J. Eur. Math. Soc. 24, 4327–4351 (2022) DOI 10.4171/JEMS/1194

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Fourier uniqueness pairs of powers of integers

Received October 11, 2019; revised July 14, 2020

Abstract. We prove, under certain conditions on (α, β), that each Schwartz function f such that $f(\pm n^\alpha) = \widehat{f}(\pm n^\beta) = 0$ for all $n \geq 0$ must vanish identically, complementing a series of recent results involving uncertainty principles, such as the pointwise interpolation formulas by Radchenko and Viazovska and the Meyer–Guinnand construction of self-dual crystaline measures.

Keywords. Fourier transform, Fourier uniqueness pair, uncertainty principle

1. Introduction

Given an integrable function $f : \mathbb{R} \to \mathbb{C}$, we define its Fourier transform by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{2\pi i x\cdot\xi}\, \mathrm{d}x. \tag{1.1}$$

Let us consider the following classical problem in Fourier analysis:

Question 1.1. Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$, what conditions can we impose on two sets $A, \widehat{A} \subset \mathbb{R}$ to ensure that the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function?

Inspired by the notion of Heisenberg uniqueness pairs introduced by Hedenmalm and Montes–Rodrígues in [10] (see also [9, 12]), we refer to such pair of sets $(A, \widehat{A})$ as a *Fourier uniqueness pair* for $\mathcal{C}$ for a natural reason: the values of $f(x)$ for $x \in A$ and $\widehat{f}(\xi)$ for $\xi \in B$ determine at most one function $f \in \mathcal{C}$. For simplicity, when $A = \widehat{A}$, we will say that A is a Fourier uniqueness set for $\mathcal{C}$.

Perhaps the most classical result which answers such a question is the celebrated Shannon–Whittaker interpolation formula, which states that a function $f \in L^2(\mathbb{R})$ whose Fourier transform $\widehat{f}$ is supported on the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$ is given by the formula

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\delta}\right) \operatorname{sinc}(\delta x - k),$$

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Mathematics Subject Classification (2020): 42A38

where convergence holds both in the $L^2(\mathbb{R})$ sense and uniformly on the real line, and $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. This means that the pair $\frac{1}{\delta}\mathbb{Z}$ and $\mathbb{R}\backslash[-\frac{\delta}{2}, \frac{\delta}{2}]$ forms a Fourier uniqueness pair for the collection $\mathcal{C} = L^2(\mathbb{R})$. More recently, Radchenko and Viazovska [16] obtained a related interpolation formula for Schwartz functions: there are even functions $a_k \in \mathcal{S}(\mathbb{R})$ such that, for any given even function $f : \mathbb{R} \to \mathbb{C}$ that belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, one has the following identity:

$$f(x) = \sum_{k=0}^{\infty} f(\sqrt{k}) a_k(x) + \sum_{k=0}^{\infty} \widehat{f}(\sqrt{k}) \widehat{a_k}(x), \tag{1.2}$$

where the right-hand side converges absolutely. This interpolation result has as immediate consequence: the set $\sqrt{\mathbb{Z}_+}$ of square roots of non-negative integers is a Fourier uniqueness set for the collection of even[1] Schwartz functions.

The two theorems we just presented to motivate our question are, in fact, also instances of the intimate relationship between interpolation and summation formulas. Indeed, as previously mentioned, the Shannon–Whittaker interpolation formula is directly related to the Poisson summation formula

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

and the result by Radchenko and Viazovska is, in fact, a by-product of the development of several summation formulas, having relationship to modular forms and the sphere packing problem (see, for instance, [6, 7, 17]). In fact, the lower bound for the Fourier analysis problem corresponding to the sphere packing problem (see [4]) is directly related to the Poisson summation formula for lattices: if $\Lambda \subset \mathbb{R}^n$ is a lattice with fundamental region having volume 1, then

$$\sum_{\lambda\in\Lambda} f(\lambda) = \sum_{\lambda^*\in\Lambda^*} \widehat{f}(\lambda^*),$$

where Λ^* denotes the dual lattice of Λ. Also, in [5], the authors need a summation formula stemming from an Eisenstein series E_6, which implies, in particular, that for each radial Schwartz function $f : \mathbb{R}^{12} \to \mathbb{C}$, there exists constants $c_j > 0$ such that

$$f(0) - \sum_{j\geq 1} c_j f(\sqrt{2j}) = -\widehat{f}(0) + \sum_{j\geq 1} c_j \widehat{f}(\sqrt{2j}).$$

These concepts seem to be all tethered to the notion of *crystaline measures* and self-duality, as discussed in [13–15]. A crystaline measure is essentially a tempered distribution with locally finite support whose Fourier transform has these same properties. For instance, Poisson summation implies that

$$\delta_{\mathbb{Z}} = \widehat{\delta_{\mathbb{Z}}},$$

which shows that the usual delta distribution at the integers is not only a crystaline measure, but also a *self-dual* one with respect to the Fourier transform. Meyer then discusses other examples of crystaline measures with certain self-duality properties, and, simil-

[1] In [16], the authors also have results for functions which are not even, but we chose to present this version to keep technicalities to a minimum.

arly to the strategy used by Radchenko and Viazovska, uses modular forms to construct explicit examples of non-zero crystaline measures μ supported in $\{\pm\sqrt{k+a}, k \in \mathbb{Z}\}$, for $a \in \{9, 24, 72\}$. It is interesting to point out that Meyer calls out the readers attention to the highly unexplored problem of analysing when there is a non-zero crystaline measure μ such that both itself and its Fourier transform have support on a given locally finite set $\{\lambda_k : k \in \mathbb{Z}\}$.

Back to Fourier uniqueness pairs, while both the Shannon–Whittaker and Radchenko–Viazovska results provide Fourier uniqueness pairs by means of interpolation identities, such explicit formulas are not always available and usually depend on special properties of the sets involved, which are somewhat rigid. In the case of the Shannon–Whittaker formula, the set $\frac{1}{\delta}\mathbb{Z}$ plays an special role because of the Poisson summation formula. In the case of the Radchenko–Viazovska interpolation, the set $\sqrt{\mathbb{Z}_+}$ becomes important due to special properties of certain modular forms involved in their proofs. Perturbing these sets breaks down the proofs of these theorems, and sometimes even the existence of such interpolation formulas. Nevertheless, the Fourier uniqueness pair property is inherently less rigid as a condition than an interpolation formula, which might lead to uniqueness results even in the absence of possible interpolation formulas.

For instance, define a set $\Lambda \subset \mathbb{R}$ to be uniformly separated if there is a number $\delta = \delta(\Lambda) > 0$ such that $|\lambda - \lambda'| > \delta$ whenever $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$. Given a uniformly separated set Λ, we define its lower density and upper density, respectively, as the numbers

$$\mathcal{D}^-(\Lambda) = \liminf_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{|\Lambda \cap [x-R, x+R]|}{2R},$$
$$\mathcal{D}^+(\Lambda) = \limsup_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{|\Lambda \cap [x-R, x+R]|}{2R}.$$

When these numbers coincide, we call it the density of Λ. As a corollary of the work of Beurling [2] and Kahane [11] about sampling sets, any pair Λ and $\mathbb{R}\backslash[-2\pi\delta, 2\pi\delta]$ forms uniqueness sets for $L^2(\mathbb{R})$ if Λ is uniformly separated and $\mathcal{D}^-(\Lambda) > \delta$. This means: any uniformly separated set that is more dense than $\frac{1}{\delta}\mathbb{Z}$ produces a pair of uniqueness sets for $L^2(\mathbb{R})$, and one can readily see that this condition, at least in terms of density, is essentially sharp just by analysing subsets of $\frac{1}{\delta}\mathbb{Z}$.

Another instance of this density situation has to do with the aforementioned Heisenberg uniqueness pairs. In [10], the authors study pairs of sets (Γ, Λ), where $\Gamma \subset \mathbb{R}^2$, which is a finite disjoint union of smooth curves, and $\Lambda \subset \mathbb{R}^2$, which have the following property: whenever a measure μ supported in Γ, which is absolutely continuous with respect to the arc length measure of Γ, has Fourier transform $\widehat{\mu}$ equal to zero on the set Λ, then $\mu = 0$. If a pair (Γ, Λ) has this property, it is called a Heisenberg uniqueness pair. One of the main results of [10] is the following: Let $\Gamma = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ be the hyperbola, and $\Lambda_{\alpha,\beta}$ be the lattice cross

$$(\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α and β are positive numbers. Then $(\Gamma, \Lambda_{\alpha\beta})$ forms a Heisenberg uniqueness pair if and only if $\alpha\beta \leq 1$. This provides yet another example of the interplay between con-

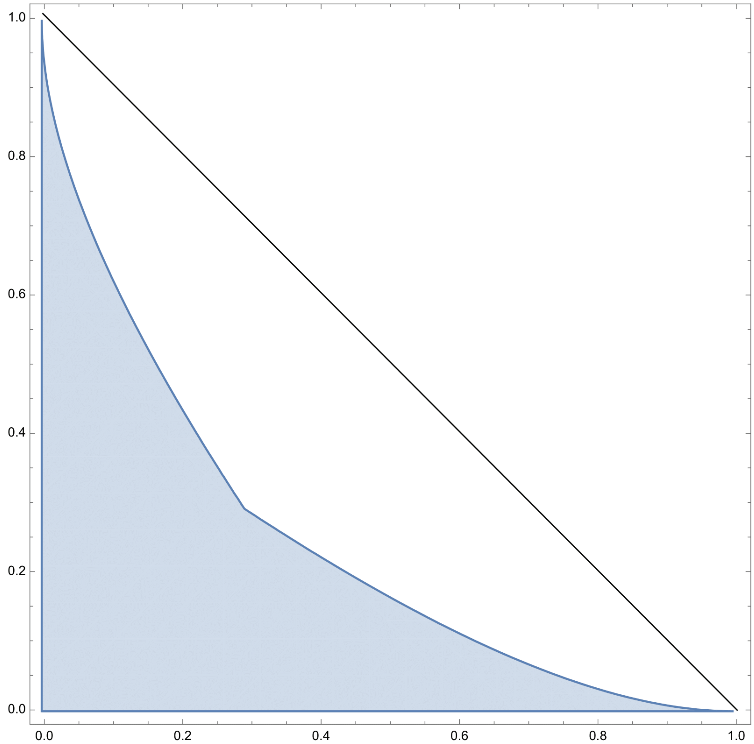

Fig. 1. In blue, the closure of the region A, with the line $\alpha + \beta = 1$ in black.

centration and uniqueness properties: there is a threshold of concentration one needs to ask in order to maintain the uniqueness property, and increasing the concentration does not affect the uniqueness property.

