J. Eur. Math. Soc. 25, 1045–1090 (2023) DOI 10.4171/JEMS/1204

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Smooth stationary water waves with exponentially localized vorticity

Received November 25, 2019

Abstract. We study stationary capillary-gravity waves in a two-dimensional body of water that rests above a flat ocean bed and below vacuum. This system is described by the Euler equations with a free surface. A great deal of recent activity has focused on finding waves with nontrivial vorticity ω. There are now many results on the existence of solutions to this problem for which the vorticity is non-vanishing at infinity, and several authors have constructed waves with ω having compact support. Our main theorem states that there are large families of stationary capillary-gravity waves that carry finite energy and exhibit an exponentially localized distribution of vorticity. They are solitary waves in the sense that the free surface is asymptotically flat. Remarkably, while their amplitude is small, the kinetic energy is $O(1)$. In this and other respects, they are strikingly different from previously known rotational water waves.

To construct these solutions, we exploit a previously unobserved connection between the steady water wave problem on the one hand and singularly perturbed elliptic PDE on the other. Indeed, our result expands the study of spike-layer solutions to free boundary problems with physical relevance.

Keywords. Water waves, vorticity, spike solution, free boundary problem

1. Introduction

We consider waves in a two-dimensional body of water that has finite depth. Mathematically, they are modeled as solutions to the incompressible Euler equation

$$\partial_t v + (v \cdot \nabla)v + \nabla p + g e_2 = 0 \tag{1.1a}$$

on the evolving fluid domain

$$\Omega(t) = \{(x_1, x_2) \in \mathbb{R}^2 : -1 < x_2 < 1 + \eta(t, x_1)\}. \tag{1.1b}$$

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Mathematics Subject Classification (2020): Primary 35Q35; Secondary 35R35

Here, the water density is assumed to be of constant value 1, $v = v(t, \cdot)\colon \Omega(t) \to \mathbb{R}^2$ is the velocity, $p = p(t, \cdot)\colon \Omega(t) \to \mathbb{R}$ is the pressure, $g > 0$ is the constant gravitational acceleration, and $e_2 = (0, 1)$. Notice that the water is bounded below by a rigid and perfectly flat bed at $\{x_2 = -1\}$. The upper boundary, given by the graph of $1 + \eta$, represents the interface between the water and a region of air which is treated as vacuum. An important feature of this problem is that η is one of the unknowns in the system. For solitary waves, η vanishes as $|x_1| \to \infty$, and hence the asymptotic depth is normalized to be 2.

The kinematic boundary conditions state that the velocity field does not penetrate the bed:

$$v_2 = 0 \quad \text{on } x_2 = -1, \tag{1.1c}$$

and, along the free surface, we have

$$\partial_t \eta = -v_1 \partial_{x_1} \eta + v_2 \quad \text{on } x_2 = 1 + \eta(t, x_1). \tag{1.1d}$$

Lastly, on the surface we impose the dynamic condition according to the Young–Laplace law that

$$p = \alpha^2 \kappa \quad \text{on } x_2 = 1 + \eta(t, x_1), \tag{1.1e}$$

where $\alpha > 0$ is a constant measuring the surface tension and

$$\kappa = -\frac{\partial_{x_1}^2 \eta}{(1 + (\partial_{x_1} \eta)^2)^{3/2}} \tag{1.2}$$

is the signed curvature. Because $g, \alpha > 0$, we always presume that surface tension is present on the interface and that gravity acts in the bulk. Solutions of (1.1) are therefore called *capillary-gravity waves*.

A stationary water wave is a solution to (1.1) that is independent of time. More generally, one can consider steady or traveling waves, which are solutions that become time independent after shifting to a moving frame of reference. These are among the oldest and most important examples of nonlinear wave phenomena studied in mathematics.

Perhaps the central object of interest for this paper is the *vorticity*

$$\omega := \nabla^\perp \cdot v = \partial_{x_1} v_2 - \partial_{x_2} v_1, \tag{1.3}$$

which is the third component of $\nabla \times (v_1, v_2, 0)$. The earliest rigorous constructions of steady water waves were given by Levi-Civita [29] and Nekrasov [32], who worked in the irrotational regime where ω vanishes identically. This assumption permits several elegant reformulations of the problem that are far more tractable; see, for example, the survey [41]. However, beginning in the early 2000s, substantial inroads have been made in the rigorous analysis of rotational water waves. With a few exceptions, these results pertain to waves without interior stagnation, meaning that the streamlines (the integral curves of v) are never closed, and hence the vorticity does not vanish at infinity.

In practice, though, many of the effects that generate vorticity are *local* – wind blowing over a section of the water or a boundary layer caused by an immersed body, for example. This naturally leads us to seek waves for which ω is concentrated in the near field. A completely different analytical approach is necessary to treat this situation, however. Consequently, there are comparatively very few rigorous results for waves with localized vorticity, and those that are available concern either periodic waves or waves with compactly supported vorticity; see the overview below.

