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Charles Collot · Tej-Eddine Ghoul · Slim Ibrahim · Nader Masmoudi

On singularity formation for the two-dimensional unsteady Prandtl system around the axis

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Abstract. We consider the two-dimensional unsteady Prandtl system. For a special class of outer Euler flows and solutions of the Prandtl system, the trace of the tangential derivative of the tangential velocity along the transversal axis solves a closed one-dimensional equation. First, we give a precise description of singular solutions for this reduced problem. A stable blow-up pattern is found, in which the blow-up point is ejected to infinity in finite time, and the solutions form a plateau with growing length. Second, in the case where, for a general analytic solution, this trace of the derivative on the axis follows the stable blow-up pattern, we show persistence of analyticity around the axis up to the blow-up time, and establish a universal lower bound of $(T-t)^{7/4}$ for its radius of analyticity.

Keywords. Prandtl's equations, blow-up, singularity, self-similarity, stability, analyticity, blowup rate

1. Introduction

We consider the two-dimensional unsteady Prandtl boundary layer equations:

$$\begin{cases} u_t - u_{yy} + uu_x + vu_y = -p_x^E, & (t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad u|_{y\to\infty} = u^E, \end{cases} \tag{1.1}$$

Charles Collot: Department of Mathematics, New York University in Abu Dhabi, Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates; cc5786@nyu.edu

Tej-Eddine Ghoul: NYUAD Research Institute, New York University Abu Dhabi, PO Box 129188, Abu Dhabi, United Arab Emirates; teg6@nyu.edu

Slim Ibrahim: Department of Mathematics and Statistics, University of Victoria, 3800 Finnerty Road, Victoria, B.C., Canada V8P 5C2; ibrahims@uvic.ca

Nader Masmoudi: NYUAD Research Institute, New York University Abu Dhabi, PO Box 129188, Abu Dhabi, United Arab Emirates; and Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA; masmoudi@cims.nyu.edu

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where $\vec{u} = (u, v)$ is the velocity field, and u^E and p^E are the traces at the boundary of the tangential component of the underlying inviscid velocity field and the pressure. Prandtl in [32] introduced this model to describe the behaviour of a fluid close to a physical boundary for high Reynolds numbers. He obtained this model as a formal limit of the Navier–Stokes equation when the viscosity goes to zero. He proposed the appearance of a boundary layer where the viscosity is still effective, describing the solution between the boundary and the interior part where the dynamics is inviscid. The leading order term in the expansion in the boundary layer solves (1.1); see for example [27,34,35] for more on the derivation of the system.

1.1. On singularity formation for the 2-dimensional Prandtl equations

In this paper we are interested in the formation of a singularity in the Prandtl system. The fact that a singularity can appear in this system is a physical phenomenon that is called the unsteady separation. Van Dommelen and Shen [38] obtained the first reliable numerical result in this direction, and explained how the separation is linked to the formation of singularity. They described the singularity as being a consequence of particles squashed in the streamwise direction, with a compensating expansion in the normal direction of the boundary. We refer to [7,14,21,33] and references therein for additional numerical results on the singularity formation.

Singularity formation is one problem out of many others regarding the Prandtl boundary layer system. The system is locally well-posed in the analytical setting [24,26,34], or Gevrey setting [9]. Under monotonicity assumptions, the well-posedness holds in Sobolev regularity [1,28,31] and weak solutions also exist globally [41]. Note that the solutions we consider here do not satisfy the monotonicity assumption. In this case, the equation can be ill-posed in Sobolev regularity [15]. Similar instabilities prevent Prandtl's system from being a good approximation of the Navier–Stokes equations for high Reynolds numbers in certain cases [18]. Indeed, monotonicity and/or Gevrey regularity in the tangential x-variable are necessary to ensure that this approximation holds. We refer to [16,34] and the references therein. Finally, let us mention that the Goldstein singularity in the steady case has recently been constructed in [8].

The precise description of the formation of singularity is still an open problem. However, E and Engquist [10] proved that blow-up can happen. They make some symmetry assumptions and consider a trivial inviscid flow in the outer region ($u^E = p^E = 0$). In this case, the trace of the tangential derivative of the horizontal component of the velocity along the transversal axis solves a closed one-dimensional equation (1.3). They proved existence of blow-up for this reduced problem. Their approach is by contradiction and does not provide any information about the mechanism that leads to the singularity. For a more general class of nontrivial inviscid outer flows (u^E, p^E) but still with a suitable assumption of symmetry, such a reduction remains possible, and the corresponding one-dimensional problem still admits blow-up solutions as shown in [25]. The authors of [25] also use a convexity argument that does not give details about the singularity.

In this paper, our first results are a complete description of the mechanism that leads to the singularity for the reduced one-dimensional problem, including the case of nontrivial inviscid flows in the outer region. In particular, we prove the existence of a stable blow-up pattern, and other unstable ones.

Our approach is inspired by the description of the so-called ODE blow-up for the semi-linear heat equation (see [2, 17, 20, 29] in particular). Note that the incompressibility condition generates difficulties through the appearance of a nonlocal nonlinear transport term. Actually, this nonlocal term will induce two new effects; the singular point is ejected to infinity in finite time, and the solution forms a plateau with a growing length. Another difficulty comes from the boundary: the blow-up is not localised near a single point but happens on a large zone. We perform a careful treatment near the boundary to show that the solution stays bounded in its vicinity.

The reduced one-dimensional problem (1.3) with a different domain and boundary conditions also appears in a special class of infinite energy solutions to the Navier–Stokes equations [13]. The authors proved the existence of a similar stable blow-up pattern to the one we describe here, for a particular class of solutions. Their approach is based on parabolic methods and maximum principles, allowing for a nonperturbative argument, but requires many special assumptions. In particular, their argument does not seem to apply to the problem that we consider in the present paper. In addition, our approach based on energy methods is more robust, since it allows us to prove the stability of the fundamental profile, to construct unstable blow-ups and to derive weighted estimates.

One may wonder how the one-dimensional reduction is related to the full two-dimensional problem. From the numerics in [14] it seems that for certain solutions with symmetries the blow-up indeed happens on the vertical axis. However, for other solutions, such as the singularity considered by Van Dommelen and Shen, the numerics show that another singularity appears before the one on the vertical axis. Our second result shows that for analytic solutions, if the solution of the reduced one-dimensional problem blows up with the aforementioned stable blow-up pattern, then the solution exists up to this blow-up time in a suitable neighbourhood of the vertical axis with a universal lower bound on its local analyticity radius. This justifies that the one-dimensional profile constructed in Theorem 1 describes blowing up solutions for the two-dimensional Prandtl system (1.1).

In [4] we treated a two-dimensional Burgers model with transverse viscosity. This corresponds to a simplified version of the Prandtl system with a trivial flow at infinity, $u^E = p^E = 0$, and no vertical velocity, $v = 0$. A similar one-dimensional reduction can be made. More interestingly we were able to prove that the one-dimensional problem captures the main features of the two-dimensional singularity. As a result we obtained a complete description of the mechanism that leads to singularity for the two-dimensional problem.

In the present work, we show that the viscosity is asymptotically negligible during the singularity formation. This indicates that the full 2-d blow-up could correspond to leading order to that of the inviscid Prandtl equations. This has been proposed for the Van Dommelen and Shen singularity in [3, 11, 37]. In the recent paper [5], Collot, Ghoul and Masmoudi studied the self-similar blow-up profiles of the inviscid 2-d Prandtl equations.

In particular, they show that there exists one of the form

$$u(t,x,y) = (T-t)^{1/2}\Theta\left(\frac{x}{(T-t)^{3/2}}, \frac{y}{(T-t)^{-1/2}}\right)$$

where T is the blow-up time, and the profile $\Theta(X,Y)$ satisfies $\partial_X\Theta(0,Y) = -\sin^2(Y/2)\mathbb{1}_{0\leq Y\leq 2\pi}$. Our main result in Theorem 1 shows that this is precisely the profile of the reduced one-dimensional equation. Therefore our result can be understood as a partial stability result for the profile Θ. In a forthcoming paper, we will pursue its stability analysis for the full two-dimensional viscous Prandlt system.

1.2. A first result on the blow-up of the derivative along the vertical axis

Without loss of generality, we consider a trivial vanishing outer flow $u^E = p^E = 0$. Our result adapts straightforwardly to more general outer flows, as they just generate additional lower order terms; see comments below. Consider an initial datum $u_0(x,y)$ of the horizontal component of the velocity field for the Prandtl equation that is odd in x. Consequently, the corresponding solution $u(t,x,y)$ is also odd in x and

$$u(t,0,y) = u_{xx}(t,0,y) = 0.$$

This allows one to consider only the dynamic of the tangential derivative of u along the y-axis. To do so, we set

$$\xi(t,y) = -u_x(t,0,y), \tag{1.2}$$

which obeys the following equation for $y\in[0,\infty)$:

$$\begin{cases} \xi_t - \xi_{yy} - \xi^2 + (\int_0^y \xi)\xi_y = 0, \\ \xi(t,0) = 0, \\ \xi(0,y) = \xi_0(y). \end{cases} \tag{1.3}$$

The local well-posedness for the above equation is standard: see for example Proposition 4.1 which adapts the result of [40]. In particular, solutions for initial data in $L^1([0,\infty))$ exist, are instantaneously regularised and the following blow-up criterion holds: If the maximal time T of existence of the solution is finite, then

$$\limsup_{t\uparrow T} \|\xi(t,\cdot)\|_{L^\infty([0,\infty))} = \infty. \tag{1.4}$$

Our first main result is the precise description of the singularity formation for the reduced one-dimensional problem (1.3).

Theorem 1 (Stable blow-up for (1.3)). *There exists $\lambda_0^* \gg 1$ such that for all $\lambda_0 \geq \lambda_0^*$, there exists an $\epsilon(\lambda_0) > 0$ with the following property. For an initial datum of the form*

$$\xi_0(y) = \lambda_0^2\cos^2\left(\frac{y-\lambda_0\pi}{2\lambda_0}\right)\mathbb{1}_{0\leq y\leq 2\lambda_0\pi} + \tilde{\xi}_0(y) \quad \textit{with} \quad \|\tilde{\xi}_0\|_{L^1([0,\infty))} \leq \epsilon(\lambda_0), \tag{1.5}$$

the unique solution to (1.3) *blows up at some time*[1] $T > 0$ *with*

$$\xi(t,y) = \lambda^2(t)\cos^2\left(\frac{y-y^*(t)}{2\lambda(t)\mu(t)}\right)\mathbb{1}_{-\pi\le\frac{y-y^*}{\lambda\mu}\le\pi} + \tilde{\xi},$$

where, for some $\mu_\infty > 0$,

$$\begin{aligned}\lambda(t) &= \frac{1}{\sqrt{T-t}} + O((T-t)^{3/2}), \quad \mu(t) = \mu_\infty + O((T-t)),\\ y^*(t) &= \frac{\mu_\infty \pi}{\sqrt{T-t}} + O((T-t)^{-1/4}),\end{aligned} \tag{1.6}$$

and

$$\|\tilde{\xi}\|_{L^\infty} \le (T-t)^{-1+1/8}. \tag{1.7}$$

Moreover, on any compact set, the solution remains uniformly regular up to time T*, so that for any* $y \in [0,\infty)$*, the limit* $\lim_{t\uparrow T}\xi(t,y) = \xi^*(y)$ *exists and satisfies*

$$\xi^*(y) \sim \frac{y^2}{4\mu_\infty^2} \quad \textit{as } y \to \infty. \tag{1.8}$$

Remark 1.1. Our analysis could be extended to show the existence of other unstable blow-up dynamics for (1.3). We show in Proposition 3.2 that there exists a countable family of blow up profiles $(G_k)_{k\ge1}$, with $G_1(Z) = \cos^2(Z/2)\mathbb{1}_{-\pi\le Z\le\pi}$. We thus mention here as an open problem to show the existence of solutions to (1.3) blowing up with a G_k profile for $k \ge 2$ according to:

$$\xi(t,y) = (T-t)^{-1}G_k\left(\frac{y-y^*(t)}{\mu_\infty(T-t)^{\frac{1}{2k}-1}}\right)\mathbb{1}_{-a_k\le\frac{y-y^*(t)}{\lambda\mu}\le a_k} + \text{l.o.t.},$$

where $a_k > 0$ is defined in Proposition 3.2, $y^*(t) = \mu_\infty a_k(T-t)^{\frac{1}{2k}-1}$, and $\mu_\infty > 0$. A sketch of proof is given in arXiv:1808.05967v1.

Let us make the following comments on the results of Theorem 1.

1. On the implication for Prandtl's boundary layer. Our result shows that the blow-up does not happen at the boundary, nor at a finite distance from it, but the singularity is ejected to infinity. This fact is rarely emphasised, but can be seen in numerical results: see [14] for example. This suggests that the boundary layer should interact with the outer Euler flow in connection with other high order boundary layer models like the Triple Deck model [22], which has been proposed to describe flow regimes where Prandtl theory is expected to fail.

Moreover, Prandtl's equations are derived neglecting the viscosity effects in the horizontal direction x. Since the x-derivative becomes unbounded in our result, the approximation of the Navier–Stokes equations by the Prandtl system is invalid just before the singularity formation.

[1] Note that our proof will show $T \to 0$ as $\lambda_0 \to \infty$.

2. On the symmetry assumptions and the stable singularity formation. The reduction to the one-dimensional problem (1.3) breaks down in the general case without symmetry assumptions. Hence our stability result in Theorem 1 should be understood within the symmetry class of odd solutions. Actually, the stable 2-d singularity is expected to be nonsymmetrical from [3, 11, 37, 38]. In particular, the blow-up scales in the transversal y direction are different from the one of Theorem 1 (see [14]).

3. On more general outer flows. Our results could be extended to other nontrivial outer flows satisfying suitable symmetry assumptions (e.g. u^E odd and p^E even in x). Indeed, this will just induce the presence of new terms that are of lower order asymptotically during singularity formation, and will not perturb the blow-up mechanism. Hence the statement of Theorem 1 would remain true. This is the case, for example, of the impulsively started cylinder [38] $u^E = \kappa \sin x$ and $p^E = (\kappa^2/4)\cos(2x)$, for which the reduced equation (1.3) becomes

$$\begin{cases} \xi_t - \xi_{yy} - \xi^2 + (\int_0^y \xi)\xi_y = -\kappa^2, \\ \xi(t,0) = 0, \quad \xi(t,y) \xrightarrow[y\to\infty]{} -\kappa. \end{cases} \tag{1.9}$$

4. Displacement thickness. The displacement thickness δ^* is a quantity that measures the effect of the Prandtl layer on the outer Eulerian flow. It is defined as

$$\delta^*(t,x) = \int_0^\infty \left(1 - \frac{u(t,x,y)}{u^E(t,x)}\right) dy$$

(see for example [36, 38]). For the aforementioned flow $u^E(t,x) = \kappa \sin x$, we have $\delta^*(t,0,\kappa) = \int_0^\infty (1 + \frac{\xi(t,y)}{\kappa})\, dy$ (using L'Hôpital's rule). Kukavica, Vicol and Wang [25] proved the existence of blow-up solutions to (1.9), by establishing that a quantity similar to $\delta^*(t,0)$ could blow up in finite time. For $u^E = 0$, the analogous quantity is $\int_0^\infty \xi(t,y)\, dy$ (which is $\lim_{\kappa\to 0} \kappa\delta^*(t,0,\kappa)$). For initial data more localised than L^1, we find that this quantity blows up as $t \uparrow T$ and we give an equivalent (see Proposition 1.2).

1.3. *A second result on a general quantitative persistence of analyticity around the vertical axis up to the blow-up time*

In what follows, as in Section 1.2, we restrict ourselves to solutions of (1.1) that are odd in x, with vanishing outer flow $u^E = p^E = 0$ (again, this second assumption is for simplicity only). We consider higher order derivatives restricted to the vertical axis and introduce, for $i \geq 0$,

$$\xi_i(t,y) := \partial_x^{2i+1} u(t,0,y) \tag{1.10}$$

(hence $\xi = -\xi_0$). They solve the following system for $i \geq 0$ and $y \in [0,\infty)$:

$$\begin{cases} \partial_t \xi_i = \partial_{yy}\xi_i - \sum_{j=0}^i \binom{2i+1}{2j+1} \xi_j \xi_{i-j} + \sum_{j=0}^i \binom{2i+1}{2j} (\partial_y^{-1}\xi_j)\partial_y \xi_{i-j}, \\ \xi_i(t,0) = 0, \\ \xi_i(0,y) = \partial_x^{2i+1} u(0,0,y). \end{cases} \tag{1.11}$$

Our second result describes solutions u to (1.1) around the axis $\{x = 0\}$, combining the study of (1.11) and an analytic extension. It shows that if u_0 is any initial datum to (1.1) that is analytic in x around the axis $\{x = 0\}$ at time $t = 0$, and such that $\partial_x u|_{x=0}$, defined as the solution to (1.3), blows up at any time T satisfying the properties in the conclusion of Theorem 1, then there is a local analytic solution up to time T on a two-dimensional set around the vertical axis, with a radius of analyticity greater than $(T-t)^{7/4}$. This justifies the blow-up profile of Theorem 1 on a two-dimensional set with universal size (i.e. regardless[2] of other information on u_0 other than $\partial_x u_0|_{x=0}$), establishing a bound for the blow-up rate for the analyticity radius. Moreover, this set is causal regarding the finite speed of propagation of the Prandtl equations. Other singularities of u might form before time T, but this shows that they cannot happen too close to the vertical axis.

Given a function $\tau \in \mathcal{C}^0([0,T],(0,\infty))$, we introduce the set

$$E_{T,\tau} := \{(t,x,y) \in [0,T) \times \mathbb{R} \times [0,\infty) : |x| \le \tau(t)(T-t)^{7/4}\}. \tag{1.12}$$

Note that $\tau \ge \tau^* > 0$ for some $\tau^* > 0$. Writing $\langle a \rangle = \sqrt{1+a^2}$, we have

Theorem 2. *Assume $p^E = u^E = 0$. Assume that $u_0 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is odd in x, and analytic in x on the set $\{|x| < \delta\}$ for some $\delta > 0$, and satisfies the following hypotheses:*

(a) (Analytic bound on the axis at initial time) *There exist $C_0, \tau_0 > 0$ such that for all $i \ge 0$, $\partial_x^{2i+1} u_0 \in C([0,\infty))$ with*

$$|\partial_x^{2i+1} u_0(0,y)| \le C_0 \tau_0^{-2i-1} (2i+1)! \langle y \rangle^{-2} \quad \textit{for all } y \ge 0. \tag{1.13}$$

(b) (Stable blow-up behaviour on the axis) *There exist $T, \mu, \iota, C_0' > 0$ such that the solution ξ to (1.3) with initial datum $\xi_0(y) = -\partial_x u_0(0,y)$ blows up at time T with*

$$\xi(t,y) = \frac{1}{T-t} \cos^2\left(\frac{y - \mu\pi(T-t)^{-1/2}}{2\mu(T-t)^{-1/2}} \right) \mathbb{1}_{-\pi \le \frac{y-\mu\pi(T-t)^{-1/2}}{\mu(T-t)^{-1/2}} \le \pi} + \tilde{\xi}(t,y),$$

where for all $t \in [0,T)$,

$$|\tilde{\xi}(t,y)| + (T-t)^{-\frac{1}{2}} |\partial_y \tilde{\xi}(t,y)| \le C_0' (T-t)^{-1+\iota} \langle y\sqrt{T-t} \rangle^{-2} \quad \textit{for } y \in [0,\infty), \tag{1.14}$$

$$|\xi(t,y)| + |\partial_y \xi(t,y)| \le C_0' \quad \textit{for } y \in [0,1/2]. \tag{1.15}$$

Then there exists $\tau \in C^0([0,T],(0,\infty))$ and a function $u \in C(E_{T,\tau})$ (see (1.12)) with $u \in C^\infty(E_{T,\tau} \cap \{t > 0\})$ such that:

(i) *u is a classical solution to (1.1) on $E_{T,\tau} \cap \{t > 0\}$ and $u = u_0$ on $E_{t,\tau} \cap \{t = 0\}$.*

[2] More precisely, this set is $\{|x| \le \tau(T-t)^{7/4}\}$, and higher order derivatives than $\partial_x u_0|_{x=0}$ only influence τ.

(ii) *There exist* $C_1, \tau_1 > 0$ *such that for all* $t \in [0, T)$ *and* $y \geq 0$*,*

$$|\partial_x^3 u(t, 0, y)| \leq C_1 (T-t)^{-4}, \tag{1.16}$$

$$|\partial_x^{2i+1} u(t, 0, y)| \leq C_1 (T-t)^{-\frac{7}{2}i - \frac{1}{8}} \tau_1^{-2i-1} (2i+1)! \quad \textit{for } i \geq 2. \tag{1.17}$$

(iii) *The set* $E_{T,\tau}$ *is causal in the sense that at its boundary,*

$$|u|_{\{x = \pm\tau(t)(T-t)^{7/4}\}}| < \left| \frac{d}{dt} (\tau(t)(T-t)^{7/4}) \right|. \tag{1.18}$$

1. Uniqueness. Assume that u_0 is everywhere x-analytic, with $|\partial_x^i u_0(x, y)| \leq \bar{C} i! \bar{\tau}^{-i} \langle y \rangle^{-2}$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, for some $\bar{C}, \bar{\tau} > 0$. In this case, there exists $T_0 > 0$ and an everywhere x-analytic solution $\bar{u}$ to (1.1) on $[0, T_0] \times \mathbb{R} \times \mathbb{R}_+$, as proved in [24]. Then the solution u of Theorem 2 coincides with $\bar{u}$ as long as it is defined, i.e. $u = \bar{u}$ on $E_{T,\tau} \cap \{t \leq T_0\}$. This is because both solutions can be obtained by the same Picard iteration scheme.

2. On the assumptions. Note that there are no conditions imposed on the parameters T_0, T, μ, ι, C_0, C_0' and τ_0. Thus $\xi(t = 0)$ can, at the initial time, be away from the blow-up regime, in the sense that both T and $\tilde{\xi}(t = 0)$ can be arbitrarily large. The existence of solutions satisfying (b) is obtained as an easy extension of the proof of Theorem 1. We shall prove that for initial data in the space $\mathcal{B}$ with norm $\|f\|_{\mathcal{B}} = \sup_{y \geq 0} (|f(y)| + |\partial_y f(y)|) \langle y \rangle^2$:

Proposition 1.2. *There exists an open set in* $\mathcal{B}$ *of initial data* ξ_0 *such that the solution* ξ *to* (1.3) *satisfies assumption* (b) *of Theorem* 2*. Moreover,* $\int_0^\infty \xi(t, y)\, dy \sim (T-t)^{-3/2} \mu\pi$ *as* $t \uparrow T$*.*

3. Optimality of the lower bound. We believe that the exponent $7/4$ is optimal. This value comes from optimal bounds for the linearised dynamics induced by assumption (b), and from certain nonlinear bounds for what we identify as the worst terms, which we believe are optimal; see the formal computation in Section 5.3.1. This value was critical for the analysis and reaching it required a delicate treatment.

4. Causality. Prandtl's equations have finite speed of propagation along the tangential direction (see for example [23]). The inequality (1.18) states that at the boundary of $E_{T,\tau}$, the vector field $\partial_t + u\partial_x$ points outward.

1.4. Strategy of the proof and organisation of the paper

The proof of Theorems 1 relies on a perturbative bootstrap argument around the blow-up profile. The maximum of the solution is the most sensitive location, where the viscosity effects are nonnegligible at the parabolic scale. There, the dynamic is given by an elliptic operator with compact resolvent (3.1) in a suitable weighted space, as in [2, 17, 20, 29]. A decomposition of the solution into eigenmodes allows us to derive modulation equations for the parameters and decay for the remainder due to a spectral gap. In the midrange zone, away from the maximum but still on the support of the blow-up profile, the viscosity is negligible and we face a singularly perturbed problem (4.42). We use a new Lyapunov functional with an adapted weight and take derivatives with a suitable vector field, which are the main technical novelties of the present paper. Finally, the solution is studied near the boundary via a no blow-up argument inspired by [17, 19, 30].

The proof of Theorem 2 relies on the study of all x- and y-derivatives $\xi_{i,k} = \partial_y^k \xi_i$. Analyticity in y is first obtained by a parabolic regularisation argument. Then, linear bounds for the dynamics of $\xi_{i,k} = \partial_y^k \xi_i$ are showed, using the maximum principle and an explicit treatment of a nonlocal term. Then, a suitable analytic norm based on a weighted L^∞ space is defined. It is controlled using a bootstrap type argument. The analytic norm controls the nonlinear effects, including what we think are the worst ones, for which y-derivatives act as forcing terms for x-derivatives. To control boundary terms at $y = 0$, we rely, classically, on the fact that controlling t-derivatives allows one to control y-derivatives for parabolic equations. Implementing this argument is delicate around the blow-up time T, and we use the fact that we are away from the blow-up zone $y \sim \mu\pi(T-t)^{-1/2}$ to obtain smallness in certain terms.

The paper is organised as follows. In Section 3, we give a heuristic argument for the derivation of the blow-up profiles and some of their properties in Proposition 3.2. Section 4 is devoted to the proof Theorem 1. A bootstrap argument is described in Section 4.3, and Proposition 4.7 states the stability result in renormalised variables. The analysis near the maximum is in Section 4.4, the modulation equations and the interior Lyapunov functional are established in Lemmas 4.9 and 4.10. The midrange zone $y \sim (T-t)^{-1/2}$ is analyzed in Section 4.5; the exterior Lyapunov functionals are established in Lemmas 4.12 and 4.13. The solution is studied on compact sets in the original variable in Lemma 4.15. The main Proposition 4.7 is proved in Section 4.7, allowing us to prove Theorem 1 in the same subsection, and Proposition 1.2 in Section 4.8.

Theorem 2 is proved in Section 5. Linear bounds are first established in Proposition 5.1. Then the third order derivative and higher order derivatives for the full problem are bounded in Propositions 5.6 and 5.12 respectively, yielding the proof of Theorem 2 in Section 5.3.1. The proof of Theorem 2 uses the fact that solutions to (1.11) become instantaneously analytic in y, which is proved in Section 6.

2. Notation

Let

$$\rho(Y) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\, e^{-\frac{3Y^2}{4}}. \tag{2.1}$$

For a function h defined on some half-line $[Y_0, \infty)$ we will write, with abuse of notation,

$$\|h\|_{L^2_\rho}^2 = \int_{Y_0}^{\infty} h^2(Y)\rho(Y)\, dY, \quad \|h\|_{H^1_\rho}^2 = \int_{Y_0}^{\infty} (h^2(Y) + |\partial_Y h(Y)|^2)\rho(Y)\, dY, \tag{2.2}$$

and the value of Y_0 (being the image of the boundary $y = 0$ in (1.3) in the original variables y by a change of variable) will always be clear from the context. We denote the primitive of a function integrated from the origin by

$$\partial_y^{-1} h(y) = \int_0^y h(\tilde{y})\, d\tilde{y}, \quad \partial_Y^{-1} h(Y) = \int_0^Y h(\tilde{Y})\, d\tilde{Y}, \quad \partial_Z^{-1} h(Z) = \int_0^Z h(\tilde{Z})\, d\tilde{Z},$$

the integration being with respect to the variables y, Y or Z to be defined later on. Note that the origin will not be preserved by the change of variables: $y = 0$ does not correspond to $Y = 0$ and the integrals do not start from the same point. Recall the Hermite polynomials:

$$h_0 = 1, \quad h_1 = \sqrt{3}\, Y, \quad h_2 = 3Y^2 - 2. \tag{2.3}$$

The heat kernel will be denoted by

$$K_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}}. \tag{2.4}$$

We write $A \leq CB$ if $A, B \geq 0$ with a positive constant C that is independent of all other parameters at stake in the analysis; we call such a constant "universal". Its value may vary from one line to another. We also write $A \lesssim B$ if $A \leq CB$, and $O(B)$ means a quantity that is $\lesssim B$. We write $C(K)$ for example to indicate that the constant depends only on some parameter K. Finally, $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

3. Formal analysis and blow-up profiles

In this section we formally derive the blow-up profile for (1.3). This approach relying on matched asymptotics is inspired by [2, 12, 13, 20, 29, 39]. Let us first perform a formal computation for the effect of the viscosity near the maximum of the solution, and to obtain suitable self-similar variables. Assume that the solution to (1.3) blows up at time T, with its maximum at a point $y^*(t)$, and that the speed of this point is given by the transport part of the equation: $y^*_t = \partial_y^{-1}\xi(y^*)$. We then use parabolic self-similar variables

$$Y = \frac{y - y^*}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad f(s, Y) = (T - t)\xi(t, y),$$

and find that f solves, assuming that one can neglect the boundary condition,

$$f_s + f + \frac{Y}{2}\partial_Y f - f^2 + \partial_Y^{-1} f \partial_Y f - \partial_{YY} f = 0.$$

An obvious solution of the above equation is the constant (in space-time) solution $f = 1$, which corresponds to $\xi = 1/(T - t)$ in the original variables (which solves (1.3) but does not satisfy the boundary condition). Assuming that 1 is a good approximation of the solution for some large zone in the variable Y, we compute the evolution of the correction $\varepsilon = f - 1$:

$$\varepsilon_s + \mathscr{L}\varepsilon = \mathrm{NL}, \quad \mathscr{L}\varepsilon := -\varepsilon + \tfrac{3}{2} Y \partial_Y \varepsilon - \varepsilon_{YY}, \quad \mathrm{NL} = \varepsilon^2 - \partial_Y^{-1}\varepsilon \partial_Y \varepsilon. \tag{3.1}$$

The linearised operator $\mathscr{L}$ is well known.

Proposition 3.1. *The operator $\mathscr{L} : H^2_\rho \to L^2_\rho$ is essentially self-adjoint with compact resolvent. Its spectrum is $\{-1 + 3i/2 \colon i = 0, 1, 2, \dots\}$, with associated eigenfunctions*

$$h_i(Y) = H_i(\sqrt{3}\,Y) = \sum_{j=0}^{[i/2]} \frac{i!}{j!(i-2j)!} 3^{\frac{i-2j}{2}} (-1)^j Y^{i-2j}$$

where H_j is a Hermite polynomial.

Proof. Changing variables and setting $u(Y) = w(z)$, $z = \sqrt{3}\,Y$ gives $\mathscr{L}u = -3(\hat{\mathscr{L}}w)(z)$ where $\hat{\mathscr{L}} := \partial_{zz} - z\partial_z + 1/3$ and the result follows from the corresponding result on $\tilde{\mathscr{L}}$ whose eigenbasis consists of Hermite polynomials (see [29]). ∎

From Proposition 3.1 one sees that the linearised dynamics has one unstable direction, and an infinite number of stable modes. The unstable direction corresponds to the constant in space mode 1, and is related to a symmetry of the equation: invariance by time translation. One can assume that the blow-up time has been chosen well, so that this mode is not excited. Neglecting the nonlinear effects, one can assume from Proposition 3.1 that one mode dominates:

$$\varepsilon(s, Y) \approx Ce^{(1-\frac{3}{2}i)s} h_i(Y), \quad i \geq 1.$$

From the behaviour at infinity of the polynomials h_i, the fact that $1 + \varepsilon$ is maximal near the origin implies that $C = -c < 0$ and that $i = 2k$ is an even positive integer (the modes associated to odd integers are related to another symmetry of the equation: invariance by space translation). Therefore, $\varepsilon(s, Y) \approx -ce^{(1-3k)s} h_{2k}(Y) \approx -ce^{(1-3k)s} Y^{2k}$ for Y large. The correction ε then starts to be of the same size as the leading order term 1 in the zone

$$|Y| \sim e^{(\frac{3}{2}-\frac{1}{2k})s}, \quad \text{i.e.} \quad y - y^* \sim (T-t)^{-1+\frac{1}{2k}}.$$

This suggests introducing the new variables

$$Z := \frac{Y}{e^{(\frac{3}{2}-\frac{1}{2k})s}} = (T-t)^{1-\frac{1}{2k}}(y - y^*), \quad F(s, Z) := f(s, Y),$$

and F solves

$$F_s + F - F^2 + \left(-\left(1 - \frac{1}{2k}\right)Z + \int_0^Z F(s, \tilde{Z})\, d\tilde{Z}\right)\partial_Z F - e^{-(3-\frac{1}{k})s}\partial_{ZZ} F = 0.$$

Assuming that F is the correct rescaled unknown, the viscosity is asymptotically negligible and F should converge to a stationary solution of the self-similar inviscid equation, we obtain

$$F - F^2 + \left(-\left(1 - \frac{1}{2k}\right)Z + \int_0^Z F(\tilde{Z})\, d\tilde{Z}\right)\frac{d}{dZ}F = 0. \tag{3.2}$$

In other words, in the renormalised variables, F should tend to a self-similar solution of (1.3) without viscosity and boundary, which is

$$\psi_t - \psi^2 + \left(\int_{-\infty}^y \psi\right)\partial_y \psi = 0. \tag{3.3}$$

This equation admits a four-parameter group of symmetries: invariance by space and time translation and a two-parameter scaling group. Namely, if $\psi(t,x)$ is a solution then so is

$$\frac{1}{\lambda}\psi\left(\frac{t-t_0}{\lambda},\frac{y-y_0}{\mu}\right), \quad (t_0,y_0,\mu,\lambda)\in\mathbb{R}^2\times(0,\infty)^2.$$

This contains the action of scaling subgroups of the form $\lambda^{2k/(2k-1)}\psi(\lambda^{2k/(2k-1)}t, y/\lambda)$ for $k\geq 0$. The following proposition describes the solutions to Equation (3.2), and is essentially taken from [13].

Proposition 3.2. *Let $k\in\mathbb{N}$. Equation (3.2) admits a one-parameter family of solutions*

$$G_k(Z/\mu), \quad \mu>0. \tag{3.4}$$

For $k\geq 2$, G_k is even, compactly supported on $[-a_k,a_k]$ with $a_k=\pi/(2k\sin(\pi/2k))$, positive and increasing on $(-a_k,0)$, of class $C^{1+1/(2k-1)-\epsilon}$ on $\mathbb{R}$, and satisfies the asymptotic expansions

$$G_k(Z)\sim(2k-1)^{1+\frac{1}{2k-1}}(Z+a_k)^{1+\frac{1}{2k-1}} \quad \text{as } Z\to -a_k,$$
$$G_k(Z)=1-Z^{2k}+O(Z^{4k}) \quad \text{as } Z\to 0.$$

For $k=1$ one has the explicit formula, with a different scaling than for $k\geq 2$ to ease notation:

$$G_1(Z)=\cos^2(Z/2)\mathbb{1}_{-\pi\leq Z\leq\pi}. \tag{3.5}$$

Remark 3.3. As will be clear from the proof of Proposition 3.2 provided below, we have $\int_0^{a_k}F(Z)\,dZ=(1-1/(2k))a_k$. Using this fact, one sees that equation (3.2) admits other solutions of the form $G_k((Z-\mu a_k)/\mu)$. It also admits the trivial solutions 0 and 1. We claim that all other bounded solutions of (3.2) can be obtained by gluing a finite or an infinite number of these solutions, when they attain 1 or 0. For example, the function

$$F(Z)=\begin{cases}1 & \text{for } Z\leq 0,\\ G_k(Z) & \text{for } 0\leq Z\leq a_k,\\ G_k\left(\frac{Z-\mu a_k-a_k}{\mu}\right) & \text{for } a_k\leq Z\end{cases}$$

is also a solution with the same regularity.

The solutions G_k of (3.2) are also well defined for $k>0$ and $k\notin\mathbb{N}$. There is then a continuum of blow-up profiles for equation (3.3), but we expect that adding viscosity would prevent appearance of nonsmooth blow-up profiles.

Proof of Proposition 3.2. We perform a change of variables on $[0,\infty)$:

$$\frac{d\xi}{dZ}=\frac{\xi}{-(1-\frac{1}{2k})Z+\int_0^Z G(\tilde Z)\,d\tilde Z}, \quad H(\xi):=G(Z),$$

so that equation (3.2) becomes

$$H-H^2+\xi\partial_\xi H=0$$

whose solution is $H = (1+\xi)^{-1}$ (renormalising the constant of integration). Notice that the function 1 is the only constant solution to (3.2), and that for nonconstant solutions, there should be a nonempty neighbourhood such that $-(1-\frac{1}{2k})Z + \int_0^Z G(\tilde{Z})\,d\tilde{Z} \neq 0$. This justifies the above change of variables. Unwinding the transformation one finds

$$\frac{dZ}{d\xi} = \frac{1}{\xi}\Big[-\Big(1-\frac{1}{2k}\Big)Z + \int_0^Z G(\tilde{Z})\,d\tilde{Z}\Big],$$

which gives

$$\frac{d^2Z}{d\xi^2} = -\frac{1}{\xi}\frac{dZ}{d\xi} - \Big(1-\frac{1}{2k}\Big)\frac{1}{\xi}\frac{dZ}{d\xi} + \frac{1}{\xi}\frac{dZ}{d\xi}F(Z) = \frac{dZ}{d\xi}\Big[-\Big(2-\frac{1}{2k}\Big)\frac{1}{\xi} + \frac{1}{\xi+\xi^2}\Big]$$

and hence

$$\frac{d}{dZ}\Big(\log\frac{dZ}{d\xi}\Big) = -\Big(2-\frac{1}{2k}\Big)\frac{1}{\xi} + \frac{1}{\xi+\xi^2}.$$

Integration yields

$$\log\frac{dZ}{d\xi} = C + \log\big(\xi^{-(2-\frac{1}{2k})}\big) + \log\xi - \log(\xi+1)$$

with an integration constant C. Because of the invariance of the equations by scaling, we can without loss of generality consider

$$\frac{dZ}{d\xi} = \frac{\xi^{-(1-\frac{1}{2k})}}{1+\xi}, \quad Z(0) = 0.$$

Since $Z(0) = 0$, one deduces that

$$\lim_{\xi\to\infty} Z(\xi) = \int_0^\xi \frac{\tau^{-(1-\frac{1}{2k})}}{1+\tau}\,d\tau = \frac{\pi}{\sin\big(\frac{\pi}{2k}\big)}.$$

and that as $\xi \to 0$,

$$Z = 2k\xi^{\frac{1}{2k}}(1 + O(\xi)),$$

while as $\xi \to \infty$,

$$Z = \frac{\pi}{\sin\big(\frac{\pi}{2k}\big)} - \frac{\xi^{-1+\frac{1}{2k}}}{1-\frac{1}{2k}}(1 + O(\xi^{-1})).$$

Near the origin $Z \sim 0$, this yields

$$\xi = \Big(\frac{Z}{2k}\Big)^{2k}(1 + O(Z^{2k}))$$

and as $Z \to \frac{\pi}{\sin(\frac{\pi}{2k})}$,

$$\xi = \Big(1-\frac{1}{2k}\Big)^{-\frac{2k}{2k-1}}\Big(\frac{\pi}{\sin\big(\frac{\pi}{2k}\big)} - Z\Big)^{-\frac{2k}{2k-1}}\Big(1 + O\Big(\frac{\pi}{\sin\big(\frac{\pi}{2k}\big)} - Z\Big)^{\frac{2k}{2k-1}}\Big).$$

Therefore near the origin $Z \sim 0$ we have $G(Z) = 1 - (2k)^{-2k} Z^{2k} + O(Z^{4k})$, and near $Z \sim \pi/\sin(\pi/(2k))$,

$$G(Z) = \left(1 - \frac{1}{2k}\right)^{\frac{2k}{2k-1}} \left(\frac{\pi}{\sin\left(\frac{\pi}{2k}\right)} - Z\right)^{\frac{2k}{2k-1}} (1 + O((a_k - Z)^{\frac{2k}{2k-1}})).$$

For $k \geq 2$, we finally define $G_k(Z) = G(2kZ)$ where G is defined above. Then G_k also solves (3.2) by scaling invariance, its support is $[-a_k, a_k]$ for $a_k = \pi/(2k \sin(\pi/(2k)))$, and it has the desired asymptotic behaviour near $-a_k$ and 0. The computation in the case $k = 1$ is more explicit, and gives $Z = 2\tan^{-1}\sqrt{\xi}$, that is, $Z = \frac{1}{\tan^2(Z/2)+1} = \cos^2(Z/2)$. Hence the result follows. ■

From Proposition 3.2 and Remark 3.3, equation (3.3) admits a family of backward self-similar profiles for $k \in \mathbb{N}$ which are smooth on their support:

$$\psi(t, y) = \frac{1}{T-t} G_k\left((y - y^*(t)) \frac{(T-t)^{1-\frac{1}{2k}}}{\mu}\right), \quad y^*(t) = \frac{\mu a_k}{(T-t)^{1-\frac{1}{2k}}} + y_0^*, \quad \mu > 0.$$

They blow up in finite time and their support, which is $y \in [y_0^*, y_0^* + 2a_k/(\mu(T-t)^{1-\frac{1}{2k}})]$, is growing to infinity. The formal analysis we just performed indicates that they could be at the heart of the blow-up phenomenon.

4. Equation on the axis

In this section we aim at proving Theorem 1. First, let us give the following local well-posedness result which is an adaptation of [40]. Note that if ξ solves (1.3), then $\lambda^2 \xi(\lambda^2 t, \lambda y)$ is also a solution. The scaling transformation $h \mapsto \lambda^2 h(\lambda y)$ is an isometry on $L^{1/2}([0, \infty))$ and (1.3) is then said to be $L^{1/2}$-critical.

Proposition 4.1 (Local well-posedness). *Let $\xi_0 \in L^1([0, \infty))$. Then there exists $T(\|\xi_0\|_{L^1}) > 0$ and a unique solution of the Duhamel formulation of (1.3) such that $\xi \in C([0, T], L^1([0, \infty)))$, $\xi(0, \cdot) = \xi_0(\cdot)$ and*[3] *$\|\partial_y \xi(t)\|_{L^1} \lesssim t^{-1/2}$. Moreover, $\xi \in C^\infty((0, T] \times [0, \infty))$ and for each $k \in \mathbb{N}$, $\partial_y^k \xi \in C((0, T], L^1([0, \infty)))$. For any $k \in \mathbb{N}$ and $0 < T_1 \leq T$, the solution map is locally uniformly continuous from L^1 into $C([T_1, T], W^{k,1}[0, \infty))$.*

Solutions associated to initial data of the form (1.5) are thus well-defined and we now turn to the proof of Theorem 1. We will sometimes use an alternative formula for the profile:

$$\begin{aligned} G_1(Z) &= \cos^2\left(\frac{Z}{2}\right) \mathbb{1}_{-\pi \leq Z \leq \pi} = \left(\frac{1}{2} + \frac{1}{2}\cos Z\right) \mathbb{1}_{-\pi \leq Z \leq \pi} \\ &= 1 - \frac{Z^2}{4} + \frac{Z^4}{48} + O(|Z|^6) \quad \text{as } Z \to 0. \end{aligned} \tag{4.1}$$

[3] With a multiplicative constant that depends on $\|\xi_0\|_{L^1([0,\infty))}$.

The proof of Theorem 1 relies on a bootstrap argument performed near the blow-up profile. First we explain how to suitably decompose a solution near the blow-up profile and then set up the bootstrap procedure. The fact that such solutions satisfy the properties of Theorem 1 is then showed at the end of this section.

4.1. *Adapted geometrical decomposition and renormalised flow*

The following lemma states that in a suitable neighbourhood of the set of self-similar profiles, there exists a unique way to project the solution onto this set using adapted orthogonality conditions.

Lemma 4.2 (Geometrical decomposition). *There exist $\lambda^*, \delta, K > 0$ such that for all $\lambda_0 \geq \lambda^*$ and $Y_0 \leq -\lambda_0^2$, for any regular $\varepsilon \in B_{L^2_\rho}(\delta\lambda_0^{-4})$ with $\varepsilon(Y_0) = -G_1(Y_0/\lambda_0^2)$, there exist $(\lambda, \mu, \tilde{Y}_0) \in (0,\infty)^2 \times \mathbb{R}$ such that the following decomposition holds:*

$$G_1\left(\frac{Y}{\lambda_0^2}\right) + \varepsilon(Y) = \lambda^2 G_1\left(\frac{Y-\tilde{Y}_0}{\lambda^2\mu}\right) + \tilde{\varepsilon}(Y-\tilde{Y}_0) \quad \text{with } \tilde{\varepsilon} \perp h_0, h_1, h_2 \text{ in } L^2_\rho.$$

Moreover, these are the only such parameters satisfying $|\lambda - 1|\lambda_0^4 + |\mu| + |\tilde{Y}_0| \leq K$. This defines a mapping $\varepsilon \mapsto (\lambda, \mu, \tilde{Y}_0)$, which is of class C^1 in L^2_ρ.

Remark 4.3. One has to keep track of the free boundary in the Y variable, and we made a slight abuse of notation in Lemma 4.2. Indeed, the space L^2_ρ to which ε belongs is given by (2.2) with boundary at Y_0, whereas the space L^2_ρ to which $\tilde{\varepsilon}$ belongs, and in which it satisfies the orthogonality condition, is defined by (2.2) with boundary at $Y_0 - \tilde{Y}_0$.

The proof of the above lemma is a standard combination of the implicit function theorem and a Taylor expansion of G_1 near the origin. It is relegated to Appendix B.

For a function $\xi : [0, T) \times [0, \infty) \to \mathbb{R}$, given parameters $(\lambda, \mu, y^*) \in C^1([0, T), (0,\infty)^2 \times \mathbb{R})$, we define two renormalisations. The first one is the parabolic self-similar renormalisation close to the blow-up point:

$$s = s_0 + \int_{t_0}^{t} \lambda^2(\tilde{t})\, d\tilde{t}, \quad Y = \lambda(y - y^*), \quad f(s, Y) = \frac{1}{\lambda^2}\xi(t, y). \tag{4.2}$$

The second one is the renormalisation associated to the leading order of the profile:

$$Z = \frac{y - y^*}{\lambda\mu} = \frac{Y}{\lambda^2\mu}, \quad F(s, Z) = \frac{1}{\lambda^2}\xi(t, y) = f(s, Y). \tag{4.3}$$

The function ξ solves (1.3) if and only if the functions f and F solve the equations

$$\begin{cases} f_s + \frac{\lambda_s}{\lambda}(2 + Y\partial_Y)f - f^2 + \partial_Y^{-1} f \partial_Y f + (\int_{-\lambda y^*}^{0} f - \lambda y_s^*)\partial_Y f - \partial_{YY} f = 0, \\ f(s, -(\pi + a)\lambda^2\mu) = 0, \end{cases} \tag{4.4}$$

and

$$\begin{cases} F_s + \frac{\lambda_s}{\lambda}(2 - Z\partial_Z)F - \frac{\mu_s}{\mu} Z\partial_Z F - F^2 + \partial_Z^{-1} F \partial_Z F \\ \quad + \left(\int_{-\frac{y^*}{\lambda\mu}}^0 F - \frac{y_s^*}{\lambda\mu}\right)\partial_Z f - \frac{1}{\lambda^4\mu^2}\partial_{ZZ} F = 0, \\ F(s, -(\pi + a)) = 0, \end{cases} \tag{4.5}$$

respectively. Since λ will behave like $(T-t)^{-1/2}$, and the blow-up point will behave like $\pi\mu(T-t)^{-1/2}$, we introduce the correction a:

$$y^* = \lambda\mu(\pi + a). \tag{4.6}$$

We adopt the following different notation for the remainder:

$$\begin{aligned} &f(s,Y) = G_1(Z) + \varepsilon(s,Y), \\ &F(s,Z) = G_1(Z) + u(s,Z), \quad \text{so that } \varepsilon(s,Y) = u(s,Z). \end{aligned} \tag{4.7}$$

4.2. *The weighted norm and derivative outside the blow-up point*

To control the solution, we need a special weight and a special vector field to take derivatives, both adapted to the linearised operator in the Z-variable. We refer to Section 4.5 and Lemma 4.11 for the motivation regarding these choices. Let $q : \mathbb{R} \to [0,\infty)$ be an even function with the following properties: $q \in C^2((0,\infty))$, $q(0) = 0$, $q' > 0$ on $(0,\pi)$ with $\lim_{Z\downarrow 0} q'(Z) > 0$, $q'(\pi) = 0$, $q''(\pi) < 0$, and $q(Z) = q(\pi) = 1$ for $Z \geq \pi$. Define a weight w on $(0,\infty) \times \mathbb{R}^*$ by

$$w(s,Z) := \begin{cases} \frac{1+\cos Z}{(1-\cos Z)\sin^4 Z}\frac{1}{\sin(-Z)}4(\pi+Z)^3\frac{1}{s^{q(Z)}} & \text{if } Z \in (-\pi,0), \\ \frac{1+\cos Z}{(1-\cos Z)\sin^4 Z}\frac{1}{\sin Z}4(\pi-Z)^3\frac{1}{s^{q(Z)}} & \text{if } Z \in (0,\pi), \\ \frac{1}{s} & \text{if } |Z| \geq \pi. \end{cases} \tag{4.8}$$

Note that the weight $w(s,\cdot)$ is even, of class C^1 on $(0,\infty)$, and C^2 on $(0,\pi)$ and (π,∞). To take derivatives in a suitable way, we will use the vector field $A\partial_Z$, where

$$A(Z) := \begin{cases} -1 & \text{for } Z \leq -\pi/2, \\ \sin Z & \text{for } -\pi/2 \leq Z \leq \pi/2, \\ 1 & \text{for } \pi/2 \leq Z. \end{cases} \tag{4.9}$$

Note that one has the following sizes for $s > 0$ and $Z \in [-\pi,\pi]$:

$$w \approx \frac{1}{|Z|^7 s^{q(Z)}}, \quad |A| \approx |Z|. \tag{4.10}$$

4.3. *The bootstrap regime*

The solution we will construct will be close to the blow-up profile in the following sense. At the initial time we require the following bounds, involving parameters which will be fixed later on. Note that Lemma 4.2 will imply the uniqueness of the decomposition used below:

$$f(s,Y) = G_1\left(\frac{Y}{\lambda^2\mu}\right) + \varepsilon(s,Y), \quad \varepsilon \perp_\rho (h_0,h_1,h_2), \tag{4.11}$$

Definition 4.4 (Initial closeness). Let $M \gg 1$, $s_0 \gg 1$ with $M^3 e^{-s_0} \ll 1$, $0 < \nu \ll 1$ and $\xi_0 \in C^\infty([0,\infty),\mathbb{R})$ with $\xi_0(0) = \partial_{yy}\xi_0(0) = 0$. We say that ξ_0 is *initially* (at $t = t_0$, i.e. $s = s_0$) *close to the blow-up profile* if there exist $\lambda_0 > 0$, $a_0 \in \mathbb{R}$ and $\mu_0 > 0$ such that the following properties are satisfied. In the variables (4.2) one has the decomposition (4.11) with $(s, \varepsilon(s), \lambda, \mu) = (s_0, \varepsilon_0, \lambda_0, \mu_0)$, where the remainder and the parameters satisfy:

(i) *Initial values of the modulation parameters:*

$$\tfrac{1}{2} e^{\frac{s_0}{2}} < \lambda_0 < 2e^{\frac{s_0}{2}}, \quad 1/2 < \mu_0 < 2, \quad |a_0| < e^{-\frac{1}{2}s_0}. \tag{4.12}$$

(ii) *Initial smallness of the remainder in the parabolic variables:*

$$\|\varepsilon_0\|_{L^2_\rho} < e^{-\frac{7}{2}s_0}, \quad \|\varepsilon_0\|_{H^3(|Y|\le M^3)} < e^{-\frac{7}{2}s_0}. \tag{4.13}$$

(iii) *Initial smallness of the remainder in the inviscid self-similar variables:*

$$\begin{aligned} &\int_{-\pi-a_0}^{-Me^{-s_0}} u^2 w\, dZ + \int_{Me^{-s_0}}^{\infty} u^2 w\, dZ < e^{-2(\frac{1}{2}-\nu)s_0}, \\ &\int_{-\pi-a_0}^{-Me^{-s_0}} |A\partial_Z u|^2 w\, dZ + \int_{Me^{-s_0}}^{\infty} |A\partial_Z u|^2 w\, dZ < e^{2\nu s_0}. \end{aligned} \tag{4.14}$$

(iv) *Initial regularity close to the origin $y = 0$ in the original variables:*

$$\|\xi_0\|_{W^{1,\infty}([0,2])} < 1. \tag{4.15}$$

We aim at proving that solutions which are initially close to the blow-up profile in the sense of Definition 4.4 will stay close to this blow-up profile up to modulation. The proximity at later times is defined as follows.