By comparing the aforementioned interpolation theorems to the considerations in [15] about crystaline measures, one is naturally lead towards the following modified version of Meyer's question: if a sequence is "more concentrated than $\sqrt{\mathbb{Z}}$", does it define a Fourier uniqueness set? For which notion of "more concentrated" could such a result possibly hold? We obtain partial progress towards this problem.

Theorem 1.2. *Let* $0 < \alpha, \beta < 1$ *and* $f \in \mathcal{S}(\mathbb{R})$. *Then:*

(A) *If* $f(\pm \log(n+1))) = 0$ *and* $\widehat{f}(\pm n^{\alpha}) = 0$ *for every* $n \in \mathbb{N}$, *then* $f \equiv 0$.

(B) *Let* $(\alpha, \beta) \in A$, *where*

$$A = \left\{(\alpha, \beta) \in [0,1]^2 : \alpha + \beta < 1, \text{ and either } \alpha < 1 - \frac{\beta}{1-\alpha-\beta} \right.$$
$$\left. \text{or } \beta < 1 - \frac{\alpha}{1-\alpha-\beta} \right\}.$$

If $f(\pm n^{\alpha}) = 0$ *and* $\widehat{f}(\pm n^{\beta}) = 0$ *for every* $n \in \mathbb{N}$, *then* $f \equiv 0$. *(See Figure 1.)*

Theorem 1.2 will follow by complex analytic considerations. We will prove that f and $\widehat{f}$ actually have better decay than usual Schwartz functions by using the fact that the sequence of zeros of f and $\widehat{f}$ grows at a certain rate, as well as the information we can obtain about the zeros of their derivatives. Once the decay is obtained, we prove either f or $\widehat{f}$ admits an analytic extension of finite order, and conclude f is the zero function by invoking the converse of Hadamard's theorem about growth of zeros of an entire function of finite order. It will also become clear from the proof that the condition on the exponents (α, β) on part (ii) of Theorem 1.2 is a barrier of our method. We postpone a more detailed discussion about sharpness of our results to the final section of this paper.

Lastly, in order to better compare our results with the ones in [15] and [16] we state the diagonal case of Theorem 1.2.

Corollary 1.3. *Let* $\alpha < 1 - \frac{\sqrt{2}}{2}$. *Then, if* $f \in \mathcal{S}(\mathbb{R})$ *is such that* $f(\pm n^\alpha) = \widehat{f}(\pm n^\alpha) = 0$ *for each* $n \in \mathbb{N}$, *one has* $f \equiv 0$.

1.1. Organisation and notation

This article is organised as follows. In Section 2, we mention a couple of basic ideas associating the denseness of zeros of a function and its pointwise decay. In Section 3, we prove the first assertion in Theorem 1.2, and in Section 4 we work upon the ideas in the previous Section to prove the second part of Theorem 1.2. Finally, in Section 5 we make remarks, mention some corollaries of our methods and state conjectures based on the proofs presented.

Throughout this manuscript, we will use Vinogradov's modified notation $A \lesssim B$ or $A = O(B)$ to denote the existence of an absolute constant $C > 0$ such that $A \le C \cdot B$. If we allow C to explicitly depend upon a parameter τ, we will write $A \lesssim_\tau B$. In general, C will denote an absolute constant that may change from line to line or from paragraph to paragraph in the argument. We adopt (1.1) as our normalisation for the Fourier transform. Finally, we warn the reader that in some of the computations the constants obtained are not necessarily the sharpest possible, but are always sufficient for our purposes.

We begin by pointing that we can assume without loss of generality in Theorem 1.2 that f is real-valued and $\widehat{f}$ is real-valued or purely imaginary-valued. In fact, if we decompose $f = (f_1 + f_2) + i(f_3 + f_4)$, where f_1 and f_3 are odd functions and f_2 and f_4 are even functions, then all these functions are of Schwartz class, have the zeros prescribed in the statement of Theorem 1.2 and they are real-valued functions whose Fourier transform is real-valued or purely imaginary-valued. Proving the result for such functions therefore implies the results for any complex-valued function, therefore from this point on we assume such conditions.

2. Preliminaries

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pose $f = (f_1 + f_2) + i(f_3 + f_4)$, where f_1 and f_3 are odd functions and f_2 and f_4 are even functions, then all these functions are of Schwartz class, have the zeros prescribed in the statement of Theorem 1.2 and they are real-valued functions whose Fourier transform is real-valued or purely imaginary-valued. Proving the result for such functions therefore implies the results for any complex-valued function, therefore from this point on we assume such conditions.

2.1. Zeros of Schwartz functions and decay

After that remark that allows us to deal with real functions, we state some properties of such functions under the condition that they are differentiable.

(I) By the mean value theorem, between two zeros of the kth derivative of a real function, there is a zero of the $(k+1)$st derivative. This means that, as long as there is a sequence $\{a_m\}_{m\in\mathbb{Z}}$ of zeros f such that

$$\lim_{m\to-\infty} a_m = -\infty, \quad \lim_{m\to\infty} a_m = +\infty,$$

by an induction argument, for each $k \geq 1$ there is a sequence $\{a_m^{(k)}\}_{m\in\mathbb{Z}}$ such that:

(i) $a_{m-1}^{(k)} < a_m^{(k)} < a_{m+1}^{(k)}$ and

$$\lim_{m\to+\infty} a_m^{(k)} = +\infty, \quad \lim_{m\to-\infty} a_m^{(k)} = -\infty.$$

(ii) $f^{(k)}(a_m^{(k)}) = 0$, for every $m \in \mathbb{Z}$.

(iii) For all $m \in \mathbb{N}$, $[a_m^{(k)}, a_{m+1}^{(k)}]$ is contained in the interval $[a_m, a_{m+k+1}]$. This, in particular, implies the following bound on gaps of consecutive zeros:

$$|a_{m+1}^{(k)} - a_m^{(k)}| \leq |a_{m+k+1} - a_m|.$$

In case $m \in \mathbb{Z}\backslash\mathbb{N}$, we can ensure instead that $[a_{m-1}^{(k)}, a_m^{(k)}]$ is contained in $[a_{m-k-1}, a_m]$, with a similar control on the gap between consecutive zeros.

(II) As, by the remark in the beginning of the section, we may suppose that $\widehat{f}$ is either purely imaginary or real, then in a completely analogous manner we can reproduce the procedure above for $\widehat{f}$. More details on this will be given throughout the text.

Given a function $g \in \mathcal{S}(\mathbb{R})$, we will use the following notation:

$$I_k(g) = \int_{\mathbb{R}} |g(y)||y|^k \,\mathrm{d}y.$$

The integrals $I_k(f)$ and $I_k(\widehat{f})$ will play an important role because of the following observation: whenever a point x lies in an interval of the form $[a_{m+1}^{(k)}, a_m^{(k)}]$, Fourier inversion implies

$$\begin{aligned} |f^{(k)}(x)| &= |f^{(k)}(x) - f(a_m^{(k)})| \\ &= \left| \int_{\mathbb{R}} \widehat{f}(y)(2\pi i y)^k [e^{-2\pi i y x} - e^{-2\pi i y a_m^{(k)}}] \,\mathrm{d}y \right| \\ &\leq (2\pi)^{k+1} I_{k+1}(\widehat{f})|x - a_m^{(k)}| \\ &\leq (2\pi)^{k+1} I_{k+1}(\widehat{f})|a_{m+1}^{(k)} - a_m^{(k)}|. \end{aligned} \tag{2.1}$$

This means that the rate at which the zeros of the derivatives accumulate at infinity provides extra decay for each derivative itself. We will use this observation iteratively to improve decay bounds on our functions.

2.2. *Fourier transforms of functions with strong decay*

In addition to connecting location of zeros to decay of functions, we need to connect decay of a function to properties of its Fourier transform. The next lemma is going to be of crucial importance for us throughout the proof.

Lemma 2.1. *Let $f \in \mathcal{S}(\mathbb{R})$ be such that there exist two constants $C > 0$ and $A > 1$ for which $|f(x)| \lesssim e^{-C|x|^A}$ for all $x \in \mathbb{R}$. Then its Fourier transform $\widehat{f}$ can be extended to the whole complex plane as an analytic function with order at most $\frac{A}{A-1}$. That is, for all $\varepsilon > 0$,*

$$|\widehat{f}(z)| \lesssim_\varepsilon e^{|z|^{\frac{A}{A-1}+\varepsilon}}.$$

Proof. Let $z = \xi + i\eta \in \mathbb{C}$. Without loss of generality, in what follows we assume that $\eta < 0$. We simply write

$$\widehat{f}(z) = \int_{\mathbb{R}} e^{2\pi i z\cdot x} f(x)\,\mathrm{d}x.$$

By the decay property of f, it is easy to see that this integral is well-defined for each $z \in \mathbb{C}$, and Morera's theorem tells us that this extension is, in fact, entire. For the assertion about its order, we have the trivial bound

$$|\widehat{f}(z)| \le \int_{\mathbb{R}} e^{-2\pi\eta x} e^{-C|x|^A}\,\mathrm{d}x.$$

