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New solution of a problem of Kolmogorov on width asymptotics in holomorphic function spaces

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Abstract. Given a domain D in $\mathbb{C}^n$ and a compact subset K of D , the set $\mathcal{A}_K^D$ of all restrictions of functions holomorphic on D the modulus of which is bounded by 1 is a compact subset of the Banach space $C(K)$ of continuous functions on K . The sequence $(d_m(\mathcal{A}_K^D))_{m \in \mathbb{N}}$ of Kolmogorov m -widths of $\mathcal{A}_K^D$ provides a measure of the degree of compactness of the set $\mathcal{A}_K^D$ in $C(K)$ and the study of its asymptotics has a long history, essentially going back to Kolmogorov's work on ϵ -entropy of compact sets in the 1950s. In the 1980s Zakharyuta showed that for suitable D and K the asymptotics

$$\lim_{m \rightarrow \infty} \frac{-\log d_m(\mathcal{A}_K^D)}{m^{1/n}} = 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}, \quad (1)$$

where $C(K, D)$ is the Bedford–Taylor relative capacity of K in D , is implied by a conjecture, now known as Zakharyuta's Conjecture, concerning the approximability of the regularised relative extremal function of K and D by certain pluricomplex Green functions. Zakharyuta's Conjecture was proved by Nivoche in 2004 thus settling (1) at the same time.

We shall give a new proof of the asymptotics (1) for D strictly hyperconvex and K non-pluripolar which does not rely on Zakharyuta's Conjecture. Instead we proceed more directly by a two-pronged approach establishing sharp upper and lower bounds for the Kolmogorov widths. The lower bounds follow from concentration results of independent interest for the eigenvalues of a certain family of Toeplitz operators, while the upper bounds follow from an application of the Bergman–Weil formula together with an exhaustion procedure by special holomorphic polyhedra.

Keywords. Kolmogorov widths, Kolmogorov numbers, Kolmogorov ϵ -entropy, pluripotential theory, capacity, Toeplitz operators, Bergman spaces, Bergman–Weil formula

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1. Introduction

1.1. Kolmogorov's problem

In the 1930s approximation theory received a new impetus, when Kolmogorov [18] introduced the concept of the width (or diameter) of a compact set: given a normed linear space $(X, \|\cdot\|)$ and $C \subset X$ a compact subset, then, for any $m \in \mathbb{N}$, the *Kolmogorov m -width* of C in X is the quantity

$$d_m(C, X) = \inf_{\dim L < m} \sup_{x \in C} \inf_{y \in L} \|x - y\|,$$

where the outermost infimum is taken over subspaces L of X . The sequence $(d_m(C, X))_{m \in \mathbb{N}}$ provides a measure of how well the compact subset C can be approximated by finite-dimensional subspaces of X .

In the 1950s, Kolmogorov returned to approximation theory through his work on complexity theory, such as his study [20] of Vitushkin's work on Hilbert's 13th problem about the complexity of function spaces, in which he proved that the space of analytic functions of n variables is "larger" than the space of analytic functions of m variables when $n > m$. Drawing inspiration from Shannon's information theory [34], Kolmogorov [19] introduced the concept of the ϵ -entropy of a compact set C in a metric space X . Given any $\epsilon > 0$, there is a covering of C by subsets of X with diameters not exceeding 2ϵ . Denoting the smallest cardinality of such a covering by $N_\epsilon(C, X)$, the ϵ -entropy $H_\epsilon(C, X)$ of C in X is given by¹

$$H_\epsilon(C, X) = \log N_\epsilon(C, X).$$

The family $(H_\epsilon(C, X))_{\epsilon > 0}$ thus quantifies the degree of compactness of C .

The determination of entropies and widths of classes of functions has several purposes. Firstly, it can produce new invariants making it possible to distinguish and classify function sets in infinite-dimensional spaces. The meaning of the fundamental concept of "number of variables" often manifests itself in this way. Secondly, computations of widths and entropies foster the creation of new methods of approximation. Thirdly, it produces stimuli for computational mathematics by giving directions for the creation of the most expedient algorithms to solve practical problems. A more thorough discussion of this circle of ideas can be found in the original papers [22, 28, 36], in the monograph [37], or in the selected works of Kolmogorov on information theory [21], containing a reprint of [22].

For a fixed compact set C in X , the computation of the exact value of $H_\epsilon(C, X)$ for each $\epsilon > 0$ is a rather difficult task. The determination of the corresponding asymptotic order as ϵ tends to zero, however, turns out to be more manageable even in infinite-dimensional spaces. The first results in this direction are due to Kolmogorov and Tikhomirov [22], who determined the asymptotic order of the ϵ -entropy of analytic functions of n variables defined on a bounded domain in $\mathbb{R}^n$ extending analytically to some domain in $\mathbb{C}^n$.

¹Here 'log' denotes the natural logarithm, whereas in the original definition the logarithm to base 2 was used.

More precisely, suppose we are given a domain D in $\mathbb{C}^n$ containing a compact set K . Write $H^\infty(D)$ for the Banach space of all bounded, holomorphic functions in D endowed with the sup-norm and $C(K)$ for the space of all continuous functions on the compact set K equipped with the sup-norm. By Montel's theorem, the set $\mathcal{A}_K^D$ of restrictions of functions in the unit ball of $H^\infty(D)$ is a compact subset of $C(K)$. Kolmogorov and Tikhomirov [22] showed that, under natural assumptions on D and K , the asymptotic order of $H_\epsilon(\mathcal{A}_K^D) = H_\epsilon(\mathcal{A}_K^D, C(K))$ is $(\log(\epsilon^{-1}))^{n+1}$, that is, there is a constant $M > 1$ such that

$$M^{-1} < \frac{H_\epsilon(\mathcal{A}_K^D)}{(\log(\epsilon^{-1}))^{n+1}} < M \quad (\forall \epsilon > 0).$$

The existence of the limit

$$\lim_{\epsilon \rightarrow 0} \frac{H_\epsilon(\mathcal{A}_K^D)}{(\log(\epsilon^{-1}))^{n+1}},$$

and its precise value, however, remained open.

In the one-dimensional case, the problem of showing the existence of the limit was raised by Kolmogorov (see [21, p. 134]) and solved in the late 1950s, with various generalisations throughout the 1960s and 1970s. The quantity central to this quest turned out to be the capacity $C(K, D)$ of the compact set K relative to the domain D , which we now recall. Suppose that we can solve the Dirichlet problem on $D \setminus K$. Let u be the relative extremal function for K in D , that is, the unique harmonic function in $D \setminus K$, continuous on the closure of $D \setminus K$, equal to 0 in ∂D and equal to -1 on ∂K . With Γ denoting a smooth contour separating K from ∂D , and n the normal to Γ directed from K to ∂D , the relative capacity $C(K, D)$ is given by

$$C(K, D) = \int_{\Gamma} \frac{\partial u(z)}{\partial n} |dz|.$$

The solution of Kolmogorov's problem in dimension 1 can now be formulated as follows (see [2, 13, 25, 39]):

Let ∂D have positive logarithmic capacity (that is, ∂D is non-polar) and let $\mathbb{C} \setminus D$ have a countable set of connected components. Then

$$\lim_{\epsilon \rightarrow 0} \frac{H_\epsilon(\mathcal{A}_K^D)}{(\log(\epsilon^{-1}))^2} = \frac{C(K, D)}{2\pi}.$$

The solution of Kolmogorov's problem for functions of several variables remained elusive, although the solution for one variable makes it possible to solve it for several variables in special cases, for instance when K and D are Cartesian products of one-dimensional sets (see [43, 44]).

In the 1980s, with the development of pluripotential theory, in particular with the introduction of different types of extremal plurisubharmonic functions with respect to the complex Monge–Ampère operator, the setting of the problem could be formulated precisely. For this we need to recall some definitions and properties relating to relative extremal functions and relative capacity.

If D is an open set in $\mathbb{C}^n$ and E a subset of D , the *relative extremal function* for E in D (see [4, 6, 15, 16, 32, 35, 41]) is defined as

$$u_{E,D}(z) = \sup \{v(z) : v \text{ is psh on } D, v|_E \leq -1, v \leq 0\} \quad (z \in D).$$

Here and in the following, we write ‘psh’ for ‘plurisubharmonic’, a notion that replaces the notion of subharmonicity in one variable. It turns out that the upper semi-continuous regularisation $u_{E,D}^*$ of $u_{E,D}$ is psh on D . In one variable, $u_{E,D}^*$ is closely related to harmonic measure.

In several variables, the natural context for the study of this function is provided by hyperconvex domains: a domain D in $\mathbb{C}^n$ is said to be *hyperconvex* if there exists a continuous plurisubharmonic exhaustion function $\varrho : D \rightarrow (-\infty, 0)$. In one dimension, a domain D is hyperconvex if and only if we are able to solve the Dirichlet problem on D . In several variables, every hyperconvex domain is pseudoconvex (or holomorphically convex) and every pseudoconvex domain is the union of an increasing sequence of hyperconvex domains. Note that if D is an open set and E a non-pluripolar relatively compact subset of D , then D is hyperconvex if and only if for any point $w \in \partial D$ we have $\lim_{z \rightarrow w} u_{E,D}(z) = 0$.

For later use, we note that if D is a bounded hyperconvex open set and $K \subset D$ is a compact set, then we say that K is *regular* in D if $u_{K,D}^*$ is a continuous function.

In one complex variable, $u_{K,D}$ is harmonic in $D \setminus K$ and $\Delta u_{K,D}$ is a positive measure supported on K . In several variables, the Laplace operator is replaced by the *complex Monge–Ampère operator* $(dd^c)^n$, defined as the n th exterior power of $dd^c = 2i\partial\bar{\partial}$, that is,

$$(dd^c)^n = \underbrace{dd^c \wedge \cdots \wedge dd^c}_{n \text{ times}}.$$

Here, $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$.

In one variable the complex Monge–Ampère operator is equal to the Laplace operator Δ , since $dd^c = \Delta dx \wedge dy$ in $\mathbb{R}^2$ or $\mathbb{C}$. In several variables, if D is a hyperconvex domain in $\mathbb{C}^n$ containing a compact subset K , then $u_{K,D}^*$ is maximal on $D \setminus K$ and $(dd^c u_{K,D}^*)^n = 0$ on $D \setminus K$. In this case, $(dd^c u_{K,D}^*)^n$ is well defined and turns out to be a positive measure supported on K [17, Section 4.5].

The *relative capacity* of K in D (see [5, 6]) is defined as

$$C(K, D) = \sup \left\{ \int_K (dd^c u)^n : u \in \text{PSH}(D, (-1, 0)) \right\},$$

where $\text{PSH}(\Omega, I)$ denotes the set of all psh functions on a domain Ω in $\mathbb{C}^n$ with values in an interval $I \subset [-\infty, +\infty)$.

It turns out that for D a hyperconvex domain in $\mathbb{C}^n$ and K a compact subset of D , the relative extremal function and the relative capacity are related as follows [17, Section 4.6]:

$$C(K, D) = \int_D (dd^c u_{K,D}^*)^n = \int_K (dd^c u_{K,D}^*)^n.$$

Note that $C(K, D) < \infty$ by the Chern–Levine–Nirenberg estimate (see, for example, [17, Proposition 3.4.2]). Note also that the previous definition of $C(K, D)$ in dimension 1 coincides with the more general one given above, since

$$\int_{\Gamma} \frac{\partial u(z)}{\partial n} |dz| = \int_K \Delta u \, dx \, dy$$

by Green’s formula.

In the 1980s, using the generalisation of the notion of relative capacity to higher dimensions given above, Zakharyuta [43] formulated a more precise version of Kolmogorov’s problem.

Kolmogorov’s problem for the ϵ -entropy. For D a domain in $\mathbb{C}^n$ and K a compact subset of D show that

$$\lim_{\epsilon \rightarrow 0} \frac{H_{\epsilon}(\mathcal{A}_K^D)}{(\log(\epsilon^{-1}))^{n+1}} = \frac{2C(K, D)}{(2\pi)^n(n+1)!}. \quad (2)$$

In [43] Zakharyuta sketched an argument to show that Kolmogorov’s problem could be solved provided that a certain conjecture, now known as Zakharyuta’s Conjecture, could be established. More detailed accounts of this reduction were provided in [44–46]. Zakharyuta’s Conjecture concerns the approximability of the regularised relative extremal function of K and D by pluricomplex Green functions in D with logarithmic poles in K . The precise setting of this conjecture as well as the definitions and properties of these functions can be found in [29, 30].

This conjecture was proved in the one-dimensional case by Skiba and Zakharyuta [42] and in the multi-dimensional case by Nivoche [29, 30] under the hypothesis that the domain D be bounded and hyperconvex and that the compact set K be regular in D , a rather natural setting, since in this case, the relative extremal function and the pluricomplex Green function are continuous psh functions tending to zero on the boundary of the domain.

As mentioned earlier, there is a close connection between Zakharyuta’s Conjecture and Kolmogorov’s problem. Indeed, in order to solve Kolmogorov’s problem, it is sufficient to prove that Zakharyuta’s Conjecture is true, as was shown in [42] for $n = 1$ and in [43] (see also [45, 46]) for $n > 1$, provided that K be regular in D with non-zero Lebesgue measure, and that the domain D be strictly hyperconvex, a rather natural notion that is defined as follows.

A domain $D \in \mathbb{C}^n$ is said to be *strictly hyperconvex* if there exists a bounded domain Ω and a continuous exhaustion function $\varrho \in \text{PSH}(\Omega, (-\infty, 1))$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$. Note that any strictly pseudoconvex domain is strictly hyperconvex.

All in all, it follows from [43] and [30] that Kolmogorov’s problem for the ϵ -entropy is solved for D strictly hyperconvex and K regular in D with positive Lebesgue measure.

1.2. A new solution of Kolmogorov's problem

In this paper we will provide a new self-contained solution of Kolmogorov's problem, which does not rely on Zakharyuta's Conjecture, but instead proceeds more directly, and, at the same time, makes the required assumptions explicit and transparent.

It turns out that Kolmogorov's problem on the asymptotics of $H_\epsilon(\mathcal{A}_K^D)$ as ϵ tends to zero is in fact equivalent to the following problem on the asymptotics of the Kolmogorov widths $d_m(\mathcal{A}_K^D) = d_m(\mathcal{A}_K^D, C(K))$ as m tends to infinity.