Definition 4.5 (Trapped solutions). We say that a solution $f(s,Y) = F(s,Z)$ is *trapped on* $[s_0,s_1]$, with $s_0 < s_1 < \infty$, if it satisfies the properties of Definition 4.4 at time s_0 with the parameters ν and M and if, for $K \gg 1$ and $0 < \nu' \ll \nu$, and for all $s \in [s_0,s_1]$, $f(s,\cdot)$ can be decomposed as in (4.7) and (4.11) with:

(i) *Values of the modulation parameters:*

$$\frac{1}{K} e^{\frac{s}{2}} < \lambda < K e^{\frac{s}{2}}, \quad \frac{1}{K} < \mu < K, \quad |a| < K e^{-(\frac{1}{2}-2\nu)s}. \tag{4.16}$$

(ii) *Smallness of the remainder in parabolic variables:*

$$\|\varepsilon\|_{L^2_\rho} < K e^{-\frac{7}{2}s}, \quad \|\varepsilon\|_{H^3(|Y|\le M^2)} < K e^{-(\frac{7}{2}-\nu')s}. \tag{4.17}$$

(iii) *Smallness of the remainder in the inviscid self-similar variables:*

$$\begin{aligned}&\int_{-\pi-a}^{-Me^{-s}} u^2 w\, dZ + \int_{Me^{-s}}^{\infty} u^2 w\, dZ < K^2 e^{-2(\frac{1}{2}-\nu)s},\\ &\int_{-\pi-a}^{-Me^{-s}} |A\partial_Z u|^2 w\, dZ + \int_{Me^{-s}}^{\infty} |A\partial_Z u|^2 w\, dZ < K^2 e^{2\nu s}.\end{aligned} \tag{4.18}$$

Remark 4.6. Lemma 4.2 and the regularity of the flow (Proposition 4.1) imply that the parameters of Definition 4.5 are uniquely determined and are in $C^1([s_0, s_1])$. In particular, the renormalisations (4.2) and (4.3) are indeed well-defined.

The heart of the paper is the following bootstrap proposition.

Proposition 4.7. *There exist universal constants $K, M, s_0^* \gg 1$ and $0 < \nu' \ll \nu \ll 1$ such that for any $s_0 \geq s_0^*$, any solution which is initially close to the blow-up profile in the sense of Definition 4.4 is trapped on $[s_0, \infty)$ in the sense of Definition 4.5.*

Lemma 4.2 and a standard continuity argument imply that for s_0 large enough, any solution which is initially close to the blow-up profile in the sense of Definition 4.4 is trapped in the sense of Definition 4.5 on some interval $[s_0, s_1]$ with $s_1 > s_0$. Letting $s^* > s_0$ be the supremum of times $s_1 > s_0$ such that the solution is trapped on $[s_0, s_1]$, the purpose now is to show that $s^* = \infty$. The strategy is to study the trapped regime via several lemmas and show that the solutions cannot escape from the open set defined by Definition 4.5. The proof of Proposition 4.7 is then given at the end of this section.

Note that the constants K, M, s_0^*, ν', ν and η (defined in Lemma 4.10) will be adjusted during the proof: we will always be able to prove various lemmas by choosing M large enough depending on K and then choosing s_0^* large enough depending on K and M. First, note that one has pointwise control of the remainder for trapped solutions.

Lemma 4.8. *There exists $\nu^* > 0$ such that for any K and $0 < \nu, \nu' < \nu^*$, for any M large enough, there exists an s_0^* such that if u is trapped on $[s_0, s_1]$ with $s_0^* \leq s_0$, then for all $s \in [s_0, s_1]$,*

$$\|\varepsilon\|_{L^\infty} = \|u\|_{L^\infty} \lesssim K s e^{-(\frac{1}{4}-\nu)s}. \tag{4.19}$$

Proof. First, Sobolev embedding together with (4.17) implies

$$\|\varepsilon\|_{L^\infty(|Z|\leq e^{-s}M^2)} \leq C(K, M) e^{-(\frac{7}{2}-\nu')s} \leq e^{-(\frac{1}{4}-\nu)s}$$

for s_0 large depending on K, M. Let $E := \{Z : -\pi - a \leq Z \leq -Me^{-s}\} \cup \{Z : Me^{-s} \leq Z\}$. Then from (4.10), we have $w \gtrsim s^{-1}$ and $|A|w \gtrsim s^{-1}$ on E, implying

$$\begin{aligned}\|u\|_{L^2(E)}^2 &\lesssim s\int_{-\pi-a}^{-Me^{-s}} u^2 w + s\int_{Me^{-s}}^{\infty} u^2 w,\\ \|\partial_Z u\|_{L^2(E)}^2 &\lesssim s\int_{-\pi-a}^{-Me^{-s}} |A\partial_Z u|^2 w + s\int_{Me^{-s}}^{\infty} |A\partial_Z u|^2 w.\end{aligned}$$

Therefore, Agmon's inequality and (4.18) give

$$\|u\|_{L^\infty(E)} \le C\|u\|_{L^2(E)}^{\frac12}(\|u\|_{L^2(E)}+\|\partial_Z u\|_{L^2(E)})^{\frac12} \le CKs^{\frac12}e^{-(\frac14-\nu)s}.$$

Hence, as for M large enough depending on K the two zones $|Z|\ge Me^{-s}$ and $|Y|\le M^2$ cover the whole space, we have $\|u\|_{L^\infty} \lesssim Kse^{-(\frac14-\nu)s}+e^{-s/4}\lesssim Kse^{-(1/4-\nu)s}$ for s_0 large enough. ∎

4.4. *Analysis near the blow-up point*

This subsection is devoted to the study of the solution near y^* in parabolic variables (4.2). This is the most sensitive zone, in which the blow-up parameters are selected. The remainder is dissipated away from this point, until it reaches the outside region $|Z|\gtrsim 1$ where another dynamics takes place (see next subsection). The analysis near the blow-up point is a consequence of the blow-up profile structure, the linear structure (Proposition 3.1) and the orthogonality conditions (4.11). The measure $\rho = ce^{-3Y^2/4}$ decreases very fast because of the transport part of the operator $\mathscr{L}$ which is unbounded and pushes the characteristics away from the origin. Therefore, the analysis here is poorly affected by the exterior dynamics. From (4.4), (4.7), (3.2) and (4.6) we infer that ε solves

$$\begin{cases} \varepsilon_s+\mathscr{L}\varepsilon+\tilde{\mathscr{L}}\varepsilon+\mathrm{Mod}+\mathrm{NL}-\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1(Z)=0, \\ \varepsilon(s,-(\pi+a)\lambda^2\mu)=-G_1(-\pi-a), \end{cases} \tag{4.20}$$

where $\mathscr{L}$ is defined by (3.1), and where the small linear term, the modulation term and the nonlinear term are

$$\tilde{\mathscr{L}}\varepsilon := 2(1-G_1(Z))\varepsilon+(\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\partial_Y\varepsilon+\frac{1}{\lambda^2\mu}\partial_ZG_1(Z)\partial_Y^{-1}\varepsilon, \tag{4.21}$$

$$\begin{aligned} \mathrm{Mod}(Y) :=& -\frac{\mu_s}{\mu}Z\partial_ZG_1(Z)+\left(\frac{\lambda_s}{\lambda}-\frac12\right)\big((2-Z\partial_Z)G_1(Z)+(2+Y\partial_Y)\varepsilon\big) \\ &+\left(\int_{-(\pi+a)\lambda^2\mu}^0 f\,dY-\lambda y_s^*\right)\left(\frac{1}{\lambda^2\mu}\partial_ZG_1(Z)+\partial_Y\varepsilon\right), \\ \mathrm{NL} :=& -\varepsilon^2+\partial_Y^{-1}\varepsilon\partial_Y\varepsilon. \end{aligned}$$

The parameters evolve according to the following dynamics.

Lemma 4.9 (Modulation equations). *For ν small enough, and K,ν',M such that Lemma 4.8 holds true, there exists s_0^* such that for a solution trapped on $[s_0,s_1]$ with $s_0\ge s_0^*$,*

$$\left|\frac{\lambda_s}{\lambda}-\frac12+\frac{1}{4\lambda^4\mu^2}\right| \le C(K)(\lambda^{-8}+\lambda^{-4}\|\varepsilon\|_{L^2_\rho}+\|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}), \tag{4.22}$$

$$\left|\frac{\mu_s}{\mu}-\frac{1}{2\lambda^4\mu^2}\right| \le C(K)(\lambda^{-8}+\|\varepsilon\|_{L^2_\rho}+\lambda^4\|\varepsilon\|_{L^\infty_\rho}\|\varepsilon\|_{L^2_\rho}), \tag{4.23}$$

$$\left|\int_{-\lambda y^*}^{0} f\,dY-\lambda y^*_s\right|\leq C(K)(e^{-e^s}+\|\varepsilon\|_{L^2_\rho}+\lambda^4\|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}), \tag{4.24}$$

$$\left|a_s+\frac{a}{2}-\int_{-\pi-a}^{-\pi}G_1\,dZ-\frac{1}{\lambda^2\mu}\int_{-\lambda y^*}^{0}\varepsilon\,dY\right|$$
$$\leq C(K)(\lambda^{-4}+\|\varepsilon\|_{L^2_\rho}+\lambda^4\|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}). \tag{4.25}$$

and we have the bound[4]

$$e^{2s}\left|\frac{\lambda_s}{\lambda}-\frac{1}{2}+\frac{1}{4\lambda^4\mu^2}\right|+\left|\frac{\mu_s}{\mu}\right|+\left|\int_{-\lambda y^*}^{0} f\,dY-\lambda y^*_s\right|\leq e^{-\frac{13}{8}s}. \tag{4.26}$$

To simplify notation, we define

$$m_1=\frac{\lambda_s}{\lambda}-\frac{1}{2},\quad m_2=\frac{\mu_s}{\mu},\quad m_3=\int_{-\lambda y^*}^{0} f\,dY-\lambda y^*_s. \tag{4.27}$$

Observe that m_1 is the difference between the evolution of λ and the expected self-similar law, while m_3 is the difference between the speed of the blow-up point and the value of the transport part of the equation at this point.

Proof of Lemma 4.9. This is a direct and standard computation using the definition of the geometrical decomposition and the spectral structure of the linearised dynamics. First we differentiate the orthogonality conditions (4.11) for $i=0,1,2$ using the boundary condition (4.20):

$$0=\frac{d}{ds}\left(\int_{-\lambda y^*}^{\infty}\varepsilon h_i\rho\,dY\right)=-\frac{d}{ds}(\lambda y^*)(h_i\rho)(-\lambda y^*)G_1(-\pi-a)+\int_{-\lambda y^*}^{\infty}\varepsilon_s h_i\rho\,dY.$$

Thanks to (4.16) and (4.6), one has $\lambda y^*\gtrsim e^s$ and therefore $|\rho(\lambda y^*)|\leq e^{-e^{3s/2}}$ when s_0 is large enough. Hence, as $|\int_{-(\pi+a)\lambda^2\mu}^{0} f\,dY|\lesssim\lambda^2\mu\lesssim e^s$ from (4.7) and (4.19), the above identity can be rewritten as

$$\int_{-\lambda y^*}^{\infty}\varepsilon_s h_i\rho\,dY=O(e^{-e^s}(1+|m_1|+|m_3|)). \tag{4.28}$$

We now estimate the contribution of each term when inserting (4.20) in the above identity.

Step 1. *The linear and small linear terms.* Performing integration by parts and thanks to the orthogonality (4.11) and Proposition 3.1, using the boundary condition (4.20) and (4.81) (the latter estimate states boundedness at the boundary, its proof is given later on), we get

$$\begin{aligned}\int_{-\lambda y^*}^{\infty}h_i\mathscr{L}\varepsilon\rho\,dY&=(\partial_Y\varepsilon\rho h_i)(-\lambda y^*)-(\varepsilon\rho\partial_Y h_i)(-\lambda y^*)+\int_{-\lambda y^*}^{\infty}\mathscr{L}h_i\varepsilon\rho\,dY\\&=(\partial_Y\varepsilon\rho h_i)(-\lambda y^*)+(\rho\partial_Y h_i)(-\lambda y^*)G_1(-\pi-a)\\&=O(e^{e^{-s}}(1+|\partial_Y\varepsilon(-\lambda y^*)|))=O(e^{-e^s}).\end{aligned} \tag{4.29}$$

[4] There is indeed no constant factor in front of $e^{-13s/8}$.

The small linear term is evaluated as follows. First, using Cauchy–Schwarz, and since $|1-G_1(Z)| \lesssim Z^2 \lesssim \lambda^{-4} Y^2$, one has

$$\left| \int_{-\lambda y^*}^{\infty} h_i (1-G_1(Z)) \varepsilon \rho \, dY \right| \lesssim \lambda^{-4} \|\varepsilon\|_{L^2_\rho}.$$

Similarly, since $|(\lambda^2 \mu \partial_Z^{-1} G_1(Z) - Y)| + |Y| |\partial_Y ((\lambda^2 \mu \partial_Z^{-1} G_1(Z) - Y))| \lesssim \lambda^{-4} |Y|^3$, we get

$$\left| \int_{-\lambda y^*}^{\infty} h_i (\lambda^2 \mu \partial_Z^{-1} G_1(Z) - Y) \partial_Y \varepsilon \rho \, dY \right| \lesssim \lambda^{-4} \|\varepsilon\|_{L^2_\rho}.$$

Using Cauchy–Schwarz one estimates

$$\left| \int_0^Y \varepsilon(s, \tilde{Y}) \, d\tilde{Y} \right| \leq \|\varepsilon\|_{L^2_\rho} \left(\int_0^Y e^{\frac{3}{4} \tilde{Y}^2} \, d\tilde{Y} \right)^{\frac{1}{2}} \lesssim \|\varepsilon\|_{L^2_\rho} \frac{e^{\frac{3Y^2}{8}}}{(1+|Y|)^{\frac{1}{2}}}, \tag{4.30}$$

which implies the bound, since $|\partial_Z G_1(Z)| \lesssim \lambda^{-4} |Y|$,

$$\left| \int_{-\lambda y^*}^{\infty} h_i \frac{1}{\lambda^2 \mu} \partial_Z G_1(Z) \partial_Y^{-1} \varepsilon \rho \, dY \right| \lesssim \lambda^{-4} \|\varepsilon\|_{L^2_\rho}.$$

From (4.21) this gives the bound for the small linear term for $i = 1, 2, 3$:

$$\left| \int_{-\lambda y^*}^{\infty} h_i \tilde{\mathscr{L}} \varepsilon \rho \, dY \right| \lesssim \lambda^{-4} \|\varepsilon\|_{L^2_\rho}. \tag{4.31}$$

Step 2. *The modulation term.* We first rewrite it performing a Taylor expansion of G_1 from (4.1) near the origin and using (2.3):

$$\begin{aligned} \mathrm{Mod} = {} & m_2 \left(\frac{1}{\lambda^4 \mu^2} \left(\frac{1}{6} h_2(Y) + \frac{1}{3} h_0(Y) \right) + \mu^{-4} \lambda^{-8} r_2(Y) \right) \\ & + m_1 \big(2 h_0(Y) + \mu^{-4} \lambda^{-8} r_1(Y) + (2 + Y \partial_Y) \varepsilon \big) \\ & + m_3 \left(-\frac{1}{\lambda^4 \mu^2} \frac{1}{2\sqrt{3}} h_1(Y) + \mu^{-4} \lambda^{-8} r_3(Y) + \partial_Y \varepsilon \right) \end{aligned} \tag{4.32}$$

where $r_1(Y) = \mu^4 \lambda^8 ((2 - Z \partial_Z) G_1(Z) - 2)$ and $r_2(Y) = -\mu^4 \lambda^8 Z (\partial_Z G_1(Z) + Z/2)$ are even functions which are $O(Y^4)$, and $r_3(Y) = \mu^3 \lambda^6 (\partial_Z G_1(Z) + Z/2)$ is an odd function that is $O(Y^3)$. We recall that h_{2i} and h_{2i+1} are even and odd functions respectively and form an almost orthogonal family: $\int_{-\lambda y^*}^{\infty} h_i h_j \rho = -\int_{-\infty}^{-\lambda y^*} h_i h_j \rho = O(e^{-e^{3s/2}})$. From (4.11) and (2.3), one has $\int_{-\lambda y^*}^{\infty} p \varepsilon = 0$ for any polynomial p of degree 2. Let $\mathrm{Mod}_i := \int_{-\lambda y^*}^{\infty} h_i \, \mathrm{Mod} \, \rho$ for $i = 0, 1, 2$. Using the previous remarks, (4.11) and the boundary condition (4.20) we obtain

$$\begin{aligned} \mathrm{Mod}_0 = {} & m_2 \frac{\|h_0\|_{L^2_\rho}^2 + O(\lambda^{-4})}{3 \lambda^4 \mu^2} + m_1 \big(2 \|h_0\|_{L^2_\rho}^2 + O(\lambda^{-8}) + (Y\rho)(-\lambda y^*) G_1(-\pi - a) \big) \\ & + m_3 \left(-\mu^{-4} \lambda^{-8} \int_{-\infty}^{-\lambda y^*} r_3 \rho + \rho(-\lambda y^*) G_1(-\pi - a) \right), \\ = {} & m_2 \left(\frac{\|h_0\|_{L^2_\rho}^2}{3 \lambda^4 \mu^2} + O(\lambda^{-8}) \right) + m_1 (2 \|h_0\|_{L^2_\rho}^2 + O(\lambda^{-8})) + m_3 O(e^{-e^s}), \end{aligned} \tag{4.33}$$

where for the last bound we use the fact that $\lambda y^* \gtrsim e^s$ and $\rho = Ce^{-3Y^2/4}$; similarly

$$\begin{aligned}
\mathrm{Mod}_1 &= m_1\Big(O(e^{-e^s}) - \mu^{-4}\lambda^{-8}\int_{-\infty}^{-\lambda y^*} h_1 r_1 \rho + \frac{Y^2\rho}{\sqrt{3}}(-\lambda y^*)G_1(-\pi-a) + O(\|\varepsilon\|_{L^2_\rho})\Big)\\
&\quad + m_2\Big(O(e^{-e^s}) - \mu^{-4}\lambda^{-8}\int_{-\infty}^{-\lambda y^*} h_1 r_1\rho\Big)\\
&\quad + m_3\Big(-\frac{\|h_1\|^2_{L^2_\rho} + O(\lambda^{-4})}{2\sqrt{3}\,\lambda^4\mu^2} - \frac{Y\rho}{\sqrt{3}}(-\lambda y^*)G_1(-\pi-a)\Big)\\
&= m_2 O(e^{-e^s}) + m_1 O(e^{-e^s} + \|\varepsilon\|_{L^2_\rho}) - m_3\frac{\|h_1\|^2_{L^2_\rho} + O(\lambda^{-4})}{2\sqrt{3}\,\lambda^2\mu},
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\mathrm{Mod}_2 &= m_2\Big(\frac{\|h_2\|^2_{L^2_\rho}}{6\lambda^4\mu^2} + O(\lambda^{-8})\Big)\\
&\quad + m_1\big(O(\lambda^{-8}) + (Yh_2\rho)(-\lambda y^*)G_1(-\pi-a) + O(\|\varepsilon\|_{L^2_\rho})\big)\\
&\quad + m_3\Big(O(e^{-e^s}) - \mu^{-4}\lambda^{-8}\int_{-\infty}^{-\lambda y^*} h_2 r_3\rho + (h_2\rho)(-\lambda y^*)G_1(-\pi-a) + O(\|\varepsilon\|_{L^2_\rho})\Big)\\
&= m_2\frac{\|h_2\|^2_{L^2_\rho} + O(\lambda^{-4})}{6\lambda^4\mu^2} + m_1 O(\lambda^{-8} + \|\varepsilon\|_{L^2_\rho}) + m_3 O(e^{-e^s} + \|\varepsilon\|_{L^2_\rho}).
\end{aligned} \tag{4.35}$$

Step 3. *The nonlinear term.* Since $|h_i| \lesssim (1+Y^2)$ for $i = 0,1,2$ we estimate

$$\left|\int_{-\lambda y^*}^{\infty} \varepsilon^2 h_i \rho\, dY\right| \lesssim \|\varepsilon\|_{L^2_\rho}\|\varepsilon\|_{L^\infty}.$$

Integrating by parts, using (4.20) for the boundary term and $|\partial_Y^{-1}\varepsilon| \le Y\|\varepsilon\|_{L^\infty}$ we get

$$\left|\int_{-\lambda y^*}^{\infty} h_i \partial_Y\varepsilon\partial_Y^{-1}\varepsilon\rho\, dY\right| \lesssim \|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho} + O(e^{-e^s}).$$

Therefore, for $i = 0,1,2$,

$$\left|\int_{-\lambda y^*}^{\infty} h_i\, \mathrm{NL}\, \rho\, dY\right| \lesssim \|\varepsilon\|_{L^2_\rho}\|\varepsilon\|_{L^\infty} + O(e^{-e^s}). \tag{4.36}$$

Step 4. *The error term.* Finally, using a Taylor expansion from (4.1) we get

$$\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1(Z) = \frac{1}{\lambda^4\mu^2}\Big(-\Big(\frac12 - \frac{1}{6\lambda^4\mu^2}\Big)h_0 + \frac{1}{12\lambda^4\mu^2}h_2 + \Big(\partial_{ZZ}G_1 + \frac12 - \frac{Z^2}{4}\Big)\Big). \tag{4.37}$$

This gives (since this term is an even function and h_1 is an odd function):

$$\int_{-\lambda y^*}^{\infty} \frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1(Z)h_i\rho\, dY = \begin{cases} -\frac{1}{\lambda^4\mu^2}\big(\frac12 - \frac{1}{6\lambda^4\mu^2}\big)\|h_0\|^2_{L^2_\rho} + O(\lambda^{-12}) & \text{if } i = 0,\\ O(e^{-e^s}) & \text{if } i = 1,\\ \frac{1}{12\lambda^8\mu^4}\|h_2\|^2_{L^2_\rho} + O(\lambda^{-12}) & \text{if } i = 2. \end{cases} \tag{4.38}$$

Step 5. *End of the proof.* We collect the estimates (4.29), (4.31), (4.33)–(4.36) and (4.38) and insert them in (4.28) using (4.20) to obtain

$$m_2 \frac{1+O(\lambda^{-4})}{3\lambda^4\mu^2} + m_1(2+O(\lambda^{-8})) + m_3 O(e^{-e^s})$$
$$= -\frac{1}{\lambda^4\mu^2}\left(\frac{1}{2} - \frac{1}{6\lambda^4\mu^2}\right) + O(\lambda^{-12}) + O(\lambda^{-4}\|\varepsilon\|_{L^2_\rho} + \|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}),$$
$$m_2 O(e^{-e^s}) + m_1 O(e^{-e^s} + \|\varepsilon\|_{L^2_\rho}) - \frac{1+O(\lambda^{-4})}{\lambda^4\mu^2} m_3$$
$$= O(e^{-e^s}) + O(\lambda^{-4}\|\varepsilon\|_{L^2_\rho} + \|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}),$$
$$m_2 \frac{1+O(\lambda^{-4})}{6\lambda^4\mu^2} + m_1 O(\lambda^{-8} + \|\varepsilon\|_{L^2_\rho}) + m_3 O(e^{-e^s} + \|\varepsilon\|_{L^2_\rho})$$
$$= \frac{1}{12\lambda^8\mu^4} + O(\lambda^{-12}) + O(\lambda^{-4}\|\varepsilon\|_{L^2_\rho} + \|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}).$$

These three estimates, together with the fact that $\|\varepsilon\|_{L^2_\rho} \lesssim e^{-7s/2}$ and $\lambda \approx e^{s/2}$ obtained from (4.17) and (4.16), imply (4.22)–(4.24). The fourth inequality (4.25) is obtained from (4.22)–(4.24), since from (4.6) and $\int_{-\pi}^0 G_1 = \pi/2$,

$$\int_{-\lambda y^*}^0 f\,dY - \lambda y_s^*$$
$$= \lambda^2\mu\left[\int_{-\pi-a}^{-\pi} G_1\,dZ + \frac{1}{\lambda^2\mu}\int_{-(\pi+a)\lambda^2\mu}^0 \varepsilon\,dY - a_s - \frac{a}{2} - ((m_1+m_2)(\pi+a)\right].$$

Summing (4.22), (4.23) and (4.24), we obtain

$$\lambda^4\left|\frac{\lambda_s}{\lambda} - \frac{1}{2} + \frac{1}{4\lambda^4\mu^2}\right| + \left|\frac{\mu_s}{\mu}\right| + \left|\int_{-\lambda y^*}^0 f\,dY - \lambda y_s^*\right| \lesssim \lambda^{-4} + \|\varepsilon\|_{L^2_\rho} + \lambda^4\|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}.$$

The right-hand side is, from (4.17) and (4.16), $\leq C(K)(e^{-2s} + e^{-7s/2} + se^{-(3/2+1/4-\nu)s})$, and hence (4.26) holds if $\nu < 1/4$, for s_0 large depending on K. ■

The decay of the remainder ε is encoded by the following Lyapunov functional.

Lemma 4.10 (Interior Lyapunov functional). *There exist universal $C, \eta^* > 0$ such that for any $0 < \eta < \eta^*$ the following holds. For K, ν, ν', M such that Lemma 4.8 holds, there exists s_0^* such that for a solution that is trapped on $[s_0, s_1]$ with $s_0 \geq s_0^*$,*

$$\frac{d}{ds}\left(\frac{1}{2}\|\varepsilon\|_{L^2_\rho}^2\right) + \left(\frac{7}{2} - Ce^{-\eta s}\right)\|\varepsilon\|_{L^2_\rho}^2 + e^{-\eta s}\|\partial_Y\varepsilon\|_{L^2_\rho}^2 \leq C\|\varepsilon\|_{L^2_\rho}\lambda^{-12} + Ce^{-e^s}. \tag{4.39}$$

Proof. This is a direct computation relying on the spectral gap that absorbs the nonlinear effects, the modulation equations established previously, and the rapid decay of the measure ρ. First, from (4.20) and (4.26), one computes

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}\|\varepsilon\|_{L^2_\rho}^2\right) &= \frac{1}{2}\frac{d}{ds}\int_{-\lambda y^*}^{\infty}\varepsilon^2\rho\, dY \\ &= -\frac{1}{2}(\varepsilon^2\rho)(-\lambda y^*)\frac{d}{ds}(-\lambda y^*)+\int_{-\lambda y^*}^{\infty}\left(-\mathscr{L}\varepsilon-\tilde{\mathscr{L}}\varepsilon-\text{Mod}-\text{NL}+\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1\right)\varepsilon\rho\, dY \\ &= O(e^{-e^s}(1+\|\partial_Y\varepsilon\|_{L^2_\rho}^2))+\int_{-\lambda y^*}^{\infty}\left(-\mathscr{L}\varepsilon-\tilde{\mathscr{L}}\varepsilon-\text{Mod}-\text{NL}+\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1\right)\varepsilon\rho\, dY.\end{aligned} \tag{4.40}$$

Step 1. *The linear term.* First, we prove the dissipative spectral gap estimate

$$\int_{-\lambda y^*}^{\infty}|\partial_Y\varepsilon|^2\rho\, dY \geq \frac{9}{2}(1-Ce^{-\eta s})\int_{-\lambda y^*}^{\infty}\varepsilon^2\rho\, dY + 2e^{-\eta s}\int_{-\lambda y^*}^{\infty}|\partial_Y\varepsilon|^2\rho\, dY - Ce^{-e^s} \tag{4.41}$$

for some universal constant $C>0$. We use analytical results on the whole space $\mathbb{R}$, with scalar product $\langle u,v\rangle=\int_{\mathbb{R}}uv\rho$ (only for the next few lines). Define the extension

$$\tilde{\varepsilon}:=\begin{cases}\varepsilon(-\lambda y^*) & \text{for } Y\leq-\lambda y^*,\\ \varepsilon(Y) & \text{for } Y\geq-\lambda y^*.\end{cases}$$

Then $\tilde{\varepsilon}\in H^1_\rho$. Define the projection on higher modes by

$$\bar{\varepsilon}:=\tilde{\varepsilon}-\frac{\langle\tilde{\varepsilon},h_0\rangle}{\|h_0\|_{L^2_\rho}^2}h_0-\frac{\langle\tilde{\varepsilon},h_1\rangle}{\|h_1\|_{L^2_\rho}^2}h_1-\frac{\langle\tilde{\varepsilon},h_2\rangle}{\|h_2\|_{L^2_\rho}^2}h_2.$$

Then from the orthogonality (4.11), since $\varepsilon(-\lambda y^*)=-G_1(-\pi-a)$ from the Dirichlet boundary condition, one infers that

$$\langle\tilde{\varepsilon},h_i\rangle=-\frac{1}{2}\sqrt{\frac{3}{\pi}}\int_{-\infty}^{-\lambda y^*}h_iG_1(-\pi-a)e^{-\frac{3}{4}Y^2}\, dY=O(e^{-e^s})$$

as $\lambda y^*\gtrsim e^s$. This implies that

$$\int_{-\lambda y^*}^{\infty}\varepsilon^2\rho\, dY\leq\|\bar{\varepsilon}\|_{L^2_\rho}^2+Ce^{-e^s},\qquad\int_{-\lambda y^*}^{\infty}|\partial_Y\varepsilon|^2\rho\, dY\geq\|\partial_Y\bar{\varepsilon}\|_{L^2_\rho}^2-Ce^{-e^s}.$$

As $\bar{\varepsilon}\in H^1_\rho$ with $\bar{\varepsilon}\perp h_i$ for $i=0,1,2$, one has the spectral gap estimate from (4.41):

$$\|\partial_Y\bar{\varepsilon}\|_{L^2_\rho}^2\geq\tfrac{9}{2}\|\bar{\varepsilon}\|_{L^2_\rho}^2.$$

The above two estimates imply (4.41). Therefore, the linear term gives, from the boundary condition (4.20) and the definition (2.1),

$$\begin{aligned}-\int_{-\lambda y^*}^{\infty}\mathscr{L}\varepsilon\varepsilon\rho\, dY &= \int_{-\lambda y^*}^{\infty}\varepsilon^2\rho\, dY-\int_{-\lambda y^*}^{\infty}|\partial_Y\varepsilon|^2\rho\, dY+(\partial_Y\varepsilon\rho)(-\lambda y^*)G_1(-\pi-a)\\ &\leq-\left(\frac{7}{2}-Ce^{-\eta s}\right)\int_{-\lambda y^*}^{\infty}\varepsilon^2\rho\, dY-2e^{-\eta s}\int_{-\lambda y^*}^{\infty}|\partial_Y\varepsilon|^2\rho\, dY+Ce^{-e^s}.\end{aligned}$$

Step 2. *The small linear term.* Recall (4.21). One computes, using Poincaré (A.1) and the fact that $|G_1(Z)-1|\lesssim \lambda^{-4}Y^2$,

$$\left|\int_{-\lambda y^*}^{\infty}(1-G_1(Z))\varepsilon^2\rho\, dY\right|\le C\lambda^{-4}\|\varepsilon\|^2_{H^1_\rho}.$$

Next, by integrating by parts, the boundary condition (4.20) together with the fact that $\lambda y^*\gtrsim e^s$ gives (note that the boundary term at $Y=\infty$ is zero from the exponential decay of ρ)

$$\begin{aligned}
&\int_{-\lambda y^s}^{\infty}\varepsilon(\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\partial_Y\varepsilon\rho\, dY\\
&=-\Big[(\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\frac{\varepsilon^2}{2}\rho\Big](-\lambda y^*)-\frac{1}{2}\int_{-\lambda y^*}^{\infty}\varepsilon^2\partial_Y\big((\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\rho\big)\, dY\\
&=O(e^{-e^s})-\frac{1}{2}\int_{-\lambda y^*}^{\infty}\varepsilon^2\partial_Y\big((\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\rho\big)\, dY
\end{aligned}$$

Then, notice that for $|Y|\le e^{3s/4}$,

$$\left|\partial_Y\big((\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\rho\big)\right|\lesssim \lambda^{-4}|Y|^2(1+|Y|)^2\rho\lesssim e^{-\frac{s}{2}}|Y|^2\rho.$$

Hence, applying (4.19), (A.1), and splitting in two zones $E=\{Y:|Y|\le e^{3s/4}\}$ and $E'=[-\lambda y^*,\infty)\setminus E$ we get

$$\begin{aligned}
\left|\int_{-\lambda y^s}^{\infty}\varepsilon(\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\partial_Y\varepsilon\rho\right|&\lesssim e^{-e^s}+\left|\int_E\varepsilon^2\partial_Y\big((\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\rho\big)\right|\\
&\quad+\left|\int_{E'}\varepsilon^2\partial_Y\big((\lambda^2\mu\partial_Z^{-1}G_1(Z)-Y)\rho\big)\right|\lesssim e^{-e^s}+e^{-\frac{s}{2}}\|\varepsilon\|^2_{H^1_\rho}.
\end{aligned}$$

For the last term, using (4.30), since $|\frac{1}{\lambda^2\mu}\partial_Z G_1(Z)|\lesssim\lambda^{-4}|Y|$ one has

$$\begin{aligned}
&\left|\int_{-\lambda y^*}^{\infty}\varepsilon\frac{1}{\lambda^2\mu}\partial_Z G_1(Z)\partial_Y^{-1}\varepsilon\rho\, dY\right|\lesssim\|\varepsilon\|_{L^2_\rho}\lambda^{-4}\int_{-\lambda y^*}^{\infty}|\varepsilon|\,|Y|\frac{e^{-\frac{3Y^2}{8}}}{(1+|Y|)^{\frac12}}\, dY\\
&\lesssim\|\varepsilon\|_{L^2_\rho}\lambda^{-4}\int_E|\varepsilon|\,|Y|\frac{e^{-\frac{3Y^2}{8}}}{(1+|Y|)^{\frac12}}\, dY+\|\varepsilon\|_{L^2_\rho}\lambda^{-4}\int_{E'}|\varepsilon|\,|Y|\frac{e^{-\frac{3Y^2}{8}}}{(1+|Y|)^{\frac12}}\, dY\\
&\lesssim\|\varepsilon\|_{L^2_\rho}\lambda^{-4}\big\||Y|\varepsilon\big\|_{L^2_\rho}\left(\int_{|Y|\le e^{3s/4}}\frac{dY}{1+|Y|}\right)^{\frac12}+O(e^{-e^s})\lesssim\|\varepsilon\|^2_{H^1_\rho}\lambda^{-3}+O(e^{-e^s}),
\end{aligned}$$

where we have used (A.1) and (4.19). Therefore, putting all the above estimates together, as $\lambda\approx e^{s/2}$ we get

$$\left|\int_{-\lambda y^*}^{\infty}\varepsilon\tilde{\mathscr{L}}\varepsilon\rho\, dY\right|\lesssim e^{-e^s}+e^{-\frac{s}{2}}\|\varepsilon\|^2_{H^1_\rho}.$$

Step 3. *The modulation term.* Recall (4.27). We use the decomposition (4.32) and the orthogonality (4.11) to obtain first, with $r_1(Y)=O(Y^4)$, $r_2(Y)=O(Y^4)$ and $r_3(Y)=O(|Y|^3)$,

$$\int_{-\lambda y^*}^{\infty}\varepsilon\,\mathrm{Mod}\,\rho\,dY=\mu^{-4}\lambda^{-8}\int_{-\lambda y^*}^{\infty}\varepsilon(m_1r_1+m_2r_2+m_3r_3)\rho\,dY$$
$$+m_1\int_{-\lambda y^*}^{\infty}((2+Y\partial_Y)\varepsilon)\varepsilon\rho\,dY+m_3\int_{-\lambda y^*}^{\infty}\partial_Y\varepsilon\varepsilon\rho\,dY.$$

For the first line, using Cauchy–Schwarz with (4.22)–(4.24), (4.16) and (4.19), we get

$$\left|\mu^{-4}\lambda^{-8}\int_{-\lambda y^*}^{\infty}\varepsilon(m_1r_1+m_2r_2+m_3r_3)\rho\,dY\right|\lesssim\lambda^{-12}\|\varepsilon\|_{L^2_\rho}+e^{-2s}\|\varepsilon\|^2_{H^1_\rho}.$$

For the second line, use Poincaré (A.1) and (4.26):

$$\left|m_1\int_{-\lambda y^*}^{\infty}((2+Y\partial_Y)\varepsilon)\varepsilon\rho\,dY+m_3\int_{-\lambda y^*}^{\infty}\partial_Y\varepsilon\varepsilon\rho\,dY\right|\lesssim e^{-\frac{13}{8}s}\|\varepsilon\|^2_{H^1_\rho}.$$

The two inequalities above then give

$$\left|\int_{-\lambda y^*}^{\infty}\varepsilon\,\mathrm{Mod}\,\rho\,dY\right|\lesssim\|\varepsilon\|_{L^2_\rho}\lambda^{-12}+e^{-\frac{13}{8}s}\|\varepsilon\|^2_{H^1_\rho}.$$

Step 4. *The nonlinear term.* A direct L^∞ estimate gives

$$\left|\int_{-\lambda y^*}^{\infty}\varepsilon^3\rho\,dY\right|\lesssim\|\varepsilon\|_{L^\infty}\|\varepsilon\|^2_{L^2_\rho}.$$

For the other nonlinear term one first performs an integration by parts, then a brute force bound for the boundary term, the same estimate as above for the second term, and (A.1):

$$\int_{-\lambda y^*}^{\infty}\varepsilon\partial_Y\varepsilon\partial_Y^{-1}\varepsilon\rho\,dY=-\frac12(\varepsilon^2\partial_Y^{-1}\varepsilon\rho)(-\lambda y^*)-\int_{-\lambda y^*}^{\infty}\frac{\varepsilon^3}{2}\rho\,dY$$
$$-\frac12\int_{-\lambda y^*}^{\infty}\varepsilon^2\partial_Y^{-1}\varepsilon\partial_Y\rho\,dY$$
$$=O(e^{-e^s})+O(\|\varepsilon\|_{L^\infty}\|\varepsilon\|^2_{H^1_\rho}).$$

Step 5. *The error term.* Using the decomposition (4.37), the orthogonality (4.11) and $|\partial_{ZZ}G_1+\frac12-\frac{Z^2}{4}|\lesssim Z^4\approx\lambda^{-8}Y^4$ one obtains

$$\left|\int_{-\lambda y^*}^{\infty}\varepsilon\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1(Z)\rho\,dY\right|=\frac{1}{\lambda^4\mu^2}\left|\int_{-\lambda y^*}^{\infty}\varepsilon\left(\partial_{ZZ}G_1+\frac12-\frac{Z^2}{4}\right)\rho\,dY\right|$$
$$\lesssim\lambda^{-12}\|\varepsilon\|_{L^2}.$$

Step 6. *End of the proof.* Collecting all the estimates of Steps 1–5 one finally deduces from (4.40) that

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}\|\varepsilon\|_{L^2_\rho}\right) \le & -\left(\frac{7}{2}-Ce^{-\eta s}\right)\|\varepsilon\|^2_{L^2_\rho}-2e^{-\eta s}\|\partial_Y\varepsilon\|^2_{L^2_\rho}\\ & + C(e^{-\frac{s}{2}}+\|\varepsilon\|_{L^\infty})\|\varepsilon\|^2_{H^1_\rho}+C\|\varepsilon\|_{L^2_\rho}\lambda^{-12}+Ce^{-e^s}\\ \le & -\left(\frac{7}{2}-Ce^{-\eta s}\right)\|\varepsilon\|^2_{L^2_\rho}-e^{-\eta s}\|\partial_Y\varepsilon\|^2_{L^2_\rho}+C\|\varepsilon\|_{L^2_\rho}\lambda^{-12}+Ce^{-e^s}\end{aligned}$$

if η has been chosen small enough and s_0 large enough, where we have used (4.19). ■

4.5. Analysis outside the blow-up point in the inviscid self-similar zone

This subsection is devoted to the study of the solution outside the blow-up point $y^*(t)$ and we switch to the Z-variable (4.3). The aim is to find decay for u, which receives information from the boundaries $Z=-\pi-a$ and $Z=0$. We first explain the linear estimate which explains the choice of the weight w and then prove full energy estimates. In view of the decomposition (4.7) and (3.2), we rewrite (4.5) as

$$\begin{cases} u_s+\mathcal{H}u-\frac{1}{\lambda^4\mu^2}\partial_{ZZ}u+\tilde{\mathcal{H}}u+\mathrm{NL}+\psi=0,\\ u(s,-(\pi+a))=-G_1(-(\pi+a)),\end{cases} \tag{4.42}$$

where the leading order linearised operator is

$$\mathcal{H}u := \mathcal{T}\partial_Z u+Vu+\partial_Z^{-1}u\partial_Z G_1, \tag{4.43}$$

with the transport and the potential term being defined by

$$\mathcal{T}(Z) := -Z/2+\partial_Z^{-1}G_1 = \begin{cases} -(Z/2+\pi/2) & \text{for } Z\le -\pi,\\ \frac{1}{2}\sin Z & \text{for } -\pi\le Z\le\pi,\\ -(Z/2-\pi/2) & \text{for } \pi\le Z,\end{cases} \tag{4.44}$$

$$V(Z) := 1-2G_1(Z) = \begin{cases} 1 & \text{for } Z\le -\pi,\\ -\cos Z & \text{for } -\pi\le Z\le\pi,\\ 1 & \text{for } \pi\le Z,\end{cases} \tag{4.45}$$

the small linear term is given by

$$\tilde{\mathcal{H}}u := m_1(2-Z\partial_Z)u-m_2Z\partial_Z u+m_3'\partial_Z u,\quad m_3'=\frac{m_3}{\lambda^2\mu}, \tag{4.46}$$

where m_1, m_2 and m_3 are defined in (4.27), the nonlinear term is

$$\mathrm{NL} := -u^2+\partial_Z^{-1}u\partial_Z u,$$

and the error term is

$$\psi(s,Z) := -\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1(Z)+m_1(2-Z\partial_Z)G_1(Z)-m_2Z\partial_Z G_1(Z)+m_3'\partial_Z G_1(Z).$$

Thanks to (4.26) the parameters m_1, m_2 and m_3' satisfy

$$e^{2s}\left|m_1-\frac{1}{4\lambda^4\mu^2}\right|+|m_2|+e^s|m_3'|\leq e^{-\frac{13}{8}s}. \tag{4.47}$$

4.5.1. Linear analysis. We claim that the dynamics of equation (4.42) is driven to leading order by the transport and potential terms, and that the nonlocal, viscosity and nonlinear terms are negligible. From a direct check, the eigenvalue problem

$$\mathcal{T}\partial_Z\phi_\beta+V\phi_\beta=\beta\phi_\beta$$

admits a solution for all $\beta\in\mathbb{R}$, of the form

$$\phi_\beta(Z) := \begin{cases}\phi_\beta^{\text{int}}(Z) & \text{for } Z\in(-\pi,\pi)\setminus\{0\},\\ \phi_\beta^{\text{ext}}(Z) & \text{for } Z\in(-\infty,-\pi)\cup(\pi,\infty),\end{cases} \tag{4.48}$$

where

$$\phi_\beta^{\text{int}}(Z)=\left(\frac{1-\cos Z}{1+\cos Z}\right)^\beta\sin^2 Z,\quad \phi_\beta^{\text{ext}}(Z)=\begin{cases}(-(Z+\pi))^{2(1-\beta)} & \text{for } Z<-\pi,\\ (Z-\pi)^{2(1-\beta)} & \text{for } Z>\pi.\end{cases}$$

Note that $\phi_\beta(Z)\sim Z^{2(1+\beta)}$ as $Z\to 0$ and $\phi_\beta(\pi+Z)\sim|Z|^{2(1-\beta)}$ as $Z\to 0$. The reduced operator $\mathcal{T}\partial_Z+V$ satisfies the following comparison-type L^∞ weighted bound:

$$\left\|\frac{e^{-s(\mathcal{T}\partial_Z+V)}v_0}{\phi_\beta}\right\|_{L^\infty}\leq e^{-\beta s}\left\|\frac{v_0}{\phi_\beta}\right\|_{L^\infty},$$

which can be showed by differentiating along the characteristics. The above bound shows how cancellations near the origin for u_0 are crucial for decay since ϕ_β cancels at the origin for positive β. Our aim for the full linear problem is to perform a weighted Sobolev energy estimate which mimics the above estimate. We will modify the weight $1/\phi_\beta$ according to three principles: (1) any multiplication by a weight which is decreasing along the vector field $\mathcal{T}(Z)\partial_Z$ preserves the spectral gap estimate, (2) the nonlocal part can be treated as a perturbation of the transport and potential terms, (3) the viscosity is negligible if one is sufficiently away from the origin. These are the reasons behind the specific choice of w in (4.8). The exponent $1/2$ for the underlying eigenfunction $\phi_{1/2}$ is made by optimising two constraints: to optimise the decay in the above inequality, and to minimise the size of the boundary terms in the lemma below. We claim the following decay, at the linear level, of a Lyapunov functional with weight w. We state it on the left of the origin but the analogue holds true on the right as well.