In order to prove that the expression on the right-hand side above is $\lesssim_\varepsilon e^{|z|^{\frac{A}{A-1}+\varepsilon}}$, we split the real line as

$$\mathbb{R} = A_\eta \cup B_\eta \cup C_\eta,$$

where

$$\begin{aligned}
A_\eta &= \left\{x \in \mathbb{R} : \left|x - \left(\frac{2\pi|\eta|}{CA}\right)^{1/(A-1)}\right| \le K_A\left(\frac{2\pi|\eta|}{CA}\right)^{1/(A-1)}\right\},\\
B_\eta &= \left\{x \in \mathbb{R} : x > (K_A+1)\left(\frac{2\pi|\eta|}{CA}\right)^{1/(A-1)}\right\},\\
C_\eta &= \left\{x \in \mathbb{R} : x < (1-K_A)\left(\frac{2\pi|\eta|}{CA}\right)^{1/(A-1)}\right\},
\end{aligned}$$

and rewrite our integral as

$$\begin{aligned}
\int_{\mathbb{R}} e^{-2\pi\eta x} e^{-C|x|^A}\,\mathrm{d}x &= \int_{A_\eta} e^{-2\pi\eta x} e^{-C|x|^A}\,\mathrm{d}x + \int_{B_\eta} e^{-2\pi\eta x} e^{-C|x|^A}\,\mathrm{d}x\\
&\quad + \int_{C_\eta} e^{-2\pi\eta x} e^{-C|x|^A}\,\mathrm{d}x\\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

On the interval over which we integrate in I_1, $-2\pi\eta x - C|x|^A$ is at most (an absolute constant depending on A times) $|\eta|^{\frac{A}{A-1}}$. This holds because the center of the interval A_η is the critical point of $-2\pi\eta x - C|x|^A$ where this function attains its maximum. As we know that $|A_\eta| \lesssim_A |\eta|^{\frac{1}{A-1}}$,

$$|I_1| \lesssim |\eta|^{\frac{1}{A-1}} e^{C_A|\eta|^{\frac{A}{A-1}}} \tag{2.2}$$

follows. On either the interval defining I_2 or on the one defining I_3, we see that, for $K_A > 0$ large enough depending on A,

$$-2\pi\eta x - C|x|^A \leq -\tilde{C}_A|x|^A.$$

Therefore,

$$|I_2| + |I_3| \lesssim \int_{|\eta|^{\frac{1}{A-1}}}^{+\infty} e^{-C'_A|x|^A}\, \mathrm{d}x \lesssim e^{-C''_A|\eta|^{\frac{A}{A-1}}}. \tag{2.3}$$

One readily notices that (2.2) together with (2.3) implies the result then. ■

As an immediate corollary, we obtain the following statement, which will be particularly useful in Section 3.

Corollary 2.2. *Let $f \in \mathcal{S}(\mathbb{R})$ be such that, for each $A > 1$, there is a constant $C_A > 0$ such that $|f(x)| \lesssim_A e^{-C_A|x|^A}$ for all $x \in \mathbb{R}$. Then its Fourier transform can be extended to the whole complex plane as an analytic function with order at most* 1.

3. Proof of Theorem 1.2, Part (A)

3.1. Obtaining decay for f

The first idea is to exploit the considerations in Section 2.1 to obtain decay for f. We must, however, obtain simultaneously bounds on the Fourier transform to somehow improve the decay on f we obtain at each step. The following result is the key ingredient to this iteration scheme.

Lemma 3.1. *Let $f \in \mathcal{S}(\mathbb{R})$, and assume that $f(\pm \log(n+1)) = 0$ and $\widehat{f}(\pm n^\alpha) = 0$ for every $n \in \mathbb{N}$, where $\alpha \in (0,1)$. Then, for $|x| > \log(k+1)$ and $|\xi| > (2j+1)^\alpha$, one has*

$$\begin{aligned} |f(x)| &\leq (k+2)(2\pi)^{k+1}((k+1)!)^2 I_{k+1}(\widehat{f})e^{-(k+1)|x|} = \tau_k e^{-(k+1)|x|}, \\ |\widehat{f}(\xi)| &\leq (j+1)!(2^{2-\alpha}\pi)^{j+1}\alpha^{j+1} I_{j+1}(f)|\xi|^{(\frac{\alpha-1}{\alpha})\cdot(j+1)} = \widehat{C}_j|\xi|^{(\frac{\alpha-1}{\alpha})\cdot(j+1)}. \end{aligned} \tag{3.1}$$

Proof. We first prove the assertion about $\widehat{f}$, as it will be also of interest to Lemma 4.1 in the next section. We start out by using the notation from Section 2.1; that is, we denote the sequence of zeros of the kth derivative of $\widehat{f}$ to be $\{a_m^{(k)}\}_{m\in\mathbb{Z}}$.

Let $\xi \geq 0$. First we consider n such that $\xi \in [n^\alpha, (n+1)^\alpha]$. This implies $n^{\alpha-1} \leq 2^{1-\alpha}\xi^{\frac{\alpha-1}{\alpha}}$. By inequality (2.1), we have

$$\begin{aligned} |\widehat{f}(\xi)| &\leq 2\pi|(n+1)^\alpha - n^\alpha|I_1(f) \leq 2\pi\alpha n^{\alpha-1} I_1(f) \\ &\leq 2^{2-\alpha}\pi\alpha\xi^{\frac{\alpha-1}{\alpha}} I_1(f). \end{aligned} \tag{3.2}$$

This gives us a preliminary bound on $\widehat{f}$. By observation (I) (i), as long as $\xi > (2j+1)^\alpha$, we can conclude there is $n \geq j+1$ such that $\xi \in [a_n^{(j)}, a_{n+1}^{(j)}] \subset [n^\alpha, (n+j+1)^\alpha]$. This means that

$$n^{\alpha-1} \leq 2^{1-\alpha}\xi^{\frac{\alpha-1}{\alpha}},$$

and therefore

$$\begin{aligned} |[\widehat{f}]^{(j)}(\xi)| &\leq (2\pi)^{j+1}|a_{n+1}^{(j)} - a_n^{(j)}| I_{j+1}(f) \\ &\leq (2\pi)^{j+1}|(n+j+1)^\alpha - n^\alpha| I_{j+1}(f) \\ &\leq \alpha(j+1)(2\pi)^{j+1} n^{\alpha-1} I_{j+1}(f) \\ &\leq 2^{1-\alpha}\alpha(j+1)(2\pi)^{j+1}\xi^{\frac{\alpha-1}{\alpha}} I_{j+1}(f). \end{aligned} \tag{3.3}$$

By induction, one can iterate this process and obtain decay of the order of $\xi^{(\frac{\alpha-1}{\alpha})j}$ for $\xi > (2j+1)^\alpha$. More precisely, suppose that, for given $k \in \{0, 1, \ldots, j-1\}$ we have

$$\begin{aligned} |[\widehat{f}]^{(j-k)}(\xi)| \leq \alpha^{k+1}&(j+1)\cdot j\cdots(j-k+1) \\ &\cdot (2\pi)^{j+1}(2^{1-\alpha})^{k+1} I_{j+1}(f)\xi^{(\frac{\alpha-1}{\alpha})(k+1)}, \end{aligned} \tag{3.4}$$

whenever $\xi > (2j+1)^\alpha$. Using the fundamental theorem of calculus, we have, whenever $y \in [a_m^{(j-k)}, a_{m+1}^{(j-k)})$,

$$\begin{aligned} |[\widehat{f}]^{(j-k+1)}(y)| &\leq \left| \int_y^{a_{m+1}^{(j-k)}} [\widehat{f}]^{(j-k)}(\xi)\,\mathrm{d}\xi \right| \\ &\leq |a_{m+1}^{(j-k)} - a_m^{(j-k)}| \alpha^{k+1}(j+1)\cdot j\cdots(j-k+1)(2\pi)^{j+1} \\ &\qquad \cdot (2^{1-\alpha})^{k+1} I_{j+1}(f) y^{(\frac{\alpha-1}{\alpha})(k+1)} \\ &\leq \alpha^{k+2}(j+1)j\cdots(j-k)(2\pi)^{j+1}(2^{1-\alpha})^{k+1} I_{j+1}(f) y^{(\frac{\alpha-1}{\alpha})(k+2)}. \end{aligned}$$

The last two inequalities follow from the hypotheses we have made on $[\widehat{f}]^{(j-k)}$ and the bounds we have on the gaps $|a_m^{(k)} - a_{m+1}^{(k)}|$ in terms of gaps of zeros of $\widehat{f}$, as long as $y > (2j+1)^\alpha$. Of course, (3.4) holds for $k = 0$, and thus, by the aforementioned argument, also for all $k = 1, \ldots, j$. In other words,

$$|\widehat{f}(\xi)| \leq (j+1)!(2^{2-\alpha}\pi)^{j+1}\alpha^{j+1} I_{j+1}(f)\xi^{(\frac{\alpha-1}{\alpha})(j+1)}, \tag{3.5}$$

as long as $\xi > (2j+1)^\alpha$. Applying the same analysis for negative $\xi \in [a_{m-1}^{(k)}, a_m^{(k)}]$, together with the observation we made about control on the gaps of the zeros of the kth derivative of $\widehat{f}$ in case $m \leq 0$ (see Section 2.1) yields the desired result for $\widehat{f}$.