Kolmogorov's problem for m -widths. For D a domain in $\mathbb{C}^n$ and K a compact subset of D show that

$$\lim_{m \rightarrow \infty} \frac{-\log d_m(\mathcal{A}_K^D)}{m^{1/n}} = 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}. \quad (3)$$

The two problems of Kolmogorov are equivalent in the sense that if the limit (2) exists then so does the limit (3) and *vice versa*. For a complete proof of this fact, see [47], which uses ideas from [28] and [25].

Our approach to solve Kolmogorov's problem for m -widths, and hence for the ϵ -entropy, will proceed as follows. In Section 2, we will establish sharp lower bounds for the Kolmogorov widths by studying the eigenvalue distribution of a family of Toeplitz operators defined on a family of Bergman spaces. Unfortunately, this method does not appear to provide sharp upper bounds.

Therefore, another method will be used in Section 3 to establish sharp upper bounds for the Kolmogorov widths, first in the special case where D and K are special holomorphic polyhedra using the Bergman–Weil formula, then in the general case exploiting refinements of approximation arguments from [30]. Curiously enough, this method does not seem to provide sharp lower bounds.

All in all, we shall establish the following result.

Theorem 1.1. *Let D be a domain in $\mathbb{C}^n$ and K a compact subset of D .*

(i) *If D is strictly hyperconvex and K is non-pluripolar, then*

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(\mathcal{A}_K^D)}{m^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

(ii) *If D is bounded hyperconvex, then*

$$\liminf_{m \rightarrow \infty} \frac{-\log d_m(\mathcal{A}_K^D)}{m^{1/n}} \geq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

In particular, if D is strictly hyperconvex and K is non-pluripolar, then

$$\lim_{m \rightarrow \infty} \frac{-\log d_m(\mathcal{A}_K^D)}{m^{1/n}} = 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

1.3. Preliminaries and notation

For D an open subset of $\mathbb{C}^n$ and K a compact subset of D we write

$$\begin{aligned}\mathcal{O}(D) &= \{f : D \rightarrow \mathbb{C} : f \text{ holomorphic on } D\}, \\ \mathcal{O}(K) &= \{f : K \rightarrow \mathbb{C} : f \text{ holomorphic on a neighbourhood of } K\}.\end{aligned}$$

We use $H^\infty(D)$ for the Banach space of bounded holomorphic functions on D equipped with the supremum norm on D and $A(K)$ for the completion of $\mathcal{O}(K)$ in the Banach space $C(K)$ of continuous functions on K .

For the derivation of our bounds on the Kolmogorov widths $d_m(\mathcal{A}_K^D)$ in Sections 2 and 3 it will be convenient to introduce the following generalisation of the Kolmogorov widths to bounded operators on a Banach space.

For $T : X \rightarrow Y$ a bounded operator with X and Y Banach spaces, we associate with it the sequence $(d_m(T))_{m \in \mathbb{N}}$ of *Kolmogorov numbers* given by

$$d_m(T) = \inf_{\dim L < m} \sup_{\|x\|_X \leq 1} \inf_{y \in L} \|Tx - y\|_Y,$$

where the outermost infimum is taken over subspaces L of Y . It turns out that $\lim_{m \rightarrow \infty} d_m(T) = 0$ if and only if T is compact (see, for example, [10, Proposition 2.2.1]), so the sequence $(d_m(T))_{m \in \mathbb{N}}$ provides a measure of the degree of compactness of T .

In order to link the two notions observe that if we define the *canonical mapping*

$$J : H^\infty(D) \rightarrow A(K), \quad Jf = f|_K,$$

then $\mathcal{A}_K^D$ is the image under J of the unit ball $B_{H^\infty(D)}$ in $H^\infty(D)$ and

$$d_m(J) = d_m(J(B_{H^\infty(D)}), A(K)) = d_m(\mathcal{A}_K^D) \quad (\forall m \in \mathbb{N}),$$

so in order to investigate Kolmogorov's problem on m -widths it suffices to study the asymptotics of the Kolmogorov numbers of the canonical mapping $J : H^\infty(D) \rightarrow A(K)$. Note that the canonical mapping is in fact an embedding if D is a domain and K is a set of uniqueness (which is, for example, the case if K is non-pluripolar).

For later use, we associate two more sequences with the bounded operator $T : X \rightarrow Y$ between Banach spaces: the sequence $(c_m(T))_{m \in \mathbb{N}}$ of *Gelfand numbers*

$$c_m(T) = \inf_{\text{codim } L < m} \sup_{\substack{x \in L \\ \|x\|_X \leq 1}} \|Tx\|_Y,$$

where the infimum is taken over closed subspaces L of X , which, like the sequence of Kolmogorov numbers, also quantifies compactness of T ; and the sequence $(a_m(T))_{m \in \mathbb{N}}$ of *approximation numbers* given by

$$a_m(T) = \inf \{\|T - F\|_{X \rightarrow Y} : F : X \rightarrow Y \text{ with } \text{rank}(F) < m\},$$

which quantifies the degree of approximability of T by operators of finite rank. It turns out (see, for example, [10, (2.2.12) and Proposition 2.4.6]) that

$$d_m(T) \leq a_m(T) \leq \sqrt{2m} d_m(T) \quad (\forall m \in \mathbb{N}), \quad (4)$$

and (see, for example, [10, (2.3.15) and Proposition 2.3.4])

$$c_m(T) \leq a_m(T) \leq \sqrt{2m} c_m(T) \quad (\forall m \in \mathbb{N}), \quad (5)$$

which implies that the Kolmogorov numbers decay at a certain stretched exponential speed precisely if the approximation numbers and, in turn, the Gelfand numbers do, the speeds being the same in all cases.

We also note that all three sequences enjoy the following submultiplicativity property. If W and Z are Banach spaces and $S : W \rightarrow X$ and $R : Y \rightarrow Z$ are bounded operators, then

$$s_m(RTS) \leq \|R\|_{Y \rightarrow Z} s_m(T) \|S\|_{W \rightarrow X} \quad (\forall m \in \mathbb{N}), \quad (6)$$

where $\|R\|_{Y \rightarrow Z}$ and $\|S\|_{W \rightarrow X}$ denote the operator norms of R and S , respectively, and where $s_m(T)$ denotes any of the Kolmogorov, Gelfand or approximation numbers (see, for example, [31, Theorems 2.3.3, 2.4.3, 2.5.3]).

Finally, we note that in the particular case where X and Y are Hilbert spaces, then the Kolmogorov numbers, Gelfand numbers and approximation numbers coincide (see, for example, [31, Theorem 2.11.9]).

2. Lower bound for the Kolmogorov widths

In the original formulation of Kolmogorov's problem of the asymptotics of the m -widths $d_m(\mathcal{A}_K^D)$, all spaces are equipped with the supremum norm. In the case where D is a strictly hyperconvex domain containing a holomorphically convex and regular compact subset K of positive Lebesgue measure, Zakharyuta [43, 44] (see also [45, 46], for a more recent exposition) and Aytuna [1] have developed a rather sophisticated theory which guarantees that the asymptotics of Kolmogorov widths of natural spaces of holomorphic functions associated with D and K coincide, including Banach spaces equipped with the supremum norm as well as Hilbert spaces with a weighted L^2 -norm, where the weight arises from a bounded psh function on D . Indeed, in this case, there exist pairs (H_0, H_1) of Hilbert spaces which are adherent to the pair $(A(K), H^\infty(D))$ with norms weaker than the supremum norm (see for instance [46, Lemma 4.12 and Corollary 4.13]).

In Subsections 2.1–2.3, we will develop an L^2 -approach for the asymptotics of Kolmogorov m -widths for bounded hyperconvex D and $K \subset D$ compact with positive Lebesgue measure, which relies on studying the eigenvalue asymptotics of a sequence of compact Toeplitz operators on certain Bergman spaces, which in turn relies on asymptotic bounds for Bergman kernels. In Subsection 2.4, we shall connect the L^2 -bounds with the usual supremum norm bounds in the original formulation of Kolmogorov's problem, under the slightly stronger hypothesis that D be strictly hyperconvex. We will not rely on the theory developed by Zakharyuta and Aytuna, preferring instead to use simple

bounds for Kolmogorov widths coupled with approximation arguments to obtain sharp lower bounds for the m -widths $d_m(\mathcal{A}_K^D)$. Finally, in Subsection 2.5 we will generalise the result from Subsection 2.4 to allow K to be non-pluripolar.

2.1. Spectral asymptotics of Toeplitz operators

Let D be a domain in $\mathbb{C}^n$ and let $L^\infty(D)$ denote the Banach space of complex-valued essentially bounded functions on D equipped with the essential supremum norm $\|\cdot\|_\infty$. For $\varphi : D \rightarrow \mathbb{R}$ a bounded measurable function, we write $L_\varphi^2(D)$ for the weighted L^2 -space of Lebesgue measurable functions on D equipped with the norm

$$\|f\|_{L_\varphi^2(D)}^2 = \int_D |f|^2 e^{-2\varphi} dm = \int_D |f|^2 dm_\varphi,$$

where m denotes $2n$ -dimensional Lebesgue measure on $\mathbb{C}^n$, and $dm_\varphi = e^{-2\varphi} dm$.

The corresponding weighted Bergman space will be denoted by $H_\varphi^2(D)$, that is,

$$H_\varphi^2(D) = \{f \in L_\varphi^2(D) : f \text{ holomorphic on } D\}.$$

Note that since φ is bounded, the weighted Bergman space $H_\varphi^2(D)$ and the (unweighted) standard Bergman space $H_0^2(D)$, or simply $H^2(D)$, are isomorphic as Banach spaces. Thus, point-evaluation $f \mapsto f(z)$ is continuous on $H_\varphi^2(D)$ for every $z \in D$ and $H_\varphi^2(D)$ is a reproducing kernel Hilbert space, the kernel of which we denote by $B_{D,\varphi}$ or simply B_φ if the domain D is understood. Thus, $B_\varphi : D \times D \rightarrow \mathbb{C}$ with

$$f(z) = \int_D B_\varphi(z, \zeta) f(\zeta) dm_\varphi(\zeta) \quad (\forall f \in H_\varphi^2(D), \forall z \in D).$$

In particular, for each $\zeta \in D$, the function $z \mapsto B_\varphi(z, \zeta)$ is holomorphic and $\overline{B_\varphi(z, \zeta)} = B_\varphi(\zeta, z)$. Moreover, the reproducing kernel can be written

$$B_\varphi(z, \zeta) = \sum_m e_m(z) \overline{e_m(\zeta)},$$

where (e_m) is an orthonormal basis of $H_\varphi^2(D)$ and the sum converges uniformly on compact subsets of $D \times D$.

Using the Bergman kernel B_φ , the orthogonal projection

$$P_\varphi : L_\varphi^2(D) \twoheadrightarrow H_\varphi^2(D),$$

known as *Bergman projection* in this context, can be written

$$P_\varphi f(z) = \int_D B_\varphi(z, \zeta) f(\zeta) dm_\varphi(\zeta).$$

Note that if J_φ denotes the natural embedding of $H_\varphi^2(D)$ in $L_\varphi^2(D)$ then P_φ is the adjoint of J_φ , that is,

$$P_\varphi^* = J_\varphi.$$

For χ in $L^\infty(D)$ we write M_χ for the corresponding multiplication operator on $L^2_\varphi(D)$, that is,

$$M_\chi : L^2_\varphi(D) \rightarrow L^2_\varphi(D), \quad M_\chi f = \chi \cdot f.$$

Note that M_χ is bounded with operator norm $\|M_\chi\| = \|\chi\|_\infty$ and that M_χ is the zero operator precisely when the support of χ is a Lebesgue null set. Ultimately, this is the reason why we require K to have non-zero Lebesgue measure for this and the following three subsections.

The compression of M_χ to $H^2_\varphi(D)$, denoted by $T_{\chi,\varphi}$, is known as *Toeplitz operator* with symbol χ in this context, that is,

$$T_{\chi,\varphi} : H^2_\varphi(D) \rightarrow H^2_\varphi(D), \quad T_{\chi,\varphi} = P_\varphi M_\chi J_\varphi.$$

Clearly, for $f \in H^2_\varphi(D)$ we have

$$T_{\chi,\varphi} f(z) = \int_D B_\varphi(z, \zeta) \chi(\zeta) f(\zeta) dm_\varphi(\zeta).$$

We shall now collect some properties of Toeplitz operators on $H^2_\varphi(D)$ which are fairly standard but difficult to find in the literature in the stated generality. We start with positivity.

Lemma 2.1. *If $\chi \in L^\infty(D)$ is real-valued and non-negative then $T_{\chi,\varphi}$ is a bounded, self-adjoint, positive operator. In particular, the spectrum of $T_{\chi,\varphi}$ is contained in $[0, \|\chi\|_\infty]$.*

Proof. Since M_χ is bounded, the operator $T_{\chi,\varphi}$ is also bounded. Moreover, as χ is real-valued and non-negative, there is a real-valued $\psi \in L^\infty(D)$ with $\chi = \psi^2$. Thus $M_\psi = M_\psi^*$, and we have

$$T_{\chi,\varphi} = P_\varphi M_\psi^2 J_\varphi = (M_\psi J_\varphi)^* M_\psi J_\varphi.$$

Hence $T_{\chi,\varphi}$ is self-adjoint and positive, and its spectrum is contained in $[0, \infty)$. Moreover, as $\|T_{\chi,\varphi}\| \leq \|M_\chi\| = \|\chi\|_\infty$, the remaining assertion follows. ■

Next we turn to compactness properties of $T_{\chi,\varphi}$.