Lemma 4.11. *Let λ, μ and a satisfy (4.16) and $\nu > 0$. Assume that u solves*

$$u_s + \mathcal{H}u - \frac{1}{\lambda^4\mu^2}\partial_{ZZ}u = 0. \tag{4.49}$$

Let $Z_1 := -(\pi + a)$ and $Z_2 := -Me^{-s}$. Then for any $K > 0$, for $M > 0$ large enough depending on K, and s_0 large enough depending on K, M, one has the estimate

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}\int_{Z_1}^{Z_2} u^2 w\,dZ\right) + \left(\frac{1}{2} - \frac{\nu}{4}\right)\int_{Z_1}^{Z_2} u^2 w\,dZ + \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z u|^2 w\,dZ \\ \leq C(K,M)e^{6s}u^2(Z_2) + C(K,M)e^{4s}|\partial_Z u|^2(Z_2) + u^2(Z_1)(e^{-(\frac{1}{2}-\nu)s} + |a_s|) \\ + C|\partial_Z u|^2(Z_1)e^{-2s} + \frac{Ce^{2s}}{M^2}\left(\int_{Z_2}^{0}|u|\,dZ\right)\left(\int_{Z_1}^{Z_2} u^2 w\,dZ\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. One first records the identity

$$\begin{aligned}&\frac{d}{ds}\left(\frac{1}{2}\int_{Z_1}^{Z_2} u^2 w\,dZ\right) \\ &= \int_{Z_1}^{Z_2} uu_s w\,dZ + \frac{1}{2}\int_{Z_1}^{Z_2} u^2 w_s\,dZ + \frac{a_s}{2}(u^2w)(Z_1) + Me^{-s}(u^2w)(Z_2).\end{aligned} \tag{4.50}$$

We compute from (4.8) that

$$\int_{Z_1}^{Z_2} u^2 w_s\,dZ = -\frac{1}{s}\int_{Z_1}^{Z_2} u^2 w q(Z)\,dZ \leq 0, \tag{4.51}$$

and we recall that

$$u_s = -Vu - \mathcal{T}\partial_Z u + \frac{1}{2}\left(\int_0^Z u(s,\tilde{Z})\,d\tilde{Z}\right)\sin Z\,\mathbb{1}_{-\pi\leq Z\leq\pi} + \frac{1}{\lambda^4\mu^2}\partial_{ZZ}u.$$

Step 1. *The potential, transport and dissipative effects.* Integrating by parts one finds

$$-\int_{Z_1}^{Z_2} u\mathcal{T}\partial_Z u w\,dZ = \frac{1}{2}(u^2\mathcal{T}w)(Z_1) - \frac{1}{2}(u^2\mathcal{T}w)(Z_2) + \int_{Z_1}^{Z_2} u^2\frac{1}{2}\partial_Z(\mathcal{T}w)\,dZ.$$

One then computes that for $-\pi < Z < 0$, from (4.44) and (4.48),

$$\begin{aligned}\frac{1}{2}\partial_Z(\mathcal{T}w) &= \frac{1}{4}\partial_Z\left(-\frac{1+\cos Z}{(1-\cos Z)\sin^4 Z}\,\frac{4(\pi+Z)^3}{s^{q(Z)}}\right) = \frac{1}{4}\partial_Z\left(-\frac{1}{\phi_{1/2}^2}4(\pi+Z)^3\frac{1}{s^{q(Z)}}\right) \\ &= -w\frac{\frac{1}{2}\sin Z\partial_Z\phi_{1/2}}{\phi_{1/2}} - \frac{1}{\phi_{1/2}^2}\frac{3(\pi+Z)^2}{s^{q(Z)}} + \frac{1}{\phi_{1/2}^2}\frac{(\pi+Z)^3}{s^{q(Z)}}\log s\,\partial_Z q \\ &\leq -w\frac{\frac{1}{2}\sin Z\,\partial_Z\phi_{1/2}}{\phi_{1/2}},\end{aligned}$$

and that for $Z \leq -\pi$,

$$\frac{1}{2}\partial_Z(\mathcal{T}w) = -\frac{1}{4}w.$$

Therefore, on $(-\pi, 0)$ one has from (4.48) the inequality behind the inviscid spectral gap:

$$-Vu^2 w + u^2\frac{1}{2}\partial_Z(\mathcal{T}w) \leq -u^2 w\frac{1}{\phi_{1/2}}(V\phi_{1/2} + \mathcal{T}\partial_Z\phi_{1/2}) = -\frac{1}{2}u^2 w,$$

and on $(-\infty, -\pi]$ from (4.45) one has

$$-Vu^2 w + u^2\frac{1}{2}\partial_Z(\mathcal{T}w) = -u^2 w - \frac{1}{4}wu^2 = -\frac{5}{4}wu^2.$$

Therefore, from the two inequalities above, on the whole ray $(-\infty, 0)$,

$$-Vu^2 w + u^2\frac{1}{2}\partial_Z(\mathcal{T}w) \leq -\frac{1}{2}wu^2.$$

That is why for the part involving the operator $\mathcal{T}\partial_Z + V$ one has

$$\int_{Z_1}^{Z_2} u(-Vu - \mathcal{T}\partial_Z u)w\,dZ \leq -\frac{1}{2}\int_{Z_1}^{Z_2} u^2 w\,dZ + \frac{1}{2}(u^2\mathcal{T}w)(Z_1) - \frac{1}{2}(u^2\mathcal{T}w)(Z_2).$$

We now turn to the dissipative effects. Integrating by parts yields

$$\begin{aligned}\int_{Z_1}^{Z_2} u\partial_{ZZ}uw = &\left(\frac{1}{2}u^2\partial_Z w - u\partial_Z uw\right)(Z_1) - \left(\frac{1}{2}u^2\partial_Z w - u\partial_Z uw)(Z_2\right)\\ &-\int_{Z_1}^{Z_2}|\partial_Z u|^2 w + \frac{1}{2}\int_{Z_1}^{Z_2} u^2 w.\end{aligned}$$

The function $\partial_{ZZ}w$, from (4.8), is supported in $(-\pi, 0)$ where one has the bound

$$\begin{aligned}|\partial_{ZZ}w| &\lesssim |Z|^{-7}\partial_{ZZ}(s^{-q(Z)}) + |Z|^{-8}\partial_Z(s^{-q(Z)}) + |Z|^{-9}s^{-q(Z)}\\ &\lesssim |Z|^{-9}s^{-q(Z)}(1 + Z^2\log^2 s + |Z|\log s)\end{aligned}$$

so that for s large enough depending on M, and for $Z \leq -Me^{-s}$,

$$|e^{-2s}\partial_{ZZ}w| \lesssim w/M^2.$$

From (4.16), the above identity becomes, for s large enough (since $\partial_Z w \geq 0$ near the origin),

$$\begin{aligned}\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} u\partial_{ZZ}uw\,dZ \leq &\, Ce^{-2s}\left|\frac{1}{2}u^2\partial_Z w - u\partial_Z uw\right|(Z_1) + Ce^{-2s}|u\partial_Z uw|(Z_2)\\ &- \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2}|\partial_Z u|^2 w\,dZ + \frac{C(K)}{M^2}\int_{Z_1}^{Z_2} u^2 w\,dZ.\end{aligned}$$

Hence

$$
\begin{aligned}
&\int_{Z_1}^{Z_2} u\Big(-Vu-\mathcal{T}\partial_Z u+\frac{1}{\lambda^4\mu^2}\partial_{ZZ}u\Big)w+\frac12\int_{Z_1}^{Z_2}u^2w_s+\frac{a_s}{2}(u^2w)(Z_1)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+Me^{-s}(u^2w)(Z_2)\\
&\le-\frac12\int_{Z_1}^{Z_2}u^2w-\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2}|\partial_Z u|^2w+\frac{C(K)}{M^2}\int_{Z_1}^{Z_2}u^2w+\frac{a_s}{2}(u^2w)(Z_1)\\
&\quad+Me^{-s}(u^2w)(Z_2)+\frac12(u^2\mathcal{T}w)(Z_1)-\frac12(u^2\mathcal{T}w)(Z_2)\\
&\quad+Ce^{-2s}\Big|\frac12u^2\partial_Z w-u\partial_Z uw\Big|(Z_1)+C(K,M)e^{-2s}|u\partial_Z uw|(Z_2)\\
&\le-\frac12\int_{Z_1}^{Z_2}u^2w-\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2}|\partial_Z u|^2w+\frac{C(K)}{M^2}\int_{Z_1}^{Z_2}u^2w+C(K,M)e^{6s}u^2(Z_2)\\
&\quad+C(K,M)e^{4s}|\partial_Z u|^2(Z_2)+Cu^2(Z_1)(e^{-(\frac12-\nu)s}+|a_s|)+C|\partial_Z u|^2(Z_1)e^{-2s},
\end{aligned}
\tag{4.52}
$$

where we have used (4.16) and the fact that $|w(Z_2)|\lesssim Z_2^{-7}\lesssim e^{7s}$, $|w(Z_1)|\lesssim 1$, $|\mathcal{T}(Z_1)|\lesssim|\pi+Z_1|\lesssim|a|\lesssim e^{-(1/2-\nu)s}$, $|\mathcal{T}(Z_2)|\lesssim|Z_2|\lesssim e^{-s}$, and $|\partial_Z w(Z_1)|\lesssim 1$.

Step 2. *The nonlocal term.* Using Cauchy–Schwarz one has, for $Z\in(-\pi,0)$,

$$
\left|\int_0^Z u(s,\tilde Z)\,d\tilde Z\right|\le\int_{Z_2}^0|u|\,dZ+\Big(\int_{Z_1}^{Z_2}u^2w\,dZ\Big)^{\frac12}\Big(\int_Z^0 w^{-1}(s,\tilde Z)\,d\tilde Z\Big)^{\frac12}. \tag{4.53}
$$

One computes that for $Z\in(-\pi,0)$,

$$
|w^{-1}(s,Z)|\lesssim|Z|^7s^{q(Z)}=|Z|^7e^{q(Z)\log s},
$$

from which we infer, by the assumptions on q in Section 4.2,

$$
\begin{aligned}
\int_Z^0 w^{-1}(s,\tilde Z)\,d\tilde Z&\lesssim\int_Z^0|\tilde Z|^7e^{q(\tilde Z)\log s}\,d\tilde Z\lesssim|Z|^7\int_Z^0\frac{1}{\log s\,\partial_Z q}\frac{d}{dZ}(e^{q(Z)\log s})\\
&\lesssim\frac{|Z|^7}{|\pi+Z|\log s}e^{q(Z)\log s}.
\end{aligned}
\tag{4.54}
$$

Therefore

$$
\Big(\int_Z^0 w^{-1}(s,\tilde Z)\,d\tilde Z\Big)\sin^2 Z\lesssim\frac{|Z|^9|\pi+Z|}{\log s}s^{q(Z)},
$$

which yields

$$
\begin{aligned}
&\int_{Z_1}^{Z_2}\Big(\int_Z^0 w^{-1}(s,\tilde Z)\,d\tilde Z\Big)\sin^2 Z\,\mathbb{1}_{-\pi\le Z\le 0}w\,dZ\\
&\qquad\qquad\lesssim\int_{-\pi}^0\frac{|Z|^9|\pi+Z|}{\log s}s^{q(Z)}\frac{1}{|Z|^7}\frac{1}{s^{q(Z)}}\,dZ\lesssim\frac{1}{\log s}.
\end{aligned}
$$

One also has

$$\int_{Z_2}^{Z_1} \sin^2 Z\, \mathbb{1}_{0\le Z\le \pi} w\, dZ \approx \int_{Z_2}^{Z_1} \frac{dZ}{|Z|^5 s^{q(Z)}} \lesssim \frac{e^{4s}}{M^4}.$$

Thus the contribution of the nonlocal term is estimated as follows:

$$\begin{aligned}
&\left|\int_{Z_2}^{Z_1} u\left(\int_0^Z u(s,\tilde Z)\, d\tilde Z\right) \sin Z\, \mathbb{1}_{-\pi\le Z\le 0} w\, dZ\right| \\
&\quad\lesssim \left|\int_{Z_2}^{Z_1} u\left(\int_{Z_2}^0 u\right) \sin Z\, \mathbb{1}_{-\pi\le Z\le 0} w\, dZ\right| + \left|\int_{Z_2}^{Z_1} u\left(\int_{Z_2}^Z u\right) \sin Z\, \mathbb{1}_{-\pi\le Z\le 0} w\, dZ\right| \\
&\quad\lesssim \frac{1}{\log s}\int_{Z_2}^{Z_1} u^2 w\, dZ + \frac{e^{2s}}{M^2}\left(\int_{Z_2}^0 |u|\, dZ\right)\left(\int_{Z_2}^{Z_1} u^2 w\, dZ\right)^{\frac12}.
\end{aligned}$$

Step 3. *End of the proof.* The above identity, (4.52), and (4.50) finally yield

$$\begin{aligned}
&\frac{d}{ds}\left(\frac12 \int_{Z_1}^{Z_2} u^2 w\right) \\
&\le -\frac12 \int_{Z_1}^{Z_2} u^2 w - \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z u|^2 w \\
&\quad + \frac{C(K)}{M^2}\int_{Z_1}^{Z_2} u^2 w + C(K,M) e^{6s} u^2(Z_2) + C(K,M) e^{4s} |\partial_Z u|^2(Z_2) \\
&\quad + C u^2(Z_1)(e^{-(\frac12-\nu)s} + |a_s|) + C|\partial_Z u|^2(Z_1) e^{-2s} \\
&\quad + \frac{1}{\log s}\int_{Z_2}^{Z_1} u^2 w + \frac{e^{2s}}{M^2}\left(\int_{Z_2}^0 |u|\right)\left(\int_{Z_2}^{Z_1} u^2 w\right)^{\frac12} \\
&\le \left(-\frac12 + \frac{C(K)}{M^2} + \frac{C}{\log s}\right)\int_{Z_1}^{Z_2} u^2 w + C(K,M) e^{6s} u^2(Z_2) + C(K,M) e^{4s} |\partial_Z u|^2(Z_2) \\
&\quad + C u^2(Z_1)(e^{-(\frac12-\nu)s} + |a_s|) + C|\partial_Z u|^2(Z_1) e^{-2s} \\
&\quad + \frac{Ce^{2s}}{M^2}\left(\int_{Z_2}^0 |u|\right)\left(\int_{Z_1}^{Z_2} u^2 w\right)^{\frac12} - \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z u|^2 w,
\end{aligned}$$

which ends the proof of the lemma for M and s_0 large enough. ∎

4.5.2. Exterior Lyapunov estimates. We now study the functional (4.11) for the full problem. First, let us estimate the function at the boundaries, $Z_1 = -\pi - a$ and $Z_2 = -Me^{-s}$. From (4.17) and Sobolev near the maximum,

$$\begin{aligned}
&u^2(Z_2) = \varepsilon^2(-M\lambda^2\mu e^{-s}) \le C\|\varepsilon\|_{H^2(|Y|\le M^2)} \le Ce^{-(7-2\nu')s}, \\
&(\partial_Z u)^2(Z_2) \le Ce^{-(5-2\nu')s}.
\end{aligned} \tag{4.55}$$

From the boundary condition (4.5), the decomposition (4.7), (3.5) and (4.16), at the origin in the original variables we get

$$u^2(Z_1) = G_1^2(-\pi - a) \le Ca^4 \le Ce^{-(2-8\nu)s}. \tag{4.56}$$

Finally, from (4.81), (4.7) and (4.16),

$$|\partial_Z u(Z_1)| \le |\partial_Z F(Z_1)| + |\partial_Z G_1(Z_1)| \le \lambda^{-1}\mu|\partial_y \xi(0)| + C|a| \le Ce^{-(\frac{1}{2}-2\nu)s}. \tag{4.57}$$

One has the following energy estimate for the function in the Z-variable outside the maximum.

Lemma 4.12 (Exterior Lyapunov functional on the left). *There exists $\nu^* > 0$ such that for any $K > 0$, $0 < \nu, \nu' \le \nu^*$, there exists $M^* > 0$ such that for $M \ge M^*$, there exist s_0^* and $C(K, M)$ such that if the solution is trapped on $[s_0, s_1]$ with $s_0 \ge s_0^*$ then*

$$\begin{aligned} &\frac{d}{ds}\left(\frac{1}{2}\int_{Z_1}^{Z_2} u^2 w\, dZ\right) + \left(\frac{1}{2} - \frac{\nu}{2}\right)\int_{Z_1}^{Z_2} u^2 w\, dZ \\ &\quad \le C(K, M)\left(e^{6s}u^2(Z_2) + e^{4s}|\partial_Z u|^2(Z_2) + \left(\int_{Z_1}^{Z_2} u^2 w\, dZ\right)^{\frac{1}{2}} e^{-\frac{5}{8}s} + e^{-(2+\frac{1}{6})s}\right). \end{aligned} \tag{4.58}$$

Proof. One first computes from (4.42) the identity

$$\begin{aligned} \frac{d}{ds}\left(\frac{1}{2}\int_{Z_1}^{Z_2} u^2 w\right) &= \int_{Z_1}^{Z_2} u\left(-\mathscr{H}u + \frac{\partial_{ZZ}u}{\lambda^4\mu^2} - \tilde{\mathscr{H}}u - \mathrm{NL} - \psi\right)w \\ &\quad + \int_{Z_1}^{Z_2} \frac{u^2}{2}w_s + \frac{a_s}{2}(u^2 w)(Z_1) + \frac{Me^{-s}}{2}(u^2 w)(Z_2). \end{aligned} \tag{4.59}$$

Step 1. *The leading order linear terms.* From (4.56), (4.91) and (4.57),

$$u^2(Z_1)(e^{-(\frac{1}{2}-\nu)s} + |a_s|) + |\partial_Z u|^2(Z_1)e^{-2s} \lesssim e^{(\frac{5}{2}-10\nu)s} + e^{(5-4\nu)s} \lesssim e^{(\frac{5}{2}-10\nu)s}.$$

From (4.3), (4.16) and (4.17), as $\lambda^2 e^{-s}\mu M \approx 1$,

$$e^{2s}\int_{Z_2}^{0} |u|\, dZ = \frac{e^{2s}}{\lambda^2\mu}\int_{-\lambda^2 e^{-s}\mu M}^{0} |\varepsilon|\, dY \lesssim e^{s}\|\varepsilon\|_{L^2_\rho} \lesssim e^{-\frac{5}{2}s}.$$

We now apply Lemma 4.11 and insert the above two inequalities:

$$\begin{aligned} &\int_{Z_1}^{Z_2} u\left(-\mathscr{H}u + \frac{1}{\lambda^4\mu^2}\partial_{ZZ}u\right)w + \frac{1}{2}\int_{Z_1}^{Z_2} u^2 w_s + \frac{a_s}{2}(u^2 w)(Z_1) + \frac{Me^{-s}}{2}(u^2 w)(Z_2) \\ &\quad \le \left(-\frac{1}{2} + \frac{\nu}{4}\right)\int_{Z_1}^{Z_2} u^2 w + C(K, M)e^{6s}u^2(Z_2) + C(K, M)e^{4s}|\partial_Z u|^2(Z_2) \\ &\qquad + e^{-(\frac{5}{2}-10\nu)s} + e^{-\frac{5}{2}s}\left(\int_{Z_1}^{Z_2} u^2 w\right)^{\frac{1}{2}} - \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z u|^2 w. \end{aligned} \tag{4.60}$$

Step 2. *The small linear term.* Recall (4.46); then

$$-\int_{Z_1}^{Z_2} u\tilde{\mathscr{H}}uw\, dZ = -\int_{Z_1}^{Z_2} u\big(m_3'\partial_Z u + m_1(2 - Z\partial_Z)u - m_2 Z\partial_Z u\big)w\, dZ.$$

Integrating by parts, one has

$$\int_{Z_1}^{Z_2} u\partial_Z uw\,dZ = \frac{1}{2}(u^2w)(Z_2) - \frac{1}{2}(u^2w)(Z_1) - \frac{1}{2}\int_{Z_1}^{Z_2} u^2\partial_Z w\,dZ.$$

We recall that $\partial_Z w$ is supported on $(-\pi, 0)$, and that for $|Z| \gtrsim e^{-s}$ we have

$$|\partial_Z w| \lesssim |Z|^8 s^{q(Z)}(1+|Z|\log s) \lesssim e^s w.$$

Therefore, since $w(Z_1) \lesssim 1$ and $w(Z_2) \lesssim e^{7s}$, using (4.47) we get

$$\left|\int_{Z_1}^{Z_2} um_3'\partial_Z uw\,dZ\right| \lesssim e^{\frac{35}{8}s}u^2(Z_2) + e^{-\frac{21}{8}s}u^2(Z_1) + e^{-\frac{13}{8}s}\int_{Z_1}^{Z_2} u^2w\,dZ.$$

The same strategy applies for the other term, and as $|\partial_Z(Zw)| \lesssim e^{s/2}w$, this gives, using (4.47),

$$\left|\int_{Z_1}^{Z_2} u\big(m_1(2-Z\partial_Z)u - m_2Z\partial_Z u\big)w\right| \\ \lesssim e^{\frac{35}{8}s}u^2(Z_2) + e^{-\frac{13}{8}s}u^2(Z_1) + e^{-\frac{9}{8}s}\int_{Z_1}^{Z_2} u^2w.$$

In conclusion, for the small linear term, using (4.56), (4.18) and (4.55) as $0 < \nu' \ll \nu$ one has

$$\begin{aligned}\left|\int_{Z_1}^{Z_2} u\tilde{\mathcal{H}}uw\,dZ\right| &\lesssim e^{\frac{35}{8}s}u^2(Z_2) + e^{-\frac{13}{8}s}u^2(Z_1) + e^{-\frac{9}{8}s}\int_{Z_1}^{Z_2} u^2w \\ &\lesssim e^{5s}u^2(Z_2) + e^{-(\frac{29}{8}-8\nu)s} + e^{-s}\int_{Z_1}^{Z_2} u^2w. \end{aligned} \tag{4.61}$$

Step 3. *The nonlinear term.* For the nonlinear term one recalls the identity

$$\int_{Z_1}^{Z_2} u\,\mathrm{NL}\,w\,dZ = \int_{Z_1}^{Z_2} u(-u^2 + \partial_Z^{-1}u\partial_Z u)w\,dZ.$$

The first term is estimated by brute force:

$$\left|\int_{Z_1}^{Z_2} u^3w\,dZ\right| \le \|u\|_{L^\infty}\int_{Z_1}^{Z_2} u^2w\,dZ.$$

For the second, we integrate by parts and obtain

$$\int_{Z_1}^{Z_2} u\partial_Z^{-1}u\partial_Z uw = \frac{1}{2}(u^2\partial_Z^{-1}uw)(Z_2) - \frac{1}{2}(u^2\partial_Z^{-1}uw)(Z_1) - \frac{1}{2}\int_{Z_1}^{Z_2} u^3w \\ - \frac{1}{2}\int_{Z_1}^{Z_2} \partial_Z^{-1}u\partial_Z wu^2.$$

In conclusion, the contribution of the nonlinear term is, using (4.19) and (4.56) for s_0 large enough, and $|\partial_Z^{-1} u| \leq |Z| \, \|u\|_{L^\infty}$ and $|Z\partial_Z w| \lesssim \log(s) w$,

$$\left| \int_{Z_1}^{Z_2} u \, \mathrm{NL} \, w \, dZ \right| \lesssim \|u\|_{L^\infty} e^{6s} u^2(Z_2) + \|u\|_{L^\infty} u^2(Z_1) + \log(s) \|u\|_{L^\infty} \int_{Z_1}^{Z_2} w u^2 \, dZ$$
$$\lesssim e^{(6-\frac{1}{8})s} u^2(Z_2) + e^{-(2+\frac{1}{4}-9\nu)s} + \frac{\nu}{4} \int_{Z_1}^{Z_2} w u^2 \, dZ. \tag{4.62}$$

Step 4. *The error term.* Recall (4.27). The function ψ is supported on $[-\pi, \pi]$, with the estimate from (4.1)

$$|\psi(s, Z)| = \left| -\frac{1}{\lambda^4 \mu^2} \partial_{ZZ} G_1(Z) + m_1 (2 - Z\partial_Z) G_1(Z) - m_2 Z \partial_Z G_1(Z) + m_3' \partial_Z G_1(Z) \right|$$
$$\lesssim \left| m_1 + \frac{1}{4\lambda^4 \mu^2} \right| + Z^2 \left(\frac{1}{\lambda^4} + |m_1| + |m_2| \right) + |m_3'| \, |Z|.$$

Since $w \lesssim |Z|^{-7}$, using (4.47) and (4.16), for s_0 large one has

$$\int_{Z_1}^{Z_2} \psi^2 w \, dZ \lesssim e^{6s} \left| m_1 + \frac{1}{4\lambda^4 \mu^2} \right|^2 + e^{2s} \left(\frac{1}{\lambda^4} + \left| \frac{\lambda_s}{\lambda} - \frac{1}{2} \right| + \left| \frac{\mu_s}{\mu} \right| \right)^2 + e^{4s} |m_3'|^2$$
$$\lesssim e^{-(\frac{13}{4}-2)s} + C(K) e^{-2s} \lesssim e^{-\frac{5}{4}s}.$$

By Cauchy–Schwarz, one has proved that for the error term we have

$$\left| \int_{Z_1}^{Z_2} u \psi w \right| \lesssim \left(\int_{Z_1}^{Z_2} u^2 w \right)^{\frac{1}{2}} e^{-\frac{5}{8}s}. \tag{4.63}$$

Step 5. *End of the proof.* Collecting the estimates (4.60)–(4.63) and inserting them in (4.59) yields the desired energy estimate (4.58). ∎

A similar energy estimate also holds for the adapted derivative of u, $A\partial_Z u$ where A is defined by (4.9), to the left of the origin. This vector field is chosen because its commutator with $\mathcal{T}$ vanishes for $Z \in [-\pi/2, \pi/2]$, and has a good sign for $|Z| > \pi/2$. Before stating the estimate, let us investigate the size of the boundary terms. From Sobolev embedding and (4.17), since $|A| \sim |Z|$ and $|\partial_Z A| \lesssim 1$ near the origin,

$$|A\partial_Z u|^2(Z_2) \leq |Y \partial_Y \varepsilon|^2(-M\lambda^2 \mu e^{-s}) \leq C \|\varepsilon\|^2_{H^2(|Y| \leq M^2)}$$
$$\leq C(K, M) e^{-(7-2\nu')s}, \tag{4.64}$$

$$(\partial_Z (A\partial_Z u))^2(Z_2) \leq (|\partial_Z u|^2 + |Z\partial_{ZZ} u|^2)(Z_2)$$
$$\leq \lambda^4 \mu^2 (|\partial_Y \varepsilon|^2 + |Y \partial_{YY} \varepsilon|^2)(-M\lambda^2 \mu e^{-s})$$
$$\leq C(K, M) e^{-(5-2\nu')s}. \tag{4.65}$$

Since $A = 1$ near $-\pi$, from (4.57) we get

$$|A\partial_Z u|^2(Z_1) \leq C |\partial_Z u|^2(Z_1) \leq C e^{-(1-4\nu)s}. \tag{4.66}$$

Now we write $\partial_Z(A\partial_Z u) = A\partial_{ZZ}u$ since $|\partial_Z A(-\pi - a)| = 0$. Since $\partial_{yy}\xi(0) = 0$ from the boundary condition in (1.3), formulas (4.7) and estimates (4.16) imply

$$\begin{aligned} |\partial_Z(A\partial_Z u)(Z_1)| &= |\partial_{ZZ}u(Z_1)| = |\partial_{ZZ}(F - G_1)(Z_1)| \\ &\le |\lambda^2\mu^2\partial_{yy}\xi(0)| + |\partial_{ZZ}G_1|(-\pi - a) \le 1/2. \end{aligned} \tag{4.67}$$

We perform the same weighted energy estimate outside the maximum for $A\partial_Z u$ as we did for u.

Lemma 4.13 (Exterior Lyapunov functional on the left for the derivative). *Let $Z_1 = -\pi - a$, $Z_2 = -Me^{-s}$ and $v = A\partial_Z u$. There exists $\nu^* > 0$ such that for any $K > 0$ and $0 < \nu, \nu' < \nu^*$, there exists $M^* > 0$ such that for any $M \ge M^*$ there exists s_0^* such that if the solution is trapped on $[s_0, s_1]$ with $s_0 \ge s_0^*$ then*

$$\frac{d}{ds}\left(\frac{1}{2}\int_{Z_1}^{Z_2} v^2 w\,dZ\right) - \frac{\nu}{2}\int_{Z_1}^{Z_2} v^2 w\,dZ + \frac{1}{2\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z v|^2 w\,dZ \le e^{-\frac{1}{4}s}. \tag{4.68}$$

Proof. In this proof, the constant C might depend on K and M. One first computes the evolution equation for $v = A\partial_Z u$ from (4.42):

$$\begin{aligned} 0 = {} & v_s + (\mathcal{T}\partial_Z + V)v + \frac{A\partial_Z\mathcal{T} - \mathcal{T}\partial_Z A}{A}v \\ & - \frac{1}{\lambda^4\mu^2}(\partial_{ZZ}v + [A\partial_Z, \partial_{ZZ}]u) + \tilde{\mathcal{H}}v + [A\partial_Z, \tilde{\mathcal{H}}]u \\ & + \tilde{\mathrm{NL}} + A\partial_Z\psi + Au\partial_Z G_1 + \partial_Z^{-1}uA\partial_{ZZ}G_1 \end{aligned} \tag{4.69}$$

where

$$\tilde{\mathrm{NL}} = -\left(2u + \partial_Z^{-1}u\frac{\partial_Z A}{A}\right)v + \partial_Z^{-1}u\partial_Z v.$$

First, one has the following identity for the energy estimate:

$$\begin{aligned} \frac{d}{ds}\left(\frac{1}{2}\int_{Z_1}^{Z_2} v^2 w\,dZ\right) = {} & \int_{Z_1}^{Z_2} vv_s w\,dZ + \frac{1}{2}\int_{Z_1}^{Z_2} v^2 w_s\,dZ + \frac{a_s}{2}(v^2 w)(Z_1) \\ & + \frac{Me^{-s}}{2}(v^2 w)(Z_2). \end{aligned} \tag{4.70}$$

Step 1. *The leading order linear terms.* From (4.52), inserting (4.64)–(4.67) one gets

$$\begin{aligned} & \int_{Z_1}^{Z_2} v\left(-Vv - \mathcal{T}\partial_Z v + \frac{1}{\lambda^4\mu^2}\partial_{ZZ}v\right)w + \frac{1}{2}\int_{Z_1}^{Z_2} u^2 w_s \\ & \qquad + \frac{a_s}{2}(v^2 w)(Z_1) + \frac{Me^{-s}}{2}(v^2 w)(Z_2) \\ & \le -\frac{1}{2}\int_{Z_1}^{Z_2} v^2 w - \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z v|^2 w + \frac{\nu}{4}\int_{Z_1}^{Z_2} v^2 w \\ & \quad + Ce^{6s}v^2(Z_2) + Ce^{4s}|\partial_Z v|^2(Z_2) + Cv^2(Z_1)e^{-(\frac{1}{2}-\nu)s} + C|\partial_Z v|^2(Z_1)e^{-2s} \\ & \le -\frac{1}{2}\int_{Z_1}^{Z_2} v^2 w - \frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} |\partial_Z v|^2 w + \frac{\nu}{4}\int_{Z_1}^{Z_2} v^2 w + Ce^{-(1-2\nu')s}. \end{aligned} \tag{4.71}$$

Then, for the commutator with A and the transport $\mathcal{T}$, a direct computation shows that since $A = 2\mathcal{T}$ for $|Z| \leq \pi/2$, and $A = -1$ for $Z \leq -\pi/2$, for all $Z \leq 0$ we have

$$\frac{A\partial_Z \mathcal{T} - \mathcal{T}\partial_Z A}{A} = \partial_Z \mathcal{T} \mathbb{1}_{Z \leq -\frac{\pi}{2}} \geq -\frac{1}{2}\mathbb{1}_{Z \leq -\pi/2},$$

which implies

$$-\int_{Z_1}^{Z_2} v \frac{A\partial_Z \mathcal{T} - \mathcal{T}\partial_Z A}{A} v w \leq \frac{1}{2}\int_{Z_1}^{Z_2} v^2 w. \tag{4.72}$$

Step 2. *The small linear term and other commutators.* For the small linear term, from (4.61), inserting (4.64), (4.66) and (4.18), for s_0 large enough we have

$$\begin{aligned}\left|\int_{Z_1}^{Z_2} v\tilde{\mathcal{H}} v w \, dZ\right| &\lesssim e^{\frac{35}{8}s} v^2(Z_2) + e^{-\frac{13}{8}s} v^2(Z_1) + e^{-\frac{9}{8}s}\int_{Z_1}^{Z_2} v^2 w \\ &\leq C(K, M)(e^{-(\frac{21}{8}-2\nu')s} + e^{-(\frac{21}{8}-4\nu)s} + e^{-(\frac{9}{8}-2\nu)s}) \leq e^{-s}.\end{aligned} \tag{4.73}$$

Next, we turn to the commutator with the dissipative term:

$$[A\partial_Z, \partial_{ZZ}]u = \left(-\frac{\partial_{ZZ}A}{A} + \frac{2(\partial_Z A)^2}{A^2}\right)v - 2\frac{\partial_Z A}{A}\partial_Z v.$$

Since, for $Z \geq Me^{-s}$,

$$\left|\frac{\partial_{ZZ}A}{A}\right| + \left|\frac{(\partial_Z A)^2}{A^2}\right| \leq \frac{C}{Z^2} \leq \frac{Ce^{2s}}{M^2},$$

for the first term one has

$$\left|\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} v^2\left(-\frac{\partial_{ZZ}A}{A}v + \frac{2(\partial_Z A)^2}{A^2}\right) w \, dZ\right| \leq \frac{C(K)}{M^2}\int_{Z_1}^{Z_2} v^2 w \, dZ.$$

For the second term, one first integrates by parts:

$$\begin{aligned}-\int_{Z_1}^{Z_2} 2v\frac{\partial_Z A}{A}\partial_Z v w \, dZ &= \left(v^2\frac{\partial_Z A}{A}w\right)(Z_1) - \left(v^2\frac{\partial_Z A}{A}w\right)(Z_2) \\ &\quad + \int_{Z_1}^{Z_2} v^2 \partial_Z\left(\frac{\partial_Z A}{A}w\right) dZ \\ &= -\left(v^2\frac{\partial_Z A}{A}w\right)(Z_2) + \int_{Z_1}^{Z_2} v^2\partial_Z\left(\frac{\partial_Z A}{A}w\right) dZ\end{aligned}$$

since $\partial_Z A(Z_1) = 0$. From a direct inspection,

$$\left|\partial_Z\left(\frac{\partial_Z A}{A}w\right)\right| \leq \frac{Cw}{Z^2} \leq \frac{Ce^{2s}}{M^2}w.$$

Therefore

$$\left|\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} 2v\frac{\partial_Z A}{A}\partial_Z v w \, dZ\right| \leq Ce^{6s}v^2(Z_2) + \frac{C(K)}{M^2}\int_{Z_1}^{Z_2} v^2 w \, dZ.$$

We have proved that for the commutator with the dissipative term, for M large enough depending on K, using (4.64) we get

$$\left|\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} v[A\partial_Z,\partial_{ZZ}]uw\right| \leq Ce^{6s}v^2(Z_2)+\frac{C(K)}{M^2}\int_{Z_1}^{Z_2} v^2w$$

$$\leq Ce^{-(1-2\nu')s}+\frac{\nu}{8}\int_{Z_1}^{Z_2} v^2w. \tag{4.74}$$

Next, the commutator with the small linear term is

$$[A\partial_Z,\tilde{H}]u=\left(-m_3'\frac{\partial_Z A}{A}-m_1\left(1-\frac{Z\partial_Z A}{A}\right)+m_2\frac{Z\partial_Z A}{A}\right)v.$$

Since $|\partial_Z A/A|\lesssim 1/Z\lesssim e^s$ for $|Z|\geq Me^{-s}$, this implies using (4.47) that

$$\left|\int_{Z_1}^{Z_2} v[A\partial_Z,\tilde{H}]uw\right| \lesssim (|m_1|+|m_2|+e^s m_3')\int_{Z_1}^{Z_2} v^2w\lesssim e^{-\frac{13}{8}s}\int_{Z_1}^{Z_2} v^2w. \tag{4.75}$$

Step 3. *The nonlinear term.* Since $|\partial_Z A/A|\lesssim 1/Z$ one has

$$\left|\int_{Z_1}^{Z_2} v(u+\partial_Z^{-1}u\frac{\partial_Z A}{A})vw\,dZ\right|\lesssim \|u\|_{L^\infty}\int_{Z_1}^{Z_2} v^2w\,dZ.$$

For the other term, integration by parts gives

$$\begin{aligned}&\left|\int_{Z_1}^{Z_2} v\partial_Z^{-1}u\partial_Z vw\,dZ\right|\\ &\quad=\left|\frac12(\partial_Z^{-1}uv^2w)(Z_1)-\frac12(\partial_Z^{-1}uv^2w)(Z_2)+\int_{Z_1}^{Z_2} v^2\partial_Z(\partial_Z^{-1}uw)\,dZ\right|\\ &\quad\lesssim \|u\|_{L^\infty}v^2(Z_1)+\|u\|_{L^\infty}e^{6s}v^2(Z_2)+\log(s)\|u\|_{L^\infty}\int_{Z_1}^{Z_2} v^2w\,dZ,\end{aligned}$$

where we have used the fact that $|\partial_Z w|\lesssim \log(s)Z^{-1}w$. Thus we have shown that for the nonlinear term, using (4.19), (4.64) and (4.66), as $0<\nu'\ll\nu$,

$$\left|\int_{Z_1}^{Z_2} v\,\tilde{\mathrm{NL}}\,w\,dZ\right|\lesssim \|u\|_{L^\infty}v^2(Z_1)+\|u\|_{L^\infty}e^{6s}v^2(Z_2)+\log(s)\|u\|_{L^\infty}\textstyle\int_{Z_1}^{Z_2} v^2w\,dZ$$

$$\lesssim e^{-(1+\frac14-5\nu)s}+\frac{\nu}{8}\int_{Z_1}^{Z_2} v^2w\,dZ. \tag{4.76}$$

Step 4. *The error term.* Recall (4.27). Since $|A|\lesssim|Z|$ for $|Z|\leq\pi$ with $A(-\pi)=-1$, and since $\partial_{ZZ}G_1$ has limit 0 and $1/2$ on the left and on the right of $-\pi$ respectively, one first computes

$$\begin{aligned}&A\partial_Z\psi(s,Z)\\ &\quad=A\partial_Z\left(-\frac{1}{\lambda^4\mu^2}\partial_{ZZ}G_1(Z)+m_1(2-Z\partial_Z)G_1(Z)-m_2Z\partial_Z G_1+m_3'\partial_Z G_1(Z)\right)\\ &\quad=\frac12\delta_{\{Z=-\pi\}}+O\left(Z^2\left(\frac{1}{\lambda^4}+|m_1|+|m_2|\right)+|m_3'|\,|Z|\right).\end{aligned}$$

Since $w \lesssim |Z|^7$ one has, by (4.47),

$$\int_{Z_1}^{Z_2} \left| O\left(Z^2\left(\frac{1}{\lambda^4}+|m_1|+|m_2|\right)+|m_3'|Z\right)\right|^2 w\,dZ \lesssim e^{2s}\left(\frac{1}{\lambda^4}+|m_1|+|m_2|\right)^2 + e^{4s}|m_3'|^2 \lesssim e^{-\frac{5}{4}s}.$$

For the Dirac term, either one has $a<0$ and then $-\pi<Z_1$ in which case there is nothing to estimate since

$$\int_{Z_1}^{Z_2} v\delta_{\{Z=-\pi\}}\,dZ = 0;$$

or, if $Z_1 \le -\pi$, we use Sobolev embedding (since $w \approx s^{-1}$ near $-\pi$) to find

$$\begin{aligned}\frac{1}{\lambda^4\mu^2}\int_{Z_1}^{Z_2} v\delta_{\{Z=-\pi\}}w &= \frac{1}{\lambda^4\mu^2} w(-\pi)v(-\pi) \le \frac{C}{\lambda^4\mu^2}\left(\left(\int_{Z_1}^{Z_2} v^2 w\right)^{\frac12} + \left(\int_{Z_1}^{Z_2}(\partial_Z v)^2 w\right)^{\frac12}\right)\\ &\le \frac{C}{\lambda^4\mu^2}\left(\int_{Z_1}^{Z_2} v^2 w\right)^{\frac12} + \frac{C\kappa}{\lambda^4\mu^2}\int_{Z_1}^{Z_2}(\partial_Z v)^2 w + \frac{C}{\kappa\lambda^4}.\end{aligned}$$

Using Cauchy–Schwarz, one sees that for the error term, in both cases $Z_1 \le \pi$ or $Z_1 > \pi$, for κ small enough, using (4.16) and (4.18) for the last inequality, one has

$$\begin{aligned}\left|\int_{Z_1}^{Z_2} vA\partial_Z\psi w\right| &\lesssim \left(e^{-\frac58 s} + \frac{C}{\lambda^4\mu^2}\right)\left(\int_{Z_1}^{Z_2} v^2 w\right)^{\frac12} + \frac{1}{2\lambda^4\mu^2}\int_{Z_1}^{Z_2}(\partial_Z v)^2 w + \frac{C}{\kappa\lambda^4}\\ &\lesssim e^{-(\frac58-\nu)s} + \frac{1}{2\lambda^4\mu^2}\int_{Z_1}^{Z_2}(\partial_Z v)^2 w. \end{aligned} \tag{4.77}$$

Step 5. *The remaining lower order terms.* For the first term, from (4.18) one has

$$\left|\int_{Z_1}^{Z_2} vAu\partial_Z G_1 w\,dZ\right| \lesssim \left(\int_{Z_1}^{Z_2} u^2 w\,dZ\right)^{\frac12}\left(\int_{Z_1}^{Z_2} v^2 w\,dZ\right)^{\frac12} \lesssim e^{-(\frac12-2\nu)s} \tag{4.78}$$

since $A\partial_Z G_1$ is bounded. For the last term, from (4.53) one has

$$\begin{aligned}&|\partial_Z^{-1} uA\partial_{ZZ}G_1|\\ &\lesssim \left(\int_{Z_2}^{0}|u|\,d\tilde Z\right)|Z|\mathbb{1}_{-\pi\le Z\le 0} + \left(\int_{Z_1}^{Z_2} u^2 w\,d\tilde Z\right)^{\frac12}\left(\int_{Z}^{0} w^{-1}\,d\tilde Z\right)^{\frac12}|Z|\mathbb{1}_{-\pi\le Z\le 0}\\ &\lesssim \left(\int_{Z_2}^{0}|u|\,d\tilde Z\right)|Z|\mathbb{1}_{-\pi\le Z\le 0} + \sqrt{s}\left(\int_{Z_1}^{Z_2} u^2 w\,d\tilde Z\right)^{\frac12}|Z|^5\mathbb{1}_{0\le Z\le\pi},\end{aligned}$$

where we have used the fact that $w \approx |Z|^{-7} s^{-q(Z)}$ for $-\pi \le Z < 0$, and that q is maximal at $-\pi$ with $q(-\pi) = 1$. One then computes that

$$\int_{Z_1}^{Z_2} Z^2 w\, dZ \lesssim \int_{Z_1}^{Z_2} Z^{-5}\, dZ \lesssim e^{4s}, \quad \int_{Z_1}^{Z_2} Z^{10} w\, dZ \lesssim 1.$$

Therefore

$$\int_{Z_1}^{Z_2} |\partial_Z^{-1} u A \partial_{ZZ} G_1|^2 w\, dZ \lesssim e^{4s} \left(\int_{Z_2}^{0} |u|\, dZ \right)^2 + \int_{Z_1}^{Z_2} u^2 w\, dZ,$$

which, by Cauchy–Schwarz, gives for the last lower order term, using (4.17) and (4.18),

$$\begin{aligned}
&\left| \int_{Z_1}^{Z_2} v \partial_Z^{-1} u A \partial_{ZZ} G_1 w\, dZ \right| \\
&\quad \lesssim \left(\int_{Z_1}^{Z_2} v^2 w\, dZ \right)^{\frac12} \left(e^{2s} \left(\int_{Z_2}^{0} |u|\, dZ \right) + s \left(\int_{Z_1}^{Z_2} u^2 w\, dZ \right)^{\frac12} \right) \\
&\quad \lesssim e^{\nu s} (e^{s} \left(\int_{M^2}^{0} |\varepsilon|\, dY) + s e^{-(\frac12 - \nu)s} \right) \lesssim e^{\nu s} (e^{s} e^{-\frac72 s} + s e^{-(\frac12-\nu)s}) \lesssim e^{-(\frac12 - 2\nu)s}.
\end{aligned}$$

Step 6. *End of the proof.* In conclusion, from the identities (4.69), (4.70), collecting the estimates (4.71)–(4.78) and the above inequality we get

$$\begin{aligned}
&\frac{d}{ds} \left(\frac12 \int_{Z_1}^{Z_2} v^2 w\, dZ \right) \\
&\le -\frac12 \int_{Z_1}^{Z_2} v^2 w\, dZ - \frac{1}{\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w\, dZ + \frac{\nu}{4} \int_{Z_1}^{Z_2} v^2 w\, dZ + C e^{-(1-2\nu')s} \\
&\quad + \frac12 \int_{Z_1}^{Z_2} v^2 w\, dZ + e^{-s} + C e^{-(1-2\nu)s} + \frac{\nu}{8} \int_{Z_1}^{Z_2} v^2 w\, dZ + C e^{-\frac{13}{8}s} \int_{Z_1}^{Z_2} v^2 w\, dZ \\
&\quad + C e^{-(1+\frac14 - 5\nu)s} + \frac{\nu}{8} \int_{Z_1}^{Z_2} v^2 w + C e^{-(\frac58 - \nu)s} + \frac{1}{2\lambda^4 \mu^2} \int_{Z_1}^{Z_2} (\partial_Z v)^2 w\, dZ \\
&\quad + C e^{-(\frac12 - 2\nu)s} \\
&\le \frac{\nu}{2} \int_{Z_1}^{Z_2} v^2 w\, dZ - \frac{1}{2\lambda^4 \mu^2} \int_{Z_1}^{Z_2} |\partial_Z v|^2 w\, dZ + C(K, M) e^{-(\frac12 - 2\nu)s},
\end{aligned}$$

which is the desired differential inequality (4.68) for ν small enough and s_0 large enough. ∎

The same analysis can be done to the right of the origin. The analogues of Lemmas 4.12 and 4.13 hold and their proofs are exactly the same.

Lemma 4.14 (Exterior Lyapunov functionals on the right). *Let $Z_3 = M e^{-s}$. There exists $\nu^* > 0$ such that for any $K > 0$ and $0 < \nu, \nu' \le \nu^*$, there exists $M^* > 0$ exist such that* *for $M \geq M^*$, there exist s_0^* and $C(K,M)$ such that if the solution is trapped on $[s_0,s_1]$ with $s_0 \geq s_0^*$ then, with $v = A\partial_Z u$,*

$$\frac{d}{ds}\left(\frac{1}{2}\int_{Z_3}^{\infty} u^2 w\,dZ\right) + \left(\frac{1}{2} - \frac{\nu}{2}\right)\int_{Z_3}^{\infty} u^2 w\,dZ$$
$$\leq C(K,M)\left(e^{6s}u^2(Z_3) + e^{4s}|\partial_Z u|^2(Z_3) + e^{-(2+\frac{1}{6})s} + \left(\int_{Z_3}^{\infty} u^2 w\,dZ\right)^{\frac{1}{2}} e^{-\frac{5}{8}s}\right), \tag{4.79}$$

$$\frac{d}{ds}\left(\frac{1}{2}\int_{Z_3}^{\infty} v^2 w\,dZ\right) - \frac{\nu}{2}\int_{Z_3}^{\infty} v^2 w\,dZ + \frac{1}{2\lambda^4\mu^2}\int_{Z_3}^{\infty} |\partial_Z v|^2 w\,dZ \leq e^{-\frac{1}{4}s}. \tag{4.80}$$

Proof. The proof follows exactly the same lines as the proofs of Lemmas 4.12 and 4.13, since everything is symmetric except the boundary condition, and we safely skip it. The only difference is that in this case the only boundary terms come from Z_3. ∎

4.6. Analysis close to the origin

This subsection is devoted to the analysis of the solution in the original variables, on compact sets and in particular close to the origin. Since the blow-up happens at infinity, eventually the nonlinear effects become weak and the solution stays regular. We state it in a perturbative way and track precisely the constants, so that this can be used both to derive uniform estimates at the origin, and to derive the asymptotics (1.8) for the profile at blow-up time.

Lemma 4.15 (No blow-up on compact sets). *Let $0 \leq s_0 \leq s_1$, $b > 0$, $N, L, L' \geq 1$, $q \in 2\mathbb{N}$. Assume that s is given by (4.2) with λ satisfying (4.16). Let ξ solve (1.3) on $[0, t(s_1)] \times [0, 2N]$, with $\xi \in C^3([0,t(s_1)] \times [0,2N])$, and such that*

$$\xi_0(t(s_0)) = by^2 + \tilde{\xi}(t(s_0)), \quad \|\tilde{\xi}(t(s_0))\|_{L^\infty([0,2N])} \leq L, \quad \|\partial_y \tilde{\xi}(t(s_0))\|_{L^2([0,2N])} \leq L',$$

and for all $t \in [t(s_0), t(s_1)]$,

$$\|\xi(t)\|_{L^\infty([0,2N])} \leq e^{(1-\frac{1}{8})s}, \quad \|\partial_y \xi(t)\|_{L^2([0,2N])} \leq e^s.$$

Then, writing $\xi = by^2 + \tilde{\xi}$, for all $t \in [t(s_0), t(s_1)]$ one has

$$\|\tilde{\xi}\|_{L^q([0,N])} \lesssim LN^{\frac{1}{q}} + N^{2+\frac{1}{q}} e^{-\frac{s_0}{16}}, \quad \|\partial_y \tilde{\xi}\|_{L^2([0,N])} \lesssim L' + N^{\frac{3}{2}} e^{-\frac{s_0}{8q}}.$$

Corollary 4.16. *There exists a universal $C > 0$ such that for any K, ν, ν', M, there exists s_0^* such that if the solution is trapped on $[s_0, s_1]$ with $s_0 \geq s_0^*$, then for all $t \in [t(s_0), t(s_1)]$,*

$$\|\xi(t,\cdot)\|_{W^{1,\infty}([0,1/2])} \leq C. \tag{4.81}$$

Proof. From (4.19), (4.16) and (4.3) we infer that for s_0 large enough and for all $s \in [s_0, s_1]$,

$$\|\xi\|_{L^\infty([0,2])} \leq e^{(1-\frac{1}{8})s}.$$

Hence from Lemma 4.15, using (4.15), one finds that for all $t \in [0, t(s_1)]$,

$$\|\xi\|_{L^q([0,1])} \lesssim 1, \quad \|\partial_y \xi\|_{L^2([0,1])} \lesssim 1.$$

The desired bound (4.81) then follows from a standard parabolic regularity result. We do not prove it here and refer to the proof of Lemma 4.18 for a similar strategy. ∎

Proof of Lemma 4.15. The proof relies on a standard localised bootstrap argument similar to that in [17]. The fact that we performed such an argument close to the anticipated profile at blow-up time is inspired by [19, 30].

