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Interacting helical vortex filaments in the three-dimensional Ginzburg–Landau equation

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Abstract. For each given $n \geq 2$, we construct a family of entire solutions $u_\varepsilon(z,t)$, $\varepsilon > 0$, with helical symmetry to the three-dimensional complex-valued Ginzburg–Landau equation

$$\Delta u + (1 - |u|^2)u = 0, \quad (z,t) \in \mathbb{R}^2 \times \mathbb{R} \simeq \mathbb{R}^3.$$

These solutions are $2\pi/\varepsilon$-periodic in t and have n helix-vortex curves, with asymptotic behavior, as $\varepsilon \to 0$,

$$u_\varepsilon(z,t) \approx \prod_{j=1}^{n} W\big(z - \varepsilon^{-1} f_j(\varepsilon t)\big),$$

where $W(z) = w(r)e^{i\theta}$, $z = re^{i\theta}$, is the standard degree $+1$ vortex solution of the planar Ginzburg–Landau equation $\Delta W + (1 - |W|^2)W = 0$ in $\mathbb{R}^2$ and

$$f_j(t) = \frac{\sqrt{n-1}\, e^{it} e^{2i(j-1)\pi/n}}{\sqrt{|\log \varepsilon|}}, \quad j = 1, \dots, n.$$

Existence of these solutions was previously conjectured by del Pino and Kowalczyk (2008), $\mathbf{f}(t) = (f_1(t), \dots, f_n(t))$ being a rotating equilibrium point for the renormalized energy of vortex filaments derived there,

$$\mathcal{W}_\varepsilon(\mathbf{f}) := \pi \int_0^{2\pi} \left(\frac{|\log \varepsilon|}{2} \sum_{k=1}^{n} |f_k'(t)|^2 - \sum_{j \neq k} \log|f_j(t) - f_k(t)| \right) \mathrm{d}t,$$

corresponding to that of a planar logarithmic n-body problem. The modulus of these solutions converges to 1 as $|z|$ goes to infinity uniformly in t, and the solutions have nontrivial dependence on t, thus negatively answering the Ginzburg–Landau analogue of the Gibbons conjecture for the Allen–Cahn equation, a question originally formulated by H. Brezis.

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1. Introduction

This paper deals with constructing entire solutions to the complex Ginzburg–Landau equation in the Euclidean space $\mathbb{R}^N$,

$$\Delta u + (1 - |u|^2)u = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $u: \mathbb{R}^N \to \mathbb{C}$ is a complex-valued function and $N \geq 2$. It is convenient for our purposes to introduce a small parameter $\varepsilon > 0$ and consider the equivalent scaled version of (1.1) given by

$$\varepsilon^2 \Delta u + (1 - |u|^2)u = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

When regarded in a bounded region $\Omega \subset \mathbb{R}^N$, equation (1.2) corresponds to the Euler–Lagrange equation for the functional

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2, \tag{1.3}$$

which for $N = 2, 3$ is often considered as a model for the energy arising in the standard Ginzburg–Landau theory of superconductivity when no external applied magnetic field is present. In that setting, the complex-valued state of the system u corresponds to a critical point of J_ε in which $|u|^2$ represents the density of the superconductive property of the sample Ω (Cooper pairs of electrons). The function u is expected to stay away from zero except on a lower-dimensional zero set, the vortex set, corresponding to *defects* where superconductivity is not present.

In their pioneering work [8], Bethuel–Brezis–Hélein analyzed in dimension $N = 2$ the behavior as $\varepsilon \to 0$ of a global minimizer u_ε of J_ε when subject to a boundary condition $g: \partial\Omega \to \mathbb{S}^1$ of degree $k \geq 1$. They established that, away from a finite number of distinct points $a_1, \dots, a_k \in \Omega$, one has (up to subsequences)

$$u_\varepsilon(x) \approx e^{i\varphi(x)} \prod_{j=1}^{k} \frac{x - a_j}{|x - a_j|},$$

where $\varphi(x)$ is a real harmonic function and the k-tuple $(a_1, \dots, a_k)$ minimizes a functional of points, the *renormalized energy* that measures through Green's function the mutual interaction between the points and the boundary. Using the results in [32, 36, 41], one gets the validity of the global approximation

$$u_\varepsilon(x) \approx e^{i\varphi(x)} \prod_{j=1}^{k} W\left(\frac{x - a_j}{\varepsilon}\right), \tag{1.4}$$

where $W(z)$ is the *standard degree* $+1$ *vortex solution* of equation (1.1) for $N = 2$, namely its unique solution of the form

$$W(z) = e^{i\theta} w(r), \quad z = re^{i\theta}, \tag{1.5}$$

where $w > 0$ solves

$$\begin{cases} w'' + \dfrac{w'}{r} - \dfrac{w}{r^2} + (1 - w^2)w = 0 & \text{in } (0, \infty), \\ \quad w(0^+) = 0, \quad w(+\infty) = 1; \end{cases} \tag{1.6}$$

see [10,25]. Thus, before reaching the limit, the vortex set of u_ε is constituted by exactly k distinct points, each with local degree $+1$. The mechanism of vortex formation in the two-dimensional Ginzburg–Landau model from the action of an external constant magnetic field has been extensively studied; see [38] and references therein. Critical points of the renormalized energy are in fact in correspondence with other critical points of J_ε in (1.3) of the form (1.4) for small ε, as it has been found in [3, 14, 17, 29, 34]. In the higher-dimensional case $N \geq 3$ and with suitable boundary conditions and energy levels, the vortex set of minimizers and more general critical points have been described when $\varepsilon \to 0$ in [2,9,26,30,31,35,37] as a codimension-2 set with a generalized minimal submanifold structure. In dimension $N = 3$, defects should typically assume the form of curves with a winding number associated: these are called *vortex filaments*. The basic *degree* $+1$ *vortex line* is the solution u of (1.1) for $N = 3$ given by

$$u(z, t) = W(z), \quad (z, t) \in \mathbb{R}^2 \times \mathbb{R} \simeq \mathbb{R}^3,$$

with $W(z)$ specified in (1.5). Its zero set is of course the t-axis, and a transversal winding number $+1$ is associated to it. In dimension $N = 3$, under Neumann boundary conditions, it was found in [33] a local minimizer with energy formally corresponding to multiple vortex lines collapsing onto a segment. Motivated by this work, in [16], an expression for the renormalized energy for the interaction of nearly parallel "degree +1 vortex lines" collapsing onto the t-axis was derived. Considering n curves

$$t \mapsto (f_i(t), t), \quad 1 \leq i \leq n, \quad \mathbf{f} = (f_1, \ldots, f_n),$$

which for simplicity we assume 2π-periodic, we look for an approximate solution of the form

$$u_\varepsilon(z, t) \approx W_\varepsilon(z, t; \mathbf{f}) := e^{i\varphi(z,t)} \prod_{j=1}^{n} W\left(\frac{z - f_j(t)}{\varepsilon}\right). \tag{1.7}$$

In the cylinder $\Omega = \mathcal{C} = B_R(0) \times (0, 2\pi)$, with φ harmonic matching lateral zero Neumann boundary conditions, it is found in [16] that

$$I_\varepsilon(\mathbf{f}) := J_\varepsilon(W_\varepsilon(\,\cdot\,; \mathbf{f})) \approx 2\pi \times n\pi |\log \varepsilon| + \mathcal{W}_\varepsilon(\mathbf{f}), \tag{1.8}$$

where

$$\mathcal{W}_\varepsilon(\mathbf{f}) := \pi \int_0^{2\pi} \left(|\log \varepsilon| \frac{1}{2} \sum_{k=1}^{n} |f_k'(t)|^2 - \sum_{j \neq k} \log|f_j(t) - f_k(t)| \right) \mathrm{d}t. \tag{1.9}$$

Equilibrium location of these curves should then correspond to an approximate critical point of the functional I_ε and hence of $\mathcal{W}_\varepsilon$, which is the action associated to the n-logarithmic body problem in $\mathbb{R}^2$. This energy also appears in related problems in fluid dynamics; see e.g. [27, 28]. If we set

$$\mathbf{f}(t) = \frac{1}{\sqrt{|\log \varepsilon|}}\tilde{\mathbf{f}}(t), \quad \tilde{\mathbf{f}} = (\tilde{f}_1, \ldots, \tilde{f}_n),$$

this corresponds to a 2π-periodic solution of the ODE system

$$-\tilde{f}_k''(t) = 2\sum_{i\neq k} \frac{\tilde{f}_k(t) - \tilde{f}_i(t)}{|\tilde{f}_k(t) - \tilde{f}_i(t)|^2}. \tag{1.10}$$

The following n-tuple $\tilde{\mathbf{f}}^0$ is a standard rotating solution of system (1.10):

$$\tilde{f}_k^0(t) = \sqrt{n-1}e^{it}e^{2i(k-1)\pi/n}, \quad k = 1, \ldots, n. \tag{1.11}$$

It is shown in [16] that the functional I_ε in (1.8) does have a 2π-periodic critical point $\mathbf{f}^\varepsilon(t)$ such that

$$\mathbf{f}^\varepsilon(t) = \mathbf{f}^0(t) + \frac{o(1)}{\sqrt{|\log \varepsilon|}}, \quad \mathbf{f}^0(t) := \frac{1}{\sqrt{|\log \varepsilon|}}\tilde{\mathbf{f}}^0(t) \tag{1.12}$$

uniformly as $\varepsilon \to 0$, and it is conjectured the existence of a solution $u_\varepsilon(z,t)$ to the system

$$\varepsilon^2 \Delta u + (1 - |u|^2)u = 0, \quad (z,t) \in \mathbb{R}^3, \tag{1.13}$$

which is 2π-periodic in t and has the approximate form (1.7) for $\mathbf{f}$ as in (1.12). The recent work [13] has established a rigorous connection, in the sense of Γ-convergence, between minimizers of functional (1.9) and minimizers in cylinders with suitable Dirichlet boundary condition, thus providing evidence towards the conjecture in [16]. In this paper, we prove this conjecture.

Theorem 1. *For every $n \geq 2$ and for ε sufficiently small, there exists a solution $u_\varepsilon(z,t)$ of (1.13), 2π-periodic in the t-variable, with the following asymptotic profile:*

$$u_\varepsilon(z,t) = \prod_{k=1}^{n} W\left(\frac{z - f_k^\varepsilon(t)}{\varepsilon}\right) + \varphi_\varepsilon(z,t),$$

where $f_k^\varepsilon(t)$ is 2π-periodic with the asymptotic behavior (1.12) and

$$|\varphi_\varepsilon(z,t)| \leq \frac{C}{|\log \varepsilon|}.$$

Besides, we have

$$\lim_{|z|\to+\infty} |u_\varepsilon(z,t)| = 1 \quad \textit{uniformly in } t. \tag{1.14}$$

We are also able to construct another family of solutions to (1.13). Until now, we have dealt only with vortex filaments of degree $+1$. However, it is believed that, in presence of several vortex filaments of different degrees $d_k \in \mathbb{Z}$, the energy governing the interaction of the filaments is

$$\mathcal{W}_\varepsilon(\mathbf{f}) := \pi \int_0^{2\pi} \left(|\log \varepsilon| \frac{1}{2} \sum_{k=1}^{n} |f_k'(t)|^2 - \sum_{j \neq k} d_j d_k \log |f_j(t) - f_k(t)| \right) \mathrm{d}t. \tag{1.15}$$

There exist special critical points of (1.15) analogous to (1.11) for $d_1 = -1$ and $d_k = +1$ for $k = 2, \dots, n$ when $n \geq 5$. These critical points can be written as

$$g_1^0(t) = 0, \quad g_k^0(t) = \sqrt{n-4}\, e^{it} e^{2i(k-1)\pi/(n-1)}, \quad k = 2, \dots, n, \quad n \geq 5. \tag{1.16}$$

From these solutions, we can obtain the following theorem.

Theorem 2. *For every $n \geq 5$ and for ε sufficiently small, there exists a solution $u_\varepsilon(z,t)$ of (1.13), 2π-periodic in the t-variable, with the following asymptotic profile:*

$$u_\varepsilon(z,t) = \overline{W}(z) \prod_{k=2}^{n} W\left(\frac{z - g_k^\varepsilon(t)}{\varepsilon} \right) + \varphi_\varepsilon(z,t),$$

where $g_k^\varepsilon(t)$ is 2π-periodic with $g_k^\varepsilon(t) = g_k(t) + o_\varepsilon(1)/\sqrt{|\log \varepsilon|}$, g_k defined by (1.16) and

$$|\varphi_\varepsilon(z,t)| \leq \frac{C}{|\log \varepsilon|}.$$

Besides, we have

$$\lim_{|z| \to +\infty} |u_\varepsilon(z,t)| = 1 \quad \textit{uniformly in } t.$$

The proofs of both theorems give a precise answer to the existence question, with an accurate description of the solution. They take specific advantage of the geometric setting: the configuration predicted is one of multiple helix vortex curves periodically winding around each other. The Ginzburg–Landau equation has a *screw-driving* symmetry which we take advantage of to reduce the original problem to a planar one. For constructions of solutions with helical vortex structures, we refer also to [11, 42].

We observe that, in terms of the parameterless equation (1.1) for $N = 3$, what we find is a family of entire solutions $u_\varepsilon(z,t)$, $2\pi/\varepsilon$-periodic in t, with approximate form

$$u_\varepsilon(z,t) \approx \prod_{j=1}^{n} W\big(z - \varepsilon^{-1} f_j^\varepsilon(\varepsilon t) \big).$$

Equation (1.1) is the complex-valued version of the Allen–Cahn equation of phase transitions,

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N, \tag{1.17}$$

where $u: \mathbb{R}^N \to \mathbb{R}$. The Allen–Cahn model describes transitions of two phases between the values -1 and $+1$ essentially separated by a thick wall, which for small ε should lie close to a minimal hypersurface. Solutions with screw-driving symmetry whose zero level set is precisely a helicoid have been built in [22] (and extended to the fractional case in [12]). Solutions with multiple interfaces whose interactions are governed by mechanical systems (Toda systems) have been built in [1, 18, 19, 21]. The celebrated *De Giorgi conjecture* states that, at least up to dimension $N = 8$, solutions of (1.17) which are monotone in one direction must have one-dimensional symmetry, namely their level sets must be parallel hyperplanes; see [4, 20, 24, 40] and references therein. A variant of this statement is the *Gibbons conjecture*: an entire solution u of (1.17) such that

$$\lim_{|x_N|\to+\infty} |u(x', x_N)| = 1 \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}$$

must necessarily be a function of x_N only. This fact has been proven for any $N \geq 2$. See [5, 7, 23]. The analogous question for the Ginzburg–Landau equation in $\mathbb{R}^N$, $N \geq 3$, originally formulated by H. Brezis, is whether or not a solution $u(z,t)$, $(z,t) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, with

$$\lim_{|z|\to+\infty} |u(z,t)| = 1 \quad \text{uniformly in } t \in \mathbb{R}^{N-2}, \tag{1.18}$$

must necessarily be a function of z. This turns out to be false since the solutions in Theorem 1 satisfy (1.18). We observe that solutions $W_n(z)$ of (1.1) with total degree n, $|n| \geq 1$, of the form

$$W_n(z) = e^{in\theta} w_n(r), \quad w_n(0) = 0, \quad w_n(+\infty) = 1, \quad z = re^{i\theta},$$

are known to exist for each $n \geq 2$; see [10,25]. The solutions in Theorem 1 have transversal total degree equal to $n \geq 2$. A natural question is whether or not Brezis' statement holds true under the additional assumption of total transversal degree equal to ± 1. See [32, 36, 41] for the corresponding question in dimension $N = 2$, and [39] for a conjecture on the symmetry of entire solutions of (1.1) when $N = 3$.

We will devote the rest of this paper to the proof of Theorem 1. The proof of Theorem 2 follows the same lines. As we have mentioned, the key observation is that the invariance under screw-driving symmetry allows to reduce the problem to one in the plane, for which the solution to be found has a finite number of vortices with degree 1. For simplicity, we treat only the case $n = 2$ in the following, but the arguments can be easily adapted. We will look for solutions that are close to the approximation

$$u_d(x,y,t) = W\left(\frac{x - d\cos t}{\varepsilon}, \frac{y - d\sin t}{\varepsilon}\right) W\left(\frac{x + d\cos t}{\varepsilon}, \frac{y + d\sin t}{\varepsilon}\right) \tag{1.19}$$

when ε is small. Here d is a parameter of size $1/\sqrt{|\log \varepsilon|}$.

It can be observed that the zero set of u_d has the shape of a double helix and the function $\tilde{u}_d := e^{2it} u_d$ is screw-symmetric (see Definition 1). Thus $\tilde{u}_d$ can be expressed as a function of two variables, which reduces the problem to a two-dimensional case. We will look for screw-symmetric perturbations of $\tilde{u}_d$. Our approach will be based on the method in [17], devised to build up solutions with isolated vortices when $N = 2$ using a Lyapunov–Schmidt reduction. A major difficulty that we need to overcome is the presence of very large terms in the error of approximation. In the two-dimensional case, we immediately obtain errors that are of size $O(\varepsilon)$, while here it is $O(1/|\log \varepsilon|)$.

This is a major difficulty since the vortex-location adjustment arises at essentially ε-order. This is overcome by carefully decomposing the error created by the nonlinearity in "odd" small and "even" large Fourier modes parts. The even part has at main order no effect in the solvability conditions needed in the linear theory we devise in Proposition 5.1. These steps are rather delicate, and we will carry them out in detail in what follows.

2. Reduction to a two-dimensional problem by using screw-symmetry

As a first step, we reduce our three-dimensional problem to a related two-dimensional one. To do so, we work with a particular type of symmetry. To define this symmetry, it is convenient to use cylindrical coordinates $(r,\theta,t)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$ and to work with functions that are 2π-periodic in the second variable.

Definition 1. We say that a function u is screw-symmetric if $u(r,\theta+h,t+h)=u(r,\theta,t)$ for any $h\in\mathbb{R}$.

Notice that this condition is equivalent to $u(r,\theta,t+h)=u(r,\theta-h,t)$ for any $h\in\mathbb{R}$, and then a screw-symmetric function can be expressed as a function of two-variables. Indeed, for any $(r,\theta,t)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$,

$$u(r,\theta,t)=u(r,\theta-t,0)=:U(r,\theta-t).$$

Writing the standard vortex of degree one in polar coordinates (r,θ), i.e., $W(z)=w(r)e^{i\theta}$, we can see that the approximation u_d defined in (1.19) satisfies

$$u_d(r,\theta,t+h)=e^{2ih}u_d(r,\theta-h,t)$$

for any h in $\mathbb{R}$. That is, u_d is not screw-symmetric, but $\tilde{u}_d(r,\theta,t):=e^{-2it}u_d(r,\theta,t)$ is. Hence we can write u_d as $u_d=e^{2it}\tilde{u}_d$, with $\tilde{u}_d$ a screw-symmetric function.

This suggests to look for solutions u of (1.2) that can be written as

$$u(r,\theta,t)=e^{2it}\tilde{u}(r,\theta,t)$$

with $\tilde{u}$ screw-symmetric. Thus $\tilde{u}(r,\theta,t)=U(r,\theta-t)$, $U\colon\mathbb{R}^+\times\mathbb{R}$ being 2π-periodic in the second variable. Denoting $U=U(r,s)$, we can see that

$$\begin{aligned}
\partial_r u &= e^{2it}\partial_r U(r,s), &\qquad \partial^2_{rr}u &= e^{2it}\partial^2_{rr}U(r,s),\\
\partial_\theta u &= e^{2it}\partial_s U(r,s), &\qquad \partial^2_{\theta\theta}u &= e^{2it}\partial^2_{ss}U(r,s),\\
\partial_t u &= [2iU-\partial_s U]e^{2it}, &\qquad \partial^2_{tt}u &= [\partial^2_{ss}U-4i\partial_s U-4U]e^{2it}.
\end{aligned}$$

Recalling that the Laplacian in cylindrical coordinates is expressed by $\partial^2_{rr}+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial^2_{\theta\theta}+\partial^2_{tt}$, we conclude that u is a solution of (1.2) if and only if U is a solution of

$$\begin{aligned}
\varepsilon^2\Big(\partial^2_{rr}U+\frac{1}{r}\partial_r U+\frac{1}{r^2}\partial^2_{ss}U+\partial^2_{ss}U-4i\partial_s U-4U\Big)&\\
+(1-|U|^2)U=0 \quad \text{in } \mathbb{R}^*_+\times\mathbb{R}.&
\end{aligned}$$

We will also work in rescaled coordinates, that is, we define $V(r,s) := U(\varepsilon r, s)$, and we search for a solution to the equation

$$\begin{aligned}\partial^2_{rr} V + \frac{1}{r}\partial_r V + \frac{1}{r^2}\partial^2_{ss} V \\ + \varepsilon^2(\partial^2_{ss}V - 4i\partial_s V - 4V) + (1-|V|^2)V = 0 \quad \text{in } \mathbb{R}^*_+ \times \mathbb{R}.\end{aligned} \tag{2.1}$$

From now on, we will work in the plane $\mathbb{R}^2$, and we will use the notation $z = x_1 + ix_2 = re^{is}$. We denote by Δ the Laplace operator in two dimensions, meaning

$$\Delta = \partial^2_{x_1x_1} + \partial^2_{x_2x_2} = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{ss}.$$

Then equation (2.1) can be written as

$$\Delta V + \varepsilon^2(\partial^2_{ss}V - 4i\partial_s V - 4V) + (1-|V|^2)V = 0 \quad \text{in } \mathbb{R}^2,$$

and the approximate solution (1.19) in the new coordinates is given by

$$V_d(z) = W(z-\tilde{d})W(z+\tilde{d}), \tag{2.2}$$

where

$$\tilde{d} := \frac{d}{\varepsilon} = \frac{\hat{d}}{\varepsilon\sqrt{|\log\varepsilon|}}$$

for some new parameter $\hat{d} = O(1)$.

3. Formulation of the problem

3.1. Additive-multiplicative perturbation

Let

$$S(v) := \Delta v + \varepsilon^2(\partial^2_{ss}v - 4i\partial_s v - 4v) + (1-|v|^2)v.$$

The equation to be solved can be written as

$$S(v) = 0. \tag{3.1}$$

Recall the notation $z = re^{is} = x_1 + ix_2$ and $\Delta = \partial^2_{x_1x_1} + \partial^2_{x_2x_2}$. When using the coordinates (x_1, x_2), equation (3.1) is posed in $\mathbb{R}^2$, while if we use polar coordinates (r,s), the domain for (3.1) is $r > 0$, $s \in \mathbb{R}$ with periodicity.

Following del Pino–Kowalczyk–Musso [17], we look for a solution to (3.1) of the form

$$v = \eta V_d(1 + i\psi) + (1-\eta)V_d e^{i\psi}, \tag{3.2}$$

where V_d is the ansatz (2.2) and ψ is the new unknown. The cut-off function η in (3.2) is defined as

$$\eta(z) = \eta_1(|z-\tilde{d}|) + \eta_1(|z+\tilde{d}|), \quad z \in \mathbb{C} = \mathbb{R}^2,$$

and $\eta_1 \colon \mathbb{R} \to [0,1]$ is a smooth cut-off function such that

$$\eta_1(t) = 1 \text{ for } t \le 1 \quad \text{and} \quad \eta_1(t) = 0 \text{ for } t \ge 2. \tag{3.3}$$

The reason for the form of the perturbation term in (3.2) is the same as in [17]. On the one hand, the nonlinear terms behave better for the norms that we consider when using the multiplicative ansatz, but near the vortices, an additive ansatz is better since it allows the position of the vortex to be adjusted.

Our objective here is to rewrite (3.1) in the form

$$\mathcal{L}^\varepsilon \psi + R + \mathcal{N}(\psi) = 0$$

and identify the linear operator $\mathcal{L}^\varepsilon$, the error R and the nonlinear terms $\mathcal{N}(\psi)$.

