such transversality condition implies that T is the first time for which $|b_+(t)| = K^5\delta_0^2$ and moreover that T is continuous in the variable $b_+(0)$ (see e.g. [7, 8] for a similar argument). Now, the image of the continuous map

$$b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2] \mapsto b_+(T) \in \{-K^5\delta_0^2, K^5\delta_0^2\}$$

is exactly $\{-K^5\delta_0^2, K^5\delta_0^2\}$ (since the image of $-K^5\delta_0^2$ is $-K^5\delta_0^2$ and the image of $K^5\delta_0^2$ is $K^5\delta_0^2$), which is a contradiction.

As a consequence, provided the constraints in (6.8)–(6.10) are all fulfilled, there exists at least one value of $b_+(0) \in (-K^5\delta_0^2, K^5\delta_0^2)$ such that $T = \infty$.

Finally, we easily see that to satisfy (6.8)–(6.10), it is sufficient first to fix $K > 0$ large enough, depending only on C_5, C_6 and C_7, and then to choose $\delta_0 > 0$ small enough.

6.3. Uniqueness and Lipschitz regularity

The following proposition implies both the uniqueness of the choice of $h(\boldsymbol{\varepsilon}) = b_+(0)$, for a given $\boldsymbol{\varepsilon} \in \mathcal{A}_0$, and the Lipschitz regularity of the graph $\mathcal{M}$ defined from the resulting map $\boldsymbol{\varepsilon} \in \mathcal{A}_0 \mapsto h(\boldsymbol{\varepsilon})$. It is thus sufficient to complete the proof of Theorem 2.

Proposition 4. *There exist $C, \delta > 0$ such if $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$ are two even solutions of (1.1) satisfying*

$$\|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1\times L^2} < \delta, \quad \|\tilde{\boldsymbol{\phi}}(t) - (Q, 0)\|_{H^1\times L^2} < \delta \quad \textit{for all } t \geq 0, \tag{6.11}$$

then, decomposing

$$\boldsymbol{\phi}(0) = (Q,0) + \boldsymbol{\varepsilon} + b_+(0)\boldsymbol{Y}_+, \quad \tilde{\boldsymbol{\phi}}(0) = (Q,0) + \tilde{\boldsymbol{\varepsilon}} + \tilde{b}_+(0)\boldsymbol{Y}_+$$

with $\langle \boldsymbol{\varepsilon}, \boldsymbol{Z}_+\rangle = \langle \tilde{\boldsymbol{\varepsilon}}, \boldsymbol{Z}_+\rangle = 0$, *it holds*

$$|b_+(0) - \tilde{b}_+(0)| \le C\delta^{\frac{1}{2}} \|\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}\|_{H^1\times L^2}. \tag{6.12}$$

Proof. We use the decomposition and the notation of Section 2.1 for the two solutions $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$ satisfying (6.11). In particular, from (2.5), there exists $C_0 > 0$ such that for all $t \ge 0$,

$$\|u_1(t)\|_{H^1} + \|\tilde{u}_1(t)\|_{H^1} + \|u_2(t)\|_{L^2} + \|\tilde{u}_2(t)\|_{L^2} + |b_\pm(t)| + \tilde{b}_\pm(t)| \le C_0\delta. \tag{6.13}$$

We denote

$$\begin{aligned} &\check{a}_1 = a_1 - \tilde{a}_1, \quad \check{a}_2 = a_2 - \tilde{a}_2, \quad \check{b}_+ = b_+ - \tilde{b}_+, \quad \check{b}_- = b_- - \tilde{b}_-,\\ &\check{u}_1 = u_1 - \tilde{u}_1, \quad \check{u}_2 = u_2 - \tilde{u}_2,\\ &\check{N} = N - \tilde{N}, \quad \check{N}^\perp = N^\perp - \tilde{N}^\perp, \quad \check{N}_0 = N_0 - \tilde{N}_0. \end{aligned}$$

Then, from (2.6) and (2.7), the equations of $(\check{u}_1, \check{u}_2, \check{b}_+, \check{b}_-)$ write

$$\begin{cases} \dot{\check{b}}_+ = \nu_0 \check{b}_+ + \dfrac{\check{N}_0}{2\nu_0}, \\ \dot{\check{b}}_- = -\nu_0 \check{b}_- - \dfrac{\check{N}_0}{2\nu_0}, \end{cases} \quad \text{and} \quad \begin{cases} \dot{\check{u}}_1 = \check{u}_2, \\ \dot{\check{u}}_2 = -L\check{u}_1 + \check{N}^\perp. \end{cases} \tag{6.14}$$

We claim that

$$|\check{N}_0| + \|\check{N}^\perp\|_{L^2} \le C\delta\big(|\check{b}_+| + |\check{b}_-| + \|\check{u}_1\|_{H^1}\big). \tag{6.15}$$