Another important quantity associated to the system is the total energy E defined by

$$E = \frac{1}{2}\int_{\Omega} |v|^2 \, dx + \int_{\mathbb{R}} \left(\frac{1}{2} g\eta^2 + \alpha^2 \left(\sqrt{1 + (\partial_{x_1}\eta)^2} - 1 \right) \right) dx_1. \tag{1.4}$$

The first term on the right-hand side represents the kinetic energy, while the second is gravitational potential energy, and the third is the surface energy. It is well-known that E is conserved by sufficiently smooth solutions of the time-dependent problem. It is physically desirable, therefore, to construct waves that carry a finite amount of total energy, which in particular means that v must be in $L^2(\Omega)$.

As the main contribution of this paper, we prove the existence of large families of solitary stationary water waves with a *smooth*, *highly localized vorticity* and a *finite energy* $E < \infty$: in a perturbed disk around the origin the vorticity is large and negative, and outside it is positive and exponentially decaying. Qualitatively, this represents an entirely novel species of water wave that we call a *vortex spike*. Our method establishes a connection between singularly perturbed elliptic equations and physical problems with free boundaries. This application to water waves is at once quite natural and yet completely new.

1.1. Main theorem

We now state the result more precisely. In two dimensions, divergence free vector fields can be represented through a stream function, namely,

$$v = \nabla^{\perp}\Psi := (-\partial_{x_2}\Psi, \partial_{x_1}\Psi).$$

One can easily confirm from (1.3) that $\omega = \Delta\Psi$.

As mentioned above, our interest is in *smooth finite energy stationary* waves with *spatially highly localized vorticity*. For the momentum equation (1.1a), we see ω satisfies

$$\partial_t \omega + v \cdot \nabla\omega = 0 \quad \text{in } \Omega(t), \tag{1.5}$$

and hence the vorticity is transported by the Lagrangian flow. In terms of the stream function, for the stationary case this becomes

$$\nabla^{\perp}\Psi \cdot \nabla\Delta\Psi = v \cdot \nabla\omega = 0 \quad \text{in } \Omega. \tag{1.6}$$

The kinematic boundary conditions (1.1c)–(1.1d) imply that Ψ is a constant along each component of $\partial\Omega$. Without loss of generality, we take

$$\Psi|_{\partial\Omega} = 0; \tag{1.7}$$

see Section 1.3 for more discussion about this. At the same time, the dynamic condition (1.1e) can be expressed in terms of Ψ as the well-known Bernoulli equation

$$\tfrac{1}{2}|\nabla\Psi|^2 + gx_2 + \alpha^2\kappa = g \quad \text{on } x_2 = 1 + \eta(x_1). \tag{1.8}$$

Together, (1.6)–(1.8) are equivalent to the (stationary) Euler equations (1.1). We seek to construct waves for which the stream function and the vorticity will have the leading order forms

$$\begin{aligned} \Psi(x) &= U\left(\frac{x - x_*}{\delta}\right) + \cdots \in H^k(\Omega) \cap H_0^1(\Omega), \\ \omega(x) &= \frac{1}{\delta^2}\Delta U\left(\frac{x - x_*}{\delta}\right) + \cdots, \end{aligned} \tag{1.9}$$

where $0 < \delta \ll 1$, x_* is roughly the location of the vorticity to be determined in the proof which will turn out to be very close to the origin in our coordinate system, and U is a smooth solution to (1.6) on the whole of $\mathbb{R}^2$, exponentially decaying as $|x| \to \infty$. It is well-known that (1.6) is satisfied provided that $\omega = \gamma(\Psi)$ for some *vorticity function* γ. We therefore construct Ψ as the solution to

$$\Delta\Psi = \frac{1}{\delta^2}\gamma(\Psi) \quad \text{in } \Omega, \tag{1.10}$$

with U a solution to

$$\Delta U = \gamma(U) \quad \text{in } \mathbb{R}^2. \tag{1.11}$$

We will assume that γ satisfies the following:

(A) $\gamma \in C^{k_0}(\mathbb{R}, \mathbb{R})$, $k_0 \geq 2$, $\gamma(0) = 0$, $\gamma'(0) = 1$, and (1.11) has a nontrivial radial solution $U \in C^{k_0+2}(\mathbb{R}^2)$ satisfying $U(x), \nabla U(x) \to 0$ as $|x| \to \infty$, and

(B) the kernel of $-\Delta + \gamma'(U)\colon H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is equal to span $\{\partial_{x_1}U, \partial_{x_2}U\}$.