It will be convenient to write $S = S_0 + S_1$, where

$$S_0(v) := \Delta v + (1 - |v|^2)v, \quad S_1(v) := \varepsilon^2(\partial_{ss}^2 v - 4i\,\partial_s v - 4v). \tag{3.4}$$

We have

$$\begin{aligned} S_0(V_d + \phi) &= S_0(V_d) + L_0(\phi) + N_0(\phi), \\ S_1(V_d + \phi) &= S_1(V_d) + S_1(\phi), \end{aligned}$$

where

$$\begin{aligned} L_0(\phi) &:= \Delta\phi + (1 - |V_d|^2)\phi - 2\,\mathrm{Re}(\overline{V}_d\phi)V_d, \\ N_0(\phi) &:= -2\,\mathrm{Re}(\overline{V}_d\phi)\phi - |\phi|^2(V_d + \phi). \end{aligned} \tag{3.5}$$

Rewrite (3.2) as

$$\begin{aligned} v = V_d + \phi, \qquad \phi &:= iV_d\psi + \gamma(\psi), \\ \gamma(\psi) &:= (1-\eta)V_d(e^{i\psi} - 1 - i\psi). \end{aligned}$$

Then

$$\begin{aligned} S_0(v) &= S_0(V_d) + L_0(iV_d\psi) + L_0(\gamma(\psi)) + N_0(iV_d\psi + \gamma(\psi)), \\ S_1(v) &= S_1(V_d) + S_1(iV_d\psi) + S_1(\gamma(\psi)). \end{aligned}$$

We compute

$$L_0(iV_d\psi) = iV_d\left[\tilde{L}_0\psi + \frac{S_0(V_d)}{V_d}\psi\right],$$

where

$$\tilde{L}_0(\psi) = \Delta\psi + 2\frac{\nabla V_d \nabla\psi}{V_d} - 2i|V_d|^2\,\mathrm{Im}(\psi),$$

and so

$$\begin{aligned} S_0(v) = iV_d\Big[-i\frac{S_0(V_d)}{V_d} + \tilde{L}_0(\psi) + \frac{S_0(V_d)}{V_d}\psi \\ - \frac{i}{V_d}L_0(\gamma(\psi)) - \frac{i}{V_d}N_0(iV_d\psi + \gamma(\psi))\Big]. \end{aligned} \tag{3.6}$$

We note that, far from the vortices, we have

$$S_0(v) = S_0(V_d e^{i\psi}) = iV_d e^{i\psi}\left[-i\frac{S_0(V_d)}{V_d} + \tilde{L}_0(\psi) + \tilde{N}_0(\psi)\right], \tag{3.7}$$

where

$$\tilde{N}_0(\psi) := i(\nabla\psi)^2 + i|V_d|^2(e^{-2\,\mathrm{Im}(\psi)} - 1 + 2\,\mathrm{Im}(\psi)).$$

Similarly, we compute

$$S_1(iV_d\psi) = iV_d\left(\frac{S_1(V_d)}{V_d}\psi + \tilde{L}_1(\psi)\right), \tag{3.8}$$

where

$$\tilde{L}_1(\psi) := \varepsilon^2\left(\partial_{ss}^2\psi + \frac{2\partial_s V_d}{V_d}\partial_s\psi - 4i\partial_s\psi\right).$$

Far away from the vortices, we have

$$S_1(V_d e^{i\psi}) = iV_d e^{i\psi}\left[-i\frac{S_1(V_d)}{V_d} + \tilde{L}_1(\psi) + \varepsilon^2 i(\partial_s\psi)^2\right]. \tag{3.9}$$

We let

$$\tilde{\eta}(z) = \eta_1(|z-\tilde{d}|-1) + \eta_1(|z+\tilde{d}|-1)$$

with η_1 defined in (3.3). Then we write $S(v)=0$ as

$$\begin{aligned}
0 = \tilde{\eta} iV_d\Bigg[&-i\frac{S_0(V_d)}{V_d} + \tilde{L}_0(\psi) + \frac{S_0(V_d)}{V_d}\psi - \frac{i}{V_d}L_0(\gamma(\psi)) - \frac{i}{V_d}N_0(iV_d\psi + \gamma(\psi))\\
&- i\frac{S_1(V_d)}{V_d} + \tilde{L}_1(\psi) + \frac{S_1(V_d)}{V_d}\psi - \frac{i}{V_d}S_1(\gamma(\psi))\Bigg]\\
&+ (1-\tilde{\eta})iV_d e^{i\psi}\Bigg[-i\frac{S_0(V_d)}{V_d} + \tilde{L}_0(\psi) + \tilde{N}_0(\psi)\\
&- i\frac{S_1(V_d)}{V_d} + \tilde{L}_1(\psi) + \varepsilon^2 i(\partial_s\psi)^2\Bigg],
\end{aligned}$$

that is, we use expressions (3.6), (3.8) near the vortices and (3.7), (3.9) far from them. Hence $S(v)=0$ is equivalent to $\mathcal{L}^\varepsilon(\psi) + R + \mathcal{N}(\psi) = 0$, where

$$\begin{aligned}
\mathcal{L}^\varepsilon(\psi) &:= (\tilde{L}_0 + \tilde{L}_1)(\psi) + \tilde{\eta}\frac{S(V_d)}{V_d}\psi,\\
R &:= -i\frac{S(V_d)}{V_d},\\
\mathcal{N}(\psi) &:= \tilde{\eta}\left(\frac{1}{\tilde{\eta} + (1-\tilde{\eta})e^{i\psi}} - 1\right)\frac{S(V_d)}{V_d}\psi\\
&\quad - \frac{i}{V_d}\frac{\tilde{\eta}}{\tilde{\eta} + (1-\tilde{\eta})e^{i\psi}}\{L_0(\gamma(\psi)) + S_1(\gamma(\psi)) + N_0(iV_d\psi + \gamma(\psi))\}\\
&\quad + \frac{(1-\tilde{\eta})e^{i\psi}}{\tilde{\eta} + (1-\tilde{\eta})e^{i\psi}}\{\tilde{N}_0(\psi) + \varepsilon^2 i(\partial_s\psi)^2\}.
\end{aligned} \tag{3.10}$$

Note that explicitly

$$\begin{aligned}
\mathcal{L}^\varepsilon(\psi) = \Delta\psi &+ 2\frac{\nabla V_d \nabla\psi}{V_d} - 2i|V_d|^2\,\mathrm{Im}(\psi)\\
&+ \varepsilon^2\left(\partial_{ss}^2\psi + \frac{2\partial_s V_d}{V_d}\partial_s\psi - 4i\partial_s\psi\right) + \tilde{\eta}\frac{S(V_d)}{V_d}\psi
\end{aligned} \tag{3.11}$$

and that, for $|z \pm \tilde{d}| \geq 3$, the nonlinear terms take the form

$$\begin{aligned}\mathcal{N}(\psi) &= \tilde{N}_0(\psi) + \varepsilon^2 i(\partial_s \psi)^2 \\ &= i(\nabla\psi)^2 + i|V_d|^2(e^{-2\,\mathrm{Im}(\psi)} - 1 + 2\,\mathrm{Im}(\psi)) + \varepsilon^2 i(\partial_s\psi)^2.\end{aligned}$$

3.2. Another form of the equation near each vortex

In order to analyze the equation near each vortex, it will be useful to write it in a translated variable. Namely, we set $\tilde{d}_j := (-1)^{1+j}\tilde{d}$ for $j = 1, 2$, and we define $\tilde{z} := z - \tilde{d}_j$ and the function $\phi_j(\tilde{z})$ through the relation

$$\phi_j(\tilde{z}) := iW(\tilde{z})\psi(z), \quad |\tilde{z}| < \tilde{d}. \tag{3.12}$$

That is, so close to the vortices so that $\eta \equiv 1$,

$$\phi(z) = iV_d\psi(z) = \phi_j(\tilde{z})\alpha_j(z), \quad \text{where } \alpha_j(z) = \frac{V_d(z)}{W(z - \tilde{d}_j)}.$$

Hence, in the translated variable, the unknown (3.2) becomes, in $|\tilde{z}| < \tilde{d}$,

$$v(z) = \alpha_j(\tilde{z})\left(W(\tilde{z}) + \phi_j(\tilde{z}) + (1 - \eta_1(\tilde{z}))W(\tilde{z})\left[e^{\frac{\phi_j(\tilde{z})}{W(\tilde{z})}} - 1 - \frac{\phi_j(\tilde{z})}{W(\tilde{z})}\right]\right).$$

We set $E := S(V_d)$. For ϕ_j, ψ linked through formula (3.12), we define

$$\begin{aligned}L_j^\varepsilon(\phi_j)(\tilde{z}) := iW(\tilde{z})\mathcal{L}^\varepsilon(\psi)(\tilde{z} + \tilde{d}_j) &= \frac{L_d^\varepsilon(\phi)(z)}{\alpha_j(z)} + (\eta_1 - 1)\frac{E(z)}{V_d(z)}\phi_j(\tilde{z}) \\ &= \frac{L_d^\varepsilon(\phi_j\alpha_j)(z)}{\alpha_j(z)} + (\eta_1 - 1)\frac{E(z)}{V_d(z)}\phi_j(\tilde{z}),\end{aligned} \tag{3.13}$$

with L_d^ε defined by

$$L_d^\varepsilon(\phi) := \Delta\phi + \varepsilon^2(\partial_{ss}^2\phi - 4i\partial_s\phi - 4\phi) + (1 - |V_d|^2)\phi - 2\,\mathrm{Re}(\overline{V_d}\phi)V_d.$$

Let us also define

$$\begin{aligned}S_2(V) &:= \partial_{rr}^2 V + \frac{1}{r}\partial_r V + \frac{1}{r^2}\partial_{ss}V + \varepsilon^2(\partial_{ss}^2 V - 4i\partial_s V - 4V), \\ S_3(V) &:= \partial_{rr}^2 V + \frac{1}{r}\partial_r V + \frac{1}{r^2}\partial_{ss}V + \varepsilon^2(\partial_{ss}^2 V - 4i\partial_s V).\end{aligned}$$

Notice that

$$E(z) = S_2(\alpha_j(z)W(\tilde{z})) + (1 - |W|^2|\alpha_j|^2)W(\tilde{z})\alpha_j(z),$$

and thus, using the equation satisfied by W,

$$\begin{aligned}E &= WS_2(\alpha_j) + (1 - |W|^2|\alpha_j|^2)\alpha_j W + 2\nabla\alpha_j\nabla W + 2\varepsilon^2\partial_s\alpha_j\partial_s W + \alpha_j S_3(W) \\ &= WS_3(\alpha_j) - 4\varepsilon^2 W\alpha_j + (1 - |W|^2|\alpha_j|^2)\alpha_j W + 2\nabla\alpha_j\nabla W + 2\varepsilon^2\partial_s\alpha_j\partial_s W \\ &\quad + \alpha_j[\varepsilon^2(\partial_{ss}^2 W - 4i\partial_s W) - (1 - |W|^2)W].\end{aligned}$$

This allows us to conclude

$$
\begin{aligned}
L_j^\varepsilon(\phi_j) = L^0(\phi_j) &+ \varepsilon^2(\partial_{ss}^2\phi_j - 4i\,\partial_s\phi_j - 4\phi_j) + 2(1-|\alpha_j|^2)\,\mathrm{Re}(\overline{W}\phi_j)W \\
&- \left(2\frac{\nabla\alpha_j}{\alpha_j}\frac{\nabla W}{W} + 2\varepsilon^2\frac{\partial_s\alpha_j}{\alpha_j}\frac{\partial_s W}{W} + \varepsilon^2\frac{(\partial_{ss}^2 W - 4i\,\partial_s W)}{W} + 4\varepsilon^2\right)\phi_j \\
&+ 2\frac{\nabla\alpha_j}{\alpha_j}\nabla\phi_j + 2\varepsilon^2\frac{\partial_s\alpha_j}{\alpha_j}\partial_s\phi_j + \tilde{\eta}\frac{E_j}{V_d^j}\phi_j,
\end{aligned}
\tag{3.14}
$$

where $V_d^j = V_d(\tilde{z} + d_j)$, $E_j = S(V_d^j)$ and L^0 is the linear operator defined by (3.5).

Let us point out that, for $|\tilde{z}| < 2$,

$$
\begin{aligned}
&|\alpha_j(\tilde{z})| = 1 + O_\varepsilon(\varepsilon^2|\log\varepsilon|), \\
&\nabla\alpha_j(\tilde{z}) = O_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|}), \quad \Delta\alpha_j = O_\varepsilon(\varepsilon^2|\log\varepsilon|).
\end{aligned}
\tag{3.15}
$$

With this in mind, we can see that the linear operator L_j^ε is a small perturbation of L^0.

3.3. *Symmetries assumptions on the perturbation*

We end this section by making use of the symmetries of the problem. Using the notation $z = x_1 + ix_2 = re^{is}$, we remark that V_d satisfies

$$
V_d(-x_1, x_2) = \overline{V}_d(x_1, x_2) \quad \text{and} \quad V_d(x_1, -x_2) = \overline{V}_d(x_1, x_2).
$$

We also remark that these symmetries are compatible with the solution operator S, that is, if $S(V) = 0$ and $U(z) = \overline{V}(-x_1, x_2)$, then $S(U) = 0$, and the same for $U(z) = \overline{V}(x_1, -x_2)$. Thus we look for a solution V satisfying

$$
V(-x_1, x_2) = \overline{V}(x_1, x_2), \quad V(x_1, -x_2) = \overline{V}(x_1, x_2),
$$

which drives to ask

$$
\psi(x_1, -x_2) = -\overline{\psi}(x_1, x_2), \quad \psi(-x_1, x_2) = -\overline{\psi}(x_1, x_2). \tag{3.16}
$$

4. Error estimates of the approximated solution

In this section, we compute the error of the approximation V_d defined as $R = -iS(V_d)/V_d$ in (3.10).

In order to measure the size of the error of our approximation, we fix $0 < \alpha, \sigma < 1$. We recall that

$$
\tilde{d}_j := (-1)^{1+j}\tilde{d}, \tag{4.1}
$$

and we denote

$$
\rho_1 e^{i\theta_1} := re^{is} - \tilde{d}, \quad \rho_2 e^{i\theta_2} := re^{is} + \tilde{d},
$$

polar coordinates around each vortex. We define

$$
R_\varepsilon = \frac{\alpha_0}{\varepsilon|\log\varepsilon|^{1/2}}, \tag{4.2}
$$

where $\alpha_0 > 0$ is a small constant that will be fixed later, and the norm

$$
\begin{aligned}
\|h\|_{**} := \sum_{j=1}^{2} \|V_d h\|_{C^\alpha(\rho_j<3)} \\
&+ \sup_{\substack{\rho_1>2 \\ \rho_2>2}} \left[\frac{|\mathrm{Re}(h)|}{\rho_1^{-2} + \rho_2^{-2} + \varepsilon^2} + \frac{|\mathrm{Im}(h)|}{\rho_1^{-2+\sigma} + \rho_2^{-2+\sigma} + \varepsilon^{\sigma-2}} \right] \\
&+ \sup_{\substack{2<|z-\tilde{d}_1|<2R_\varepsilon \\ 2<|z-\tilde{d}_2|<2R_\varepsilon}} \frac{[\mathrm{Re}(h)]_{\alpha, B_{|z|/2}(z)}}{|z-\tilde{d}_1|^{-2-\alpha} + |z-\tilde{d}_2|^{-2-\alpha}} \\
&+ \sup_{\substack{2<|z-\tilde{d}_1|<2R_\varepsilon \\ 2<|z-\tilde{d}_2|<2R_\varepsilon}} \frac{[\mathrm{Im}(h)]_{\alpha, B_1(z)}}{|z-\tilde{d}_1|^{-2+\sigma} + |z-\tilde{d}_2|^{-2+\sigma}},
\end{aligned}
\tag{4.3}
$$

where $\|f\|_{C^\alpha(D)} = \|f\|_{C^{0,\alpha}(D)}$ and where we have used the notation

$$
[f]_{\alpha,D} := \sup_{\substack{x,y\in D \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \tag{4.4}
$$

$$
\|f\|_{C^{k,\alpha}(D)} := \sum_{j=0}^{k} \|D^j f\|_{L^\infty(D)} + [D^k f]_{\alpha,D}. \tag{4.5}
$$

Proposition 4.1. *Let V_d be given by* (2.2)*, and denote*

$$
S(V_d) = E = iV_d R = iV_d(R_1 + iR_2).
$$

Then

$$
\|R\|_{**} \leq \frac{C}{|\log \varepsilon|}.
$$

Proof. Let us write $V_d = W^a W^b$, where $W^a(z) := W(z - \tilde{d})$ and $W^b(z) := W(z + \tilde{d})$. We want to estimate $E := S(V_d)$, i.e., how *far* our approximation is to be a solution.

By symmetry, it suffices to compute the error in the region $(x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}$. We recall that $\Delta(fg) = g\Delta f + f\Delta g + 2\nabla f \nabla g$, and thus, with S_0 defined in (3.4),

$$
\begin{aligned}
S_0(V_d) = {} & (W^a_{x_1x_1} + W^a_{x_2x_2})W^b + (W^b_{x_1x_1} + W^b_{x_2x_2})W^a \\
& + 2(W^a_{x_1}W^b_{x_1} + W^a_{x_2}W^b_{x_2}) + (1 - |W^aW^b|^2)W^aW^b.
\end{aligned}
$$

Using the fact that $\Delta W + (1 - |W|^2)W = 0$ in $\mathbb{R}^2$, we conclude that

$$
\begin{aligned}
S_0(V_d) = {} & 2(W^a_{x_1}W^b_{x_1} + W^a_{x_2}W^b_{x_2}) \\
& + (1 - |W^aW^b|^2 + |W^a|^2 - 1 + |W^b|^2 - 1)W^aW^b.
\end{aligned}
\tag{4.6}
$$

We estimate the size of this error separately in two different regions, near the vortices and far from them. Notice first that, since we work in the half-plane $\mathbb{R}^+ \times \mathbb{R}$, we have

$$
\rho_2 \geq \tilde{d} \geq \frac{C}{\varepsilon\sqrt{|\log \varepsilon|}}
$$

for some $C > 0$ of order 1.

Step 1: Estimate of $S_0(V_d)$ near one vortex, i.e., when $|re^{is} - \tilde{d}| < 3$. Writing $W = W(\rho e^{i\theta})$, we have

$$\begin{aligned} W_{x_1} &= e^{i\theta}\left(w'(\rho)\cos\theta - i\,\frac{w(\rho)}{\rho}\sin\theta\right), \\ W_{x_2} &= e^{i\theta}\left(w'(\rho)\sin\theta + i\,\frac{w(\rho)}{\rho}\cos\theta\right). \end{aligned}$$

We define $w_1 := w(\rho_1)$ and $w_2 := w(\rho_2)$, and we obtain

$$\begin{aligned} W^a_{x_1}W^b_{x_1} &= e^{i(\theta_1+\theta_2)}\Big\{w_1'w_2'\cos\theta_1\cos\theta_2 - \frac{w_1w_2}{\rho_1\rho_2}\sin\theta_1\sin\theta_2 \\ &\qquad - i\left[\frac{w_1'w_2}{\rho_2}\cos\theta_1\sin\theta_2 + \frac{w_2'w_1}{\rho_1}\cos\theta_2\sin\theta_1\right]\Big\}, \\ W^a_{x_2}W^b_{x_2} &= e^{i(\theta_1+\theta_2)}\Big\{w_1'w_2'\sin\theta_1\sin\theta_2 - \frac{w_1w_2}{\rho_1\rho_2}\cos\theta_1\cos\theta_2 \\ &\qquad + i\left[\frac{w_1'w_2}{\rho_2}\sin\theta_1\cos\theta_2 + \frac{w_2'w_1}{\rho_1}\cos\theta_1\sin\theta_2\right]\Big\}. \end{aligned}$$

Since $w'(\rho) = 1/\rho^3 + O(1/\rho^4)$ when $\rho \to +\infty$ (Lemma A.1) and $\rho_2 \geq C/(\varepsilon\sqrt{|\log\varepsilon|})$, we can see that

$$\|W^a_{x_1}W^b_{x_1} + W^a_{x_2}W^b_{x_2}\|_{L^\infty(\rho_1<3)} \leq C\varepsilon\sqrt{|\log\varepsilon|}$$

when ε is small and for some $C > 0$. Using now that $w(\rho) = 1 - 1/(2\rho^2) + O(1/\rho^4)$ when $\rho \to +\infty$, we obtain

$$\|(1 - |W^aW^b|^2 + |W^a|^2 - 1 + |W^b|^2 - 1)W^aW^b\|_{L^\infty(\rho_1<3)} \leq C\varepsilon^2|\log\varepsilon|,$$

and thus

$$\|E_0\|_{L^\infty(\rho_1<3)} = \|S_0(V_d)\|_{L^\infty(\rho_1<3)} \leq C\varepsilon\sqrt{|\log\varepsilon|}. \tag{4.7}$$

Similarly,

$$\|\nabla E_0\|_{L^\infty(\rho_1<3)} \leq C\varepsilon\sqrt{|\log\varepsilon|}. \tag{4.8}$$

Step 2: Estimate of $S_0(V_d)$ *far away* from the vortices, i.e., when $|re^{is} - \tilde{d}| > 2$. We write $E_0 = S_0(V_d) = iV_d(R_0^1 + iR_0^2)$ with

$$\begin{aligned} R_0^1 &= 2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\left(\frac{w_1'}{\rho_2w_1} - \frac{w_2'}{\rho_1w_2}\right) \\ &= 2\sin(\theta_1-\theta_2)\left(\frac{w_1'}{\rho_2w_1} - \frac{w_2'}{\rho_1w_2}\right), \\ R_0^2 &= 2\cos(\theta_1-\theta_2)\left(\frac{-w_1'w_2'}{w_1w_2} + \frac{1}{\rho_1\rho_2}\right) - \left(1 - w_1^2w_2^2 + w_2^2 - 1 + w_1^2 - 1\right). \end{aligned}$$

Using that $\rho_1 \leq \rho_2$ and $\rho_2 \geq C/(\varepsilon\sqrt{|\log\varepsilon|})$, along with Lemma A.1, we conclude

$$|R_0^1| \leq C\varepsilon\sqrt{|\log\varepsilon|}\,\frac{1}{\rho_1^3}. \tag{4.9}$$

Using again Lemma A.1, we obtain

$$
\begin{aligned}
1 - w_1^2 w_2^2 + w_2^2 - 1 + w_1^2 - 1 &= 1 - w_1^2 + O\left(\frac{1}{\rho_2^2}\right) w_1^2 + O\left(\frac{1}{\rho_2^2}\right) + w_1^2 - 1 \\
&\le C \frac{1}{\rho_2} \frac{1}{\rho_1},
\end{aligned}
$$

and hence

$$
|R_0^2| \le C \frac{1}{\rho_2 \rho_1} \le C(\varepsilon |\log \varepsilon|^{1/2})^{\sigma} \frac{1}{\rho_1^{2-\sigma}} \quad \text{for all } 0 < \sigma < 1. \tag{4.10}
$$

To see that the previous inequality holds, we can distinguish the cases $2 < \rho_1 < \tilde{d} < \rho_2$ and $\tilde{d} < \rho_1 < \rho_2$. We remark that we also have

$$
|\nabla R_0^1| \le \frac{C\varepsilon \sqrt{|\log \varepsilon|}}{\rho_1^4}, \quad |\nabla R_0^2| \le \frac{C(\varepsilon \sqrt{|\log \varepsilon|})^{\sigma}}{\rho_1^{3-\sigma}}.
$$

Step 3: Estimates of $S_1(V_d)$. We recall that $w_1 := w(\rho_1)$ and $w_2 := w(\rho_2)$. Thus we have $V_d(r,s) =: w_1 w_2 e^{i(\theta_1+\theta_2)}$ with

$$
\begin{aligned}
\rho_1 &= \sqrt{(r\cos s - \tilde{d})^2 + r^2 \sin^2 s}, & \rho_2 &= \sqrt{(r\cos s + \tilde{d})^2 + r^2 \sin^2 s}, \\
e^{i\theta_1} &= \frac{(r\cos s - \tilde{d}) + ir\sin s}{\rho_1}, & e^{i\theta_2} &= \frac{(r\cos s + \tilde{d}) + ir\sin s}{\rho_2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\partial_s V_d &= [\partial_s \rho_1 w_1' w_2 + \partial_s \rho_2 w_2' w_1 + i\partial_s(\theta_1 + \theta_2) w_1 w_2] e^{i(\theta_1+\theta_2)}, \\
\partial_{ss}^2 V_d &= \big\{\big[\partial_{ss}^2 \rho_1 w_1' w_2 + \partial_{ss}^2 \rho_2 w_2' w_1 + (\partial_s \rho_1)^2 w_1'' w_2 + (\partial_s \rho_2)^2 w_2'' w_1 \\
&\qquad + 2\partial_s \rho_1 \partial_s \rho_2 w_1' w_2' - [\partial_s(\theta_1 + \theta_2)]^2 w_1 w_2\big] \\
&\quad + i[2\partial_s(\theta_1 + \theta_2)(\partial_s \rho_1 w_1' w_2 + \partial_s \rho_2 w_2' w_1) \\
&\qquad + \partial_{ss}^2(\theta_1 + \theta_2) w_1 w_2]\big\} e^{i(\theta_1+\theta_2)},
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\varepsilon^2(\partial_{ss}^2 V_d - 4i\partial_s V_d - 4V_d) \\
&\quad = \varepsilon^2 \big\{(\partial_s \rho_1)^2 w_1'' w_2 + (\partial_s \rho_2)^2 w_2'' w_1 + 2(\partial_s \rho_1)(\partial_s \rho_2) w_1' w_2' \\
&\qquad\qquad + \partial_{ss}^2 \rho_1 w_1' w_2 + \partial_{ss}^2 \rho_2 w_2' w_1 \\
&\qquad\qquad - \big([\partial_s(\theta_1 + \theta_2)]^2 - 4\partial_s(\theta_1 + \theta_2) + 4\big) w_1 w_2\big\} e^{i(\theta_1+\theta_2)} \\
&\qquad + i\varepsilon^2 \big\{\partial_{ss}^2(\theta_1 + \theta_2) w_1 w_2 \\
&\qquad\qquad + \big(2[\partial_s(\theta_1 + \theta_2)] - 4\big)[\partial_s \rho_1 w_1' w_2 + \partial_s \rho_2 w_2' w_1]\big\} e^{i(\theta_1+\theta_2)}.
\end{aligned}
$$

We also need to compute the following derivatives:

$$
\partial_s \rho_1 = \frac{r\tilde{d}\sin s}{\rho_1} = \tilde{d}\sin\theta_1, \quad \partial_s \rho_2 = \frac{-r\tilde{d}\sin s}{\rho_2} = -\tilde{d}\sin\theta_2,
$$

$$\partial_{ss}^2 \rho_1 = \frac{r\tilde{d}\cos s}{\rho_1} - \frac{r^2\tilde{d}^2\sin^2 s}{\rho_1^3} = \tilde{d}\cos\theta_1 + \tilde{d}^2\frac{\cos^2\theta_1}{\rho_1},$$
$$\partial_{ss}^2 \rho_2 = \frac{-r\tilde{d}\cos s}{\rho_2} - \frac{r^2\tilde{d}^2\sin^2 s}{\rho_2^3} = -\tilde{d}\cos\theta_2 + \tilde{d}^2\frac{\cos^2\theta_2}{\rho_2}.$$