Lemma 2.2. *If $\chi \in L^\infty(D)$ has compact support in the open set $D \subset \mathbb{C}^n$, then there are positive constants c_1 and c_2 such that*

$$d_m(T_{\chi,\varphi}) \leq c_1 \exp(-c_2 m^{1/n}) \quad (\forall m \in \mathbb{N}). \quad (7)$$

In particular, $T_{\chi,\varphi}$ is trace class with

$$\text{Tr}(T_{\chi,\varphi}) = \int_D B_\varphi(z, z) \chi(z) dm_\varphi(z), \quad (8)$$

and

$$\text{Tr}(T_{\chi,\varphi}^2) = \int_D \int_D |B_\varphi(z, \zeta)|^2 \chi(z) \chi(\zeta) dm_\varphi(z) dm_\varphi(\zeta). \quad (9)$$

Proof. Since χ has compact support we can choose a domain U containing the support of χ such that the closure of U is a compact subset of D . We can now write

$$T_{\chi,\varphi} = P_\varphi M_\chi J_\varphi = P_\varphi \widetilde{M}_\chi J_U J_{D,U}, \quad (10)$$

where $J_{D,U} : H_\varphi^2(D) \rightarrow H_\varphi^2(U)$ and $J_U : H_\varphi^2(U) \rightarrow L_\varphi^2(U)$ denote the canonical embeddings and $\widetilde{M}_\chi$ denotes the operator of multiplication by χ , albeit considered as an operator from $L_\varphi^2(U)$ to $L_\varphi^2(D)$. Note that $\widetilde{M}_\chi$ is well defined and bounded since the support of χ is contained in U .

Now, since U is compactly contained in D , standard arguments (see, for example, [3, Theorem 4.7]) show that the Kolmogorov numbers of $J_{D,U}$ enjoy a stretched exponential bound, that is, there are positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$d_m(J_{D,U}) \leq \tilde{c}_1 \exp(-\tilde{c}_2 m^{1/n}) \quad (\forall m \in \mathbb{N}). \quad (11)$$

Moreover, using the factorisation (10) and the fact that $P_\varphi \widetilde{M}_\chi J_U$ is bounded we have

$$d_m(T_{\chi,\varphi}) \leq \|P_\varphi \widetilde{M}_\chi J_U\| d_m(J_{D,U}) \quad (\forall m \in \mathbb{N}),$$

which, together with (11), yields (7). In particular, the operator $T_{\chi,\varphi}$ is trace class, since its Kolmogorov numbers, and hence its singular values, are summable.

Given an orthonormal basis (e_m) of $H_\varphi^2(D)$ we then have, using properties of the reproducing kernel stated earlier,

$$\begin{aligned} \text{Tr}(T_{\chi,\varphi}) &= \sum_m \int_D \int_D B_\varphi(z, \zeta) \chi(\zeta) e_m(\zeta) \overline{e_m(z)} dm_\varphi(\zeta) dm_\varphi(z) \\ &= \int_D \int_D B_\varphi(z, \zeta) \chi(\zeta) B_\varphi(\zeta, z) dm_\varphi(\zeta) dm_\varphi(z) \\ &= \int_D B_\varphi(\zeta, \zeta) \chi(\zeta) dm_\varphi(\zeta), \end{aligned}$$

and (8) is proven. For the proof of the second trace formula we observe that we can also write $T_{\chi,\varphi}^2$ as an integral operator,

$$T_{\chi,\varphi}^2 f(z) = \int_D K_{\chi,\varphi}(z, \zeta) f(\zeta) dm_\varphi(\zeta),$$

with kernel

$$K_{\chi,\varphi}(z, \zeta) = \int_D B_\varphi(z, \zeta') \chi(\zeta') B_\varphi(\zeta', \zeta) \chi(\zeta) dm_\varphi(\zeta'),$$

so using the same arguments as before, we have

$$\text{Tr}(T_{\chi,\varphi}^2) = \int_D \int_D B_\varphi(\zeta', \zeta) \chi(\zeta) B_\varphi(\zeta, \zeta') \chi(\zeta') dm_\varphi(\zeta) dm_\varphi(\zeta')$$

and (9) follows by using the symmetry of the reproducing kernel. ■

Suppose now that the symbol χ is non-negative with compact support in D . By the preceding two lemmas, the associated Toeplitz operator $T_{\chi,\varphi}$ is positive and compact. If f is an eigenfunction of $T_{\chi,\varphi}$ with positive eigenvalue λ , then

$$\lambda \|f\|_{H_{\varphi}^2(D)}^2 = (T_{\chi,\varphi} f, f)_{H_{\varphi}^2(D)} = (M_{\chi} J_{\varphi} f, J_{\varphi} f)_{L_{\varphi}^2(D)} = \int_D \chi |f|^2 dm_{\varphi}.$$

Thus, if χ is the characteristic function of a compact subset K of D , the eigenvalue λ measures the concentration of the mass of the corresponding eigenfunction f to K .

In order to investigate this further we shall write $(\lambda_m(T_{\chi,\varphi}))_{m \in \mathbb{N}}$ for the corresponding eigenvalue sequence arranged in non-increasing order, so that

$$\lambda_1(T_{\chi,\varphi}) \geq \lambda_2(T_{\chi,\varphi}) \geq \lambda_3(T_{\chi,\varphi}) \geq \cdots,$$

with each eigenvalue repeated according to its algebraic multiplicity.

In the following we shall be interested in the behaviour of the eigenvalue sequence of $T_{\chi,k\varphi}$ when k tends to infinity. Of fundamental importance for this study is the following result, essentially due to Engliš (see [12, Theorem 1]), which gives an asymptotic expansion for the Bergman kernels $B_{k\varphi}$ as k tends to infinity, provided that φ is smooth and strictly psh.

Theorem 2.3. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and φ be a strictly psh and C^∞ function on D . Then the following holds.*

(i) *We have*

$$k^{-n} B_{k\varphi}(z, z) dm_{k\varphi}(z) \xrightarrow{k \rightarrow \infty} \frac{1}{(2\pi)^n n!} (dd^c \varphi)^n(z)$$

pointwise in D , with the left hand side being locally uniformly bounded in D for every k .

(ii) *We have*

$$k^{-n} |B_{k\varphi}(z, \zeta)|^2 dm_{k\varphi}(z) dm_{k\varphi}(\zeta) \xrightarrow{k \rightarrow \infty} \frac{1}{(2\pi)^n n!} (dd^c \varphi)^n|_{z=\zeta}$$

weakly as positive measures on $D \times D$, that is, for every $g \in C_c(D \times D)$,

$$\begin{aligned} k^{-n} \int_D \int_D g(z, \zeta) |B_{k\varphi}(z, \zeta)|^2 dm_{k\varphi}(z) dm_{k\varphi}(\zeta) \\ \xrightarrow{k \rightarrow \infty} \frac{1}{(2\pi)^n n!} \int_D g(z, z) (dd^c \varphi)^n(z). \end{aligned}$$

Remark 2.4. Assertion (i) is due to Engliš. Its proof in [12] is based on Fefferman's asymptotic expansion of the Bergman kernel of a Forelli–Rudin domain over D .

Analogues of assertions (i) and (ii) can be found in the paper [27] of Lindholm (Theorems 10 and 11) for Toeplitz operators on Fock spaces over $\mathbb{C}^n$. The method of proving these results is inspired by an approach of Landau [24], based on studying functions concentrated on compact sets, and using L^2 -techniques to obtain precise size estimates of the Bergman kernel both on and off the diagonal. Lindholm's proof of the analogue of (ii) in [27, Theorem 11] is easily adapted to the case of bounded pseudoconvex domains in $\mathbb{C}^n$.

Using the previous theorem we are now able to prove a crucial result concerning the asymptotics of the number of eigenvalues of $T_{\chi,k\varphi}$ greater than a fixed threshold as k tends to infinity. The proposition and the main idea of its proof are inspired by an analogous result of Lindholm [27, Theorem 13].

Proposition 2.5. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and φ be a strictly psh and C^∞ function on D . Let $\chi \in L^\infty(D)$ be non-zero, non-negative and have compact support in D . Then, for any $\gamma \in (0, 1)$, we have*

$$\lim_{k \rightarrow \infty} \#\{m \in \mathbb{N} : \lambda_m(T_{\chi,k\varphi}) > \gamma \|\chi\|_\infty\} \cdot k^{-n} = \frac{1}{(2\pi)^n n!} \int_D \frac{\chi}{\|\chi\|_\infty} (dd^c \varphi)^n.$$

Proof. We start by noting that it suffices to show the result under the additional assumption that $\|\chi\|_\infty = 1$. The general case then follows by observing that $\lambda_m(T_{c\chi,\varphi}) = c\lambda_m(T_{\chi,\varphi})$ for any $c \geq 0$ and any $m \in \mathbb{N}$.

Suppose now that $\|\chi\|_\infty = 1$. Fix $\gamma \in (0, 1)$. We need to show that

$$\lim_{k \rightarrow \infty} \#\{m \in \mathbb{N} : \lambda_m(T_{\chi,k\varphi}) > \gamma\} \cdot k^{-n} = \frac{1}{(2\pi)^n n!} \int_D \chi (dd^c \varphi)^n. \quad (12)$$

This will follow from the remarkable fact that both $\text{Tr}(T_{\chi,k\varphi})$ and $\text{Tr}(T_{\chi,k\varphi}^2)$ have the same asymptotics as k tends to infinity. More precisely, by combining Theorem 2.3 and Lemma 2.2, we have

$$\lim_{k \rightarrow \infty} k^{-n} \text{Tr}(T_{\chi,k\varphi}) = \frac{1}{(2\pi)^n n!} \int_D \chi (dd^c \varphi)^n, \quad (13)$$

$$\lim_{k \rightarrow \infty} k^{-n} \text{Tr}(T_{\chi,k\varphi}^2) = \frac{1}{(2\pi)^n n!} \int_D \chi (dd^c \varphi)^n. \quad (14)$$

In order to simplify notation we shall use the shorthand

$$\lambda_m(k) = \lambda_m(T_{\chi,k\varphi})$$

so that $\text{Tr}(T_{\chi,k\varphi}) = \sum_m \lambda_m(k)$ and $\text{Tr}(T_{\chi,k\varphi}^2) = \sum_m \lambda_m(k)^2$. Before proceeding we note that by Lemma 2.1 we have $\lambda_m(k) \leq \|\chi\|_\infty = 1$ for all m and k . Moreover we have

$$\begin{aligned} \lambda_1(k) &= \sup \{(T_{\chi,k\varphi} f, f) : \|f\|_{H_{k\varphi}^2(D)} = 1\} \\ &\geq \frac{\int_D \chi \cdot \chi_D dm_{k\varphi}}{\int_D \chi_D dm_{k\varphi}} = \frac{\int_D \chi dm_{k\varphi}}{m_{k\varphi}(D)} > 0. \end{aligned}$$

Thus, for all k we have

$$0 < \sum_m \lambda_m(k)^2 \leq \sum_m \lambda_m(k).$$

Combining the above with the equality of the limits (13) and (14) we see that for any $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that for any $k \geq k_\delta$ we have

$$1 - \delta \leq \frac{\sum_m \lambda_m(k)^2}{\sum_m \lambda_m(k)} \leq 1. \quad (15)$$

We shall now establish the following two bounds: for every $k \geq k_\delta$,

$$\#\{m : \lambda_m(k) > \gamma\} \geq \left(1 - \frac{\delta}{1 - \gamma}\right) \sum_m \lambda_m(k), \quad (16)$$

and for every $k \geq k_\delta$ and every $\gamma' \in (\gamma, 1)$

$$\#\{m : \lambda_m(k) > \gamma\} \leq \left(\frac{1}{\gamma'} + \frac{\delta}{\gamma(1 - \gamma')}\right) \sum_m \lambda_m(k). \quad (17)$$

In order to see this fix $k \geq k_\delta$ and define, for every $\beta \in (0, 1)$,

$$S_\beta = \frac{\sum_{m: \lambda_m \leq \beta} \lambda_m(k)}{\sum_m \lambda_m(k)}.$$

It now follows from (15) that

$$\begin{aligned} (1 - \delta) \sum_m \lambda_m(k) &\leq \sum_{m: \lambda_m > \gamma} \lambda_m(k)^2 + \sum_{m: \lambda_m \leq \gamma} \lambda_m(k)^2 \\ &\leq \sum_{m: \lambda_m > \gamma} \lambda_m(k) + \gamma \cdot \sum_{m: \lambda_m \leq \gamma} \lambda_m(k) = (1 - S_\gamma) \sum_m \lambda_m(k) + \gamma S_\gamma \sum_m \lambda_m(k), \end{aligned}$$

so $S_\gamma \leq \frac{\delta}{1 - \gamma}$. Hence

$$\#\{m : \lambda_m(k) > \gamma\} \geq \sum_{m: \lambda_m > \gamma} \lambda_m(k) \geq \left(1 - \frac{\delta}{1 - \gamma}\right) \sum_m \lambda_m(k),$$

and (16) is proven. For (17) we note that for $\gamma < \gamma' < 1$ we have

$$\begin{aligned} \#\{m : \lambda_m(k) > \gamma\} &= \#\{m : \lambda_m(k) > \gamma'\} + \#\{m : \gamma' \geq \lambda_m(k) > \gamma\} \\ &\leq \frac{1}{\gamma'} \sum_{m: \lambda_m > \gamma'} \lambda_m(k) + \frac{1}{\gamma} \sum_{m: \gamma' \geq \lambda_m > \gamma} \lambda_m(k) \\ &\leq \frac{1}{\gamma'} \sum_m \lambda_m(k) + \frac{1}{\gamma} S_{\gamma'} \sum_m \lambda_m(k) \end{aligned}$$

and (17) follows.