Indeed, by Taylor formula, for any $v, \tilde{v}$, it holds (recall that $\alpha > 1$)

$$\begin{aligned} &\big|f(Q+v) - f(Q) - f'(Q)v - \big[f(Q+\tilde{v}) - f(Q) - f'(Q)\tilde{v}\big]\big| \\ &\quad\lesssim |v - \tilde{v}|\big(|v| + |\tilde{v}|\big)\big(Q^{2\alpha-1} + |v|^{2\alpha-1} + |\tilde{v}|^{2\alpha-1}\big) \\ &\quad\lesssim |v - \tilde{v}|(|v| + |\tilde{v}|). \end{aligned}$$

Using this inequality for $\check{N} = N - \tilde{N}$, where N is defined in (2.8), and (6.13), we obtain

$$|\check{N}| \lesssim \big(|\check{a}_1|Y_0 + |\check{u}_1|\big)\big(Y_0|a_1| + Y_0|\tilde{a}_1| + |u_1| + |\tilde{u}_1|\big).$$

Using the Cauchy–Schwarz inequality and again (6.13), we find $\|\check{N}\|_{L^2} \lesssim \delta(|\check{a}_1| + |\check{u}_1|)$ and estimate (6.15) follows.

Let

$$\beta_+ = \check{b}_+^2, \quad \beta_- = \check{b}_-^2, \quad \beta_c = \langle L\check{u}_1, \check{u}_1\rangle + \langle \check{u}_2, \check{u}_2\rangle.$$

By (6.14) and (6.15) (and the coercivity property (6.2) for $\check{u}_1$) we have, for some $K > 0$,

$$|\dot{\beta}_c| + |\dot{\beta}_+ - 2\nu_0\beta_+| + |\dot{\beta}_- + 2\nu_0\beta_-| \le K\delta(\beta_c + \beta_+ + \beta_-). \tag{6.16}$$

For the sake of contradiction, assume that the following holds:

$$0 \le K\delta\big(\beta_c(0) + \beta_+(0) + \beta_-(0)\big) < \frac{\nu_0}{10}\beta_+(0). \tag{6.17}$$

We introduce the following bootstrap estimate:

$$K\delta(\beta_c + \beta_+ + \beta_-) \le \nu_0\beta_+. \tag{6.18}$$

Define

$$T = \sup\{t > 0 : (6.18) \text{ holds}\} > 0.$$

We work on the interval $[0, T]$. Note that from (6.16) and (6.18), it holds

$$\dot\beta_+ \ge 2\nu_0\beta_+ - K\delta(\beta_c + \beta_+ + \beta_-) \ge \nu_0\beta_+. \tag{6.19}$$

In particular, by standard arguments, β_+ is positive and increasing on $[0, T]$.

Next, by (6.16) and (6.18),

$$\dot\beta_c \le \nu_0\beta_+ \le \dot\beta_+$$

and thus, by integration,

$$\beta_c(t) \le \beta_c(0) + \beta_+(t) - \beta_+(0) \le \beta_c(0) + \beta_+(t).$$

Therefore, by (6.17), for δ small enough,

$$K\delta\beta_c(t) \le K\delta(\beta_c(0) + \beta_+(t)) \le \frac{\nu_0}{10}\beta_+(0) + K\delta\beta_+(t) \le \frac{\nu_0}{5}\beta_+(t).$$

Then, by (6.16) and (6.18),

$$\dot\beta_- \le -2\nu_0\beta_- + \nu_0\beta_+,$$

and so by integration and (6.17),

$$\beta_-(t) \le e^{-2\nu_0 t}\beta_-(0) + \nu_0\beta_+(t)e^{-2\nu_0 t}\int_0^t e^{2\nu_0 s}\,ds \le \beta_-(0) + \frac{1}{2}\beta_+(t).$$

Therefore, for δ small enough,

$$K\delta\beta_-(t) \le K\delta(\beta_-(0) + \beta_+(t)) \le \frac{\nu_0}{10}\beta_+(0) + K\delta\beta_+(t) \le \frac{\nu_0}{5}\beta_+(t).$$

Last, it is clear that for δ small, it holds $K\delta\beta_+ \le \frac{\nu_0}{5}\beta_+$.

Therefore, we have proved that, for all $t \in [0, T]$,

$$K\delta\big(\beta_c(t) + \beta_+(t) + \beta_-(t)\big) \le \frac{3}{5}\nu_0\beta_+(t).$$

By a continuity argument, this means that $T = +\infty$. By the exponential growth (6.19) and $\beta_+(0) > 0$, we obtain a contradiction with the global bound (6.13) on $|b_+|$.