In order to obtain the asserted bound for f, we run the same scheme of proof, paying attention to the fact that, if $\{b_m^{(k)}\}_{m\in\mathbb{Z}}$ denotes the sequence of zeros of the kth derivative of f, in the sense of Section 2.1, then $[b_m^{(k)}, b_{m+1}^{(k)}] \subset [\log(m+1), \log(m+k+2)]$, and

$$\begin{aligned} |b_m^{(k)} - b_{m+1}^{(k)}| &\leq \log\left(1 + \frac{k+1}{m+1}\right) \leq \frac{k+1}{m+1} \\ &\leq (k+1)\cdot(k+2)\cdot e^{-\log(m+k+2)}. \end{aligned}$$

If $x \geq 0$ belongs to the interval $[b_m^{(k)}, b_{m+1}^{(k)}]$, then the expression above is bounded by $(k+1)\cdot(k+2)\cdot e^{-x}$. This implies, in particular, that

$$|f^{(k)}(x)| \leq (2\pi)^{k+1} I_{k+1}(\widehat{f})(k+1)\cdot(k+2)e^{-|x|},$$

as long as $x \geq \log(1+k)$. In a completely analogous fashion to what we did for $\widehat{f}$, one may use this bound to, inductively, attain the bounds

$$|f^{(k-j)}(x)| \leq (2\pi)^{k+1} I_{k+1}(\widehat{f})(k+2)((k+1)\cdot k \cdots (k-j+2))^2 \cdot (k-j+1)e^{-(j+1)|x|}$$

for $x \geq \log(1+k)$ and $j \in \{0, 1, \ldots, k\}$. The same analysis, together with the observations mentioned in Section 2.1, implies the desired decay for $x < 0$ as well. We leave out the details to the induction procedure, for they essentially only replicate equations (3.2)–(3.5). ■

We now describe, in a concise way, the iteration scheme to be undertaken. In order to do so, let $f \in \mathcal{S}(\mathbb{R})$ satisfy the same assumptions as in Lemma 3.1; that is, it has zeros at $\pm\log(1+n)$ and its Fourier transform has zeros at $\pm n^\alpha$, for some $\alpha \in (0,1)$. Since $f \in \mathcal{S}(\mathbb{R})$, there is a constant $D > 0$ such that

$$|\widehat{f}(\xi)| \leq D.$$

Hence, the estimates in Lemma 3.1 for $\widehat{f}$ imply

$$\begin{aligned} I_k(\widehat{f}) &\leq D \int_{|\xi|\leq(1+2j)^\alpha} |\xi|^k \mathrm{d}\xi + \widehat{C_j} \int_{|\xi|\geq(1+2j)^\alpha} |\xi|^{k+(\frac{\alpha-1}{\alpha})j} \mathrm{d}\xi \\ &\leq 2D\frac{1}{k+1}(1+2j)^{\alpha(k+1)} + \widehat{C_j}\frac{1}{(\frac{1-\alpha}{\alpha})j-k+1}(1+2j)^{\alpha(k+(\frac{\alpha-1}{\alpha})j+1)}, \end{aligned}$$

as long as we choose $j \geq \frac{(k+2)\alpha}{1-\alpha}$. Choosing $j = j(k)$ to be the smallest integer greater than $\frac{(k+2)\alpha}{1-\alpha}$ implies

$$\begin{aligned} I_k(\widehat{f}) &\leq 2D\frac{1}{k+1}\left(3+\frac{2(k+2)\alpha}{1-\alpha}\right)^{\alpha(k+1)} + \widehat{C_j}\left(1+\frac{2(k+2)\alpha}{1-\alpha}\right)^{-\alpha} \\ &\leq A_\alpha\big(k^{\alpha(k+1)-1} + \widehat{C}_j\big) \\ &= A_\alpha\big(k^{\alpha(k+1)-1} + (j+1)!(2^{2-\alpha}\pi)^{j+1}\alpha^j I_j(f)\big). \end{aligned} \tag{3.6}$$

We also observe that the bound (3.1) in Lemma 3.1 for f with $k = 1$ implies

$$I_j(f) \leq \mathcal{C}(f)\int_{\mathbb{R}} e^{-|x|}|x|^j \,\mathrm{d}x \lesssim_f j!, \tag{3.7}$$

where the implicit constant depends only on f, but otherwise does not depend on any other parameter. Putting (3.6), (3.7) together with (3.1), we obtain

$$\begin{aligned} |f(x)| &\leq k(2\pi)^k((k+1)!)^3 I_k(\widehat{f})e^{-k|x|} \\ &\leq k(2\pi)^k((k+1)!)^3 A_\alpha\big(k^{\alpha(k+1)-1} + (j+1)!(2\pi\alpha)^j I_j(f)\big)e^{-k|x|} \\ &\leq e^{O(k\log k)-k|x|} \end{aligned} \tag{3.8}$$

for $|x| \geq \log(k+1)$, where by $O(k \log k)$ we denote an expression that is bounded by $C_\alpha k \log(k+1)$, for some constant depending on α. Equation (3.8) implies, as $k \leq e^{|x|} - 1$ can be chosen arbitrarily, that for each $A \gg 1$, there is $c_A > 0$ such that

$$|f(x)| \lesssim_{f,A} e^{-c_A |x|^A}.$$

3.2. Viewing $\widehat{f}$ as an entire function

The final part of the argument uses complex analysis to derive a contradiction. In fact, by Corollary 2.2, $\widehat{f}$ is an entire function of order at most 1. The converse to Hadamard's factorisation theorem then predicts that the sum of reciprocals of zeros of $\widehat{f}$ raised to $1+\varepsilon$ should converge, no matter which value of $\varepsilon > 0$ we choose. But we know that $\{\pm n^\alpha\}_{n \geq 0}$ is contained in the set of zeros of $\widehat{f}$, therefore

$$\sum_{n \geq 0} \frac{1}{n^{(1+\varepsilon)\alpha}} < +\infty.$$

This is a clear contradiction, as long as $\alpha < 1$. The contradiction came from assuming that $\widehat{f} \not\equiv 0$, and thus we have proved the first part of Theorem 1.2.

4. Proof of Theorem 1.2, Part (B)

Assume, throughout this section, that $f \in \mathcal{S}(\mathbb{R})$ satisfies the hypothesis of Theorem 1.2, Part (B). That is, $f(\pm n^\alpha) = \widehat{f}(\pm n^\beta) = 0$ holds for all $n \in \mathbb{N}$, where $(\alpha, \beta) \in A$ are indices belonging to the range described in the introduction.

4.1. Obtaining simultaneous decay

The first key step of the proof is, analogously to the proof of Part (A), obtaining enough decay on either f or $\widehat{f}$ in order extend the other as an analytic function. One of the key estimates for that will be an iteration scheme of inequality (2.1), which is the content of the next lemmata.

Lemma 4.1. *Let $f \in \mathcal{S}(\mathbb{R})$ and assume that $f(\pm n^\alpha) = 0$ and $\widehat{f}(\pm n^\beta) = 0$ for every $n \in \mathbb{N}$, where $0 < \alpha, \beta < 1$. Then, for $|x| > (2k+1)^\alpha$ and $|\xi| > (2j+1)^\beta$, one has*

$$|f(x)| \leq (k+1)!(2^{2-\alpha}\pi)^{k+1}\alpha^k I_k(\widehat{f})|x|^{(\frac{\alpha-1}{\alpha})k} = C_k |x|^{(\frac{\alpha-1}{\alpha})k},$$
$$|\widehat{f}(\xi)| \leq (j+1)!(2^{2-\beta}\pi)^{j+1}\beta^j I_j(f)|\xi|^{(\frac{\beta-1}{\beta})j} = \widehat{C}_j |\xi|^{(\frac{\beta-1}{\beta})j}.$$

The proof of this lemma is identical to that of the second assertion in Lemma 3.1, and we therefore omit it. Lemma 4.1 means that one can obtain very good decay for $f(x)$ for large values of x by sacrificing the potentially large number

$$C_k = (k+1)!(2^{2-\alpha}\pi)^{k+1}\alpha^k I_k(\widehat{f}) = B_k I_k(\widehat{f}).$$

Thus, we need some device in order to control the growth of these numbers in terms of k. The number B_k is easy to estimate by using Stirling's formula. Indeed,

$$\begin{aligned} B_k &\le C e^{-(k+1)+(k+\frac{3}{2})\log(k+1)+k\log(2\pi\alpha)} \\ &\le c_\alpha e^{k\log k+(\log(2\pi\alpha)+1)k+\frac{3}{2}\log k}. \end{aligned} \tag{4.1}$$

Meanwhile, the number $I_k(\widehat{f})$, although finite due to the fact that the functions we are interested in belong to the Schwartz class, might grow at an undesirable rate. Our next step is to use Lemma 4.1 in order to produce the appropriate control over the growth of such integrals.

Lemma 4.2. *Let* $f \in \mathcal{S}(\mathbb{R})$ *and assume that* $f(\pm n^\alpha) = 0$ *and* $\widehat{f}(\pm n^\beta) = 0$ *for every* $n \in \mathbb{N}$, *where* $0 < \alpha + \beta < 1$. *Then there exists* $\tau = \tau(\alpha, \beta) > 0$ *such that*

$$I_k(f) \lesssim_{f,\alpha,\beta} e^{\tau k\log k + O(k)}.$$

Proof. From the proof of Theorem 1.2, Part (A), and generally the remarks made after the proof of Lemma 3.1,

$$\begin{aligned} I_k(\widehat{f}) &\le 2D\frac{1}{k+1}\left(3 + \frac{2(k+2)\beta}{1-\beta}\right)^{\beta(k+1)} + \widehat{C_j}\left(1 + \frac{2(k+2)\beta}{1-\beta}\right)^{-\beta} \\ &\le A_\beta\big(k^{\beta(k+1)-1} + \widehat{C}_j\big), \end{aligned}$$

where $|\widehat{f}| \le D$ pointwise. We can now apply the same inequality to $I_k(f)$, and obtain

$$I_k(f) \le A_\alpha\big(k^{\alpha(k+1)-1} + C_{\widehat{j}}\big), \tag{4.2}$$

where $\widehat{j} = \widehat{j}(k)$ is the smallest integer larger than $\frac{(k+2)\alpha}{1-\alpha}$. Keeping in mind that

$$\begin{aligned} C_k &= (k+1)!(2^{2-\alpha}\pi)^{k+1}\alpha^k I_k(\widehat{f}) = B_k I_k(\widehat{f}), \\ \widehat{C}_j &= (j+1)!(2^{2-\beta}\pi)^{j+1}\beta^j I_j(f) = \widehat{B}_j I_j(f), \end{aligned}$$

one can iterate inequalities (3.6) and (4.2) within each other. More precisely,

$$\begin{aligned} I_k(f) &\le A_\alpha\big(k^{\alpha(k+1)-1} + B_{\widehat{j}(k)} I_{\widehat{j}(k)}(\widehat{f})\big) \\ &\le A_\alpha\big(k^{\alpha(k+1)-1} + B_{\widehat{j}(k)} A_\beta\big(\widehat{j}(k)^{\beta(\widehat{j}(k)+1)-1} + \widehat{C}_{j(\widehat{j}(k))}\big)\big) \\ &= A_\alpha\big(k^{\alpha(k+1)-1} + B_{\widehat{j}(k)} A_\beta\big(\widehat{j}(k)^{\beta(\widehat{j}(k)+1)-1} + \widehat{B}_{j(\widehat{j}(k))} I_{j(\widehat{j}(k))}(f)\big)\big). \end{aligned}$$

This chain of inequalities amounts to the following inequality:

$$I_k(f) \le G(k) + H(k) I_{j(\widehat{j}(k))}(f), \tag{4.3}$$

where

$$\begin{aligned} G(k) &= A_{\alpha,\beta}\big(k^{\alpha(k+1)-1} + B_{\widehat{j}(k)}\widehat{j}(k)^{\beta(\widehat{j}(k)+1)-1}\big), \\ H(k) &= A_{\alpha,\beta} B_{\widehat{j}(k)} \widehat{B}_{j(\widehat{j}(k))}. \end{aligned} \tag{4.4}$$

An observation in order is that, from the way we defined $j(n), \widehat{j}(n)$ for $n \in \mathbb{N}$,

$$0 < \rho(k) - \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\beta}{1-\beta}\right)k \lesssim_\beta 1, \tag{4.5}$$

where we let $\rho(k) = j(\widehat{j}(k))$. Therefore, we let

$$\gamma = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\beta}{1-\beta}\right) < 1 \iff \alpha + \beta < 1.$$

Since we assumed that $\alpha + \beta < 1$, we have $\gamma < 1$ and inequality (4.3) translates directly to

$$I_k(f) \le G(k) + H(k)I_{\rho(k)}(f). \tag{4.6}$$

The bound (4.6) can be successively iterated, as we are roughly "decreasing the degree" of the integral I_k to $\gamma k + c_\beta$, where we let $c_\beta > 0$ be the constant appearing on the right-hand side of (4.5).