Now, combining (13) and (16) gives, for any $\delta > 0$,

$$\liminf_{k \rightarrow \infty} \#\{m : \lambda_m(k) > \gamma\} \cdot k^{-n} \geq \left(1 - \frac{\delta}{1 - \gamma}\right) \frac{1}{(2\pi)^n n!} \int_D \chi(dd^c \varphi)^n,$$

which, since $\delta > 0$ was arbitrary, yields

$$\liminf_{k \rightarrow \infty} \#\{m : \lambda_m(k) > \gamma\} \cdot k^{-n} \geq \frac{1}{(2\pi)^n n!} \int_D \chi(dd^c \varphi)^n. \quad (18)$$

Similarly, combining (13) and (17) gives, for any $\delta > 0$ and any $\gamma' \in (\gamma, 1)$,

$$\limsup_{k \rightarrow \infty} \sharp\{m : \lambda_m(k) > \gamma\} \cdot k^{-n} \leq \left(\frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')} \right) \frac{1}{(2\pi)^n n!} \int_D \chi (dd^c \varphi)^n;$$

but since $\delta > 0$ and $\gamma < \gamma' < 1$ were arbitrary, the above implies

$$\limsup_{k \rightarrow \infty} \sharp\{m : \lambda_m(k) > \gamma\} \cdot k^{-n} \leq \frac{1}{(2\pi)^n n!} \int_D \chi (dd^c \varphi)^n. \quad (19)$$

Combining (18) and (19) the limit (12) follows. ■

2.2. Lower bounds for Kolmogorov widths

We shall now use the results of the previous subsection to study the asymptotics of the Kolmogorov numbers of certain embedding operators. To be precise, let D be a domain in $\mathbb{C}^n$ and let K be a compact subset of D with positive $2n$ -dimensional Lebesgue measure m . For $\varphi \in L^\infty(D)$ we write $L_\varphi^2(K)$ for the weighted L^2 -space of Lebesgue measurable functions on K equipped with the norm

$$\|f\|_{L_\varphi^2(K)}^2 = \int_K |f|^2 e^{-2\varphi} dm = \int_K |f|^2 dm_\varphi.$$

Clearly, the natural embedding

$$J_{\varphi,K,D} : H_\varphi^2(D) \hookrightarrow L_\varphi^2(K)$$

given by restricting functions in $H_\varphi^2(D)$ to K is continuous. Moreover, this embedding is closely related to a Toeplitz operator on $H_\varphi^2(D)$, provided that its symbol is the indicator function of K , as the following lemma shows.

Lemma 2.6. *Let $D \subset \mathbb{C}^n$ be a domain, K a compact subset of D with positive Lebesgue measure and $\varphi \in L^\infty(D)$. If χ is the indicator function of K , then*

$$T_{\chi,\varphi} = J_{\varphi,K,D}^* J_{\varphi,K,D}.$$

In particular, we have

$$\lambda_m(T_{\chi,\varphi}) = d_m(J_{\varphi,K,D})^2 \quad (\forall m \in \mathbb{N}).$$

Proof. Let $f, g \in H_\varphi^2(D)$. Then

$$\begin{aligned} (T_{\chi,\varphi} f, g)_{H_\varphi^2(D)} &= (P_\varphi M_\chi J_\varphi f, g)_{H_\varphi^2(D)} = (M_\chi J_\varphi f, J_\varphi g)_{L_\varphi^2(D)} \\ &= \int_D \chi f \bar{g} dm_\varphi = \int_K f \bar{g} dm_\varphi \end{aligned}$$

and

$$(J_{\varphi,K,D}^* J_{\varphi,K,D} f, g)_{H_\varphi^2(D)} = (J_{\varphi,K,D} f, J_{\varphi,K,D} g)_{L_\varphi^2(K)} = \int_K f \bar{g} dm_\varphi,$$

so $T_{\chi,\varphi} = J_{\varphi,K,D}^* J_{\varphi,K,D}$ as claimed.

In order to finish the proof, we recall that if T is a compact operator on a Hilbert space, then, by definition, its m th singular value is given by $\lambda_m(T^*T)^{1/2}$, which in turn is equal to the m th Kolmogorov number of T ; see, for example, [31, Theorem 2.11.9]. Thus

$$\lambda_m(T^*T)^{1/2} = d_m(T) \quad (m \in \mathbb{N}),$$

and the remaining assertion follows. $\blacksquare$

In order to apply the results of the previous subsection we also require the following simple relationship between the Kolmogorov numbers of $J_{k\varphi, K, D}$ and $J_{0, K, D}$, that is, the natural embedding of the standard, unweighted Bergman space $H^2(D)$ in the unweighted Lebesgue space $L^2(K)$.

Lemma 2.7. *Let $D \subset \mathbb{C}^n$ be a domain, K a compact subset of D with $m(K) > 0$ and $\varphi \in L^\infty(D)$. If $\varphi \leq 0$ on D and $\varphi \geq -1$ on K , then*

$$d_m(J_{k\varphi, K, D}) \leq e^k d_m(J_{0, K, D}) \quad (\forall k, m \in \mathbb{N}).$$

Proof. Let

$$I_{k, K} : L^2_0(K) \rightarrow L^2_{k\varphi}(K) \quad \text{and} \quad I_{k, D} : H^2_0(D) \rightarrow H^2_{k\varphi}(D)$$

denote the respective identities. Then

$$J_{k\varphi, D, K} = I_{k, K} J_{0, K, D} I_{k, D}^{-1},$$

and so

$$d_m(J_{k\varphi, D, K}) \leq \|I_{k, K}\| d_m(J_{0, K, D}) \|I_{k, D}^{-1}\|.$$

Note now that $\|I_{k, K}\| \leq e^k$, since, using $\varphi \geq -1$ on K , we have

$$\|I_{k, K} f\|_{L^2_{k\varphi}(K)}^2 = \int_K |f|^2 e^{-2k\varphi} dm \leq e^{2k} \int_K |f|^2 dm = e^{2k} \|f\|_{L^2_0(K)}^2.$$

Furthermore, $\|I_{k, D}^{-1}\| \leq 1$, since, using $\varphi \leq 0$ on D , we have

$$\|I_{k, D}^{-1} f\|_{H^2_0(D)}^2 = \int_D |f|^2 dm \leq \int_D |f|^2 e^{-2k\varphi} dm = \|f\|_{H^2_{k\varphi}(D)}^2,$$

and the assertion follows. $\blacksquare$

Combining the previous two lemmas with Proposition 2.5 we obtain the following result.

Proposition 2.8. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and K a compact subset of D with $m(K) > 0$. If φ is strictly psh and C^∞ on D with $\varphi \leq 0$ on D and $\varphi \geq -1$ on K , then*

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(J_{0, K, D})}{m^{1/n}} \leq 2\pi \left(\frac{n!}{\int_K (dd^c \varphi)^n} \right)^{1/n}.$$

Proof. We start by observing that $\int_K (dd^c \varphi)^n > 0$. Let now χ be the indicator function of K and fix $\gamma \in (0, 1)$. Using Lemmas 2.6 and 2.7 it follows that

$$e^{2k} d_m(J_{0,K,D})^2 \geq d_m(J_{k\varphi,K,D})^2 = \lambda_m(T_{\chi,k\varphi}) \quad (\forall k, m \in \mathbb{N}),$$

hence

$$\#\{m \in \mathbb{N} : d_m(J_{0,K,D}) > \gamma e^{-k}\} \geq \#\{m \in \mathbb{N} : \lambda_m(T_{\chi,k\varphi}) > \gamma^2\}.$$

Applying Proposition 2.5 to the above we see that

$$\liminf_{k \rightarrow \infty} \#\{m \in \mathbb{N} : d_m(J_{0,K,D}) > \gamma e^{-k}\} \cdot k^{-n} \geq C, \quad (20)$$

where

$$C = \frac{1}{(2\pi)^n n!} \int_K (dd^c \varphi)^n.$$

Write d_m for $d_m(J_{0,K,D})$ and fix $C' \in (0, C)$. Using (20) it follows that there is $k' \in \mathbb{N}$ such that

$$\#\{m \in \mathbb{N} : d_m > \gamma e^{-k}\} \geq C' k^n \quad (\forall k \geq k'),$$

so

$$d_{\lfloor k^n C' \rfloor} \geq \gamma e^{-k} \quad (\forall k \geq k'), \quad (21)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. After possibly enlarging k' we shall assume that $(k')^n C' \geq 1$. Now, for any $m \geq \lfloor (k')^n C' \rfloor$ choose $k_m \geq k'$ to be the largest $k \in \mathbb{N}$ such that

$$\lfloor k^n C' \rfloor \leq m < \lfloor (k+1)^n C' \rfloor.$$

Then $\lim_{m \rightarrow \infty} k_m = \infty$ and we have, for every $m \geq \lfloor (k')^n C' \rfloor$,

$$d_m \geq d_{\lfloor (k_m+1)^n C' \rfloor} \geq \gamma e^{-(k_m+1)},$$

which follows from (21) together with the observation that $k_m \geq k'$, and

$$m \geq \lfloor k_m^n C' \rfloor > k_m^n C' - 1,$$

and so

$$\frac{-\log d_m}{m^{1/n}} \leq \frac{k_m + 1 - \log \gamma}{(k_m^n C' - 1)^{1/n}}.$$

This implies

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m}{m^{1/n}} \leq \frac{1}{(C')^{1/n}}.$$

But since $C' < C$ was arbitrary, the assertion follows. ■

2.3. Sharp lower bounds for Kolmogorov widths with respect to L^2 -topologies

The bounds for the Kolmogorov numbers for the natural embedding $H^2(D) \hookrightarrow L^2(K)$ for a bounded pseudoconvex domain D in $\mathbb{C}^n$ and K a compact subset of D with positive Lebesgue measure were obtained under the additional assumption that φ be strictly psh

and smooth on D . In order to make the connection with the relative capacity $C(K, D)$ we want to choose φ in Proposition 2.8 to be the upper semicontinuous regularisation $u_{K,D}^*$ of the relative extremal function $u_{K,D}$. However, even if D is bounded and hyperconvex, and K is regular in D , then $u_{K,D} = u_{K,D}^*$ is, in general, merely continuous on the closure $\overline{D}$ and not necessarily smooth there. Nevertheless, using results of Cegrell [11], we will be able to approximate $u_{K,D}^*$ by smooth functions with the desired properties.

In order to state these results we need some more notation. Let $\text{PSH}^-(D)$ denote the set of non-positive psh functions on D and $\mathcal{E}_0(D)$ the class of bounded psh functions ψ such that $\lim_{z \rightarrow \xi} \psi(z) = 0$ for all $\xi \in \partial D$ and $\int_D (dd^c \psi)^n < +\infty$. Note that if a function $\varphi \in \mathcal{E}_0(D)$ is continuous in D , then φ is actually continuous up to the boundary.

Theorem 2.9 ([11]). *Let D be a hyperconvex domain in $\mathbb{C}^n$. For every $u \in \text{PSH}^-(D)$, there is a decreasing sequence $(\psi_j)_{j \in \mathbb{N}}$ of functions in $\mathcal{E}_0 \cap C^\infty(D)$ such that $\psi_j \rightarrow u$ as $j \rightarrow \infty$, pointwise in D .*

Corollary 2.10 ([11]). *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. Then there is a strictly psh exhaustion function $\psi \in \mathcal{E}_0 \cap C^\infty(D)$ for D .*

Recall that for any compact subset K in an open set D , the relative capacity $C(K, D)$ is defined as follows:

$$C(K, D) = \sup \left\{ \int_K (dd^c \varphi)^n : \varphi \in \text{PSH}(D, (-1, 0)) \right\}.$$

Using the previous results we obtain the following characterisation of the relative capacity $C(K, D)$ in the case where D is a bounded hyperconvex domain.

Corollary 2.11. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ containing a compact subset K . Then*

$$C(K, D) = \sup \left\{ \int_K (dd^c \varphi)^n : \varphi \in \text{SPSH}(D, (-1, 0)) \cap C^\infty(D) \right\}.$$

Here, $\text{SPSH}(D, (-1, 0))$ denotes the collection of all strictly psh functions on D with values in $(-1, 0)$.

Proof. First, we approximate $u_{K,D}^*$ by a decreasing sequence $(\psi_j)_{j \in \mathbb{N}}$ of functions in $\mathcal{E}_0 \cap C^\infty(D)$ such that $\psi_j \rightarrow u_{K,D}^*$ pointwise on D as $j \rightarrow \infty$ (see Theorem 2.9). Since $u_{K,D}^* \geq -1$ in D , each function ψ_j satisfies $\psi_j \geq -1$ in D .

Secondly, let ψ be the function of Corollary 2.10 and replace each function ψ_j by the function

$$\varphi_{j,\epsilon} = \frac{\psi_j + \epsilon\psi}{1 + \epsilon M},$$

where $\epsilon > 0$ is small and $M > 0$ is chosen so that $-M \leq \psi \leq 0$ on $\overline{D}$, which is possible since ψ is bounded on $\overline{D}$. Each $\varphi_{j,\epsilon}$ is a strictly psh exhaustion function in $\mathcal{E}_0 \cap C^\infty(D)$ for D and $\varphi_{j,\epsilon} \geq -1$ in D .

Since, for fixed ϵ , the sequence $(\varphi_{j,\epsilon})_{j \in \mathbb{N}}$ decreases in D to $\frac{u_{K,D}^* + \epsilon\psi}{1 + \epsilon M}$, which is bounded in D , we have

$$\lim_{j \rightarrow \infty} (dd^c \varphi_{j,\epsilon})^n = \left(dd^c \left(\frac{u_{K,D}^* + \epsilon\psi}{1 + \epsilon M} \right) \right)^n,$$

in the sense of weak convergence of measures, and in particular

$$\lim_{j \rightarrow \infty} \int_K (dd^c \varphi_{j,\epsilon})^n = \int_K \left(dd^c \left(\frac{u_{K,D}^* + \epsilon\psi}{1 + \epsilon M} \right) \right)^n.$$

Moreover, since $\frac{u_{K,D}^* + \epsilon\psi}{1 + \epsilon M}$ converges uniformly to $u_{K,D}^*$ on $\overline{D}$ when $\epsilon \rightarrow 0$, we have

$$\lim_{\epsilon \rightarrow 0} \int_K \left(dd^c \left(\frac{u_{K,D}^* + \epsilon\psi}{1 + \epsilon M} \right) \right)^n = \int_K (dd^c u_{K,D}^*)^n = C(K, D). \quad \blacksquare$$

Combining the results of this and the previous subsections we now obtain the following sharp lower bound for the Kolmogorov numbers of the natural embedding $H^2(D) \hookrightarrow L^2(K)$.

Theorem 2.12. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ containing a compact subset K with positive Lebesgue measure. Then*

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(H^2(D) \hookrightarrow L^2(K))}{m^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

Proof. Follows from Proposition 2.8 and Corollary 2.11. ■

2.4. From L^2 to L^∞

So far we have obtained lower bounds for the Kolmogorov widths of holomorphic functions in spaces carrying L^2 -topologies. We shall now consider bounds with respect to spaces arising from supremum norm topologies, as in the original formulation of Kolmogorov's problem. We shall achieve this by using simple bounds for the Kolmogorov numbers coupled with approximation arguments.