Step 1. *The bootstrap procedure.* Let $1 < \alpha_1 < 2$, $0 < \kappa < 1$ with $\kappa \neq 1 - 1/(16q)$, $L_1 = LN^{\frac{1}{q}} + N^{2+\frac{1}{q}} e^{-\frac{s_0}{16}}$, and assume that for $t \in [t(s_0), t(s_1)]$ one has the bound

$$\int_{y \le 2N} |\tilde{\xi}|^q \, dy \le L_1^q e^{q(1-\kappa)s}. \tag{4.82}$$

We claim that then, for all $t \in [0, t(s_1)]$,

$$\int_{y \le \alpha_1 N} |\tilde{\xi}|^q \, dy \lesssim \begin{cases} L_1^q e^{q(1-\kappa-\frac{1}{16q})s} & \text{if } \kappa < 1 - 1/(8q), \\ L_1^q & \text{if } 1 - 1/(8q) < \kappa. \end{cases} \tag{4.83}$$

To prove this claim, we write $\xi = by^2 + \tilde{\xi}$. Then $\tilde{\xi}$ solves

$$\tilde{\xi}_t - \partial_{yy}\tilde{\xi} + \partial_y^{-1}\xi \partial_y \tilde{\xi} - \xi^2 + 2b\partial_y^{-1}\xi y - 2b = 0, \quad \tilde{\xi}(t, 0) = 0.$$

Let $0 < \alpha \ll 1$ and χ be a smooth cut-off function with $\chi(y) = 1$ for $y \le 1 + \alpha$ and $\chi(y) = 0$ for $y \ge 1 + 2\alpha$, set $\chi_1 = \chi(\frac{y}{\alpha_1 N})$, and let $v := \chi_1 \tilde{\xi}$. Then v solves

$$v_t - \partial_{yy}v + \partial_y^{-1}\xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\xi} - \chi_1 \xi^2 + 2b\partial_y^{-1}\xi \chi_1 y - 2b\chi_1 + \partial_{yy}\chi_1 \tilde{\xi} - \partial_y^{-1}\xi \partial_y \chi_1 \tilde{\xi} = 0.$$

One then has the following identity for an L^q energy estimate:

$$0 = \frac{d}{dt}\left(\frac{1}{q}\int v^q \, dy\right) + (q-1)\int v^{q-2}|\partial_y v|^2 \, dy + \int v^{q-1}\left(\partial_y^{-1}\xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\xi} - \chi_1 \xi^2 + 2b\partial_y^{-1}\xi \chi_1 y - 2b\chi_1 + \partial_{yy}\chi_1 \tilde{\xi} - \partial_y^{-1}\xi \partial_y \chi_1 \tilde{\xi}\right) dy.$$

We now estimate all terms. For the first one, integration by parts gives, using $|v| \lesssim |\tilde{\xi}|$,

$$\left|\int v^{q-1}\partial_y^{-1}\xi \partial_y v dy\right| = \frac{1}{q}\left|\int v^q \xi \, dy\right| \lesssim \|\xi\|_{L^\infty([0,2N])} \int_{y \le 2N} |\tilde{\xi}|^q \, dy \lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{8})s}.$$

For the second term, integrating by parts, applying the Hölder and Young inequalities and $|v| \lesssim |\tilde{\xi}|$ we get

$$\left|\int v^{q-1}\partial_y\chi_1\partial_y\tilde{\xi}\right| \le \frac{1}{2}\int |\partial_y v|^2 v^{q-2} + C\int_{y\le 2N}|\tilde{\xi}|^q \le \frac{1}{2}\int |\partial_y v|^2 v^{q-2} + CL_1^q e^{q(1-\kappa)s}.$$

For the third term, since $|v| \lesssim \tilde{\xi}$ and $\xi^2 \lesssim |\xi|(|\tilde{\xi}| + y^2|)$, from Hölder and (4.82) we obtain

$$\begin{aligned}
&\left|\int v^{q-1}\chi_1\xi^2\right| \\
&\quad\lesssim \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}\tilde{\xi}^q + \|\xi\|_{L^\infty([0,2N])}\left(\int_{y\le 2N} y^{2q}\,dy\right)^{\frac{1}{q}}\left(\int_{y\le 2N}|\tilde{\xi}|^q\right)^{1-\frac{1}{q}} \\
&\quad\lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{8})s} + e^{(1-\frac{1}{8})s}N^{2+\frac{1}{q}}L_1^{q-1}e^{(q-1)(1-\kappa)s} \lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{16})s}
\end{aligned}$$

since $e^{-\frac{1}{16}}N^{2+\frac{1}{q}} \le L_1$. For the fourth term, since $|\partial_y^{-1}\xi y| \le \|\xi\|_{L^\infty([0,2N])}y^2$ and $|v| \lesssim |\tilde{\xi}|$, we get

$$\begin{aligned}
\left|\int v^{q-1}\partial_y^{-1}\xi\chi_1 y\right| &\le \|\xi\|_{L^\infty([0,2N])}\left(\int_{y\le 2N} y^{2q}\,dy\right)^{\frac{1}{q}}\left(\int_{y\le 2N}|\tilde{\xi}|^q\right)^{1-\frac{1}{q}} \\
&\lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{16})s}.
\end{aligned}$$

For the next two terms we have

$$\left|\int v^{q-1}(-2b\chi_1 + \partial_{yy}\chi_1\tilde{\xi})\,dy\right| \lesssim \int_{y\le 2N}\tilde{\xi}^q dy \lesssim L_1^q e^{q(1-\kappa)s}.$$

Finally, for the last term, as $\partial_y\chi_1 \lesssim N^{-1}$, one has $|\partial_y^{-1}\xi\partial_y\chi_1| \lesssim \|\xi\|_{L^\infty([0,2N])}$ and

$$\left|\int v^{q-1}\partial_y^{-1}\xi\partial_y\chi_1\tilde{\xi}\,dy\right| \le \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}\tilde{\xi}^q\,dy \lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{8})s}.$$

Collecting all the above estimates gives

$$\frac{d}{dt}\left(\int v^q\,dy\right) \lesssim L_1^q e^{(q(1-\kappa)+1-\frac{1}{16})s}.$$

We reintegrate the above over time, using the relation $ds/dt = \lambda^2 \approx e^s$ from (4.16):

$$\begin{aligned}
\int v^q &\lesssim \int |\tilde{\xi}(s_0)|^q + L_1^q\int_{s_0}^{s} e^{(q(1-\kappa)-\frac{1}{16})s'}\,ds' \\
&\lesssim \begin{cases} L^q N + L_1^q e^{(q(1-\kappa)-\frac{1}{16})s} & \text{if } \kappa < 1-\frac{1}{16q}, \\ L^q N + L_1^q e^{(q(1-\kappa)-\frac{1}{16})s_0} & \text{if } \kappa > 1-\frac{1}{16q}, \end{cases} \\
&\lesssim \begin{cases} L_1^q e^{(q(1-\kappa)-\frac{1}{16})s} & \text{if } \kappa < 1-\frac{1}{16q}, \\ L_1^q & \text{if } \kappa > 1-\frac{1}{16q}, \end{cases}
\end{aligned}$$

since $L_1 = LN^{\frac{1}{q}} + N^{2+\frac{1}{q}} e^{-\frac{s_0}{8q}}$ (the case $\kappa = 1 - 1/(16q)$ produces a harmless log which can be avoided by choosing slightly different parameters without affecting the result). This ends the proof of (4.83) and of the claim.

Step 2. *Uniform-in-time L^q bound.* We iterate Step 1 for a sequence of intervals $[0, \alpha_1 N]$, $\dots, [0, \alpha_k N]$ and parameters $\kappa_1, \dots, \kappa_k$. Note that this is possible from the initial bounds. At each iteration, if one is not in the second case the gain in (4.83) is $\kappa_i = \kappa_{i-1} + 1/(16q)$. Hence we only need a finite number of iterations depending on the choice of q to reach the second case, yielding

$$\int_{y \le N} |\tilde{\xi}|^q dy \lesssim L_1^q = L^q N + N^{2q+1} e^{-\frac{s_0}{16}}.$$

Step 3. *The bootstrap procedure for the derivative.* Let $1 < \alpha_1 < 2$, $0 \le \kappa < 2$ with $\kappa \neq 2 - 1/8$, $L_1 = L' + N^{3/2} e^{-\frac{s_0}{8q}}$, and assume that for $t \in [t(s_0), t(s_1)]$,

$$\int_{y \le 2N} |\partial_y \tilde{\xi}|^2 dy \le L_1^2 e^{(2-\kappa)s}. \tag{4.84}$$

We claim that then for all $t \in [0, t(s_1)]$,

$$\int_{y \le \alpha_1 N} |\tilde{\xi}|^2 dy \lesssim \begin{cases} L_1^2 e^{(2-\kappa-\frac{1}{8})s} & \text{if } \kappa < 2 - 1/8, \\ L_1^q & \text{if } 2 - 1/8 < \kappa. \end{cases} \tag{4.85}$$

We now prove this claim. Let $\zeta := \partial_y \xi$. Then ζ solves

$$\zeta_t - \xi\zeta + \partial_y^{-1}\xi \partial_y \zeta - \partial_{yy}\zeta = 0.$$

We write $\zeta = h + \tilde{\zeta}$ with h smooth such that $h = 2by$ for $y \ge 1$, $h(0) = h'(0) = h''(0) = 0$. Then $\tilde{\zeta}$ solves

$$\tilde{\zeta}_t - \partial_{yy}\tilde{\zeta} + \partial_y^{-1}\xi \partial_y \tilde{\zeta} - \xi\zeta + \partial_y^{-1}\xi \partial_y h - \partial_{yy} h = 0, \quad \partial_y \tilde{\zeta}(t, 0) = 0.$$

Let $0 < \alpha \ll 1$ and χ be a smooth cut-off function with $\chi(y) = 1$ for $y \le 1 + \alpha$ and $\chi(y) = 0$ for $y \ge 1 + 2\alpha$, set $\chi_1 = \chi(\frac{y}{\alpha_1 N})$ and let $v := \chi_1 \tilde{\zeta}$. Then v solves

$$\begin{aligned} v_t - \partial_{yy} v + \partial_y^{-1}\xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\zeta} - \chi_1 \xi\zeta + \partial_y^{-1}\xi \chi_1 \partial_y h - 2b\partial_{yy} h \\ + \partial_{yy}\chi_1 \tilde{\zeta} - \partial_y^{-1}\xi \partial_y \chi_1 \tilde{\zeta} = 0. \end{aligned}$$

An L^2 energy identity then reads

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int v^2\right) + \int |\partial_y v|^2 \\ &+ \int v(\partial_y^{-1}\xi \partial_y v + 2\partial_y \chi_1 \partial_y \tilde{\zeta} - \chi_1 \xi\zeta + \partial_y^{-1}\xi \chi_1 \partial_y h - 2b\partial_{yy} h + \partial_{yy}\chi_1 \tilde{\zeta} - \partial_y^{-1}\xi \partial_y \chi_1 \tilde{\zeta}) = 0. \end{aligned}$$

We now estimate all terms. For the first one, integration by parts gives, using $|v| \lesssim |\tilde{\zeta}|$,

$$\left|\int v\partial_y^{-1}\xi\partial_y v\, dy\right| = \frac{1}{2}\left|\int v^2\xi\, dy\right| \lesssim \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}|\tilde{\zeta}|^2\, dy \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s}.$$

For the second one, integrating by parts, applying the Hölder and Young inequalities and $|v| \lesssim |\tilde{\zeta}|$, we get

$$\left|\int v\partial_y\chi_1\partial_y\tilde{\zeta}\, dy\right| \le \frac{1}{2}\int|\partial_y v|^2 dy + C\int_{y\le 2N}|\tilde{\zeta}|^2 dy \le \frac{1}{2}\int|\partial_y v|^2 dy + CL_1^2 e^{(2-\kappa)s}.$$

For the third term, since $|v\xi\zeta| \lesssim |\tilde{\zeta}|^2|\xi| + y|\xi|$, we obtain

$$\begin{aligned}\left|\int v\chi_1\xi\zeta\right| &\lesssim \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}\tilde{\zeta}^2 + \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}y^2\\ &\lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.\end{aligned}$$

Similarly for the fourth term, since $|\partial_y^{-1}\xi\partial_y h| \le \|\xi\|_{L^\infty([0,2N])}y$ and $|v| \lesssim |\tilde{\zeta}|$, we find

$$\begin{aligned}\left|\int v\partial_y^{-1}\xi\chi_1\partial_y h\right| &\le \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}\tilde{\zeta}^2 + \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}y^2\\ &\lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.\end{aligned}$$

Finally, for the next two terms,

$$\left|\int v(-\partial_{yy}h\chi_1 + \partial_{yy}\chi_1\tilde{\zeta})\, dy\right| \lesssim \int_{y\le 2N}\tilde{\zeta}^2\, dy \lesssim L_1^2 e^{(2-\kappa)s}.$$

Finally, for the last term, as $\partial_y\chi_1 \lesssim N^{-1}$, one has $|\partial_y^{-1}\xi\partial_y\chi_1| \lesssim \|\xi\|_{L^\infty([0,2N])}$ and

$$\begin{aligned}\left|\int v\partial_y^{-1}\xi\partial_y\chi_1\tilde{\zeta}\right| &\le \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}\tilde{\zeta}^2 + \|\xi\|_{L^\infty([0,2N])}\int_{y\le 2N}y^2\, dy\\ &\lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.\end{aligned}$$

Collecting all the above estimates gives

$$\frac{d}{dt}\left(\int v^2\, dy\right) \lesssim L_1^2 e^{(2-\kappa+1-\frac{1}{8})s} + N^3 e^{(1-\frac{1}{8})s}.$$

We reintegrate the above identity over time, using the relation $ds/dt = \lambda^2 \approx e^s$ from (4.16):

$$\begin{aligned}\int v^2 &\lesssim \int|\tilde{\zeta}(s_0)|^2 + L_1^2\int_{s_0}^s e^{(2-\kappa-\frac{1}{8})s'}\, ds' + N^3\int_{s_0}^s e^{-\frac{1}{8}s}\\ &\lesssim \begin{cases} L'^2 + L_1^2 e^{(2-\kappa-\frac{1}{8})s} + N^3 e^{-\frac{1}{8}s_0} & \text{if } \kappa < 2-1/8,\\ L'^2 + L_1^2 e^{(2-\kappa-\frac{1}{8})s_0} + N^3 e^{-\frac{1}{8}s_0} & \text{if } \kappa > 2-1/8,\end{cases}\\ &\lesssim \begin{cases} L_1^2 e^{(2-\kappa-\frac{1}{8})s} & \text{if } \kappa < 2-1/8,\\ L_1^2 & \text{if } \kappa > 2-1/8,\end{cases}\end{aligned}$$

since $L_1 = L' + N^3 e^{-s_0/8}$. This ends the proof of (4.85).

Step 4. *Uniform-in-time L^2 bound for the derivative.* Again, as in Step 2, we iterate Step 3 for a finite sequence of intervals $[0, \alpha_1 N], \dots, [0, \alpha_k N]$ and finally obtain

$$\int_{y \le N} |\partial_y \tilde{\xi}|^2 dy \lesssim L_1^2 = L'^2 + N^3 e^{-\frac{s_0}{8}}. \qquad \blacksquare$$

4.7. End of the proof of Proposition 4.7 and proof of Theorem 1

In this subsection we reintegrate over time the modulation equations and the various energy estimates, to show that the various upper bounds describing the bootstrap cannot be saturated. We first reintegrate the modulation equations and Lyapunov functionals.

Lemma 4.17. *There exists $\nu^* > 0$ such that for any $\nu < \nu^*$, for ν' small enough and then for η small enough, for any K, M such that Lemmas 4.8, 4.12, 4.13 and 4.14 hold true, the following holds for s_0 large enough. For a solution that is trapped on $[s_0, s_1]$, at time $s \in [s_0, s_1]$,*

$$\|\varepsilon\|_{L^2_\rho}^2 \le 2e^{-\frac{7}{2}s}, \quad \int_{s_0}^{s} e^{(7-\eta)\tilde{s}} \|\partial_Y \varepsilon(\tilde{s})\|_{L^2_\rho}^2 \, d\tilde{s} \le 2, \tag{4.86}$$

$$\frac{1}{2e} \le \mu \le 2e, \quad \frac{1}{4} e^{\frac{s}{2}} \le \lambda \le \frac{9}{4} e^{\frac{s}{2}}, \quad |a| \le 2e^{-(\frac{1}{2}-2\nu)s}, \tag{4.87}$$

$$\mu = \mu_\infty (1 + O(e^{-s})), \quad \lambda = e^{\frac{s}{2}} \tilde{\lambda}_\infty (1 + O(e^{-2s})), \tag{4.88}$$

$$\begin{aligned} &\int_{Z_1}^{Z_2} u^2 w \, dZ + \int_{Z_3}^{\infty} u^2 w \, dZ \le 4e^{-(1-2\nu)s}, \\ &\int_{Z_1}^{Z_2} |A \partial_Z u|^2 w \, dZ + \int_{Z_3}^{\infty} |A \partial_Z u|^2 w \, dZ \le 4e^{2\nu s}. \end{aligned} \tag{4.89}$$

Proof. **Step 1.** *Interior Lyapunov functional and energy dissipation.* We rewrite (4.39) as

$$\frac{d}{ds} \left(e^{7s} \|\varepsilon\|_{L^2_\rho}^2 \right) + e^{(7-\eta)s} \|\partial_Y \varepsilon\|_{L^2_\rho}^2 \le C e^{(7-\eta)s} \|\varepsilon\|_{L^2_\rho}^2 + C e^{7s} \|\varepsilon\|_{L^2_\rho} \lambda^{-12} + C e^{7s - e^s}.$$

Inserting the bounds (4.16) and (4.17) and integrating over time using (4.13) gives

$$e^{7s} \|\varepsilon\|_{L^2_\rho}^2 - 1 + \int_{s_0}^{s} e^{(7-\eta)\tilde{s}} \|\partial_Y \varepsilon(\tilde{s})\|_{L^2_\rho}^2 \le \int_{s_0}^{s} \left(C(K) e^{-\eta \tilde{s}} + C(K) e^{-\frac{1}{2}\tilde{s}} + C e^{7\tilde{s} - e^{\tilde{s}}} \right) d\tilde{s} \le 1$$

for s_0 large enough depending on K, which implies the desired estimates (4.86).

Step 2. *Law for μ.* We integrate the inequality (4.26) over time to find that for s_0 large enough,

$$|\log \mu(s) - \log \mu(s_0)| \le \int_{s_0}^{s} e^{-\frac{13}{8}\tilde{s}} \, d\tilde{s} \le 1,$$

which using (4.12) gives indeed $(2e)^{-1} \le \mu \le 2e$, and if the solution is trapped for all times then

$$\mu(s)=\mu(s_0)\exp\Big(\int_{s_0}^{s}O(e^{-\frac{13}{8}\tilde s})\,d\tilde s\Big)=\mu(s_0)\exp\Big(\Big(\int_{s_0}^{\infty}-\int_{s}^{\infty}\Big)O(e^{-\frac{13}{8}\tilde s})\,d\tilde s\Big)$$
$$=\mu_\infty(1+O(e^{-\frac{13}{8}s})),$$

where we have set $\mu_\infty:=\mu(s_0)\exp(\int_{s_0}^{\infty}O(e^{-\frac{13}{8}\tilde s})\,d\tilde s)$.

Step 3. *Law for* λ. We rewrite as in Step 2 the equation for λ in (4.26) using (4.16):

$$\left|\frac{\lambda_s}{\lambda}-\frac{1}{2}\right|\le C(K)e^{-2s}. \tag{4.90}$$

This can be written alternatively as $|\frac{d}{ds}(e^{-s/2}\lambda)|\le C(K)e^{-5s/2}$, which when reintegrated over time using (4.12) gives

$$|e^{-\frac{s}{2}}\lambda-e^{-\frac{s_0}{2}}\lambda(s_0)|\le C(K)\int_{s_0}^{s}e^{-5\tilde s/2}\,d\tilde s,$$

which with (4.12) yields $1/4\le e^{-s/2}\lambda\le 9/4$ for s_0 large enough, implying the bound for λ in (4.87). If the solution is trapped for all times, this gives

$$\lambda=e^{\frac{s}{2}}\Big(e^{-\frac{s_0}{2}}\lambda_0+\int_{s_0}^{s}O(e^{-5\tilde s/2})\,d\tilde s\Big)=e^{\frac{s}{2}}\Big(e^{-\frac{s_0}{2}}\lambda_0+\Big(\int_{s_0}^{\infty}-\int_{s}^{\infty}\Big)O(e^{-5\tilde s/2})\,d\tilde s\Big)$$
$$=e^{\frac{s}{2}}\tilde\lambda_\infty(1+O(e^{-\frac{5}{2}s})),$$

where we have set $\tilde\lambda_\infty=e^{-s_0/2}\lambda_0+\int_{s_0}^{\infty}O(e^{-5\tilde s/2})\,d\tilde s$.

Step 4. *Law for* a. We rewrite the equation for a in (4.25) and insert the bounds (4.16), (4.17) and (4.19), using $G(-\pi+Z)=O(Z^2)$ as $Z\to 0$:

$$\begin{aligned}\left|\frac{d}{ds}(e^{\frac{s}{2}}a)\right|&\lesssim e^{\frac{s}{2}}\Big(\Big|\int_{-\pi-a}^{-\pi}G_1\,dZ\Big|+\Big|\int_{-\pi-a}^{0}u\,dZ\Big|+\lambda^{-4}+\|\varepsilon\|_{L^2_\rho}+\lambda^4\|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2_\rho}\Big)\\
&\lesssim e^{\frac{s}{2}}|a|^3+e^{\frac{s}{2}}\Big|\int_{-\pi-a}^{-Me^{-s}}u\,dZ+\int_{-Me^{-s}}^{0}u\,dZ\Big|\\
&\quad+C(K)(e^{-\frac{3}{2}s}+e^{-3s}+se^{-(1+\frac{1}{4}-\nu)s})\\
&\lesssim e^{-(1-6\nu)s}+e^{\frac{s}{2}}\Big(s\Big(\int_{-\pi-a}^{-Me^{-s}}wu^2\,dZ\Big)^{\frac{1}{2}}+e^{-s}\int_{CM}^{0}|\varepsilon|\,dY\Big)\\
&\lesssim e^{-(1-6\nu)s}+e^{\frac{s}{2}}\big(se^{-(\frac{1}{2}-\nu)s}+e^{-\frac{9}{2}s}\big)\lesssim se^{\nu s},\end{aligned}$$

for s_0 large enough. This implies in particular the following bound for a_s using (4.16):

$$|a_s|\lesssim e^{-(\frac{1}{2}-2\nu)s}. \tag{4.91}$$

Reintegrating this estimate over time gives, using (4.12),

$$|a|=e^{-\frac{s}{2}}\Big|a_0e^{\frac{s}{2}}+\int_{s_0}^{s}O(\tilde se^{\nu\tilde s})\,d\tilde s\Big|\le 2e^{-(\frac{1}{2}-2\nu)s}.$$

Step 5. *Exterior energy functionals.* We insert the bounds (4.18) and (4.55) in (4.58):

$$\begin{aligned}
&\frac{d}{ds}\left(e^{(1-\nu)s}\int_{Z_1}^{Z_2} u^2 w\right) + \left(\frac{1}{2}-\frac{\nu}{2}\right)\int_{Z_1}^{Z_2} u^2 w \\
&\le C(K,M)e^{(1-\nu)s}\left(e^{6s}u^2(Z_2)+e^{4s}|\partial_Z u|^2(Z_2)+e^{-(2+\frac{1}{6})s}+\left(\int_{Z_1}^{Z_2} u^2 w\right)^{\frac{1}{2}} e^{-\frac{5}{8}s}\right) \\
&\lesssim C(K,M)(e^{(2\nu'-\nu)s}+e^{-(1+\frac{1}{6}-\nu)s}+e^{-(\frac{1}{8}-2\nu)s}) \lesssim C(K,M)e^{(2\nu'-\nu)s},
\end{aligned}$$

where the $e^{-(\nu-2\nu')s}$ is the worst term, due to the boundary condition at Z_2. Indeed, we optimised the weight w to match the exterior decay with the interior decay, hence the choice of $\beta = 1/2$ for the eigenfunction (4.48) in the weight (4.8). Reintegrating the above inequality over time using (4.86) and (4.14) yields, since $0<\eta \ll \nu' \ll \nu \ll 1$, for s_0 large enough,

$$\begin{aligned}
\int_{Z_1}^{Z_2} u^2 w &\le e^{-(1-\nu)s}\left[e^{(1-\nu)s_0}\int_{Z_1(s_0)}^{Z_2(s_0)} u^2 w + C(K,M)\int_{s_0}^{s} e^{-(\nu-2\nu')s}\right] \\
&\le e^{-(1-2\nu)s}(e^{\nu(s_0-s)}+C(K,M)e^{-\nu s}) \le 2e^{-(1-2\nu)s}.
\end{aligned}$$

The differential inequality on the right (4.79) can be reintegrated over time the same way, giving $\int_{Z_3}^{\infty} u^2 w \le 2e^{-(1-2\nu)s}$. These two bounds imply the first bound in (4.89). We now turn to the derivative. We write (4.68) as

$$\left|\frac{d}{ds}\left(e^{-\nu s}\int_{Z_1}^{Z_2} |A\partial_Z u|^2 w\, dZ\right)\right| \le e^{-\frac{1}{4}s}.$$

Note that compared to the differential inequality for u, the above inequality for $A\partial_Z u$ is better. Indeed, the fact that $A \sim Z$ near the origin improves the control of the boundary term at Z_2, and $A\partial_Z$ kills the worst component of the error near the origin. Reintegrating the above inequality over time using (4.86) and (4.14) yields

$$\begin{aligned}
\int_{Z_1}^{Z_2} |A\partial_Z u|^2 &\le e^{2\nu s}\left(e^{-\nu s}e^{\nu s_0}\int_{Z_1(s_0)}^{Z_2(s_0)} |A\partial_Z u(s_0)|^2 + Ce^{-\nu s}\int_{s_0}^{s} e^{-\frac{1}{4}\tilde{s}}\, d\tilde{s}\right) \\
&\le e^{2\nu s}(e^{\nu(s_0-s)}+Ce^{-\nu s}) \le 2e^{2\nu s}.
\end{aligned}$$

The same bound can also be proved the same way for the right derivative at the origin, implying the last bound in (4.89). ∎

We now bootstrap the last bound and control ε on $[-M^2, M^2]$ using parabolic regularity.

Lemma 4.18. *There exists $\nu^* > 0$ such that for any $\nu < \nu^*$, for ν' small enough, for K, M such that Lemma 4.17 holds true, for a solution that is trapped on $[s_0, s_1]$ for s_0 large enough,*

$$\|\varepsilon(s_1)\|^2_{H^3(|Y|\le M^2)} \le 10e^{-(7-\nu')s_1}. \tag{4.92}$$

Proof. The proof is a classical application of parabolic regularity: ε evolves according to a parabolic equation, its size and the size of the forcing terms are precisely $e^{-7s/2}$, hence this bound propagates for higher order derivatives due to the smoothing effect of the heat kernel. In this proof, the constants C might depend on M and K unless explicitly mentioned. We rewrite (4.20) as

$$\varepsilon_s - \partial_{YY}\varepsilon + \tilde{V}\varepsilon + \tilde{\mathcal{T}}\partial_Y\varepsilon = \mathcal{F},$$

where

$$\tilde{V} := 2\frac{\lambda_s}{\lambda} - 2G_1 - \varepsilon, \quad \tilde{\mathcal{T}} := \frac{\lambda_s}{\lambda}Y + \int_{(-\pi-a)\lambda^2\mu}^{0} f - \lambda y_s^* + \lambda^2\mu\partial_Z^{-1}G_1 + \partial_Y^{-1}\varepsilon,$$

$$\mathcal{F} := \left(m_2 - \frac{1}{2\lambda^4\mu^2}\right)Z\partial_Z G_1 + \left(m_1 + \frac{1}{4\lambda^4\mu^2}\right)(2 - Z\partial_Z)G_1 + m_3\frac{1}{\lambda^2\mu}\partial_Z G_1 - \frac{1}{\lambda^4\mu^2}\left(\partial_{ZZ}G_1 + \frac{1}{4}Z\partial_Z G_1 + \frac{1}{2}G_1\right).$$

Note that from (4.26), (4.8) and (4.17) one has, for a universal $C > 0$,

$$\|\tilde{\mathcal{T}}\|_{W^{1,\infty}(|Y|\le M^3)} + \|\tilde{V}\|_{W^{1,\infty}(|Y|\le M^3)} \le C \tag{4.93}$$

We now let $\varepsilon^1 := \partial_Y\varepsilon$. It solves

$$\varepsilon_s^1 - \partial_{YY}\varepsilon^1 + (\tilde{V} + \partial_Y\tilde{\mathcal{T}})\varepsilon^1 + \tilde{\mathcal{T}}\partial_Y\varepsilon^1 = -\partial_Y\tilde{V}\varepsilon + \partial_Y\mathcal{F}. \tag{4.94}$$

Let $M^2 < M_1 < M_2 < M^3$, let χ be a cut-off function with $\chi = 1$ for $Y \le M_1$ and $\chi = 0$ for $Y \ge M_2$, and let $v = \chi\varepsilon^1$. Then v solves

$$v_s - \partial_{YY}v + (\tilde{V} + \partial_Y\tilde{\mathcal{T}})v + \tilde{\mathcal{T}}\partial_Y v = -\partial_{YY}\chi\varepsilon^1 - 2\partial_Y\chi\partial_Y\varepsilon^1 - \tilde{\mathcal{T}}\partial_Y\chi\varepsilon^1 - \chi\partial_Y\tilde{V}\varepsilon + \chi\mathcal{F}.$$

We then apply a standard energy identity:

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{2}\int v^2\,dY\right) &+ \int |\partial_Y v|^2\,dY \\ &= \int (-\partial_{YY}\chi\varepsilon^1 - 2\partial_Y\chi\partial_Y\varepsilon^1 - \tilde{\mathcal{T}}\partial_Y\chi\varepsilon^1 - \chi\partial_Y\tilde{V}\varepsilon + \chi\mathcal{F})v\,dY \\ &\quad - \int ((\tilde{V} + \partial_Y\tilde{\mathcal{T}})v + \tilde{\mathcal{T}}\partial_Y v)v\,dY.\end{aligned}$$

Let $0 < \kappa \ll 1$. Integrating by parts and using Young's inequality one finds, since $|v| \lesssim \varepsilon^1$,

$$\begin{aligned}\left|\int (-\partial_{YY}\varepsilon^1 - 2\partial_Y\chi\partial_Y\varepsilon^1)v\,dY\right| &\le \frac{C}{\kappa}\int_{|Y|\le M_3} |\varepsilon^1|^2 + \kappa C\int |\partial_Y v|^2\,dY \\ &\le C\|\partial_Y\varepsilon\|_{L^2_\rho}^2 + \frac{1}{4}\int |\partial_Y v|^2\,dY\end{aligned}$$

for κ small enough. Similarly, integrating by parts, using the Young inequality, (4.17) and (4.93) gives

$$\begin{aligned}\left|\int \tilde{\mathcal{T}}\partial_Y\chi\varepsilon^1 v\right| &= \left|\int \tilde{\mathcal{T}}\partial_Y\chi\partial_Y\varepsilon v\right| \\ &\le \frac14\int|\partial_Y v|^2 + C\|\tilde{\mathcal{T}}\|_{W^{1,\infty}(|Y|\le M_2)}\int_{|Y|\le M_2}\varepsilon^2 + C\int_{|Y|\le M_2}|\varepsilon^1|^2 \\ &\le \frac14\int|\partial_Y v|^2 + C\|\varepsilon\|^2_{L^2_\rho} + C\|\partial_Y\varepsilon\|^2_{L^2_\rho} \le \frac14\int|\partial_Y v|^2 + Ce^{-7s} + C\|\partial_Y\varepsilon\|^2_{L^2_\rho}.\end{aligned}$$

Next, from Cauchy–Schwarz, (4.17) and (4.93), and the Young inequality,

$$\begin{aligned}\left|\int \chi\partial_Y\tilde{V}\varepsilon v\right| &\le C\|\partial_Y\tilde{V}\|_{L^\infty(|Y|\le M^3)}\|v\|_{L^2}\|\varepsilon\|_{L^2(|Y|\le M^3)} \le Ce^{-\frac72 s}\|v\|_{L^2} \\ &\le Ce^{-7s} + C\|v\|^2_{L^2}.\end{aligned}$$

For the error, we recall the cancellation $\partial_{ZZ}G_1 + \frac14 Z\partial_Z G_1 + \frac12 G_1 = O(|Z|^4)$ and $|\partial_Z G_1| = O(|Z|)$ as $Z\to 0$, which implies using (4.47) that

$$\int \chi^2\mathcal{F}^2\,dY \le Ce^{-\frac{29}{4}s},$$

which by Cauchy–Schwarz and Young yields

$$\left|\int \chi\mathcal{F}v\,dY\right| \le Ce^{-\frac{29}{8}}\|v\|_{L^2} \le Ce^{-\frac{29}{4}s} + C\|v\|^2_{L^2}.$$

Integrating by parts and using (4.93) we get

$$\begin{aligned}\left|\int((\tilde{V}+\partial_Y\tilde{\mathcal{T}})v + \tilde{\mathcal{T}}\partial_Y v)v\right| &\le \|v\|^2_{L^2}(\|\tilde{V}\|_{W^{1,\infty}(|Y|\le M^3)} + \|\tilde{\mathcal{T}}\|_{W^{1,\infty}(|Y|\le M^3)}) \\ &\le C\|v\|^2_{L^2}.\end{aligned}$$

Let $0<\eta\ll\nu_1\ll\nu'$. Collecting all the estimates above, and since $|v|\lesssim|\varepsilon^1|$, one has the energy estimate

$$\frac{d}{ds}\left(e^{(7-\nu_1)s}\int v^2\right) + \frac12 e^{(7-\nu_1)s}\int|\partial_Y v|^2 \le Ce^{(7-\nu_1)s}\|\partial_Y\varepsilon\|^2_{L^2_\rho} + Ce^{-\nu_1 s}.$$

Reintegrated over time, using (4.86) and (4.13), this gives, for s_0 large enough,

$$e^{(7-\nu_1)s}\int v^2\,dY + \frac12\int_{s_0}^s e^{(7-\nu_1)s'}\int|\partial_Y v|^2\,dY\,ds' \le e^{(7-\nu_1)s_0}\int v_0^2\,dY + 1 \le 2.$$

Therefore, $\|v(\tilde{s})\|_{L^2} \le 2e^{-(7/2-\nu_1)\tilde{s}}$. We have thus proved the following pointwise bound for $\partial_Y\varepsilon$ and integrated bound for $\partial_{YY}\varepsilon$:

$$\begin{aligned}\forall s\in[s_0,s_1],\quad &\int_{|Y|\le M_1}|\partial_Y\varepsilon|^2\,dY \le 10e^{-(\frac72-\nu_1)s}, \\ &\int_{s_0}^s e^{(7-\nu_1)s'}\int_{|Y|\le M_1}|\partial_{YY}\varepsilon|^2\,dY\,ds' \le 2.\end{aligned}$$

Let now

$$M^2 < M_4 < M_3 < M_1.$$

We claim that we can differentiate equation (4.94) and, with the same arguments, obtain the analogue of the above estimates for $\partial_{YY}\varepsilon$, with an exponent ν_2 such that $\nu_1 \ll \nu_2 \ll \nu'$. Indeed, the only crucial arguments to derive the above bounds were the pointwise-in-time boundedness (4.17) of $\|\varepsilon\|_{L^2_\rho}$ and the dissipation estimate (4.86) for $\|\partial_Y \varepsilon\|_{L^2_\rho}$, and we have just obtained the analogues for $\partial_Y \varepsilon$, so that the same strategy can be applied. Then, another iteration yields the analogue of the above bounds for $\partial_Y^{(3)}\varepsilon$ for $|Y| \le M_4$ for an exponent $\nu_2 \ll \nu_3 \ll \nu'$, which ends the proof of the lemma. ∎

All the bounds of the bootstrap and the modulation equations have been investigated previously. We can now end the proof of Proposition 4.7.

Proof of Proposition 4.7. Let an initial datum satisfy the properties of Definition 4.4 at time s_0. Let $\tilde{s}$ be the supremum of times such that the solution is trapped on $[s_0, \tilde{s}]$. Assume for contradiction that $\tilde{s} < \infty$. Then from the local well-posedness Proposition 4.1 and the blow-up criterion (1.4), the solution can be extended beyond time $\tilde{s}$. Hence, from the definition of $\tilde{s}$ and Definition 4.5 and a continuity argument, one of the inequalities (4.16), (4.17) or (4.18) must be an equality at time $\tilde{s}$. This is however impossible for K large enough from (4.86), (4.87), (4.89) and (4.92), which is the desired contradiction. Hence $\tilde{s} = \infty$, which proves Proposition 4.7. ∎

Theorem 1 is a direct consequence of Proposition 4.7 and we can now give its proof.

Proof of Theorem 1. For an initial datum of the form (1.5), let $s_0 = 2\log(\lambda_0^2)$. Then for $\epsilon(\lambda_0) > 0$ small enough, thanks to the smoothing effect of the equation (see Proposition 4.1), $\tilde{\xi}_0$ is instantaneously regularised, and $\xi(t^*)$ is initially trapped in the sense of Definition 4.4. Applying Proposition 4.7, the solution is then trapped for all times in the sense of Definition 4.5. Since $ds/dt = \lambda^2$ and λ satisfies (4.88),

$$\frac{dt}{ds} = e^{-s}\tilde{\lambda}_\infty^{-2}(1 + O(e^{-2s})).$$

Reintegrating the above equation, we find that there exists $T > 0$ such that

$$T - t = e^{-s}\tilde{\lambda}_\infty^{-2}(1 + O(e^{-2s})).$$

This implies $e^{-s} = \lambda_\infty^2 (T-t) + O((T-t)^3)$. The identities (1.6) are then consequences of (4.88). From (4.19), $\tilde{x}(t,y) = u(s,Z)$ and (4.7) one infers that

$$\|\tilde{\xi}\|_{L^\infty} = \lambda^2 \|u\|_{L^\infty} \lesssim e^{-s} e^{-(\frac14 - \nu)s} \le C(T-t)^{1-\frac18},$$

which proves (1.7). We now investigate the existence and asymptotic behaviour of the blow-up profile at time T. The existence of a limit $\xi(t,y) \to \xi^*(y)$ as $t \uparrow T$ follows from Lemma 4.15 and a standard parabolic bootstrap argument. We now use Lemma 4.15 more carefully to find the asymptotic of the profile at blow-up time. For $y^*\geq e^{(1/2-1/16)s_0}$ we define the following adapted time, which now depends on the point that we consider:

$$s_0(y^*)=\left(\frac12-\frac1{16}\right)^{-1}\log(y)=\log(y^\alpha),\quad \alpha:=\left(\frac12-\frac1{16}\right)^{-1}=\frac{16}{7},$$

so $y^*=e^{(\frac12-\frac1{16})s_0(y)}$. For $s\geq s_0(y)$ and $y\in[0,2y^*]$, one has

$$Z(y)=\frac{y-y^*}{\lambda\mu}=-\pi-a+\frac{y}{\lambda\mu}=-\pi+O(e^{-\frac{s_0}{16}}).$$

Therefore one can apply the Taylor expansion of G_1 near the origin for s_0 large enough. Using (4.87), (4.88) and (4.19), for $s\geq s_0(y^*)$ we have

$$\begin{aligned}\lambda^2(s_0)G_1(Z(y))&=\frac14\left(-a+\frac{y}{\lambda\mu}\right)^2\lambda^2+\lambda^2O\left(\left|-a+\frac{y}{\lambda\mu}\right|^4\right)\\&=\frac{y^2}{4\mu_\infty^2}+O((y^*)^{2-\frac1{16}})\leq e^{(1-\frac18)s_0}\leq e^{(1-\frac18)s},\end{aligned}$$

and

$$|\lambda^2(s_0)u(s_0,Z(y))|\leq C\lambda^2(s_0)e^{-\frac16 s_0}\leq Ce^{(1-\frac16)s_0}=C\cdot(y^*)^{\frac{5\alpha}{6}}=C\cdot(y^*)^{2-\frac{2}{21}}.$$

The above two estimates imply that, writing $\xi=\frac{y^2}{4\mu_\infty^2}+\tilde\xi$, at time s_0^* on $[0,y^*]$ we have

$$\xi(t(s_0(y)),y)=\frac{y^2}{4\mu_\infty^2}+O((y^*)^{2-\frac1{16}}),\quad\text{i.e.}\quad \|\tilde\xi(s_0(y^*))\|_{L^\infty([0,2y^*])}\leq C\cdot(y^*)^{2-\frac1{16}},$$

and that for $s\geq s_0(y^*)$,

$$\|\xi\|_{L^\infty([0,2y^*])}\lesssim e^{(1-\frac18)s}.$$

Moreover, from (4.18), changing variables one gets

$$\begin{aligned}\|\partial_y(\lambda^2u(s,Z(y)))\|_{L^2([0,2y^*])}&\lesssim\lambda^{\frac32}\|\partial_Zu(s,Z)\|_{L^2([0,2y^*/\lambda])}\lesssim e^{\frac34 s}se^{2\nu}\leq e^s,\\ \|\partial_y(\lambda^2G_1(s,Z(y)))\|_{L^2([0,2y^*])}&\lesssim\lambda^{\frac32}\|\partial_ZG_1(s,Z)\|_{L^2([0,2y^*/\lambda])}\lesssim e^{\frac34 s}\leq e^s,\\ \|\partial_y(y^2)\|_{L^2([0,2y^*])}&\lesssim\lambda^{\frac32}\|\partial_ZG_1(s,Z)\|_{L^2([0,2y^*/\lambda])}\lesssim(y^*)^{\frac32}\leq e^s,\end{aligned}$$

for s_0 large enough, so that for $s\geq s_0(y^*)$,

$$\|\partial_y\tilde\xi\|_{L^2([0,2y^*])}\leq e^s.$$

We apply Lemma 4.15 to find that for all $t\geq t(s_0(y^*))$,

$$\|\tilde\xi\|_{L^q([0,y^*])}\lesssim(y^*)^{2-\frac1{16}}(y^*)^{\frac1q}+(y^*)^{2+\frac1q}(y^*)^{-\frac{\alpha}{16}}\lesssim(y^*)^{2-\frac1{16}+\frac1q},$$

and for some fixed constant $c>0$,

$$\|\partial_y\tilde\xi\|_{L^2([0,y^*])}\lesssim(y^*)^\alpha+(y^*)^{\frac32}e^{-\frac{s_0}{8q}}\lesssim(y^*)^c.$$

We apply the following interpolated Sobolev inequality:

$$\|h\|_{L^\infty} \lesssim \|h\|_{L^q}^{1-\frac{2}{q+2}} \|\partial_y h\|_{L^2}^{\frac{2}{q+2}},$$

implying that for all $t \geq t(s_0(y^*))$,

$$\|\tilde{\xi}\|_{L^\infty([0,y^*])} \lesssim (y^*)^{2-\frac{1}{16}+\frac{c}{q}} \lesssim (y^*)^{2-\frac{1}{32}}$$

for q large enough. Thus, since this remains true in the limit at time T we have showed that for $y^* \geq e^{(\frac{1}{2}-\frac{1}{16})s_0}$,

$$\xi(y^*) = \frac{(y^*)^2}{4\mu_\infty^2} + O((y^*)^{2-\frac{1}{32}}),$$

which ends the proof of (1.8). ■

4.8. Localised initial data

We now prove Proposition 1.2. It is obtained from the analysis of the previous subsections by controlling an additional weighted norm. We introduce $Z^* = Me^{-s}$.

Lemma 4.19. *Fix any ν, ν', K such that Lemma 4.8 holds true, and assume a solution is trapped on $[s_0, \infty)$. Then for M large enough, there exists $\delta_0 > 0$ such that for s_0 large enough:*

(i) *If* $\sup_{Z\geq\pi} |F(s_0, Z)|Z^2 \leq \delta_0$ *then* $\sup_{Z\geq\pi} |F(s, Z)|Z^2 \leq e^{-\frac{1}{8}s}$ *for $s \geq s_0$.*

(ii) *If* $\sup_{Z\geq-(\pi+a(s_0))} |\partial_Z u(s_0, Z)|\langle Z\rangle^2 \leq \delta_0$ *then* $\sup_{Z\geq-(\pi-a(s_0))} |\partial_Z u(s, Z)|\langle Z\rangle^2 \leq e^{-\frac{1}{50}s}$ *for $s \geq s_0$.*

Proof of Proposition 1.2. Let

$$\|f\|_* = \sup_{Z\geq\pi} |f(Z)|Z^2 + \sup_{Z\geq-(\pi+a(s_0))} |\partial_Z f(Z)|\langle Z\rangle^2.$$

Let an initial datum $\xi_0 \in \mathcal{B}$ for (1.3) satisfy the conditions of Proposition 4.7 and $\|u(s_0)\|_* \leq \delta_0/2$. Consider the open set of initial datum satisfying $\|\tilde{\xi}_0 - \xi_0\|_{\mathcal{B}} < \bar{\delta}$ with corresponding variable $\tilde{u}$. Then $\tilde{\xi}_0$ satisfies the conditions of Proposition 4.7 and $\|\tilde{u}(s_0)\|_* \leq \delta$. Hence $\tilde{\xi}$ satisfies the conclusions of Theorem 1 from its proof done in Section 4.7, and the bounds (1.14) and (1.15) are then consequences of Lemma 4.19 and of (4.81). ■

Proof of Lemma 4.19. Recall (4.27) and (4.46). Let $m_4 = \frac{1}{\lambda^4\mu^2}$. Note that one has the following estimates, from (4.87), (4.88), (4.26), (4.17) (using Sobolev embedding), for s_0 large enough:

$$m_1 = O(e^{-\frac{3}{2}s}), \quad m_2 = O(e^{-\frac{3}{2}s}), \quad m_3' = O(e^{-\frac{5}{2}s}), \quad m_4 = O(e^{-2s}), \tag{4.95}$$

$$\|u\|_{L^\infty} \leq e^{-\frac{s}{8}}, \quad \|u\|_{L^\infty[-Z^*,Z^*]} + e^{-s}\|\partial_Z u\|_{L^\infty[-Z^*,Z^*]} \leq e^{-3s}. \tag{4.96}$$

Step 1. *Proof of (i).* We rewrite equation (4.5) as $(\partial_s + \mathcal{M})F = 0$, with the elliptic operator $\mathcal{M} = 2\frac{\lambda_s}{\lambda} - F + (\partial_Z^{-1} F - \frac{\lambda_s}{\lambda} Z - m_2 Z + m_3')\partial_Z - m_4 \partial_{ZZ}$. We compute that $F' = e^{-\frac{1}{8}s} Z^{-2}$ is a supersolution on $[\pi, \infty)$. Indeed, from (4.95), (4.96), and since $G_1(Z) = 0$ for $Z \geq \pi$ and $\int_0^\pi G_1 = \pi/2$, we have

$$
\begin{aligned}
Z^2 e^{\frac{1}{8}s}(\partial_s + \mathcal{M})F' &= -\frac{1}{8} + 1 - u + 2\left(\frac{Z}{2} - \frac{\pi}{2} - \partial_Z^{-1} u\right) Z^{-1} + 4m_1 + 2m_2 - \frac{2m_3'}{Z} - \frac{6m_4}{Z^4} \\
&= \frac{7}{8} + (Z - \pi)Z^{-1} + o_{s_0 \to \infty}(1) > 0,
\end{aligned}
$$

where the $o()$ is uniform for $(s, Z) \in [s_0, \infty) \times [\pi, \infty)$. At the boundary $|F(s, \pi)| \leq F'(s, \pi)$ for all $s \geq s_0$ for s_0 large enough from (4.96). Then (i) is a consequence of the parabolic comparison principle.