Since estimate (6.17) is contradicted, and since it holds

$$\boldsymbol{\varepsilon} = \boldsymbol{u}(0) + b_-(0)\boldsymbol{Y}_-, \quad \tilde{\boldsymbol{\varepsilon}} = \tilde{\boldsymbol{u}}(0) + \tilde b_-(0)\boldsymbol{Y}_- \quad \text{with } \langle \boldsymbol{u}(0), \boldsymbol{Y}_-\rangle = \langle \tilde{\boldsymbol{u}}(0), \boldsymbol{Y}_-\rangle = 0,$$

we have proved (6.12). ■

Funding. M. Kowalczyk was partially funded by Chilean research grants FONDECYT 1170164. C. Muñoz was partially funded by Chilean research grants FONDECYT 1150202. M. Kowalczyk and C. Muñoz were partially funded by project France-Chile ECOS-Sud C18E06 and CMM Conicyt PIA AFB170001. Part of this work was done while C. Muñoz and M. Kowalczyk were visiting the CMLS at École Polytechnique, France. Part of this work was done while C. Muñoz was visiting the Departamento de Matemáticas Aplicadas de Granada, UGR, Spain.

References

[1] Bambusi, D., Cuccagna, S.: On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential. Amer. J. Math. **133**, 1421–1468 (2011) Zbl 1237.35115 MR 2843104

[2] Bates, P. W., Jones, C. K. R. T.: Invariant manifolds for semilinear partial differential equations. In: Dynamics Reported, Vol. 2, Dynam. Report. Ser. Dynam. Systems Appl. 2, Wiley, Chichester, 1–38 (1989) Zbl 0674.58024 MR 1000974

[3] Bates, P. W., Lu, K., Zeng, C.: Approximately invariant manifolds and global dynamics of spike states. Invent. Math. **174**, 355–433 (2008) Zbl 1157.37013 MR 2439610

[4] Bizoń, P., Chmaj, T., Szpak, N.: Dynamics near the threshold for blowup in the one-dimensional focusing nonlinear Klein–Gordon equation. J. Math. Phys. **52**, 103703, 11 (2011) Zbl 1272.35174 MR 2894613

[5] Cazenave, T., Haraux, A.: An Introduction to Semilinear Evolution Equations. Oxford Lecture Ser. Math. Appl. 13, The Clarendon Press, Oxford University Press, New York (1998) Zbl 0926.35049 MR 1691574

[6] Chang, S.-M., Gustafson, S., Nakanishi, K., Tsai, T.-P.: Spectra of linearized operators for NLS solitary waves. SIAM J. Math. Anal. **39**, 1070–1111 (2007/08) Zbl 1168.35041 MR 2368894

[7] Côte, R., Martel, Y., Merle, F.: Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations. Rev. Mat. Iberoam. **27**, 273–302 (2011) Zbl 1273.35234 MR 2815738

[8] Côte, R., Muñoz, C.: Multi-solitons for nonlinear Klein–Gordon equations. Forum Math. Sigma **2**, Paper No. e15, 38 (2014) Zbl 1301.35126 MR 3264254

[9] Côte, R., Muñoz, C., Pilod, D., Simpson, G.: Asymptotic stability of high-dimensional Zakharov–Kuznetsov solitons. Arch. Ration. Mech. Anal. **220**, 639–710 (2016) Zbl 1334.35276 MR 3461359

[10] Cuccagna, S.: A survey on asymptotic stability of ground states of nonlinear Schrödinger equations. In: Dispersive Nonlinear Problems in Mathematical Physics, Quad. Mat. 15, Dept. Math., Seconda Univ. Napoli, Caserta, 21–57 (2004) Zbl 1130.35360 MR 2231327

[11] Cuccagna, S., Pelinovsky, D. E.: The asymptotic stability of solitons in the cubic NLS equation on the line. Appl. Anal. **93**, 791–822 (2014) Zbl 1457.35067 MR 3180019

[12] Delort, J.-M.: Existence globale et comportement asymptotique pour l'équation de Klein–Gordon quasi linéaire à données petites en dimension 1. Ann. Sci. Éc. Norm. Supér. (4) **34**, 1–61 (2001) Zbl 0990.35119 MR 1833089

[13] Delort, J.-M.: Semiclassical microlocal normal forms and global solutions of modified one-dimensional KG equations. Ann. Inst. Fourier (Grenoble) **66**, 1451–1528 (2016) Zbl 1377.35200 MR 3494176

[14] Gravejat, P., Smets, D.: Asymptotic stability of the black soliton for the Gross–Pitaevskii equation. Proc. Lond. Math. Soc. (3) **111**, 305–353 (2015) Zbl 1326.35346 MR 3384514

[15] Kenig, C. E., Martel, Y.: Asymptotic stability of solitons for the Benjamin–Ono equation. Rev. Mat. Iberoam. **25**, 909–970 (2009) Zbl 1247.35133 MR 2590690