We start with the following simple observation which allows us to change the topology on the target space.

Lemma 2.13. *Let D be a domain in $\mathbb{C}^n$ and K a compact subset of D with positive Lebesgue measure. Then*

$$d_m(H^2(D) \hookrightarrow A(K)) \geq \frac{1}{\sqrt{m(K)}} d_m(H^2(D) \hookrightarrow L^2(K)) \quad (\forall m \in \mathbb{N}).$$

Proof. Let

$$J_1 : H^2(D) \hookrightarrow A(K), \quad J_2 : A(K) \hookrightarrow L^2(K), \quad J : H^2(D) \hookrightarrow L^2(K)$$

denote the natural embeddings. Then $J = J_2 J_1$. Clearly, J_2 is bounded with $\|J_2\| \leq \sqrt{m(K)}$, so

$$d_m(J) = d_m(J_2 J_1) \leq \|J_2\| d_m(J_1) \quad (\forall m \in \mathbb{N}),$$

and the assertion follows. $\blacksquare$

In order to change the topology on the original space we use a similar argument.

Lemma 2.14. *Let D and D' be domains in $\mathbb{C}^n$ with $D \subset\subset D'$ and let K be a compact subset of D . Then there is a constant $C > 0$, depending on D and D' only, and not on m , such that*

$$d_m(H^\infty(D) \hookrightarrow A(K)) \geq C d_m(H^2(D') \hookrightarrow A(K)) \quad (\forall m \in \mathbb{N}).$$

Proof. Let

$$J_1 : H^2(D') \hookrightarrow H^\infty(D), \quad J_2 : H^\infty(D) \hookrightarrow A(K), \quad J : H^2(D') \hookrightarrow A(K)$$

denote the natural embeddings. Then $J = J_2 J_1$. Now J_1 is bounded by a standard result for Bergman spaces (see, for example, [23, Lemma 1.4.1]) and clearly non-zero, so

$$d_m(J) = d_m(J_2 J_1) \leq d_m(J_2) \|J_1\| \quad (\forall m \in \mathbb{N}),$$

and the assertion follows with $C = \|J_1\|^{-1}$. $\blacksquare$

Before stating and proving the main result of this subsection, we require one more result, a convergence lemma for strictly hyperconvex domains D . For such domains, using the same notation as in Subsection 1.1, it follows that there exists a bounded domain Ω and an exhaustion function $\varrho \in \text{PSH}(\Omega, (-\infty, 1)) \cap C(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$ and for all real numbers $c \in [0, 1]$, the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected. Given any integer $j \geq 1$, we now define

$$D_j = \{z \in \Omega : \varrho(z) < 1/j\}. \quad (22)$$

This decreasing sequence $(D_j)_j$ of bounded hyperconvex domains satisfies the following lemma which is a generalisation of Lemma 2.6 in [30].

Lemma 2.15. *Let D be a strictly hyperconvex domain in $\mathbb{C}^n$ and let K be a compact subset of D . If $(D_j)_j$ denotes the sequence of bounded hyperconvex domains defined in (22), then the increasing sequence $(u_{K, D_j}^*)_j$ converges quasi-everywhere in D to $u_{K, D}^*$ and the increasing sequence $(C(K, D_j))_j$ of capacities converges to the capacity $C(K, D)$.*

Proof. First we note that since $K \subset D \subset D_j$, we have $-1 \leq u_{K, D_j} \leq u_{K, D}$ and $u_{K, D_j}^* \leq u_{K, D}^* \leq 0$ in D as well as $C(K, D_j) \leq C(K, D)$ (see [6] or [17, p. 120]).

We now collect some simple properties satisfied by the functions u_{K,D_j} and $u_{K,D}$. There exists a positive constant c sufficiently large such that $cQ \leq -1$ in K and $c(Q - 1/j) \leq u_{K,D_j}$ in D_j , for any j . The sequence $(u_{K,D_j}^*)_j$ is increasing in D , since the sequence $(D_j)_j$ is decreasing. Let v be the function defined on D by

$$v := \lim_{j \rightarrow \infty} u_{K,D_j}^*.$$

Now v^* is psh in D . Moreover, v and v^* are equal quasi-everywhere in D (that is, they are equal except perhaps on a pluripolar set, see [6]). As $u_{K,D_j}^* \in \text{PSH}(D, (-1, 0))$, we have $v^* \in \text{PSH}(D, (-1, 0))$.

Since each u_{K,D_j}^* satisfies $u_{K,D_j}^* \leq u_{K,D}^*$ in D , it follows that v^* also satisfies $v^* \leq u_{K,D}^*$ in D .

Moreover, the sequence of positive measures $((dd^c u_{K,D_j}^*)^n)_j$ converges to the positive measure $(dd^c v^*)^n$ in the weak*-topology (see [6] or [17, p. 125]). In particular, since for any j , $(dd^c u_{K,D_j}^*)^n = 0$ in $D_j \setminus K$, we have $(dd^c v^*)^n = 0$ in $D \setminus K$.

Note that $v^*(w)$ converges to 0 when $w \rightarrow z$, for any $z \in \partial D$. Indeed, for any j we have

$$-c/j \leq \liminf_{w \rightarrow z} u_{K,D_j}^*(w) \leq \liminf_{w \rightarrow z} v^*(w) \leq \limsup_{w \rightarrow z} v^*(w) \leq 0.$$

Now recall that

$$C(K, D) = \int_K (dd^c u_{K,D}^*)^n = \int_D (dd^c u_{K,D}^*)^n$$

and that the same equalities are satisfied for $C(K, D_j)$ and u_{K,D_j}^* . Thus, using the Comparison Theorem of Bedford and Taylor (see [6] or [17, p. 126]) we deduce that

$$\begin{aligned} C(K, D) &= \int_D (dd^c u_{K,D}^*)^n \leq \int_D (dd^c v^*)^n = \int_K (dd^c v^*)^n \\ &= \lim_{j \rightarrow \infty} \int_K (dd^c u_{K,D_j}^*)^n = \lim_{j \rightarrow \infty} C(K, D_j) \leq C(K, D), \end{aligned}$$

and that v^* is identical to $u_{K,D}^*$ in D . Since $v = v^*$ quasi-everywhere in D , the proof is complete. $\blacksquare$

The following is the main result of this subsection, a sharp lower bound for the asymptotics of the Kolmogorov numbers $d_m(H^\infty(D) \hookrightarrow A(K))$ under the hypotheses of Theorem 2.12, except that we require a bit more regularity for D , namely that D be not just bounded and hyperconvex but strictly hyperconvex.

Theorem 2.16. *Let D be a strictly hyperconvex domain in $\mathbb{C}^n$ containing a compact subset K with positive Lebesgue measure. Then*

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(H^\infty(D) \hookrightarrow A(K))}{m^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

Proof. Combining Lemmas 2.13 and 2.14, applied to $K \subset D \subset D_j$ with D_j defined in (22), we obtain, for any j ,

$$\begin{aligned} d_m(H^\infty(D) \hookrightarrow A(K)) &\geq C_j d_m(H^2(D_j) \hookrightarrow A(K)) \\ &\geq \frac{C_j}{\sqrt{m(K)}} d_m(H^2(D_j) \hookrightarrow L^2(K)), \end{aligned}$$

where $C_j = \|J_{H^2(D_j) \hookrightarrow H^\infty(D)}\|^{-1}$.

Applying Theorem 2.12 to bound $d_m(H^2(D_j) \hookrightarrow L^2(K))$ from below, we deduce, for any j ,

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(H^\infty(D) \hookrightarrow A(K))}{m^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D_j)} \right)^{1/n},$$

and the assertion now follows from Lemma 2.15. ■

2.5. Generalisation to non-pluripolar K

In the previous subsection we have obtained lower bounds for the Kolmogorov numbers of the embedding $J : H^\infty(D) \hookrightarrow A(K)$ for D strictly hyperconvex and $K \subset D$ compact with positive Lebesgue measure. We shall now explain how to obtain Theorem 2.16 in the more general case where the compact set K is only assumed to be non-pluripolar, thus finishing the proof of the first half of our main theorem (Theorem 1.1). We shall achieve this by choosing suitable external approximations of the holomorphically convex hull of K in D .

We start with the following simple observation. If D is hyperconvex (but not necessarily bounded) and K is a non-pluripolar compact subset of D , which we do not assume to be holomorphically convex in D , then the relative extremal function $u_{K,D}$ is lower semicontinuous on D [17, Corollary 4.5.11]. Thus, the upper level sets $\{z \in D : u_{K,D}(z) > -1 + c\}$ are open in D for any real number $0 \leq c < 1$. Since, for any $w \in \partial D$, the relative extremal function $u_{K,D}(z)$ tends to 0 as z tends to w , it follows that, for any real number $0 \leq c < 1$,

$$K_c = \{z \in D : u_{K,D}(z) \leq -1 + c\} \quad (23)$$

is a compact subset of D , which, for $0 < c < 1$, has positive Lebesgue measure, since it contains the open set $\{z \in D : u_{K,D}^* < -1 + c\}$.

We also note that

$$\bigcap_{0 < c < 1} K_c = \hat{K}_D,$$

where $\hat{K}_D$ is the holomorphically convex hull of K in the open hyperconvex set D . Moreover, $u_{K,D} = u_{\hat{K}_D,D}$ and $C(K, D) = C(\hat{K}_D, D)$.

Lemma 2.17. *Let D be a hyperconvex domain in $\mathbb{C}^n$ and let K be a non-pluripolar compact subset of D . Then, for any real number $0 < c < 1$, we have*

$$\max \left\{ \frac{u_{K,D}^*}{1-c}, -1 \right\} = u_{K_c,D}^* \quad \text{on } D$$

and

$$C(K_c, D) = \frac{C(K, D)}{(1-c)^n},$$

where K_c is given by (23).

Proof. We start by observing that the relative extremal function $u_{K_c,D}$ for K_c in D satisfies

$$\max \left\{ \frac{u_{K,D}}{1-c}, -1 \right\} \leq u_{K_c,D} \quad \text{on } D. \quad (24)$$

Indeed, any psh function w appearing in the definition of $u_{K,D}$ satisfies $w \leq u_{K,D}$ on D , so $w \leq -1 + c$ on K_c . Thus $\max \left\{ \frac{w}{1-c}, -1 \right\}$ is a negative psh function on D which is less than or equal to -1 on K_c , from which, using the definition of $u_{K_c,D}$, we conclude that $\max \left\{ \frac{w}{1-c}, -1 \right\} \leq u_{K_c,D}$ on D , and (24) follows.

Let v_c denote the following psh function on D :

$$v_c = \max \left\{ \frac{u_{K,D}^*}{1-c}, -1 \right\}.$$

Now, v_c is a negative psh function on D , which is greater than or equal to -1 on $D \setminus K_c$. Indeed, on $D \setminus K_c$, we have $u_{K,D} > -1 + c$ and thus $u_{K,D}^* > -1 + c$ as well. In addition, v_c tends to 0 as z tends to w , for any $w \in \partial D$.

Since $(dd^c u_{K,D}^*)^n = 0$ on $D \setminus K$, we also have $(dd^c v_c)^n = 0$ on $D \setminus K_c$. By the maximality of the function v_c , we deduce that $v_c \geq w$ in $D \setminus K_c$ for any psh function w which appears in the definition of $u_{K_c,D}$. Thus $v_c \geq u_{K_c,D}$ and even $v_c \geq u_{K_c,D}^*$ on $D \setminus K_c$, since v_c is psh.

Combining the bounds in the previous two paragraphs we deduce that

$$v_c = u_{K_c,D}^* \quad \text{on } D \setminus K_c.$$

Summarising, we have

$$\max \left\{ \frac{u_{K,D}^*}{1-c}, -1 \right\} = u_{K_c,D}^* \quad \text{on } D \setminus K_c$$

and

$$\max \left\{ \frac{u_{K,D}}{1-c}, -1 \right\} = u_{K_c,D} = -1 \quad \text{on } K_c.$$

We know that $u_{K,D} = u_{K,D}^*$ and $u_{K_c,D} = u_{K_c,D}^*$ except for a pluripolar set in D . As two psh functions which are equal except for a pluripolar set are in fact equal everywhere, we finally deduce that

$$\max \left\{ \frac{u_{K,D}^*}{1-c}, -1 \right\} = u_{K_c,D}^* \quad \text{on } D,$$

and the proof is complete. ■

We are finally able to prove the main result of this section, the first half of our main theorem.

Theorem 2.18. *Let D be a strictly hyperconvex domain in $\mathbb{C}^n$ and K a non-pluripolar compact subset of D . Then*

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(H^\infty(D) \hookrightarrow A(K))}{m^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

Proof. Fix c with $0 < c < 1$. Let K_c be as in (23) and define the canonical embeddings

$$J : H^\infty(D) \hookrightarrow A(K), \quad J_c : H^\infty(D) \hookrightarrow A(K_c).$$

We start by observing that the Two Constants Theorem (see, for example, [17, Proposition 4.5.6]) implies that for any $f \in H^\infty(D)$ with $\|f\|_{H^\infty(D)} \leq 1$ we have

$$\|f\|_{A(K_c)} \leq \|f\|_{A(K)}^{1-c},$$

which implies the following relation between the Gelfand numbers of the embeddings J and J_c :

$$c_m(J_c) \leq c_m(J)^{1-c} \quad (m \in \mathbb{N}).$$

Thus

$$\limsup_{m \rightarrow \infty} \frac{-\log c_m(J)}{m^{1/n}} \leq \limsup_{m \rightarrow \infty} \frac{1}{1-c} \frac{-\log c_m(J_c)}{m^{1/n}},$$

and so, using (4) and (5), it follows that

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(J)}{m^{1/n}} \leq \limsup_{m \rightarrow \infty} \frac{1}{1-c} \frac{-\log d_m(J_c)}{m^{1/n}}.$$

Now, since K_c has positive Lebesgue measure, Theorem 2.16 implies

$$\limsup_{m \rightarrow \infty} \frac{1}{1-c} \frac{-\log d_m(J_c)}{m^{1/n}} \leq \frac{2\pi}{1-c} \left(\frac{n!}{C(K_c, D)} \right)^{1/n},$$

while Lemma 2.17 gives

$$\frac{2\pi}{1-c} \left(\frac{n!}{C(K_c, D)} \right)^{1/n} = 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

Thus, all in all, we have

$$\limsup_{m \rightarrow \infty} \frac{-\log d_m(J)}{m^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n},$$

and the proof is finished. ■

3. Upper bound for the Kolmogorov widths

Our strategy for determining sharp upper bounds for the Kolmogorov widths is based on the Bergman–Weil formula coupled with an approximation argument. More precisely, we shall start by letting K and D be special holomorphic polyhedra in $\mathbb{C}^n$ (a notion we shall recall below), in which case sharp upper bounds for the Kolmogorov numbers $d_n(H^\infty(D) \hookrightarrow A(K))$ can be obtained using the Bergman–Weil integral formula. Next, for any pair (K, D) , where K is a regular compact subset of a strictly hyperconvex domain D , we will simultaneously approximate $\hat{K}_D$ externally and D internally by two special holomorphic polyhedra defined by the same holomorphic mapping in such a way that the relative capacity of the approximations will converge to the relative capacity $C(K, D)$. This will allow us to deduce a sharp upper bound for the Kolmogorov widths in the case where K is a regular compact subset in a strictly hyperconvex domain D . From this, we will deduce an upper bound for the Kolmogorov widths of a general pair (K, D) , also termed a *condenser* in this section, where K is any compact subset of a bounded and hyperconvex domain D in $\mathbb{C}^n$.