Step 2. *Proof of (ii).* Let $\Omega_1 = [-\pi + a, -Z^*] \cup [Z^*, \infty)$ and $\Omega_2 = [-Z^*, Z^*]$. For χ a smooth cut-off with $\chi(y) = 1$ for $y \leq -1$ and $\chi(y) = 0$ for $y \geq 0$, we define $\chi^*(s, Z) = \chi(e^{s/2}(Z - \pi))\chi(e^{s/2}(-Z - \pi))$. After smoothing the profile near the points $\pm\pi$, we decompose $\partial_Z F$ as follows:

$$
\partial_Z F = \chi^* \partial_Z G_1 + \bar{F}.
$$

Since $\partial_Z u = (\chi^* - 1)\partial_Z G_1 + \bar{F}$, recalling (4.1), it is sufficient to prove (ii) for $\bar{F}$ in order to prove it for $\partial_Z u$. On Ω_1 we find from (4.5), (4.26), (4.95) and (4.96) that one has $(\partial_s + \bar{\mathcal{M}} + \tilde{\mathcal{M}})\bar{F} = \mathcal{E}$ where

$$
\begin{aligned}
\bar{\mathcal{M}} &= \frac{1}{2} - G_1 + \mathcal{T}\partial_Z, \quad \mathcal{E} = -(\partial_s + \bar{\mathcal{M}} + \tilde{\mathcal{M}})(\chi^* \partial_Z G_1), \\
\tilde{\mathcal{M}} &= m_1(2 - 2Z\partial_Z) - m_2 Z \partial_Z - u + (\partial_Z^{-1} u + m_3')\partial_Z - m_4 \partial_{ZZ} \\
&= O(e^{-\frac{1}{8}s}) + O(e^{-\frac{1}{8}s}|Z|)\partial_Z + O(e^{-2s})\partial_{ZZ}.
\end{aligned}
$$

We claim that there exists a smooth positive function $\tilde{w}$ on $\mathbb{R} \setminus \{0\}$ such that

$$
\begin{aligned}
\tilde{w}(Z) &= \left|\frac{\sin(Z/2)}{\cos(Z/2)}\right|^{\frac{1}{10}} |\sin Z| \quad \text{for } |Z| \leq \pi/2, \\
\tilde{w}(Z) &= 1/Z^2 \quad \text{for } |Z| \geq \pi + 1, \quad \bar{\mathcal{M}}\tilde{w} \geq \tfrac{1}{20}\tilde{w}.
\end{aligned}
\tag{4.97}
$$

We relegate the proof of this fact to Step 3. We now show that $\bar{F}'(s, Z) = e^{-\frac{1}{50}s}\tilde{w}(Z)$ is a supersolution on Ω_1. Note first that $|\partial_Z^i \tilde{w}| \lesssim Z^{-i}\tilde{w}$ for $i = 1, 2$. Hence on Ω_1,

$$
\begin{aligned}
\tilde{\mathcal{M}}\tilde{w} &= O(e^{-\frac{1}{8}s}\tilde{w}) + O(e^{-\frac{1}{8}s}|Z|)O(|Z|^{-1}\tilde{w}) + O(e^{-2s})O(|Z|^{-2}\tilde{w}) \\
&= O((e^{-\frac{1}{8}s} + M^{-2})\tilde{w}),
\end{aligned}
$$

as $|Z| \geq Me^{-s}$. This and (4.97) imply that on Ω_1, for M large enough and then s_0 large enough,

$$
(\partial_s + \bar{\mathcal{M}} + \tilde{\mathcal{M}})\bar{F}' \geq \tfrac{1}{50}\bar{F}'.
$$

Next, since $\bar{\mathcal{M}}(\partial_Z G_1) = 0$, from (4.1) we obtain $(\partial_s + \bar{\mathcal{M}})(\chi^* \partial_Z G_1) = O(e^{-s/2})$ and it has support in $[-\pi, -\pi + e^{-s/2}] \cup [\pi - e^{-s/2}, \pi]$. From (4.1) and (4.96) we also find that $\tilde{\mathcal{M}}(\chi^* \partial_Z G_1) = O(e^{-\frac{1}{8}s}|Z|)$ and it has support in $[-\pi, \pi]$. Therefore, since $\tilde{w}(Z) \sim |Z|^{1+\frac{1}{10}}$ as $Z \to 0$, and since $e^{-\frac{1}{8}s} \le e^{-\frac{1}{40}s}|Z|^{\frac{1}{10}} M^{-\frac{1}{10}}$ on Ω_1, we infer that on Ω_1,

$$|\mathcal{E}| \le \tfrac{1}{100} \bar{F}'$$

for s_0 large enough. The above two inequalities imply that $(\partial_s + \bar{\mathcal{M}} + \tilde{\mathcal{M}})\bar{F}' - \mathcal{E} \ge 0$ on Ω_1. At the boundary, $|\bar{F}(s, \pm Z^*)| \le \bar{F}'(s, \pm Z^*)$ from (4.96), $|\bar{F}(s, -\pi - a)| \le Ce^{-s/2} \le \bar{F}'(s, -\pi - a)$ from (4.81), and $|\bar{F}(s_0)| \le F'(s_0)$ for δ_0 small enough. Hence $|\bar{F}| \le \bar{F}'$ on Ω_1 from parabolic comparison (using similarly $-\bar{F}'$ as a subsolution). This bound on Ω_1 and the bound (4.96) on Ω_2 show (ii).

Step 3. *Existence proof for* (4.97). For example, we choose

$$\tilde{w}(Z) = \left|\frac{\sin(Z/2)}{\cos(Z/2)}\right|^{\frac{1}{10}} |\sin Z| \bar{w}(Z)$$

on $(0, \pi)$, with $\bar{w}(Z) = 1$ for $0 < Z \le \pi/2$ so that on $(0, \pi)$,

$$\bar{\mathcal{M}}\tilde{w} = \frac{1}{20}\tilde{w} + \frac{1}{2}\left|\frac{\sin(Z/2)}{\cos(Z/2)}\right|^{\frac{1}{10}} |\sin Z|(\sin Z)\partial_Z \bar{w}.$$

We choose $\bar{w} > 0$ to be an increasing function of Z such that $\tilde{w}$ is smooth on $(0, \pi]$ all the way up to π with $\tilde{w}(\pi) = 1$, and so $\bar{\mathcal{M}}\tilde{w} \ge \frac{1}{20}\tilde{w}$ on $(0, \pi]$ from the above identity. Next, on $[\pi, \infty)$ we choose $\tilde{w}$ to be any smooth extension that is a nonincreasing function of Z with $\tilde{w}(Z) = Z^{-2}$ for $Z \ge \pi + 1$. Then $\bar{\mathcal{M}}\tilde{w} = \frac{1}{2}\tilde{w} - \frac{1}{2}(Z - \pi)\partial_Z \tilde{w} \ge \frac{1}{2}\tilde{w}$ on $[\pi, \infty)$. Hence the desired properties hold on $(0, \infty)$. We finally extend $\tilde{w}$ to $(-\infty, 0)$ by even symmetry. ∎

5. Application to the two-dimensional Prandtl system

Here we prove Theorem 2. Recalling that ξ_i is defined by (1.10), we introduce

$$\xi_{i,k}(t, y) := \partial_y^k \xi_i(t, y) \tag{5.1}$$

(i.e. $\xi_{i,k} = \partial_x^{2i+1}\partial_y^k u|_{x=0}$). First, we apply Proposition 6.1 using (1.13), and find that there exist $T_0, \tau_0', C_0'' > 0$, and $(\xi_i)_{i\ge 0} \in C([0, T_0] \times [0, \infty))$ with $(\xi_i)_{i\ge 0} \in C^\infty((0, T_0] \times [0, \infty))$ a classical solution to (1.11) on $(0, T_0]$, such that

$$|\xi_i(t, y)| \le C_0''(\tau_0')^{-2i-1}(2i + 1)!\langle y\rangle^{-2} \quad \text{for all } (t, y) \in [0, T_0] \times [0, \infty), \tag{5.2}$$

$$|\xi_{i,k}(T_0, y)| \le C_0''(\tau_0')^{-2i-k-1}(2i + k + 1)!\langle y\rangle^{-2} \quad \text{for all } y \in [0, \infty). \tag{5.3}$$

Thus now our aim is to control $(\xi_i)_{i\ge 0}$ from T_0 up to time T.

We first establish linear estimates in Section 5.1, then study $\xi_{1,0}$ in Section 5.2, and then all remaining derivatives in Section 5.3. Theorem 2 is proved in Section 5.3.1.

Throughout this section, we assume that all the hypotheses of Theorem 2, (5.2) and (5.3) hold true. In particular, the parameters $T, T_0, C_0, C_0', C_0'', \iota, \tau_0, \tau_0'$ are independent of all other forthcoming parameters, since they are fixed a priori. We shall denote by C a constant that may vary from line to line, but that depends solely on those parameters. Since the precise value of μ will never play a role, we assume

$$\mu = 1$$

without loss of generality. We perform the following renormalisations:

$$\begin{gathered} \mathsf{s} = -\log(T-t), \quad \mathsf{s}_0 = -\log(T-T_0), \quad z = (T-t)^{\frac{1}{2}} y - \pi, \\ \xi(t,y) = (T-t)^{-1} \mathsf{F}(\mathsf{s},z), \quad \mathsf{F}(\mathsf{s},z) = G_1(z) + \mathsf{u}(\mathsf{s},z), \\ \xi_{i,k}(t,y) = (T-t)^{\frac{k}{2}-3i-1} F_{i,k}(\mathsf{s},z), \end{gathered}$$

and will use the notation

$$(\partial_z^{-1} f)(z) = \int_0^z f, \quad (\partial^{-1} f)(z) = \int_{-\pi}^z f,$$

so that $\partial^{-1} f = \int_{-\pi}^0 f + \partial_z^{-1} f$. The evolution equation for $F_{i,k}$ for $i+k \geq 1$ is, from (1.11),

$$\begin{aligned} \partial_s F_{i,k} + \mathcal{L}_{i,k} F_{i,k} &= \delta_{k\neq 0}\delta_{i\neq 0}(2i+1) F_{0,k+1} \partial^{-1} F_{i,0} \\ &\quad + \sum_{j=1}^{i-1} \binom{2i+1}{2j} (\partial^{-1} F_{j,0}) F_{i-j,k+1} - \sum_{(j,l)\in E^1_{i,k}} \binom{2i+1}{2j+1}\binom{k}{l} F_{j,l} F_{i-j,k-l} \\ &\quad + \sum_{(j,l)\in E^2_{i,k}} \binom{2i+1}{2j}\binom{k}{l+1} F_{j,l} F_{i-j,k-l} \end{aligned} \tag{5.4}$$

where $E^1_{i,k} = \{(j,l) : 0 \leq j \leq i, 0 \leq l \leq k\} \setminus \{(0,0),(i,k)\}$ and $E^2_{i,k} = \{(j,l) : 0 \leq j \leq i, 0 \leq l \leq k-1\} \setminus \{(0,0)\}$, with Kronecker notation $\delta_{p\neq 0} = 0$ if $p = 0$ and $\delta_{p\neq 0} = 1$ if $p \geq 1$ and similarly for $\delta_{p=0}$, and where the linearised operator is

$$\begin{aligned} \mathcal{L}_{i,k} F_{i,k} = &-\left(\frac{k}{2} - 3i - 1 + (2i-k+2)\mathsf{F}\right) F_{i,k} - \left(\frac{z}{2} - \partial^{-1}\mathsf{F} + \frac{\pi}{2}\right)\partial_z F_{i,k} \\ &- e^{-2s}\partial_{zz} F_{i,k} + \delta_{k=0}\delta_{i\neq 0}(2i+1)\partial_z \mathsf{F} \partial^{-1} F_{i,k}. \end{aligned} \tag{5.5}$$

Note that above, from the assumptions of Theorem 2, F is well-defined for all times $\mathsf{s} \geq \mathsf{s}_0$. The quantity $\mathcal{L}_{i,k} F_{i,k}$ is then linear with respect to $F_{i,k}$, but the coefficients of $\mathcal{L}_{i,k}$ involve F. We introduce a weight w and the associated weighted space:

$$\mathsf{w}(z) = \begin{cases} 1 & \text{for } -\pi \leq z \leq \pi, \\ \langle z-\pi \rangle^{-2} & \text{for } z \geq \pi, \end{cases} \qquad \|f\|_{L^\infty_{\mathsf{w}}} := \sup_{z\geq -\pi} \frac{|f(z)|}{\mathsf{w}(z)},$$

so that hypothesis (b) in Theorem 2 implies, with $0 < \iota \leq 1/8$ without loss of generality,

$$\|\mathsf{u}\|_{L^\infty_{\mathsf{w}}} + \|\partial_z \mathsf{u}\|_{L^\infty_{\mathsf{w}}} \leq C_0 e^{-\iota \mathsf{s}}. \tag{5.6}$$

5.1. Linear bounds

The semigroup generated by the linear part is denoted by $S_{i,k}$. That is, we write $v(\mathsf{s}) = S_{i,k}(\mathsf{s}_1, \mathsf{s})(v_0)$ for the solution v on $[-\pi, \infty)$ of

$$\partial_{\mathsf{s}} v + \mathcal{L}_{i,k} v = 0, \quad v(\mathsf{s}, -\pi) = 0, \quad v(\mathsf{s}_1, z) = v_0. \tag{5.7}$$

Proposition 5.1. *Assume hypothesis* (b) *in Theorem 2 holds, and for any $\eta > 0$ set*

$$c_{i,k} := \max\left(-i - \frac{k}{2} + 1, \frac{k}{2} - 3i - 1\right) + \eta\langle i\rangle. \tag{5.8}$$

Then there exist $C, \mathsf{K} > 0$ depending on η, T, C_0 and ι but independent of i and k such that for any $i + k \geq 1$ and $\mathsf{s}_2 \geq \mathsf{s}_1 \geq \mathsf{s}_0$,

$$\|S_{i,k}(\mathsf{s}_1, \mathsf{s}_2)(v_0)\|_{L^\infty_{\mathsf{w}}} \leq C e^{c_{i,k}(\mathsf{s}_2 - \mathsf{s}_1)} \left(\frac{(1 + e^{-\frac{\iota}{2}\mathsf{s}_1}) e^{\mathsf{K}e^{-\mathsf{s}_1}}}{(1 + e^{-\frac{\iota}{2}\mathsf{s}_2}) e^{\mathsf{K}e^{-\mathsf{s}_2}}} \right)^{a_{i,k}} \|v_0\|_{L^\infty_{\mathsf{w}}}, \tag{5.9}$$

where $a_{i,k}$ is defined by (5.35). *Moreover, one can take $C = 1$ if $k \geq 1$.*

Remark 5.2. On the right-hand side of (5.9), $e^{c_{i,k}(\mathsf{s}_2 - \mathsf{s}_1)}$ is the sharp leading factor. The factor $1 + e^{-\frac{\iota}{2}\mathsf{s}}$ controls lower order terms close to the blow-up time. The blow-up time T is arbitrary, hence there is a transient regime between $t = 0$ and a time close to T, in which ξ has not yet entered its asymptotic regime described by (b) in Theorem 2 (i.e. $\tilde{\xi}$ may be large). The $e^{Ke^{-\mathsf{s}}}$ factor controls the solution in this transient regime. Together with the exponent $a_{i,k}$, they could have been chosen differently, but such formulation will be easier to use in what follows.

To prove Proposition 5.1, when $k \geq 1$ we decompose $\mathcal{L}_{i,k}$ as

$$\mathcal{L}_{i,k} = \mathcal{L}'_{i,k} + \tilde{\mathcal{L}}'_{i,k},$$

where the leading order and lower order linear operators are ($\mathcal{T}$ being defined in (4.44))

$$\begin{aligned} \mathcal{L}'_{i,k} &= 3i + 1 - \frac{k}{2} - (2i + 2 - k)G_1 + \mathcal{T}(z)\partial_z, \\ \tilde{\mathcal{L}}'_{i,k} &= -(2i + 2 - k)\mathsf{u} + (\partial^{-1}\mathsf{u})\partial_z - e^{-2s}\partial_{zz}. \end{aligned} \tag{5.10}$$

For $k = 0$, there is a nonlocal term in (5.5). We will write $\partial_z \mathsf{F}\partial^{-1} = \partial_z \mathsf{F} \int_{-\pi}^0 + \partial_z \mathsf{F}\partial_z^{-1}$ as the sum of a projection onto $\partial_z \mathsf{F}$ and of a nonlocal term that we treat perturbatively. $\partial_z \mathsf{F}$ is indeed a stable eigenfunction at leading order for $\mathcal{L}_{i,0}$, as the next lemma will show. Let

$$\chi^*(\mathsf{s}, z) = \chi(e^{\frac{\mathsf{s}}{2}}(z - \pi))\chi(e^{\frac{\mathsf{s}}{2}}(-z - \pi)) \tag{5.11}$$

where χ is a smooth cut-off with $\chi(y) = 1$ for $y \leq -1$ and $\chi(y) = 0$ for $y \geq 0$.

Lemma 5.3. *For any $i \geq 1$, $\partial_z G_1$ satisfies the identity*

$$(\mathcal{L}'_{i,0} + (2i + 1)\partial_z G_1 \partial^{-1})\partial_z G_1 = \left(3i + \frac{1}{2}\right)\partial_z G_1. \tag{5.12}$$

Moreover, assume (5.6) *and set* $\phi(\mathsf{s},z)=\chi^*(\mathsf{s},z)\partial_z G_1(z)$, *with* χ^* *given by* (5.11). *Then for all* $i\geq 1$ (*the constant in the* $O()$ *being universal and uniform*),

$$\mathsf{R}_i := \partial_{\mathsf{s}}\phi + (\mathcal{L}_{i,0}+(2i+1)\partial_z G_1\partial^{-1})\phi - \left(3i+\frac{1}{2}\right)\phi = O(ie^{-\iota\mathsf{s}}) \tag{5.13}$$

and R_i *has compact support in* $[-\pi,\pi]$.

Proof. Differentiating equation (3.2) yields the identity

$$\frac{1}{2}\partial_z G_1 - G_1\partial_z G_1 + \left(-\frac{z}{2}+\partial_z^{-1}G_1\right)\partial_{zz}G_1 = 0.$$

In turn, this directly implies (5.12) as $\int_{-\pi}^0 G_1=\pi/2$. Using (5.12) and (5.6), we next make the following computation which proves (5.13) :

$$\begin{aligned}|\mathsf{R}_i| = \Bigg|&\partial_{\mathsf{s}}\phi + (3i+1-(2i+2)\mathsf{F})(\phi-\partial_z G_1) - \left(\frac{z}{2}-\partial_z^{-1}\mathsf{F}+\frac{\pi}{2}\right)\partial_z(\phi-\partial_z G_1)\\ &- e^{-2\mathsf{s}}\partial_{zz}\phi + (2i+1)(\partial_z\mathsf{F})\partial^{-1}(\phi-\partial_z G_1) + \left(3i+\frac{1}{2}\right)(\phi-\partial_z G_1)\\ &-(2i+2)\mathsf{u}\partial_z G_1 + (\partial^{-1}\mathsf{u})\partial_z\partial_z G_1 + (2i+1)(\partial_z\mathsf{u})\partial_z^{-1}\partial_z G_1\Bigg|\\ &\lesssim e^{-\frac{\mathsf{s}}{2}} + ie^{-\frac{\mathsf{s}}{2}} + e^{-\frac{\mathsf{s}}{2}} + e^{-\frac{3}{2}\mathsf{s}} + ie^{-\mathsf{s}} + ie^{-\frac{\mathsf{s}}{2}} + ie^{-\iota\mathsf{s}} + e^{-\iota\mathsf{s}} + ie^{-\iota\mathsf{s}}.\end{aligned}$$

■

In the case $k=0$ we thus decompose a solution v of (5.7) with ϕ defined in Lemma 5.3:

$$v(\mathsf{s},z)=b(\mathsf{s})\phi(\mathsf{s},z)+v'(\mathsf{s},z)\quad\text{with}\quad\begin{cases}\frac{db}{d\mathsf{s}}=-(3i+\frac{1}{2})b-(2i+1)\int_{-\pi}^0 v',\\ b(\mathsf{s}_1)=0.\end{cases} \tag{5.14}$$

We obtain the following evolution equation for v' using (5.5) and Lemma 5.3:

$$(\partial_{\mathsf{s}}+\mathcal{L}'_{i,0}+\tilde{\mathcal{L}}'_{i,0}+\hat{\mathcal{L}}'_{i,0})v' = b\mathsf{R}_i \tag{5.15}$$

where the leading order and lower order elliptic operators $\mathcal{L}'_{i,0}$ and $\tilde{\mathcal{L}}'_{i,0}$ are given by (5.10) with $k=0$, and where the nonlocal operator is

$$\hat{\mathcal{L}}'_{i,0}v' = (2i+1)\partial_z G_1\partial_z^{-1}v' + (2i+1)\partial_z\mathsf{u}\partial^{-1}v' + (2i+1)(1-\chi^*)\partial_z G_1\int_{-\pi}^0 v'.$$

We first study the dynamics generated by $\mathcal{L}'_{i,k}+\tilde{\mathcal{L}}'_{i,k}$. We write $\tilde{v}(\mathsf{s})=\tilde{S}_{i,k}(\mathsf{s}_1,\mathsf{s})(\tilde{v}_0)$ for the solution $\tilde{v}$ on $[-\pi,\infty)$ of

$$\partial_{\mathsf{s}}\tilde{v}+\mathcal{L}'_{i,k}\tilde{v}+\tilde{\mathcal{L}}'_{i,k}\tilde{v}=0,\quad \tilde{v}(\mathsf{s},-\pi)=0,\quad \tilde{v}(\mathsf{s}_1,z)=\tilde{v}_0. \tag{5.16}$$

Let $\tilde{\mathsf{w}}:[-\pi,\infty)\to(0,\infty)$ be a function that satisfies the following properties (note that it is possible to construct explicitly such a weight $\tilde{\mathsf{w}}$ for any $\eta>0$):

(i) $\tilde{\mathsf{w}}$ is C^2, nonincreasing on $[-\pi,0]$, nondecreasing on $[\pi,2\pi]$, $\tilde{\mathsf{w}}(0)=1$, $\tilde{\mathsf{w}}(z)=\tilde{\mathsf{w}}(\pi)\langle z-\pi\rangle^{-2}$ for $z\geq\pi$.

(ii) For all $z\in[-\pi,\pi]$, $|\partial_z^{-1}\tilde{\mathsf{w}}(z)|\leq\eta^2\tilde{\mathsf{w}}(z)$.

We introduce the space with the norm $\|f\|_{L^\infty_{\tilde{\mathsf{w}}}}=\sup_{z\geq-\pi}|f(z)|\tilde{\mathsf{w}}^{-1}(z)$.

Lemma 5.4. *For any $\eta>0$, there exist s^* and $\mathsf{K}>0$ such that for all $i+k\geq1$ and $\mathsf{s}_2\geq\mathsf{s}_1\geq\mathsf{s}_0$,*

$$\|\tilde{S}_{i,k}(\mathsf{s}_1,\mathsf{s}_2)(\tilde{v})\|_X\leq e^{c_{i,k}(\mathsf{s}_2-\mathsf{s}_1)}\left(\frac{(1+e^{-\frac{\iota}{2}\mathsf{s}_1})e^{\mathsf{K}e^{-\mathsf{s}_1}}}{(1+e^{-\frac{\iota}{2}\mathsf{s}_2})e^{\mathsf{K}e^{-\mathsf{s}_2}}}\right)^{a_{i,k}}\|\tilde{v}\|_X \quad \text{if } k\geq1 \text{ and } i+k\geq2, \tag{5.17}$$

$$\|\tilde{S}_{i,k}(\mathsf{s}_1,\mathsf{s}_2)(\tilde{v})\|_X\leq e^{c_{i,k}(\mathsf{s}_2-\mathsf{s}_1)}\left(\frac{1+e^{-\frac{\iota}{2}\mathsf{s}_1}}{1+e^{-\frac{\iota}{2}\mathsf{s}_2}}\right)^{a_{i,k}}\|\tilde{v}\|_X \quad \text{if } \mathsf{s}_2\geq\mathsf{s}_1\geq\mathsf{s}^*, \tag{5.18}$$

where X denotes either $L^\infty_{\tilde{\mathsf{w}}}$ or L^∞_{w}, and $c_{i,k}$ and $a_{i,k}$ are defined in (5.8) and (5.35).

Proof. Let $\mathbf{w}$ denote either w or $\tilde{\mathsf{w}}$, and let $h(\mathsf{s})$ be either 1 or $e^{-a_{i,k}\mathsf{K}e^{-\mathsf{s}}}$. We prove that

$$\mathsf{S}(\mathsf{s},z):=e^{c_{i,k}\mathsf{s}}(1+e^{-\frac{\iota}{2}\mathsf{s}})^{-a_{i,k}}h(\mathsf{s})\mathbf{w}(z) \tag{5.19}$$

is a supersolution for the parabolic operator $\partial_{\mathsf{s}}+\mathcal{L}'_{i,k}+\tilde{\mathcal{L}}'_{i,k}$. We compute

$$\begin{aligned} R&:=\partial_{\mathsf{s}}\mathsf{S}-\left(\frac{k}{2}-3i-1+(2i-k+2)\mathsf{F}\right)\mathsf{S}-\left(\frac{z}{2}-\partial^{-1}\mathsf{F}-\frac{\pi}{2}\right)\partial_z\mathsf{S}-e^{-2s}\partial_{zz}\mathsf{S}\\ &=\mathsf{S}\Bigg[c_{i,k}+\frac{\iota a_{i,k}}{2}e^{-\frac{\iota}{2}\mathsf{s}}+\frac{\partial_{\mathsf{s}}h}{h}-\frac{k}{2}+3i+1-(2i-k+2)\mathsf{F}\\ &\qquad+(\mathcal{T}(z)+\partial^{-1}\mathsf{u})\frac{\partial_z\mathbf{w}}{\mathbf{w}}-e^{-2\mathsf{s}}\frac{\partial_{zz}\mathbf{w}}{\mathbf{w}}\Bigg]. \end{aligned} \tag{5.20}$$

On the interval $[\pi,\infty)$, using $G_1(z)=0$, $\partial_{\mathsf{s}}h\geq0$, $\mathcal{T}(z)=(z-\pi)/2$ and (5.6), we get

$$\begin{aligned} R\geq\mathsf{S}\Bigg[&c_{i,k}+\frac{\iota a_{i,k}}{2}e^{-\frac{\iota}{2}\mathsf{s}}-\frac{k}{2}+3i+1+O(\langle i+k\rangle e^{-\iota\mathsf{s}})\\ &-\left(\frac{z-\pi}{2}+O(\langle z\rangle e^{-\iota\mathsf{s}})\right)\frac{\partial_z\mathbf{w}}{\mathbf{w}}-e^{-2\mathsf{s}}\frac{\partial_{zz}\mathbf{w}}{\mathbf{w}}\Bigg]. \end{aligned}$$

We have $c_{i,k}-k/2+3i+1\geq\eta\langle i\rangle$. On $[\pi,\infty)$, $\partial_z\mathsf{w}\leq0$ and $|\partial_z^j\mathsf{w}|\lesssim\mathsf{w}\langle z\rangle^{-j}$ for $j=1,2$. Hence

$$R\geq\mathsf{S}\left[\eta\langle i\rangle+\frac{\iota a_{i,k}}{2}e^{-\frac{\iota}{2}\mathsf{s}}-C\langle i+k\rangle e^{-\iota\mathsf{s}}-Ce^{-2\mathsf{s}}\langle z\rangle^{-2}\right]>0$$

for s larger than some s^* depending on η, C_0, ι and T, but independent of i and k.

On $[-\pi,\pi]$, we have $0\leq G_1\leq1$ so that $c_{i,k}-k/2+3i+1-(2i-k+2)G_1\geq\eta\langle i\rangle$ from (5.8). As $\mathcal{T}(z)$ is nonpositive on $[-\pi,0]$ and nonnegative on $[0,\pi]$, and $\mathbf{w}$ is nonincreasing on $[-\pi, 0]$ and nondecreasing on $[0, \pi]$, we get $\mathcal{T}(z)\partial_z \mathbf{w} \geq 0$. Hence from (5.20), using (5.6) and $\partial_{\mathsf{s}} h \geq 0$,

$$R \geq \mathsf{S}\left[\eta\langle i\rangle + \frac{\iota a_{i,k}}{2}e^{-\frac{\iota}{2}\mathsf{s}} - (2i-k+2)\mathsf{u} + \partial^{-1}\mathsf{u}\frac{\partial_z \mathbf{w}}{\mathbf{w}} - e^{-2\mathsf{s}}\frac{\partial_{zz}\mathbf{w}}{\mathbf{w}}\right]$$
$$\geq \mathsf{S}\left[\eta\langle i\rangle + \frac{\iota a_{i,k}}{2}e^{-\frac{\iota}{2}\mathsf{s}} - C\langle i+k\rangle e^{-\iota \mathsf{s}} - Ce^{-2\mathsf{s}}\right] > 0$$

for s large enough as $\langle i\rangle + a_{i,k} \gtrsim \langle i+k\rangle$. We conclude that $R > 0$ on $[-\pi, \infty)$ for s large enough. Hence there exists s^* such that S is a supersolution for $\mathsf{s} \geq \mathsf{s}^*$. The bound (5.18) is then a consequence of the maximum principle. Assume now $i+k \geq 2$ and $h(\mathsf{s}) = e^{-a_{i,k}\mathsf{K}e^{-\mathsf{s}}}$. We have proved that S is a supersolution for $\mathsf{s} \geq \mathsf{s}^*$. For $\mathsf{s}^* \geq \mathsf{s} \geq \mathsf{s}_0$ we have $\frac{\partial_{\mathsf{s}} h}{h} = Ka_{i,k}e^{-\mathsf{s}} \geq cK\langle i+k\rangle$ for some $c > 0$ depending on s^*, since $a_{i,k} \gtrsim \langle i+k\rangle$. All other terms in (5.20) are $O(\langle i+k\rangle)$ with some uniform constant, hence for all $\mathsf{s} \in [\mathsf{s}_0, \mathsf{s}^*]$ and $z \geq -\pi$,

$$R \geq \mathsf{S}\left[\frac{\partial_{\mathsf{s}} h}{h} - C\langle i+k\rangle\right] \geq \mathsf{S}[cK\langle i+k\rangle - C\langle i+k\rangle] > 0$$

for K large enough depending on s^*. Hence S is a supersolution for all $\mathsf{s} \geq \mathsf{s}_0$, proving (5.17) by the maximum principle. ∎

We now study the dynamics of (5.14) for $k = 0$, setting $b = 0$. We write $\hat{v}(\mathsf{s}) = \hat{S}_{i,0}(\mathsf{s}_1, \mathsf{s})(\hat{v}_0)$ for the solution $\hat{v}$ on $[-\pi, \infty)$ of

$$\partial_{\mathsf{s}}\hat{v} + \mathcal{L}'_{i,0}\hat{v} + \tilde{\mathcal{L}}'_{i,0}\hat{v} + \hat{\mathcal{L}}'_{i,0}\hat{v} = 0, \quad \hat{v}(\mathsf{s}, -\pi) = 0, \quad \hat{v}(\mathsf{s}_1, z) = \hat{v}_0. \tag{5.21}$$

Lemma 5.5. *For any $\eta > 0$, if $\mathsf{s}_2 \geq \mathsf{s}_1$ are large enough, then for all $i \geq 1$,*

$$\|\hat{S}_{i,0}(\mathsf{s}_1, \mathsf{s}_2)(\hat{v})\|_{L^\infty_{\mathsf{w}}} \leq C(\eta)e^{c_{i,0}(\mathsf{s}_2-\mathsf{s}_1)}\left(\frac{1+e^{-\frac{\iota}{2}\mathsf{s}_1}}{1+e^{-\frac{\iota}{2}\mathsf{s}_2}}\right)^{a_{i,0}}\|\hat{v}\|_{L^\infty_{\mathsf{w}}}, \tag{5.22}$$

where $c_{i,0}$ and $a_{i,0}$ are defined in (5.8) and (5.35).

Proof. We reason with a parameter $\eta' > 0$, and let $\tilde{\mathsf{w}}' = \tilde{\mathsf{w}}[\eta']$ and $c'_{i,0} = c_{i,0}[\eta']$. Using the assumption (ii) on $\tilde{\mathsf{w}}'$, the bound (5.6) and that G_1 vanishes outside $[-\pi, \pi]$ we get for η' small enough, and then s large enough,

$$\|\hat{\mathcal{L}}'_{i,0}\hat{v}\|_{L^\infty_{\tilde{\mathsf{w}}'}} \leq (Ci\eta'^2 + C(\eta')ie^{-\iota \mathsf{s}} + C(\eta')ie^{-\frac{\mathsf{s}}{2}})\|\hat{v}\|_{L^\infty_{\tilde{\mathsf{w}}'}} \leq \eta' i\|\hat{v}\|_{L^\infty_{\tilde{\mathsf{w}}'}}.$$

Duhamel gives $\hat{S}_{i,0}(\mathsf{s}_1, \mathsf{s}_2)(\hat{v}) = \tilde{S}_{i,0}(\mathsf{s}_1, \mathsf{s}_2)(\hat{v}) - \int_{\mathsf{s}_1}^{\mathsf{s}_2}\tilde{S}_{i,0}(\mathsf{s}, \mathsf{s}_2)(\hat{\mathcal{L}}'_{i,0}(S_{i,0}(\mathsf{s}_1, \mathsf{s})(\hat{v})))d\mathsf{s}$. Set $\Phi(\mathsf{s}) = \left(\frac{1+e^{-\frac{\iota}{2}\mathsf{s}}}{1+e^{-\frac{\iota}{2}\mathsf{s}_1}}\right)^{a_{i,0}}\|\hat{S}_{i,0}(\mathsf{s}_1, \mathsf{s})(\hat{v})\|_{L^\infty_{\tilde{\mathsf{w}}'}}$. The above bound and (5.18) imply

$$\Phi(\mathsf{s}_2) \leq e^{c'_{i,0}(\mathsf{s}_2-\mathsf{s}_1)}\|\hat{v}\|_{L^\infty_{\tilde{\mathsf{w}}'}} + i\eta'\int_{\mathsf{s}_1}^{\mathsf{s}_2} e^{c'_{i,0}(\mathsf{s}_2-\mathsf{s})}\Phi(\mathsf{s})\,d\mathsf{s}.$$

Gronwall then gives $\Phi(\mathsf{s}) \leq e^{(c'_{i,0}+i\eta')(\mathsf{s}_2-\mathsf{s}_1)}\|\hat{v}\|_{L^\infty_{\tilde{\mathsf{w}}'}}$. This proves the lemma, upon noticing that $c_{i,k}[\eta'] + i\eta' \leq c_{i,k}[\eta]$ for η' small enough, and that the weights $\tilde{\mathsf{w}}'$ and w are equivalent. ∎

Proof of Proposition 5.1. **Step 1.** We claim that for any $\eta > 0$, there exist $\mathsf{s}^*, C' > 0$ such that

$$\left\| S_{i,k}(\mathsf{s}_1,\mathsf{s}_2)(v_0)\right\|_{L^\infty_{\mathsf{w}}} \leq C' e^{c_{i,k}(\mathsf{s}_2-\mathsf{s}_1)}\left(\frac{1+e^{-\frac{\iota}{2}\mathsf{s}_1}}{1+e^{-\frac{\iota}{2}\mathsf{s}_2}}\right)^{a_{i,k}} \|v_0\|_{L^\infty_{\mathsf{w}}} \quad \text{for } \mathsf{s}_2 \geq \mathsf{s}_1 \geq \mathsf{s}^*, \tag{5.23}$$

for all $i+k \geq 1$, and that one can take $C' = 1$ if $k \geq 1$. For $k \geq 1$, this is Lemma 5.18. So we only need to prove the above inequality for $k = 0$. Let $\eta' > 0$, and $c'_{i,0} = c_{i,0}[\eta']$. Recall (5.14) and (5.15). We write by Duhamel $b(\mathsf{s}) = (2i+1)e^{-(3i+\frac{1}{2})\mathsf{s}} \int_{\mathsf{s}_1}^{\mathsf{s}} e^{(3i+\frac{1}{2})\mathsf{s}'} \int_{-\pi}^{0} v'(\mathsf{s}')\, d\mathsf{s}'$ and $v'(\mathsf{s}) = \hat{S}_{i,0}(\mathsf{s}_1,\mathsf{s})(v_0) + \int_{\mathsf{s}_1}^{\mathsf{s}} b(\mathsf{s}')\hat{S}_{i,0}(\mathsf{s}',\mathsf{s})(\mathsf{R}_i(\mathsf{s}'))\, d\mathsf{s}'$. Recall that (5.13) gives $\|\mathsf{R}_i\|_{L^\infty_{\mathsf{w}}} \leq Cie^{-\iota\mathsf{s}}$. We take s_1 large enough, and apply the linear estimate (5.22) with parameter η' to get

$$|b(\mathsf{s})| \leq Ci\int_{\mathsf{s}_1}^{\mathsf{s}} e^{-(3i+\frac{1}{2})(\mathsf{s}-\mathsf{s}')}\|v'(\mathsf{s}')\|_{L^\infty_{\mathsf{w}}}\, d\mathsf{s}',$$

$$\begin{aligned}\|v'(\mathsf{s})\|_{L^\infty_{\mathsf{w}}} &\leq Ce^{c'_{i,k}(\mathsf{s}-\mathsf{s}_1)}\left(\frac{1+e^{-\frac{\iota}{2}\mathsf{s}_1}}{1+e^{-\frac{\iota}{2}\mathsf{s}}}\right)^{a_{i,k}} \|v_0\|_{L^\infty_{\mathsf{w}}}\\ &\quad + Ci\int_{\mathsf{s}_1}^{\mathsf{s}} e^{c'_{i,k}(\mathsf{s}-\mathsf{s}')}\left(\frac{1+e^{-\frac{\iota}{2}\mathsf{s}'}}{1+e^{-\frac{\iota}{2}\mathsf{s}}}\right)^{a_{i,k}} e^{-\iota\mathsf{s}'}|b(\mathsf{s}')|\, d\mathsf{s}'.\end{aligned}$$

Consider the function $\Phi(\mathsf{s}) = (\frac{1+e^{-\frac{\iota}{2}\mathsf{s}}}{1+e^{-\frac{\iota}{2}\mathsf{s}_1}})^{a_{i,k}}(\|v'(\mathsf{s})\|_{L^\infty_{\mathsf{w}}} + \eta'|b(\mathsf{s})|)$. Take s_1 large enough so that $e^{-\iota\mathsf{s}'} \leq \eta'^2$ for $\mathsf{s}' \geq \mathsf{s}_1$. Note that for all $i+k \geq 1$ one has $c'_{i,k} \geq -3i - \frac{1}{2}$. Hence Φ satisfies the integral inequality $\Phi(\mathsf{s}) \leq Ce^{c'_{i,k}(\mathsf{s}-\mathsf{s}_1)}\|v_0\|_{L^\infty_{\mathsf{w}}} + Ci\eta'\int_{\mathsf{s}_1}^{\mathsf{s}} e^{c'_{i,k}(\mathsf{s}-\mathsf{s}')}\Phi(\mathsf{s}')\, d\mathsf{s}'$. Hence $\Phi(\mathsf{s}) \leq Ce^{(c'_{i,k}+Ci\eta')(\mathsf{s}-\mathsf{s}_1)}\|v_0\|_{L^\infty_{\mathsf{w}}}$ by the Gronwall lemma. This shows (5.23), on taking η' small so that $c_{i,k}[\eta'] + Ci\eta' \leq c_{i,k}[\eta]$.

Step 2. Fix $\eta > 0$, s^* as in Step 1, and $\mathsf{s}^* \geq \mathsf{s}_2 \geq \mathsf{s}_1 \geq -\log T$. The control of (5.7) on $[-\log T, s^*]$ is direct since this is a linear equation with bounded coefficients, over a *finite* interval. Indeed, the functions $|\mathsf{F}|, |\partial_z\mathsf{F}| \leq C\mathsf{w}$ are uniformly bounded from (5.6). Then, (5.7) is a linear parabolic equation, with variable coefficients in the elliptic part that are uniformly bounded by $C\langle i+k\rangle$, and with a nonlocal operator $v \mapsto \delta_{i\neq 0}(2i+1)\partial_z\mathsf{F}\partial^{-1}v$ that is bounded from L^∞_{w} onto L^∞_{w} with operator norm $\leq C\langle i\rangle$. As a result, we have a classical linear bound using a standard Gronwall argument: there exists $C > 0$ depending on T, s^*, ι and C_0 such that $\left\|S_{i,k}(\mathsf{s}_1,\mathsf{s}_2)(v_0)\right\|_{L^\infty_{\mathsf{w}}} \leq e^{C\langle i+k\rangle(\mathsf{s}_2-\mathsf{s}_1)}\|v_0\|_{L^\infty_{\mathsf{w}}}$. Since for K', C'' large enough, $e^{C\langle i+k\rangle(\mathsf{s}_2-\mathsf{s}_1)} \leq C''e^{a_{i,k}\mathsf{K}'(e^{-\mathsf{s}_1}-e^{-\mathsf{s}_2})}$ uniformly for $\mathsf{s}^* \geq \mathsf{s}_2 \geq \mathsf{s}_1 \geq -\log T$, we get

$$\|S_{i,k}(\mathsf{s}_1,\mathsf{s}_2)(v_0)\|_{L^\infty_{\mathsf{w}}} \leq C''e^{a_{i,k}\mathsf{K}'(e^{-\mathsf{s}_1}-e^{-\mathsf{s}_2})}\|v_0\|_{L^\infty_{\mathsf{w}}}.$$

The above estimate on $[-\log T, \mathsf{s}^*]$ and (5.23) on $[\mathsf{s}^*, \infty)$ directly imply (5.9) for all $\mathsf{s}_2 \geq \mathsf{s}_1 \geq -\log T$ (upon using them after writing $S_{i,k}(\mathsf{s}_1,\mathsf{s}_2) = S_{i,k}(\mathsf{s}^*,\mathsf{s}_2)\circ S_{i,k}(\mathsf{s}_1,\mathsf{s}^*)$ if $\mathsf{s}_1 \leq \mathsf{s}^* \leq \mathsf{s}_2$, and up to taking $\mathsf{K}, C > 0$ large enough depending on $\mathsf{K}', C', C'', \mathsf{s}^*, T$). ■

5.2. Control of the third order tangential derivative on the axis

We first control $\xi_{1,0}$ (equivalently, $F_{1,0}$). This is because the growth of this function as $t \to T$ will be responsible for the $(T-t)^{7/4}$ bound of the radius of analyticity, and because the bound below is critical for the linearised analysis due to the presence of a nontrivial kernel, thus requiring a more careful treatment.

Proposition 5.6. *Assume hypothesis* (b) *in Theorem* 2 *and* $\|\xi_{1,0}(0)\|_{L^\infty(\langle y\rangle^{-2})} < \infty$. *Then the solution* $\xi_{1,0}$ *of* (5.4) *is defined for all* $t \in [0, T)$. *Moreover, there exists* $C_2 > 0$ *such that for all* $t \in [0, T)$ *and* $\mathsf{s} \geq \mathsf{s}_0$,

$$\|\xi_{1,0}(t)\|_{L^\infty([0,1/4])} \leq C_2 \quad \text{and} \quad \|F_{1,0}(\mathsf{s})\|_{L^\infty_{\mathsf{w}}} \leq C_2. \tag{5.24}$$

The proof is decomposed into several steps, and Proposition 5.6 is proved at the end of this subsection. The existence up to time T is straightforward, since $F_{1,0}$ solves a linear equation

$$(\partial_{\mathsf{s}} + \mathcal{L}_{1,0})F_{1,0} = 0 \iff (\partial_{\mathsf{s}} + \mathcal{N} + \tilde{\mathcal{N}} - e^{-2s}\partial_{zz})F_{1,0} = 0, \tag{5.25}$$

where the leading and lower order linear operators are

$$\mathcal{N} F_{1,0} = 4(1 - G_1(z))F_{1,0} + \mathcal{T}(z)\partial_z F_{1,0} + 3\partial_z G_1(z)\partial^{-1} F_{1,0},$$
$$\tilde{\mathcal{N}} F_{1,0} = -4\mathsf{u}F_{1,0} + \partial^{-1}\mathsf{u}\partial_z F_{1,0} + 3\partial_z \mathsf{u}\partial^{-1} F_{1,0}.$$

Applying Proposition 5.1, for any $\eta > 0$, as $c_{1,0} = \langle 1\rangle\eta$ and $a_{1,0} = 0$, we obtain $\|F_{1,0}(\mathsf{s})\|_{L^\infty} \leq Ce^{\langle 1\rangle\eta\mathsf{s}}$. By taking a smaller η in this inequality and s large enough we get

$$\|F_{1,0}(\mathsf{s})\|_{L^\infty(\mathsf{w})} \leq e^{\eta\mathsf{s}} \quad \text{for any } \eta > 0, \text{ for } \mathsf{s} \text{ large enough depending on } \eta. \tag{5.26}$$

This is almost the second bound in (5.24) we wish to prove. The problem with improving to $\eta = 0$ above is the presence of a nontrivial kernel.

Lemma 5.7 ([5, Proposition 6 (vi)]). *There exists a* C^1 *solution* Q *to* $\mathcal{N} Q = 0$ *on* $[-\pi, \infty)$ *that has the following properties: the support of* Q *is* $[-\pi, \pi]$, *and* Q *restricted to* $[-\pi, \pi]$ *is smooth;* Q *is positive on* $(-\pi, \pi)$ *with* $Q(0) = 1$*; and there exist positive constants* c *and* c' *such that* $Q(z) \sim c(z + \pi)^8$ *as* $z \downarrow -\pi$ *and* $Q(z) \sim c'(\pi - z)$ *as* $z \uparrow \pi$.

To improve (5.26), we prove boundedness in a parabolic neighbourhood of a particular characteristic of the transport operator, and extend this local bound to a global one.