Now we can check that

$$\partial_s\theta_1 = 1 + \frac{\tilde{d}}{\rho_1^2}(r\cos s - \tilde{d}) = 1 + \frac{\tilde{d}\cos\theta_1}{\rho_1},$$
$$\partial_s\theta_2 = 1 - \frac{\tilde{d}}{\rho_2^2}(r\cos s + \tilde{d}) = 1 - \frac{\tilde{d}\cos\theta_2}{\rho_2}$$
$$\partial_{ss}^2\theta_1 = \frac{-\tilde{d}r\sin s}{\rho_1^4}\big(\rho_1^2 + 2\tilde{d}(r\cos s - \tilde{d})\big) = \frac{-\tilde{d}\sin\theta_1}{\rho_1} - \frac{2\tilde{d}^2\sin\theta_1\cos\theta_1}{\rho_1^2},$$
$$\partial_{ss}^2\theta_2 = \frac{\tilde{d}r\sin s}{\rho_2^4}\big(\rho_2^2 - 2\tilde{d}(r\cos s + \tilde{d})\big) = \frac{\tilde{d}\sin\theta_2}{\rho_2} - \frac{2\tilde{d}^2\sin\theta_2\cos\theta_2}{\rho_2^2},$$

and besides,

$$[\partial_s(\theta_1+\theta_2)]^2 - 4\partial_s(\theta_1+\theta_2) + 4 = \tilde{d}^2\left(\frac{\cos\theta_1}{\rho_1} - \frac{\cos\theta_2}{\rho_2}\right)^2,$$
$$\partial_{ss}^2(\theta_1+\theta_2) = \tilde{d}\left(\frac{\sin\theta_2}{\rho_2} - \frac{\sin\theta_1}{\rho_1}\right) - 2\tilde{d}^2\left(\frac{\sin\theta_1\cos\theta_1}{\rho_1^2} + \frac{\sin\theta_2\cos\theta_2}{\rho_2^2}\right),$$
$$2\partial_s(\theta_1+\theta_2) - 4 = 2\tilde{d}\left(\frac{\cos\theta_1}{\rho_1} - \frac{\cos\theta_2}{\rho_2}\right).$$

Hence we obtain

$$\begin{aligned} S_1(V_d) = \Bigg\{ &\frac{\hat{d}^2}{|\log\varepsilon|}\Bigg(w_1''w_2\sin^2\theta_1 + w_2''w_1\sin^2\theta_2 - w_1'w_2'\sin\theta_1\sin\theta_2 \\ &\qquad + \frac{\cos^2\theta_1}{\rho_1}w_1'w_2 + \frac{\cos^2\theta_2}{\rho_2}w_2'w_1 - \left(\frac{\cos\theta_1}{\rho_1} - \frac{\cos\theta_2}{\rho_2}\right)^2 w_1w_2\Bigg) \\ &+ \frac{\varepsilon\hat{d}}{\sqrt{|\log\varepsilon|}}(\cos\theta_1 w_1'w_2 - \cos\theta_2 w_2'w_1) \\ &+ i\Bigg[\frac{\varepsilon\hat{d}}{\sqrt{|\log\varepsilon|}}\left(\frac{\sin\theta_2}{\rho_2} - \frac{\sin\theta_1}{\rho_1}\right)w_1w_2 \\ &\qquad - \frac{2\hat{d}^2}{|\log\varepsilon|}\left(\frac{\sin\theta_1\cos\theta_1}{\rho_1^2} + \frac{\sin\theta_2\cos\theta_2}{\rho_2^2}\right)w_1w_2 \\ &\qquad + \frac{2\hat{d}^2}{|\log\varepsilon|}\left(\frac{\cos\theta_1}{\rho_1} - \frac{\cos\theta_2}{\rho_2}\right) \\ &\qquad\qquad \times(\sin\theta_1 w_1'w_2 - \sin\theta_2 w_2'w_1)\Bigg]\Bigg\} e^{i(\theta_1+\theta_2)}. \end{aligned} \tag{4.11}$$

The conclusion of the proof follows from (4.7), (4.8), (4.9), (4.10) and Lemma 4.1. We also use the symmetry of the problem. ∎

Lemma 4.1. *Let $S_1(V_d) = iV_d R_1 = iV_d(R_1^1 + iR_1^2)$. In the half-plane $\mathbb{R}^+ \times \mathbb{R}$, we have*

$$\|S_1(V_d)\|_{L^\infty(\rho_1<3)} \le \frac{C}{|\log \varepsilon|}, \quad \|\nabla S_1(V_d)\|_{L^\infty(\rho_1<3)} \le \frac{C}{|\log \varepsilon|}, \tag{4.12}$$

and for $\rho_1 > 2$,

$$\begin{aligned} |R_1^1| &\le \frac{C}{|\log \varepsilon|}\frac{1}{\rho_1^2}, \quad |\nabla R_1^1| \le \frac{C}{|\log \varepsilon|}\frac{1}{\rho_1^3}, \\ |R_1^2| &\le \frac{C}{|\log \varepsilon|}\frac{1}{\rho_1^2}, \quad |\nabla R_1^2| \le \frac{C}{|\log \varepsilon|}\frac{1}{\rho_1^3}. \end{aligned} \tag{4.13}$$

Proof. By using Lemma A.1, we see that

$$\|S_1(V_d)\|_{L^\infty(\rho_1<3)} \le \frac{C}{|\log \varepsilon|}, \quad \|\nabla S_1(V_d)\|_{L^\infty(\rho_1<3)} \le \frac{C}{|\log \varepsilon|}.$$

For $\rho_1 > 2$, we have

$$\begin{aligned} -R_1^2 = {} & \frac{\hat{d}^2}{|\log \varepsilon|}\left(\frac{w_1''}{w_1}\sin^2\theta_1 + \frac{\cos^2\theta_1}{\rho_1}\frac{w_1'}{w_1} - \left(\frac{\cos\theta_1}{\rho_1} - \frac{\cos\theta_2}{\rho_2}\right)^2\right) \\ & + \frac{\hat{d}^2}{|\log \varepsilon|}\left(\frac{w_2''}{w_2}\sin^2\theta_2 - w_1'w_2'\sin\theta_1\sin\theta_2 + \frac{\cos^2\theta_2}{\rho_2}\frac{w_2'}{w_2}\right) \\ & + \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}}\left(\cos\theta_1\frac{w_1'}{w_1} - \cos\theta_2\frac{w_2'}{w_2}\right). \end{aligned}$$

By using Lemma A.1 and the fact that $\rho_2 \ge \rho_1 > 2$, we can see that

$$\frac{\hat{d}^2}{|\log \varepsilon|}\left|\frac{w_1''}{w_1}\sin^2\theta_1 + \frac{\cos^2\theta_1}{\rho_1}\frac{w_1'}{w_1} - \left(\frac{\cos\theta_1}{\rho_1} - \frac{\cos\theta_2}{\rho_2}\right)^2\right| \le \frac{C}{|\log \varepsilon|}\frac{1}{\rho_1^2},$$

and

$$\frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}}\left|\cos\theta_1\frac{w_1'}{w_1} - \cos\theta_2\frac{w_2'}{w_2}\right| \le \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}\frac{1}{\rho_1^3}.$$

Besides, by using also that $\rho_2 \ge \tilde{d} \ge \hat{d}/(\varepsilon\sqrt{|\log \varepsilon|})$, we observe that

$$\frac{\hat{d}^2}{|\log \varepsilon|}\left|\frac{w_2''}{w_2}\sin^2\theta_2 - w_1'w_2'\sin\theta_1\sin\theta_2 + \frac{\cos^2\theta_2}{\rho_2}\frac{w_2'}{w_2}\right| \le C\frac{\varepsilon}{\sqrt{|\log \varepsilon|}}\frac{1}{\rho_1^2}.$$

Thus we obtain (4.12) and the third estimate in (4.13). By differentiating, we can also obtain the fourth estimate.

Now, for $\rho_1 > 2$, we see that

$$\begin{aligned} R_1^1 = {} & \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}}\left(\frac{\sin\theta_2}{\rho_2} - \frac{\sin\theta_1}{\rho_1}\right) - \frac{2\hat{d}^2}{|\log \varepsilon|}\left(\frac{\sin\theta_1\cos\theta_1}{\rho_1^2} + \frac{\sin\theta_2\cos\theta_2}{\rho_2^2}\right) \\ & + \frac{2\hat{d}^2}{|\log \varepsilon|}\left[\frac{\cos\theta_1\sin\theta_1 w_1'}{\rho_1 w_1} - \frac{\cos\theta_1\sin\theta_2 w_2'}{\rho_1 w_2}\right] \\ & - \frac{2\hat{d}^2}{|\log \varepsilon|}\frac{\cos\theta_2}{\rho_2}\left(\sin\theta_1\frac{w_1'}{w_1} - \sin\theta_2\frac{w_2'}{w_2}\right). \end{aligned}$$

By using Lemma A.1 and the fact that $\rho_2 \geq \rho_1$, we can see that the two first estimates of (4.13) hold. Actually, to prove the first estimate, the only difficult term to handle is

$$\frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}}\left(\frac{\sin\theta_1}{\rho_1} - \frac{\sin\theta_2}{\rho_2}\right).$$

In the region $(\mathbb{R}^+ \times \mathbb{R}) \cap \{\rho_1 < C/(\varepsilon\sqrt{|\log \varepsilon|})\}$, we have $\varepsilon\sqrt{|\log\varepsilon|}/\rho_1 < C/\rho_1^2$. By using this and that $\rho_2 > \rho_1$, we find that

$$\left|\frac{\tilde{d}\varepsilon}{\sqrt{|\log \varepsilon|}} w_1 w_2 \left(\frac{\sin\theta_1}{\rho_1} - \frac{\sin\theta_2}{\rho_2}\right)\right| \leq \frac{C}{|\log\varepsilon|}\frac{1}{1+\rho_1^2} \tag{4.14}$$

in $(\mathbb{R}^+ \times \mathbb{R}) \cap \{\rho_1 < 1/(\varepsilon\sqrt{|\log\varepsilon|})\}$.

Now we use $\rho_1 \sin\theta_1 = \rho_2 \sin\theta_2 = r \sin s$ to obtain

$$\left(\frac{\sin\theta_1}{\rho_1} - \frac{\sin\theta_2}{\rho_2}\right) = \frac{\sin\theta_1}{\rho_1}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right).$$

But $\rho_2^2 = \rho_1^2 + 4\tilde{d} r \cos s = \rho_1^2 + 4\tilde{d}\rho_1\cos\theta_1 + \tilde{d}^2$. Thus, when $|4\tilde{d}\rho_1\cos\theta_1 + \tilde{d}^2| < 1$, which is true when $\rho_1 \geq C/(\varepsilon\sqrt{|\log\varepsilon|})$ for an appropriate constant $C > 0$, we find that

$$\frac{\rho_1^2}{\rho_2^2} = 1 + 4\frac{\hat{d}}{\rho_1}\cos\theta_1 + O\left(\frac{\tilde{d}^2}{\rho_1^2}\right).$$

Thus

$$\left|\frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} w_1 w_2 \left(\frac{\sin\theta_1}{\rho_1} - \frac{\sin\theta_2}{\rho_2}\right)\right| \leq \frac{C}{|\log\varepsilon|}\frac{1}{\rho_1^2} \tag{4.15}$$

in $(\mathbb{R}^+ \times \mathbb{R}) \cap \{\rho_1 > C/(\varepsilon\sqrt{|\log\varepsilon|})\}$. Combining estimates (4.14) and (4.15) and differentiating, we arrive at the conclusion. ∎

Recall the polar coordinates ρ_j, θ_j about $\tilde{d}_j$ defined by the relation $z = \rho_j e^{i\theta_j} + \tilde{d}_j$. We can decompose a function h satisfying $h(\overline{z}) = -\overline{h}(z)$ in Fourier series in θ_j as

$$h = \sum_{k=0}^{\infty} h^{k,j}, \tag{4.16}$$

$$h^{k,j}(\rho_j, \theta_j) := h_1^{k,j}(\rho_j)\sin(k\theta_j) + i h_2^{k,j}(\rho_j)\cos(k\theta_j), \quad h_1^{k,j}(\rho_j), h_2^{k,j}(\rho_j) \in \mathbb{R},$$

and define

$$h^{e,j} := \sum_{k \text{ even}} h^{k,j}, \quad h^{o,j} := \sum_{k \text{ odd}} h^{k,j}.$$

The definitions above can also be expressed by the following. Let $\mathcal{R}_j$ denote the reflection across the line $\mathrm{Re}(z) = \tilde{d}_j$. Since $\tilde{d}_j \in \mathbb{R}$, we have

$$\mathcal{R}_j z = 2\tilde{d}_j - \mathrm{Re}(z) + i\,\mathrm{Im}(z). \tag{4.17}$$

Then $h^{e,j}$ and $h^{o,j}$ have the symmetries

$$h^{o,j}(\mathcal{R}_j z) = \overline{h^{o,j}(z)}, \quad h^{e,j}(\mathcal{R}_j z) = -\overline{h^{e,j}(z)},$$

and we can define equivalently

$$h^{o,j}(z) := \frac{1}{2}[h(z) + \overline{h(\mathcal{R}_j z)}], \quad h^{e,j}(z) := \frac{1}{2}[h(z) - \overline{h(\mathcal{R}_j z)}].$$

It is convenient to consider a global function h^o defined as follows. We introduce cut-off functions $\eta_{j,R}$ as

$$\eta_{j,R}(z) := \eta_1\left(\frac{|z-\tilde{d}_j|}{R}\right), \tag{4.18}$$

where $\eta_1\colon \mathbb{R} \to [0,1]$ is a smooth function such that $\eta_1(t) = 1$ for $t \le 1$ and $\eta_1(t) = 0$ for $t \ge 2$. Given $h\colon \mathbb{C} \to \mathbb{C}$, consider R_ε defined in (4.2) with $\alpha_0 > 0$ fixed small enough so that $R_\varepsilon \le \frac{1}{2}\tilde{d}$ and

$$\begin{aligned} h^o &:= \eta_{1,R_\varepsilon} h^{o,1} + \eta_{2,R_\varepsilon} h^{o,2}, \\ h^e &:= h - h^o. \end{aligned} \tag{4.19}$$

For a complex function $h = h_1 + ih_2$, we introduce the new seminorm

$$|h|_{\sharp\sharp} := \sum_{j=1}^{2} \|V_d h\|_{C^{0,\alpha}(\rho_j<4)} + \sup_{\substack{2<\rho_1<R_\varepsilon \\ 2<\rho_2<R_\varepsilon}} \left[\frac{|h_1|}{\rho_1^{-1}+\rho_2^{-1}} + \frac{|h_2|}{\rho_1^{-1+\sigma}+\rho_2^{-1+\sigma}}\right],$$

where $0 < \alpha, \sigma < 1$ are constants to be selected later. We then have the following proposition.

Proposition 4.2. *Let V_d be given by (2.2), and denote $S(V_d) = E = iV_d R$. Then we can write $R = R^o + R^e$ and $R^o = R^o_\alpha + R^o_\beta$ with R^o defined analogously to (4.19) and $R^o(\mathcal{R}_j z) = \overline{R^o(z)}$ in $B_{R_\varepsilon}(\tilde{d}) \cup B_{R_\varepsilon}(-\tilde{d})$,*

$$|R^o_\alpha|_{\sharp\sharp} \le C\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}, \quad \|R^o_\beta\|_{**} \le C\varepsilon\sqrt{|\log\varepsilon|}, \quad \|R^e\|_{**} + \|R^o\|_{**} \le \frac{C}{|\log\varepsilon|}.$$

Proof. The conclusion follows using the expression of $S_1(V_d)$ given by (4.11). More precisely, we define

$$\begin{aligned} r^{o,1} &:= -i\frac{\hat{d}^2}{|\log\varepsilon|}\left(\frac{2\cos\theta_1\cos\theta_2}{\rho_1\rho_2} + \frac{\cos^2\theta_2}{\rho_2^2}\right) \\ &\quad + \left\{\frac{\hat{d}\varepsilon}{\sqrt{|\log\varepsilon|}}\frac{-\sin\theta_1}{\rho_1} - \frac{2\hat{d}^2}{|\log\varepsilon|}\frac{\sin\theta_2\cos\theta_2}{\rho_2^2}\right\}, \\ r^{o,2} &:= -i\frac{\hat{d}^2}{|\log\varepsilon|}\left(\frac{2\cos\theta_1\cos\theta_2}{\rho_1\rho_2} + \frac{\cos^2\theta_1}{\rho_1^2}\right) \\ &\quad + \left\{\frac{\hat{d}\varepsilon}{\sqrt{|\log\varepsilon|}}\frac{+\sin\theta_2}{\rho_2} - \frac{2\hat{d}^2}{|\log\varepsilon|}\frac{\sin\theta_1\cos\theta_1}{\rho_1^2}\right\}, \\ R^{j,o}_\alpha &:= \frac{1}{2}[r^{o,1}(z) + \overline{r^{o,2}(\mathcal{R}_j z)}], \quad j = 1,2, \\ R^o_\alpha &:= \eta_{1,R_\varepsilon} R^{1,o}_\alpha + \eta_{2,R_\varepsilon} R^{2,o}_\alpha, \end{aligned}$$

with R_ε defined by (4.2). We can check that R^o_α and $R^o_\beta := R^o - R^o_\alpha$ satisfy the desired properties. ∎

In the last step of the proof of Theorem 1, when we aim at canceling the Lyapunov–Schmidt coefficient, we will need the following.

Lemma 4.2. *In the region $B(\tilde{d}, \tilde{d})$, we have*

$$S_1(V_d) = \frac{\hat{d}}{|\log \varepsilon|} W^a_{x_2 x_2} W^b + \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} W^a_{x_1} W^b + G$$

with

$$\mathrm{Re} \int_{B(\tilde{d}, \frac{\hat{d}}{\varepsilon\sqrt{|\log \varepsilon|}})} W^a_{x_2 x_2} \overline{W}^a_{x_1} = 0,$$
$$\mathrm{Re} \int_{B(\tilde{d}, \frac{\hat{d}}{\varepsilon\sqrt{|\log \varepsilon|}})} \frac{G}{W^b} \overline{W}^a_{x_1} = O_\varepsilon\left(\frac{\varepsilon}{\sqrt{|\log \varepsilon|}}\right).$$

Proof. It suffices to observe that

$$\begin{aligned} S_1(V_d) = E_1 = \frac{\hat{d}^2 w_2 e^{i(\theta_1+\theta_2)}}{|\log \varepsilon|} \Bigg(& w_1'' \sin^2\theta_1 + \cos^2\theta_1 \left(\frac{w_1'}{\rho_1} - \frac{w_1}{\rho_1^2}\right) \\ & + 2i \cos\theta_1 \sin\theta_1 \left(\frac{w_1'}{\rho_1} - \frac{w_1}{\rho_1^2}\right)\Bigg) \\ & + \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} \left(w_1' w_2 \cos\theta_1 - i w_1 w_2 \frac{\sin\theta_1}{\rho_1}\right) e^{i(\theta_1+\theta_2)} + G, \end{aligned}$$

where

$$\begin{aligned} G := {} & \frac{\hat{d}\varepsilon e^{i(\theta_1+\theta_2)}}{\sqrt{|\log \varepsilon|}} w_1 w_2' \cos\theta_2 \\ & + \frac{\hat{d}^2 e^{i(\theta_1+\theta_2)}}{|\log \varepsilon|} \Bigg(w_2'' w_1 \sin^2\theta_2 + w_1' w_2' \sin\theta_1 \sin\theta_2 + \frac{\cos^2\theta_2}{\rho_2} w_2' w_1 \\ & \qquad + \left(\frac{2\cos\theta_1 \cos\theta_2}{\rho_1 \rho_2} + \frac{\cos^2\theta_2}{\rho_2^2}\right) w_1 w_2 \Bigg) \\ & + i e^{i(\theta_1+\theta_2)} \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} \frac{\sin\theta_2}{\rho_2} w_1 w_2 \\ & - i e^{i(\theta_1+\theta_2)} \Bigg\{ \frac{2\hat{d}^2}{|\log \varepsilon|} \frac{\sin\theta_2 \cos\theta_2}{\rho_2^2} w_1 w_2 \\ & \qquad + \frac{2\hat{d}^2}{|\log \varepsilon|} \frac{\cos\theta_2}{\rho_2} (\sin\theta_1 w_1' w_2 + \sin\theta_2 w_2' w_1) \\ & \qquad + \frac{2\hat{d}^2}{|\log \varepsilon|} \frac{\cos\theta_1}{\rho_1} \sin\theta_2 w_2' w_1 \Bigg\}. \end{aligned}$$

∎

5. A projected linear problem

Given h satisfying symmetries (3.16) and appropriate decay, our aim in this section is to solve the linear equation

$$\begin{cases} \mathcal{L}^{\varepsilon}(\psi) = h + c \displaystyle\sum_{j=1}^{2} \frac{\chi_j}{iW(z-\tilde{d}_j)}(-1)^j W_{x_1}(z-\tilde{d}_j) \quad \text{in } \mathbb{R}^2, \\ \operatorname{Re} \displaystyle\int_{B(0,4)} \chi \overline{\phi_j} W_{x_1} = 0 \quad \text{with } \phi_j(z) = iW(z)\psi(z+\tilde{d}_j), \\ \psi \text{ satisfies symmetry (3.16)}, \end{cases} \tag{5.1}$$

where

$$\chi(z) := \eta_1\left(\frac{|z|}{2}\right), \quad \chi_j(z) := \eta_1\left(\frac{\rho_j}{2}\right) = \eta_1\left(\frac{|z-\tilde{d}_j|}{2}\right)$$

with η_1 a smooth cut-off function such that $\eta_1(t) = 1$ if $t \le 1$ and $\eta_1(t) = 0$ if $t \ge 2$.

Thanks to the symmetries imposed on ψ and h, it suffices to use one reduced parameter c and not six as it should be the case when working with two vortices, since the linearized operator around each has three elements in its kernel.

In order to find estimates on the solution of (5.1), we introduce some norms, for which we use the following notation. Let $\tilde{d}_j$, $j = 1, 2$, denote the center of each vortex as in (4.1). We recall that (ρ_j, θ_j) are polar coordinates around $\tilde{d}_j$, that is, $z = \rho_j e^{i\theta_j} + \tilde{d}_j$.

We will define two sets of norms. The first one is the following: given $\alpha, \sigma \in (0, 1)$ and $\psi: \mathbb{C} \to \mathbb{C}$, we define

$$\|\psi\|_* := \sum_{j=1}^{2} \|V_d \psi\|_{C^{2,\alpha}(\rho_j<3)} + \|\operatorname{Re}(\psi)\|_{1,*} + \|\operatorname{Im}(\psi)\|_{2,*},$$

where, with $\operatorname{Re}\psi = \psi_1, \operatorname{Im}\psi = \psi_2$,

$$\begin{aligned} \|\psi_1\|_{1,*} := {} & \sup_{\substack{\rho_1>2\\ \rho_2>2}} |\psi_1| + \sup_{\substack{2<\rho_1<\frac{2}{\varepsilon}\\ 2<\rho_2<\frac{2}{\varepsilon}}} \frac{|\nabla\psi_1|}{\rho_1^{-1} + \rho_2^{-1}} + \sup_{r>\frac{1}{\varepsilon}} \left[\frac{1}{\varepsilon}|\partial_r\psi_1| + |\partial_s\psi_1|\right] \\ & + \sup_{\substack{2<\rho_1<R_\varepsilon\\ 2<\rho_2<R_\varepsilon}} \frac{|D^2\psi_1|}{\rho_1^{-2} + \rho_2^{-2}} + \sup_{\substack{2<|z-\tilde{d}_1|<R_\varepsilon\\ 2<|z-\tilde{d}_2|<R_\varepsilon}} \frac{[D^2\psi_1]_{\alpha, B_{|z|/2}(z)}}{|z-\tilde{d}_1|^{-2-\alpha} + |z-\tilde{d}_2|^{-2-\alpha}}, \end{aligned}$$

$$\begin{aligned} \|\psi_2\|_{2,*} := {} & \sup_{\substack{\rho_1>2\\ \rho_2>2}} \frac{|\psi_2|}{\rho_1^{-2+\sigma} + \rho_2^{-2+\sigma} + \varepsilon^{\sigma-2}} + \sup_{\substack{2<\rho_1<\frac{2}{\varepsilon}\\ 2<\rho_2<\frac{2}{\varepsilon}}} \frac{|\nabla\psi_2|}{\rho_1^{-2+\sigma} + \rho_2^{-2+\sigma}} \\ & + \sup_{r>\frac{1}{\varepsilon}} [\varepsilon^{\sigma-2}|\partial_r\psi_2| + \varepsilon^{\sigma-1}|\partial_s\psi_2|] + \sup_{\substack{2<\rho_1<R_\varepsilon\\ 2<\rho_2<R_\varepsilon}} \frac{|D^2\psi_2|}{\rho_1^{-2+\sigma} + \rho_2^{-2+\sigma}} \\ & + \sup_{\substack{2<|z-\tilde{d}_1|<R_\varepsilon\\ 2<|z-\tilde{d}_2|<R_\varepsilon}} \frac{[D^2\psi_2]_{\alpha, B_1(z)}}{|z-\tilde{d}_1|^{-2+\sigma} + |z-\tilde{d}_2|^{-2+\sigma}}. \end{aligned}$$

Here we have used notation (4.4)–(4.5). We recall also that the norm for the right-hand side h of (5.1) is defined by (4.3). One of the main results in this section is the following.

Proposition 5.1. *If h satisfies (3.16) and $\|h\|_{**} < +\infty$, then, for $\varepsilon > 0$ sufficiently small, there exists a unique solution $\psi = T_\varepsilon(h)$ to (5.1) with $\|\psi\|_* < \infty$. Furthermore, there exists a constant $C > 0$ depending only on $\alpha, \sigma \in (0,1)$ such that this solution satisfies*

$$\|\psi\|_* \le C\|h\|_{**}.$$

The proof of Proposition 5.1 is in Section 5.1.