[16] Kopylova, E., Komech, A. I.: On asymptotic stability of kink for relativistic Ginzburg–Landau equations. Arch. Ration. Mech. Anal. **202**, 213–245 (2011) Zbl 1256.35146 MR 2835867

[17] Kopylova, E. A., Komech, A. I.: On asymptotic stability of moving kink for relativistic Ginzburg–Landau equation. Comm. Math. Phys. **302**, 225–252 (2011) Zbl 1209.35134 MR 2770013

[18] Kowalczyk, M., Martel, Y., Muñoz, C.: Kink dynamics in the ϕ^4 model: Asymptotic stability for odd perturbations in the energy space. J. Amer. Math. Soc. **30**, 769–798 (2017) Zbl 1387.35419 MR 3630087

[19] Kowalczyk, M., Martel, Y., Muñoz, C.: Nonexistence of small, odd breathers for a class of nonlinear wave equations. Lett. Math. Phys. **107**, 921–931 (2017) Zbl 1384.35109 MR 3633030

[20] Kowalczyk, M., Martel, Y., Muñoz, C.: On asymptotic stability of nonlinear waves. In: Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2016–2017, Ed. Éc. Polytech., Palaiseau, Exp. No. XVIII, 27 (2017) MR 3790944

[21] Krieger, J., Nakanishi, K., Schlag, W.: Global dynamics above the ground state energy for the one-dimensional NLKG equation. Math. Z. **272**, 297–316 (2012) Zbl 1263.35002 MR 2968226

[22] Krieger, J., Schlag, W.: Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. J. Amer. Math. Soc. **19**, 815–920 (2006) Zbl 1281.35077 MR 2219305

[23] Lindblad, H., Soffer, A.: Scattering for the Klein–Gordon equation with quadratic and variable coefficient cubic nonlinearities. Trans. Amer. Math. Soc. **367**, 8861–8909 (2015) Zbl 1328.35201 MR 3403074

[24] Martel, Y.: Linear problems related to asymptotic stability of solitons of the generalized KdV equations. SIAM J. Math. Anal. **38**, 759–781 (2006) Zbl 1126.35055 MR 2262941

[25] Martel, Y., Merle, F.: A Liouville theorem for the critical generalized Korteweg–de Vries equation. J. Math. Pures Appl. (9) **79**, 339–425 (2000) Zbl 0963.37058 MR 1753061

[26] Martel, Y., Merle, F.: Asymptotic stability of solitons of the gKdV equations with general nonlinearity. Math. Ann. **341**, 391–427 (2008) Zbl 1153.35068 MR 2385662

[27] Martel, Y., Merle, F., Nakanishi, K., Raphaël, P.: Codimension one threshold manifold for the critical gKdV equation. Comm. Math. Phys. **342**, 1075–1106 (2016) Zbl 1336.35315 MR 3465440

[28] Martel, Y., Merle, F., Raphaël, P.: Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton. Acta Math. **212**, 59–140 (2014) Zbl 1301.35137 MR 3179608

[29] Matveev, V. B., Salle, M. A.: Darboux Transformations and Solitons. Springer Ser. Nonlinear Dyn., Springer, Berlin (1991) Zbl 0744.35045 MR 1146435

[30] Nakanishi, K., Schlag, W.: Invariant Manifolds and Dispersive Hamiltonian Evolution Equations. Zur. Lect. Adv. Math., European Mathematical Society (EMS), Zürich (2011) Zbl 1235.37002 MR 2847755

[31] Pego, R. L., Weinstein, M. I.: Asymptotic stability of solitary waves. Comm. Math. Phys. **164**, 305–349 (1994) Zbl 0805.35117 MR 1289328

[32] Raphaël, P., Rodnianski, I.: Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang–Mills problems. Publ. Math Inst. Hautes Études Sci. **115**, 1–122 (2012) Zbl 1284.35358 MR 2929728

[33] Rodnianski, I., Sterbenz, J.: On the formation of singularities in the critical O(3) σ-model. Ann. of Math. (2) **172**, 187–242 (2010) Zbl 1213.35392 MR 2680419

[34] Schlag, W.: Stable manifolds for an orbitally unstable nonlinear Schrödinger equation. Ann. of Math. (2) **169**, 139–227 (2009) Zbl 1180.35490 MR 2480603

[35] Soffer, A., Weinstein, M. I.: Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. Invent. Math. **136**, 9–74 (1999) Zbl 0910.35107 MR 1681113

[36] Sterbenz, J.: Dispersive decay for the 1D Klein–Gordon equation with variable coefficient nonlinearities. Trans. Amer. Math. Soc. **368**, 2081–2113 (2016) Zbl 1339.35191 MR 3449234

[37] Sulem, C., Sulem, P.-L.: The Nonlinear Schrödinger Equation. Appl. Math. Sci. 139, Springer, New York (1999) Zbl 0928.35157 MR 1696311