Lemma 5.8. *There exists a solution* $z^*(\mathsf{s})$ *of* $\partial_{\mathsf{s}} z^* = -\frac{z^*}{2} + \int_{-\pi}^{z^*} \mathsf{F}(\mathsf{s}, z)\, dz - \frac{\pi}{2}$ *such that*

$$|z^*| \leq Ce^{-\iota\mathsf{s}}. \tag{5.27}$$

Proof. For an initial time $\mathsf{s}_1 \geq \mathsf{s}_0$, using $\int_{-\pi}^0 G_1 = \pi/2$ and (5.6) the ODE becomes

$$\partial_{\mathsf{s}} z^* = -\frac{z^*}{2} + \int_0^{z^*} G_1 + \int_{-\pi}^{z^*} \mathsf{u} = \frac{1}{2}z^* + O(z^{*3}) + O(e^{-\iota\mathsf{s}}), \quad z^*(\mathsf{s}_1) = z_0^*. \tag{5.28}$$

Consider for $M > 0$ the sets I^-_{M,s_1} and I^+_{M,s_1} defined by

$$I^{\pm}_{M,\mathsf{s}_1} = \{z_0^* : |z_0^*| \le Me^{-\iota \mathsf{s}_1},\ \exists \mathsf{s}_2 \ge \mathsf{s}_1,\ |z^*(\mathsf{s})| < Me^{-\iota \mathsf{s}} \text{ for } \mathsf{s}_1 \le \mathsf{s} < \mathsf{s}_2 \\ \text{and } z^*(\mathsf{s}_2) = \pm Me^{-\iota \mathsf{s}_2}\}.$$

Then for M large enough and then for s_1 large enough the following holds true. For $z_0^* \in I^{\pm}_{M,\mathsf{s}_1}$, at time s_2, by (5.28),

$$\partial_s(|e^{\iota \mathsf{s}} z^*|)(\mathsf{s}_2) = M/2 + O(M^3 e^{-2\iota \mathsf{s}_2}) + O(1) > 0.$$

This inequality, by continuity of the flow of the ODE (5.28), implies that both I^-_{M,s_1} and I^+_{M,s_1} are open in $[-Me^{-\iota \mathsf{s}_1}, Me^{-\iota \mathsf{s}_1}]$. They are moreover disjoint by definition, and nonempty as they contain $-Me^{-\iota \mathsf{s}_1}$ and $Me^{-\iota \mathsf{s}_1}$ respectively. Hence, by connectedness, there exists $z_0^* \in [-Me^{-\iota \mathsf{s}_1}, Me^{-\iota \mathsf{s}_1}]$ with $z_0^* \notin I^-_{M,\mathsf{s}_1} \cup I^+_{M,\mathsf{s}_1}$. The solution to (5.28) with data z_0^* at time s_1 then satisfies the conclusions of the lemma by the definitions of I^-_{M,s_1} and I^+_{M,s_1}. ■

Lemma 5.9. *Let z^* satisfy the conclusion of Lemma 5.8. Then there exists $d \in \mathbb{R}$ such that for any $\mathsf{M} > 0$ and $0 < \iota' < \iota$, for all large enough s and all $z \in [z^* - \mathsf{M}e^{-s}, z^* + \mathsf{M}e^{-s}]$,*

$$|F_{1,0}(\mathsf{s}, z) - d| \le e^{-\iota' \mathsf{s}}. \tag{5.29}$$

Proof. We switch to the following parabolic variables:

$$\mathsf{Y} = \frac{z - z^*}{T - t}, \quad F_{1,0}(\mathsf{s}, z) = f_{1,0}(\mathsf{s}, \mathsf{Y}), \quad \mathsf{Y}^* = \frac{\pi + z^*}{T - t}. \tag{5.30}$$

Then $f_{1,0}$ solves the following equation on $[-\mathsf{Y}^*, \infty)$ with Dirichlet boundary condition:

$$\partial_{\mathsf{s}} f_{1,0} + \left(\frac{\mathsf{Y}}{2} + \partial_{\mathsf{Y}}^{-1}\mathsf{F}(\mathsf{s}, z)\right)\partial_{\mathsf{Y}} f_{1,0} - \partial_{\mathsf{YY}} f_{1,0} = (4\mathsf{F} - 4)F_{1,0} - 3\left(\int_{-\pi}^{z} F_{1,0}\right)\partial_z \mathsf{F},$$

the right-hand side being a function of the space variable z. Set $d(\mathsf{s}) = \int_{\mathbb{R}} \tilde{\chi} f_{1,0}\rho(\mathsf{Y})\,d\mathsf{Y}$, where $\rho(\mathsf{Y}) = e^{-3\mathsf{Y}^2/4}$ and $\tilde{\chi}(\mathsf{s}, \mathsf{Y}) = \chi(e^{-\mathsf{s}/2}\mathsf{Y})$ for χ a smooth cut-off, $\chi(y) = 1$ for $|y| \le 1$ and $\chi(y) = 0$ for $|y| \ge 0$. Notice that the support of $\tilde{\chi}$ is strictly inside $[-\mathsf{Y}^*, \infty)$ for s large, justifying that the integral for d is on $\mathbb{R}$. Note that $\partial_{\mathsf{Y}}\rho(\mathsf{Y}) = -\frac{3}{2}\mathsf{Y}\rho(\mathsf{Y})$. Then integrating by parts, using the exponential decay of ρ and (5.26) to upper bound by $Ce^{-e^{\mathsf{s}}}$ all boundary terms due to $\tilde{\chi}$,

$$\partial_{\mathsf{s}} d = \int_{\mathbb{R}} \tilde{\chi}\Big(\Big(4\mathsf{F}(\mathsf{s}, z) - 4 + \rho^{-1}\partial_{\mathsf{Y}}\big(\rho\partial_{\mathsf{Y}}^{-1}(\mathsf{F}(\mathsf{s}, z) - 1)\big)\Big)F_{1,0} - 3\Big(\int_{-\pi}^{z} F_{1,0}\Big)\partial_z \mathsf{F}\Big)\rho\,d\mathsf{Y} \\ + O(e^{-e^{\mathsf{s}}}). \tag{5.31}$$

Above, we note that, using (5.30), (5.27) and (5.6), one gets

$$|\mathsf{F}(\mathsf{s}, z) - 1| \le |G_1(z) - G_1(z^*)| + |G_1(z^*) - 1| + |\mathsf{u}| \le C\mathsf{Y}e^{-\mathsf{s}} + Ce^{-\iota \mathsf{s}}, \tag{5.32}$$

$$|\partial_z \mathsf{F}(\mathsf{s}, z)| \le |\partial_z G_1(z) - \partial_z G_1(z^*)| + |\partial_z G_1(z^*)| + |\partial_z \mathsf{u}| \le C\mathsf{Y}e^{-\mathsf{s}} + Ce^{-\iota \mathsf{s}}. \tag{5.33}$$

Let $0 < \iota'' < \iota$, and let s be large so that $|F_{1,0}| \leq e^{-\iota''\mathsf{s}}$ from (5.26). Inserting (5.32) and (5.33) in (5.31) gives $|\partial_{\mathsf{s}} d| \leq Ce^{-(\iota-\iota'')\mathsf{s}}$. Hence there exists $d_\infty \in \mathbb{R}$ with $|d - d_\infty| \leq Ce^{-(\iota-\iota'')\mathsf{s}}$. Set now $\bar{f}_{1,0} = f_{1,0} - d$. It solves the equation

$$\partial_{\mathsf{s}} \bar{f}_{1,0} + \left(\frac{\mathsf{Y}}{2} + \partial_{\mathsf{Y}}^{-1}\mathsf{F}(\mathsf{s}, z)\right)\partial_{\mathsf{Y}} \bar{f}_{1,0} - \partial_{\mathsf{YY}} \bar{f}_{1,0} = (4\mathsf{F} - 4)F_{1,0} - 3\left(\int_{-\pi}^{z} F_{1,0}\right)\partial_z \mathsf{F} - \partial_{\mathsf{s}} d.$$

We compute the following energy estimate by integrating by parts, using the exponential decay of ρ, that d is bounded and (5.26) to upper bound all boundary terms due to $\tilde{\chi}$ by $Ce^{-e^{\mathsf{s}}}$:

$$\begin{aligned}\frac{d}{d\mathsf{s}}\frac{1}{2}\left(\int_{\mathbb{R}} |\tilde{\chi}\bar{f}_{1,0}|^2\rho\right) = &-\int_{\mathbb{R}} |\partial_{\mathsf{Y}}(\tilde{\chi}\bar{f}_{1,0})|^2\rho + \int_{\mathbb{R}} \tilde{\chi}^2\partial_{\mathsf{Y}}(\rho\partial_{\mathsf{Y}}^{-1}(\mathsf{F}(\mathsf{s}, z) - 1))|\bar{f}_{1,0}|^2 \\ &+ \int_{\mathbb{R}} \tilde{\chi}^2 \bar{f}_{1,0}\left((4\mathsf{F} - 4)F_{1,0} - 3\left(\int_{-\pi}^{z} F_{1,0}\right)\partial_z \mathsf{F} - \partial_s d\right)\rho + O(e^{-e^{\mathsf{s}}}).\end{aligned}$$

Above, since $\int_{\mathbb{R}} \tilde{\chi}\bar{f}_{1,0}\rho = 0$, we obtain the coercivity $\int_{\mathbb{R}} |\partial_{\mathsf{Y}}(\tilde{\chi}\bar{f}_{1,0})|^2\rho \geq \frac{3}{2}\int_{\mathbb{R}} |\tilde{\chi}\bar{f}_{1,0}|^2\rho$ from Proposition 3.1 (note that $1 = h_0$). Bounding the remaining terms by using the fact that d is bounded, that $|\bar{f}_{1,0}| = |f_{1,0} - d| \leq Ce^{\iota''\mathsf{s}}$, (5.26), (5.32) and (5.33) we get

$$\frac{d}{d\mathsf{s}}\frac{1}{2}\left(\int_{\mathbb{R}} |\tilde{\chi}\bar{f}_{1,0}|^2\rho\right) \leq -\frac{3}{2}\int_{\mathbb{R}} |\tilde{\chi}\bar{f}_{1,0}|^2\rho + Ce^{-(\iota-2\iota'')\mathsf{s}}.$$

Reintegrating this inequality gives $\int_{\mathbb{R}} |\tilde{\chi}\bar{f}_{1,0}|^2\rho \leq Ce^{-(\iota-2\iota'')\mathsf{s}}$. Hence $\|\bar{f}_{1,0}\|_{L^2([-2\mathsf{M},2\mathsf{M}])} \leq Ce^{-(\iota-2\iota'')\mathsf{s}}$. A standard application of parabolic regularisation gives $\|\bar{f}_{1,0}\|_{L^\infty([-\mathsf{M},\mathsf{M}])} \leq Ce^{-(\iota-2\iota'')\mathsf{s}} \leq e^{-\iota'\mathsf{s}}$ (upon choosing ι'' small depending on ι', and then s large). This and the bound on d show the lemma upon renaming d_∞ as d. ■

Lemma 5.10. *Let (z^*, d) be given by Lemmas 5.8 and 5.9, and let Q be as defined in Lemma 5.7. Then*

$$\lim_{\mathsf{s}\to\infty} \|F_{1,0}(\mathsf{s}, z) - dQ(z)\|_{L^\infty_{\mathsf{w}}} = 0.$$

Proof. We regularise the element of the kernel Q near the points $\pm\pi$ and decompose

$$F_{1,0}(\mathsf{s}, z) = d\chi^*(\mathsf{s}, z)Q(z) + \bar{F}_{1,0}(\mathsf{s}, z),$$

where χ^* was defined in (5.11). Then $\bar{F}_{1,0}$ solves

$$(\partial_{\mathsf{s}} + \mathcal{N}'' + \hat{\mathcal{N}}'' - \gamma_2\partial_{zz})\bar{F}_{1,0} = \mathsf{E}(\mathsf{s}, z),$$

where the elliptic linear operator, the nonlocal linear operator and the error are

$$\mathcal{N}'' = 4(1 - G_1(z)) + \left(\partial_z^{-1}G_1 + \int_{-\pi}^{z} \mathsf{u} - \frac{z}{2}\right)\partial_z, \quad \hat{\mathcal{N}}'' = 3\partial_z G_1 \partial_z^{-1},$$

$$\mathsf{E} = -d\left((\partial_{\mathsf{s}} + \tilde{\mathcal{N}} - e^{-2s}\partial_{zz})(\chi^* Q) + \mathcal{N}((\chi^* - 1)Q)\right) + 4\mathsf{u}\bar{F}_{1,0} - 3\partial_z u\int_{-\pi}^{z} F_{1,0}.$$

Since d is bounded, from the asymptotic behaviour of Q near $\pm\pi$ in Lemma 5.7 and (5.6) we get

$$\|\mathsf{E}\|_{L^\infty_{\bar{\mathsf{w}}}} \le e^{-\frac{3\iota}{4}\mathsf{s}}. \tag{5.34}$$

We introduce the domain $\bar{\Omega} = [-\pi, z^* - \mathsf{M}e^{-s}] \cup [z^* + \mathsf{M}e^{-s}, \infty]$ and let s_1 be large.

Step 1. Let $\bar{\mathsf{w}} : (-\infty,\infty) \to (0,\infty)$ be a function that satisfies all the following properties (note that it is possible to construct explicitly such a weight $\bar{\mathsf{w}}$ for any $\iota > 0$, along the same lines as in Section 4.5.1):

(i) $\bar{\mathsf{w}}$ is an even C^2 solution of the differential inequality $\mathcal{N}\bar{\mathsf{w}} \ge \frac{\iota}{4}\bar{w}$ on $\mathbb{R}\setminus\{0\}$.

(ii) $\bar{\mathsf{w}}(z) = |z|^{\iota/2}$ for $|z|$ small enough and $\bar{\mathsf{w}}(z) = \bar{\mathsf{w}}(\pi)\langle Z-\pi\rangle^{-2}$ for $z \ge \pi$.

(iii) For all $z \in [-\pi,\pi]$, $|\partial_z^{-1}\bar{\mathsf{w}}(z)| \le \iota^2\bar{\mathsf{w}}(z)$.

Then we claim that there exists $\mathsf{M} > 0$ such that for s large enough, $\bar{\mathsf{S}}(\mathsf{s}, z) = e^{-\frac{\iota}{8}\bar{s}}\bar{w}(z-z^*)$ satisfies $(\partial_s + \mathcal{N}'' - e^{-2s}\partial_{zz})\bar{\mathsf{S}} \ge \frac{\iota}{10}\bar{\mathsf{S}}$ on $\bar{\Omega}$. This is a direct computation. We indeed compute using the evolution equation for z^* that, for s large enough,

$$\begin{aligned}
&(\partial_s + \mathcal{N}'' - e^{-2s}\partial_{zz})\bar{\mathsf{S}}\\
&= -\frac{\iota}{8}\bar{\mathsf{S}} + e^{-\frac{\iota}{4}\mathsf{s}}\left(4(1-G_1(z-z^*)\bar{\mathsf{w}}(z-z^*) + \left(\int_{z^*}^{z} G_1(\tilde{z}-z^*)\,d\tilde{z} - \frac{z-z^*}{2}\right)\partial_z\bar{\mathsf{w}}(z-z^*)\right)\\
&\quad - e^{-2s}\partial_{zz}\bar{\mathsf{S}} + \int_{z^*}^{z} u\partial_z\bar{\mathsf{S}} + 4(G_1(z-z^*) - G_1(z))\bar{\mathsf{S}} + \int_{z^*}^{z}(G_1(\tilde{z}) - G_1(\tilde{z}-z^*))\,d\tilde{z}\,\partial_z\bar{\mathsf{S}}\\
&\ge -\frac{\iota}{8}\bar{\mathsf{S}} + \frac{\iota}{4}\bar{\mathsf{S}} + O(\mathsf{M}^{-2}\bar{\mathsf{S}}) + O(e^{-\iota\mathsf{s}}|\bar{\mathsf{S}}|) \ \ge \frac{\iota}{10}\bar{\mathsf{S}}
\end{aligned}$$

where we have used property (i) for the second term, $e^{-2s}|\partial_{zz}\bar{\mathsf{S}}| \lesssim e^{-2s}|z-z^*|^{-2}\bar{\mathsf{S}} \lesssim \mathsf{M}^{-2}\bar{\mathsf{S}}$ for $|z-z^*| \ge \mathsf{M}e^{-s}$ for the third one, and (5.6) and (5.27) for the remaining terms.

Step 2. Let $\bar{F}^1_{1,0}$ solve $(\partial_s + \mathcal{N}'' - e^{-2s}\partial_{zz})\bar{F}^1_{1,0} = \mathsf{E}$ on $\bar{\Omega}$, with boundary conditions $\bar{F}^1_{1,0}(-\pi) = 0$, $\bar{F}^1_{1,0}(z^* \pm \mathsf{M}e^{-s}) = \bar{F}_{1,0}(z^* \pm \mathsf{M}e^{-s})$ and $\bar{F}^1_{1,0}(\mathsf{s}_1) = \bar{F}_{1,0}(\mathsf{s}_1)$. From the behaviour of $\bar{\mathsf{w}}$ near 0, we have $\bar{\mathsf{S}} \ge C(\mathsf{M})e^{-\frac{\iota}{2}\mathsf{s}}\mathsf{w}$ uniformly on $\bar{\Omega}$ for a constant $C(\mathsf{M}) > 0$. Hence by Step 1 and (5.34) we get $(\partial_s + \mathcal{N}'' - e^{-2s}\partial_{zz})\bar{\mathsf{S}} \ge |\mathsf{E}|$. Moreover, $\bar{\mathsf{S}} \ge |\bar{F}_{1,0}|$ at the boundary 0 and $z^* \pm Me^{-s}$ from (5.29). Hence by parabolic comparison, for some $C(\mathsf{s}_1) > 0$ and $\mathsf{s} \ge \mathsf{s}_1$ large,

$$|\bar{F}^1_{1,0}| \le C(\mathsf{s}_1)|\bar{\mathsf{S}}| \le C(\mathsf{s}_1)e^{-\frac{\iota}{8}\mathsf{s}}\bar{\mathsf{w}}.$$

Step 3. Let $\bar{F}_{1,0} = \bar{F}^1_{1,0} + \bar{F}^2_{1,0}$. Then $\bar{F}^2_{1,0}$ solves $(\partial_s + \mathcal{N}'' + \hat{\mathcal{N}}'' - e^{-2s}\partial_{zz})\bar{F}^2_{1,0} = -\hat{\mathcal{N}}''\bar{F}^1_{1,0}$ on $\bar{\Omega}$ with Dirichlet boundary conditions and zero initial datum at time s_1. From Step 1, the solution to $(\partial_s + \mathcal{N}'' - e^{-2s}\partial_{zz})v = 0$ on $\bar{\Omega}$ with Dirichlet boundary conditions satisfies $\|v(\mathsf{s})\|_{L^\infty(\bar{\mathsf{w}})} \le e^{-\frac{\iota}{8}(\mathsf{s}-\mathsf{s}_2)}\|v(\mathsf{s}_2)\|_{L^\infty(\bar{\mathsf{w}})}$ for $\mathsf{s} \ge \mathsf{s}_2$ large enough. Hence, reasoning as in Step 1 of the proof of Proposition 5.1, one finds that the solution to $(\partial_s + \mathcal{N}'' + \hat{\mathcal{N}}'' - e^{-2s}\partial_{zz})u = 0$ on $\bar{\Omega}$ with Dirichlet boundary conditions satisfies $\|u(\mathsf{s})\|_{L^\infty(\bar{\mathsf{w}})} \le e^{-\frac{\iota}{16}(\mathsf{s}-\mathsf{s}_2)}\|u(\mathsf{s}_0)\|_{L^\infty(\bar{\mathsf{w}})}$ for ι small enough. This linear bound and the bound for $\bar{F}^1_{1,0}$ obtained in Step 2 show that for $\mathsf{s} \geq \mathsf{s}_1$,

$$|\bar{F}^2_{1,0}| \leq C|\bar{\mathsf{S}}| \leq Ce^{-\frac{\iota}{16}\mathsf{s}}\bar{\mathsf{w}}$$

for s_1 large enough. This bound and the one from Step 2 prove the lemma since $\bar{F}_{1,0} = \bar{F}^1_{1,0} + \bar{F}^2_{1,0}$, and since $\bar{\mathsf{w}} \leq C\mathsf{w}$ for some constant $C > 0$. ■

We can now end the proof of Proposition 5.6.

Proof of Proposition 5.6. The second bound is a direct consequence of Lemma 5.10. As for the first one, from (1.15) we find that $\xi_{1,0}$ solves a linear parabolic equation on $[0, 1/2]$, with variable coefficients involving ξ and $\partial_y\xi$, but which are uniformly bounded on $[0, T] \times [0, 1/2]$ from (1.15). Hence this first bound is obtained via a standard parabolic bootstrap. ■

5.3. *Control of higher order derivatives*

5.3.1. The analytic norm and formal explanations. In this subsection we will use the bound (5.24) obtained for the third order tangential derivative, and the constant C_2 is now considered as a universal constant. Our aim is to control the following seminorm for derivatives in an analytical setting (we recall that i denotes $2i+1$ derivatives in the tangential variable):

$$\sup_{i+k\geq 2} \tau^{a_{i,k}}\hat{\tau}^{a_{i,k}}\bar{\tau}^{b_{i,k}}\frac{\langle i+k\rangle^3}{(2i+1+k)!}\|F_{i,k}(\mathsf{s})\|_{L^\infty_{\mathsf{w}}}$$

where $0 < \bar{\tau}, \hat{\tau} \leq 1$ are constants, 3 is a correction exponent,[5] the other exponents are

$$a_{i,k} = \begin{cases} 0 & \text{if } i+k \leq 1, \\ i+k-\frac{7}{4} & \text{otherwise,} \end{cases} \qquad b_{i,k} = \begin{cases} 0 & \text{if } i = 0, \\ 2i-1 & \text{if } k \geq 1 \text{ and } i \geq 1, \\ 2i-2 & \text{if } k = 0 \text{ and } i \geq 1, \end{cases} \tag{5.35}$$

and for $\mathsf{K} > 0$ a constant such that Proposition 5.1 holds true, we have

$$\tau(s) = e^{-\frac{1}{2}\mathsf{s}}(1+e^{-\frac{\iota}{2}\mathsf{s}})^2e^{\mathsf{K}e^{-\mathsf{s}}} = (T-t)^{\frac{1}{2}}\tilde{\tau}, \quad \tilde{\tau}(t) = (1+(T-t)^{\frac{\iota}{2}})^2e^{\mathsf{K}(T-t)}. \tag{5.36}$$

Let us now explain formally why the above seminorm will remain bounded, and the role of the parameters. For this, let us only keep the term with $j = 1$ in the first line of (5.4):

$$\partial_s F_{i,k} + \mathcal{L}_{i,k}F_{i,k} = \binom{2i+1}{2}(\partial^{-1}F_{1,0})F_{i-1,k+1} + \cdots.$$

Since $\partial^{-1}F_{1,0} = O(1)$ from Proposition 5.6, this means that the evolution of $F_{i,k}$ has a forcing term that is $O(F_{i-1,k+1})$; that of $F_{i-1,k+1}$ has an $O(F_{i-2,k+2})$ forcing; and so on. At the end of this chain, we see that $F_{0,k+i}$ is in a sense forcing the evolution of $F_{i,k}$.

[5] There is no need to optimise the value of the exponent 3.

The optimal bound on the linear evolution of $F_{0,k+i}$ is $F_{0,k+i} = O(e^{-\frac{1}{2}(k+i)\mathsf{s}+O(1)})$ from Proposition 5.1. Hence we formally infer that $F_{i,k} = O(e^{-\frac{1}{2}(k+i)\mathsf{s}+O(1)})$ and in particular $F_{i,0} = O(e^{-\frac{i}{2}\mathsf{s}+O(1)})$. Back in the original variables, this gives $\xi_{i,0} = O((T-t)^{-\frac{3}{2}(2i+1)}F_{i,0}) = O((T-t)^{-\frac{7}{2}i+O(1)})$, hence a radius of analyticity of $(T-t)^{7/4}$ in the x direction.

We separate the radius of analyticity into three parts that will play different roles. First, $\tau^{a_{i,k}}$ is the time dependent part: it encodes the above expected temporal bound, and is compatible with the linear estimates of Proposition 5.1. Next, $\hat{\tau}$ is the constant part, and taking it small enough allows us to control the second line in (5.4). Finally, $\bar{\tau}^{b_{i,k}}$ gives a different estimate for ∂_x and ∂_y derivatives ; this anisotropy in the norm allows one to control the first line in (5.4).

Finally, let us mention that certain short time analytical results as [24] only require the control of a finite number of ∂_y derivatives, relying on parabolic regularising effects. However, here the viscosity is negligible as $s \to \infty$, and ∂_y derivatives are forcing ∂_x derivatives as explained above, requiring us to control an infinite number of ∂_y derivatives.

The heart of the analysis is to control the analytic norm using a bootstrap argument. We introduce the weight $\omega(t,y) = \mathsf{w}(y\sqrt{T-t}-\pi)$ and the space with the norm $\|f\|_{L^\infty_\omega} = \sup_{y\geq 0}|f(y)|\omega^{-1}(s,y)$.

Definition 5.11. Let $\mathsf{L}, \hat{\tau}, \bar{\tau}, \mathsf{K} > 0$. We say for $T_0 < t_1 < T$ that u is *in the analytic trap on* $[T_0, t_1]$ if $(\xi_{i,k})_{i,k\in\mathbb{N}}$ is a C^∞ solution of (1.11) on $[T_0,t_1]\times[0,\infty)$ such that, initially,

$$\|\xi_{i,k}(T_0,\cdot)\|_{L^\infty_\omega} \leq (T-T_0)^{-\frac{7}{2}i}\,\hat{\tau}^{-a_{i,k}}\,\bar{\tau}^{-b_{i,k}}\,\tilde{\tau}^{-a_{i,k}}\,\frac{(2i+k+1)!}{\langle i+k\rangle^3} \quad \text{for } i+k\geq 2, \tag{5.37}$$

and for all $t\in[T_0,t_1]$, setting $\tilde{L} = L/(2(T-T_0)^{1/8})$, one has

$$\|\xi_{i,k}(t,\cdot)\|_{L^\infty_\omega} \leq \mathsf{L}(T-t)^{-\frac{7}{2}i-\frac{1}{8}}\,\hat{\tau}^{-a_{i,k}}\,\bar{\tau}^{-b_{i,k}}\,\tilde{\tau}^{-a_{i,k}}\,\frac{(2i+k+1)!}{\langle i+k\rangle^3} \quad \text{for } i+k\geq 2, \tag{5.38}$$

$$|\xi_{i,k}(t,0)| \leq \tilde{\mathsf{L}}(T-t)^{-\frac{7}{2}i}\,\hat{\tau}^{-a_{i,k}}\,\bar{\tau}^{-b_{i,k}}\,\tilde{\tau}^{-a_{i,k}}\,\frac{(2i+k+1)!}{\langle i+k\rangle^3} \quad \text{for } i+k\geq 0. \tag{5.39}$$

Proposition 5.12. *For any constants T_0, T, μ, ι, C_0, C_0' and τ_0 in the hypotheses of Theorem 2, and C_2 in the inequality (5.24), there exist $\mathsf{L}^*, \mathsf{K} > 0$ such that for any $\mathsf{L}\geq \mathsf{L}^*$, there exist $\hat{\tau}^*, \bar{\tau}^* > 0$ such that, for any $0<\bar{\tau}<\bar{\tau}^*$ and $0<\hat{\tau}<\hat{\tau}^*$, if u is in the analytic trap on $[T_0,t_1]$ in the sense of the previous definition, then at time t_1, for all $i+k\geq 2$,*

$$\|\xi_{i,k}(t_1,\cdot)\|_{L^\infty_{\mathsf{w}}} \leq \frac{3}{4}\mathsf{L}(T-t)^{-\frac{7}{2}i-\frac{1}{8}}\,\hat{\tau}^{-a_{i,k}}\,\bar{\tau}^{-b_{i,k}}\,\tilde{\tau}^{-a_{i,k}}\,\frac{(2i+k+1)!}{\langle i+k\rangle^3}, \tag{5.40}$$

$$|\xi_{i,k}(t_1,0)| \leq \frac{1}{2}\tilde{\mathsf{L}}(T-t)^{-\frac{7}{2}i}\,\hat{\tau}^{-a_{i,k}}\,\bar{\tau}^{-b_{i,k}}\,\tilde{\tau}^{-a_{i,k}}\,\frac{(2i+k+1)!}{\langle i+k\rangle^3}. \tag{5.41}$$

Proof. The inequality (5.40) is proved in Proposition 5.17. The inequality (5.41) for k even is proved in Corollary 5.15 and for k odd in Lemma 5.16. ∎

Remark 5.13. The bounds (5.40) and (5.41) improve (5.38) and (5.39) by factors < 1. This is used to prove Theorem 2 as follows: for a solution starting in the analytic trap (Definition 5.11), the bounds (5.38) and (5.39) can never be saturated, showing that the solution remains in this trap up to the blow-up time T.

The bound (5.40), valid up to the blow-up time, is used to prove Theorem 3. The bounds (5.38) and (5.41) however are only used as additional estimates to prove Proposition 5.12.

The above proposition directly implies Theorem 2.

Proof of Theorem 2. Proof of (ii). Fix all constants in Definition 5.11 such that Proposition 5.12 holds true. Then (5.37) is satisfied because of (5.3), by choosing possibly a smaller coefficient $\hat{\tau}$.

Let now $\ell \in \mathbb{N}$ and define $t^*(\ell)$ as the supremum of times $t_1 > T_0$ such that $(\xi_{i,k})_{i+k\leq \ell}$ satisfies the estimates (5.38) and (5.39) in $[T_0, t_1]$. Assume $t^*(\ell) < T$ for contradiction. Notice that $(\xi_{i,k})_{i+k\leq \ell}$ solves a closed system of equations of the form

$$\partial_t \xi_{i,k} - \partial_{yy}\xi_{i,k} - \partial_y^{-1}\xi\partial_y\xi_{i,k} = f_{i,k}((\xi_{i',k'})_{i'+k'\leq \ell})$$

from (1.11). Notice that, as a consequence, the proof of the bounds (5.40) and (5.41) for $i+k\leq \ell$ only relies on the use of the bounds (5.38) and (5.39) for $i+k\leq \ell$. Thus, by definition of $t^*(\ell)$, the bounds (5.40) and (5.41) hold true for $i+k\leq \ell$ at any time $t_1 < t^*$, hence at time t^* as well by continuity. By a continuity argument and because of propagation of regularity, using the fact that (5.40) and (5.41) strictly improve (5.38) and (5.39), we deduce that $(\xi_{i,k})_{i+k\leq \ell}$ satisfies (5.38) and (5.39) on $[t^*, t^*+\delta]$ for some $\delta > 0$, contradicting the definition of t^*.

Hence $t^*(\ell) = T$. Letting $\ell \to \infty$, we infer that $(\xi_{i,k})_{i,k\geq 0}$, on $[0, T)$, satisfies (5.38) and (5.39). Thus, $(\xi_{i,0})_{i\geq 0}$ satisfies (1.16) and (1.17), as a consequence of (5.2), (5.24) and (5.40).

Proof of (i). We now define $u(t,x,y) = \sum_{i=0}^{\infty} x^{2i+1}\frac{\xi_{i,0}(t,y)}{(2i+1)!}$. The convergence in E_{T,τ^*} for τ^* independent of time small enough is a direct consequence of (1.16). Moreover, the trace of all x-derivatives of u on the vertical axis $\{x=0\}$ solves the corresponding trace of Prandtl's equations. Hence u solves Prandtl's equations on E_{T,τ^*} by uniqueness of analytic extensions.

Proof of (iii). We set τ to be constant on $[T-\delta', T]$ and then for $x = \pm\tau(T-t)^{7/4}$ for $T-\delta' \leq t \leq T$, using (1.14), (1.16) and (1.17), we bound

$$\begin{aligned}
|u| &\leq \tau(T-t)^{\frac{7}{4}}|\xi_0(t,y)| + \tau^3(T-t)^{\frac{21}{4}}\frac{|\xi_1(t,y)|}{6} + \sum_{i=2}^{\infty}\tau^{2i+1}(T-t)^{\frac{7(2i+1)}{4}}\frac{|\xi_i(t,y)|}{(2i+1)!} \\
&\leq \tau(T-t)^{\frac{7}{4}}\frac{(1+o_{\delta\to 0}(1))}{T-t} + C_1\tau^3(T-t)^{\frac{5}{4}} + C_1\sum_{i=2}^{\infty}\frac{\tau^{2i+1}}{\tau_1^{2i+1}}(T-t)^{\frac{13}{8}} \\
&< \frac{7}{4}\tau(T-t)^{\frac{3}{4}}
\end{aligned}$$

if τ and δ' have been chosen small enough. This shows (1.18) on $[T-\delta, T)$. Since on $[0, T-\delta']$, ξ_0 and ξ_1 remain bounded, it suffices to take τ decreasing fast enough on $[0, T-\delta]$ to obtain (1.18) on this interval. ∎

We turn to the proof of Proposition 5.12. We use the following throughout this section. For any $\mathsf{K}>0$, one has $0\le \tau\hat\tau\le 1$ on $[T_0, T)$ for $\hat\tau$ small enough. The function $\tilde\tau$ satisfies

$$\tilde\tau \ge 1 \quad \text{and} \quad \tilde\tau \text{ is decreasing on } [T_0, T]. \tag{5.42}$$

We shall use the following properties of the exponents for any $i, i', k, k' \ge 0$:

$$a_{i,k} \le a_{i',k'} \quad \text{and} \quad b_{i,k} \le b_{i',k'} \quad \text{if } i \le i' \text{ and } k \le k', \tag{5.43}$$

$$a_{i,k} + a_{i',k'} \le a_{i+i',k+k'}, \qquad b_{i,k} + b_{i',k'} \le b_{i+i',k+k'}, \tag{5.44}$$

$$a_{i,k} \le a_{i,k+1} - 1/4 \quad \text{if } i+k \ge 1, \qquad a_{i,k} = a_{i,k+1} - 1 \quad \text{if } i+k \ge 2, \tag{5.45}$$

$$a_{i,k} + a_{i',k'} \le a_{i+i',k+k'} - 1/4 \quad \text{if } i+k \ge 1 \text{ and } i'+k' \ge 1; \tag{5.46}$$

moreover, if $i \ge 1$ then

$$a_{i,0} + a_{i',k+1} \le a_{i+i',k}, \qquad \text{and if } k \ge 1 \text{ or } i' \ge 1 \text{ then } b_{i,0} + b_{i',k+1} \le b_{i+i',k} - 1. \tag{5.47}$$

5.3.2. Analytic control at the boundary. The aim now is to prove (5.41). We rely on the fact that the control of ∂_y^{2m} derivatives is similar to that of ∂_t^m derivatives for parabolic equations, the latter having the advantage of preserving Dirichlet boundary conditions. However, this equivalence degenerates as one approaches the blow-up time T. We need to exploit two gains coming from the fact that near the boundary one is away from the blow-up zone: first the bound (1.15), and then the fact that ∂_y^{-1} lose a $(T-t)^{-1/2}$ factor for $y \sim (T-t)^{-1/2}$ but not for $y = O(1)$.

Lemma 5.14 (Improved estimates at the boundary for even derivatives). *For any* $\mathsf{L}, \mathsf{K}, \bar\tau^* > 0$, *there exists* $\hat\tau^* > 0$ *such that the following holds true for any* $0 < \bar\tau \le \bar\tau^*$ *and* $0 < \hat\tau \le \hat\tau^*$. *Assume* (1.15), (5.24) *and* (5.39) *in Definition* 5.11. *Then for any* $m \ge 1$ *and any* i,

$$\partial_t^m \xi_{i,0} = \partial_y^{2m}\xi_{i,0} + \xi_i^m, \tag{5.48}$$

where for some universal $C > 0$, *for all* k *and* $t \in [T_0, t_1]$,

$$|\partial_y^k \xi_i^m(t,0)| \le C\tilde{\mathsf{L}}(T-t)^{-\frac{7}{2}i}\bar\tau^{-b_{i,k+2m-2}}\hat\tau^{-a_{i,k+2m-2}}\tilde\tau^{-a_{i,k+2m-2}}\frac{(2i+k+2m+1)!}{\langle i+k+m\rangle^3}, \tag{5.49}$$

with the convention that for all i, $a_{i,k} = b_{i,k} = 0$ *for* $k = -1$ *and* $k = -2$.

Corollary 5.15. *With the same hypotheses, for a universal* $C > 0$, *for any* $i, k \ge 0$ *with* k *even,*

$$|\xi_{i,k}(t,0)| \le \hat\tau\tilde\tau\mathsf{L}(T-t)^{-\frac{7}{2}i}\bar\tau^{-b_{i,k}}\hat\tau^{-a_{i,k}}\tilde\tau^{-a_{i,k}}\frac{(2i+k+1)!}{\langle i+k\rangle^3}, \tag{5.50}$$

$$|\partial_t\xi_{i,k}(t,0)| \le \mathsf{L}(T-t)^{-\frac{7}{2}i}\bar\tau^{-b_{i,k}}\hat\tau^{-a_{i,k}}\tilde\tau^{-a_{i,k}}\frac{(2i+k+3)!}{\langle i+k\rangle^3}. \tag{5.51}$$

Proof. The boundary condition $u|_{y=0}=0$ implies (5.50) for all $i\geq 0$ for $k=0$. By time differentiation, and from (1.11) for $\xi_{0,0}$, one obtains $\xi_{0,2}(t,0)=0$, hence (5.50) for $(i,k)=(0,2)$. By differentiation again, $\partial_t^k\xi_{i,0}(t,0)=0$ for all k, hence $\partial_y^{2k}\xi_{i,0}(t,0)=-\xi_i^k(t,0)$. So (5.50) is then a direct consequence of (5.49), since $a_{i,k-2}\leq a_{i,k}-1$ for any i if $k\geq 2$ and $(i,k)\neq(0,2)$. Next, we write $\partial_t\xi_{i,2k}=\xi_{i,2k+2}+\partial_y^{2k}\xi_i^1$, so that $\partial_t\xi_{i,2k}(t,0)=-\xi_i^{k+1}(t,0)+\partial_y^{2k}\xi_i^1(t,0)$ at the boundary, and (5.51) is again obtained from (5.49). ∎

Proof of Lemma 5.14. We set

$$\xi_{i,k,m}=\partial_t^m\xi_{i,k}.$$

By induction we obtain from (1.11) the recurrence identity

$$\begin{aligned}\partial_t^m\xi_{i,0}&=\partial_y^{2m}\xi_{i,0}\\&\quad+\sum_{n=0}^{m-1}\partial_y^{2m-2n-2}\partial_t^n\left(-\sum_{j=0}^{i}\binom{2i+1}{2j+1}\xi_{j,0}\xi_{i-j,0}+\sum_{j=0}^{i}\binom{2i+1}{2j}(\partial_y^{-1}\xi_{j,0})\xi_{i-j,1}\right).\end{aligned}$$

We now reason by induction on $m\geq 0$ to prove (5.49). For $m=0$ the bound is trivial since $\xi_i^0=0$ for all i. We now assume the desired bound holds true for all $m'\leq m$, for all i and k. Note that if $m\geq 1$ then $a_{i,k+2m-2}\leq a_{i,k+2m}-1$ so that

$$\hat{\tau}^{-a_{i,k+2m-2}}\tilde{\tau}^{-a_{i,k+2m-2}}\leq(\hat{\tau}\tilde{\tau})\hat{\tau}^{-a_{i,k+2m}}\tilde{\tau}^{-a_{i,k+2m}}.$$

Note also that if $m=0$ then $\xi_i^m=\xi_i^0=0$. In particular, the identity (5.48) and the bounds (5.39) and (5.49) give, for $m'\leq m$ and $\hat{\tau}$ small enough,

$$|\xi_{i,k,m'}(t,0)|\leq 2\tilde{\mathrm{L}}(T-t)^{-\frac{7}{2}i}\tilde{\tau}^{-b_{i,k+2m}}\hat{\tau}^{-a_{i,k+2m}}\tilde{\tau}^{-a_{i,k+2m}}\frac{(2i+k+2m+1)!}{\langle i+k+m\rangle^3}. \tag{5.52}$$

To prove the desired bound for $m+1$ we first obtain the following identity from the recurrence identity using the Leibniz rule and the fact that ∂_y^{-1} terms vanish at the boundary (with the convention that $\binom{a}{b}=0$ if $b>a$):

$$\begin{aligned}\partial_y^k\xi_i^{m+1}(t,0)&=\sum_{n=0}^{m}\sum_{j=0}^{i}\sum_{p=0}^{n}\sum_{l=0}^{k+2m-2n}\binom{n}{p}\xi_{j,l,p}(t,0)\xi_{i-j,k+2m-2n-l,n-p}(t,0)\\&\quad\times\left(\binom{2i+1}{2j}\binom{k+2m-2n}{l+1}-\binom{2i+1}{2j+1}\binom{k+2m-2n}{l}\right).\end{aligned} \tag{5.53}$$

Note that in the sum, if $(j,l,p)\in\{(0,0,0),(i,2m-2n+k,n)\}$ then the term is zero because $\xi_{0,0,0}(t,0)=u_x(t,0)=0$ from the Prandtl boundary condition $u|_{y=0}=0$. Therefore we assume $(j,l,p)\notin\{(0,0,0),(i,2m-2n+k,n)\}$ without loss of generality. Introducing $r=2j+l+2p+1$ we bound, using (5.52),

$$
\begin{aligned}
&|\xi_{j,l,p}(t,0)\xi_{i-j,2m-2n+k-l,n-p}(t,0)|\\
&\quad\le C\tilde{\mathsf{L}}^2(T-t)^{-\frac{7}{2}j-\frac{7}{2}(i-j)}\bar{\tau}^{-b_{j,2p+l}-b_{i-j,2(m-p)+k-l}}\\
&\qquad\cdot\hat{\tau}^{-a_{j,2p+l}-a_{i-j,2(m-p)+k-l}}\tilde{\tau}^{-a_{j,2p+l}-a_{i-j,2(m-p)+k-l}}\\
&\qquad\cdot\frac{(2j+2p+l+1)!}{\langle j+p+l\rangle^3}\frac{(2(i-j)+2(m-p)+k-l+1)!}{\langle i-j+m-p+k-l\rangle^3}\\
&\quad\le\tilde{\mathsf{L}}(T-t)^{-\frac{7}{2}i}\bar{\tau}^{-b_{i,2m+k}}\hat{\tau}^{-a_{i,2m+k}}\tilde{\tau}^{-a_{i,k+2m}}\frac{r!}{\langle r\rangle^3}\frac{(2i+2m+k+2-r)!}{\langle 2i+k+2m+1-r\rangle^3},
\end{aligned}
\tag{5.54}
$$

where in the last bound we have used (5.44) and (5.46) for the exponents, and $C\tilde{\mathsf{L}}\tilde{\tau}\hat{\tau}\le 1$ for $\hat{\tau}$ small enough. We recall the estimate, for some universal $C>0$,

$$
\sum_{r=0}^{2i+2m+k+1}\binom{2i+2m+k+1}{r}\frac{r!}{\langle r\rangle^3}\frac{(2i+k+2m+2-r)!}{\langle 2i+k+2m+1-r\rangle^3}\le C\frac{(2i+k+2m+2)!}{\langle i+k+m\rangle^3}.
$$

Using the inequality $\binom{n}{p}\le\binom{2n}{2p}$, (D.3) with $(A_1,A_2,A_3,r_2)=(2i+1,k+2m-2n,2n,2p+2j+l+1)$, and the above inequality, we get

$$
\begin{aligned}
&\sum_{j=0}^{i}\sum_{p=0}^{n}\sum_{l=0}^{k+2m-2n}\frac{r!}{\langle r\rangle^3}\frac{(2i+2m+k+2-r)!}{\langle 2i+k+2m+1-r\rangle^3}\binom{n}{p}\\
&\qquad\cdot\left|\binom{2i+1}{2j}\binom{k+2m-2n}{l+1}-\binom{2i+1}{2j+1}\binom{k+2m-2n}{l}\right|\\
&\qquad\le\sum_{r=0}^{2i+2m+k+1}\binom{2i+2m+k+1}{r}\frac{r!}{\langle r\rangle^3}\frac{(2i+k+2m+2-r)!}{\langle 2i+k+2m+1-r\rangle^3}\\
&\qquad\le C\frac{(2i+k+2m+2)!}{\langle i+k+m\rangle^3}.
\end{aligned}
\tag{5.55}
$$

Inserting (5.54) in the identity (5.53), then using (5.55) and the inequality $\sum_{n=0}^{m}1\le\langle 2i+k+2m+3\rangle$, we get the upper bound

$$
|\partial_y^k\xi_i^{m+1}(t,0)|\le C\tilde{\mathsf{L}}(T-t)^{-\frac{7}{2}i}\bar{\tau}^{-b_{i,k+2m}}\hat{\tau}^{-a_{i,k+2m}}\tilde{\tau}^{-a_{i,k+2m}}\frac{(2i+k+2(m+1)+1)!}{\langle i+k+m\rangle^3}.
$$

Thus (5.49) holds true for $m+1$, for any i and k. It thus holds true for any i,k,m by induction. ■

Lemma 5.16 (Improved estimates at the boundary for odd derivatives). *Assume that the bounds (1.15), (5.24), (5.38), (5.50) and (5.51) are satisfied. Then for any $\mathsf{L}>0$, there exists $\hat{\tau}^*>0$ small enough such that for all $0<\hat{\tau}\le\hat{\tau}^*$, all k odd (with $k\ge 3$ if $i=0$) and $t\in[T_0,t_1]$,*

$$
|\xi_{i,k}(t,0)|\le\frac{\tilde{\mathsf{L}}}{2}\tilde{\tau}^{-a_{i,k}}(T-t)^{-\frac{7}{2}i}\bar{\tau}^{-b_{i,k}}\hat{\tau}^{-a_{i,k}}\frac{(2i+k+1)!}{\langle i+k\rangle^3},
$$

Proof. Assume k is even, with $k \geq 2$ if $i = 0$. Let $\chi : [0, \infty) \to \mathbb{R}$ be a smooth cut-off function with $\chi(y) = 1$ for $|y| \leq 1/8$ and $\chi(y) = 0$ for $|y| \geq 1/4$. Set $\zeta_{i,k} = \chi \xi_{i,k}$. Then from (1.11) we infer the evolution equation of $\zeta_{i,k}$:

$$\begin{aligned}
&\partial_t \zeta_{i,k} - \partial_{yy} \zeta_{i,k} \\
&= \underbrace{-\sum_{j=0}^{i} \sum_{l=0}^{k} \binom{2i+1}{2j+1} \binom{k}{l} \chi \xi_{j,l} \xi_{i-j,k-l} + \sum_{j=0}^{i} \sum_{l=0}^{k-1} \binom{2i+1}{2j} \binom{k}{l+1} \chi \xi_{j,l} \xi_{i-j,k-l}}_{I} \\
&\quad + \underbrace{\sum_{j=1}^{i} \binom{2i+1}{2j} \chi (\partial_y^{-1} \xi_{j,0}) \xi_{i-j,k+1}}_{II} + \underbrace{\chi (\partial_y^{-1} \xi_{0,0}) \xi_{i,k+1} - 2\partial_y \chi \xi_{i,k+1} - \partial_{yy} \chi \xi_{i,k}}_{III}.
\end{aligned}$$

We decompose $\zeta_{i,k}(t, y) = \xi_{i,k}(t, 0)\chi(y) + \eta_{i,k}(t, y) + \eta'_{i,k}(t, y)$ where

$$\begin{cases} \partial_t \eta_{i,k} - \partial_{yy} \eta_{i,k} = \xi_{i,k}(t, 0)\partial_{yy} \chi - \partial_t \xi_{i,k}(t, 0)\chi, \\ \eta_{i,k}(T_0, y) = \chi(y)(\xi_{i,k}(T_0, y) - \xi_{i,k}(T_0, 0)), \quad \eta_{i,k}(t, 0) = 0, \end{cases}$$

$$\begin{cases} \partial_t \eta'_{i,k} - \partial_{yy} \eta'_{i,k} = I + II + III, \\ \eta'_{i,k}(T_0, y) = 0, \quad \eta'_{i,k}(t, 0) = 0. \end{cases}$$

The first term $\eta_{i,k}$. Recall (6.3), and that η is given by the representation formula (6.15). For the first part, as $\partial_{yy} \chi = 0$ on $[0, 1/8]$, $\overline{\partial_{yy} \chi}$ is a smooth function so that from (5.50) and (5.42),

$$\begin{aligned}
&\left\| \partial_y \int_{T_0}^{t} K_{t-t'} * \left(\xi_{i,k}(t', 0) \overline{\partial_{yy} \chi} \right) dt' \right\|_{L^\infty} = \left\| \int_{T_0}^{t} \xi_{i,k}(t', 0) K_{t-t'} * (\partial_y \overline{\partial_{yy} \chi})\, dt' \right\|_{L^\infty} \\
&\qquad \leq C \|\xi_{i,k}(\cdot, 0)\|_{L^\infty([T_0,t])} \leq C \hat{\tau} \tilde{\tau} \mathsf{L} (T - t)^{-\frac{7}{2} i} \bar{\tau}^{-b_{i,k}} \hat{\tau}^{-a_{i,k}} \tilde{\tau}^{-a_{i,k}} \frac{(2i + k + 1)!}{\langle i + k \rangle^3}.
\end{aligned}$$

For the second part, we let $\bar{t} = \max(t - \langle i + k \rangle^{-2}, T_0)$, decompose the time integral and integrate by parts:

$$\begin{aligned}
\int_{T_0}^{t} \partial_{t'} \xi_{i,k}(t', 0) K_{t-t'} * \bar{\chi}\, dt' &= \xi_{i,k}(\bar{t}, 0) K_{t-\bar{t}} * \bar{\chi} - \xi_{i,k}(T_0, 0) K_{t-T_0} * \bar{\chi} \\
&\quad + \int_{T_0}^{\bar{t}} \xi_{i,k}(t', 0) \partial_t K_{t-t'} * \bar{\chi} + \int_{\bar{t}}^{t} \partial_{t'} \xi_{i,k}(t', 0) K_{t-t'} \bar{\chi}.
\end{aligned}$$

We estimate the first line. It is zero if $\bar{t} = T_0$ so we assume $t > T_0 + \langle i + k \rangle^{-2}$. For the first term on the first line we have, using (C.2) and (5.50),

$$\begin{aligned}
\left| \partial_y \left(\xi_{i,k}(\bar{t}, 0) K_{\langle i+k \rangle^{-2}} * \bar{\chi} \right) \right| &\leq C \langle i + k \rangle |\xi_{i,k}(\bar{t}, 0)| \\
&\leq C \hat{\tau} \tilde{\tau} \mathsf{L} (T - t)^{-\frac{7}{2} i} \bar{\tau}^{-b_{i,k}} \hat{\tau}^{-a_{i,k}} \tilde{\tau}^{-a_{i,k}} \frac{(2i + k + 2)!}{\langle i + k \rangle^3}.
\end{aligned}$$

The second term on the first line enjoys the same estimate. For the first term on the second line, using (C.2), (5.50) and (5.42) we get

$$\left|\partial_y \int_{T_0}^{\bar{t}} \xi_{i,k}(t',0)\partial_t K_{t-t'} * \bar{\chi}\, dt'\right| \lesssim \int_{T_0}^{\bar{t}} \frac{1}{(t-t')^{3/2}} |\xi_{i,k}(t',0)|\, dt'$$
$$\lesssim \langle i+k\rangle \|\xi_{i,k}(\cdot,0)\|_{L^\infty([T_0,t])} \le C\mathsf{L}(T-t)^{-\frac{7}{2}i} \bar{\tau}^{-b_{i,k}} \hat{\tau}^{-a_{i,k}+1} \tilde{\tau}^{-a_{i,k}} \frac{(2i+k+2)!}{\langle i+k\rangle^3}.$$

For the second term on the second line, from (5.24), (5.51), (5.42) and (C.2),

$$\left|\partial_y \int_{\bar{t}}^{t} \partial_{t'}\xi_{i,k}(t',0) K_{t-t'}\bar{\chi}\, dt'\right| \le C \int_{\max(t-\langle i+k\rangle^{-2},T_0)}^{t} \frac{|\partial_{t'}\xi_{i,k}(t',0)|}{\sqrt{t-t'}}\, dt'$$
$$\le C\langle i+k\rangle^{-1} \|\partial_t \xi_{i,k}(\cdot,0)\|_{L^\infty[T_0,t]} \le C\mathsf{L}(T-t)^{-\frac{7}{2}i} \bar{\tau}^{-b_{i,k}} \hat{\tau}^{-a_{i,k}} \tilde{\tau}^{-a_{i,k}} \frac{(2i+k+2)!}{\langle i+k\rangle^3}.$$