Although the existence and estimate in Proposition 5.1 are sufficient to solve a nonlinear projected problem, the estimates for ψ are too weak to enable us to solve the reduced problem. This means that they are too weak to justify that we can choose the parameter d such that the Lyapunov–Schmidt coefficient c in (5.1) vanishes. In order to address this difficulty, we use that the largest part of the error and ψ have a symmetry that makes them orthogonal to the kernel. To state the extra (partial) symmetry involved, let us consider $\psi\colon \mathbb{C} \to \mathbb{C}$. Recall the polar coordinates ρ_j, θ_j about $\tilde{d}_j$ defined by the relation $z = \rho_j e^{i\theta_j} + \tilde{d}_j$. We can decompose ψ in Fourier series in θ_j as in (4.16) and define

$$\psi^{e,j} := \sum_{k \text{ even}} \psi^{k,j}, \quad \psi^{o,j} := \sum_{k \text{ odd}} \psi^{k,j}.$$

The intuitive idea is that $\psi^{o,j}$ is not orthogonal to the kernel near $\tilde{d}_j$ but small, while $\psi^{e,j}$ is large but orthogonal to the kernel near $\tilde{d}_j$ by symmetry. With $\mathcal{R}_j$ defined in (4.17), we have

$$\psi^{o,j}(\mathcal{R}_j z) = \overline{\psi^{o,j}(z)},$$
$$\psi^{e,j}(\mathcal{R}_j z) = -\overline{\psi^{e,j}(z)},$$

and we can define equivalently

$$\psi^{o,j}(z) := \frac{1}{2}[\psi(z) + \overline{\psi(\mathcal{R}_j z)}],$$
$$\psi^{e,j}(z) := \frac{1}{2}[\psi(z) - \overline{\psi(\mathcal{R}_j z)}].$$

It is convenient to consider a global function ψ^o defined as follows: with R_ε given by (4.2) and $\eta_{j,R}$ defined in (4.18), we set

$$\psi^o := \eta_{1,\frac{1}{2}R_\varepsilon}\psi^{o,1} + \eta_{2,\frac{1}{2}R_\varepsilon}\psi^{o,2}, \tag{5.2}$$

That is, ψ^o represents the *odd* part of ψ around each vortex $\tilde{d}_j$, localized with a cut-off function.

The part of ψ that will be small, namely ψ^o, will be estimated in norms that allow for growth up to a certain distance. We do this because that part arises from terms in the error R^o that are small, but decay slowly. To capture this behavior, we define

$$|\psi|_\sharp := \sum_{j=1}^{2} |\log\varepsilon|^{-1}\|V_d\psi\|_{C^{2,\alpha}(\rho_j<3)} + |\mathrm{Re}(\psi)|_{\sharp,1} + |\mathrm{Im}(\psi)|_{\sharp,2},$$

where

$$|\psi_1|_{\sharp,1} := \sup_{\substack{2<\rho_1<R_\varepsilon \\ 2<\rho_2<R_\varepsilon}} \Bigg[\frac{|\psi_1|}{\rho_1 \log(2R_\varepsilon/\rho_1) + \rho_2 \log(2R_\varepsilon/\rho_2)} + \frac{|\nabla\psi_1|}{\log(2R_\varepsilon/\rho_1) + \log(2R_\varepsilon/\rho_2)} \Bigg], \tag{5.3}$$

$$|\psi_2|_{\sharp,2} := \sup_{\substack{2<\rho_1<R_\varepsilon \\ 2<\rho_2<R_\varepsilon}} \Bigg[\frac{|\psi_2| + |\nabla\psi_2|}{\rho_1^{-1+\sigma} + \rho_2^{-1+\sigma} + \rho_1^{-1}\log(2R_\varepsilon/\rho_1) + \rho_2^{-1}\log(2R_\varepsilon/\rho_2)} \Bigg], \tag{5.4}$$

and we recall

$$|h|_{\sharp\sharp} := \sum_{j=1}^{2} \|V_d h\|_{C^{0,\alpha}(\rho_j<4)} + \sup_{\substack{2<\rho_1<R_\varepsilon \\ 2<\rho_2<R_\varepsilon}} \Bigg[\frac{|h_1|}{\rho_1^{-1} + \rho_2^{-1}} + \frac{|h_2|}{\rho_1^{-1+\sigma} + \rho_2^{-1+\sigma}} \Bigg].$$

Proposition 5.2. *Suppose that h satisfies symmetries (3.16) and $\|h\|_{**} < \infty$. Suppose furthermore that h^o defined by (4.19) is decomposed as $h^o = h^o_\alpha + h^o_\beta$, where $|h^o_\alpha|_{\sharp\sharp} < \infty$ and h^o_α, h^o_β satisfy*

$$h^o_k(\mathcal{R}_j z) = \overline{h^o_k(z)}, \quad |z - \tilde{d}_j| < R_\varepsilon, \quad j = 1,2,\ k = \alpha, \beta,$$

and have support in $B_{2R_\varepsilon}(\tilde{d}_1) \cup B_{2R_\varepsilon}(\tilde{d}_2)$. Let us write $\psi = \psi^e + \psi^o$ with ψ^o defined by (5.2). Then ψ^o can be decomposed as $\psi^o = \psi^o_\alpha + \psi^o_\beta$, with each function supported in $B_{R_\varepsilon}(\tilde{d}_1) \cup B_{R_\varepsilon}(\tilde{d}_2)$ and satisfying

$$|\psi^o_\alpha|_\sharp \lesssim |h^o_\alpha|_{\sharp\sharp} + \varepsilon|\log\varepsilon|^{1/2}(\|h^o_\alpha\|_{**} + \|h - h^o\|_{**}) \tag{5.5}$$

$$\|\psi^o_\beta\|_* \lesssim \|h^o_\beta\|_{**}, \tag{5.6}$$

$$\|\psi^o_\alpha\|_* + \|\psi^o_\beta\|_* \lesssim \|h\|_{**} + \|h^o_\alpha\|_{**} + \|h^o_\beta\|_{**}$$

and

$$\psi^o_k(\mathcal{R}_j z) = \overline{\psi^o_k(z)}, \quad |z - \tilde{d}_j| < R_\varepsilon, \quad j = 1,2,\ k = \alpha, \beta.$$

The proof of Proposition 5.2 is in Section 5.2.

5.1. *First a priori estimate and proof of Proposition 5.1*

Here we obtain a priori estimates for solutions to

$$\begin{cases} \mathcal{L}^\varepsilon(\psi) = h \quad \text{in } \mathbb{R}^2, \\ \operatorname{Re} \displaystyle\int_{B(0,4)} \chi_j \overline{\phi_j} W_{x_1} = 0 \quad \text{with } \phi_j(z) = iW(z)\psi(z + \tilde{d}_j), \\ \psi \text{ satisfies symmetry (3.16).} \end{cases} \tag{5.7}$$

Lemma 5.1. *There exists a constant $C > 0$ such that, for all ε sufficiently small and any solution ψ of (5.7) with $\|\psi\|_* < \infty$, one has*

$$\|\psi\|_* \le C\|h\|_{**}. \tag{5.8}$$

Proof. To prove Lemma 5.1, we will use first weaker norms. For $\psi : \mathbb{C} \to \mathbb{C}$, we define

$$\|\psi\|_{*,0} := \sum_{j=1}^{2} \|V_d \psi\|_{L^\infty(\rho_j<3)} + \|\mathrm{Re}(\psi)\|_{1,*,0} + \|\mathrm{Im}(\psi)\|_{2,*,0}$$

where

$$\|\psi_1\|_{1,*,0} := \sup_{\substack{\rho_1>2\\ \rho_2>2}} |\psi_1| + \sup_{\substack{2<\rho_1<\frac{2}{\varepsilon}\\ 2<\rho_2<\frac{2}{\varepsilon}}} \frac{|\nabla\psi_1|}{\rho_1^{-1}+\rho_2^{-1}} + \sup_{r>\frac{1}{\varepsilon}} \left[\frac{1}{\varepsilon}|\partial_r\psi_1| + |\partial_s\psi_1|\right],$$

$$\|\psi_2\|_{2,*,0} := \sup_{\substack{\rho_1>2\\ \rho_2>2}} \frac{|\psi_2|}{\rho_1^{-2+\sigma}+\rho_2^{-2+\sigma}+\varepsilon^{\sigma-2}} + \sup_{\substack{2<\rho_1<\frac{2}{\varepsilon}\\ 2<\rho_2<\frac{2}{\varepsilon}}} \frac{|\nabla\psi_2|}{\rho_1^{-2+\sigma}+\rho_2^{-2+\sigma}}$$
$$+ \sup_{r>\frac{1}{\varepsilon}} [\varepsilon^{\sigma-2}|\partial_r\psi_2| + \varepsilon^{\sigma-1}|\partial_s\psi_2|].$$

In the expressions above, the gradient of ψ_j is $(\partial_{x_1}\psi_j, \partial_{x_2}\psi_j)$, where $z=(x_1,x_2)$. Since $z = x_1 + ix_2 = re^{is} = \rho_1 e^{i\theta_1} + \tilde{d} = \rho_2 e^{i\theta_2} - \tilde{d}$, we have

$$|\nabla\psi_j|^2 = (\partial_{x_1}\psi_j)^2 + (\partial_{x_2}\psi_j)^2 = (\partial_r\psi_j)^2 + \frac{1}{r^2}(\partial_s\psi_j)^2$$
$$= (\partial_{\rho_1}\psi_j)^2 + \frac{1}{\rho_1^2}(\partial_{\theta_1}\psi_j)^2 = (\partial_{\rho_2}\psi_j)^2 + \frac{1}{\rho_2^2}(\partial_{\theta_2}\psi_j)^2.$$

We define also the norm for the right-hand side $h = h_1 + ih_2$ of (5.1),

$$\|h\|_{**,0} := \sum_{j=1}^{2} \|V_d h\|_{L^\infty(\rho_j<3)} + \sup_{\substack{\rho_1>2\\ \rho_2>2}} \left[\frac{|\mathrm{Re}(h)|}{\rho_1^{-2}+\rho_2^{-2}+\varepsilon^2} + \frac{|\mathrm{Im}(h)|}{\rho_1^{-2+\sigma}+\rho_2^{-2+\sigma}+\varepsilon^{2-\sigma}}\right].$$

We claim that there exists a constant $C>0$ such that, for all ε sufficiently small and any solution of (5.7), one has

$$\|\psi\|_{*,0} \le C\|h\|_{**,0}. \tag{5.9}$$

To prove this, we assume by contradiction that there exist $\varepsilon_n \to 0$ and $\psi^{(n)}, h^{(n)}$ solutions of (5.7) such that

$$\|\psi^{(n)}\|_{*,0} = 1, \quad \|h^{(n)}\|_{**,0} = o_n(1). \tag{5.10}$$

We first work near the vortices $\tilde{d}_j$, and work with the function

$$\phi_j^{(n)}(z) = iW(z)\psi^{(n)}(z+\tilde{d}_j).$$

The uniform bounds (5.10) imply directly that $\|\nabla\phi_j^{(n)}\|_{L^\infty(\mathbb{R}^2\setminus(B(\tilde{d}_1,2)\cup B(\tilde{d}_2,2)))}$ is uniformly bounded. In the region $B(\tilde{d}_1,2)\cup B(\tilde{d}_2,2)$, the equation can be rewritten in the form

$$\Delta\phi^{(n)} + \varepsilon_n^2[y^2\partial_{xx}^2\phi^{(n)} - 2xy\partial_{xy}^2\phi^{(n)} + x^2\partial_{yy}^2\phi^{(n)}$$
$$- 4i(x\partial_y\phi^{(n)} - y\partial_x\phi^{(n)}) - 4\phi^{(n)}]$$
$$+ (1-|V_d|^2)\phi^{(n)} - 2\,\mathrm{Re}(\bar{V}_d\phi^{(n)})V_d = iV_d h^{(n)} \tag{5.11}$$

with $\phi^{(n)} = i V_d \psi^{(n)}$. We have used the expression in Cartesian coordinates of the operator $\varepsilon^2(\partial_{ss}^2 - 4i\partial_s - 4)$. We remark that the linear operator in (5.11) is uniformly elliptic. The uniform bounds (5.10) and standard elliptic estimates imply that

$$\|\nabla \phi^{(n)}\|_{L^\infty(B(\tilde{d}_1,2)\cup B(\tilde{d}_2,2))}$$

is uniformly bounded. As a consequence, $\|\nabla \phi_j^{(n)}\|_{L^\infty(\mathbb{R}^2)}$ is uniformly bounded for $j = 1, 2$. We then can apply Arzela–Ascoli's theorem to extract a subsequence such that $\phi_j^{(n)} \to \phi_0$ in $C^0_{\text{loc}}(\mathbb{R}^2)$. Passing to the limit in (5.7) (we use (3.14) and (3.15)), we conclude that $L^0(\phi_0) = 0$ in $\mathbb{R}^2$, with L^0 defined in (3.5). Moreover, ϕ_0 inherits the symmetry $\phi_0(\bar{z}) = \overline{\phi_0(z)}$. From the estimate $\|\psi^{(n)}\|_{*,0} = 1$, we deduce that $\phi_0 \in L^\infty(\mathbb{R}^2)$ and that $\psi_1 = \operatorname{Re}(\phi_0/iW)$, $\psi_2 = \operatorname{Im}(\phi_0/iW)$ satisfy

$$|\psi_1| + |z||\nabla\psi_1| \le C, \quad |\psi_2| + |\nabla\psi_2| \le \frac{C}{|z|^{2-\sigma}}, \quad |z| > 1.$$

By Lemma A.2, we deduce that $\phi_0 = c_1 W_{x_1}$ for some $c_1 \in \mathbb{R}$.

On the other hand, we can pass to the limit in the orthogonality condition

$$\operatorname{Re} \int_{B(0,4)} \chi \bar{\phi}_j^{(n)} W_{x_1} = 0$$

and obtain necessarily $c_1 = 0$. Hence $\phi_j^{(n)} \to 0$ in $C^0_{\text{loc}}(\mathbb{R}^2)$. Therefore,

$$\psi^{(n)} \to 0 \quad \text{uniformly on compact subsets of } \{\rho_1 \ge 2,\ \rho_2 \ge 2\}. \tag{5.12}$$

Next we derive estimates *far away* from the vortices. In the following we drop the superscript n for simplicity. In $\{\rho_1 > 2\} \cap \{\rho_2 > 2\}$, we have that $\psi^{(n)} = \psi$ solves

$$h = \Delta\psi + 2\frac{\nabla V_d \nabla\psi}{V_d} - 2i|V_d|^2\psi_2 + \varepsilon^2\partial_{ss}^2\psi + \varepsilon^2\left(2\frac{\partial_s V_d}{V_d} - 4i\right)\partial_s\psi,$$

which for $\psi_1 = \operatorname{Re}(\psi)$, $\psi_2 = \operatorname{Im}(\psi)$ translates into the following system:

$$\begin{aligned} h_1 = \Delta\psi_1 &+ \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\nabla\psi_1 - \nabla(\theta_1+\theta_2)\nabla\psi_2 + \varepsilon^2\partial_{ss}^2\psi_1 \\ &+ 2\varepsilon^2\left[\left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2}\right)\partial_s\psi_1 - \partial_s(\theta_1+\theta_2)\partial_s\psi_2\right] + 4\varepsilon^2\partial_s\psi_2, \end{aligned} \tag{5.13}$$

$$\begin{aligned} h_2 = \Delta\psi_2 &+ \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\nabla\psi_2 + \nabla(\theta_1+\theta_2)\nabla\psi_1 - 2|V_d|^2\psi_2 + \varepsilon^2\partial_{ss}^2\psi_2 \\ &+ 2\varepsilon^2\left[\left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\partial_s\psi_2 + \partial_s(\theta_1+\theta_2)\partial_s\psi_1\right] - 4\varepsilon^2\partial_s\psi_1. \end{aligned} \tag{5.14}$$

We start by estimating ψ_2. Since ψ_2 satisfies

$$\psi_2(x_1,-x_2) = \psi_2(x_1,x_2) \quad \text{and} \quad \psi_2(-x_1,x_2) = \psi_2(x_1,x_2),$$

it is sufficient to obtain estimates for ψ_2 in the quadrant $\{x_1 > 0,\ x_2 > 0\}$.

Let $R > 0$ be large fixed and $D_R = \{x_1 > 0,\ x_2 > 0\} \cap \{\rho_1 > R\}$. By the symmetries, ψ_2 satisfies a homogeneous Neumann boundary condition at $x_1 = 0$ or $x_2 = 0$.

In D_R, we have $|V_d|^2 \geq c > 0$ for some fixed positive constant c. We consider (5.14) in D_R and rewrite it as $\Delta\psi_2 + \varepsilon^2 \partial_{ss}\psi_2 - 2|V_d|^2\psi_2 = p_2$ in D_R, where

$$
\begin{aligned}
p_2 = h_2 &- \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\nabla\psi_2 - \nabla(\theta_1 + \theta_2)\nabla\psi_1 \\
&- 2\varepsilon^2\left[\left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2}\right)\partial_s\psi_2 + \partial_s(\theta_1 + \theta_2)\partial_s\psi_1\right] + 4\varepsilon^2\partial_s\psi_1.
\end{aligned}
$$

We use polar coordinates (ρ_1, θ_1) around $\tilde{d}$ and (ρ_2, θ_2) around $-\tilde{d}$, that is,

$$
re^{is} = \rho_1 e^{i\theta_1} + \tilde{d} = \rho_2 e^{i\theta_2} - \tilde{d}.
$$

From this, we get

$$
\begin{aligned}
\partial_r &= \frac{1}{r}(\rho_1 + \tilde{d}\cos\theta_1)\partial_{\rho_1} - \frac{\tilde{d}\sin\theta_1}{r\rho_1}\partial_{\theta_1}, \quad \partial_s = \tilde{d}\sin\theta_1\partial_{\rho_1} + \left(1 + \frac{\tilde{d}\cos\theta_1}{\rho_1}\right)\partial_{\theta_1}, \\
\partial_r &= \frac{1}{r}(\rho_2 - \tilde{d}\cos\theta_2)\partial_{\rho_2} + \frac{\tilde{d}\sin\theta_2}{r\rho_2}\partial_{\theta_2}, \quad \partial_s = -\tilde{d}\sin\theta_2\partial_{\rho_2} + \left(1 - \frac{\tilde{d}\cos\theta_2}{\rho_2}\right)\partial_{\theta_2}.
\end{aligned}
$$

With these expressions and the asymptotic behavior stated in Lemma A.1, we see that

$$
\begin{aligned}
\left|\left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\nabla\psi_2\right| &\leq \frac{C}{R^3}\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right)\|\psi_2\|_{2,*,0}, \\
|\nabla(\theta_1 + \theta_2)\nabla\psi_1| &\leq C(R^{-\sigma} + \varepsilon^{\sigma})\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right)\|\psi_1\|_{1,*,0}, \\
\varepsilon^2\left|\left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2}\right)\partial_s\psi_2\right| &\leq C(R^{-3} + \varepsilon)\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right)\|\psi_2\|_{2,*,0}, \\
\varepsilon^2|\partial_s(\theta_1 + \theta_2)\partial_s\psi_1| &\leq C(R^{-\sigma} + \varepsilon^{\sigma})\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right)\|\psi_1\|_{1,*,0}, \\
\varepsilon^2|\partial_s\psi_1| &\leq C(R^{-\sigma} + \varepsilon^{\sigma})\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right)\|\psi_1\|_{1,*,0}
\end{aligned}
$$

Since we assumed $\|\psi\|_{*,0} = 1$, we get

$$
|p_2| \leq C(\|h\|_{**,0} + R^{-\sigma} + \varepsilon^{\sigma})\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right).
$$

We use a barrier of the form

$$
\mathcal{B}_2 = M\left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma}\right)
$$

with $M = C(\|h\|_{**,0} + R^{-\sigma} + \varepsilon^{\sigma} + \|\psi_2\|_{L^\infty(B_R(\tilde{d}))})$ and $C > 0$ is a large fixed constant. Note that

$$
\begin{aligned}
\partial_{ss}^2\mathcal{B}_2 &= \frac{\partial^2\mathcal{B}_2}{\partial\rho_1^2}\tilde{d}^2\sin(\theta_1)^2 + \frac{\partial\mathcal{B}_2}{\partial\rho_1}\tilde{d}\cos(\theta_1)\left(1 + \frac{\tilde{d}}{\rho_1}\cos(\theta_1)\right) \\
&= M(\sigma - 2)(\sigma - 3)\frac{\tilde{d}^2\sin(\theta_1)^2}{\rho_1^{4-\sigma}} + M(\sigma - 2)\tilde{d}\frac{\cos(\theta_1)}{\rho_1^{3-\sigma}}\left(1 + \frac{\tilde{d}}{\rho_1}\cos(\theta_1)\right),
\end{aligned}
$$

and so

$$\Delta \mathcal{B}_2 + \varepsilon^2 \partial_{ss} \mathcal{B}_2 - 2|V_d|^2 \mathcal{B}_2 \leq -\tilde{c} M \left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma} \right) \quad \text{in } D_R$$

for some fixed $\tilde{c} > 0$. Thanks to a comparison principle in D_R (a slight variant of Lemma A.5) and standard elliptic estimates, we get

$$|\psi_2| \leq C \left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma} \right) (\|h\|_{**,0} + R^{-\sigma} + \varepsilon^{\sigma} + \|\psi_2\|_{L^\infty(B_R(\tilde{d}))}) \quad \text{in } D_R. \tag{5.15}$$

Standard elliptic estimates imply

$$|\nabla \psi_2| \leq C \left(\frac{1}{\rho_1^{2-\sigma}} + \varepsilon^{2-\sigma} \right) (\|h\|_{**,0} + R^{-\sigma} + \varepsilon^{\sigma} + \|\psi_2\|_{L^\infty(B_R(\tilde{d}))}) \quad \text{in } D_R \cap \left\{ \rho_1 \leq \frac{2}{\varepsilon} \right\}. \tag{5.16}$$

For points in D_R with $\rho_1 > 1/\varepsilon$, we use the scaling $\tilde{\psi}(\tilde{r}, s) = \psi(\varepsilon^{-1} r, s)$, and we get the estimate

$$\varepsilon^{-1} |\partial_r \psi| + |\partial_s \psi| \leq C \varepsilon^{2-\sigma} (\|h\|_{**} + R^{-\sigma} + \varepsilon^{\sigma} + \|\psi_2\|_{L^\infty(B_R(\tilde{d}))}) \tag{5.17}$$

for points in D_R with $\rho_1 > 1/\varepsilon$.

Combining (5.15), (5.16) and (5.17), we get

$$\|\psi_2\|_{2,*,0} \leq C(\|h\|_{**,0} + R^{-\sigma} + \varepsilon^{\sigma} + \|\psi_2\|_{L^\infty(B_R(\tilde{d}))}). \tag{5.18}$$

We next estimate ψ_1. We also use the symmetries satisfied by ψ_1, that is,

$$\psi_1(-x_1, x_2) = -\psi_1(x_1, x_2), \quad \psi_1(x_1, -x_2) = -\psi_1(x_1, x_2),$$

to look at the equation for ψ_1 in the quadrant $\{x_1 > 0,\ x_2 > 0\}$. Let us rewrite (5.13) as $\Delta \psi_1 + \varepsilon^2 \partial_{ss}^2 \psi_1 = p_1$, where

$$p_1 = h_1 - \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2} \right) \nabla \psi_1 + \nabla(\theta_1 + \theta_2) \nabla \psi_2 - 2\varepsilon^2 \left[\left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2} \right) \partial_s \psi_1 - \partial_s(\theta_1 + \theta_2) \partial_s \psi_2 \right] - 4\varepsilon^2 \partial_s \psi_2.$$

We have, in D_R,

$$\left| \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2} \right) \nabla \psi_1 \right| \leq \frac{C}{R \rho_1^2} \|\psi_1\|_{1,*,0},$$

$$|\nabla(\theta_1 + \theta_2) \nabla \psi_2| \leq C R^{\sigma - 1} \left(\frac{1}{\rho_1^2} + \varepsilon^2 \right) \|\nabla \psi_2\|_{2,*,0},$$

$$2\varepsilon^2 \left| \left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2} \right) \partial_s \psi_1 \right| \leq \frac{C}{R^2} \left(\frac{1}{\rho_1^2} + \varepsilon^2 \right) \|\psi_1\|_{1,*,0},$$

$$2\varepsilon^2 |\partial_s(\theta_1 + \theta_2) \partial_s \psi_2| \leq C \left(\varepsilon^{1-\sigma} + R^{\sigma - 1} \right) \left(\varepsilon^2 + \frac{1}{\rho_1^2} \right) \|\psi_2\|_{2,*,0},$$

$$\varepsilon^2 |\partial_s \psi_2| \leq C \left(\varepsilon^{1-\sigma} + R^{\sigma - 1} \right) \left(\varepsilon^2 + \frac{1}{\rho_1^2} \right) \|\psi_2\|_{2,*,0}.$$

Using that $\|\psi\|_{*,0} = 1$, we get

$$|p_1| \le C(\|h\|_{**,0} + R^{\sigma-1} + \varepsilon^{1-\sigma})\Big(\frac{1}{\rho_1^2} + \varepsilon^2\Big).$$

We use the comparison principle with the barrier

$$\mathcal{B}_1 = M\theta_1(\pi - \theta_1)$$

with $M = C(\|h\|_{**,0} + R^{\sigma-1} + \varepsilon^{1-\sigma} + \|\psi_1\|_{L^\infty(B_R(\tilde d))})$ and C a large fixed constant. We note that

$$\begin{aligned}\partial_{ss}^2\mathcal{B}_1 &= -\tilde d \sin\theta_1 \frac{\tilde d\cos\theta_1}{\rho_1^2} M(\pi - 2\theta_1)\\ &\quad + \Big(1 + \frac{\tilde d\cos\theta_1}{\rho_1}\Big)\Big[-\frac{\tilde d\sin\theta_1}{\rho_1} M(\pi - 2\theta_1) - 2\Big(1 + \frac{\tilde d\cos\theta_1}{\rho_1}\Big)M\Big]\end{aligned}$$

From this, we get

$$\Delta\mathcal{B}_1 + \varepsilon^2\partial_{ss}^2\mathcal{B}_1 \le -\tilde c M\Big(\frac{1}{\rho_1^2} + \varepsilon^2\Big)$$

for some fixed $\tilde c > 0$.