3.1. Upper bounds in case K and D are special holomorphic polyhedra

We start by recalling the notions of holomorphic polyhedron and special holomorphic polyhedron. Let Ω be an open subset of $\mathbb{C}^n$ and N a positive integer. A *holomorphic polyhedron* of type N in Ω is a finite union of relatively compact connected components of the subset of Ω of the form

$$\{z \in \Omega : |f_j(z)| < 1 \text{ for all } j \in \{1, \dots, N\}\},$$

where each $f_j : \Omega \rightarrow \mathbb{C}$ is holomorphic.

Clearly, a holomorphic polyhedron of type N is also of type $N + 1$, so the minimal type of a given polyhedron is of particular interest. Note that if Ω is a holomorphically convex domain in $\mathbb{C}^n$ and $\mathcal{P}$ a holomorphic polyhedron of type N in Ω then $N \geq n$. Thus, in this case there is a non-trivial lower bound for the type of a holomorphic polyhedron, and polyhedra of this minimal type play a special role: a holomorphic polyhedron of type n in a holomorphically convex domain Ω in $\mathbb{C}^n$ is called a *special holomorphic polyhedron* (see Bishop [8]).

For the rest of this subsection we shall focus on condensers of the form $(K, D) = (\overline{\mathcal{U}_b}, \mathcal{U}_a)$, where both $\mathcal{U}_a$ and $\mathcal{U}_b$ are special holomorphic polyhedra obtained as follows.

Let $F = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{C}^n$ denote a holomorphic mapping on a pseudoconvex open set Ω in $\mathbb{C}^n$. For $a \in (0, \infty)^n$ write

$$\mathcal{P}_a = \{z \in \Omega : |f_k(z)| < a_k \text{ for all } k \in \{1, \dots, n\}\},$$

and assume that the open set $\mathcal{P}_a$ has relatively compact connected components. We now define the special holomorphic polyhedron $\mathcal{U}_a$ to be a finite union of such components.

Given $b \in (0, \infty)^n$ with $b_k < a_k$ for $k \in \{1, \dots, n\}$ we define a second special holomorphic polyhedron $\mathcal{U}_b$ as the finite union of connected components of the open set

$$\mathcal{P}_b = \{z \in \Omega : |f_k(z)| < b_k \text{ for all } k \in \{1, \dots, n\}\},$$

which belong to $\mathcal{U}_a$.

In this case we shall call the condenser $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$ a *special holomorphic polyhedral condenser* and F the *underlying mapping*.

Note that the underlying mapping F is finite and proper from $\mathcal{U}_a$ onto the polydisc $P(O, a) := \{z \in \mathbb{C}^n : |z_j| < a_j, 1 \leq j \leq n\}$. Moreover, $F|_{\mathcal{U}_a}$ is an unramified covering over the open set $P(O, a)$. We shall refer to the number of sheets of this covering as the *multiplicity* of F .

It turns out that for special holomorphic polyhedral condensers there is a simple formula for the corresponding relative capacity.

Proposition 3.1. *Let $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$ be a special holomorphic polyhedral condenser with underlying mapping $F = (f_1, \dots, f_n)$ in $\mathbb{C}^n$. Then the corresponding relative extremal function is given explicitly by*

$$u_{\overline{\mathcal{U}_b}, \mathcal{U}_a}(z) = u_{\overline{P(O, b)}, P(O, a)}(F(z)) = \sup_{1 \leq k \leq n} \frac{\log(|f_k(z)|/a_k)}{\log(a_k/b_k)},$$

and its relative capacity is

$$C(\overline{\mathcal{U}_b}, \mathcal{U}_a) = \frac{(2\pi)^n m_0}{\prod_{k=1}^n \log(a_k/b_k)},$$

where m_0 is the multiplicity of F .

Proof. This result is an easy consequence of [17, Proposition 4.5.14] and [30, Lemma 4.1], since the holomorphic mapping F is proper and surjective from $\mathcal{U}_a$ onto the open polydisc $P(O, a)$, and from $\mathcal{U}_b$ onto the open polydisc $P(O, b)$, respectively. ■

3.1.1. A sequence of finite rank operators (J_m) approximating J . Let us recall that in order to find upper bounds for the Kolmogorov widths $d_m(\mathcal{A}_K^D)$, it suffices to provide upper bounds for the approximation numbers $a_m(J)$ of the canonical identification

$$J : H^\infty(D) \rightarrow A(K), \quad f \mapsto Jf = f|_K.$$

As detailed in Subsection 1.3, this follows since

$$d_m(\mathcal{A}_K^D) = d_m(J) \leq a_m(J) \quad (\forall m \in \mathbb{N}).$$

Specialising to the case where $(K, D) = (\overline{\mathcal{U}_b}, \mathcal{U}_a)$ is a special holomorphic polyhedral condenser we shall now construct a sequence of finite rank operators $J_m : H^\infty(D) \rightarrow A(K)$ which approximate J at a certain stretched exponential speed. The main tool will be the Bergman–Weil integral formula originally due to Weil [38] and Bergman [7], which we turn to shortly.

Before doing so, we briefly recall multi-index notation. For $z \in \mathbb{C}^n$ and $\nu \in \mathbb{Z}^n$ with $z = (z_1, \dots, z_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ we write $z^\nu = \prod_{k=1}^n z_k^{\nu_k}$. We use the symbol $\mathcal{J}$ for $(1, \dots, 1)$, so that $z^{\mathcal{J}} = \prod_{k=1}^n z_k$.

Suppose now that $F = (f_1, \dots, f_n)$ is the underlying mapping of the special holomorphic polyhedral condenser $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$, that is, $F : \Omega \rightarrow \mathbb{C}^n$ is a holomorphic mapping

on a pseudoconvex open set Ω in $\mathbb{C}^n$ containing $\mathcal{U}_a$ and $\overline{\mathcal{U}_b}$. It turns out that there is a holomorphic function $G : \Omega \times \Omega \rightarrow \mathbb{C}^{n \times n}$ such that

$$f_i(\zeta) - f_i(z) = \sum_{j=1}^n G_{ij}(\zeta, z)(\zeta_j - z_j) \quad (\forall i \in \{1, \dots, n\}, \forall \zeta, z \in \Omega).$$

The existence of G for pseudoconvex Ω is a non-trivial matter and is originally due to Hefer [14] (see also [33, Paragraphs 30 and 50]).

If $\partial^* \mathcal{U}_a$ denotes the distinguished boundary of $\mathcal{U}_a$, that is,

$$\partial^* \mathcal{U}_a = \{z \in \Omega : |f_k(z)| = a_k \text{ for all } k \in \{1, \dots, n\}\},$$

considered as an n -dimensional surface with a suitable orientation, then the Bergman–Weil integral formula on $\mathcal{U}_a$ can be stated as follows (see [33, Paragraph 30]): for any $g \in H^\infty(\mathcal{U}_a)$ and any $z \in \mathcal{U}_a$ we have

$$g(z) = \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{(F(\zeta) - F(z))^J} d\zeta,$$

where $\det(G(\zeta, z))$ is the determinant of the square matrix $G(\zeta, z) = (G(\zeta, z)_{ij})_{1 \leq i, j \leq n}$ and $d\zeta$ is the n -form $d\zeta_1 \wedge \dots \wedge d\zeta_n$. Here, the star at the top of the integral sign indicates that integration is to be taken over any $\partial^* \mathcal{U}_{a'}$ where $a' \in (0, \infty)^n$ with $a'_k < a_k$ for any k , and a' is chosen such that $z \in \mathcal{U}_{a'}$. It is not difficult to see that the integral $\int_{\partial^* \mathcal{U}_a}^*$ does not depend on this choice. Note that if $g \in A(\overline{\mathcal{U}_a})$ then $\int_{\partial^* \mathcal{U}_a}^*$ is in fact the classical integration over $\partial^* \mathcal{U}_a$.

An important property of the Bergman–Weil integral representation is that its kernel is holomorphic in z . This implies that we can write the canonical identification $J : H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})$ as an infinite series of operators, all of which, as we shall see later, are finite rank. More precisely, the following holds. For $g \in H^\infty(\mathcal{U}_a)$ and $z \in \overline{\mathcal{U}_b}$, we have

$$\begin{aligned} (Jg)(z) &= g(z) = \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{(F(\zeta) - F(z))^J} d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^J} \sum_{l=1}^{\infty} \frac{F(z)^{v(l)}}{F(\zeta)^{v(l)}} d\zeta \\ &= \sum_{l=1}^{\infty} F(z)^{v(l)} \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^{J+v(l)}} d\zeta, \end{aligned} \quad (25)$$

where $v : \mathbb{N} \rightarrow \mathbb{N}_0^n$ could, in principle, be any bijection, but we shall fix it so as to facilitate obtaining effective bounds for the approximation numbers of $J : H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})$.

In order to achieve this, let $\alpha \in (0, 1)^n$ be given by

$$\alpha_k = b_k/a_k \quad \text{for } k \in \{1, \dots, n\}.$$

Now choose $\nu : \mathbb{N} \rightarrow \mathbb{N}_0^n$ so that $m \mapsto \gamma_m := \alpha^{\nu(m)}$ is decreasing. In other words, the bijection ν is chosen to provide a non-increasing rearrangement $(\gamma_m)_{m \in \mathbb{N}}$ of the set $\{\alpha^\nu : \nu \in \mathbb{N}_0^n\}$.

As a first step towards bounding the approximation numbers $a_m(J)$ we need to bound the rate of decay of $(\gamma_m)_{m \in \mathbb{N}}$. For this we require the following auxiliary result, the short proof of which is adapted from [9, proof of Lemma 2.4].

Lemma 3.2. *Let $\beta \in (0, \infty)^n$ and let $N_\beta : [0, \infty) \rightarrow \mathbb{N}_0$ denote the counting function*

$$N_\beta(r) = \#\left\{\nu \in \mathbb{N}_0^n : \sum_{k=1}^n \nu_k \beta_k \leq r\right\}.$$

Then

$$\frac{1}{n!} \frac{r^n}{\prod_{k=1}^n \beta_k} \leq N_\beta(r) \leq \frac{1}{n!} \frac{(r + \sum_{k=1}^n \beta_k)^n}{\prod_{k=1}^n \beta_k} \quad (\forall r \geq 0). \quad (26)$$

Proof. Fix $r \geq 0$ and define the sets

$$\begin{aligned} \mathcal{N}_\beta(r) &= \left\{\nu \in \mathbb{N}_0^n : \sum_{k=1}^n \nu_k \beta_k \leq r\right\}, \\ \mathcal{S}_\beta(r) &= \left\{\xi \in [0, \infty)^n : \sum_{k=1}^n \xi_k \beta_k \leq r\right\}, \\ \mathcal{C}_\nu &= \{\xi \in \mathbb{R}^n : \nu_k \leq \xi_k \leq \nu_k + 1 \ \forall k \in \{1, \dots, n\}\} \quad (\nu \in \mathbb{N}_0^n), \\ \mathcal{M}_\beta(r) &= \bigcup_{\nu \in \mathcal{N}_\beta(r)} \mathcal{C}_\nu. \end{aligned}$$

It is not difficult to see that

$$\mathcal{S}_\beta(r) \subset \mathcal{M}_\beta(r) \subset \mathcal{S}_\beta\left(r + \sum_{k=1}^n \beta_k\right),$$

from which the inequalities (26) readily follow, by computing the volumes of the respective sets. ■

Remark 3.3. The lemma above implies that

$$N_\beta(r) \sim \frac{1}{n!} \frac{r^n}{\prod_{k=1}^n \beta_k} \quad \text{as } r \rightarrow \infty.$$

This asymptotics also follows easily from Karamata's Tauberian theorem (see, for example, [26, Lemma 6.1]). The lemma above, however, provides completely explicit bounds valid for all $r \geq 0$.

We now have the following upper bound for $(\gamma_n)_{n \in \mathbb{N}}$.

Lemma 3.4. *Let $\alpha \in (0, \infty)^n$ and let $(\gamma_m)_{m \in \mathbb{N}}$ denote a non-increasing rearrangement of the set $\{\alpha^v : v \in \mathbb{N}_0^n\}$. Writing*

$$c = n! \prod_{k=1}^n \log \alpha_k^{-1}$$

we have

$$\begin{aligned} \gamma_m &\leq \frac{1}{\prod_{k=1}^n \alpha_k} \exp(-(cm)^{1/n}) \quad (\forall m \in \mathbb{N}), \\ \sum_{l=m+1}^{\infty} \gamma_l &\leq \frac{1}{\prod_{k=1}^n \alpha_k \log \alpha_k^{-1}} \sum_{k=0}^{n-1} \frac{(cm)^{k/n}}{k!} \exp(-(cm)^{1/n}) \quad (\forall m \in \mathbb{N}_0). \end{aligned}$$

Proof. We start by observing that

$$m \leq \#\{v \in \mathbb{N}_0^n : \alpha^v \geq \gamma_m\} = N_\beta(\log \gamma_m^{-1}) \quad (\forall m \in \mathbb{N}),$$

where N_β is the counting function from Lemma 3.2 with $\beta \in (0, +\infty)^n$ given by $\beta_k = \log \alpha_k^{-1}$ for $1 \leq k \leq n$. The first bound now follows from the upper bound in Lemma 3.2.