From (5.44), (5.45), and the initial bound (5.37), the above estimates imply

$$\|\partial_y \eta_{i,k}\|_{L^\infty} \le (1 + C(\hat{\tau}\tilde{\tau})^{\frac{1}{4}}\mathsf{L})(T-t)^{-\frac{7}{2}i} \bar{\tau}^{-b_{i,k+1}} \hat{\tau}^{-a_{i,k+1}} \tilde{\tau}^{-a_{i,k+1}} \frac{(2i+k+2)!}{\langle i+k\rangle^3}. \tag{5.56}$$

The second term $\eta'_{i,k}$. For I, first from (1.15), (5.24) and (5.38) we have the bound

$$\begin{aligned}
&\|\xi_{j,l}\xi_{i-j,k-l}\|_{L^\infty([0,1/4])} \\
&\quad \le (T-t)^{-\frac{7}{2}j-\frac{1}{8}-\frac{7}{2}(i-j)-\frac{1}{8}} \mathsf{L}^2 \bar{\tau}^{-b_{j,l}-b_{i-j,k-l}} \hat{\tau}^{-a_{j,l}-a_{i-j,k-l}} \tilde{\tau}^{-a_{j,l}-a_{i-j,k-l}} \\
&\qquad \cdot \frac{(2j+l+1)!}{\langle j+l\rangle^3} \frac{(2i-2j+k-l+1)!}{\langle i-j+k-l\rangle^3} \\
&\quad \le (T-t)^{-\frac{7}{2}i-\frac{1}{4}} \mathsf{L}^2 \bar{\tau}^{-b_{i,k}} \tilde{\tau}^{-a_{i,k}} \hat{\tau}^{-a_{i,k}} \frac{(r)!}{\langle r\rangle^3} \frac{(2i+k+2-r)!}{\langle 2i+k+1-r\rangle^3},
\end{aligned}$$

where we have used (5.44) and set $r = 2j + l + 1$. Therefore, using the bound (5.55) with $n = m = 0$ we get

$$\|I\|_{L^\infty([0,1/4])} \le (T-t)^{-\frac{7}{2}i-\frac{1}{4}} \mathsf{L}^2 \bar{\tau}^{-b_{i,k}} \hat{\tau}^{-a_{i,k}} \tilde{\tau}^{-a_{i,k}} \frac{(2i+k+2)!}{\langle i+k\rangle^3}.$$

We turn to *II*; using the bounds (1.15), (5.24) and (5.38), the inequalities (5.47) for the exponents since $j \ge 1$ in the sum in the definition of *II*, we estimate

$$\begin{aligned}
&\|(\partial_y^{-1}\xi_{j,0})\xi_{i-j,k+1}\|_{L^\infty([0,1/4])} \\
&\quad \le (T-t)^{-\frac{7}{2}j-\frac{1}{8}-\frac{7}{2}(i-j)-\frac{1}{8}} \mathsf{L}^2 \bar{\tau}^{-b_{j,0}-b_{i-j,k+1}} \hat{\tau}^{-a_{j,0}-a_{i-j,k+1}} \\
&\qquad \cdot \tilde{\tau}^{-a_{j,0}-a_{i-j,k+1}} \frac{(2j+1)!}{\langle j\rangle^3} \frac{(2i-2j+k+2)!}{\langle i-j+k+1\rangle^3} \\
&\quad \le C(T-t)^{-\frac{7}{2}i-\frac{1}{4}} \mathsf{L}^2 \bar{\tau}^{-b_{i,k}} \hat{\tau}^{-a_{i,k}} \tilde{\tau}^{-a_{i,k}} \frac{(2j+1)!}{\langle j\rangle^3} \frac{(2i+k+2-2j)!}{\langle i-j+k\rangle^3}.
\end{aligned}$$

As $\sum_{j=0}^{i}\binom{2i+1}{2j}\frac{(2j+1)!}{\langle j\rangle^3}\frac{(2i+k+2-2j)!}{\langle i-j+k\rangle^3}\leq C\frac{(2i+k+2)!}{\langle i+k\rangle^3}$, we conclude that II enjoys the same estimate as I,

$$\|II\|_{L^\infty([0,1/4])}\leq C(T-t)^{-\frac72 i-\frac14}\mathsf{L}^2\bar\tau^{-b_{i,k}}\hat\tau^{-a_{i,k}}\tilde\tau^{-a_{i,k}}\frac{(2i+k+2)!}{\langle i+k\rangle^3}.$$

Therefore, by (5.42) and (C.2),

$$\begin{aligned}&\left|\partial_y\int_{T_0}^t K_{t-t'}*\overline{(I+II)}\,dt'\right|\\&\leq C\mathsf{L}^2\bar\tau^{-b_{i,k}}\hat\tau^{-a_{i,k}}\frac{(2i+k+2)!}{\langle i+k\rangle^3}\int_{T_0}^t\frac{1}{\sqrt{t-t'}}(T-t')^{-\frac72 i-\frac14}\tilde\tau^{-a_{i,k}}(t')\,dt'\\&\leq C(\hat\tau\tilde\tau)^{\frac14}\mathsf{L}^2(T-t)^{-\frac72 i}\bar\tau^{-b_{i,k+1}}\hat\tau^{-a_{i,k+1}}\tilde\tau^{-a_{i,k+1}}(t)\frac{(2i+k+2)!}{\langle i+k\rangle^3}\end{aligned}\tag{5.57}$$

where we have used (5.44) and (5.45). We turn to III. Let $r_0>0$ be fixed small in a universal way. Let χ_0 be a smooth function on $[0,\infty)$ such that $\chi_0(y)=1$ on $[0,r_0]$ and $\chi_0(y)=0$ for $y\geq 2r_0$. We decompose $\chi\partial_y^{-1}\xi_{0,0}\xi_{i,k+1}=\chi_0\partial_y^{-1}\xi_{0,0}\xi_{i,k+1}+(\chi-\chi_0)\partial_y^{-1}\xi_{0,0}\xi_{i,k+1}$. Since from (1.15) we have $|\partial_y^{-1}\xi_{0,0}|\leq Cy$, we deduce from (5.38), (5.42) and (C.2) that

$$\begin{aligned}&\left|\partial_y\int_{T_0}^t K_{t-t'}*\overline{(\chi_0\partial_y^{-1}\xi_{0,0}\xi_{i,k+1})}\,dt'\right|\\&\qquad\leq Cr_0\mathsf{L}\bar\tau^{-b_{i,k+1}}\hat\tau^{-a_{i,k+1}}\frac{(2i+k+2)!}{\langle i+k\rangle^3}\int_{T_0}^t\frac{\tilde\tau^{-a_{i,k+1}}(t')(T-t')^{-\frac72 i-\frac18}\,dt'}{\sqrt{t-t'}}\\&\qquad\leq Cr_0\mathsf{L}(T-t)^{-\frac72 i}\bar\tau^{-b_{i,k+1}}\hat\tau^{-a_{i,k+1}}\tilde\tau^{-a_{i,k+1}}\frac{(2i+k+2)!}{\langle i+k\rangle^3}.\end{aligned}$$

For the other term we write

$$(\chi-\chi_0)\partial_y^{-1}\xi_{0,0}\xi_{i,k+1}=\partial_y((\chi-\chi_0)\partial_y^{-1}\xi_{0,0}\xi_{i,k})+\partial_y\chi_0\partial_y^{-1}\xi_{0,0}\xi_{i,k}+(\chi-\chi_0)\xi_{0,0}\xi_{i,k}.$$

Notice that all terms are supported away from the origin, at distance r_0 from it. Thus using (C.2), (1.15), (5.38), (5.42) and integration by parts we obtain

$$\begin{aligned}&\left|\int_{T_0}^t\partial_y\big(K_{t-s}*\overline{(\chi-\chi_0)\partial_y^{-1}\xi_{0,0}\xi_{i,k+1}}\big)(0)\,ds\right|\leq C(r_0)\int_{T_0}^t\|\xi_{i,k}\|_{L^\infty}\,ds\\&\qquad\leq(\hat\tau\tilde\tau)^{\frac14}C(r_0)\mathsf{L}(T-t)^{-\frac72 i}\tilde\tau^{-a_{i,k+1}}\bar\tau^{-b_{i,k+1}}\hat\tau^{-a_{i,k+1}}\frac{(2i+k+1)!}{\langle i+k\rangle^3},\end{aligned}$$

where we have used (5.44) and (5.45). The other terms in III can be treated in the same way. Hence

$$\begin{aligned}&\left|\int_{T_0}^t\partial_y(K_{t-s}*\overline{III})(0)\,ds\right|\\&\leq(C(r_0)(\hat\tau\tilde\tau)^{\frac14}+Cr_0)\mathsf{L}\bar\tau^{-b_{i,k+1}}\hat\tau^{-a_{i,k+1}}\frac{(2i+k+1)!}{\langle i+k\rangle^3}\tilde\tau^{-a_{i,k+1}}(T-t)^{-\frac72 i}.\end{aligned}\tag{5.58}$$

Conclusion. Gathering the estimates (5.56)–(5.58), we have proved that

$$|\partial_y \xi_{i,k}(0)| \le \big(\mathsf{L}^{-1} + C(r_0)(\hat\tau\tilde\tau)^{\frac14} + Cr_0 + C(\hat\tau\tilde\tau)^{\frac14}\mathsf{L}\big) \cdot \mathsf{L}\bar\tau^{-b_{i,k+1}}\hat\tau^{-a_{i,k+1}}\frac{(2i+k+1)!}{\langle i+k\rangle^3}\tilde\tau^{-a_{i,k+1}}(T-t)^{-\frac72 i}$$

which is the desired estimate upon taking $\mathsf{L} \ge 2$, $r_0 > 0$ small enough in a universal way, and then $\hat\tau$ small enough depending on r_0, K, L and T. ∎

5.3.3. Analytic analysis in the blow-up zone. Our aim here is to prove (5.40). Note that (5.38) is equivalent to, for $i+k \ge 2$,

$$\|F_{i,k}(\mathsf{s})\|_{L^\infty_{\mathsf{w}}} \le \mathsf{L}\tau^{-a_{i,k}}\bar\tau^{-b_{i,k}}\tilde\tau^{-a_{i,k}}\frac{(2i+k+1)!}{\langle i+k\rangle^3}. \tag{5.59}$$

Recall the evolution equation (5.4) for $F_{i,k}$, and Proposition 5.1 for the linear evolution.

Proposition 5.17. *Assume* (a) *and* (b) *in Theorem* 2*, and that* (5.37)*,* (5.39) *and* (5.59) *hold on* $[\mathsf{s}_0,\mathsf{s}_1]$*. Then*

$$\|F_{i,k}(\mathsf{s}_1)\|_{L^\infty_{\mathsf{w}}} \le \frac34\mathsf{L}\tau^{-a_{i,k}}\bar\tau^{-b_{i,k}}\tilde\tau^{-a_{i,k}}\frac{(2i+k+1)!}{\langle i+k\rangle^3}. \tag{5.60}$$

Proof. Note that combining the assumption (1.14) in Theorem 2, (5.24) and (5.59), we get, for all $i+k \ge 1$,

$$\|F_{i,k}(\mathsf{s})\|_{L^\infty_{\mathsf{w}}} \le \mathsf{L}\tau^{-a_{i,k}}\bar\tau^{-b_{i,k}}\tilde\tau^{-a_{i,k}}\frac{(2i+k+1)!}{\langle i+k\rangle^3} \tag{5.61}$$

if L has been chosen large enough. We fix $i+k \ge 2$ and recall $\mathsf{s}_0 = -\log(T-T_0)$.

Step 1. *The case $k \ge 1$.* Assume $k \ge 1$. From (5.4) we write, with $S_{i,k}$ being the semi-group (5.7),

$$F_{i,k}(\mathsf{s}_1) = \tilde F_{i,k}(\mathsf{s}_1) + \int_{\mathsf{s}_0}^{\mathsf{s}_1} S_{i,k}(\mathsf{s},\mathsf{s}_1)(I+II)\,d\mathsf{s},$$

where $\tilde F_{i,k}$ solves the free evolution with the same boundary conditions as $F_{i,k}$:

$$\partial_{\mathsf{s}}\tilde F_{i,k} + \mathcal{L}_{i,k}F_{i,k} = 0, \quad \tilde F_{i,k}(\mathsf{s},-\pi) = F_{i,k}(\mathsf{s},-\pi), \quad \tilde F_{i,k}(\mathsf{s}_0,z) = F_{i,k}(\mathsf{s}_0,z),$$

and the second term is obtained via the Duhamel formula with forcing terms

$$I = \sum_{j=1}^{i}\binom{2i+1}{2j}(\partial^{-1}F_{j,0})F_{i-j,k+1},$$

$$II = -\sum_{E^1_{i,k}}\binom{2i+1}{2j+1}\binom{k}{l}F_{j,l}F_{i-j,k-l} + \sum_{E^2_{i,k}}\binom{2i+1}{2j}\binom{k}{l+1}F_{j,l}F_{i-j,k-l}.$$

The free evolution term. Let

$$e_{i,k} = \frac{\mathsf{L}}{2}\frac{(2i+k+1)!}{\langle i+k\rangle^3}\hat{\tau}^{-a_{i,k}}\bar{\tau}^{-b_{i,k}} \quad \text{and} \quad \tilde{\mathsf{S}}(\mathsf{s},z) := e_{i,k}\tau^{-a_{i,k}}(\mathsf{s})\mathsf{w}(z).$$

Then $\mathsf{S} = e_{i,k}(1+e^{-\frac{L}{2}\mathsf{s}})^{-a_{i,k}}e^{(a_{i,k}/2-c_{i,k})\mathsf{s}}\mathsf{S}$ where S was defined in (5.19) (with $h(\mathsf{s}) = e^{-\mathsf{K}a_{i,k}e^{-\mathsf{s}}}$). For $i+k\geq 2$, from the definitions (5.8) and (5.35) of $c_{i,k}$ and $a_{i,k}$ we compute

$$c_{i,k} - \frac{a_{i,k}}{2} = \max\left(-\frac{3}{2}i - k + \frac{15}{8}, -\frac{7}{2}i - \frac{1}{8}\right) + \eta\langle i\rangle.$$

Therefore, there exist $\eta^* > 0$ and $c > 0$ independent of i and k such that for all $0 < \eta \leq \eta^*$ and $i+k\geq 2$,

$$-\frac{1}{c}\langle i\rangle \leq c_{i,k} - \frac{a_{i,k}}{2} \leq -c\langle i\rangle < 0. \tag{5.62}$$

As a result, since S was proved to be a supersolution for $\partial_{\mathsf{s}} + \mathscr{L}_{i,k}$ in the proof of Lemma 5.4 for all $\mathsf{s}\geq \mathsf{s}_0$, we see that $\tilde{\mathsf{S}}$ is also a supersolution for $\partial_{\mathsf{s}} + \mathscr{L}_{i,k}$. At the boundary $\{\mathsf{s}=\mathsf{s}_0\}$ or $\{z=-\pi\}$ we have $|\tilde{F}_{i,k}|\leq \tilde{\mathsf{S}}$ from (5.41) (proved in the previous subsubsection) and (5.37). Hence $|\tilde{F}_{i,k}|\leq\tilde{\mathsf{S}}$ for all $\mathsf{s}\geq\mathsf{s}_0$ and $z\geq -\pi$ by the maximum principle. This yields the bound

$$\|\tilde{F}_{i,k}(\mathsf{s}_1)\|_{L^\infty_{\mathsf{w}}} \leq \frac{1}{2}\mathsf{L}\tau^{-a_{i,k}}(\mathsf{s}_1)\bar{\tau}^{-b_{i,k}}\tilde{\tau}^{-a_{i,k}}\frac{(2i+k+1)!}{\langle i+k\rangle^3}. \tag{5.63}$$

The first term I. Note that this term is zero if $i=0$ so we assume $i\geq 1$. From $\|\partial^{-1}fg\|_{L^\infty_{\mathsf{w}}}\lesssim \|f\|_{L^\infty_{\mathsf{w}}}\|g\|_{L^\infty_{\mathsf{w}}}$ and (5.61) we have

$$\|(\partial^{-1}F_{j,0})F_{i-j,k+1}\|_{L^\infty_{\mathsf{w}}} \lesssim \mathsf{L}^2\bar{\tau}^{-b_{i,k}+1}\tau^{-a_{i,k}}\hat{\tau}^{-a_{i,k}}\frac{(2j+1)!}{\langle j\rangle^3}\frac{(2i-2j+k+2)!}{\langle i-j+k\rangle^3},$$

where we have used (5.47) as $j,k\geq 1$. We then compute

$$\begin{aligned}\frac{(2i+1)!}{(2i-2j+1)!}&\frac{(2i-2j+k+2)!}{\langle i-j+k\rangle^3}\\ &= \frac{(2i+1+k)!}{\langle i+k\rangle^3}\frac{\langle i+k\rangle^3}{\langle i-j+k\rangle^3}\frac{(2i+1)!(2i-2j+k+2)!}{(2i-2j+1)!(2i+k+1)!}.\end{aligned}$$

Since $(2i+1-m)/(2i+k+1-m)\leq 1$ for $m=0,\dots,2j-1$, we get

$$\frac{(2i+1)!(2i-2j+k+2)!}{(2i-2j+1)!(2i+k+1)!} = \frac{(2i+1)\cdots(2i-2j+2)}{(2i+k+1)\cdots(2i-2j+k+3)} \lesssim \langle i-j\rangle.$$

As a result,

$$\begin{aligned}\frac{\langle i+k\rangle^3}{(2i+1+k)!}\sum_{j=1}^{i-1}\frac{(2i+1)!}{(2j)!(2i-2j+1)!}&\frac{(2j+1)!}{\langle j\rangle^3}\frac{(2i-2j+k+2)!}{\langle i-j+k\rangle^3}\\ &\lesssim \sum_{j=1}^{i-1}\frac{\langle i+k\rangle^3\langle i-j\rangle}{\langle j\rangle^2\langle i-j+k\rangle^3} \lesssim \langle i\rangle.\end{aligned}$$

This yields the bound

$$\|I\|_{L^\infty_{\mathrm{w}}} \leq (C\bar{\tau}\mathsf{L})\mathsf{L}\bar{\tau}^{-b_{i,k}}\tau^{-a_{i,k}}\hat{\tau}^{-a_{i,k}}\langle i\rangle\frac{(2i+1+k)!}{\langle i+k\rangle^3}.$$

Using this and (5.9) we find that for the first term, for a universal constant $C>0$,

$$\begin{aligned}&\left\|\int_{\mathsf{s}_0}^{\mathsf{s}_1} S_{i,k}(\mathsf{s},\mathsf{s}_1)(I)\,d\mathsf{s}\right\|_{L^\infty_{\mathrm{w}}}\\ &\qquad\leq (C\bar{\tau}\mathsf{L})\mathsf{L}\bar{\tau}^{-b_{i,k}}\hat{\tau}^{-a_{i,k}}\langle i\rangle\frac{(2i+1+k)!}{\langle i+k\rangle^3}\int_{\mathsf{s}_0}^{\mathsf{s}_1} p(\mathsf{s},\mathsf{s}_1)\tau^{-a_{i,k}}(\mathsf{s})\,d\mathsf{s}\end{aligned}$$

where $p(\mathsf{s},\mathsf{s}_1):=e^{c_{i,k}(\mathsf{s}_1-\mathsf{s})}\left(\frac{(1+e^{-\frac{\iota}{2}\mathsf{s}})e^{\mathsf{K}e^{-\mathsf{s}}}}{(1+e^{-\frac{\iota}{2}\mathsf{s}_1})e^{\mathsf{K}e^{-\mathsf{s}_1}}}\right)^{a_{i,k}}$, so that

$$p(\mathsf{s},\mathsf{s}_1)\tau^{-a_{i,k}}(\mathsf{s})=e^{-a_{i,k}\mathsf{K}e^{-\mathsf{s}_1}}e^{c_{i,k}\mathsf{s}_1}(1+e^{-\frac{\iota}{2}\mathsf{s}_1})^{-a_{i,k}}e^{(\frac{a_{i,k}}{2}-c_{i,k})\mathsf{s}}(1+e^{-\frac{\iota}{2}\mathsf{s}})^{-a_{i,k}}.$$

Using (5.36) and integrating by parts we find that for η small independently of i and k,

$$\begin{aligned}\int_{\mathsf{s}_0}^{\mathsf{s}_1} e^{(a_{i,k}/2-c_{i,k})\mathsf{s}}(1+e^{-\frac{\iota}{2}\mathsf{s}})^{-a_{i,k}}\,d\mathsf{s}&=\int_{\mathsf{s}_0}^{\mathsf{s}_1}\partial_s\left(\frac{e^{(a_{i,k}/2-c_{i,k})s}}{a_{i,k}/2-c_{i,k}}\right)(1+e^{-\frac{\iota}{2}s})^{-a_{i,k}}\,ds\\ &\leq\frac{e^{\frac{a_{i,k}}{2}(s-\mathsf{s}_1)}}{\frac{a_{i,k}}{2}-c_{i,k}}(1+e^{-\frac{\iota}{2}\mathsf{s}_1})^{-a_{i,k}}\\ &\quad-\frac{\frac{\iota}{2}a_{i,k}}{\frac{a_{i,k}}{2}-c_{i,k}}e^{c_{i,k}\mathsf{s}_1}\int_{\mathsf{s}_0}^{\mathsf{s}_1}e^{(\frac{a_{i,k}}{2}-c_{i,k}-\frac{\iota}{2})s}(1+e^{-\frac{\iota}{2}s})^{-a_{i,k}-1}\,ds.\end{aligned}$$

Using the identity above and (5.62), we infer that there exists $C>0$ depending on T and T_0 (since $\mathsf{s}_0=-\log(T-T_0)$) and $\iota>0$ such that

$$\int_{\mathsf{s}_0}^{\mathsf{s}_1} p(\mathsf{s},\mathsf{s}_1)\tau^{-a_{i,k}}(\mathsf{s})\,d\mathsf{s}+\frac{\langle i+k\rangle}{\langle i\rangle}\int_{\mathsf{s}_0}^{\mathsf{s}_1} p(\mathsf{s},\mathsf{s}_1)\tau^{-a_{i,k}}(\mathsf{s})e^{-\frac{\iota}{2}\mathsf{s}}\,d\mathsf{s}\leq\frac{C}{\langle i\rangle}\tau^{-a_{i,k}}(\mathsf{s}_1). \tag{5.64}$$

In particular, for $\bar{\tau}$ small enough depending only on L and T,

$$\left\|\int_{\mathsf{s}_0}^{\mathsf{s}_1} S_{i,k}(\mathsf{s},\mathsf{s}_1)(I)\,d\mathsf{s}\right\|_{L^\infty_{\mathrm{w}}}\leq\frac{\mathsf{L}}{8}\bar{\tau}^{-b_{i,k}}\tau^{-a_{i,k}}(\mathsf{s}_1)\hat{\tau}^{-a_{i,k}}\frac{(2i+1+k)!}{\langle i+k\rangle^3}. \tag{5.65}$$

The second term II. From (5.61) we compute

$$\begin{aligned}&\|F_{j,l}\|_{L^\infty_{\mathrm{w}}}\|F_{i-j,k-l}\|_{L^\infty_{\mathrm{w}}}\\ &\qquad\leq(\hat{\tau}\tau)^{\frac14}\mathsf{L}^2\tau^{-a_{i,k}}\bar{\tau}^{-b_{i,k}}\hat{\tau}^{-a_{i,k}}\frac{(2j+\ell+1)!}{\langle j+\ell\rangle^3}\frac{(2i-2j+k-\ell+1)!}{\langle i-j+k-\ell\rangle^3},\end{aligned}$$

where we have used (5.44) and (5.46) as $l+j\geq 1$ and $i-j+k-l\geq 1$ in the sums. We use the identity (D.2) with $(A_1,A_2,r_1)=(2i+1,k,2j+l+1)$ to obtain

$$\|II\|_{L^\infty_{\mathrm{w}}}\leq(\hat{\tau}\tau)^{\frac14}\mathsf{L}^2\hat{\tau}^{-a_{i,k}}\tau^{-a_{i,k}}\bar{\tau}^{-b_{i,k}}\sum_{r=2}^{2i-1+k}\binom{2i+1+k}{r}\frac{r!}{\langle r\rangle^3}\frac{(2i+k-r+2)!}{\langle 2i+k+1-r\rangle^3}.$$

Hence, since $\sum_{r=0}^{2i+1+k}\langle r\rangle^{-3}\langle 2i+k+1-r\rangle^{-2}\langle i+k\rangle^3 \lesssim \langle i+k\rangle$ and $\tau = e^{-\frac{1}{2}s}\tilde{\tau}$ we get

$$\|II\|_{L^\infty_{\mathrm{w}}} \le C\langle i+k\rangle(\hat{\tau}\tilde{\tau})^{\frac{1}{4}}e^{-\frac{1}{8}s}\mathsf{L}^2\hat{\tau}^{-a_{i,k}}\tau^{-a_{i,k}}\tilde{\tau}^{-b_{i,k}}\frac{(2i+1+k)!}{\langle i+k\rangle^3}.$$

From the above bound and the linear estimate (5.9) we get

$$\left\|\int_{\mathsf{s}_0}^{\mathsf{s}_1} S_{i,k}(\mathsf{s},\mathsf{s}_1)(II)\,d\mathsf{s}\right\|_{L^\infty_{\mathrm{w}}}$$
$$\le C(\hat{\tau}\tilde{\tau})^{\frac{1}{4}}\mathsf{L}^2\hat{\tau}^{-a_{i,k}}\tilde{\tau}^{-b_{i,k}}\frac{(2i+1+k)!}{\langle i+k\rangle^3}\langle k+i\rangle\int_{\mathsf{s}_0}^{\mathsf{s}_1} p(\mathsf{s},\mathsf{s}_1)\tau^{-a_{i,k}}(\mathsf{s})e^{-\frac{1}{8}\mathsf{s}}\,d\mathsf{s}.$$

Using (5.64), for $\hat{\tau}$ small enough depending on L, K and T, from the above identity we obtain

$$\left\|\int_{\mathsf{s}_0}^{\mathsf{s}_1} S_{i,k}(\mathsf{s},\mathsf{s}_1)(II)\,d\mathsf{s}\right\|_{L^\infty_{\mathrm{w}}} \le \frac{\mathsf{L}}{8}\tilde{\tau}^{-b_{i,k}}\tau^{-a_{i,k}}(\mathsf{s}_1)\hat{\tau}^{-a_{i,k}}\frac{(2i+1+k)!}{\langle i+k\rangle^3}. \tag{5.66}$$

End of the proof. Summing the estimates (5.66), (5.66) and (5.66) shows (5.60).

Step 2. *The case $k=0$.* Note that $i \ge 2$ since $i+k \ge 2$. This case can be treated almost exactly the same way. We just point out the minor modifications.

The free evolution $\tilde{F}_{i,0}$ now satisfies the Dirichlet boundary condition at $z=-\pi$ because $F_{i,0}$ does. To estimate it, we use the linear estimate (5.9) and the initial datum estimate (5.37). The resulting bound is acceptable if L has been taken large enough depending solely on the universal constant $C>0$ in (5.9).

Next, the forcing terms I and II are treated in the same way. Note that for I the sum is taken only over $j \in \{1, i-1\}$ since the term corresponding to i is zero for $k=0$ from (5.4). The estimate (5.47) is still valid in this case, and so I is estimated in the same way. There are no changes to make to treat II. This concludes the proof of the proposition. ■

6. Analyticity in the transverse variable close to the axis

Here we show that solutions $\vec{\xi} = (\xi_i)_{i\ge 0}$ to system (1.11), rewritten as

$$\begin{cases} \partial_t\xi_i = \partial_{yy}\xi_i + H_i(\vec{\xi},\vec{\xi}) + J_i(\vec{\xi},\vec{\xi}), & \\ \xi_i(0,y) = \xi_i^0(y), & i\in\mathbb{N},\ y\in[0,\infty),\ t>0, \\ \xi_i(t,0) = 0, & \end{cases} \tag{6.1}$$

with

$$H_i(\vec{\xi},\vec{\xi}') = -\sum_{j=0}^{i}\binom{2i+1}{2j+1}\xi_j\xi'_{i-j},\quad J_i(\vec{\xi},\vec{\xi}') = \sum_{j=0}^{i}\binom{2i+1}{2j}(\partial_y^{-1}\xi_j)\partial_y\xi'_{i-j},$$

are instantaneously regularised for $t>0$ and become analytic in y, up to the boundary $y=0$. Solutions with only bounded initial data will be understood in an integral sense, and will be classical solutions for $t > 0$. Indeed, there is a representation formula for solutions to

$$\begin{cases} \partial_t \phi = \partial_{yy}\phi, \\ \phi(0, y) = \phi^0(y), \qquad y \in [0, \infty),\ t > 0. \\ \phi(t, 0) = 0, \end{cases} \tag{6.2}$$

Given a real valued function f on $[0, \infty)$, let $\bar{f}$ denote its extension to $\mathbb{R}$ by odd symmetry:

$$\bar{f}(y) = \begin{cases} f(y) & \text{for } y \geq 0, \\ -f(-y) & \text{for } y < 0. \end{cases} \tag{6.3}$$

Then the solution $\phi(t) = S(t)\phi^0$ to (6.2) is given by (with K_t being defined in (2.4))

$$(S(t)\phi^0)(y) = \int_{-\infty}^{\infty} K_t(y - \tilde{y})\bar{\phi}^0(\tilde{y})\, d\tilde{y}, \quad y > 0. \tag{6.4}$$

We shall therefore look for solutions to (6.1) in the following integral sense, using Duhamel's formula:

$$\xi_i(t, y) = S(t)\xi_i^0 + \int_0^t S(t - t')(H_i(\vec{\xi}(t'), \vec{\xi}(t')) + J_i(\vec{\xi}(t'), \vec{\xi}(t')))\, dt'. \tag{6.5}$$

Throughout this section, ω denotes the weight

$$\omega(y) = \langle y \rangle^{-2}$$

and we introduce the weighted L^∞ spaces for $\phi : [0, \infty) \to \mathbb{R}$ or $\phi : \mathbb{R} \to \mathbb{R}$ respectively:

$$\|\phi\|_{L^\infty_\omega} = \sup_{y \geq 0} \frac{|\phi(y)|}{\omega(y)} \quad \text{or} \quad \|\phi\|_{L^\infty_\omega} = \sup_{y \in \mathbb{R}} \frac{|\phi(y)|}{\omega(y)}.$$

For $\tilde{\tau} > 0$, we introduce the weighted (in y) analytic space (in x, recalling that ξ_i stands for the trace of $\partial_x^{2i+1} u$ on the axis) with the norm

$$\|\vec{\xi}^0\|_{X^0} = \sup_{i \in \mathbb{N}} \frac{\tilde{\tau}^{2i+1}}{(2i+1)!} \|\xi_i^0\|_{L^\infty_\omega}.$$

The main result of this section is the following proposition.

Proposition 6.1. *Let $\tilde{\tau} > 0$ and assume $\|\vec{\xi}^0\|_{X^0} < \infty$. Then there exist $T_0 > 0$ and a solution $\vec{\xi}$ to (6.1) on $[0, T_0]$ in the sense of (6.5) such that for each i, $\xi_i \in C([0, T_0], L^\infty_\omega)$. Moreover, we have:*

(i) (Immediate regularisation up to the boundary) *For each i, $\xi_i \in C^\infty((0, T_0] \times [0, \infty))$, and $\vec{\xi}$ is a classical solution to (6.1) on $(0, T_0] \times [0, \infty)$.*

(ii) (Analytic bounds) *There exist $C, \tau > 0$ such that for all $i \geq 0$ and $(t, y) \in [0, T_0] \times [0, \infty)$,*

$$|\xi_i(t, y)| \leq C\tau^{-2i-1}(2i+1)!\langle y \rangle^{-2}. \tag{6.6}$$

For each $\tilde{T} \in (0, T_0)$, there exist $\bar{C}, \bar{\tau} > 0$ such that for all $i, n \geq 0$ and $(t, y) \in [\tilde{T}, T_0] \times [0, \infty)$,

$$|\partial_y^n \xi_i(t, y)| \leq C \bar{\tau}^{-2i-1-n} (2i + n + 1)! \langle y \rangle^{-2}. \tag{6.7}$$

Proof. This is a direct consequence of Lemmas 6.6 and 6.7. ∎

Remark 6.2. We believe our proof of Proposition 6.1 could be adapted to show instantaneous analytic (in y) regularisation for solutions to the Prandtl system (1.1), for data that are everywhere x-analytic, and without the oddness-in-x assumption.

To simplify notation, from now on and throughout this section, in the estimates we will use quantities of the form $(2i + n)!$ instead of $(2i + n + 1)!$, and τ^i instead of τ^{2i+1}. These are equivalent, up to changing certain constants by a fixed factor, which is harmless for the analysis. We write T instead of T_0 for convenience, so T here is not the blow-up time.

6.1. *Strategy of the proof of Proposition 6.1*

For small times, we approximate the solution to (6.1) by the linear solution to (6.8), showing that it undergoes a parabolic regularisation like the linear solution does. We proceed as follows.

- We construct the solution through a Picard approximation scheme (6.32). At each iterative step, the scheme preserves C^∞ differentiability in t but not necessarily the C^∞ differentiability in y due to boundary effects. That is why we first obtain the C^∞ regularity in time.
- This C^∞ regularity in t is measured in Gevrey-2 spaces. Indeed, first the system of homogeneous linear heat equations (6.8) regularises the initial data, making it analytical in time, and so Gevrey-α for all $\alpha \geq 1$ (see Lemma 6.3). Second, for the inhomogeneous linear system (6.9) with a source term that has analyticity radius $\sqrt{t}$ (singular at initial time), we show Gevrey-2 regularity (see Lemma 6.4).
- Once a Gevrey-2 in t solution is obtained, we get its analyticity in y in Lemma 6.7 by elliptic regularity techniques applied to equation (6.1).

6.2. *Regularisation for the system of homogeneous heat equations*

Our strategy is to approximate, for small times $t > 0$, solutions to (6.1) by solutions to the system of linear heat equations

$$\begin{cases} \partial_t \xi_i = \partial_{yy} \xi_i, \\ \xi_i(0, y) = \xi_i^0(y), \qquad i \geq 0, \ y \in [0, \infty), \ t \in (0, T]. \\ \xi_i(t, 0) = 0, \end{cases} \tag{6.8}$$

Standard regularisation estimates for (6.8) rely on the above formula and on the standard heat kernel estimates given in Lemma C.1 in Appendix C.

Lemma 6.3 (Estimates for the system of homogeneous heat equations). *Let τ_0 be given by Lemma C.1. There exists $C > 0$ such that for each $\tilde{\tau} > 0$, for $0 < \tau < \min(\tilde{\tau}^2/2, \tau_0)$, given $\vec{\xi}^0$ satisfying $\|\vec{\xi}^0\|_{X^0} < \infty$, the solution $\vec{\xi} = S(t)\vec{\xi}^0$ to (6.8) satisfies, for all $i \in \mathbb{N}$, $t \in (0, 1]$ and $y \in [0, \infty)$,*

$$|\partial_t^k \xi_i(t, y)| + \sqrt{t}\,|\partial_t^k \partial_y \xi_i(t, y)| \le C\omega(y)(2i + k)!t^{-k}\tau^{-i-k}\|\vec{\xi}^0\|_{X^0}.$$

Proof. Recall from (6.4) that $\xi_i(t) = K_t * \bar{\xi}_i^0$ where $\bar{\xi}_i^0$ is defined by (6.3). Differentiating, using (C.2), then (C.1), and then $a!b! \le (a + b)!$, we obtain, for any $i, k \in \mathbb{N}$, $m = 0, 1$, $t \in (0, 1]$ and $y \ge 0$,

$$\begin{aligned}
|\partial_t^k \partial_y^m \xi_i(t, y)| &\le Ck!t^{-k-\frac{m}{2}}\tau_0^{-k} \int_{\tilde{y}\in\mathbb{R}} |\bar{\xi}_0^i(\tilde{y})| K_{\kappa t}(y - \tilde{y})\, d\tilde{y} \\
&\le Ck!t^{-k-\frac{m}{2}}\tau_0^{-k}(2i + 1)!\tilde{\tau}^{-2i}\omega(y)\|\vec{\xi}^0\|_{X^0} \\
&\le C\omega(y)t^{-k-\frac{m}{2}}\tau^{-i-k}(2i + k)!\|\vec{\xi}^0\|_{X^0}(i + 1)\frac{\tau^{i+k}}{\tau_0^k\tilde{\tau}^{2i}}.
\end{aligned}$$

This proves the lemma, because $(i + 1)\frac{\tau^{i+k}}{\tau_0^k\tilde{\tau}^{2i}}$ is uniformly bounded since $0 < \tau < \min(\tilde{\tau}^2/2, \tau_0)$. ∎

6.3. *Estimates for the system of inhomogeneous heat equations*

Nonlinear terms in (6.1) will be considered as forcing terms for a linear inhomogeneous heat equation. That is why here we study solutions to

$$\begin{cases} \phi_t = \partial_{yy}\phi + f, \\ \phi(0, y) = 0, \\ \phi(t, 0) = 0, \end{cases} \qquad y \in [0, \infty),\ t \in (0, T]. \tag{6.9}$$

We will formulate estimates in particular function spaces, in order to be able to apply them to (6.1) later on. Namely we introduce τ defined by

$$\frac{\partial_t \tau}{\tau} = -\frac{1}{\sqrt{T}\sqrt{t}} \quad (\text{i.e. } \tau(t) = \tau(0)e^{-2\sqrt{t/T}}). \tag{6.10}$$

For $i \in \mathbb{N}$ and a Sobolev correction exponent α (that we will take equal[6] to 2), define the coefficients

$$\Lambda_{i,k,\alpha}(t) = t^{-k}\tau^{-i-k}(2i + 2k)!\langle i + k\rangle^{-\alpha}.$$

For measurable functions u such that, for each $y \in [0, \infty)$, the function $(0, T] \ni t \mapsto u(t, y)$ is C^∞, we introduce the Gevrey-2 in time norms:

$$\|\phi\|_{X^i_{T,\alpha}([0,\infty))} = \sup_{t\in(0,T],\,k,i\in\mathbb{N}} \Lambda^{-1}_{i,k,\alpha}(t)\|\partial_t^k \phi(t)\|_{L^\infty_\omega}, \tag{6.11}$$

[6] The exact value 2 is not relevant. It only needs to be large enough for the inequality (D.1) to hold true.

$$\|\phi\|_{Y^i_{T,\alpha}([0,\infty))} = \sup_{t\in(0,T],\,k,i\in\mathbb{N}} t^{\frac12}\Lambda^{-1}_{i,k,\alpha}(t)\|\partial_t^k\phi(t)\|_{L^\infty_\omega}, \tag{6.12}$$

$$\|\phi\|_{Z^i_{T,2}([0,\infty))} = \|\phi\|_{X^i_{T,2}([0,\infty))} + \|\partial_y\phi\|_{Y^i_{T,1}([0,\infty))}. \tag{6.13}$$

To simplify notation, we write

$$\Lambda_{i,k} = \Lambda_{i,k,2}.$$

Lemma 6.4 (Estimates for the inhomogeneous heat equation). *There exist $C,\tau^*>0$ such that for any $0<T\le 1$, τ satisfying (6.10) with $0<\tau(0)\le\tau^*$, the following holds true. Let Θ be the mapping which to f associates the solution $\phi=\Theta(f)$ to (6.9). Then Θ satisfies the continuity estimate*

$$\|\Theta(f)\|_{Z^i_{T,2}} \le C\sqrt{T}\,\|f\|_{Y^i_{T,1}}. \tag{6.14}$$

Proof. Pick $m\in\{0,1\}$. The solution to (6.9) is given by the formula

$$u(t) = \int_0^t K_{t-t'} * h(t')\,dt', \quad h(t) = \bar f(t). \tag{6.15}$$

Recall that if $\varphi\in L^\infty_\omega$, then $\bar\varphi\in L^\infty_\omega$ with the same norm. For $k\in\mathbb{N}$, we let $\theta_k = 1-\frac{1}{2+k}$ and decompose

$$u(t) = \underbrace{\int_0^{\theta_k t} K_{t-t'} * h(t')\,dt'}_{=u_1} + \underbrace{\int_{\theta_k t}^t K_{t-t'} * h(t')\,dt'}_{=u_2}. \tag{6.16}$$

Step 1. *Estimates for u_1.* By a direct computation,

$$\partial_t^k\partial_y^m u_1 = \theta_k\sum_{p=0}^{k-1}\partial_t^{k-1-p}[\partial_y^m\partial_t^p K_{(1-\theta_k)t} * h(\theta_k t)] + \int_0^{\theta_k t}(\partial_y^m\partial_t^k K_{t-t'}) * h(t')\,dt'. \tag{6.17}$$

For the first term in (6.17), we can assume $k\ge 1$, and using the Leibniz identity we get

$$\partial_t^{k-1-p}[\partial_y^m\partial_t^p K_{(1-\theta_k)t} * h(\theta_k t)] = \sum_{l=0}^{k-1-p}\binom{k-1-p}{l}(\partial_t^l[\partial_y^m\partial_t^p K_{(1-\theta_k)t}] * \partial_t^{k-1-p-l}[h(\theta_k t)]). \tag{6.18}$$

Using (C.2) and $(1-\theta_k)^{-p-m/2} = (2+k)^{p+m/2} \le Ck^{p+m/2}$ with C independent of k and p as $p\le k-1$, we find that for $t\in(0,T]$,

$$\begin{aligned}|\partial_t^l[\partial_y^m\partial_t^p K_{(1-\theta_k)t}]| &= (1-\theta_k)^l|(\partial_y^m\partial_t^{l+p}K)_{(1-\theta_k)t}|\\ &\le C\tau_0^{-l-p}t^{-l-p-\frac m2}(1-\theta_k)^{-p-\frac m2}(l+p)!K_{(1-\theta_k)\kappa t}\\ &\le C\tau_0^{-l-p}t^{-l-p-\frac m2}k^{p+\frac m2}(l+p)!K_{(1-\theta_k)\kappa t}.\end{aligned} \tag{6.19}$$

Using the definition of $Y^i_{T,1}$, $\theta_k^{-1/2} \lesssim 1$ and then the inequality $(2i+2k-2-2p-2l)! \lesssim (2i+2k-2p-2l)!\langle i+k-p-l\rangle^{-2}$, we get

$$\begin{aligned}
|\partial_t^{k-1-p-l}[h(\theta_k t)]| &= \theta_k^{k-1-p-l}|(\partial_t^{k-1-p-l}h)(\theta_k t)| \\
&\le \theta_k^{-\frac12} t^{-\frac12}\Lambda_{i,k-1-p-l,1}(t)\|f\|_{Y^i_{T,1}}\omega \\
&\lesssim (2i+2k-2p-2l)!\langle i+k-p-l\rangle^{-3} t^{-k+p+l+\frac12}\tau^{-i-k+p+l+1}\|f\|_{Y^i_{T,1}}\omega.
\end{aligned} \tag{6.20}$$

Combining (C.1), (6.19) and (6.20), choosing $\tau(0) \le \tau_0/2$ so that $\tau \le \tau_0/2$ from (6.10), we obtain

$$\begin{aligned}
&\|\partial_t^l[\partial_y^m\partial_t^p K_{(1-\theta_k)t}] * \partial_t^{k-1-p-l}[h(\theta_k t)]\|_{L^\infty_\omega} \\
&\le C\sqrt{T}\,\|f\|_{Y^i_{T,1}}\tau^{-i-k+1}t^{-k-\frac{m}{2}}\langle i+k-p-l\rangle^{-3}k^{\frac{m}{2}}2^{-l-p}(2i+2k-2p-2l)!(l+p)!k^p.
\end{aligned}$$

Now using $a!b! \le (a+b)!$, $(l!)^{-1} \le 1$, and the fact that for each $p \le k-1$ and $l \le k-1-p$ one has $2i+2k-p-l+1 \ge k-1-p$ and $2i+2k-p+1 \ge k$, we estimate

$$\begin{aligned}
&\binom{k-1-p}{l}(2i+2k-2p-2l)!(l+p)!k^p \\
&\qquad\le \frac{(k-1-p)\dots(k-p-l)}{l!}(2i+2k-p-l)!k^p \\
&\qquad\le \frac{k-1-p}{2i+2k-p}\cdots\frac{k-p-l}{2i+2k-p-l+1}(2i+2k-p)!k^p \\
&\qquad\le (2i+2k-p)!k^p \le (2i+2k)!.
\end{aligned}$$

Inserting the above two inequalities in (6.18) we obtain

$$\begin{aligned}
&\|\partial_t^{k-1-p}[\partial_y^m\partial_t^p K_{(1-\theta_k)t} * h(\theta_k t)]\|_{L^\infty_\omega} \\
&\quad\le C\sqrt{T}\,\|f\|_{Y^i_{T,1}}\tau^{-i-k+1}t^{-k-\frac{m}{2}}(2i+2k)!k^{\frac{m}{2}}\sum_{p=0}^{k-1}\sum_{l=0}^{k-1-p}\langle i+k-p-l\rangle^{-3}2^{-l-p} \\
&\quad\le C\sqrt{T}\,\langle i+k\rangle^{-\frac12}\|f\|_{Y^i_{T,1}}\tau t^{-\frac{m}{2}}\Lambda_{i,k},
\end{aligned} \tag{6.21}$$

where we use $\sum_{p=0}^{k-1}\sum_{l=0}^{k-1-p}\langle i+k-p-l\rangle^{-3}2^{-l-p} \le C\langle i+k\rangle^{-3}$ for C independent of i,k.