Thanks to a comparison principle in D_R (a slight variant of Lemma A.5), we get

$$|\psi_1| \le C(\|h\|_{**,0} + R^{\sigma-1} + \varepsilon^{1-\sigma} + \|\psi_1\|_{L^\infty(B_R(\tilde d))}) \quad \text{in } D_R. \tag{5.19}$$

Elliptic estimates and a standard scaling give

$$\rho_1|\nabla\psi_1| \le C(\|h\|_{**,0} + R^{\sigma-1} + \varepsilon^{1-\sigma} + \|\psi_1\|_{L^\infty(B_R(\tilde d))}) \tag{5.20}$$

for points in D_R with $2 < \rho_1 < 2/\varepsilon$. To estimate the gradient for points in D_R with $\rho_1 > 1/\varepsilon$, we use the scaling $\tilde\psi(r,s) = \psi(\varepsilon^{-1}\tilde r, s)$ and see that

$$\varepsilon^{-1}|\partial_r\psi_1| + |\partial_s\psi_1| \le C(\|h\|_{**,0} + R^{\sigma-1} + \varepsilon^{1-\sigma} + \|\psi_1\|_{L^\infty(B_R(\tilde d))}) \tag{5.21}$$

in this region.

Combining (5.19), (5.20) and (5.21), we get

$$\|\psi_1\|_{1,*,0} \le C(\|h\|_{**,0} + R^{\sigma-1} + \varepsilon^{1-\sigma} + \|\psi_1\|_{L^\infty(B_R(\tilde d))}).$$

Then, using (5.18), we conclude that

$$\begin{aligned}\|\psi\|_{*,0} \le C(&\|h\|_{**,0} + R^{-\sigma} + \varepsilon^{\sigma} + R^{\sigma-1} + \varepsilon^{1-\sigma}\\ &+ \|\psi_1\|_{L^\infty(B_R(\tilde d))} + \|\psi_2\|_{L^\infty(B_R(\tilde d))}).\end{aligned}$$

Using then (5.12) and $\|h\|_{**} = o(1)$, from the previous inequality, we get $\|\psi\|_{*,0} < 1/2$ if from the start R is fixed large and we take $\varepsilon > 0$ small. This is a contradiction and proves (5.9).

The full estimate (5.8) follows from (5.9) and Schauder estimates. ∎

Proof of Proposition 5.1. We first solve the problem in bounded domains. We consider the equation

$$
\begin{cases}
\mathcal{L}^{\varepsilon}(\psi) = h + c\displaystyle\sum_{j=1}^{2} \frac{\chi_j}{iW(z-\tilde{d}_j)}(-1)^j W_{x_1}(z-\tilde{d}_j) & \text{in } B_M(0),\\
\psi = 0 & \text{on } \partial B_M(0),\\
\operatorname{Re}\displaystyle\int_{B(0,4)} \chi\overline{\phi_j} W_{x_1} = 0 \quad \text{with } \phi_j(z) = iW(z)\psi(z+\tilde{d}_j), \quad j=1,2, & \\
\psi \text{ satisfies symmetry (3.16).} &
\end{cases}
\tag{5.22}
$$

with $M > 10\tilde{d}$. We set

$$
\mathcal{H} := \Big\{\phi = iV_d\psi \in H_0^1(B_M(0),\mathbb{C}) : \operatorname{Re}\int_{B(0,4)} \chi\bar{\phi}_j W_{x_1} = 0,\ j=1,2,\ \psi \text{ satisfies (3.16)}\Big\}.
$$

We equip $\mathcal{H}$ with the inner product

$$
[\phi,\varphi] := \operatorname{Re}\int_{B_M(0)} (\nabla\phi\overline{\nabla\varphi} + \varepsilon^2\partial_s\phi\overline{\partial_s\varphi}).
$$

With this, $\mathcal{H}$ is a Hilbert space. Indeed, it is a closed subspace of $H_0^1(B_M(0),\mathbb{C})$ and $[\cdot,\cdot]$ is an inner product on $H^1(B_M(0),\mathbb{C})$ thanks to the Poincaré inequality. In terms of ϕ, the first equation of (5.22) can be rewritten as

$$
\begin{aligned}
\Delta\phi + (1-|V_d|^2)\phi - 2\operatorname{Re}(\overline{\phi}V_d)V_d + \varepsilon^2(\partial_{ss}^2\phi - 4i\partial_s\phi - 4\phi) + (\eta-1)\frac{E}{V_d}\phi\\
= iV_d h + iV_d c\sum_{j=1}^{2}\tilde{\eta}(-1)^j\chi_j(z)\frac{W_{x_1}(z-\tilde{d}_j)}{iW(z-\tilde{d}_j)}.
\end{aligned}
$$

We can express this equation in its variational form. Namely, for all $\varphi \in \mathcal{H}$,

$$
\begin{aligned}
-\operatorname{Re}\int_{B_M(0)} (\nabla\phi\overline{\nabla\varphi} + \varepsilon^2\partial_s\phi\overline{\partial_s\varphi}) + \varepsilon^2\operatorname{Re}\int_{B_M(0)} (4i\phi\overline{\partial_s\varphi} - 4\phi\overline{\varphi})\\
- 2\operatorname{Re}\int_{B_M(0)} \operatorname{Re}(\overline{\phi}V_d)V_d\overline{\varphi} + \operatorname{Re}\int_{B_M(0)} [(\eta-1)\frac{E}{V_d} + (1-|V_d|^2)]\phi\overline{\varphi}\\
= \operatorname{Re}\int_{B_M(0)} iV_d\Big(h - c\sum_{j=1}^{2}\chi_j(-1)^j\frac{W_{x_1}(z-\tilde{d}_j)}{iW(z-\tilde{d}_j)}\Big)\overline{\varphi}.
\end{aligned}
$$

We now denote by $\langle k(x)\phi,\cdot\rangle$ the linear form on $\mathcal{H}$ defined by

$$
\begin{aligned}
\langle k(x)\phi,\varphi\rangle := \varepsilon^2\operatorname{Re}\int_{B_M(0)} (4i\phi\overline{\partial_s\varphi} - 4\phi\overline{\varphi}) - 2\operatorname{Re}\int_{B_M(0)} \operatorname{Re}(\overline{\phi}V_d)V_d\overline{\varphi}\\
+ \operatorname{Re}\int_{B_M(0)}\Big[(\eta-1)\frac{E}{V_d} + (1-|V_d|^2)\Big]\phi\overline{\varphi}.
\end{aligned}
$$

In the same way, we denote by $\langle s, \cdot\rangle$ the linear form defined by

$$\langle s,\varphi\rangle := \operatorname{Re}\int_{B_M(0)} iV_d\left(h - c\sum_{j=1}^{2}\chi_j(-1)^j\frac{W_{x_1}(z-\tilde{d}_j)}{iW(z-\tilde{d}_j)}\right)\overline{\varphi}.$$

Thus the equation can be rewritten as

$$[\phi,\varphi] - \langle k(x)\phi,\varphi\rangle = \langle s,\varphi\rangle \quad \text{for all } \varphi\in\mathcal{H}.$$

By using the Riesz representation theorem, we can find a bounded linear operator K on $\mathcal{H}$ and S, an element of $\mathcal{H}$ depending linearly on s, such that the equation has the operational form

$$\phi - K(\phi) = S. \tag{5.23}$$

Besides, thanks to the compact Sobolev injections $H_0^1(B_M(0),\mathbb{C})\hookrightarrow L^2(B_M(0),\mathbb{C})$, we know that K is compact. We can then apply Fredholm's alternative to deduce the existence of ϕ such that (5.23) holds if the homogeneous equation only has the trivial solution. To prove this last point, we establish an a priori estimate on c. In order to do that, we use the following equivalent form of the equation in the region $B(\tilde{d},\tilde{d})$; with the translated variable, it becomes

$$L_j^\varepsilon(\phi_j) = h_j + c\chi W_{x_1} \quad \text{in } B(0,\tilde{d}),$$

where L_j^ε is defined in (3.13), $\phi_j(\tilde{z}) = iW(\tilde{z})\psi(z-\tilde{d}_j)$ and $h_j(\tilde{z}) = iW(\tilde{z})\psi(z-\tilde{d}_j)$ for $|\tilde{z}|<\tilde{d}$.

We can test this equation against $\overline{W}_{x_1}$ to find

$$c = -\frac{1}{c_*}\left[\operatorname{Re}\int_{B(0,\tilde{d})} h_j\overline{W}_{x_1} - \operatorname{Re}\int_{B(0,\tilde{d})} L_j^\varepsilon(\phi_j)\overline{W}_{x_1}\right],$$

with $c_* := \operatorname{Re}\int_{B(0,\tilde{d})}\chi|W_{x_1}|^2 = \operatorname{Re}\int_{B(0,R)}\chi|W_{x_1}|^2 \simeq C$ for some $C>0$ of order 1 and L_j^ε defined in (3.13). Integrating by parts, we obtain

$$\begin{aligned}\operatorname{Re}\int_{B(0,\tilde{d})} L_j^\varepsilon(\phi_j)\overline{W}_{x_1} &= \operatorname{Re}\int_{B(0,\tilde{d})}\overline{\phi}_j(L_j^\varepsilon - L^0)(W_{x_1})\\ &\quad + \operatorname{Re}\left\{\int_{\partial B(0,\tilde{d})}\frac{\partial\phi_j}{\partial\nu}\overline{W}_{x_1} - \phi_j\frac{\partial}{\partial\nu}\overline{W}_{x_1}\right\}.\end{aligned}$$

In the previous equality, we used that $L^0(W_{x_1}) = 0$. However, using the expansion of $L_j^\varepsilon - L^o$ in (3.14) and estimates (3.15), we can see that

$$\left|\operatorname{Re}\int_{B(0,\tilde{d})}\overline{\phi}_j(L_j^\varepsilon - L^0)(W_{x_1})\right| = O_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|})\|\psi\|_*. \tag{5.24}$$

By using the decay of $\phi_j, \nabla\phi_j$ and $W_{x_1}, \nabla W_{x_1}$, we can also check that

$$\left|\operatorname{Re}\left\{\int_{\partial B(0,\tilde{d})}\frac{\partial\phi_j}{\partial\nu}\overline{W}_{x_1} - \phi_j\frac{\partial}{\partial\nu}\overline{W}_{x_1}\right\}\right| = O_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|})\|\psi\|_*.$$

Therefore, we arrive at

$$c = -\frac{1}{c_*}\operatorname{Re}\int_{B(0,\tilde{d}))} h_j \overline{W}_{x_1} + O_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|})\frac{\|\psi\|_*}{c_*}.$$

To conclude the proof, we note that we can apply Lemma 5.1 to conclude that a solution of the homogeneous equation satisfies

$$\|\psi\|_* \le C\left\| c\sum_{j=1}^{2}\chi_j(z)(-1)^j\frac{W_{x_1}(z-\tilde{d}_j)}{iW(z-\tilde{d}_j)}\right\|_{**} \le C\varepsilon\sqrt{|\log\varepsilon|}\|\psi\|_*,$$

and thus $\psi = 0$. Then, for any $M > 10\tilde{d}$, we obtain the existence of a solution of (5.22) satisfying $\|\psi_M\|_* \le C\|h\|_{**}$, with C independent of M. Note that, in the previous argument, the norms $\|\cdot\|_*$, $\|\cdot\|_{**}$ are slightly adapted to deal with the fact that we work on bounded domains. We can extract a subsequence such that $\psi_M \rightharpoonup \psi$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ with ψ solving (5.1). From Lemma 5.1, we deduce $\|\psi\|_* \le C\|h\|_{**}$. ∎

5.2. Second a priori estimate and proof of Proposition 5.2

Lemma 5.2. *Let $\alpha \in (0,1)$, $\sigma \in (0,1)$. Then there exists a constant $C > 0$ such that, for all ε sufficiently small and any solution ψ of (5.7) with $\|\psi\|_* < \infty$, one has*

$$|\psi|_\sharp \le C(|h|_{\sharp\sharp} + \varepsilon|\log\varepsilon|^{1/2}\|h\|_{**}). \tag{5.25}$$

Proof. We work with the weaker seminorms

$$|\psi|_{\sharp,0} := \sum_{j=1}^{2}|\log\varepsilon|^{-1}\|V_d\psi\|_{L^\infty(\rho_j<3)} + |\mathrm{Re}(\psi)|_{\sharp,1} + |\mathrm{Im}(\psi)|_{\sharp,2},$$

where $|\cdot|_{\sharp,1}$, $|\cdot|_{\sharp,2}$ are defined in (5.3), (5.4) and

$$|h|_{\sharp\sharp,0} := \sum_{j=1}^{2}\|V_d h\|_{L^\infty(\rho_j<4)} + \sup_{\substack{2<\rho_1<R_\varepsilon\\ 2<\rho_2<R_\varepsilon}}\left[\frac{|h_1|}{\rho_1^{-1}+\rho_2^{-1}} + \frac{|h_2|}{\rho_1^{-1+\sigma}+\rho_2^{-1+\sigma}}\right].$$

We claim that there exists a constant $C > 0$ such that, for all ε sufficiently small and any solution of (5.7), one has

$$|\psi|_{\sharp,0} \le C(|h|_{\sharp\sharp,0} + \varepsilon|\log\varepsilon|^{1/2}\|h\|_{**}). \tag{5.26}$$

We argue by contradiction and assume that there exist $\varepsilon_n \to 0$ and $\psi^{(n)}, h^{(n)}$ solutions of (5.7) such that

$$|\psi^{(n)}|_{\sharp,0} = 1, \quad (|h^{(n)}|_{\sharp\sharp,0} + \varepsilon_n|\log\varepsilon_n|^{1/2}\|h^{(n)}\|_{**}) \to 0 \tag{5.27}$$

as $n \to \infty$.

We first work near the vortices and notice that, by symmetry, it is enough to consider the vortex at $+\tilde{d}$. We work with the function $\phi_j^{(n)}(z) = i|\log\varepsilon|^{-1}W(z)\psi^{(n)}(z+\tilde{d}_j)$.

Using (5.27), from Arzela–Ascoli's theorem, we can extract a subsequence such that $\tilde{\phi}_j^{(n)} \to \phi_0$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. Passing to the limit in (5.7), we see that

$$L^0(\phi_0) = 0 \quad \text{in } \mathbb{R}^2,$$

with L^0 defined in (3.5). The function ϕ_0 inherits the symmetry $\phi_0(\bar{z}) = \overline{\phi_0(z)}$ and satisfies $\phi_0 \in L^\infty_{\mathrm{loc}}$. Moreover, writing $\phi_0 = iW\psi_0$, $\psi_0 = \psi_0^1 + i\psi_0^2$, we have

$$|\psi_0^1(z)| \le |z|, \quad |\psi_0^2(z)| \le 1, \quad |z| > 2.$$

Thanks to the above estimate and Lemma A.3, we deduce $\phi_0 = c_1 W_{x_1}$ for some $c_1 \in \mathbb{R}$. On the other hand, we can pass to the limit in the orthogonality condition

$$\mathrm{Re} \int_{B(0,4)} \chi \bar{\phi}_j^{(n)} W_{x_1} = 0$$

and obtain necessarily $c_1 = 0$. Hence $\phi_j^{(n)} \to 0$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. We can also apply the same argument near $-\tilde{d}$ and get

$$\frac{\psi^{(n)}}{|\log \varepsilon_n|} \to 0$$

uniformly on compact sets of $\{\rho_1 \ge 1,\ \rho_2 \ge 1\}$ as $\varepsilon_n \to 0$.

In what follows in this proof, we work in the region

$$\tilde{D}_{R_0} = \{R_0 < \rho_1 < R_\varepsilon\} \cap \{x_2 > 0\},$$

where $R_0 > 0$ is fixed large and R_ε is given by (4.2).

We use barriers to estimate $\psi_2^{(n)}(z)$ in $\tilde{D}_R$. By the symmetries of $\psi_2^{(n)}$, we get the estimates for all $2 < \rho_1 < R_\varepsilon$. Let us write equation (5.14) as

$$\begin{aligned} \Delta\psi_2 &+ \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\nabla\psi_2 - 2|V_d|^2\psi_2 \\ &+ \varepsilon^2 \partial^2_{ss}\psi_2 + 2\varepsilon^2\left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2}\right)\partial_s\psi_2 = \tilde{p}_2, \end{aligned}$$

where

$$\tilde{p}_2 = h_2 - \nabla(\theta_1 + \theta_2)\nabla\psi_1 - 2\varepsilon^2\partial_s(\theta_1 + \theta_2)\partial_s\psi_1 + 4\varepsilon^2\partial_s\psi_1.$$

We observe that, in $\tilde{D}_{R_0}$, it holds

$$\begin{aligned} |\nabla(\theta_1 + \theta_2)\nabla\psi_1^{(n)}| &\le \frac{C}{\rho_1}\log\left(\frac{2R_\varepsilon}{\rho_1}\right)|\psi_1^{(n)}|_{\sharp,1}, \\ \varepsilon_n^2|\partial_s(\theta_1 + \theta_2)\partial_s\psi_1| &\le \frac{C}{\rho_1|\log\varepsilon_n|}\log\left(\frac{2R_\varepsilon}{\rho_1}\right)|\psi_1^{(n)}|_{\sharp,1}, \\ \varepsilon_n^2|\partial_s\psi_1^{(n)}| &\le \frac{C}{\rho_1|\log\varepsilon_n|}\log\left(\frac{2R_\varepsilon}{\rho_1}\right)|\psi_1^{(n)}|_{\sharp,1}. \end{aligned}$$

Using the a priori estimate of Lemma 5.1, we find that

$$\|\psi^{(n)}\|_* \le C\|h^{(n)}\|_{**} = o(1)\varepsilon_n^{-1}|\log\varepsilon_n|^{-1/2}. \tag{5.28}$$

Thus, writing $\psi^{(n)} = \psi_1^{(n)} + i\psi_2^{(n)}$, we have

$$|\psi_1^{(n)}(z)| \le o(1)\varepsilon_n^{-1}|\log\varepsilon_n|^{-1/2},$$
$$|\psi_2^{(n)}(z)| + |\nabla\psi_2^{(n)}(z)| \le o(1)\varepsilon_n^{-1}|\log\varepsilon_n|^{-1/2}\left(\frac{1}{\rho_1^{2-\sigma}} + \frac{1}{\rho_2^{2-\sigma}}\right)$$

for $2 < |z| < 1/\varepsilon_n$ with $o(1) \to 0$ as $n \to \infty$. We note that, for $|z - \tilde{d}_j| = R_\varepsilon$,

$$|\psi_2^{(n)}(z)| \le \frac{\|\psi^{(n)}\|_*}{R_\varepsilon^{2-\sigma}} = \frac{o(1)R_\varepsilon}{R_\varepsilon^{2-\sigma}} = o(1)$$

as $n \to \infty$ by (5.28). We use as a barrier the function

$$\tilde{\mathscr{B}}_2 = \frac{C}{\rho_1^{1+\sigma}}(|h^{(n)}|_{\sharp\sharp,0} + \|\psi_2^{(n)}\|_{L^\infty(\rho_1 = R_{\varepsilon_n})})$$
$$+ \frac{C}{\rho_1}\log\left(\frac{2R_\varepsilon}{\rho_1}\right)\left(|\psi_1^{(n)}|_{\sharp,1} + \frac{\|\psi_2^{(n)}\|_{L^\infty(\rho_1 = R_0)}}{|\log\varepsilon_n|}\right),$$

where $C > 0$ is a large fixed constant. We note that

$$\tilde{\mathscr{B}}_2 \le \frac{b_n}{\rho_1^{1-\sigma}} + \frac{1}{\rho_1}\log\left(\frac{2R_\varepsilon}{\rho_1}\right)(C|\psi_1^{(n)}|_{\sharp,1} + b_n) \tag{5.29}$$

in $\tilde{D}_{R_0}$, where $b_n \to 0$ as $n \to \infty$. By the maximum principle and elliptic estimates, we get

$$|\psi_2^{(n)}| + |\nabla\psi_2^{(n)}| \le \tilde{\mathscr{B}}_2 \tag{5.30}$$

in $\tilde{D}_{R_0}$.

Next we use barriers to estimate $\psi_1^{(n)}$ in $\tilde{D}_{R_0}$. By the symmetries of $\psi_1^{(n)}$, we get the estimates for all $2 < \rho_1 < R_\varepsilon$. Let us write equation (5.13) as

$$\Delta\psi_1 + \left(\frac{\nabla w_1}{w_1} + \frac{\nabla w_2}{w_2}\right)\nabla\psi_1 + \varepsilon^2\partial_{ss}^2\psi_1 + 2\varepsilon^2\left(\frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2}\right)\partial_s\psi_1 = p_1,$$

where $p_1 = h_1 + \nabla(\theta_1 + \theta_2)\nabla\psi_2 + 2\varepsilon^2\partial_s(\theta_1 + \theta_2)\partial_s\psi_2 - 4\varepsilon^2\partial_s\psi_2$.

We find that, in $\tilde{D}_R$, the following estimates hold:

$$|\nabla(\theta_1 + \theta_2)\nabla\psi_2^{(n)}| \le \frac{C}{\rho_1}\tilde{\mathscr{B}}_2,$$
$$\varepsilon_n^2|\partial_s(\theta_1 + \theta_2)\partial_s\psi_2^{(n)}| \le \frac{C}{\rho_1|\log\varepsilon_n|}\tilde{\mathscr{B}}_2,$$
$$\varepsilon_n^2|\partial_s\psi_2^{(n)}| \le \frac{C}{\rho_1|\log\varepsilon_n|}\tilde{\mathscr{B}}_2$$

Hence, using (5.29) and (5.30), we get

$$|p_1| \le \frac{b_n}{\rho_1} + \frac{1}{\rho_1^2}\log\left(\frac{2R_\varepsilon}{\rho_1}\right)(C|\psi_1^{(n)}|_{\sharp,1} + b_n)$$

for a new sequence $b_n \to 0$.

Using Lemma A.8 for part of the right-hand side and the supersolution

$$\log\left(\frac{2R_\varepsilon}{\rho_1}\right)(C|\psi_1^{(n)}|_{\sharp,1}+b_n),$$

we conclude that

$$\begin{aligned}|\psi_1^{(n)}(z)| &\le Cb_n\rho_1\log\left(\frac{2R_\varepsilon}{\rho_1}\right)+\log\left(\frac{2R_\varepsilon}{\rho_1}\right)(C|\psi_1^{(n)}|_{\sharp,1}+b_n)\\ &\le C\rho_1\log\left(\frac{2R_\varepsilon}{\rho_1}\right)\left(b_n+\frac{|\psi_1^{(n)}|_{\sharp,1}}{R_0}\right).\end{aligned}$$

This and standard elliptic estimates yield

$$|\psi_1^{(n)}|_{1,\sharp}\le C\left(b_n+\frac{|\psi_1^{(n)}|_{\sharp,1}}{R_0}\right).$$

Choosing $R_0>0$ large and fixed, we get $|\psi_1^{(n)}|_{1,\sharp}\to 0$ as $n\to\infty$. Using this and (5.29), (5.30), we obtain $|\psi_2^{(n)}|_{2,\sharp}\to 0$ as $n\to\infty$. This contradicts assumption (5.27), and we obtain (5.26). With this inequality and standard Schauder estimates, we deduce (5.25). ■

As an intermediate step to obtain Proposition 5.2, we consider the symmetry properties of the solution constructed in Proposition 5.1, when the right-hand side has symmetries. More precisely, let us consider the local symmetry condition

$$h(\mathcal{R}_j z)=-\overline{h(z)},\quad |z-\tilde{d}_j|<2R_\varepsilon,\quad j=1,2. \tag{5.31}$$

Lemma 5.3. *Suppose h satisfies symmetries (3.16) and (5.31). We assume $\|h\|_{**}<\infty$. Then there exist ψ^s, ψ^* such that the solution ψ to (5.1) with $\|\psi\|_*<\infty$ can be written as $\psi=\psi^s+\psi^*$ with the estimates*

$$\|\psi^s\|_*+\|\psi^*\|_*\le C\|h\|_{**},\quad |\psi^*|_\sharp\le C\varepsilon|\log\varepsilon|^{1/2}\|h\|_{**}.$$

Moreover, (ψ^s,ψ^*) define linear operators of h, ψ^s has its support in $B_{R_\varepsilon}(\tilde{d}_1)\cup B_{R_\varepsilon}(\tilde{d}_2)$ and satisfies

$$\psi^s(\mathcal{R}_j z)=-\overline{\psi^s(z)},\quad |z-\tilde{d}_j|<R_\varepsilon. \tag{5.32}$$

Proof. To construct the function ψ^s, we split the operator $\mathcal{L}^\varepsilon$ (cf. (3.11)) into a part $\mathcal{L}^\varepsilon_s$ preserving symmetry (5.32) and a remainder $\mathcal{L}^\varepsilon_r$. This splitting depends on which vortex $\tilde{d}_j$ we are considering for symmetry (5.32), and thus we write $\mathcal{L}^\varepsilon_{s,j}$, $\mathcal{L}^\varepsilon_{r,j}$, $j=1,2$. It is sufficient to consider the vortex at $\tilde{d}_1$. We set

$$\begin{aligned}\mathcal{L}^\varepsilon_{s,1}(\psi) &:= \Delta\psi+2\frac{\nabla W^a\nabla\psi}{W^a}-2i|W^a|^2\,\mathrm{Im}(\psi)\\ &\quad+\varepsilon^2\Bigg[\tilde{d}^2\partial^2_{\rho_1\rho_1}\psi\sin(\theta_1)^2+\frac{\tilde{d}^2}{\rho_1}\partial^2_{\rho_1\theta_1}\psi\sin(\theta_1)\cos(\theta_1)\\ &\qquad+\partial^2_{\theta_1\theta_1}\psi\left(1+\frac{\tilde{d}^2}{\rho_1^2}\cos^2(\theta_1)\right)\\ &\qquad+\partial_{\rho_1}\psi\frac{\tilde{d}^2}{\rho_1}\cos^2(\theta_1)-2\partial_{\theta_1}\psi\frac{\tilde{d}^2}{\rho_1^2}\sin(\theta)\cos(\theta_1)\Bigg],\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}^{\varepsilon}_{r,1}(\psi) := {} & 2\frac{\nabla W^b \nabla \psi}{W^b} - 2i(|V_d|^2 - |W^a|^2)\,\mathrm{Im}(\psi) \\
& + \varepsilon^2 \Big[2\tilde{d}\,\partial^2_{\rho_1\theta_1}\psi \,\sin(\theta_1) + 2\partial^2_{\theta_1\theta_1}\psi \frac{\tilde{d}}{\rho_1}\cos(\theta_1) \\
& \qquad + \partial_{\rho_1}\psi \tilde{d}\cos(\theta_1) - \partial_{\theta_1}\psi \frac{\tilde{d}}{\rho_1}\sin(\theta_1) \Big] \\
& + \varepsilon^2 \left(\frac{2\partial_s V_d}{V_d} - 4i \right) \left[\partial_{\rho_1}\psi \tilde{d}\sin(\theta_1) + \left(1 + \frac{\tilde{d}}{\rho_1}\cos(\theta_1)\partial_{\theta_1}\psi \right) \right].
\end{aligned}
$$

We use the same cut-off functions defined in (4.18) and solve

$$
\begin{cases}
\mathcal{L}^{\varepsilon}_{s}(\psi^{2,1}) = h\eta_{1,2R_\varepsilon} \quad \text{in } \mathbb{R}^2, \\
\mathrm{Re}\displaystyle\int_{B(0,4)} \chi\overline{\phi^{2,1}} W_{x_1} = 0 \quad \text{with } \phi^{2,1}(z) = iW(z)\psi^{2,1}(z+\tilde{d}_1), \\
\psi^{2,1} \text{ satisfies } \psi^{2,1}(\bar{z}) = -\psi^{2,1}(z).
\end{cases}
$$

This is obtained as variant of Proposition 5.1 with the same proof. Note that there is no need to project the right-hand side since it is automatically orthogonal to the kernel by symmetry, and note also that the orthogonality condition for the solution holds also by symmetry. Recall that h satisfies (5.31), and we get a solution $\psi 2,1$ satisfying (5.32) with the estimate $\|\psi^{2,1}\|_* \le C\|h\|_{**}$. In a similar way, we construct $\psi^{2,2}$ centered at the vortex $\tilde{d}_2$ and define

$$
\psi^s := \eta_{1,\frac{1}{2}R_\varepsilon}\psi^{2,1}, + \eta_{2,\frac{1}{2}R_\varepsilon}\psi^{2,2}. \tag{5.33}
$$

Note that we have the estimate $\|\psi^s\|_* \le C\|h\|_{**}$.