For the second bound we use the first bound together with a majorisation of the sum by an integral to obtain

$$\begin{aligned} \sum_{l=m+1}^{\infty} \gamma_l &\leq \frac{1}{\prod_{k=1}^n \alpha_k} \sum_{l=m+1}^{\infty} \exp(-(cl)^{1/n}) \\ &\leq \frac{1}{\prod_{k=1}^n \alpha_k} \int_m^{\infty} \exp(-(ct)^{1/n}) dt \\ &= \frac{1}{\prod_{k=1}^n \alpha_k} \left[-\frac{n!}{c} \sum_{k=0}^{n-1} \frac{(ct)^{k/n}}{k!} \exp(-(ct)^{1/n}) \right]_m^{\infty} \\ &= \frac{1}{\prod_{k=1}^n \alpha_k \log \alpha_k^{-1}} \sum_{k=0}^{n-1} \frac{(cm)^{k/n}}{k!} \exp(-(cm)^{1/n}). \quad \blacksquare \end{aligned}$$

We are now ready to define the sequence of finite rank operators $J_m : H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})$ alluded to earlier. For $m \in \mathbb{N}$ and $z \in \overline{\mathcal{U}_b}$ write

$$\begin{aligned} (J_m g)(z) &= \sum_{l=1}^m F(z)^{v(l)} \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^{J+v(l)}} d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^J} \sum_{l=1}^m \frac{F(z)^{v(l)}}{F(\zeta)^{v(l)}} d\zeta. \end{aligned} \quad (27)$$

Clearly, J_m is a well-defined operator from $H^\infty(\mathcal{U}_a)$ to $A(\overline{\mathcal{U}_b})$. Moreover, we have the following upper bound for the rate at which the sequence $(J_m)_{m \in \mathbb{N}}$ approximates J .

Lemma 3.5. *Let $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$ be a special holomorphic polyhedral condenser in $\mathbb{C}^n$ with underlying mapping F . Then*

$$\|J - J_m\|_{H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})} \leq C \left(\sum_{k=0}^{n-1} \frac{(cm)^{k/n}}{k!} \right) \exp(-(cm)^{1/n}) \quad (\forall m \in \mathbb{N}),$$

where

$$c = n! \prod_{k=1}^n \log \alpha_k^{-1},$$

$$C = \left[(2\pi)^n \prod_{k=1}^n \alpha_k \log \alpha_k^{-1} \right]^{-1} \sup_{z \in \overline{\mathcal{U}_b}} \int_{\partial^*(\mathcal{U}_a)} \frac{|\det(G(\zeta, z))|}{|F(\zeta)^J|} |d\zeta|, \quad (28)$$

and, as before, $\alpha_k = b_k/a_k$ for $1 \leq k \leq n$.

Proof. Fix $z \in \overline{\mathcal{U}_b}$. Recall that the integration $\int_{\partial^* \mathcal{U}_a}^*$ in (27) is in fact an integration over any $\partial^* \mathcal{U}_{a'}$ where $a' \in (0, \infty)^n$ with $b_k < a'_k < a_k$ for any k is chosen so that $z \in \mathcal{U}_{a'}$. Thus, for $g \in H^\infty(\mathcal{U}_a)$ with $\|g\|_{H^\infty(\mathcal{U}_a)} \leq 1$, we have, using the calculation in (25),

$$\begin{aligned} & |(Jg - J_m g)(z)| \\ & \leq \left(\frac{1}{(2\pi)^n} \int_{\partial^*(\mathcal{U}_{a'})} \frac{|\det(G(\zeta, z))|}{|F(\zeta)^J|} |d\zeta| \right) \sum_{l=m+1}^{\infty} \left(\frac{b_1}{a'_1} \right)^{v_1(l)} \cdots \left(\frac{b_n}{a'_n} \right)^{v_n(l)}. \end{aligned}$$

Since there exists $\delta > 0$ such that $b_k < b_k + \delta \leq a'_k < a_k$ for any k , the power series $\sum_{l=m+1}^{\infty} (b_1/a'_1)^{v_1(l)} \cdots (b_n/a'_n)^{v_n(l)}$ converges to $\sum_{l=m+1}^{\infty} \gamma_l$ when a' tends to a . Consequently,

$$|(Jg - J_m g)(z)| \leq \left(\frac{1}{(2\pi)^n} \int_{\partial^*(\mathcal{U}_a)} \frac{|\det(G(\zeta, z))|}{|F(\zeta)^J|} |d\zeta| \right) \sum_{l=m+1}^{\infty} \gamma_l \quad (\forall z \in \overline{\mathcal{U}_b}),$$

where we have used the fact that

$$a' \mapsto \frac{1}{(2\pi)^n} \int_{\partial^*(\mathcal{U}_{a'})} \frac{|\det(G(\zeta, z))|}{|F(\zeta)^J|} |d\zeta|$$

is continuous at a . Thus using Lemma 3.4 we have, for any $m \in \mathbb{N}$,

$$\|J - J_m\|_{H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})} \leq C \left(\sum_{k=0}^{n-1} \frac{(cm)^{k/n}}{k!} \right) \exp(-(cm)^{1/n}), \quad (29)$$

as claimed. ■

3.1.2. Upper bound for the rank of J_m . For $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$ a special holomorphic polyhedral condenser with underlying mapping F we want to use Lemma 3.5 to obtain an upper bound for the approximation numbers of $J : H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})$. For this we need to impose an extra assumption on F , and hence on the condenser: we shall call $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$

non-degenerate if O is a regular value of the underlying proper mapping F from $\mathcal{U}_a$ onto $P(O, a)$. Since F has finite multiplicity m_0 , this implies that $F|_{\mathcal{U}_a}$ has exactly m_0 distinct zeros.

Note that this is no essential restriction for our purposes, since if O is not a regular value of F , then we can replace F by $F - c$, where c is a regular value for F , which can be chosen arbitrarily small.

Lemma 3.6. *Let $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$ be a non-degenerate special holomorphic polyhedral condenser with underlying mapping F , and J_m the operator defined in (27). Then*

$$\text{rank}(J_m) \leq m_0 m \quad (\forall m \in \mathbb{N}),$$

where m_0 is the multiplicity of F .

Proof. Since the condenser is non-degenerate, the underlying mapping F has exactly m_0 distinct zeros, call them $z^{(1)}, \dots, z^{(m_0)}$. Next choose $c = (c_1, \dots, c_n)$ where each c_j is a positive real number small enough so that $\mathcal{U}_c$ has m_0 connected components $\mathcal{U}_c^k$, the closures of which are pairwise disjoint.

Now fix $z \in \mathcal{U}_c$ and let $\mathcal{U}_c^{k_0}$ denote the neighbourhood of $z^{(k_0)}$ to which z belongs. For any $m \in \mathbb{N}$ and any $g \in H^\infty(\mathcal{U}_a)$, we have

$$\begin{aligned} (J_m g)(z) &= \sum_{l=1}^m F(z)^{v(l)} \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_a}^* g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^{J+v(l)}} d\zeta \\ &= \sum_{k=1}^{m_0} \sum_{l=1}^m F(z)^{v(l)} \frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_c^k} g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^{J+v(l)}} d\zeta. \end{aligned}$$

Each integral $\int_{\partial^* \mathcal{U}_c^k} g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^{J+v(l)}} d\zeta$ is over a small neighbourhood of a zero $z^{(k)}$ of F . Since $z \in \mathcal{U}_c^{k_0} \subset \overline{\mathcal{U}_c^{k_0}}$, which is disjoint from $\overline{\mathcal{U}_c^k}$ for any $k \neq k_0$, we have

$$\int_{\partial^* \mathcal{U}_c^k} g(\zeta) \frac{\det(G(\zeta, z))}{(F(\zeta) - F(z))^J} d\zeta = 0 \quad (k \neq k_0).$$

For any $1 \leq k \leq m_0$, the underlying mapping F is a biholomorphism from $\mathcal{U}_c^k$, a neighbourhood of $z^{(k)}$, onto a neighbourhood V_k of the origin. Let π_k denote the corresponding inverse mapping, so that $F(\zeta) = w$ implies $\pi_k(w) = \zeta$. Then for any $l \geq 1$ and for any $1 \leq k \leq m_0$, we have

$$\begin{aligned} &\frac{1}{(2\pi i)^n} \int_{\partial^* \mathcal{U}_c^k} g(\zeta) \frac{\det(G(\zeta, z))}{F(\zeta)^{J+v(l)}} d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_{\partial^* P(O, c)} g(\pi_k(w)) \frac{\det(G(\pi_k(w), z))}{w^{J+v(l)}} (\text{Jac } \pi_k)(w) dw \\ &= \frac{1}{v(l)!} \frac{\partial^{v(l)}}{\partial w^{v(l)}} (g \circ \pi_k \cdot \det(G(\pi_k, z)) \cdot \text{Jac } \pi_k)(0), \end{aligned}$$

where $\text{Jac } \pi_k$ is the complex Jacobian of the holomorphic map π_k :

$$\text{Jac } \pi_k = \det \left(\frac{\partial \pi_{k,j}}{\partial w_{j'}} \right)_{1 \leq j, j' \leq n}.$$

Consequently, for any $z \in \mathcal{U}_c$,

$$(J_m g)(z) = \sum_{l=1}^m \sum_{k=1}^{m_0} \frac{1}{v(l)!} \frac{\partial^{v(l)}}{\partial w^{v(l)}} (g \circ \pi_k \cdot \det(G(\pi_k, z)) \cdot \text{Jac } \pi_k)(0) F(z)^{v(l)}.$$

But since the right hand side of this equality is a well defined holomorphic function on all of $\mathcal{U}_a$ and since $J_m g$ is in $\mathcal{A}(\overline{\mathcal{U}_b})$, the analytic continuation principle implies that the equality above holds for every $z \in \mathcal{U}_b$.

Using the product rule, it follows that the partial derivative

$$\frac{\partial^{v(l)}}{\partial w^{v(l)}} (g \circ \pi_k \cdot \det(G(\pi_k, z)) \cdot \text{Jac } \pi_k)(0)$$

involves only partial derivatives of g of the form $\frac{\partial^\mu g}{\partial \xi^\mu}(z^{(k)})$, where the multi-indices satisfy $0 \leq \mu_1 \leq v_1(l), \dots, 0 \leq \mu_n \leq v_n(l)$. Due to our choice of v , this implies that these multi-indices are of the form $v(l')$ with $1 \leq l' \leq m$.

As a result, the collection of values

$$\frac{\partial^{v(l)} g}{\partial \xi^{v(l)}}(z^{(k)}) \quad (1 \leq k \leq m_0, 1 \leq l \leq m)$$

completely determines the function $J_m(g)$ on $\overline{\mathcal{U}_b}$. Thus, for each $m \in \mathbb{N}$, the operator J_m has finite rank, bounded above by $m_0 m$. $\blacksquare$

3.1.3. Upper bound for Kolmogorov widths in the case of special holomorphic polyhedra. We are now able to state and prove the main result of this subsection: explicit and asymptotically sharp upper bounds for the Kolmogorov widths of special holomorphic polyhedral condensers.

Proposition 3.7. *Let $(\overline{\mathcal{U}_b}, \mathcal{U}_a)$ be a non-degenerate special holomorphic polyhedral condenser in $\mathbb{C}^n$ and $J : H^\infty(\mathcal{U}_a) \rightarrow A(\overline{\mathcal{U}_b})$ the canonical identification. Then for any $m \in \mathbb{N}$ with $m > m_0$ we have*

$$d_m(J) \leq a_m(J) \leq C \left(\sum_{k=0}^{n-1} \frac{1}{k!} \left(c \frac{m - m_0}{m_0} \right)^{k/n} \right) \exp \left(- \left(c \frac{m - m_0}{m_0} \right)^{1/n} \right)$$

where

$$c = n! \prod_{k=1}^n \log(a_k/b_k),$$

m_0 is the multiplicity of the underlying mapping of the condenser, and C is the explicit constant given in (28). In particular,

$$\liminf_{m \rightarrow \infty} \frac{\log d_m(\mathcal{A}_{\mathcal{U}_b}^{\mathcal{U}_a})}{m^{1/n}} \geq 2\pi \left(\frac{n!}{C(\overline{\mathcal{U}_b}, \mathcal{U}_a)} \right)^{1/n}. \quad (30)$$

Proof. The explicit upper bound follows from Lemmas 3.5 and 3.6 together with the fact that the sequence $m \mapsto a_m(J)$ is decreasing. Moreover, using Proposition 3.1, which yields

$$C(\overline{\mathcal{U}_b}, \mathcal{U}_a) = \frac{(2\pi)^n m_0}{\prod_{k=1}^n \log(a_k/b_k)},$$

the assertion (30) follows. ■

Remark 3.8. A stronger version of the asymptotics (30) above, with similar hypotheses but with a limit instead of the limit inferior and equality instead of inequality, can be found, without proof, as Proposition 5.1 in Zakharyuta's survey article [46], where it is credited to [40] in which it appears, again without proof, as a consequence of [40, Theorem 4.5].

3.2. Exhaustion of D and K by special holomorphic polyhedra

We shall now extend the sharp asymptotic upper bound obtained at the end of the previous subsection for special polyhedral condensers to more general condensers (K, D) where D is strictly hyperconvex and K a regular compact subset of D .

In this setup, we have the following refinement of [30, Theorem 3], itself a quantitative version of a lemma of Bishop [8], which provides an external approximation of $\hat{K}_D$, the holomorphically convex hull of K in D , and an internal approximation of D by two special holomorphic polyhedra defined simultaneously by the same holomorphic mapping such that the relative capacities of the approximations converge to the capacity $C(K, D) = C(\hat{K}_D, D)$.