In order to do such an iteration, we define, for each $k \in \mathbb{N}$, a sequence of numbers $\{\omega_l(k)\}_{l\in\mathbb{N}}$ associated to it as

$$\omega_{l+1}(k) = \gamma \cdot \omega_l(k) + c_\beta, \quad \omega_0(k) = k.$$

With this definition and iterating (4.6), keeping in mind (4.5), one obtains

$$\begin{aligned} I_k(f) \le \sum_{l=0}^{m-1}\left[G(\omega_l(k))\prod_{s=0}^{l-1}H(\omega_s(k))\right] \\ + H(\omega_{m-1}(k))\cdots H(\omega(k))H(k)I_{\rho^m(k)}(f). \end{aligned} \tag{4.7}$$

In order for our bounds to behave in a controlled way, we assume at this point that $A_{\alpha,\beta} = 1$ in (4.4), which is possible simply by dividing f by $A_{\alpha,\beta}$ at the cost of an extra constant depending only on α and β on the desired bounds. We estimate G using (4.1):

$$G(k) \lesssim_\alpha e^{\alpha(k+1)\log k} + e^{(1+\beta)\frac{\alpha}{1-\alpha}k\log k + O(k)} \le e^{\lambda k \log k + O(k)}, \tag{4.8}$$

where we let

$$\lambda = (1+\beta)\frac{\alpha}{1-\alpha},$$

and we have used that the bound controlling $G(k)$ as a sum of two terms in (4.8) is bounded by the maximum of the two terms. We estimate H in the same fashion:

$$\begin{aligned} H(k) &= B_{\widehat{j}(k)}\widehat{B}_{j(\widehat{j}(k))} \\ &\lesssim_\alpha e^{(\frac{\alpha}{1-\alpha})k\log k + O(k)}e^{\gamma k \log k + O(k)} \\ &\le e^{\delta k \log k + O(k)}, \end{aligned} \tag{4.9}$$

where

$$\delta = \frac{\alpha}{1-\alpha} + \gamma = \frac{\alpha}{(1-\alpha)(1-\beta)}.$$

This means, on the other hand, when translating estimates (4.8) and (4.9) to (4.7),

$$\prod_{s=0}^{l-1} H(\omega_s(k)) \leq \exp\left(\sum_{s=0}^{l-1}[\delta\gamma^s k\log k + \delta\tilde{c_\beta}\log k + \gamma^s O(k) + O(\tilde{c_\beta})]\right)$$
$$\leq e^{\delta\frac{1-\gamma^l}{1-\gamma}k\log k + O(k)},$$

whenever $k \gtrsim_\beta 1$ and $l \lesssim_\beta \log k$, where we let $\tilde{c_\beta} = \frac{c_\beta}{1-\gamma}$. Therefore, using (4.7), we obtain

$$I_k(f) \leq \left[\sum_{l=0}^{m-1} e^{\lambda\gamma^l k\log k + O(\gamma^l k)} e^{\delta\frac{1-\gamma^l}{1-\gamma}k\log k + O(k)}\right]$$
$$+ e^{\delta\frac{1-\gamma^m}{1-\gamma}k\log k + O(k)} I_{\rho^m(k)}(f)$$
$$\leq m e^{(\lambda+\delta)\frac{1}{1-\gamma}k\log k + O(k)} + e^{\delta\frac{1}{1-\gamma}k\log k + O(k)} I_{\rho^m(k)}(f).$$

We have used, in the inequalities above, the estimate $\omega_j(k) \leq \gamma^j \cdot k + \tilde{c_\beta}$ several times. Now, if we choose m to be the least integer larger than $-\frac{\log k}{\log\gamma}$, we are going to have

$$I_k(f) \leq e^{\frac{\lambda+\delta}{1-\gamma}k\log k + O(k)} \cdot J(f),$$

where

$$J(f) = \int_{\mathbb{R}} |f(x)|(1+|x|)^{\tilde{c_\beta}+1}.$$

The proof of the lemma is then complete by letting $\tau = \frac{\lambda+\delta}{1-\gamma}$. ∎

The choice of τ given by the proof of Lemma 4.2 is going to be important in the considerations below. In fact, one direct consequence of applying Lemma 4.2 to the estimates in Lemma 4.1 is that we obtain an explicit decay for $\widehat{f}$ of the form

$$\begin{aligned}|\widehat{f}(\xi)| &\leq e^{(1+\frac{\lambda+\delta}{1-\gamma})k\log k + O(k)}|\xi|^{k(\frac{\beta-1}{\beta})}\\ &= e^{(1+\frac{\lambda+\delta}{1-\gamma})k\log k + (\frac{\beta-1}{\beta})k\log|\xi| + O(k)},\end{aligned} \tag{4.10}$$

whenever $(1+2k)^\beta \leq |\xi|$. Now, if one chooses $k \sim |\xi|^{\frac{1}{\epsilon}}$, the exponent in (4.10) becomes

$$\left[\frac{1}{\epsilon}\left(1+\frac{\lambda+\delta}{1-\gamma}\right)\log|\xi| + \left(\frac{\beta-1}{\beta}\right)\log|\xi|\right]|\xi|^{\frac{1}{\epsilon}} + O(|\xi|^{\frac{1}{\epsilon}}).$$

As long as

$$\frac{1}{\epsilon}\left(1+\frac{\lambda+\delta}{1-\gamma}\right) < \frac{1-\beta}{\beta},$$

or equivalently

$$\begin{aligned}\epsilon &> \left(1+\frac{\lambda+\delta}{1-\gamma}\right)\frac{\beta}{1-\beta}\\ &= \frac{1-\alpha-\beta+(2-\beta^2)\alpha}{1-\alpha-\beta}\,\frac{\beta}{1-\beta}\\ &= \frac{1+\alpha-\beta(1+\alpha\beta)}{1-\alpha-\beta}\cdot\frac{\beta}{1-\beta},\end{aligned} \tag{4.11}$$

we can conclude that, for some $0 < \theta < 1$,

$$|\widehat{f}(\xi)| \lesssim_f e^{-(1-\theta)|\xi|^{\frac{1}{\epsilon}}}, \tag{4.12}$$

where estimate (4.11) is obviously true for some admissible large ϵ, where by "admissible" we mean a number such that $(1+2k)^{\beta} < |\xi| \sim k^{\epsilon}$; or, in other words, so that we find ourselves within the context of applying Lemma 4.1.

Although (4.12) already gives us some exponential-like decay, which, together with Lemma 2.1 allows us to extend f to the whole complex plane as an entire function in case $\epsilon < 1$, we mention that we can further *sharpen* our results by rerunning the optimisation algorithm from above. Although the details of such a procedure shall be undertaken in the next subsection, we state and prove briefly a lemma improving the magnitude of $I_k(f)$.

Lemma 4.3. *Let $f \in \mathcal{S}(\mathbb{R})$ be such that*

$$|f(x)| \le C_f e^{-(1-\theta)|x|^{\frac{1}{\delta}}}. \tag{4.13}$$

Then $I_k(f) \lesssim_{f,\delta,\theta} \Gamma(\delta(k+1))$.

Proof. By (4.13), we have

$$I_k(f) \lesssim \int_{\mathbb{R}} e^{-(1-\theta)|x|^{\frac{1}{\delta}}} |x|^k \, dx.$$

By the change variables $x \rightsquigarrow \frac{t^{\delta}}{(1-\theta)^{\delta}}$, we have

$$\begin{aligned} \int_{\mathbb{R}} e^{-(1-\theta)|x|^{\frac{1}{\delta}}} |x|^k \, dx &= \frac{2\delta}{(1-\theta)^{k(\delta+1)}} \int_0^{\infty} e^{-t} t^{\delta(k+1)-1} \, dt \\ &= \frac{2\delta}{(1-\theta)^{k(\delta+1)}} \Gamma(\delta(k+1)), \end{aligned}$$

which directly implies the assertion of the lemma. ∎

4.2. *Optimising the exponent*

It is important to point out that up to this point the only condition imposed on the pair (α, β) is that $\alpha + \beta < 1$. This means that, whenever f is a Schwartz function such that $f(\pm n^{\alpha}) = 0$ and $\widehat{f}(\pm n^{\beta}) = 0$, then inequality (4.12) holds for some small θ and ε satisfying (4.11). We now describe an iteration procedure to improve the decay obtained in the previous subsection, at the cost of extra constraints on the pair (α, β).