Let

$$
\begin{aligned}
\tilde{h} := h & - \mathcal{L}^{\varepsilon}_{s,1}(\eta_{1,\frac{1}{2}R_\varepsilon}\psi^{2,1}) - \mathcal{L}^{\varepsilon}_{r,1}(\eta_{1,\frac{1}{2}R_\varepsilon}\psi^{2,1}) \\
& - \mathcal{L}^{\varepsilon}_{s,2}(\eta_{2,\frac{1}{2}R_\varepsilon}\psi^{2,2}) - \mathcal{L}^{\varepsilon}_{r,2}(\eta_{2,\frac{1}{2}R_\varepsilon}\psi^{2,2}).
\end{aligned}
$$

Some lengthy but direct calculations show that

$$
\|\tilde{h}\|_{**} \le C\|h\|_{**}, \quad |\tilde{h}|_{\sharp\sharp} \le C\varepsilon|\log\varepsilon|^{1/2}\|h\|_{**}.
$$

Then we solve, using Proposition 5.1,

$$
\begin{cases}
\mathcal{L}^{\varepsilon}(\tilde{\psi}) = \tilde{h} + \tilde{c}\displaystyle\sum_{j=1}^{2} \frac{\chi_j}{iW(z-\tilde{d}_j)}(-1)^j W_{x_1}(z-\tilde{d}_j) \quad \text{in } \mathbb{R}^2, \\
\mathrm{Re}\displaystyle\int_{B(0,4)} \chi\overline{\tilde{\phi}_j} W_{x_1} = 0 \quad \text{with } \tilde{\phi}_j(z) = iW(z)\tilde{\psi}(z+\tilde{d}_j), \\
\tilde{\psi} \text{ satisfies symmetry (3.16)}
\end{cases}
$$

and obtain, using also Lemma 5.2,

$$
\|\tilde{\psi}\|_* \le C\|\tilde{h}\|_{**}, \quad |\tilde{\psi}|_{\sharp} \le C(|\tilde{h}|_{\sharp\sharp} + \varepsilon|\log\varepsilon|^{1/2}\|\tilde{h}\|_{**}).
$$

Finally, we set

$$\psi^* = \tilde{\psi}. \tag{5.34}$$

The functions ψ^s, ψ^* defined in (5.33), (5.34) satisfy the stated properties. ■

Proof of Proposition 5.2. Let us define $\tilde{h} := h - h^o$ so that $h = \tilde{h} + h^o_\alpha + h^o_\beta$. Let $\tilde{\psi}$, $\tilde{\psi}_\alpha$, $\tilde{\psi}_\beta$ be the solution with finite $\|\cdot\|_*$-norm of (5.1) with right-hand sides $\tilde{h}$, h^o_α, h^o_β given by Proposition 5.1. Then $\psi = \tilde{\psi} + \tilde{\psi}_\alpha + \tilde{\psi}_\beta$, and we have the estimates

$$\|\tilde{\psi}\|_* \lesssim \|\tilde{h}\|_{**}, \quad \|\tilde{\psi}_j\|_* \lesssim \|h^o_j\|_{**}, \quad j = \alpha, \beta.$$

We have $\psi^o = \tilde{\psi}^o + \tilde{\psi}^o_\alpha + \tilde{\psi}^o_\beta$. We define

$$\psi^o_\alpha := \tilde{\psi}^o + \tilde{\psi}^o_\alpha, \quad \psi^o_\beta := \tilde{\psi}^o_\beta.$$

Note that, by Lemma 5.2,

$$|\tilde{\psi}^o_\alpha|_\sharp \lesssim |\tilde{\psi}_\alpha|_\sharp \lesssim |h^o_\alpha|_{\sharp\sharp} + \varepsilon|\log\varepsilon|^{1/2}\|h^o_\alpha\|_{**}.$$

According to Lemma 5.3, we can write $\tilde{\psi} = \psi^s + \psi^*$ with ψ^s, ψ^* satisfying the properties stated in that lemma, from which we get

$$|\tilde{\psi}^o|_\sharp = |(\psi^*)^o|_\sharp \lesssim |\psi^*|_\sharp \lesssim \varepsilon|\log\varepsilon|^{1/2}\|\tilde{h}\|_{**}.$$

Therefore,

$$|\psi^o_\alpha|_\sharp \lesssim |h^o_\alpha|_{\sharp\sharp} + \varepsilon|\log\varepsilon|^{1/2}(\|h^o_\alpha\|_{**} + \|\tilde{h}\|_{**}),$$

and this proves (5.5).

On the other hand,

$$\begin{aligned}
&\|\psi^o_\alpha\|_* \le \|\tilde{\psi}^o\|_* + \|\tilde{\psi}^o_\alpha\|_* \lesssim \|\tilde{\psi}\|_* + \|\tilde{\psi}_\alpha\|_* \lesssim \|\tilde{h}\|_{**} + \|h^o_\alpha\|_{**},\\
&\|\psi^o_\beta\|_* = \|\tilde{\psi}^o_\beta\|_* \lesssim \|\tilde{\psi}_\beta\|_* \lesssim \|h^o_\beta\|_{**},
\end{aligned}$$

and from here, (5.6) follows. ■

6. A projected nonlinear problem

We consider now the nonlinear projected problem

$$\begin{cases}
\mathcal{L}^\varepsilon(\psi) = R + \mathcal{N}(\psi) + c\displaystyle\sum_{j=1}^{2} \frac{\chi_j(z)}{iW(z-\tilde{d}_j)}(-1)^j W_{x_1}(z-\tilde{d}_j) \quad \text{in } \mathbb{R}^2,\\
\operatorname{Re}\displaystyle\int_{\mathbb{R}^2} \chi\overline{\phi_j} W_{x_1} = 0 \quad \text{with } \phi_j(z) = iW(z)\psi(z+\tilde{d}_j), \quad j = 1, 2,\\
\psi \text{ satisfies (3.16)}.
\end{cases} \tag{6.1}$$

Using the operator T_ε introduced in Proposition 5.1, we can rewrite this equation in the form of a fixed-point problem as $\psi = T_\varepsilon(R + \mathcal{N}(\psi)) =: G_\varepsilon(\psi)$.

Proposition 6.1. *There exists a constant $C > 0$ depending only on $0 < \alpha, \sigma < 1$, such that, for all ε sufficiently small, there exists a unique solution ψ_ε of* (6.1) *that satisfies*

$$\|\psi_\varepsilon\|_* \le \frac{C}{|\log\varepsilon|}.$$

Furthermore, ψ_ε is a continuous function of the parameter $\hat{d} := \sqrt{|\log\varepsilon|}d$ and

$$|\psi_\varepsilon^o|_\sharp \le C\varepsilon\sqrt{|\log\varepsilon|},$$

where ψ_ε^o is defined according to (5.2).

Proof. We let

$$\mathcal{F} := \Big\{\psi : \psi \text{ satisfies (3.16)},\ \mathrm{Re}\int_{\mathbb{R}^2} \chi_j\overline{\phi_j}W_{x_1} = 0,\ j = 1,2,$$
$$\|\psi\|_* \le \frac{C}{|\log\varepsilon|},\ |\psi^o|_\sharp \le C\varepsilon\sqrt{|\log\varepsilon|}\Big\}.$$

Endowed with the norm $\|\cdot\|_*$, $\mathcal{F}$ is a closed subset of the Banach space $\{\psi : \|\psi\|_* < +\infty\}$. We will show that, for ε small enough, G_ε maps $\mathcal{F}$ into itself. Indeed, we need to check that if $\|\psi\|_* \le C/|\log\varepsilon|$, then $\|T_\varepsilon(E + \mathcal{N}(\psi))\|_* \le C/|\log\varepsilon|$.

Note first that, from Proposition 4.1,

$$\|R\|_{**} \le \frac{C}{|\log\varepsilon|}.$$

Let us now estimate the size of the nonlinear term. For $\rho_1 > 3$ and $\rho_2 > 3$, the nonlinear terms are

$$i(\nabla\psi)^2 + i|V_d|^2(e^{-2\psi_2} - 1 + 2\psi_2) + i\varepsilon^2(\partial_s\psi)^2.$$

Let us work in the right half-plane (so $\rho_1 \le \rho_2$). We start with $(\nabla\psi)^2$. For $3 < \rho_1 < 2/\varepsilon$, we have

$$|(\nabla\psi)^2| \le |\nabla\psi|^2 \le \frac{\|\psi\|_*^2}{\rho_1^2}.$$

For $r > 1/\varepsilon$, we use

$$(\nabla\psi)^2 = (\partial_r\psi)^2 + \frac{1}{r^2}(\partial_s\psi)^2$$

and estimate

$$|(\partial_r\psi)^2| = (\partial_r\psi_1)^2 + (\partial_r\psi_2)^2 \le \varepsilon^2\|\psi\|_*^2$$

and

$$\frac{1}{r^2}|(\partial_s\psi)^2| \le \frac{1}{r^2}\big((\partial_s\psi_1)^2 + (\partial_s\psi_2)^2\big) \le \frac{1}{r^2}\|\psi\|_*^2.$$

It follows that $\|i(\nabla\psi)^2\|_{**} \le C\|\psi\|_*^2$.

Next we consider $i|V_d|^2(e^{-2\psi_2} - 1 + 2\psi_2)$. We note that the real part of this function is zero. Again, we work in the right half-plane. We have, for $\rho_1 > 3$,

$$\big||V_d|^2(e^{-2\psi_2} - 1 + 2\psi_2)\big| \le C|\psi_2|^2 \le C(\rho_1^{-2+\sigma} + \varepsilon^{2-\sigma})^2\|\psi_2\|_{2,*}^2,$$

and hence
$$\|i|V_d|^2(e^{-2\psi_2}-1+2\psi_2)\|_{**}\le C\|\psi\|_*^2,$$

Finally, we consider $i\varepsilon^2(\partial_s\psi)^2$. We have, for $3<\rho_1<2/\varepsilon$,
$$|\varepsilon^2(\partial_s\psi)^2|\le \varepsilon^2\tilde d^2|\partial_{\rho_1}\psi|^2+\left(\varepsilon+\frac{\varepsilon\tilde d}{\rho_1}\right)^2|\partial_{\theta_1}\psi|^2\le C(\varepsilon^2+\rho_1^{-2})\|\psi\|_*^2.$$

For $r>1/\varepsilon$,
$$|\varepsilon^2(\partial_s\psi)^2|\le\varepsilon^2\|\psi\|_*^2.$$

It follows that $\|i\varepsilon^2(\partial_s\psi)^2\|_{**}\le\|\psi\|_*^2$.

In $\{\rho_1\le 3\}\cup\{\rho_2\le 3\}$, it can be checked that
$$\begin{aligned}|iV_d\mathcal{N}(\psi)|\le C\big(&|D^2\gamma|+|D\gamma|+|\gamma|+|\gamma+\phi||\phi|\\&+|\gamma+\phi|^2(1+|\gamma+\phi|+|\gamma|)+|E_d||\phi|+|\nabla\phi|^2\big)\end{aligned}$$

with $\gamma=(1-\eta)V_d(e^{i\psi}-1-i\psi)$. Thus we obtain that, for any $j=1,2$,
$$\|iV_d\mathcal{N}(\psi)\|_{C^\alpha(\{\rho_j<3\})}\le C\|\psi\|_*^2+|E||\phi|\le\frac{C}{|\log\varepsilon|^2}.$$

Thus, for an appropriate constant C, we have that $G_\varepsilon\colon\psi\mapsto T_\varepsilon(E+\mathcal{N}(\psi))$ maps the ball $\{\psi;\|\psi\|_*\le C/|\log\varepsilon|\}$ into itself.

Let us now see the precise estimates on the "odd parts" and "even parts". From Proposition 4.2, we know that R^o, defined as in (4.19), can be decomposed into $R^o=R^o_\alpha+R^o_\beta$ with
$$|R^o_\alpha|_{\sharp\sharp}\le\frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}\qquad\|R^o_\beta\|_{**}\le C\varepsilon\sqrt{|\log\varepsilon|}.$$

It remains to prove that
$$|\mathcal{N}(\psi)^o|_{\sharp\sharp}\le C\big((|\psi^o|_\sharp+\varepsilon|\log\varepsilon|^{1/2})\|\psi^e\|_*+|\psi^o|_\sharp^2\big).\tag{6.2}$$

In order to do that, we recall that, in the decomposition of a function f in odd and even modes, we have that, near $+\tilde d$, the function f^e is exactly π-periodic in θ_1, whereas f^o is exactly 2π-periodic in θ_1. An analogous statement is true near $-\tilde d$. Now we can express the product of two functions as
$$fg=(f^e+f^o)(g^e+g^o)=f^eg^e+f^eg^o+g^ef^o+g^of^o.$$

We see that f^eg^e is exactly π-periodic, and hence $(fg)^o=[f^eg^o+g^ef^o+f^og^o]^o$. Thus we arrive at
$$|(fg)^o|\le(|f^o||g^e|+|f^e||g^o|+|f^o||g^o|).\tag{6.3}$$

To estimate $\mathcal{N}(\psi)$, we use a change of variables $(r,s)\to(\rho_j,\theta_j)=(\rho,\theta)$, and we observe that
$$(\nabla\psi)^2=(\partial_r\psi)^2+\frac{1}{r^2}(\partial_s\psi)^2=(\partial_\rho\psi)^2+\frac{1}{\rho^2}(\partial_\theta\psi)^2,$$

and

$$\varepsilon^2(\partial_s\psi)^2 = \varepsilon^2(\partial_\theta\psi)^2 + \varepsilon^2\tilde{d}\left(\sin\theta\partial_\rho\psi\partial_\theta\psi + \frac{\cos\theta}{\rho}(\partial_\theta\psi)^2\right) + \varepsilon^2\tilde{d}^2\left(\sin^2\theta(\partial_\rho\psi)^2 + \frac{4\cos\theta\sin\theta}{\rho}\partial_\rho\psi\partial_\theta\psi + \frac{\cos^2\theta}{\rho^2}(\partial_\theta\psi)^2\right).$$

Thus, component-wise, we obtain

$$\begin{aligned}(\tilde{\mathcal{N}}(\psi))_1 &= 2(\partial_\rho\psi_1)(\partial_\rho\psi_2) + 2(\partial_\theta\psi_1)(\partial_\theta\psi_2)\left(\varepsilon^2 + \frac{1}{\rho^2}\right)\\
&\quad + \varepsilon^2\tilde{d}\left(\sin\theta[\partial_\rho\psi_1\partial_\theta\psi_2 + \partial_\theta\psi_1\partial_\rho\psi_2] + \frac{2\cos\theta}{\rho}\partial_\theta\psi_1\partial_\theta\psi_2\right)\\
&\quad + \varepsilon^2\tilde{d}^2\Big(2\sin^2\theta\partial_\rho\psi_1\partial_\rho\psi_2\\
&\qquad + \frac{4\sin\theta\cos\theta}{\rho}[\partial_\rho\psi_1\partial_\theta\psi_2 + \partial_\theta\psi_1\partial_\rho\psi_2]\\
&\qquad + \frac{2\cos^2\theta}{\rho^2}\partial_\theta\psi_1\partial_\theta\psi_2\Big),\\
(\tilde{\mathcal{N}}(\psi))_2 &= -(\partial_\rho\psi_1)^2 + (\partial_\rho\psi_2)^2 - \left(\varepsilon^2 + \frac{1}{\rho^2}\right)\big((\partial_\theta\psi_2)^2 - (\partial_\theta\psi_1)^2\big)\\
&\quad + \varepsilon^2\tilde{d}\left(\sin\theta(\partial_\rho\psi_1\partial_\theta\psi_1 + \partial_\rho\psi_2\partial_\theta\psi_2) + \frac{\cos\theta}{\rho}[(\partial_\theta\psi_2)^2 - (\partial_\theta\psi_1)^2]\right)\\
&\quad + \varepsilon^2\tilde{d}^2\Big(\sin^2\theta\big((\partial_\rho\psi_2)^2 - (\partial_\rho\psi_1)^2\big)\\
&\qquad + \frac{4\cos\theta\sin\theta}{\rho}(\partial_\rho\psi_1\partial_\theta\psi_1 + \partial_\rho\psi_2\partial_\theta\psi_2)\\
&\qquad + \frac{\cos^2\theta}{\rho^2}[(\partial_\theta\psi_2)^2 - (\partial_\theta\psi_1)^2]\Big)\\
&\quad + |V_d|^2(1 - e^{2\psi_2} - 2\psi_2).\end{aligned}$$

We define

$$\begin{aligned}\mathcal{A}_1(\psi) &:= 2(\partial_\rho\psi_1)(\partial_\rho\psi_2) + 2(\partial_\theta\psi_1)(\partial_\theta\psi_2)\left(\varepsilon^2 + \frac{1}{\rho^2}\right),\\
\mathcal{B}_1(\psi) &:= \varepsilon^2\tilde{d}\left(\sin\theta[\partial_\rho\psi_1\partial_\theta\psi_2 + \partial_\theta\psi_1\partial_\rho\psi_2] + \frac{2\cos\theta}{\rho}\partial_\theta\psi_1\partial_\theta\psi_2\right),\\
\mathcal{C}_1(\psi) &:= \varepsilon^2\tilde{d}^2\Big(2\sin^2\theta\partial_\rho\psi_1\partial_\rho\psi_2 + \frac{4\sin\theta\cos\theta}{\rho}[\partial_\rho\psi_1\partial_\theta\psi_2 + \partial_\theta\psi_1\partial_\rho\psi_2]\\
&\qquad + \frac{2\cos^2\theta}{\rho^2}\partial_\theta\psi_1\partial_\theta\psi_2\Big).\end{aligned}$$

We have $(\mathcal{N}(\psi))_1 = \mathcal{A}_1(\psi) + \mathcal{B}_1(\psi) + \mathcal{C}_1(\psi)$. Besides, we can see that

$$\frac{|\mathcal{B}_1(\psi)|}{\rho_1^{-1} + \rho_2^{-1}} \leq \frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}\left(\frac{|\nabla\psi_1|}{\rho_1^{-1} + \rho_2^{-1}} \times \frac{|\nabla\psi_2|}{\rho_1^{-1} + \rho_2^{-1}}\right) \leq \frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}\|\psi\|_*^2$$

for $3 < \rho_1 < R_\varepsilon$. Now, by using the argument that a product of two π-periodic functions is π-periodic and the products of one π-periodic function and one 2π-periodic function is 2π-periodic and (6.3), we find that

$$\frac{|[\mathcal{A}_1(\psi)+\mathcal{C}_1(\psi)]^o|}{\rho_1^{-1}+\rho_2^{-1}} \leq C(\|\psi\|_*|\psi^o|_\sharp + |\psi^o|_\sharp^2).$$

Thus we obtain

$$\frac{|(\mathcal{N}(\psi))_1^o|}{\rho_1^{-1}+\rho_2^{-1}} \leq C\left(\|\psi\|_*|\psi^o|_\sharp + |\psi^o|_\sharp^2 + \frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}\|\psi\|_*^2\right)$$

for $3 < \rho_1 < R_\varepsilon$. We also have

$$1-e^{2\psi_2}-2\psi = (1-e^{2\psi_2^e}-2\psi_2^e) + (1-e^{2\psi_2^o}-2\psi_2^o)e^{2\psi_2^e} + 2\psi_2^o(e^{2\psi_2^e}-1).$$

We notice that $1-e^{2\psi_2^e}-2\psi_2^e$ is a π-periodic function. Thus we find that

$$|(1-e^{2\psi_2}-2\psi_2)^o| \leq C(|\psi_2^o||\psi^e| + |\psi^o|^2).$$

By using again that a product of two π-periodic functions is π-periodic and the products of one π-periodic function and one 2π-periodic function is 2π-periodic and (6.3), we can obtain

$$|(\mathcal{N}(\psi))_2^o| \leq C\left(\|\psi\|_*|\psi^o|_\sharp + |\psi^o|_\sharp^2 + \frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}\|\psi\|_*^2\right)$$

We proceed in the same way to estimate the other terms in $\mathcal{N}(\psi)$ when $\rho_1 < 3$ or $\rho_2 < 3$, and we use repeatedly (6.3) to arrive at (6.2).

We now show that G_ε is a contraction for ε small enough. Indeed, if $\|\psi^j\|_* \leq C/|\log\varepsilon|$ for $j = 1, 2$, then

$$\|\mathcal{N}(\psi^1)-\mathcal{N}(\psi^2)\|_{**} \leq \frac{C}{|\log\varepsilon|}\|\psi^1-\psi^2\|_*.$$

This is mainly due to the fact that $N(\psi)$ is quadratic and cubic in ψ, and in the first and second derivatives of ψ. Then we can use $a^2-b^2 = (a-b)(a+b)$ and $a^3-b^3 = (a-b)(a^2+ab+b^2)$. We finally apply the Banach fixed-point theorem, and we find the desired solution.