Theorem 3.9. *Let D be a strictly hyperconvex domain in $\mathbb{C}^n$ and K a regular compact subset of D . Then, for every $j \in \mathbb{N}$, there is a non-degenerate special holomorphic polyhedral condenser $(\mathcal{U}_j^1, \mathcal{U}_j^2)$ with*

$$K \subset \hat{K}_D \subset \overline{\mathcal{U}_j^1} \subset \mathcal{U}_j^2 \subset D$$

such that

$$\lim_{j \rightarrow \infty} C(\overline{\mathcal{U}_j^1}, \mathcal{U}_j^2) = C(K, D).$$

Proof. We start by recalling some notation already used in Subsection 1.1, as well as in Subsection 2.4, immediately above Lemma 2.15.

Since D is strictly hyperconvex there exists a bounded domain Ω and an exhaustion function $\varrho \in \text{PSH}(\Omega, (-\infty, 1)) \cap C(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$ and, for all real numbers $c \in [0, 1]$, the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected.

For any integer $j \geq 1$, let D_j denote the bounded hyperconvex domain $\{z \in \Omega : \varrho(z) < 1/j\}$.

Since K is a regular compact subset of D the relative extremal function $u_{K,D}$ is continuous in $\overline{D}$. Let now u and u_j denote the relative extremal functions $u_{K,D}$ and u_{K,D_j} , respectively.

Lemma 2.6 in [30] then implies the following: K is regular for any D_j with $j \geq 1$, the sequence (u_j) converges uniformly on $\overline{D}$ to u , and the increasing sequence $(C(K, D_j))_{j \in \mathbb{N}}$ of capacities converges to the capacity $C(K, D)$.

For any $r \in [-1, 0]$ write $D(r)$ for the set $\{z \in D : u(z) < r\}$. As u is continuous on $\overline{D}$ and is an exhaustion function for D , each $D(r)$ is open. Moreover, for any $-1 < r < 0$ sufficiently close to 0, the open sets $D(r)$ are connected since D is, and provide an internal exhaustion of D . Similarly, for any $-1 < r < 0$ sufficiently near -1 , the open sets $D(r)$ provide an external exhaustion of $\hat{K}_D = \{z \in D : u_{K,D}(z) = -1\}$.

We shall now proceed to the construction of the special holomorphic polyhedra by a five-step process of definitions and assertions, the first two of which follow from [30, Theorem 3].

For any $\epsilon, \epsilon' > 0$ sufficiently small with $\epsilon' < \epsilon$, there exist integers $j \geq 2$ and $p \geq 2$ and holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(D_j)$ such that the following assertions hold.

- (i) $\frac{1}{p} \log |f_l(z)| \leq u_{2j}(z)$ in D_{2j} for all $1 \leq l \leq n$.
- (ii) There exist special holomorphic polyhedra $\mathcal{U}^0$ and $\mathcal{U}^2$ and a positive constant $\beta(\epsilon')$ satisfying $\beta(\epsilon') \leq \epsilon'/2$ such that

$$\begin{aligned} K \subset \hat{K}_D \subset \overline{D(-1 + \epsilon)} \subset \mathcal{U}^0 \subset D(-1 + \epsilon + \epsilon') \quad \text{and} \\ \overline{D(-\epsilon)} \subset \mathcal{U}^2 \subset D(-\epsilon + \epsilon'). \end{aligned}$$

Here, $\mathcal{U}^0$ is the finite union of the connected components of the open set

$$\left\{ z \in D : \sup_{1 \leq l \leq n} \frac{1}{p} \log |f_l(z)| < -1 + \epsilon + \beta(\epsilon') \right\}$$

that meet $\overline{D(-1 + \epsilon)}$, and $\mathcal{U}^2$ is the connected component containing $\overline{D(-\epsilon)}$ of the open set

$$\left\{ z \in D : \sup_{1 \leq l \leq n} \frac{1}{p} \log |f_l(z)| < -\epsilon + \beta(\epsilon') \right\}.$$

Using the inclusions in assertion (ii), we obtain the next assertion,

- (iii) $C(\overline{D(-1 + \epsilon)}, D(-\epsilon + \epsilon')) \leq C(\overline{\mathcal{U}^0}, \mathcal{U}^2) \leq C(\overline{D(-1 + \epsilon + \epsilon')}, D(-\epsilon)).$

Indeed,

$$C(\overline{D(-1 + \epsilon)}, D(-\epsilon + \epsilon')) \leq C(\overline{\mathcal{U}^0}, D(-\epsilon + \epsilon')) \leq C(\overline{\mathcal{U}^0}, \mathcal{U}^2)$$

and

$$C(\overline{\mathcal{U}^0}, \mathcal{U}^2) \leq C(\overline{D(-1 + \epsilon + \epsilon')}, \mathcal{U}^2) \leq C(\overline{D(-1 + \epsilon + \epsilon')}, D(-\epsilon)).$$

- (iv) Denote by $\mathcal{U}^1$ the special holomorphic polyhedron defined as the finite union of all connected components of the open set

$$\left\{ z \in D : \sup_{1 \leq l \leq n} \frac{1}{p} \log |f_l(z)| < -1 + \epsilon + \beta(\epsilon') \right\}$$

included in $\mathcal{U}^2$.

- (v) The holomorphic mapping

$$F = (f_1, \dots, f_n) : \mathcal{U}^2 \rightarrow \mathbb{C}^n$$

is proper and surjective from the bounded special holomorphic polyhedron $\mathcal{U}^2$ to the polydisc $P(0, r_2)$ and from the bounded special holomorphic polyhedron $\mathcal{U}^1$ to the polydisc $P(0, r_1)$, where $r_2 = \exp(p(-\epsilon + \beta(\epsilon')))$ and $r_1 = \exp(p(-1 + \epsilon + \beta(\epsilon')))$, respectively. This follows from [30, Proposition 3.4].

All in all, the above yields, for any $\epsilon, \epsilon' > 0$ sufficiently small with $\epsilon' < \epsilon$, three special holomorphic polyhedra $\mathcal{U}_{\epsilon, \epsilon'}^0$, $\mathcal{U}_{\epsilon, \epsilon'}^1$, and $\mathcal{U}_{\epsilon, \epsilon'}^2$ with

$$K \subset \hat{K}_D \subset \mathcal{U}_{\epsilon, \epsilon'}^0 \subset \mathcal{U}_{\epsilon, \epsilon'}^1 \subset \mathcal{U}_{\epsilon, \epsilon'}^2 \subset D.$$

By (iii) we know that $C(\overline{\mathcal{U}_{\epsilon, \epsilon'}^0}, \mathcal{U}_{\epsilon, \epsilon'}^2)$ converges to $C(\hat{K}_D, D) = C(K, D)$ when ϵ and ϵ' both tend to 0. Moreover, using an argument similar to the proof of [30, Lemma 5.2] it follows that $C(\overline{\mathcal{U}_{\epsilon, \epsilon'}^1}, \mathcal{U}_{\epsilon, \epsilon'}^2)$ also converges to $C(K, D)$ when ϵ and ϵ' both tend to 0. This, coupled with assertion (v) yields the desired sequence of special holomorphic polyhedral condensers, each of which can be chosen to be non-degenerate, by possibly replacing the underlying mapping $F_{\epsilon, \epsilon'}$ with $F_{\epsilon, \epsilon'} - c_{\epsilon, \epsilon'}$, for $c_{\epsilon, \epsilon'}$ a sufficiently small regular value of $F_{\epsilon, \epsilon'}$. ■

The hypotheses in the above theorem can be weakened further using the following lemma.

Lemma 3.10. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ containing a compact subset K . Then we can approximate $\hat{K}_D$ externally and D internally by two sequences (K_j) and (D_j) such that:*

- (i) (K_j) is a decreasing sequence of compact sets and (D_j) is an increasing sequence of domains with $\bigcap_j K_j = \hat{K}_D$, $K_{j+1} \subset \overset{\circ}{K}_j$, $\overline{D}_j \subset D_{j+1}$ and $\bigcup_j D_j = D$;
- (ii) for any j , each compact K_j is holomorphically convex and regular in D_j , which in turn is strictly hyperconvex;
- (iii) the sequence $(C(K_j, D_j))_j$ converges to $C(K, D)$.

Proof. Note that K need not be regular, that is, $u = u_{K, D}$ need not necessarily be continuous in $\overline{D}$. Now, for any $\epsilon > 0$ sufficiently small, let K^ϵ denote the compact subset of D defined by

$$K^\epsilon = \{z \in D : \text{dist}(z, \partial D) \leq \epsilon\}.$$

Using [17, Corollaries 4.5.9, 4.5.11 and Proposition 4.7.1], we deduce, successively, the following:

- (a) $u_{K^\epsilon, D}$ is continuous in D , that is, K^ϵ is regular for D ;
- (b) $(u_{K^\epsilon, D})_{\epsilon>0}$ converges pointwise and monotonically from below to u in D when ϵ decreases to 0;
- (c) $u_{K^\epsilon, D} = u_{\widehat{K_D^\epsilon}, D}$ and $C(K^\epsilon, D) = C(\widehat{K_D^\epsilon}, D)$;
- (d) $\widehat{K_D^\epsilon}$ approximates $\hat{K}_D$ in the sense that $\bigcap_{\epsilon>0} \widehat{K_D^\epsilon} = \hat{K}_D$;
- (e) $\lim_{\epsilon \rightarrow 0} C(K^\epsilon, D) = C(K, D)$.

Now, since $u_{K^\epsilon, D} =: u_\epsilon$ is continuous on $\overline{D}$ and is an exhaustion function for D , we can construct an internal exhaustion of D by strictly hyperconvex domains as follows: for any $\delta > 0$ sufficiently small, $D_\epsilon(-\delta) = \{z \in D : u_\epsilon(z) < -\delta\}$ is a strictly hyperconvex domain and $\bigcup_{\delta>0} D_\epsilon(-\delta) = D$.

We now choose $K_j = \overline{D_{\epsilon_j}(-1 + \delta_j)}$ and $D_j = D_{\epsilon_j}(-\delta_j)$, where $\epsilon_j, \delta_j > 0$ are sufficiently small with the sequences $(\epsilon_j)_j$ and $(\delta_j)_j$ strictly decreasing to zero and δ_{j+1} chosen such that

$$\sup \{u_{\epsilon_{j+1}}(z) : z \in \overline{D_{\epsilon_j}(-\delta_j)}\} < -\delta_{j+1}.$$

In this case the relative extremal function u_{K_j, D_j} for K_j and D_j is easily seen to be given explicitly by

$$u_{K_j, D_j}(z) = \max \left\{ \frac{u_{\epsilon_j}(z) + \delta_j}{1 - 2\delta_j}, -1 \right\} \quad (\forall z \in D_{\epsilon_j}(-\delta_j)),$$

while the corresponding relative capacity satisfies

$$C(K_j, D_j) = \frac{C(K^{\epsilon_j}, D)}{(1 - 2\delta_j)^n},$$

and the assertions of the lemma follow. ■

Combining Theorem 3.9 and Lemma 3.10 immediately yields the following corollary providing the approximation result required for the proof of the subsequent theorem.

Corollary 3.11. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ containing a compact subset K . Then, for every $j \in \mathbb{N}$, there is a non-degenerate special holomorphic polyhedral condenser $(\mathcal{U}_j^1, \mathcal{U}_j^2)$ with*

$$K \subset \hat{K}_D \subset \overline{\mathcal{U}_j^1} \subset \mathcal{U}_j^2 \subset D$$

such that

$$\lim_{j \rightarrow \infty} C(\overline{\mathcal{U}_j^1}, \mathcal{U}_j^2) = C(K, D).$$

All in all, we now obtain the main result of this section, furnishing the second part of our main result, Theorem 1.1.

Theorem 3.12. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ and K a compact subset of D . Then*

$$\liminf_{m \rightarrow \infty} -\frac{\log d_m(\mathcal{A}_K^D)}{m^{1/n}} \geq (2\pi) \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

Proof. Let $(\overline{\mathcal{U}}_j^1)$ and $(\mathcal{U}_j^2)$ denote the sequences of sets furnished by Corollary 3.11. Then, for every j , the pair $(\overline{\mathcal{U}}_j^1, \mathcal{U}_j^2)$ is a non-degenerate special holomorphic polyhedral condenser with

$$K \subset \hat{K}_D \subset \overline{\mathcal{U}}_j^1 \subset \mathcal{U}_j^2 \subset D.$$

Thus, we have

$$H^\infty(D) \hookrightarrow H^\infty(\mathcal{U}_j^2) \hookrightarrow A(\overline{\mathcal{U}}_j^1) \rightarrow A(K).$$

Both $H^\infty(D) \hookrightarrow H^\infty(\mathcal{U}_j^2)$ and $A(\overline{\mathcal{U}}_j^1) \rightarrow A(K)$ are easily seen to be continuous with norm 1, so

$$d_m(\mathcal{A}_K^D) \leq d_m(\mathcal{A}_{\overline{\mathcal{U}}_j^1}^{\mathcal{U}_j^2}) \quad (\forall m, j \in \mathbb{N}),$$

and we deduce

$$\liminf_{m \rightarrow \infty} -\frac{\log d_m(\mathcal{A}_K^D)}{m^{1/n}} \geq \liminf_{m \rightarrow \infty} -\frac{\log d_m(\mathcal{A}_{\overline{\mathcal{U}}_j^1}^{\mathcal{U}_j^2})}{m^{1/n}} \quad (\forall j \in \mathbb{N}).$$

But by (30) in Proposition 3.7 we have

$$\liminf_{m \rightarrow \infty} -\frac{\log d_m(\mathcal{A}_{\overline{\mathcal{U}}_j^1}^{\mathcal{U}_j^2})}{m^{1/n}} \geq (2\pi) \left(\frac{n!}{C(\overline{\mathcal{U}}_j^1, \mathcal{U}_j^2)} \right)^{1/n},$$

and since by Corollary 3.11 the relative capacities $C(\overline{\mathcal{U}}_j^1, D_j)$ converge to $C(K, D)$ when j tends to infinity, we finally deduce that

$$\liminf_{m \rightarrow \infty} -\frac{\log d_m(\mathcal{A}_K^D)}{m^{1/n}} \geq (2\pi) \left(\frac{n!}{C(K, D)} \right)^{1/n},$$

which concludes the proof. ■

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