Let $\epsilon(\widehat{f})$ denote the infimum of all $\epsilon > 0$ obtained in the previous subsection such that (4.12) holds. That is, we let

$$\epsilon(\widehat{f}) = \frac{\beta(1 + \frac{\lambda+\delta}{1-\gamma})}{1-\beta}.$$

Define $\epsilon(f)$ in the same fashion, exchanging the roles of α and β. A careful analysis of the estimates from the previous subsection implies that

$$|f(x)| \lesssim e^{-(1-\vartheta)|x|^{\frac{1}{a}}}$$

holds for $|x| \gtrsim_{\alpha,\beta} 1$ and *any* $a > \epsilon(f)$. The process that follows is a way to progressively decrease the magnitude of both $\epsilon(f)$ and $\epsilon(\widehat{f})$.

It follows from Lemmata 4.1, 4.3 and estimate (4.10) that

$$|\widehat{f}(x)| \le e^{(1+\epsilon(f))k\log k + (\frac{\beta-1}{\beta})k\log|\xi| + O(k)}.$$

Define then two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ associated to $f, \widehat{f}$ to be

$$\begin{aligned} b_0 &= \epsilon(\widehat{f}), & a_0 &= \epsilon(f), \\ b_{n+1} &= (1+a_n)\frac{\beta}{1-\beta}, & a_{n+1} &= (1+b_n)\frac{\alpha}{1-\alpha}. \end{aligned} \tag{4.14}$$

Notice that the definition of $\epsilon(f), \epsilon(\widehat{f})$ and of the sequences, together with a straightforward induction argument, implies that $a_n > \alpha, b_n > \beta$ for all $n \ge 0$. In spite of their seemingly sudden occurrence, we will see that these sequences determine, iteratively, improvements on the exponential-like behaviour of the functions f and $\widehat{f}$.

In order to set the iteration process needed to improve the decay of f and $\widehat{f}$ in motion, we will need Lemmata 4.1, 4.3 and estimate (4.12). Indeed, if we use Lemma 4.3 with $\widehat{f}$ – which satisfies the hypotheses by (4.12) –, we will obtain

$$I_{k+1}(\widehat{f}) \lesssim_{f,\alpha,\beta,\varepsilon} \Gamma(\epsilon((\widehat{f}) + \varepsilon)(k+2))$$

for each $\varepsilon > 0$. We can, thus, use this new bound on $I_k(\widehat{f})$ in the first assertion of Lemma 4.1. By doing so, and using Stirling's formula in order to bound the Γ-factor, one obtains

$$|f(x)| \le \exp\left(k\log k - k\frac{1-\alpha}{\alpha}\log|x| + (\epsilon(\widehat{f}) + \varepsilon)k\log k + O_\varepsilon(k)\right)$$

for each $\varepsilon > 0$ and whenever $|x| > (2k+1)^\alpha$. By using that $a_1 = (1+\epsilon(\widehat{f}))\frac{\alpha}{1-\alpha}$ and that $\varepsilon > 0$ was arbitrary, we readily obtain that, whenever $a > a_1$, the choice of k to be the smallest integer larger than $|x|^{\frac{1}{a}}$ is allowed (as we know already that $a_n > \alpha$ for all $n \ge 0$) in such a bound, and calculating it out gives a $C_1 > 0$ so that

$$|f(x)| \lesssim_{\alpha,\beta} e^{-C_1|x|^{\frac{1}{a}}}$$

holds on the real line. In a completely analogous manner, we perform the same process for $\widehat{f}$. By symmetry, for any $b > b_1$,

$$|\widehat{f}(\xi)| \lesssim_{\alpha,\beta} e^{-\tilde{C}_1|\xi|^{\frac{1}{b}}}$$

for some $\tilde{C}_1 > 0$. In other words, for any $\varepsilon > 0$,

$$|f(x)| \lesssim_{f,\varepsilon} e^{-C_1|x|^{\frac{1}{a_1+\varepsilon}}}, \quad |\widehat{f}(\xi)| \lesssim_{f,\varepsilon} e^{-\tilde{C}_1|\xi|^{\frac{1}{b_1+\varepsilon}}}$$

for each $x, \xi \in \mathbb{R}$. We then reiterate this procedure indefinitely: recalling the definition of the exponent sequences $\{a_n\}_{n\ge 0}, \{b_n\}_{n\ge 0}$ given in (4.14), suppose that, for some given $n \ge 0$ the inequality

$$|f(x)| \lesssim_{f,\varepsilon} e^{-C_n|x|^{\frac{1}{a_n+\varepsilon}}}, \quad |\widehat{f}(\xi)| \lesssim_{f,\varepsilon} e^{-\tilde{C}_n|\xi|^{\frac{1}{b_n+\varepsilon}}}$$

holds for some pair of constants $C_n, \tilde{C}_n > 0$ (that may also additionally depend on $\varepsilon > 0$, but not on x or ξ), whenever $\varepsilon > 0$. Lemma 4.3 applies directly to show that

$$I_{k+1}(f) \lesssim_{f,\alpha,\beta,\delta} \Gamma((a_n + \delta)(k+1)), \; I_{j+1}(\widehat{f}) \lesssim_{f,\alpha,\beta,\delta} \Gamma((b_n + \delta)(k+1)),$$

whenever $\delta > 0$. Using these bounds in Lemma 4.1, we obtain, whenever $|x| > (2k+1)^\alpha$ and $|\xi| > (2j+1)^\beta$,

$$|f(x)| \le \exp\left(k \log k - k \frac{1-\alpha}{\alpha} \log |x| + (a_n + \delta) k \log k + O_\delta(k)\right),$$
$$|\widehat{f}(\xi)| \le \exp\left(j \log j - j \frac{1-\alpha}{\alpha} \log |x| + (b_n + \delta) j \log j + O_\delta(j)\right)$$

for all $\delta > 0$. Setting k to be the smallest integer larger than $|x|^{1/(a_{n+1}+\varepsilon)}$ and j to be the smallest integer larger than $|\xi|^{1/(b_{n+1}+\varepsilon)}$, where $\varepsilon > 0$ is fixed, and carrying out the computations yields that there are constants $C_{n+1}, \tilde{C}_{n+1} > 0$ so that

$$|f(x)| \lesssim_{f,\varepsilon} e^{-C_{n+1}|x|^{\frac{1}{a_{n+1}+\varepsilon}}}, \quad |\widehat{f}(\xi)| \lesssim_{f,\varepsilon} e^{-\tilde{C}_{n+1}|\xi|^{\frac{1}{b_{n+1}+\varepsilon}}},$$

closing the inductive procedure. Notice that the constants $C_n, \tilde{C}_n$ are allowed to depend on n and even on $\varepsilon > 0$, but *not* on x or ξ.

In order to reach the best possible threshold for the exponential decay of f and $\widehat{f}$, we need still to analyse the limiting behaviour of the sequences $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}$. To that extent, we define

$$\theta_1(\alpha,\beta) = \frac{\alpha}{(1-\alpha)(1-\beta)}, \quad \theta_2(\alpha,\beta) = \frac{\beta}{(1-\alpha)(1-\beta)}.$$

A computation with (4.14) shows that we actually have

$$a_{n+2} = \theta_1 + \gamma a_n, \quad b_{n+2} = \theta_2 + \gamma b_n.$$

As $\gamma < 1$, we see that both $\{a_{2n}\}_{n\ge0}$ and $\{b_{2n}\}_{n\ge0}$ are convergent sequences, with limit

$$L_1(\alpha,\beta) = \lim_{n\to\infty} a_{2n} = \frac{\alpha}{1-\alpha-\beta}, \quad L_2(\alpha,\beta) = \lim_{n\to\infty} b_{2n} = \frac{\beta}{1-\alpha-\beta}.$$

This implies that, for all $\varepsilon > 0$, there are constants $C, \tilde{C} = C_\varepsilon, \tilde{C}_\varepsilon > 0$ so that

$$|f(x)| \lesssim_{f,\varepsilon} e^{-C|x|^{\frac{1}{L_1(\alpha,\beta)+\varepsilon}}}, \quad |\widehat{f}(\xi)| \lesssim_{f,\varepsilon} e^{-\tilde{C}|\xi|^{\frac{1}{L_2(\alpha,\beta)+\varepsilon}}}. \tag{4.15}$$

Notice that, if $\epsilon(f) > L_1(\alpha,\beta)$ and $\epsilon(\widehat{f}) > L_2(\alpha,\beta)$, it can be proven (for instance from (5.2)) that the sequences $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}$ are *decreasing*, and (4.15) is the best exponential decay we could expect for $f, \widehat{f}$ with our methods. Notice that condition (4.11) gives us that $\epsilon(\widehat{f}) > L_2(\alpha,\beta)$ as desired, which proves that the iteration scheme presented achieves, in fact, a better exponential decay for $f, \widehat{f}$ than the original one.

Remark. If we let $S_\mu^\nu(\mathbb{R})$ denote the *Gelfand–Shilov space* of Schwartz functions φ such that

$$\sup_{x\in\mathbb{R}} |\varphi(x) e^{h|x|^{\frac{1}{\nu}}}|, \quad \sup_{\xi\in\mathbb{R}} |\widehat{\varphi}(\xi) e^{k|\xi|^{\frac{1}{\nu}}}| < +\infty$$

for some $k, h > 0$, then we have actually proved that $f \in \tilde{S}^{\nu}_{\mu}(\mathbb{R}) := \bigcup_{\nu_0>\nu,\mu_0>\mu} S^{\nu_0}_{\mu_0}(\mathbb{R})$, where $\nu = L_1(\alpha, \beta)$ and $\mu = L_2(\alpha, \beta)$. These function spaces are originally defined through specific decay properties of the Schwartz seminorms $\varphi \mapsto \|x^\alpha \partial^\beta \varphi\|_\infty$, and the equivalence to the higher-order decay statement above is proved through the seminorm decay. This procedure is in many ways analogous to the one undertaken here to obtain that $f \in S^{\nu}_{\mu}(\mathbb{R})$, and the relationship between our proof and these function spaces was recently brought to our attention. For more information on Gelfand–Shilov spaces, see, for instance, [3,8] and the references therein.