By definition, the error function E, the coefficients of $\mathcal{L}^\varepsilon$ and of $\mathcal{N}^\varepsilon$ depend continuously on the parameter $\hat{d}$. Thus we also have that the operator T_ε defined in Proposition 5.1 depends continuously on $\hat{d}$. Underlying the dependence on $\hat{d}$ and using the fixed-point characterization of ψ, we find that, for $\hat{d}_1, \hat{d}_2 > 0$,

$$\begin{aligned}
\psi(\hat{d}_1)-\psi(\hat{d}_2) &= T_\varepsilon(\hat{d}_1)(E(\hat{d}_1)) - T_\varepsilon(\hat{d}_2)(E(\hat{d}_2)) \\
&\qquad + T_\varepsilon(\hat{d}_1)(\mathcal{N}(\psi(\hat{d}_1)) - T_\varepsilon(\hat{d}_2)(\mathcal{N}(\psi(\hat{d}_2)) \\
&= T_\varepsilon(\hat{d}_1)[E(\hat{d}_1)-E(\hat{d}_2)] + (T_\varepsilon(\hat{d}_1)-T_\varepsilon(\hat{d}_2))(E(\hat{d}_2)) \\
&\qquad + T_\varepsilon(\hat{d}_1)[\mathcal{N}(\psi(\hat{d}_1)) - \mathcal{N}(\psi(\hat{d}_2))] \\
&\qquad + (T_\varepsilon(\hat{d}_1)-T_\varepsilon(\hat{d}_2))(\mathcal{N}(\psi(\hat{d}_2))).
\end{aligned}$$

Thus, by using that, for ε small, $T_\varepsilon \circ \mathcal{N}$ is a contraction, we find that

$$\begin{aligned}\|\psi(\hat{d}_1) - \psi(\hat{d}_2)\|_* \le \frac{1}{1-k}\big[&\|T_\varepsilon(\hat{d}_1)[E(\hat{d}_1) - E(\hat{d}_2)]\|_* \\ &+ \|(T_\varepsilon(\hat{d}_1) - T_\varepsilon(\hat{d}_2))(E(\hat{d}_2))\|_* \\ &+ \|(T_\varepsilon(\hat{d}_1) - T_\varepsilon(\hat{d}_2))(\mathcal{N}(\psi(\hat{d}_2)))\|_*\big],\end{aligned}$$

where k is a Lipschitz constant strictly less than 1. We can conclude that $\psi = \psi_\varepsilon$ inherits the continuous dependence on $\hat{d}$. ■

7. Solving the reduced problem

The solution ψ_ε of (6.1) previously found depends continuously on $\hat{d} := \sqrt{|\log\varepsilon|}d$. We want to find $\hat{d}$ such that the Lyapunov–Schmidt coefficient in (6.1) satisfies $c = c(\hat{d}) = 0$. We let

$$\varphi_\varepsilon := \eta i V_d \psi_\varepsilon + (1-\eta) V_d e^{i\psi_\varepsilon} \quad \text{and} \quad \phi_\varepsilon := i V_d \psi_\varepsilon,$$

where η was defined in (3.3). By symmetry, we work only in $\mathbb{R}^+ \times \mathbb{R}$. From the previous section, we have found ψ_ε such that

$$iW(z)[\mathcal{L}^\varepsilon(\psi_\varepsilon) + R + \mathcal{N}(\psi_\varepsilon)](z + \tilde{d}) = c\chi W_{x_1}.$$

For R_ε defined in (4.2), we set

$$c_* := \operatorname{Re}\int_{B(0,R_\varepsilon)} \chi |W_{x_1}|^2 = \operatorname{Re}\int_{B(0,4)} \chi |W_{x_1}|^2,$$

and we remark that this quantity is of order 1. We find that

$$\begin{aligned}cc_* = \operatorname{Re}&\int_{B(0,R_\varepsilon)} iW(z)R(z+\tilde{d})\overline{W}_{x_1}(z) \\ &+ \operatorname{Re}\int_{B(0,R_\varepsilon)} iW(z)\mathcal{L}^\varepsilon(\psi_\varepsilon)(z+\tilde{d})\overline{W}_{x_1}(z) \\ &+ \operatorname{Re}\int_{B(0,R_\varepsilon)} iW\mathcal{N}(\psi_\varepsilon)(z+\tilde{d})\overline{W}_{x_1}.\end{aligned}$$

We recall that $iW(z)\mathcal{L}^\varepsilon(\psi)(z+\tilde{d}) = L_j^\varepsilon(\phi_j)$ for $j = 1$ and L_j defined in (3.13). Integrating by parts, we find

$$\begin{aligned}\operatorname{Re}\int_{B(0,R_\varepsilon)} L_j^\varepsilon(\phi_j)\overline{W}_{x_1} = \operatorname{Re}&\int_{B(0,R_\varepsilon)} \overline{\phi_j}(L_j^\varepsilon - L^0)(W_{x_1}) \\ &+ \operatorname{Re}\Big\{\int_{\partial B(0,R_\varepsilon)}\Big(\frac{\partial\phi_j}{\partial\nu}\overline{W}_{x_1} - \phi_j\frac{\partial\overline{W}_{x_1}}{\partial\nu}\Big)\Big\}.\end{aligned}$$

Proceeding like in (5.24), we conclude

$$\left|\operatorname{Re}\int_{B(0,R_\varepsilon)} L_j^\varepsilon(\phi_j)\overline{W}_{x_1}\right| \le C\varepsilon\sqrt{|\log\varepsilon|}\|\psi\|_* \le \frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}.$$

Now we estimate the inner product of W_{x_1} and $iW(z)\mathcal{N}(\psi)(z+\tilde{d})$. We use the orthogonality of the Fourier modes to write

$$
\begin{aligned}
\mathrm{Re}&\int_{B(0,R_\varepsilon)} iW(z)\mathcal{N}(\psi)(z+\hat{d})\overline{W}_{x_1}\\
&= \mathrm{Re}\int_{B(0,R_\varepsilon)} iW\overline{W}_{x_1}(\mathcal{N}(\psi))^o\\
&= \mathrm{Re}\int_{B(0,R_\varepsilon)} iw\left(w'\cos\theta-\frac{iw}{\rho}\sin\theta\right)[\mathcal{N}(\psi)_1^o+i\mathcal{N}_2^o(\psi)]\\
&= -\int_{B(0,R_\varepsilon)}\left(ww'\cos\theta\,\mathcal{N}(\psi)_2^o-\frac{w^2}{r}\mathcal{N}(\psi)_1^o\sin\theta\right).
\end{aligned}
$$

We use that

$$
\begin{aligned}
&|(\mathcal{N}(\psi))_2^o|\le|(\mathcal{N}(\psi))_2^o|_{\sharp\sharp}\le C\|\psi^e\|_*|\psi^o|_\sharp+|\psi^o|_\sharp^2\le C\varepsilon|\log\varepsilon|^{-1/2},\\
&|(\mathcal{N}(\psi))_1^o|\le C\left(\frac{|\psi_2^o|_\sharp\|\psi_1^e\|_*}{1+\rho^2}+\frac{|\psi_1^o|_\sharp\|\psi_2\|_*}{1+\rho^{2-\sigma}}+\frac{|\psi_1^o|_\sharp|\psi_2^o|_\sharp}{1+\rho^{2-\sigma}}\right)\le C\frac{\varepsilon|\log\varepsilon|^{-1/2}}{1+\rho^{2-\sigma}}
\end{aligned}
$$

to obtain

$$
\left|\mathrm{Re}\int_{B(0,R_\varepsilon)} iW(z)\mathcal{N}(\psi)(z+\tilde{d})\overline{W}_{x_1}\right|\le C\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}.
$$

Now we claim

$$
\mathrm{Re}\int_{B(0,R_\varepsilon)} iW(z)R(z+\tilde{d})\overline{W}_{x_1}=-\varepsilon\sqrt{|\log\varepsilon|}\left(\frac{a_0}{\hat{d}}-a_1\hat{d}\right)+o_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|}).
$$

We set

$$
\begin{aligned}
B_0&:=\mathrm{Re}\int_{B(0,R_\varepsilon)} iW(z)R(z+\tilde{d})^0\overline{W}_{x_1},\\
B_1&:=\mathrm{Re}\int_{B(0,R_\varepsilon)} iW(z)R(z+\tilde{d})^1\overline{W}_{x_1},
\end{aligned}
$$

where we recall that $S_0(V_d)=iV_dR^0$, $S_1(V_d)=iV_dR^1$ and S_0,S_1 are given by (3.4).

From Lemma A.1 and Lemma 4.2, we find that

$$
\begin{aligned}
B_1&=\frac{\hat{d}\varepsilon}{\sqrt{|\log\varepsilon|}}\mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}}|W_{x_1}|^2+O_\varepsilon\left(\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\right)\\
&=\hat{d}\varepsilon\sqrt{|\log\varepsilon|}a_1+o_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|}),
\end{aligned}
$$

where we set

$$
a_1:=\frac{1}{|\log\varepsilon|}\int_0^{2\pi}\int_0^{\tilde{R}_\varepsilon}\frac{w_1^2\sin^2\theta_1}{\rho_1}\,\mathrm{d}\rho_1\,\mathrm{d}\theta_1,
$$

with $\tilde{R}_\varepsilon$ which is of order $\varepsilon^{-1}|\log\varepsilon|^{-1/2}$ and which does not depend on $\hat{d}$.

Because $\lim_{\rho\to+\infty}w(\rho)=1$, we can see that $0<c<a_1<C$ for some constants $c,C>0$, and a_1 is independent of $\hat{d}$.

On the other hand, by (4.6), we have

$$B_0 = \mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} 2\frac{(W^a_{x_1}W^b_{x_1} + W^a_{x_2}W^b_{x_2})}{W^b}\overline{W}^a_{x_1} + \mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} (1-|W^aW^b|^2 + |W^a|^2 - 1 + |W^b|^2 - 1)W^a\overline{W}^a_{x_1}.$$

The second integral is equal to

$$\mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} (1-(w_1w_2)^2 + w_1^2 - 1 + w_2^2 - 1)\Big(w_1'\cos\theta_1 + \frac{iw_1}{\rho_1}\sin\theta_1\Big)w_1 = O_\varepsilon(\varepsilon^2|\log\varepsilon|),$$

where we used that $(1-(w_1w_2)^2 + w_1^2 - 1 + w_2^2 - 1) = O(\varepsilon^2|\log\varepsilon|)$ and $w'(\rho) = 1/\rho^3 + O_\rho(1/\rho^4)$. We can also see that

$$\begin{aligned}
&\mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} \frac{W^a_{x_1}W^b_{x_1}}{W^b}\overline{W}^a_{x_1} \\
&= \mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} \Big(w_1'\cos\theta_1 + i\frac{w_1}{\rho_1}\sin\theta_1\Big)\Big[w_1'\frac{w_2'}{w_2}\cos\theta_1\cos\theta_2 \\
&\qquad - \frac{w_1}{\rho_1\rho_2}\sin\theta_1\sin\theta_2 - i\Big(\frac{w_1'}{\rho_2}\cos\theta_1\sin\theta_2 + \frac{w_2'w_1}{w_2\rho_1}\cos\theta_2\sin\theta_1\Big)\Big] \\
&= -\int_{\{\rho_1<R_\varepsilon\}} w_1w_1'\cos\theta_1\sin\theta_1\sin\theta_2\frac{\mathrm{d}\rho_1}{\rho_2}\,\mathrm{d}\theta_1 \\
&\qquad + \int_{\{\rho_1<R_\varepsilon\}} w_1'w_1\cos\theta_1\sin\theta_1\sin\theta_2\frac{\mathrm{d}\rho_1}{\rho_2}\,\mathrm{d}\theta_1 + O(\varepsilon^2|\log\varepsilon|).
\end{aligned}$$

In the previous equality, we used $w_2' \le C\varepsilon^3|\log\varepsilon|^{3/2}$. Hence we get

$$\mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} \frac{W^a_{x_1}W^b_{x_1}}{W^b}\overline{W}^a_{x_1} = O_\varepsilon(\varepsilon^2|\log\varepsilon|).$$

Finally, we have

$$\begin{aligned}
&\mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} \frac{W^a_{x_2}W^b_{x_2}}{W^b}\overline{W}^a_{x_1} \\
&= \mathrm{Re}\int_{\{\rho_1<R_\varepsilon\}} \Big(w_1'\cos\theta_1 + i\frac{w_1}{\rho_1}\sin\theta_1\Big)\Big[w_1'\frac{w_2'}{w_2}\sin\theta_1\sin\theta_2 \\
&\qquad - \frac{w_1}{\rho_1\rho_2}\cos\theta_1\cos\theta_2 + i\Big(\frac{w_1'}{\rho_2}\sin\theta_1\cos\theta_2 + \frac{w_2'w_1}{w_2\rho_1}\cos\theta_1\sin\theta_1\Big)\Big] \\
&= -\int_{\{\rho_1<R_\varepsilon\}} \frac{w_1w_1'}{\rho_1\rho_2}\cos^2\theta_1\cos\theta_2\rho_1\,\mathrm{d}\rho_1\,\mathrm{d}\theta_1 \\
&\qquad - \int_{\{\rho_1<\tilde{d}\}} \frac{w_1w_1'}{\rho_1\rho_2}\sin^2\theta_1\cos\theta_2\rho_1\,\mathrm{d}\rho_1\,\mathrm{d}\theta_1 + O(\varepsilon^2|\log\varepsilon|) \\
&= \int_{\{\rho_1<R_\varepsilon\}} \frac{w_1w_1'}{\rho_2}\cos\theta_2\,\mathrm{d}\rho_1\,\mathrm{d}\theta_1 + O(\varepsilon^2|\log\varepsilon|).
\end{aligned}$$

Using the properties of w_1, w_1' and that, in this region,

$$\cos\theta_2 > 0 \quad \text{and} \quad 0 < c < \frac{\rho_2 \varepsilon \sqrt{|\log\varepsilon|}}{\hat{d}} < C$$

for some constants $c, C > 0$ and $\rho_2 \varepsilon \sqrt{|\log\varepsilon|}/\hat{d}$ is independent of $\hat{d}$ in the region $0 < \rho_1 < R_\varepsilon$, we find

$$\operatorname{Re} \int_{\{\rho_1 < R_\varepsilon\}} \frac{W^a_{x_2} W^b_{x_2}}{W^b} \overline{W^a_{x_1}} = -\frac{a_0}{\hat{d}} \varepsilon \sqrt{|\log\varepsilon|} + o_\varepsilon(\varepsilon \sqrt{|\log\varepsilon|}),$$

with $c < a_0 < C$ for some constants $c, C > 0$ and independent of $\hat{d}$.

Therefore, we conclude that

$$cc_* = \varepsilon \sqrt{|\log\varepsilon|} \left(\frac{a_0}{\hat{d}} - a_1 \hat{d} \right) + o_\varepsilon(\varepsilon \sqrt{|\log\varepsilon|}).$$

Let us point out that, in this expression, $o_\varepsilon(\varepsilon \sqrt{|\log\varepsilon|})$ is a continuous function of the parameter $\hat{d}$. By applying the intermediate value theorem, we can find $\hat{d}_0$ near $\sqrt{a_0/a_1}$ such that $c = c(\hat{d}_0) = 0$. For such $\hat{d}_0$, we obtain that $V_d + \varphi_\varepsilon$ is a solution of (1.1). To conclude the proof of Theorem 1, thanks to the helical symmetry, it suffices to show that the solutions of the two-dimensional problem we found satisfy $\lim_{|z|\to+\infty} |V_\varepsilon(z)| = 1$. But this is because, far away from the vortices, our solution takes the form $V_\varepsilon(z) = W(z - \tilde{d}) W(z + \tilde{d}) e^{i\psi_\varepsilon}$. Thus $|V_\varepsilon| = |W(z - \tilde{d}) W(z + \tilde{d})| e^{-\operatorname{Im}\psi_\varepsilon}$. Thanks to the decay estimates obtained on ψ_ε, we have

$$|\operatorname{Im}\psi| \leq \frac{C}{|\log\varepsilon|} \left(\frac{1}{1 + |z - \tilde{d}|^{2-\sigma}} + \frac{1}{1 + |z + \tilde{d}|^{2-\sigma}} \right).$$

This proves that $\lim_{|z|\to+\infty} |V_\varepsilon(z)| = 1$ and thus that the solution of the three-dimensional problem satisfies (1.14).

Appendix A.

A.1. The standard vortex and its linearized operator

As stated in the introduction, the building block used to construct our solutions to equation (2.1) is the standard vortex of degree one, W, in $\mathbb{R}^2$. It satisfies

$$\Delta W + (1 - |W|^2) W = 0 \quad \text{in } \mathbb{R}^2$$

and can be written as

$$W(x_1, x_2) = w(r) e^{i\theta}, \quad \text{where } x_1 = r\cos\theta,\ x_2 = r\sin\theta.$$

Here w is the unique solution of (1.6). In this section, we collect useful properties of w and of the linearized Ginzburg–Landau operator around W.

Lemma A.1. *The following properties hold:*

(1) $w(0) = 0$, $w'(0) > 0$, $0 < w(r) < 1$ *and* $w'(r) > 0$ *for all* $r > 0$*;*

(2) $w(r) = 1 - 1/(2r^2) + O(1/r^4)$ *for large* r*;*

(3) $w(r) = \alpha r - \alpha r^3/8 + O(r^5)$ *for* r *close to* 0 *for some* $\alpha > 0$*;*

(4) *if we define* $T(r) = w'(r) - w/r$*, then* $T(0) = 0$ *and* $T(r) < 0$ *in* $(0, +\infty)$*;*

(5) $w'(r) = 1/r^3 + O(1/r^4)$*,* $w''(r) = O(1/r^4)$*.*

For the proof of this lemma, we refer to [10, 25].

An object of special importance to construct our solution is the linearized Ginzburg–Landau operator around W, defined by

$$L^0(\phi) := \Delta\phi + (1 - |W|^2)\phi - 2\,\mathrm{Re}(\overline{W}\phi)W.$$

This operator does have a kernel, as the following result states.

Lemma A.2. *Suppose that* $\phi \in L^\infty(\mathbb{R}^2)$ *satisfies* $L^0(\phi) = 0$ *in* $\mathbb{R}^2$ *and the symmetry* $\phi(\bar{z}) = \bar{\phi}(z)$*. Assume furthermore that, when we write* $\phi = iW\psi$ *and* $\psi = \psi_1 + i\psi_2$ *with* $\psi_1, \psi_2 \in \mathbb{R}$*, we have*

$$|\psi_1| + (1 + |z|)|\nabla\psi_1| \le C, \quad |\psi_2| + |\nabla\psi_2| \le \frac{C}{1 + |z|}, \quad |z| > 1.$$

Then $\phi = c_1 W_{x_1}$ *for some real constant* c_1*.*

Proof. The equation $L_0(\phi) = 0$ in $B(0,1)^c$ translates into

$$\Delta\psi + 2\frac{\nabla W}{W}\nabla\psi - 2i|W|^2\,\mathrm{Im}\,\psi = 0 \quad \text{in } B(0,1)^c.$$

This reads

$$\begin{aligned} 0 &= \Delta\psi_1 + \frac{2w'}{w}\partial_r\psi_1 + \frac{2}{r^2}\partial_\theta\psi_2 && \text{in } B(0,1)^c,\\ 0 &= \Delta\psi_2 + \frac{2w'}{w}\partial_r\psi_2 - \frac{2}{r^2}\partial_\theta\psi_1 - 2|W|^2\psi_2 && \text{in } B(0,1)^c. \end{aligned}$$

We thus have, by using the decay assumption on ψ_1, ψ_2, that

$$\left|\Delta\psi_2 - 2|W|^2\psi_2\right| \le \frac{C}{1 + r^2} \quad \text{in } B(0,1)^c.$$

Since $|W|^2 \ge C > 0$ in $B(0,1)^c$, we can use a barrier argument and elliptic estimates to obtain

$$(1 + |z|^2)(|\psi_2| + |\nabla\psi_2|) \le C. \tag{A.1}$$

We can then use the previous estimate to obtain

$$|\Delta\psi_1| \le \frac{C}{1 + r^3} \quad \text{in } B(0,1)^c.$$

We use that $\psi_1(z = x_1 + ix_2) = 0$ for $x_2 = 0$, a barrier argument in the half-plane and elliptic estimates to obtain

$$|\psi_1| + (1 + |z|)|\nabla\psi_1| \le \frac{C}{(1 + |z|)^\alpha} \tag{A.2}$$

for any $\alpha \in (0,1)$. From (A.1) and (A.2), we get

$$|\phi(z)| + (1+|z|)|\nabla\phi| \le \frac{C}{(1+|z|)^{\alpha}}, \quad |z|>1. \tag{A.3}$$

From the fact $L^0(\phi)=0$ in $\mathbb{R}^2$, we know that

$$\mathrm{Re}\int_{B_R(0)} \bar{\phi}\Delta\phi + \int_{B_R(0)} (1-|W|^2)|\phi|^2 - 2\int_{B_R(0)} |\mathrm{Re}(\overline{W}\phi)|^2 = 0$$

for any $R>0$. Then, integrating by parts, we get

$$\int_{B_R(0)} |\nabla\phi|^2 - \mathrm{Re}\int_{\partial B_R(0)} \bar{\phi}\partial_\nu\phi - \int_{B_R(0)} (1-|W|^2)|\phi|^2 + 2\int_{B_R(0)} |\mathrm{Re}(\overline{W}\phi)|^2 = 0.$$

Using (A.3), we find $|\mathrm{Re}(\bar{\phi}\partial_\nu\phi)| \le C/(1+|z|^{2\alpha+1})$. Thus

$$\left|\mathrm{Re}\int_{\partial B_R(0)} \bar{\phi}\partial_\nu\phi\right| \le \frac{C}{R^{2\alpha}}.$$

Making $R\to\infty$, we conclude

$$\int_{\mathbb{R}^2} |\nabla\phi|^2 - \int_{\mathbb{R}^2} (1-|W|^2)|\phi|^2 + 2\int_{\mathbb{R}^2} |\mathrm{Re}(\overline{W}\phi)|^2 = 0.$$

Thanks to the decay estimates (A.3), we also have

$$\int_{\mathbb{R}^2} [|\nabla\phi|^2 + (1-|W|^2)|\phi|^2 + |\mathrm{Re}(\bar{W}\phi)|^2] < +\infty.$$

We can then apply [15, Theorem 1] to obtain that there exists $c_1, c_2 \in \mathbb{R}$ such that $\phi = c_1 W_{x_1} + c_2 W_{x_2}$. Using the symmetry assumption $\phi(\bar{z}) = \bar{\phi}(z)$, we can conclude that actually $\phi = c_1 W_{x_1}$ for some $c_1 \in \mathbb{R}$. ∎

Lemma A.3. *Suppose that $\phi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ satisfies $L^0(\phi)=0$ in $\mathbb{R}^2$ and the symmetry $\phi(\bar{z}) = \overline{\phi(z)}$. Assume furthermore that, when we write $\phi = iW\psi$ and $\psi = \psi_1 + i\psi_2$ with $\psi_1, \psi_2 \in \mathbb{R}$, we have*

$$|\psi_1| + (1+|z|)|\nabla\psi_1| \le C(1+|z|)^{\alpha}, \quad |\psi_2| + |\nabla\psi_2| \le \frac{C}{1+|z|}, \quad |z|>1,$$

for some $\alpha < 3$. Then $\phi = c_1 W_{x_1}$ for some real constant c_1.

Proof. Here we work with the change of variables $\phi = e^{i\theta}\psi$. Then $L^0(\phi)=0$ becomes

$$0 = \Delta\psi - \frac{1}{r^2}\psi + \frac{2i}{r^2}\partial_\theta\psi + (1-w^2)\psi - 2iw^2\,\mathrm{Im}(\psi).$$

Writing $\psi = \psi_1 + i\psi_2$ with $\psi_1, \psi_2 \in \mathbb{R}$, we get the system

$$\begin{cases} 0 = \Delta\psi_1 - \dfrac{1}{r^2}\psi_1 - \dfrac{2}{r^2}\partial_\theta\psi_2 + (1-w^2)\psi_1 \\ 0 = \Delta\psi_2 - \dfrac{1}{r^2}\psi_2 + \dfrac{2}{r^2}\partial_\theta\psi_1 + (1-3w^2)\psi_2, \end{cases}$$

which holds in $\mathbb{R}^2\setminus\{0\}$ with the symmetry condition $\psi_1(\bar{z}) = -\psi_1(z)$, $\psi_2(\bar{z}) = \psi_2(z)$.

We decompose in Fourier modes

$$\psi_1 = \sum_{k=1}^{\infty} \psi_{1,k}^2(r)\sin(k\theta), \quad \psi_2 = \sum_{k=0}^{\infty} \psi_{2,k}^1(r)\cos(k\theta)$$

and obtain

$$0 = \partial_{rr}\psi_{1,k}^2 + \frac{1}{r}\partial_r\psi_{1,k}^2 - \frac{k^2+1}{r^2}\psi_{1,k}^2 + 2\frac{k}{r^2}\psi_{2,k}^1 + (1-w^2)\psi_{1,k}^2, \tag{A.4}$$

$$0 = \partial_{rr}\psi_{2,k}^1 + \frac{1}{r}\partial_r\psi_{2,k}^1 - \frac{k^2+1}{r^2}\psi_{2,k}^1 + 2\frac{k}{r^2}\psi_{1,k}^2 + (1-3w^2)\psi_{2,k}^1. \tag{A.5}$$

In particular, equation (A.4) for $k=1$ can be written as

$$\partial_{rr}\psi_{1,1}^2 + \frac{1}{r}\partial_r\psi_{1,1}^2 - \frac{1}{r^2}\psi_{1,1}^2 = g_0,$$

where

$$g_0(r) = -2\frac{1}{r^2}\psi_{2,1}^1 + \left(w^2 - 1 + \frac{1}{r^2}\right)\psi_{1,1}^2 = O(r^{\alpha-4})$$

as $r\to\infty$. The variation of parameters formula yields a function

$$\psi_0(r) = -\frac{1}{r}\int_0^r \rho\int_\rho^\infty g_0(s)\,\mathrm{d}s\,\mathrm{d}\rho,$$

which satisfies

$$\partial_{rr}\psi_0 + \frac{1}{r}\partial_r\psi_0 - \frac{1}{r^2}\psi_0 = g_0 \qquad \text{for } r>1,$$

$$|\psi_0(r)| \le Cr^{\alpha-2}, \quad |\partial_r\psi_0(r)| \le Cr^{\alpha-3} \quad \text{for } r>1. \tag{A.6}$$

Hence

$$\psi_{1,1}^2(r) = \psi_0(r) + \alpha_1 r + \alpha_2 r^{-1}, \quad r>1, \tag{A.7}$$

for some $\alpha_1,\alpha_2\in\mathbb{R}$. We claim that $\alpha_1=0$. To prove this, we note that, for $k=1$, system (A.4)–(A.5) has the explicit solution

$$\bar\psi = \begin{bmatrix}\bar\psi_1\\ \bar\psi_2\end{bmatrix}, \quad \bar\psi_1 = \frac{w(r)}{r}, \quad \bar\psi_2 = -w'(r)$$

Let

$$\psi = \begin{bmatrix}\psi_{1,1}^2\\ \psi_{2,1}^1\end{bmatrix},$$

and define the Wronskian $W(r) = \psi\cdot\bar\psi_r - \psi_r\cdot\bar\psi$. We claim that

$$W(r) = \frac{c}{r} \tag{A.8}$$

for some $c\in\mathbb{R}$. To prove this, note that system (A.4)–(A.5) for ψ can be written as

$$0 = \psi_{rr} + \frac{1}{r}\psi_r + B\psi,$$

where B is the 2×2 matrix

$$B = \frac{2}{r^2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{2}{r^2}\begin{bmatrix} 1-w^2 & 0 \\ 0 & 1-3w^2 \end{bmatrix}.$$

Then

$$W_r = -\frac{1}{r}W + \psi^T(B - B^T)\bar{\psi} = -\frac{1}{r}W$$

because the matrix B is symmetric, and we get (A.8). Using decomposition (A.7), decay (A.6) and the explicit form of $\bar{\psi}$, we see that

$$W(r) = -\frac{2\alpha_1}{r}.$$

On the other hand, from the smoothness of ϕ near the origin, we get that $\psi^2_{1,1}(r)$, $\psi^1_{2,1}(r)$ and their derivatives remain bounded as $r \to 0$. Since the same is true for $\bar{\psi}$, we see that $W(r)$ is bounded as $r \to 0$, which implies that $\alpha_1 = 0$ as claimed. This in turn implies that

$$|\psi^2_{1,1}| \le C r^{\alpha'}, \quad |\partial_r \psi^2_{1,1}| \le C r^{\alpha'-1} \quad \text{for } r > 1,$$

where $\alpha' = \max(-1, \alpha - 2) < 1$. Using barriers for ODE (A.5), we get, for $k = 1$,

$$|\psi^2_{1,1}| \le C \quad \text{for } r > 1.$$

and for $k \ge 2$,

$$|\psi^2_{1,k}(r)| \le \frac{C}{k^2 r} \quad \text{for } r > 1.$$

Adding these inequalities, we see that $|\psi_1(z)| \le C$, $|z| > 1$, and then a standard scaling and elliptic estimates show that

$$|\nabla \psi_1(z)| \le \frac{C}{|z|}, \quad |z| > 1.$$

Now we can apply Lemma A.2 and conclude that $\phi = c_1 W_{x_1}$ for some constant $c_1 \in \mathbb{R}$.