For the second term in (6.17), using (C.2) and (C.1), and then (6.10), we estimate

$$\begin{aligned}
\|\partial_t^k\partial_y^m K_{t-t'} * h(t')\|_{L^\infty_\omega} &\le Ck!(t-t')^{-k-\frac{m}{2}}\tau_0^{-k}\|f\|_{L^\infty_\omega} \\
&\le Ck!(2i)!\langle i\rangle^{-1}\left(\frac{\tau(0)}{\tau_0}\right)^k\|f\|_{Y^i_{T,1}}(t-t')^{-k-\frac{m}{2}}(t')^{-\frac12}\tau^{-i-k}.
\end{aligned}$$

We estimate the following time integral if $i=k=0$:

$$\int_0^{\theta_k t}(t-t')^{-k-\frac{m}{2}}(t')^{-\frac12}\tau^{-i-k}(t')\,dt'=\int_0^{\theta_k t}(t-t')^{-\frac{m}{2}}(t')^{-\frac12}\,dt'\lesssim\sqrt{T}\,t^{-\frac{m}{2}},$$

and using (6.10) if $i+k\geq 1$, we get

$$\begin{aligned}\int_0^{\theta_k t}(t-t')^{-k-\frac{m}{2}}(t')^{-\frac12}\tau^{-i-k}(t')\,dt'&\leq(t-t\theta_k)^{-k-\frac{m}{2}}\int_0^t(t')^{-\frac12}\tau^{-i-k}(t')\,dt'\\&=t^{-k-\frac{m}{2}}(1-\theta_k)^{-k-\frac{m}{2}}\frac{\sqrt{T}}{i+k}\int_0^t\partial_{t'}(\tau^{-i-k})(t')\,dt'\\&\leq t^{-k-\frac{m}{2}}(1-\theta_k)^{-k-\frac{m}{2}}\frac{\sqrt{T}}{i+k}\tau^{-i-k}.\end{aligned}$$

Hence, using $\langle i\rangle^{-1}\langle i+k\rangle^{-1}(\frac{\tau(0)}{\tau_0})^k(1-\theta_k)^{-\frac12}\lesssim(\frac{\tau(0)}{\tau_0})^{\frac{k}{2}}\langle i+k\rangle^{-2}$ if $\tau(0)$ is small enough, we get

$$\begin{aligned}&\left\|\int_0^{\theta_k t}\partial_y^m\partial_t^kK_{t-t'}*h(t')\,dt'\right\|_{L^\infty_\omega}\\&\qquad\leq Ck!(2i)!\langle i+k\rangle^{-2}t^{-k-\frac{m}{2}}(1-\theta_k)^{-k}\tau^{-i-k}\left(\frac{\tau(0)}{\tau_0}\right)^{\frac{k}{2}}\sqrt{T}\,\|f\|_{Y^i_{T,1}}.\end{aligned}$$

Stirling's formula yields $\frac{k!}{(2k)!}\sim\frac{1}{\sqrt{2}}(\frac{e}{4})^kk^{-k}$, so that since $1-\theta_k=\frac{1}{k+2}$,

$$k!(1-\theta_k)^{-k}\sim(2k)!\frac{1}{\sqrt{2}}\left(\frac{e}{4}\right)^k\left(\frac{k+2}{k}\right)^k\sim(2k)!\frac{e^2}{\sqrt{2}}\left(\frac{e}{4}\right)^k\quad\text{as } k\to\infty,$$

and hence, for $\tau(0)/\tau_0$ small enough, using $(2k)!(2i)!\leq(2i+2k)!$ we get

$$\left\|\int_0^{\theta_k t}\partial_y^m\partial_t^kK_{t-t'}*h(t')\,dt'v\right\|_{L^\infty_\omega}\leq C\sqrt{T}\,\|f\|_{Y^i_{T,1}}t^{-\frac{m}{2}}\Lambda_{i,k}. \tag{6.22}$$

Combining (6.21) and (6.22) yields

$$\|u_1\|_{Z^i_{T,2}}\leq C\sqrt{T}\,\|f\|_{Y^i_{T,1}}. \tag{6.23}$$

Step 2. *Estimate for u_2.* We differentiate with respect to time and then integrate by parts to find

$$\begin{aligned}\partial_tu_2&=h(t)-\theta_kK_{(1-\theta_k)t}*h(\theta_kt)+\int_{\theta_kt}^t\partial_t[K_{t-t'}]*h(t')\,dt'\\&=h(t)-\theta_kK_{(1-\theta_k)t}*h(\theta_kt)-\int_{\theta_kt}^t\partial_{t'}[K_{t-t'}]*h(t')\,dt'\\&=(1-\theta_k)K_{(1-\theta_k)t}*h(\theta_kt)+\int_{\theta_kt}^tK_{t-t'}*(\partial_th)(t')\,dt'.\end{aligned}$$

Iterating the above computation, we find the following identity for all $k \in \mathbb{N}$:

$$\partial_t^k \partial_y^m u_2 = \sum_{p=0}^{k-1} \partial_t^{k-1-p}[(1-\theta_k)\partial_y^m K_{(1-\theta_k)t} * (\partial_t^p h)(\theta_k t)] + \int_{\theta_k t}^{t} \partial_y^m K_{t-t'} * (\partial_t^k h)(t')\, dt'. \tag{6.24}$$

The first term in (6.24) is estimated as the first term in (6.17) in Step 1. Namely, using Leibniz, we obtain

$$\partial_t^{k-1-p}[\partial_y^m K_{(1-\theta_k)t} * (\partial_t^p h)(\theta_k t)] = \sum_{l=0}^{k-p-1} \binom{k-p-1}{l} \partial_t^l [\partial_y^m K_{(1-\theta_k)t}] * \partial_t^{k-1-p-l}[(\partial_t^p h)(\theta_k t)]. \tag{6.25}$$

Using (6.19) with $p = 0$ one has

$$|\partial_t^l[\partial_y^m K_{(1-\theta_k)t}]| \le C\tau_0^{-l} t^{-l-\frac{m}{2}} k^{\frac{m}{2}} l! K_{(1-\theta_k)\kappa t}.$$

Using $\partial_t^{k-1-p-l}[(\partial_t^p h)(\theta_k t)] = \theta_k^{-p} \partial_t^{k-1-l}[h(\theta_k t)]$, then (6.20) (with $p + l$ replaced by l), and $\theta_k^{-p} \le \theta_k^{-k} = (1 - 1/(k+2))^{-k} \lesssim 1$, one gets

$$|\partial_t^{k-1-p-l}[(\partial_t^p h)(\theta_k t)]| \le C(2i+2k-2l)! \langle i+k-l\rangle^{-3} t^{-k+l+\frac{1}{2}} \tau^{-i-k+l+1} \|f\|_{Y^i_{T,1}} \omega.$$

Choosing $\tau \le \tau_0/2$ and using (C.1), we see that the two inequalities above give

$$\|\partial_t^l[\partial_y^m K_{(1-\theta_k)t}] * \partial_t^{k-1-p-l}[(\partial_t^p h)(\theta_k t)]\|_{L^\infty_\omega} \le C\sqrt{T}\, \|f\|_{Y^i_{T,1}} \tau^{-i-k+1} t^{-k-\frac{m}{2}} \langle i+k-l\rangle^{-3} k^{\frac{m}{2}} 2^{-l} (2i+2k-2l)! l!. \tag{6.26}$$

Using $a!b! \le (a+b)!$, $(l!)^{-1} \le 1$, and the fact that for $p \le k-1$ and $l \le k-1-p$ one has $k-1-p \le 2i+2k-l$, it follows that

$$\binom{k-1-p}{l} (2i+2k-2l)! l! \le \frac{(k-1-p)\cdots(k-p-l)}{l!} (2i+2k-l)! \le \frac{k-1-p}{2i+2k} \cdots \frac{k-p-l}{2i+2k-l+1} (2i+2k)! \le (2i+2k)!. \tag{6.27}$$

Inserting (6.26) and (6.27) in (6.25), using $\sum_{p=0}^{k-1} \sum_{l=0}^{k-1-p} \langle i+k-l\rangle^{-3} 2^{-l} \lesssim k\langle i+k\rangle^{-3}$, $1-\theta_k \le k^{-1}$ and $k^{m/2} \le \langle i+k\rangle^{1/2}$, one finds that the first term in (6.24) satisfies

$$\left\| \sum_{p=0}^{k-1} \partial_t^{k-1-l}[(1-\theta_k)\partial_y^m K_{(1-\theta_k)t} * (\partial_t^p h)(\theta_k t)] \right\|_{L^\infty_\omega} \le C\tau\sqrt{T}\, \|f\|_{Y^i_{T,1}} \langle k+i\rangle^{-\frac{1}{2}} t^{-\frac{m}{2}} \Lambda_{i,k}. \tag{6.28}$$

For the second term in (6.24) we first estimate, by (C.1),

$$\begin{aligned}\|\partial_y^m K_{t-t'} * (\partial_t^k h)(t')\|_{L^\infty_\omega} &\lesssim (t-t')^{-\frac{m}{2}} \|(\partial_t^k h)(t')\|_{L^\infty_\omega} \\ &\lesssim (t-t')^{-\frac{m}{2}} \tau^{-i-k}(2k+2i)!\langle i+k\rangle^{-1}(t')^{-k-\frac12}\|f\|_{Y^i_{T,1}},\end{aligned}$$

and hence, since $\theta_k^{-k} = (1-\frac{1}{2+k})^{-k} \to e$ as $k\to\infty$,

$$\begin{aligned}&\left\|\int_{\theta_k t}^t \partial_y^m K_{t-t'} * (\partial_t^k h)(t')\,dt'\right\|_{L^\infty_\omega} \\ &\qquad \le C(2k+2i)!\langle k+i\rangle^{-1}\|f\|_{Y^i_{T,1}} \int_{\theta_k t}^t (t-t')^{-\frac{m}{2}}(t')^{-k-\frac12}\tau^{-i-k}(t')\,dt' \\ &\qquad \le C(2k+2i)!\langle k+i\rangle^{-1}\|f\|_{Y^i_{T,1}} t^{-k} \int_{\theta_k t}^t (t-t')^{-\frac{m}{2}} t^{-\frac12}\tau^{-i-k}(t')\,dt'.\end{aligned}$$

We now estimate the above integral. For $i=k=0$, we have

$$\int_{\theta_k t}^t (t-t')^{-m/2}(t')^{-\frac12}\tau^{-i-k}(t')\,dt' \lesssim \sqrt{T}\, t^{-m/2}.$$

For $i+k\ge 1$, for $m=0$, using (6.10) we get

$$\int_{\theta_k t}^t (t')^{-\frac12}\tau^{-i-k}(t')\,dt' = \frac{\sqrt{T}}{i+k}\int_{\theta_k t}^t \partial_{t'}(\tau^{-i-k})(t')\,dt' \le \frac{\sqrt{T}}{i+k}\tau^{i+k}(t),$$

while for $m=1$, using $\tau(t')\ge\tau(t)$ for $t'\le t$ we get

$$\begin{aligned}\int_{\theta_k t}^t (t-t')^{-\frac12}(t')^{-\frac12}\tau^{-i-k}(t')\,dt' &\le \tau^{-i-k}(t)\int_{\theta_k t}^t (t-t')^{-\frac12}(t')^{-\frac12}\,dt' \\ &\le \tau^{-i-k}\int_0^1 (1-\sigma)^{-\frac12}\sigma^{-\frac12}\,d\sigma \lesssim \tau^{-i-k}.\end{aligned}$$

Combining the above four inequalities, one ends up with

$$\left\|\int_{\theta_k t}^t \partial_y^m K_{t-t'} * (\partial_t^k h)(t')\,dt'\right\|_{L^\infty_\omega} \le C\sqrt{T}\, t^{-\frac{m}{2}}\langle i+k\rangle^m \Lambda_{i,k}\|f\|_{Y^i_{T,1}}. \tag{6.29}$$

Therefore, summing (6.28) and (6.29) we find that

$$\|u_2\|_{Z^i_{T,2}} \le C\sqrt{T}\,\|f\|_{Y^i_{T,1}}. \tag{6.30}$$

Conclusion. Inserting (6.23) and (6.30) in (6.16) shows the bound (6.14). ■

6.4. Bilinear estimates

We now estimate the quadratic terms in (6.1). We introduce the $X^i_{T,2}$ and $Z^i_{T,2}$ based (defined by (6.11) and (6.13)) vector spaces with the norms

$$\|\vec{\xi}\|_{X_{T,2}([0,\infty))} = \sup_{i\in\mathbb{N}}\|\xi_i\|_{X^i_{T,2}([0,\infty))}, \qquad \|\vec{\xi}\|_{Z_{T,2}([0,\infty))} = \sup_{i\in\mathbb{N}}\|\xi_i\|_{Z^i_{T,2}([0,\infty))}.$$

The following lemma states that H_i and J_i both loose a derivative in a combinatorial sense (an $\langle i+k\rangle$ factor). Moreover, as ∂_y derivatives are regularised each with a $t^{-1/2}$ factor, J_i looses an additional $t^{-1/2}$ factor.

Lemma 6.5 (Bilinear estimates). *For some $C>0$ independent of τ and i,*

$$\|H_i(\vec{\xi},\vec{\xi}')\|_{X^i_{T,1}} \le C\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}}, \quad \|J_i(\vec{\xi},\vec{\xi}')\|_{Y^i_{T,1}} \le C\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{Z_{T,2}}. \tag{6.31}$$

Proof. We write $H_i = H_i(\vec{\xi},\vec{\xi}')$ and $J_i = J_i(\vec{\xi},\vec{\xi}')$ for simplicity. Then, by the Leibniz rule, for $t\in(0,T]$,

$$|\partial_t^k H_i| \le \sum_{j=0}^{i}\sum_{l=0}^{k}\binom{2i+1}{2j+1}\binom{k}{l}|\partial_t^l\xi_j\,\partial_t^{k-l}\xi'_{i-j}|.$$

We introduce $r=2j+2l$ and using the definition of the $X_{T,2}$ norms, we bound

$$\begin{aligned}\|\partial_t^l\xi_j\,\partial_t^{k-l}\xi'_{i-j}\|_{L^\infty_\omega} &\le \Lambda_{j,l}\|\vec{\xi}\|_{X_{T,2}}\Lambda_{i-j,k-l}\|\vec{\xi}'\|_{X_{T,2}}\\ &\lesssim \|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}}\tau^{-i-k}t^{-k}r!(2i+2k-r)!\langle r\rangle^{-2}\langle 2i+2k-r\rangle^{-2}.\end{aligned}$$

Therefore, using $\binom{k}{l}\le\binom{2k}{2l}$ and (D.2) with $A_1=2i+1$, $A_2=2k$ and $r_1=r+1$, we get

$$\begin{aligned}&\|\partial_t^k H_i\|_{L^\infty_\omega}\\ &\lesssim t^{-k}\tau^{-i-k}\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}}\sum_{j=0}^{i}\sum_{l=0}^{k}\binom{2i+1}{2j+1}\binom{k}{l}r!(2i+2k-r)!\langle r\rangle^{-2}\langle 2i+2k-r\rangle^{-2}\\ &\lesssim t^{-k}\tau^{-i-k}\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}}\sum_{r=0}^{2i+2k+1}\binom{2i+2k+1}{r+1}r!(2i+2k-r)!\langle r\rangle^{-2}\langle 2i+2k-r\rangle^{-2}\\ &\lesssim t^{-k}\tau^{-i-k}(2i+2k+1)!\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}}\sum_{r=0}^{2i+2k+1}\langle r\rangle^{-3}\langle 2i+2k-r\rangle^{-2}\\ &\lesssim t^{-k}\tau^{-i-k}(2i+2k)!\langle i+k\rangle^{-1}\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}} \lesssim \Lambda_{i,k}\langle i+k\rangle\|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{X_{T,2}},\end{aligned}$$

where we have used (D.1) with $K=2i+2k+1$. This precisely implies the first inequality in (6.31).

To prove the second inequality in (6.31), we first write

$$\partial_t^k J_i = \sum_{l=0}^{k}\sum_{j=0}^{i}\binom{k}{l}\binom{2i+1}{2j}(\partial_y^{-1}\partial_t^l\xi_j)\partial_y\partial_t^{k-l}\xi'_{i-j}.$$

Using the definition of the $X_{T,2}$ and $Z_{T,2}$ norms, and $\int_0^\infty\omega(y)\,dy<\infty$, and introducing $r=2l+2j$, we get

$$\begin{aligned}\|(\partial_y^{-1}\partial_t^l\xi_j)\,\partial_y\partial_t^{k-l}\xi'_{i-j}\|_{L^\infty_\omega} &\lesssim \Lambda_{j,l}\|\vec{\xi}\|_{X_{T,2}}\Lambda_{i-j,k-l}\langle i-j+k-l\rangle t^{-\frac12}\|\vec{\xi}'\|_{Z_{T,2}}\\ &\lesssim \|\vec{\xi}\|_{X_{T,2}}\|\vec{\xi}'\|_{Z_{T,2}}\tau^{-i-k}t^{-k-\frac12}r!(2i+2k-r)!\langle r\rangle^{-2}\langle 2i+2k-r\rangle^{-1},\end{aligned}$$

so that, since $\binom{k}{l} \le \binom{2k}{2l}$,

$$\|\partial_t^k J_i\|_{L^\infty_\omega} \lesssim \|\vec{\xi}\|_{X_{T,2}} \|\vec{\xi}'\|_{Z_{T,2}} \tau^{-i-k} t^{-k-\frac{1}{2}} \cdot \sum_{l=0}^{k} \sum_{j=0}^{i} \binom{2k}{2l} \binom{2i+1}{2j} r!(2i+2k-r)! \langle r\rangle^{-2} \langle 2i+2k-r\rangle^{-1}.$$

Using (D.2) with $(A_1, A_2, r_1) = (2i+1, 2k, r)$, and then (D.1) with $K = 2i+2k+1$, we obtain

$$\begin{aligned}
&\sum_{l=0}^{k} \sum_{j=0}^{i} \binom{2k}{2l} \binom{2i+1}{2j} r!(2i+2k-r)! \langle r\rangle^{-2} \langle 2i+2k-r\rangle^{-1} \\
&\quad \le \sum_{r=0}^{2i+2k} \binom{2i+2k+1}{r} r!(2i+2k-r)! \langle r\rangle^{-2} \langle 2i+2k-r\rangle^{-1} \\
&\quad = \sum_{r=0}^{2i+2k} (2i+2k+1)!(2i+2k-r+1)^{-1} \langle r\rangle^{-2} \langle 2i+2k-r\rangle^{-1} \\
&\quad \lesssim (2i+2k)! \langle i+k\rangle^{-1}.
\end{aligned}$$

Combining the above two inequalities shows that

$$\|\partial_t^k J_i\|_{L^\infty_\omega} \le C \Lambda_{i,k} t^{-\frac{1}{2}} \langle i+k\rangle \|\vec{\xi}\|_{X_{T,2}} \|\vec{\xi}'\|_{Z_{T,2}},$$

which is precisely the second inequality in (6.31). ■

6.5. Obtaining Gevrey-2 in time regularity by the Picard iteration scheme

The lemma below shows analytic regularisation for solutions to (6.8).

Lemma 6.6. *For any $\tilde{\tau} > 0$, there exists $\tau(0) > 0$ such that for any $\vec{\xi}^0$ satisfying $\|\vec{\xi}^0\|_{X^0} < \infty$, there exists a $T > 0$ and a solution $\vec{\xi}$ to (6.1) in the sense of (6.5) such that $\xi_i \in C([0,T], L^\infty_\omega)$ for each i. Moreover,*

$$\|\vec{\xi}\|_{Z_{T,2}} < \infty.$$

Proof. Let $\vec{\zeta} = S(t)\vec{\xi}^0$. We look for a solution of the form $\vec{\xi} = \vec{\zeta} + \vec{\tilde{\xi}}$. We consider the mapping Φ which to $\vec{v} \in Z_{T,2}$ associates the unique solution $\vec{w} = \Phi(\vec{v})$ to

$$\begin{cases} \partial_t w_i = \partial_{yy} w_i + H_i(\vec{\zeta}+\vec{v}, \vec{\zeta}+\vec{v}) + J_i(\vec{\zeta}+\vec{v}, \vec{\zeta}+\vec{v}), \\ w_i(0,y) = 0, \qquad\qquad i \in \mathbb{N},\ y \in [0,\infty),\ t \in [0,T], \\ w_i(t,0) = 0, \end{cases} \tag{6.32}$$

By Lemma 6.3 and (6.31), for all $i \in \mathbb{N}$ we have

$$\|H_i(\vec{\zeta}+\vec{v},\vec{\zeta}+\vec{v})\|_{Y_{T,1}} \le \sqrt{T}\,\|H_i(\vec{\zeta}+\vec{v},\vec{\zeta}+\vec{v})\|_{X_{T,1}} \lesssim \sqrt{T}\,(\|\vec{\zeta}\|_{X_{T,2}}+\|\vec{v}\|_{X_{T,2}})^2 \lesssim \sqrt{T}\,(C+\|\vec{v}\|_{Z_{T,2}})^2$$

where C is independent of T, and similarly for $\vec{v}' \in Z_{T,2}$, since H_i is bilinear:

$$\begin{aligned}\|H_i(\vec{\zeta}+\vec{v},\vec{\zeta}+\vec{v}) - H_i(\vec{\zeta}+\vec{v}',\vec{\zeta}+\vec{v}')\|_{Y_{T,1}} &= \|H_i(\vec{\zeta}+\vec{v},\vec{v}-\vec{v}') + H_i(\vec{v}-\vec{v}',\vec{\zeta}+\vec{v}')\|_{Y_{T,1}} \\ &\lesssim \sqrt{T}\,\|\vec{v}-\vec{v}'\|_{Z_{T,2}}(C+\|\vec{v}\|_{Z_{T,2}}+\|\vec{v}'\|_{Z_{T,2}}).\end{aligned}$$

Similarly, using again Lemma 6.3 and (6.31), we have

$$\|J_i(\vec{\zeta}+\vec{v},\vec{\zeta}+\vec{v})\|_{Y_{T,1}} \lesssim (\|\vec{\zeta}\|_{Z_{T,2}}+\|\vec{v}\|_{Z_{T,2}})^2 \lesssim (C+\|\vec{v}\|_{Z_{T,2}})^2.$$

In addition, since J_i is bilinear,

$$\begin{aligned}\|J_i(\vec{\zeta}+\vec{v},\vec{\zeta}+\vec{v}) - J_i(\vec{\zeta}+\vec{v}',\vec{\zeta}+\vec{v}')\|_{Y_{T,1}} &= \|J_i(\vec{\zeta}+\vec{v},\vec{v}-\vec{v}') + J_i(\vec{v}-\vec{v}',\vec{\zeta}+\vec{v}')\|_{Y_{T,1}} \\ &\lesssim \|\vec{v}-\vec{v}'\|_{Z_{T,2}}(C+\|\vec{v}\|_{Z_{T,2}}+\|\vec{v}'\|_{Z_{T,2}}).\end{aligned}$$

Therefore, thanks to (6.14) we deduce from the above estimates that

$$\begin{aligned}\|\Phi(\vec{v})\|_{Z_{T,2}} &\lesssim \sqrt{T}\,(C+\|\vec{v}\|_{Z_{T,2}})^2, \\ \|\Phi(\vec{v})-\Phi(\vec{v}')\|_{Z_{T,2}} &\lesssim \sqrt{T}\,\|\vec{v}-\vec{v}'\|_{Z_{T,2}}(C+\|\vec{v}\|_{Z_{T,2}}+\|\vec{v}'\|_{Z_{T,2}}).\end{aligned}$$

Thus, there exists T small enough such that Φ is a contraction on the unit ball of $Z_{T,2}$. Hence Φ has a unique fixed point $\vec{\tilde{\zeta}}$ by the Banach fixed point theorem. Then $\vec{\xi} = \vec{\zeta} + \vec{\tilde{\zeta}}$ solves the system (6.1) on $(0, T]$, and belongs to $Z_{T,2}$. ∎

6.6. *Instantaneous analytic regularisation in the transverse variable*

Thanks to Lemma (6.6), for any $0 < t_0 < T$, the solution is Gevrey-2 in time on $[t_0, T]$, with a radius of analyticity that is now *bounded from below uniformly on* $[t_0, T]$. In this subsection, τ is thus independent of time. Analyticity in the y-variable is given by the following lemma.

Lemma 6.7. *Let $\tau > 0$ and assume that $\vec{\xi}$ is a smooth (in space and time) solution to (6.1) on $[t_0, T] \times [0, \infty)$ such that for all $k, i \in \mathbb{N}$ and $m = 0, 1$,*

$$\|\partial_t^k \partial_y^m \xi_i\|_{L^\infty([t_0,T],L^\infty_\omega)} \le C\tau^{-i-k}(2i+2k+m)!\langle i+k+m\rangle^{-2}. \tag{6.33}$$

Then there exists $\tau' > 0$ such that for all $m, i \in \mathbb{N}$,

$$\|\partial_y^m \xi_i\|_{L^\infty([t_0,T],L^\infty_\omega)} \le C \cdot (\tau')^{-i-m}(2i+m)!\langle i+k+m\rangle^{-2}.$$

Proof. To shorten notation, we shall write L^∞ for $L^\infty([t_0,T],L^\infty_\omega[0,\infty))$. In the proof, $\tilde C$ denotes a constant independent of the other parameters, whose value may change from one line to another. We prove the following bound by induction on $m\in\mathbb{N}$:

$$\|\partial_t^k\partial_y^n\xi_i\|_{L^\infty}\leq C\Lambda_{i,k,n}\quad \text{for all } 0\leq n\leq m \text{ and } i,k\in\mathbb{N}, \tag{6.34}$$

where

$$\Lambda_{i,k,m}=\tau^{-k-i}(\tau')^{-m}(2i+2k+m)!\langle i+k+m\rangle^{-2}.$$

Note that if $0<\tau'\leq 1$, then (6.34) is true for $m=1$ by (6.33). We now assume it is true for $m-1\geq 1$, and aim at proving it for m.

Step 1. *$m=2p$ is even.* By induction on p, using (6.1) we get

$$\partial_y^m\xi_i=\partial_t^p\xi_i-\sum_{q=0}^{p-1}\partial_t^q\partial_y^{2(p-1-q)}(H_i+J_i),$$

and hence, for all $k\in\mathbb{N}$,

$$\partial_t^k\partial_y^m\xi_i=\partial_t^{p+k}\xi_i-\sum_{q=0}^{p-1}\partial_t^{k+q}\partial_y^{2(p-1-q)}(H_i+J_i). \tag{6.35}$$

We bound the first term on the right-hand side of (6.35) using (6.33):

$$\|\partial_t^{p+k}\xi_i\|_{L^\infty}\leq C\tau^{-i-p-k}(2i+2p+2k)!\langle i+k+p\rangle^{-2}\leq \tilde C C\left(\frac{\tau'}{\sqrt{\tau}}\right)^m\Lambda_{i,k,m}. \tag{6.36}$$

For the second and third terms in (6.35), using the Leibniz formula we get

$$\partial_t^{k+q}\partial_y^{2(p-1-q)}H_i$$
$$=-\sum_{l=0}^{k+q}\sum_{n=0}^{2(p-1-q)}\sum_{j=0}^{i}\binom{k+q}{l}\binom{2(p-1-q)}{n}\binom{2i+1}{2j+1}\partial_t^l\partial_y^n\xi_j\,\partial_t^{k+q-l}\partial_y^{2(p-1-q)-n}\xi_{i-j}, \tag{6.37}$$

$$\partial_t^{k+q}\partial_y^{2(p-1-q)}J_i$$
$$=\underbrace{\sum_{l=0}^{k+q}\sum_{n=0}^{2(p-1-q)-1}\sum_{j=0}^{i}\binom{k+q}{l}\binom{2(p-1-q)}{n+1}\binom{2i+1}{2j}\partial_t^l\partial_y^n\xi_j\,\partial_t^{k+q-l}\partial_y^{2(p-1-q)-n}\xi_{i-j}}_{=I}$$
$$+\underbrace{\sum_{l=0}^{k+q}\sum_{j=0}^{i}\binom{k+q}{l}\binom{2i+1}{2j}\partial_y^{-1}(\partial_t^l\xi_j)\,\partial_t^{k+q-l}\partial_y^{2(p-1-q)+1}\xi_{i-j}}_{=II}. \tag{6.38}$$

Using (6.34), introducing $r = 2j + 2l + n$, and using the inequality $\langle k + q - l + 2(p-1-q) - n + i - j\rangle^{-2} \lesssim \langle 2i + 2k + m - r\rangle^{-2}$ given the range of the parameters l, n and j in the sum, we obtain

$$\begin{aligned}
&\|\partial_t^l \partial_y^n \xi_j \partial_t^{k+q-l} \partial_y^{2(p-1-q)-n} \xi_{i-j}\|_{L^\infty} \le C^2 \Lambda_{j,l,n} \Lambda_{i-j,k+q-l,2(p-1-q)-n} \\
&\quad \le C^2 \tau^{-l-j} (\tau')^{-n} (2j + 2l + n)! \langle j + l + n\rangle^{-2} \times \tau^{-(k+q-l)-(i-j)} (\tau')^{-(2(p-1-q)-n)} \\
&\qquad \cdot (2(i-j) + 2(k+q-l) + 2(p-1-q) - n)! \langle 2i + 2k + m - r\rangle^{-2} \\
&\quad = C^2 \tau'^2 \left(\frac{\tau'^2}{\tau}\right)^q \tau^{-i-k} (\tau')^{-m} r! (2i + 2k + m - r - 2)! \langle r\rangle^{-2} \langle 2i + 2k + m - r\rangle^{-2}.
\end{aligned}$$

Using $\binom{k+q}{l} \le \binom{2k+2q}{2l}$ and (D.3) with $(A_1, A_2, A_3, r_2) = (2k + 2q, 2(p-1-q), 2i + 1, r + 1)$, we get

$$\begin{aligned}
&\sum_{l=0}^{k+q} \sum_{n=0}^{2(p-1-q)} \sum_{j=0}^{i} \left(\binom{k+q}{l} \binom{2(p-1-q)}{n} \binom{2i+1}{2j+1} + \binom{k+q}{l} \binom{2(p-1-q)}{n+1} \binom{2i+1}{2j} \right) \\
&\qquad\qquad \cdot r! (2i + 2k + m - r - 2)! \langle r\rangle^{-2} \langle 2i + 2k + m - r\rangle^{-2} \\
&\quad \le \sum_{r=0}^{2i+2k+m} \binom{2i + 2k + m - 1}{r + 1} r! (2i + 2k + m - r - 2)! \langle r\rangle^{-2} \langle 2i + 2k + m - r\rangle^{-2} \\
&\quad = \sum_{r=0}^{2i+2k+m} (2i + 2k + m - 1)! (r + 1)^{-1} \langle r\rangle^{-2} \langle 2i + 2k + m - r\rangle^{-2} \\
&\quad \lesssim (2i + 2k + m - 1)! \langle i + k + m\rangle^{-2},
\end{aligned}$$

where we have used (D.1) with $K = 2i + 2k + m$. Combining the three inequalities above, (6.37) and (6.38), one finds

$$\|\partial_t^{k+q} \partial_y^{2(p-1-q)} H_i + I\|_{L^\infty} \le \tilde{C} C^2 \tau'^2 \left(\frac{\tau'^2}{\tau}\right)^q \langle i + k + m\rangle^{-1} \Lambda_{i,k,m}. \tag{6.39}$$

For the second term in J_i, using (6.34), $\int_0^\infty \omega(y)\, dy < \infty$, letting $r = 2j + 2l$, and using $\langle i - j + k + q - l + 2(p-1-q) + 1\rangle^{-2} \lesssim \langle 2i + 2k + m - r\rangle^{-2}$ for the range of parameters l and j in the sum, we get

$$\begin{aligned}
&\|\partial_y^{-1}(\partial_t^l \xi_j) \partial_t^{k+q-l} \partial_y^{2(p-1-q)+1} \xi_{i-j}\|_{L^\infty_\omega} \le \tilde{C} \|\partial_t^l \xi_j\|_{L^\infty_\omega} \|\partial_t^{k+q-l} \partial_y^{2(p-1-q)+1} \xi_{i-j}\|_{L^\infty_\omega} \\
&\quad \le \tilde{C} C \tau^{-l-j} (2j + 2l)! \langle j + l\rangle^{-2} \times C \tau^{-(k+q-l)-(i-j)} \\
&\qquad \cdot (\tau')^{-(2(p-1-q)+1)} (2(i-j) + 2(k+q-l) + 2(p-1-q) + 1)! \langle 2i + 2k + m - r\rangle^{-2} \\
&\quad = \tilde{C} C^2 \tau' \left(\frac{\tau'^2}{\tau}\right)^q \tau^{-i-k} (\tau')^{-m} r! (2i + 2k + m - r - 1)! \langle r\rangle^{-2} \langle 2i + 2k + m - r\rangle^{-2}.
\end{aligned} \tag{6.40}$$

Using $\binom{k+q}{l} \leq \binom{2k+2q}{2l}$ and (D.2) with $(A_1, A_2, r_1) = (2k+2q, 2i+1, r)$ we have

$$\begin{aligned}
&\sum_{l=0}^{k+q}\sum_{j=0}^{i}\binom{k+q}{l}\binom{2i+1}{2j} r!(2i+2k+m-r-1)!\langle r\rangle^{-2}\langle i+k+m-r\rangle^{-2} \\
&\leq \sum_{r=0}^{2k+2q+2i}\binom{2k+2q+2i+1}{r} r!(2i+2k+m-r-1)!\langle r\rangle^{-2}\langle i+k+m-r\rangle^{-2} \\
&\leq (2k+2i+m-1)! \\
&\quad \cdot \sum_{r=0}^{2k+2q+2i}\frac{(2k+2q+2i+1)!}{(2k+2i+m-1)!}\frac{(2i+2k+m-1-r)!}{(2k+2q+2i+1-r)!}\langle r\rangle^{-2}\langle 2i+2k+m-r\rangle^{-2}.
\end{aligned}$$

We estimate

$$\begin{aligned}
&\frac{(2k+2q+2i+1)!}{(2k+2i+m-1)!}\frac{(2i+2k+m-1-r)!}{(2k+2q+2i+1-r)!} \\
&\qquad = \frac{2i+2k+2p-1-r}{2i+2k+2p-1}\cdots\frac{2i+2k+2q+2-r}{2i+2k+2q+2} \leq 1.
\end{aligned}$$

Combining the above two inequalities and (D.1) with $K = 2k+2q+2i$ we obtain

$$\begin{aligned}
&\sum_{l=0}^{k+q}\sum_{j=0}^{i}\binom{k+q}{l}\binom{2i+1}{2j} r!(2i+2k+m-1-r)!\langle r\rangle^{-2}\langle i+k+m-r\rangle^{-2} \\
&\qquad \leq \tilde{C}(2k+2i+m)!\langle i+k+m\rangle^{-3}.
\end{aligned}$$

Combining (6.40) and the above inequality, for the second term in (6.38) we get

$$\|II\|_{L^\infty} \leq \tilde{C} C^2 \tau' \left(\frac{\tau'^2}{\tau}\right)^q \langle i+k+m\rangle^{-1}\Lambda_{i,k,m}. \tag{6.41}$$

Combining (6.39) and (6.41) we get, for τ' small enough,

$$\|\partial_t^{k+q}\partial_y^{2(p-1-q)}(H_i + J_i)\|_{L^\infty} \leq \tilde{C} C^2 \tau' \left(\frac{\tau'^2}{\tau}\right)^q \langle i+k+m\rangle^{-1}\Lambda_{i,k,m}.$$

Therefore, for $\tau'^2 \leq \tau/2$, we have $\sum_{q=0}^{p-1}(\tau'^2/\tau)^q \leq 2$ and so from the above identity,

$$\left\|\sum_{q=0}^{p-1}\partial_t^{k+q}\partial_y^{2(p-1-q)}(H_i + J_i)\right\|_{L^\infty} \leq \tilde{C} C^2 \tau' \langle i+k+m\rangle^{-1}\Lambda_{i,k,m}.$$

Inserting the above inequality and (6.36) in (6.35) gives

$$\|\partial_t^k\partial_y^m \xi_i\|_{L^\infty} \leq C\Lambda_{i,k,m}\left(C\tilde{C}\tau' + \tilde{C}\left(\frac{\tau'}{\sqrt{\tau}}\right)^m\right) \leq C\Lambda_{i,k,m}$$

for τ' small enough, since $m \geq 2$. Therefore, (6.34) is true for m.

Step 2. *$m = 2p+1$ is even.* By the formula of Step 1 we obtain

$$\partial_t^k \partial_y^m \xi_i = \partial_t^{p+k} \partial_y \xi_i - \sum_{q=0}^{p-1} \partial_t^{k+q} \partial_y^{2(p-1-q)+1} (H_i + J_i),$$

and the same computations show the desired result. We omit the details. ∎

Appendix A. Functional analysis

Lemma A.1. *There exists $C > 0$ such that for all $-\infty \le Y_0 < 0$ and $\varepsilon : (Y_0, \infty) \to \mathbb{R}$ with $\varepsilon \in H^1_\rho$,*

$$\int_{Y_0}^{\infty} Y^2 \varepsilon^2 e^{-\frac{3Y^2}{4}} \, dY \le C \|\varepsilon\|^2_{H^1_\rho}. \tag{A.1}$$

Proof. Let first $Y_0 = -\infty$. For $\varepsilon \in C^\infty_c(\mathbb{R})$, integrating by parts yields

$$\frac{4}{3} \int_{\mathbb{R}} \varepsilon \partial_Y \varepsilon Y e^{-\frac{3Y^2}{4}} \, dY + \frac{2}{3} \int_{\mathbb{R}} \varepsilon^2 e^{-\frac{3Y^2}{4}} \, dY = \int_{\mathbb{R}} \varepsilon^2 Y^2 e^{-\frac{3Y^2}{4}} \, dY.$$

From the Cauchy–Schwarz and Young inequalities, we have $4 |\int \varepsilon \partial_Y \varepsilon Y e^{-3Y^2/4}| \le \frac{1}{2} \int Y^2 \varepsilon^2 e^{-3Y^2/4} + 8 \int |\partial_Y \varepsilon|^2 e^{-3Y^2/4}$ and we infer from the above identity that

$$\int_{\mathbb{R}} \varepsilon^2 Y^2 e^{-\frac{3Y^2}{4}} \, dY \le 4 \int_{\mathbb{R}} \varepsilon^2 e^{-\frac{3Y^2}{4}} \, dY + \frac{16}{5} \int_{\mathbb{R}} |\partial_Y \varepsilon|^2 e^{-\frac{3Y^2}{4}} \, dY.$$

By density, this proves (A.1) for all $\varepsilon \in H^1_\rho$ in case $Y_0 = -\infty$. For $-\infty < Y_0 < 0$, define the even extension: $\tilde\varepsilon(Y) = \varepsilon(Y)$ for $Y \ge Y_0$ and $\tilde\varepsilon(Y) = \varepsilon(2Y_0 - Y)$ for $Y < Y_0$, and $\tilde Y_0 = -\infty$. Then $\|\varepsilon\|^2_{H^1_{\rho, Y_0}} \le \|\tilde\varepsilon\|^2_{H^1_{\rho, \tilde Y_0}} \le 2 \|\varepsilon\|^2_{H^1_{\rho, Y_0}}$, where the second inequality holds since $\rho(Y) \le \rho(2Y_0 - Y)$ for $Y \le Y_0$. Applying (A.1) for $\tilde\varepsilon$ with $\tilde Y_0 = \infty$ then implies (A.1) for ε with Y_0. ∎

Appendix B. Geometrical decomposition

Proof of Lemma 4.2. The proof relies on a classical use of the implicit function theorem, preceded by a renormalisation procedure to obtain a result which is uniformly valid for all λ large enough. Define the mapping

$$\Phi : (\varepsilon, \lambda, \mu, \tilde Y_0) \mapsto \lambda_0^4 (\langle \tilde\varepsilon, h_0 \rangle_\rho, \langle \tilde\varepsilon, h_1 \rangle_\rho, \langle \tilde\varepsilon, h_2 \rangle_\rho),$$

where $\langle u, v \rangle_\rho = \int_{Y_0 - \tilde Y_0}^{\infty} u v \rho$ and, for $Y \ge Y_0 - \tilde Y_0$,

$$\tilde\varepsilon(Y) = G_1\left(\frac{Y + \tilde Y_0}{\lambda_0^2}\right) - (1 + \lambda_0^{-4}\lambda)^2 G_1\left(\frac{Y}{\lambda_0^2 (1 + \lambda_0^{-4}\lambda)^2 \mu}\right) + \frac{\varepsilon(Y + \tilde Y_0)}{\lambda_0^4}.$$

Then Φ is a C^2 mapping on $L^2_\rho \times (-\lambda_0^4, \infty) \times (0,\infty) \times \mathbb{R}$. Moreover, one computes that its differential at $(0,0,1,0)$ is, with $\langle u, v\rangle = \int_{Y\geq Y_0} uv\rho$,

$$J\Phi(0,0,1,0) + O(e^{-\lambda_0^2}) =$$
$$\begin{pmatrix} \langle \cdot, h_0\rangle & \langle -2G_1(\frac{Y}{\lambda_0^2}) + 2\frac{Y}{\lambda_0^2}\partial_Z G_1(\frac{Y}{\lambda_0^2}), h_0\rangle & \lambda_0^2\langle Y\partial_Z G_1(\frac{Y}{\lambda_0^2}), h_0\rangle & \lambda_0^2\langle \partial_Z G_1(\frac{Y}{\lambda_0^2}), h_0\rangle \\ \langle \cdot, h_1\rangle & \langle -2G_1(\frac{Y}{\lambda_0^2}) + 2\frac{Y}{\lambda_0^2}\partial_Z G_1(\frac{Y}{\lambda_0^2}), h_1\rangle & \lambda_0^2\langle Y\partial_Z G_1(\frac{Y}{\lambda_0^2}), h_1\rangle & \lambda_0^2\langle \partial_Z G_1(\frac{Y}{\lambda_0^2}), h_1\rangle \\ \langle \cdot, h_2\rangle & \langle -2G_1(\frac{Y}{\lambda_0^2}) + 2\frac{Y}{\lambda_0^2}\partial_Z G_1(\frac{Y}{\lambda_0^2}), h_2\rangle & \lambda_0^2\langle Y\partial_Z G_1(\frac{Y}{\lambda_0^2}), h_2\rangle & \lambda_0^2\langle \partial_Z G_1(\frac{Y}{\lambda_0^2}), h_2\rangle \end{pmatrix}$$

where the $O(e^{-\lambda_0^2})$ comes from the boundary terms. Using the Taylor expansion of G_1 one has

$$\begin{aligned} -2G_1\left(\frac{Y}{\lambda_0^2}\right) + 2\frac{Y}{\lambda_0^2}\partial_Z G_1\left(\frac{Y}{\lambda_0^2}\right) &= -2 - \frac{Y^2}{2\lambda_0^4} + O\left(\frac{Y^4}{\lambda_0^8}\right) \\ &= -\frac{1}{6\lambda_0^4}h_2(Y) - \left(2 + \frac{1}{3\lambda_0^4}\right)h_0(Y) + O\left(\frac{Y^4}{\lambda_0^8}\right), \\ \lambda_0^2 Y\partial_Z G_1\left(\frac{Y}{\lambda_0^2}\right) = -\frac{Y^2}{2} + O\left(\frac{Y^4}{\lambda_0^4}\right) &= -\frac{1}{6}h_2(Y) - \frac{1}{3}h_0(Y) + O\left(\frac{Y^4}{\lambda_0^4}\right), \\ \lambda_0^2\partial_Z G_1\left(\frac{Y}{\lambda_0^2}\right) = -\frac{Y}{2} + O\left(\frac{|Y|^3}{\lambda_0^4}\right) &= -\frac{1}{2\sqrt{3}}h_1(Y) + O\left(\frac{|Y|^3}{\lambda_0^4}\right). \end{aligned}$$

Therefore

$$J\Phi(0,0,1,0) =$$
$$\begin{pmatrix} \langle \cdot, h_0\rangle & -2\|h_0\|^2_{L^2_\rho} + O(\lambda_0^{-4}) & -\frac{1}{3}\|h_0\|^2_{L^2_\rho} + O(\lambda_0^{-4}) & O(\lambda_0^{-4}) \\ \langle \cdot, h_1\rangle & O(\lambda_0^{-4}) & O(\lambda_0^{-4}) & -\frac{1}{2\sqrt{3}}\|h_1\|^2_{L^2_\rho} + O(\lambda_0^{-4}) \\ \langle \cdot, h_2\rangle & O(\lambda_0^{-4}) & -\frac{1}{6}\|h_2\|^2_{L^2_\rho} + O(\lambda_0^{-4}) & O(\lambda_0^{-8}) \end{pmatrix}.$$

This implies that the restriction of the differential to $\{0\}\times\mathbb{R}^3$ is invertible for λ_0 large enough, with a uniform size. Moreover, one can also check similarly that the second differential of Φ is bounded near $(0,0,1,0)$, and this uniformly for large λ. Therefore the implicit function theorem applies uniformly for all $\lambda_0 \geq \lambda^*$ large enough and $Y_0 \leq -\lambda_0^2$. There exist $\delta, K > 0$ such that for each $\varepsilon \in L^2_\rho$ with $\|\varepsilon\|_{L^2_\rho} \leq \delta$, there exist unique parameters $(\lambda, \mu, \tilde{Y}_0)$ with $|\lambda| + |\mu - 1| + |\tilde{Y}_0| \leq K$ such that $\Phi(\varepsilon, \lambda, \mu, \tilde{Y}_0) = 0$. Moreover, they define C^1 functions with respect to the L^2_ρ topology.

Let $\lambda_0 \geq \lambda^*$ and $\|\varepsilon\|_{L^2_\rho} \leq \delta\lambda_0^{-4}$. The above discussion yields the existence, uniqueness, and differentiability of (λ, μ, Y_0) such that $\Phi(\lambda_0^4\varepsilon, \lambda, \mu, Y_0) = 0$. Let $(\tilde{\lambda}, \tilde{\mu}, \tilde{Y}_0) = (1 + \lambda_0^{-4}\lambda, \mu, Y_0)$. Then indeed

$$G_1\left(\frac{Y}{\lambda_0^2}\right) + \varepsilon(Y) = \tilde{\lambda}^2 G_1\left(\frac{Y - \tilde{Y}_0}{\tilde{\lambda}^2\tilde{\mu}}\right) + \tilde{\varepsilon}(Y - \tilde{Y}_0) \quad \text{with} \quad \tilde{\varepsilon} \perp h_0, h_1, h_2 \text{ in } L^2_\rho$$

and one has $|\tilde{\lambda}-1|\leq K\lambda_0^{-4}$ and $|\mu-1|+|Y_0|\leq K$. The uniqueness when requiring these bounds follows similarly, and implies the smoothness from the above discussion. This ends the proof. ■

Appendix C. Estimates for the heat kernel

Lemma C.1. *Let K_t be given by (2.4). First, for any $T>0$, there exists $C(T)>0$ such that for any $t\in[0,T]$ and $y\in\mathbb{R}$,*

$$(K_t*\omega)(y)\leq C\omega(y). \tag{C.1}$$

Second, there exist $C,\kappa,\tau_0>0$ such that for all $k\in\mathbb{N}$, $t>0$ and $y\in\mathbb{R}$,

$$|\partial_t^k K_t(y)|\leq C\tau_0^{-k}t^{-k}k!K_{\kappa t}(y),\quad |\partial_t^k\partial_y K_t(y)|\leq C\tau_0^{-k}t^{-k-\frac{1}{2}}k!K_{\kappa t}(y). \tag{C.2}$$

Proof. From a direct computation, $\int_{z\in\mathbb{R}}\omega(y-z)K_t(z)\,dz\leq C\omega(y)$ for all $y\in\mathbb{R}$ and $t\in[0,T]$ for some universal constant $C(T)>0$; we omit the details. This shows (C.1).

Below we will denote by $C>0$ some universal constant whose value may change from line to line.

Let $z\in\mathbb{C}$ denote a complex variable, and (ρ,θ) be its (radius, angle) variables. Let $\phi(z)=e^{-1/z}$. Then ϕ is an analytic function on $\mathbb{C}\setminus\{0\}$. Consider for any $t>0$ the circle $\mathcal{C}_t:=\{z\in\mathbb{C}:|z-t|=\frac{t}{10}\}$. Then, for all $z\in\mathcal{C}_t$, we have $\frac{9}{10}t\leq\rho\leq\frac{11}{10}t$ and $|\theta|\leq\theta_0$ for some $\theta_0<\pi/2$. Hence, denoting $\tilde{z}=1/z$, we see that for $z\in\mathcal{C}_t$,

$$\frac{10}{11t}\leq\tilde{\rho}\leq\frac{10}{9t}\quad\text{and}\quad|\tilde{\theta}|\leq\theta_0.$$

Therefore, there exists a constant $c_0>0$ independent of t such that $\frac{c_0}{t}\leq\Re(\frac{1}{z})\leq\frac{1}{c_0t}$ for all $z\in\mathcal{C}_t$. Consequently, for all $z\in\mathcal{C}_t$,

$$|\phi(z)|\leq e^{-\frac{c_0}{t}}.$$

Applying the Cauchy contour formula to the holomorphic function ϕ with the contour $\mathcal{C}_t$, and differentiating, one finds that for some constant $C>0$, for all $j\in\mathbb{N}$,

$$|\partial_t^j\phi(t)|\leq Ct^{-j}10^j j!e^{-\frac{c_0}{t}}.$$

Let now $y\in\mathbb{R}\setminus\{0\}$ and $\phi_y(t)=e^{-\frac{y^2}{4t}}=\phi(\frac{4t}{y^2})$. Then

$$|\partial_t^j\phi_y(t)|=\frac{4^j}{y^{2j}}\left|\partial_t^j\phi\left(\frac{4t}{y^2}\right)\right|\leq\frac{4^j}{y^{2j}}C\left(\frac{4t}{y^2}\right)^{-j}10^j j!e^{-\frac{c_0y^2}{4t}}=Ct^{-j}10^j j!e^{-\frac{c_0y^2}{4t}}.$$

Combining the above bound with the bounds $|\partial_t^j(t\mapsto1/\sqrt{t})|\leq j!t^{-j-1/2}$ and $|\partial_t^j(t\mapsto1/t^{3/2})|\leq(j+1)!t^{-j-3/2}$, using the Leibniz rule, one finds that for all $j\in\mathbb{N}$, $t>0$ and $y\in\mathbb{R}$,

$$|\partial_t^jK_t(y)|\leq Ct^{-j-\frac{1}{2}}11^j j!e^{-\frac{c_0y^2}{4t}}\quad\text{and}\quad\left|\partial_t^j\left(\frac{1}{t}K_t(y)\right)\right|\leq Ct^{-j-\frac{3}{2}}11^j j!e^{-\frac{c_0y^2}{4t}}.$$

The first bound above is precisely the first bound in (C.2), while the second bound above gives the second one in (C.2), using the fact that $\partial_y K_t = -\frac{y}{2t} K_t$ and that for any $0 < c_0' < c_0$ there exists $C > 0$ with $\frac{|y|}{\sqrt{t}} e^{-\frac{c_0 y^2}{4t}} \leq C e^{-\frac{c_0' y^2}{4t}}$. ■

Appendix D. Combinatorial estimates

Lemma D.1 (Combinatorial estimates). *There exists $C > 0$ such that for any $A \in \mathbb{N}$,*

$$\sum_{a=0}^{A} \langle a\rangle^{-2} \langle A-a\rangle^{-2} \leq C\langle A\rangle^{-2}. \tag{D.1}$$

For any $A_1, A_2, A_3 \in \mathbb{N}$, $r_1 \leq A_1 + A_2$ and $r_2 \leq A_1 + A_2 + A_3$,

$$\sum_{a_1 \leq A_1, a_2 \leq A_2, a_1+a_2=r_1} \binom{A_1}{a_1}\binom{A_2}{a_2} = \binom{A_1+A_2}{r_1}, \tag{D.2}$$

$$\sum_{a_1 \leq A_1, a_2 \leq A_2, a_3 \leq A_3, a_1+a_2+a_3=r_2} \binom{A_1}{a_1}\binom{A_2}{a_2}\binom{A_3}{a_3} = \binom{A_1+A_2+A_3}{r_2}. \tag{D.3}$$

Proof. To prove (D.1), we decompose

$$\begin{aligned}\sum_{k=0}^{K} \langle k\rangle^{-2}\langle K-k\rangle^{-2} &= \sum_{k=0}^{\lfloor K/2\rfloor} \langle k\rangle^{-2}\langle K-k\rangle^{-2} + \sum_{k=0}^{\lceil K/2\rceil} \langle k\rangle^{-2}\langle K-k\rangle^{-2} \\ &\lesssim \langle K\rangle^{-2} \sum_{k\geq 0} \langle k\rangle^{-2} \lesssim \langle K\rangle^{-2}.\end{aligned}$$

(D.2) and (D.3) are obtained from a standard counting argument. ■

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