4.3. Analytic continuation

We wish to derive a contradiction from the fact that $f \not\equiv 0$. In order to do it, we prove that either f or $\widehat{f}$ can be analytically extended with control on its order depending only on $\min\{L_1(\alpha, \beta), L_2(\alpha, \beta)\}$. Without loss of generality, let $\alpha \le \beta$. Therefore,

$$L_1(\alpha, \beta) < L_2(\alpha, \beta)$$

and, in case $\beta \le 1 - 2\alpha$, then $L_1(\alpha, \beta) < 1$, and this contains the region A described in the introduction. We then resort to Lemma 2.1, which enables us to conclude that $\widehat{f}$ is extendable as an analytic function of order at most

$$\frac{1}{1 - L_1(\alpha, \beta)}.$$

By the converse to Hadamard's factorisation theorem, we must have

$$\sum_{n\ge 0} n^{-\frac{\beta+\varepsilon}{1-L_1(\alpha,\beta)}} < +\infty$$

for each $\varepsilon > 0$. Thus, we reach an immediate contradiction if

$$\beta < 1 - L_1(\alpha, \beta).$$

As we supposed initially that $\alpha \le \beta$, elementary calculations lead to the following observation: if $(\alpha, \beta) \in A$, then each Schwartz function f such that $f(\pm n^\alpha) = \widehat{f}(\pm n^\beta) = 0$ for all $n \in \mathbb{N}$, then $f \equiv 0$. This finishes the proof of Theorem 1.2.

5. Remarks and complements

5.1. Spacing between zeros and bounds for f

In Sections 2, 3 and 4, we have seen how to obtain decay for a Schwartz function given we have information on the location of the zeros of its derivatives. A main feature, in particular, of the proof in Section 4 was that the sequence of zeros of the derivative $f^{(k)}$ satisfies $a^{(k)}_n \in [n^\alpha, (n + k + 1)^\alpha]$, which enables us to bound

$$|a^{(k)}_{n+1} - a^{(k)}_n| \le C_\alpha (k + 1)|a^{(k)}_{n+1}|^{-\frac{1-\alpha}{\alpha}} \tag{5.1}$$

if $n > k+1$. A careful look into the proofs undertaken above relates the exponent of k on the left hand side above to the iteration scheme for optimising the exponent performed in Section 4.2. Indeed, if we were able to improve the factor on the right-hand side of (5.1) from $(k+1)$ to $(k+1)^{\omega}, \omega < 1$, then the sequences a_n, b_n above would take the form

$$\begin{aligned} b_0 &= \epsilon(\widehat{f}), & a_0 &= \epsilon(f), \\ b_n &= (\omega + a_n)\frac{\beta}{1-\beta}, & a_{n+1} &= (\omega + b_n)\frac{\alpha}{1-\alpha}. \end{aligned} \tag{5.2}$$

A simple computation shows that the limit of this new sequences is strictly *smaller* than the one we obtained in Section 4.2. This yields, as a consequence, an improvement on the set A of admissible exponents for Theorem 1.2, described in the introduction. For instance, if (5.2) holds, then

$$\lim_{n\to\infty} a_n = \frac{\omega\alpha(1+(\omega-1)\beta)}{1-\alpha-\beta}, \quad \lim_{n\to\infty} b_n = \frac{\omega\beta(1+(\omega-1)\alpha)}{1-\alpha-\beta}.$$

If $\alpha \le \beta$ and ω satisfies the equation

$$\omega(1+(\omega-1)\beta) = 1-\alpha-\beta,$$

then the argument in Section 4.3 produces a contradiction whenever $\alpha+\beta < 1$, which would be the biggest regime in which one expects a version of our main theorem to hold. This raises the question whether the decay in (5.1) can be improved. Unfortunately, the answer to this question is negative. Indeed, let $a_n^{(0)} = n^{\alpha}$ as before. Consider 1the intervals $\{n \in \mathbb{N} : n^{\alpha} \in [2^j, 2^{j+1})\} = [n_j, n_{j+1})$, and define the sequence $\{a_n^{(k)}\}$ for $n \in [n_j, n_{j+1}-k)$ and $\frac{1}{j+1}2^{\frac{k}{\alpha}} < k < 2^{\frac{j}{\alpha}}$ satisfying

$$\begin{aligned} &a_{n_j}^{(k-1)} < a_{n_j}^{(k)} < (n_j+1)^{\alpha}, \\ &a_{n+1}^{(k-1)} > a_n^{(k)} > \max(a_{n+1}^{(k-1)} - 2^{-10k(1-\alpha)\frac{j}{\alpha}}, a_n^{(k-1)}). \end{aligned} \tag{5.3}$$

This satisfies, in particular, the growth requirements on the sequence from Section 2.1. For $k > 2^{\frac{j}{\alpha}}, n \in [n_j, n_{j+1})$, we let $a_n^{(k)}$ be chosen arbitrarily satisfying (I) (i) in the same subsection. The definition implies, in particular, that

$$a_{n_j+1}^{(k)} > a_{n_j+k+1}^{(0)} - \sum_{\ell\le k} 2^{-10\ell(1-\alpha)\frac{j}{\alpha}} > (n_j+k+1)^{\alpha} - c_{\alpha}2^{-10(1-\alpha)\frac{j}{\alpha}}.$$

Therefore,

$$\begin{aligned} |a_{n_j+1}^{(k)} - a_{n_j}^{(k)}| &\ge (n_j+k+1)^{\alpha} - (n_j+1)^{\alpha} - 2^{-10(1-\alpha)\frac{j}{\alpha}} \\ &\ge \alpha k \cdot (n_j+k+1)^{\alpha-1} - 2^{-10(1-\alpha)\frac{j}{\alpha}}. \end{aligned}$$

As $n_j > 2^{\frac{j}{\alpha}}$, the right-hand side is controlled from below by a constant depending on α times $k2^{-\frac{(1-\alpha)j}{\alpha}}$. As $n_{j+1} \le 2^{\frac{1}{\alpha}}2^{\frac{j}{\alpha}}$, estimate (5.1) is sharp for $k < 2^{\frac{j}{\alpha}}$. Replicating the same argument for all $j > 1$ and concatenating the sequences together implies the desired sharpness for all $k \ge 1$.

Nevertheless, a question still remaining is whether a decay better than (5.1) can hold *on average*. We have used this estimate on the gap between zeros of the kth derivative to obtain decay for $f^{(k)}$ pointwise. It could happen, though, that one obtains better decay averaging over large intervals, rather than doing pointwise evaluation. This intuitive thought is partially backed up by the fact that, for $n \in [n_j, n_{j+1} - k)$, the average gap

$$|a_{n+1}^{(k)} - a_n^{(k)}|$$

is of the same order of $2^{-(1-\alpha)\frac{j}{\alpha}}$, as long as $n - k \sim 2^{\frac{j}{\alpha}}$. We show here that this phenomenon does not happen in case the sequence of zeros $\{a_n^{(k)}\}$ has structure similar to the counterexample above. Considering the bound (2.1), we wish to bound the average of $f^{(k)}$ over the interval $[2^j, 2^{j+1})$. A computation shows that

$$\int_{2^j}^{2^{j+1}} |f^{(k)}(x)|\,\mathrm{d}x \lesssim \frac{1}{2^j}(2\pi)^k I_{k+1}(\widehat{f})\left(\sum_{l=n_j-k}^{n_{j+1}} |a_{l+1}^{(k)} - a_l^{(k)}|^2\right). \tag{5.4}$$

Notice that each of the $|a_{l+1}^{(k)} - a_l^{(k)}|$ terms is bounded by $C_\alpha \cdot (k+1)2^{-(1-\alpha)\frac{j}{\alpha}}$, for some absolute $C_\alpha > 0$. Our problem is equivalent to the following: we have a sequence of N non-negative real numbers $\{c_j\}_{j=1}^N$ such that $\sum_{j=1}^N c_j = A$ and $0 < c_j \le B$. What is the maximum of

$$\sum_{j=1}^N c_j^2, \tag{5.5}$$

and when is it attained? By fixing all but two variables, it is easy to see that the maximum of (5.5) happens when the c_j are all either B or 0. As

$$\sum_{j=1}^N c_j = A,$$

the optimal value happens when there are $\sim \frac{A}{B}$ different indices j for which $c_j = B$, and then the maximal value of (5.5) is $\sim B \cdot A$. Applying this analysis to (5.4) yields that

$$\int_{2^j}^{2^{j+1}} |f^{(k)}(x)|\,\mathrm{d}x \lesssim (2\pi)^k I_{k+1}(\widehat{f})C_\alpha \cdot (k+1)2^{-(1-\alpha)\frac{j}{\alpha}}, \tag{5.6}$$

as long as $k \le 2^{\frac{j}{\alpha}}$, which is essentially the same as we obtained before. In order to prove that there is a sequence with the behaviour described above, we define a sequence $\{a_n^{(k)}\}$ of the following form: on the interval $[n_j, n_j + k + 1)$, we define our sequence exactly as in (5.3); we then do the same construction as in (5.3) on $[n_j + k + 1, n_j + 2(k+1))$, but with $n_j + k + 1$ in place of n_j. Similarly, we do it for each of the $\sim \frac{1}{k}2^{\frac{j}{\alpha}}$ intervals of the form $[n_j + \ell(k+1), n_j + (\ell+1)(k+1))$. The sequence obtained that way will nearly maximise the square sums, in the sense that there are going to be $\sim \frac{1}{k}2^{\frac{j}{\alpha}}$ terms close to $\sim k2^{-(1-\alpha)\frac{j}{\alpha}}$, and the remaining ones will be close to zero. A computation shows that the bound (5.6) holds in the same way for this sequence.

These examples indicate that not much more can be improved in our methods in terms of the range of exponents A above without additional information about the location of the sequences of zeros $\{a_n^{(k)}\}_{k\ge 0, n\in\mathbb{Z}}$.