∎

A.2. Elliptic estimates used in the linear theory

In this subsection, we prove elliptic estimates that we needed in Section 3 to develop the linear theory. More specifically, we prove estimates of solutions to some model equations.

We use the notation $z = (x_1, x_2) = re^{is}$, and throughout this section, $\varepsilon > 0$ is a parameter. We also use

$$\Delta = \partial^2_{x_1x_1} + \partial^2_{x_2x_2} = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{ss}.$$

Furthermore, in the equations, the following term will appear:

$$\partial^2_{ss} u = x_2^2 \partial^2_{x_1x_1} u + x_1^2 \partial^2_{x_2x_2} u - 2x_1x_2\partial^2_{x_1x_2} u - x_1\partial_{x_1} u - x_2 \partial_{x_2} u.$$

We start with recalling the statement and the proof of the comparison principle in the half-plane for the operator $\Delta + \varepsilon^2 \partial^2_{ss}$ with Dirichlet boundary condition.

Lemma A.4. *Let* $u: \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ *be a bounded function which is in* $C^2(\mathbb{R} \times \mathbb{R}_+^*) \cap C^0(\overline{\mathbb{R} \times \mathbb{R}_+^*})$ *and which satisfies*

$$\begin{cases} \Delta u + \varepsilon^2 \partial_{ss}^2 u \geq 0 & \text{in } \mathbb{R} \times \mathbb{R}_+^*, \\ u \leq 0 & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

Then $u \leq 0$ *in* $\mathbb{R} \times \mathbb{R}_+^*$.

Proof. We adapt the proof of [6, Lemma 2.1].

Let us use polar coordinates $(r, s) \in (0, +\infty) \times (0, \pi)$, and let $\varphi > 0$ be the first eigenfunction of ∂_{ss}^2 in $(-\frac{\pi}{4}, \frac{5\pi}{4})$ associated to the eigenvalue $\mu > 0$, i.e.,

$$\begin{cases} \partial_{ss}^2 \varphi + \mu \varphi = 0 \quad \text{on } \left(-\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right), \\ \varphi\left(-\dfrac{\pi}{4}\right) = \varphi\left(\dfrac{5\pi}{4}\right) = 0. \end{cases}$$

We define $\beta := \sqrt{\mu}$, and we set $g(r, s) := r^\beta \varphi(s)$ in $(0, +\infty) \times (-\frac{\pi}{4}, \frac{5\pi}{4})$, and hence

$$\partial_{rr}^2 g + \frac{1}{r} \partial_r g + \left(\frac{1}{r^2} + \varepsilon^2\right) \partial_{ss}^2 g = -\mu \varepsilon^2 g \leq 0 \quad \text{in } (0, +\infty) \times \left(-\frac{\pi}{4}, \frac{5\pi}{4}\right).$$

Consider $\sigma := u/g$ in $(0, +\infty) \times (0, \pi)$ (note that $g > 0$ in this domain). Since $\Delta u + \varepsilon^2 \partial_{ss}^2 u \geq 0$, we find

$$\Delta \sigma + \varepsilon^2 \partial_{ss}^2 \sigma + \frac{2}{g}\left[\partial_r g \partial_r \sigma + \left(\frac{1}{r^2} + \varepsilon^2\right) \partial_s g \partial_g \sigma\right] + \frac{\Delta g + \varepsilon^2 \partial_{ss}^2 g}{g} \sigma \geq 0.$$

We note that $(\Delta g + \varepsilon^2 \partial_{ss}^2 g)/g\, \sigma \leq 0$ and, since u is bounded, $\limsup_{r \to +\infty} \sigma = 0$. We can thus apply the maximum principle to deduce that $\sigma \leq 0$ in $(0, +\infty) \times (0, \pi)$. Hence $u \leq 0$ as well in $(0, +\infty) \times (0, \pi)$. ∎

In the same spirit, we have the following comparison principle for the Neumann boundary condition.

Lemma A.5. *Let* $u: \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ *be a bounded function which is in* $C^2(\mathbb{R} \times \mathbb{R}_+^*) \cap C^1(\overline{\mathbb{R} \times \mathbb{R}_+^*})$*. Let* $c \geq 0$*. We assume that* u *satisfies*

$$\begin{cases} \Delta u + \varepsilon^2 \partial_{ss}^2 u - cu \geq 0 & \text{in } \mathbb{R} \times \mathbb{R}_+^*, \\ \partial_\nu u \leq 0 & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

Then $u \leq 0$ *in* $\mathbb{R} \times \mathbb{R}_+^*$.

For a function $f: \mathbb{R}^2 \to \mathbb{R}$ and $\nu \in \mathbb{N}^*, \alpha > 0$, we introduce the norms

$$\|f\|_{\nu,\alpha} := \|(1 + |z|^\nu) f\|_{L^\infty(\mathbb{R}^2)} + \sup_{z \in \mathbb{R}^2} |z|^{\nu+\alpha} [f]_{z,\alpha}$$

with

$$[f]_{z,\alpha} := \sup_{|h|<1} \frac{|f(z+h) - f(z)|}{|h|^\alpha}.$$

Our first goal is to prove the following proposition.

Proposition A.1. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that $f(\bar z)=-f(z)$ and $\|f\|_{2,\alpha}<+\infty$. Then there exists a unique bounded solution of*

$$\Delta u+\varepsilon^2\partial_{ss}^2 u=f \quad \text{in } \mathbb{R}^2,$$

which satisfies $u(\bar z)=-u(z)$ and

$$|u(z)|\le C\|f\|_{2,\alpha},\quad |\nabla u(z)|\le C\frac{\|f\|_{2,\alpha}}{1+|z|}\quad \text{for all } z \text{ in } \mathbb{R}^2, \tag{A.9}$$

$$|\varepsilon\partial_s u(z)|\le C\frac{\|f\|_{2,\alpha}}{|z|}\quad \text{for } |z|\ge\frac{C}{\varepsilon},\quad \|D^2u\|_{2,\alpha}\le C\|f\|_{2,\alpha}. \tag{A.10}$$

We first prove the following lemma.

Lemma A.6. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that $f(\bar z)=-f(z)$ and $\|f\|_{2,\alpha}<+\infty$. Let $u:\mathbb{R}^2\to\mathbb{R}$ be a bounded function such that $u(\bar z)=-u(z)$ and*

$$\Delta u+\varepsilon^2\partial_{ss}^2 u=f \quad \text{in } \mathbb{R}^2.$$

Then there exists $C>0$ independent of u, f, ε such that (A.9), (A.10) hold.

Proof. Thanks to the symmetry $u(\bar z)=-u(z)$, it is sufficient to consider the problem

$$\begin{cases}\Delta u+\varepsilon^2\partial_{ss}^2u-f(z)=0, & z\in\mathbb{R}\times\mathbb{R}_+^*,\\ \qquad u(x_1,0)=0 & \text{for all } x_1\in\mathbb{R},\end{cases} \tag{A.11}$$

which we can alternatively write as

$$\begin{cases}\Delta u+\varepsilon^2\partial_{ss}^2u-f=0, & (r,s)\in(0,+\infty)\times(0,\pi),\\ u(r,0)=u(r,\pi)=0.\end{cases}$$

Let us assume

$$|f(z)|\le\frac{1}{1+|z|^2}.$$

We want to prove that, for an absolute constant C, we have $|u(z)|\le C$. We define

$$v(z)=v(r,s):=s(\pi-s).$$

We can check that

$$\Delta v+\varepsilon^2\partial_{ss}^2v+\frac{1}{1+r^2}=\frac{-2}{r^2}-2\varepsilon^2+\frac{1}{1+r^2}<0\quad \text{for all } z=re^{is}\in\mathbb{R}\times\mathbb{R}_+^*.$$

Hence v is a positive supersolution (and $-v$ a subsolution) for (A.11) in $(0,+\infty)\times(0,\pi)$, and in this set, for any bounded solution u of (A.11), we have, from Lemma A.4,

$$|u(z)|\le|v(z)|\quad \text{in } \mathbb{R}\times\mathbb{R}_+^*.$$

The decay estimates in (A.9)–(A.10) follow by Schauder estimates and a standard scaling argument. ∎

Lemma A.7. *If u is a bounded function that satisfies*

$$\Delta u + \varepsilon^2 \partial_{ss}^2 u = 0 \quad \text{in } \mathbb{R}^2, \quad u(\bar{z}) = -u(z),$$

then $u \equiv 0$.

Proof. Suppose $u \not\equiv 0$, and assume without loss of generality that $\sup_{\mathbb{R}^2} u = 1$. By the strong maximum principle, the supremum cannot be attained in $\mathbb{R}^2 \setminus \{0\}$. Let $z_n \in \mathbb{R}^2$ be a sequence such that $u(z_n) \to 1$. Up to a subsequence, we have two possibilities: $z_n \to 0$ or $|z_n| \to \infty$.

Case $z_n \to \infty$. Let us write $z_n = R_n e^{i\sigma_n}$, where $R_n \to \infty$ and $\sigma_n \in (0, \pi)$. We express u in polar coordinates (r, s) and define $\tilde{u}_n(r, s) := u(r + R_n, s)$. Up to a subsequence, we have $\tilde{u}_n \to \tilde{u}$ uniformly in compact sets of $\mathbb{R}^2$, where $\tilde{u} \le 1$, $\tilde{u}(p) = 1$ for some point $p = (1, s)$ with $s \in [0, \pi]$, and $\partial_{rr}^2 \tilde{u} + \varepsilon^2 \partial_{ss}^2 \tilde{u} = 0$ in $\mathbb{R}^2$, with the additional condition $\tilde{u}(r, 0) = \tilde{u}(r, \pi) = 0$. This contradicts the strong maximum principle.

Case $z_n \to 0$. Let us write $z_n = R_n e^{i\sigma_n}$, where $R_n \to 0$ and $\sigma_n \in (0, \pi)$. Define $\tilde{u}_n(\zeta) := u(R_n \zeta)$. Up to a subsequence, $\tilde{u}_n \to \tilde{u}$ uniformly in compact sets of $\mathbb{R}^2$, where $\tilde{u} \le 1$ attains its maximum at some point and satisfies $\Delta \tilde{u} = 0$ in $\mathbb{R}^2$. This is a contradiction. ∎

Proof of proposition A.1. We use $v := \|f\|_{2,\alpha} s(\pi - s)$ as a supersolution to solve the problem in large half-balls centered at the origin. More precisely, for any $M > 0$, there exists a solution of

$$\begin{cases} \Delta u_M + \varepsilon^2 \partial_{ss}^2 u_M = f & \text{in } B_M^+(0), \\ u_M = 0 & \text{on } \partial B_M^+(0), \end{cases}$$

where $B_M^+(0) := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+ : |z| < M\}$. Thanks to gradient estimates (A.9), we have

$$|\nabla u_M| \le \frac{C|v|}{1 + |z|} \quad \text{in } B_M^+(0),$$

for some $C > 0$ independent of M, and thus we can apply the Arzela–Ascoli theorem to take the limit of u_M along a suitable subsequence, obtaining a solution of (A.11). The uniqueness is proved in Lemma A.7, and the estimates follow from Lemma A.6. ∎

Proposition A.1 is a model for the treatment of ψ_1, the real part of ψ in Lemma 5.1. To deal with ψ_2, we have to use an analogous proposition.

Proposition A.2. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be such that $g(\bar{z}) = g(z)$ and $\|g\|_{1,\alpha} < +\infty$ Then there exists a unique bounded $v\colon \mathbb{R}^2 \to \mathbb{R}$ such that $v(\bar{z}) = v(z)$ and $\Delta v + \varepsilon^2 \partial_{ss}^2 v - v = g$. Furthermore, there exists a constant $C > 0$ such that*

$$(1 + |z|)\big(|v(z)| + |\nabla v(z)|\big) \le C\|g\|_{1,\alpha}, \quad \|D^2 v\|_{1,\alpha} \le C\|g\|_{1,\alpha},$$
$$\big|\varepsilon |z| \partial_s v(z)\big| \le C\|g\|_{1,\alpha} \quad \textit{for } |z| > 1/\varepsilon.$$

Proof. The symmetry assumption allows us to work in the half-plane $\mathbb{R} \times \mathbb{R}_+^*$ with homogeneous Neumann condition on the boundary. We can then apply a barrier argument and rescaled Schauder estimates to prove the proposition. ∎

In the course of the linear theory for our problem, when we separate even and odd modes, we need an analogue of the following lemma.

Let us consider $R_0 > 0$ fixed and $R_0 < R_\varepsilon < \varepsilon^{-1}$, and let Ω, Ω' be the regions

$$\Omega := \{z \in \mathbb{R}^2 : R_0 < |z| < R_\varepsilon\}, \quad \Omega' := \left\{z \in \mathbb{R}^2 : 2R_0 < |z| < \frac{1}{2}R_\varepsilon\right\},$$

and recall the polar coordinates notation $z = re^{is}$, $r > 0$, $s \in \mathbb{R}$.

Lemma A.8. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(\bar{z}) = -f(z)$ and $|f(z)| \le 1/|z|$. Let u be a solution of $\Delta u + \varepsilon^2 \partial_{ss}^2 u = f$ in Ω such that $u(\bar{z}) = -u(z)$ and*

$$\begin{aligned} |u(z)| &\le R_0 |\log \varepsilon|, \quad && |z| = R_0,\\ |u(z)| &\le R_\varepsilon, && |z| = R_\varepsilon. \end{aligned}$$

Then there is C such that

$$|u(z)| \le C|z| \log\left(\frac{2R_\varepsilon}{|z|}\right) \quad \text{for all } z \in \Omega'.$$

Proof. We use a Fourier series decomposition which, thanks to the symmetries, we can take of the form

$$f(r,s) = \sum_{k \ge 1} f_k(r) \sin(ks), \quad u(r,s) = \sum_{k \ge 1} u_k(r) \sin(ks).$$

The equations on the Fourier coefficients are

$$u_k'' + \frac{1}{r} u_k' - k^2 \left(\frac{1}{r^2} + \varepsilon^2\right) u_k = f_k \quad \text{in } (R_0, R_\varepsilon).$$

We estimate each u_k using barriers. For $k = 1$, we define

$$\bar{u}_1(r) := r \log\left(\frac{3R_\varepsilon}{r}\right)$$

The function $\bar{u}_1$ satisfies

$$\bar{u}_1'' + \frac{1}{r}\bar{u}_1' - \left(\frac{1}{r^2} + \varepsilon^2\right)\bar{u}_1 < -\frac{1}{r} \quad \text{for } r < R_\varepsilon.$$

Thus we can use $\bar{u}_1$ as a barrier for u_1 in the interval (R_0, R_ε) and deduce that

$$|u_1(r)| \le r \log\left(\frac{3R_\varepsilon}{r}\right), \quad r \in (R_0, R_\varepsilon). \tag{A.12}$$

For $k \ge 2$, we use the barrier

$$\bar{u}_k(r) = C\left(\frac{r}{k^2} + C|\log \varepsilon| \left(\frac{r}{R_0}\right)^{-k} + R_\varepsilon \left(\frac{r}{R_\varepsilon}\right)^k\right),$$

where C is a large fixed constant (the last two terms in $\bar{u}_k$ solve almost the homogeneous equation and are there for the boundary conditions). By the maximum principle, $|u_k| \le \bar{u}_k$

in (R_0, R_ε), and for $r \in (2R_0, R_\varepsilon/2)$, we get

$$\sum_{k=2}^{\infty} \bar{u}_k(r) \leq C \frac{|\log \varepsilon|}{r} + Cr \leq Cr \log\left(\frac{2R_\varepsilon}{r}\right).$$

This and (A.12) imply the desired conclusion. ■

Summary of general notation and norms

For the sake of clarity, in this section, we collect the definitions of the different norms and the common notation used along the paper.

In rescaled variables, we denote the distance $\tilde{d}$ of the vortices to the origin as

$$\tilde{d} := \frac{d}{\varepsilon} = \frac{\hat{d}}{\varepsilon\sqrt{|\log \varepsilon|}},$$

where $\hat{d} = O(1)$. For every specific vortex, we write $\tilde{d}_j := (-1)^{1+j}\tilde{d}$ and

$$\rho_1 e^{i\theta_1} := re^{is} - \tilde{d}, \quad \rho_2 e^{i\theta_2} := re^{is} + \tilde{d},$$

the polar coordinates around each one. Defining $R_\varepsilon := \alpha_0/(\varepsilon|\log \varepsilon|^{1/2})$, with $\alpha_0 > 0$ a fixed small constant, the norm we require in the right-hand side for the general invertibility theory is

$$\begin{aligned}
\|h\|_{**} := &\sum_{j=1}^{2} \|V_d h\|_{C^\alpha(\rho_j<3)} \\
&+ \sup_{\substack{\rho_1>2\\ \rho_2>2}} \left[\frac{|\mathrm{Re}(h)|}{\rho_1^{-2} + \rho_2^{-2} + \varepsilon^2} + \frac{|\mathrm{Im}(h)|}{\rho_1^{-2+\sigma} + \rho_2^{-2+\sigma} + \varepsilon^{\sigma-2}}\right] \\
&+ \sup_{\substack{2<|z-\tilde{d}_1|<2R_\varepsilon\\ 2<|z-\tilde{d}_2|<2R_\varepsilon}} \frac{[\mathrm{Re}(h)]_{\alpha, B_{|z|/2}(z)}}{|z-\tilde{d}_1|^{-2-\alpha} + |z-\tilde{d}_2|^{-2-\alpha}} \\
&+ \sup_{\substack{2<|z-\tilde{d}_1|<2R_\varepsilon\\ 2<|z-\tilde{d}_2|<2R_\varepsilon}} \frac{[\mathrm{Im}(h)]_{\alpha, B_1(z)}}{|z-\tilde{d}_1|^{-2+\sigma} + |z-\tilde{d}_2|^{-2+\sigma}},
\end{aligned}$$

where $\alpha, \sigma \in (0,1)$, $\|f\|_{C^\alpha(D)} = \|f\|_{C^{0,\alpha}(D)}$ and

$$[f]_{\alpha,D} := \sup_{\substack{x,y\in D\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \quad \|f\|_{C^{k,\alpha}(D)} := \sum_{j=0}^{k} \|D^j f\|_{L^\infty(D)} + [D^k f]_{\alpha,D}.$$

Likewise, the solution $\psi\colon \mathbb{C} \to \mathbb{C}$ lies in a space determined by the norm

$$\|\psi\|_* := \sum_{j=1}^{2} \|V_d \psi\|_{C^{2,\alpha}(\rho_j<3)} + \|\mathrm{Re}(\psi)\|_{1,*} + \|\mathrm{Im}(\psi)\|_{2,*},$$

where, with $\operatorname{Re}\psi = \psi_1, \operatorname{Im}\psi = \psi_2$,

$$\begin{aligned}\|\psi_1\|_{1,*} &:= \sup_{\substack{\rho_1>2\\ \rho_2>2}} |\psi_1| + \sup_{\substack{2<\rho_1<\frac{2}{\varepsilon}\\ 2<\rho_2<\frac{2}{\varepsilon}}} \frac{|\nabla\psi_1|}{\rho_1^{-1}+\rho_2^{-1}} + \sup_{r>\frac{1}{\varepsilon}}\left[\frac{1}{\varepsilon}|\partial_r\psi_1| + |\partial_s\psi_1|\right]\\ &\quad + \sup_{\substack{2<\rho_1<R_\varepsilon\\ 2<\rho_2<R_\varepsilon}} \frac{|D^2\psi_1|}{\rho_1^{-2}+\rho_2^{-2}} + \sup_{\substack{2<|z-\tilde d_1|<R_\varepsilon\\ 2<|z-\tilde d_2|<R_\varepsilon}} \frac{[D^2\psi_1]_{\alpha,B_{|z|/2}(z)}}{|z-\tilde d_1|^{-2-\alpha}+|z-\tilde d_2|^{-2-\alpha}},\\ \|\psi_2\|_{2,*} &:= \sup_{\substack{\rho_1>2\\ \rho_2>2}} \frac{|\psi_2|}{\rho_1^{-2+\sigma}+\rho_2^{-2+\sigma}+\varepsilon^{\sigma-2}} + \sup_{\substack{2<\rho_1<\frac{2}{\varepsilon}\\ 2<\rho_2<\frac{2}{\varepsilon}}} \frac{|\nabla\psi_2|}{\rho_1^{-2+\sigma}+\rho_2^{-2+\sigma}}\\ &\quad + \sup_{r>\frac{1}{\varepsilon}}[\varepsilon^{\sigma-2}|\partial_r\psi_2| + \varepsilon^{\sigma-1}|\partial_s\psi_2|] + \sup_{\substack{2<\rho_1<R_\varepsilon\\ 2<\rho_2<R_\varepsilon}} \frac{|D^2\psi_2|}{\rho_1^{-2+\sigma}+\rho_2^{-2+\sigma}}\\ &\quad + \sup_{\substack{2<|z-\tilde d_1|<R_\varepsilon\\ 2<|z-\tilde d_2|<R_\varepsilon}} \frac{[D^2\psi_2]_{\alpha,B_1(z)}}{|z-\tilde d_1|^{-2+\sigma}+|z-\tilde d_2|^{-2+\sigma}}\end{aligned}$$

for $\alpha, \sigma \in (0,1)$.

Given a complex function $g\colon \mathbb{C}\to\mathbb{C}$ satisfying $g(\bar z) = -\bar g(z)$, we write its decomposition in even and odd Fourier modes in θ_j as $g = \sum_{k=0}^{\infty} g^{k,j}$, where

$$g^{k,j}(\rho_j,\theta_j) := g_1^{k,j}(\rho_j)\sin(k\theta_j) + i g_2^{k,j}(\rho_j)\cos(k\theta_j), \quad g_1^{k,j}(\rho_j), g_2^{k,j}(\rho_j) \in \mathbb{R}.$$

We define

$$g^{e,j} := \sum_{k \text{ even}} g^{k,j}, \quad g^{o,j} := \sum_{k \text{ odd}} g^{k,j}$$

and

$$g^o := \eta_{1,R_\varepsilon} g^{o,1} + \eta_{2,R_\varepsilon} g^{o,2}, \quad g^e := g - g^o,$$

where

$$\eta_{j,R}(z) := \eta_1\left(\frac{|z-\tilde d_j|}{R}\right),$$

and $\eta_1\colon \mathbb{R}\to[0,1]$ is a smooth function such that $\eta_1(t) = 1$ for $t \le 1$ and $\eta_1(t) = 0$ for $t \ge 2$.

Finally, in order to have specific control of the odd parts of the functions involved, we introduce the following seminorms: given $h = h_1 + ih_2$ and $\psi = \psi_1 + i\psi_2$, we denote

$$\begin{aligned}|h|_{\sharp\sharp} &:= \sum_{j=1}^{2}\|V_d h\|_{C^{0,\alpha}(\rho_j<4)} + \sup_{\substack{2<\rho_1<R_\varepsilon\\ 2<\rho_2<R_\varepsilon}}\left[\frac{|h_1|}{\rho_1^{-1}+\rho_2^{-1}} + \frac{|h_2|}{\rho_1^{-1+\sigma}+\rho_2^{-1+\sigma}}\right],\\ |\psi|_{\sharp} &:= \sum_{j=1}^{2}|\log\varepsilon|^{-1}\|V_d\psi\|_{C^{2,\alpha}(\rho_j<3)} + |\psi_1|_{\sharp,1} + |\psi_2|_{\sharp,2},\end{aligned}$$

where

$$|\psi_1|_{\sharp,1} := \sup_{\substack{2<\rho_1<R_\varepsilon \\ 2<\rho_2<R_\varepsilon}} \bigg[\frac{|\psi_1|}{\rho_1 \log(2R_\varepsilon/\rho_1) + \rho_2 \log(2R_\varepsilon/\rho_2)} + \frac{|\nabla \psi_1|}{\log(2R_\varepsilon/\rho_1) + \log(2R_\varepsilon/\rho_2)} \bigg],$$

$$|\psi_2|_{\sharp,2} := \sup_{\substack{2<\rho_1<R_\varepsilon \\ 2<\rho_2<R_\varepsilon}} \bigg[\frac{|\psi_2| + |\nabla \psi_2|}{\rho_1^{-1+\sigma} + \rho_2^{-1+\sigma} + \rho_1^{-1} \log(2R_\varepsilon/\rho_1) + \rho_2^{-1} \log(2R_\varepsilon/\rho_2)} \bigg].$$

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