5.2. *Generalisations of Theorem 1.2*

5.2.1. Conditions on the sets of zeros. One might wonder if the sequences in Theorem 1.2 being composed of powers and logarithms of integers plays an important role in our proofs, but it does not. The spacing of the zeros comes into the proofs in order to produce the first decay estimates, and for that the important piece of information that plays a role is the bound (5.1), which comes from the distance between two consecutive zeros of the derivatives of f, and the growth condition of the sequence of zeros of f and $\widehat{f}$. In other words, if $f(\pm a_n) = f(\pm b_n) = 0$, then it is sufficient to have two positive numbers η and ω such that

$$\eta \cdot \omega > 1,$$
$$|a_{k+n} - a_n| \leq Ck|a_{k+n}|^{-\eta},$$
$$|b_{k+n} - b_n| \leq Ck|b_{k+n}|^{-\omega}$$

in order to apply the same procedure as in Lemma 4.2 and obtain the initial degree of exponential decay. Now, in order to optimise the exponent as in Section 4.2, we need

$$|a_n| \leq Cn^{\frac{1}{1+\eta}}, \quad |b_n| \leq Cn^{\frac{1}{1+\omega}},$$

where $(\alpha, \beta) = (\frac{1}{1+\eta}, \frac{1}{1+\omega})$ belong to the region A in Theorem 1.2. This means our results are stable under small perturbations of the sequences of zeros. In fact, one can even delete a large number of zeros and still get the same results. One should compare, for instance, to the interpolation result (1.2) mentioned in the introduction, whose proof, to the best of our knowledge, is rigid to the fact that the interpolation nodes are the square roots of the natural numbers, and the construction of the interpolation basis itself shows that one cannot remove any term from the sequence without breaking down the final result.

5.2.2. Conditions on the functions. Another very natural question that arises from the results is if it is completely necessary to assume the functions involved are in the Schwartz class. Perhaps the result could hold with more relaxed conditions, but our proof rely heavily on finiteness of $I_k(f)$ and $I_k(\widehat{f})$ for every $k \geq 0$, and this implies, although not in a straightforward manner, that f is a Schwartz function. For the sake of completeness, we outline the proof of this fact.

First of all, by Fourier inversion and the Riemann-Lebesgue lemma, finiteness of $I_k(\widehat{f})$ implies that f is of C^∞ class with all derivatives bounded and converging to zero at infinity. Now, we only need to prove polynomial decay of all the derivatives of f, and in order for that to be true we start by proving that f has polynomial decay. For a fixed $N > 0$, we define the set

$$E_{j,N} = E_j = \{x \in [2^j, 2^{j+1}) : |x|^N f(x) > 1\}.$$

From Chebychev's inequality, we have

$$|E_j| \leq \int_{2^j}^{2^{j+1}} |f(x)||x|^N \, \mathrm{d}x \leq 2^{-jN} I_{2N}(f).$$

This means there is $y \in E_j$ and $x \in [2^j, 2^{j+1})\backslash E_j$ such that $|x-y| \le 2^{-jN} I_{2N}(f)$. By the aforementioned fact that f' is bounded, we have

$$\begin{aligned} |f(y)| &\le |f(x)-f(y)| + |f(x)| \\ &\le C_f|x-y| + |x|^{-N} \\ &\lesssim_{N,f} |y|^{-N}. \end{aligned}$$

Therefore f has polynomial decay of any order. Now, in order to propagate this decay to every derivative, we combine the fact that f'' is a bounded function and $|f(x)| \lesssim |x|^{-N}$ with a Taylor series remainder argument in order to obtain

$$|f'(x)| \lesssim |x|^{-\frac{N}{2}}.$$

This implies polynomial decay for f'. Iterating this argument with higher-order derivatives implies that f is of Schwartz class.

5.2.3. Radial versions for other dimensions. A very natural generalisation one could think of is that of asking the same question for higher dimensional functions. Of course, the notion of density would have to be redefined for general functions of several variables since one can easily construct functions that vanish along uncountable sets, such as manifolds, but if one restricts its attention to the case of radial functions, similar questions will naturally arise. In fact, if we consider $\mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ to be the class of radial Schwartz class on $\mathbb{R}^d$, in [7] the authors study interpolation formulas in this radial setting, and dimensional differences come into the fold. Indeed, one can deduce that the lattice $\mathbb{Z}^4$ forms a Fourier uniqueness set for a certain class of radial functions in dimension $d = 4$ starting from the result of Hedenmalm and Montes–Rodrígues for $d = 0$ via a duality argument, which is an analogue of the uniqueness part of the Radchenko–Viazovska interpolation, since radial functions that vanish on $\mathbb{Z}^4$ will vanish on all spheres of radius equal to the square root of an integer by Lagrange's four-square theorem. We refer the reader to [1] for more information on this beautiful connection.

These considerations motivate the question: for which exponents (α, β) does the pair $(\{n^\alpha\}_{n\in\mathbb{Z}_+}, \{n^\beta\}_{n\in\mathbb{Z}_+})$ forms a Fourier uniqueness pair for $\mathcal{S}_{\text{rad}}(\mathbb{R}^d)$? Turns out in this setting the same ideas already introduced here apply to this problem, and we outline the steps here.

Step 1. By replacing $f^{(k)}$ by the kth-order radial derivative $\partial_r^k f$, one can run the same game of intermediate zeros as in section 2.1 to get high-order polynomial decay with loss on the constants involved in terms of $I_{k,d}(f)$ and $I_{k,d}(\widehat{f})$, where

$$I_{k,d}(g) = \int_{\mathbb{R}^d} |g(x)||x|^k \, \mathrm{d}x.$$

One can also obtain analogues of Lemmata 4.2 and 4.3. More precisely, one gets the analogue of inequality (4.3) paying a dimensional constant, which means one can directly replicate Lemma 4.2 to obtain

$$|\widehat{f}(|\xi|)| \lesssim_f e^{-(1-\theta)|\xi|^{\frac{1}{\epsilon}}}.$$

Lemma 4.3 for the d-dimensional setting will read as the estimate

$$I_{k,d}(f) \lesssim_{f,\delta,\theta} \Gamma(\delta(k+d)),$$

which can be applied in the same fashion in the rest of the iteration procedures to reach the same order of decay.

Step 2. Hadamard's theorem on distribution of zeros of entire functions fails to work in the same fashion for several complex variable functions, so one cannot do the simply extend the radial functions involved to $\mathbb{C}^d$. The alternative to this is observe that the Fourier transform of a radial function can be seen as a Hankel transform. We consider the following Hankel transform:

$$\mathcal{H}_\nu(f)(\rho) := \int_0^\infty f(r) A_\nu(r\rho)\,\mathrm{d}r,$$

where $\mathcal{A}_\nu(s) = (2\pi s)^\nu J_\nu(2\pi s)$, and J_ν is a Bessel function of first kind. In this setting, if we consider $\widetilde{f}(r) = f(r)r^{d-1}$, which has the same zeros as f, then

$$\widehat{f}(\xi) = (2\pi)^{\frac{d}{2}} \mathcal{H}_{\frac{d-2}{2}}(\widetilde{f})(|\xi|).$$

By observing that the function $\mathcal{A}_{\frac{d-2}{2}}$ can be extended as a real entire function satisfying the estimate

$$|\mathcal{A}_{\frac{d-2}{2}}(\xi + i\eta)| \lesssim_d e^{2\pi|\eta|},$$

it is clear that an analogue version of Lemma 2.1 holds for the Hankel transform.

Step 3. In order to finish, we now combine the analytic extension property of the Hankel transform and its connections with the Fourier transform mention in Step 2, together with the decay mentioned in Step 1, one can invoke Hadamard's theorem in the same fashion as before and conclude f has to be the zero function, as long as $(\alpha, \beta) \in A$, where A is the set introduced in Theorem 1.2.

5.3. *Open problems*

Comparing Theorem 1.2 and (1.2), we see that there is a gap in area between the two pictures. The point $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$ considered by Radchenko and Viazovska possesses a "quasi-uniqueness" property, in the sense that there is essentially one real function who vanishes on the nodes $\pm\sqrt{n}$ and belongs to the Schwartz class. We believe that the question of denseness of the sequences $(\pm n^\alpha, \pm n^\beta)$ plays an important role in removing this rigidity condition, which is reflected on the following conjecture.

Conjecture. Let $\alpha, \beta \in (0, 1)$ be such that $\alpha + \beta < 1$. If a function $f \in \mathcal{S}(\mathbb{R})$ satisfies $f(\pm n^\alpha) = \widehat{f}(\pm n^\beta) = 0$ for all $n \geq 0$, then $f \equiv 0$.

Of course, Theorem 1.2 is partial progress towards this conjecture, but our techniques do not seem to be immediately susceptible to being generalised in order to conclude the full conjecture. On the other hand, another interesting problem that, as far as we know, is still largely unexplored is that of sequences that grow roughly as a power of an integer, but do not posses as strong tightness properties as in Section 5.2.1 above.

Question 5.1. Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta < 1$. Under which conditions does it hold that, for two sequences $(\pm c_n, \pm d_n)_{n \geq 0}$ such that

$$\lim_{n\to\infty} \frac{d_n}{n^\beta}, \lim_{n\to\infty} \frac{c_n}{n^\alpha} < +\infty$$

and a function $f \in \mathcal{S}(\mathbb{R})$ such that $f(\pm c_n) = \widehat{f}(\pm d_n) = 0$ for all $n \geq 0$, then $f \equiv 0$?

The first natural guess is that a result of that kind should hold in the same range as Conjecture 5.3, but it would already be interesting if one could prove that the uniqueness property holds under the assumptions in Theorem 1.2. Finally, our last question concerns what happens on the critical case of Theorem 1.2.

Question 5.2. Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta = 1$. Suppose $f \in \mathcal{S}(\mathbb{R})$ is a real function such that $f(\pm a n^\alpha) = \widehat{f}(\pm b n^\beta) = 0$ holds for each natural number $n \geq 0$. Under which conditions on $a, b > 0$ do we have that $f \equiv 0$?

This type of questions remains heavily unexplored even in the $\alpha = \beta = \frac{1}{2}$ case, where we believe that a combination of our present techniques with those of [16] may be useful.

Acknowledgements. We would like to especially thank Felipe Gonçalves and Danylo Radchenko for discussing these problems and ideas about Fourier uniqueness pairs with us. We would also like to thank Christoph Thiele for helpful suggestions during the development of this manuscript. We also thank Emanuel Carneiro and Lucas Oliveira for helpful comments and remarks. In addition, we would like to thank the anonymous referees for corrections, comments and remarks that improved a lot the overall exposition of this paper.

Funding. João P. G. Ramos acknowledges financial support from the Deutscher Akademischer Austauschdienst (DAAD).

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