We would like to point out that the asymptotic vanishing of U and ∇U at $|x| = \infty$ can be ensured by further asking that $U \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ or $U \in H^1(\mathbb{R}^2)$ (see Remark 3.4). Also, as a consequence of Assumption (A), U and its derivatives up to order $k_0 + 1$ decay exponentially, and so in particular $U \in H^{k_0+2}(\mathbb{R}^2)$ (see Proposition 3.1). Prototypical functions γ fulfilling Assumptions (A) and (B) are $\gamma(t) = t - |t|^p t$ for integers $p \geq 1$, but many others will do as well. Classical results for dimension $n = 2$ may be found in for example [2,3,27], and a modern summary including the nondegeneracy results in [1].

Under the above assumptions, our main theorem is as follows.

Theorem 1.1. *For any γ as in Assumptions (A) and (B), there exists $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$, there is a finite energy solution*

$$(\Psi, \eta) \in (H_0^1(\Omega) \cap H^{k_0}(\Omega)) \times H^{k_0}(\mathbb{R})$$

to the stationary water wave problem (1.7), (1.8), and (1.10). Both Ψ and η are even in x_1. Moreover, there exists a constant $C > 0$, independent of δ but depending on γ, such that for each $\delta \in (0, \delta_0)$ there exists τ with $|\tau| \leq C\delta^{-7/2}e^{-2/\delta}$ satisfying

$$|\Psi - \Psi_0|_{H^{k_0}(\Omega)} \leq C\delta^{1-2k_0}e^{-2/\delta}, \tag{1.12}$$

where

$$\Psi_0(x) = U\left(\frac{x_1}{\delta}, \frac{x_2-\tau}{\delta}\right) - U\left(\frac{x_1}{\delta}, \frac{2-x_2-\tau}{\delta}\right) - U\left(\frac{x_1}{\delta}, \frac{-2-x_2-\tau}{\delta}\right),$$

and

$$|\eta|_{H^{k_0}(\mathbb{R})} \leq C\delta^{1-k_0}e^{-2/\delta}, \quad |\eta-\eta_0|_{H^{k_0}(\mathbb{R})} \leq C\delta^{3/4-2k_0}e^{-3/\delta}, \tag{1.13}$$

with

$$\begin{aligned}\eta_0 &= -2\delta^{-2}(g-\alpha^2\partial_{x_1}^2)^{-1}\Big(\big(\partial_{x_2}U\big(\tfrac{\cdot}{\delta},\tfrac{1}{\delta}\big)\big)^2\Big)\\ &= -\frac{1}{\alpha\sqrt{g}\,\delta^2}e^{-\frac{\sqrt{g}}{\alpha}|\cdot|} * \Big(\big(\partial_{x_2}U\big(\tfrac{\cdot}{\delta},\tfrac{1}{\delta}\big)\big)^2\Big).\end{aligned}$$

We first comment on the vorticity and the surface profile given in the above theorem. On the one hand, from Proposition 3.1, Corollary 3.5 and (1.12), we see that the kinetic energy is of $O(1)$. Roughly,

$$|v|_{L^2(\Omega)} = |\nabla\Psi|_{L^2(\Omega)} = |\nabla U|_{L^2(\mathbb{R}^2)} + o(e^{-\frac{1}{2\delta}}),$$

while the corresponding vorticity is spiked in the sense that

$$\omega = \tfrac{1}{\delta^2}\gamma\big(U\big(\tfrac{\cdot}{\delta}\big)\big) + o(e^{-\frac{1}{2\delta}}), \quad |\omega|_{L^\infty(\Omega)} = O\big(\tfrac{1}{\delta^2}\big), \quad |\omega|_{L^1(\Omega)} = |\Delta U|_{L^1(\mathbb{R}^2)} + o(e^{-\frac{1}{2\delta}}).$$

On the other hand, the total vorticity is exponentially small in $0 < \delta \ll 1$:

$$\int_\Omega \omega\, dx = \int_\Omega \Delta\Psi\, dx = \int_{\partial\Omega} N\cdot\nabla\Psi\, dS = o(e^{-\frac{1}{2\delta}}).$$

By (1.12) and the definition of Ψ_0, $\chi_\Omega(x)\omega(x) \to 0$ for any $x \neq 0$ as $\delta \searrow 0$, where χ_Ω is the characteristic function of Ω. Then from the above integral estimate we can readily prove $\int_{\mathbb{R}^2} f\chi_\Omega\omega\, dx \to 0$ as $\delta \searrow 0$ for any continuous f compactly supported in $\mathbb{R}^2$. Therefore, as a measure, $\chi_\Omega\omega\, dx$ converges weakly to 0 as $\delta \searrow 0$. However, the vorticity has a rich spatial structure in a domain on the scale of $O(\delta)$ where its pointwise value is $O(\frac{1}{\delta^2})$. Moreover, as ω is $O(1)$ in $L^1(\Omega)$, these waves exhibit a highly localized but strong rotational vector field with kinetic energy of order $O(1)$.

Since ω concentrates far away from $\partial\Omega$ and the total vorticity is exponentially small, $\partial\Omega$ is only weakly impacted by the spike. This fact is reflected in the exponential smallness of η in (1.13). According to Proposition 3.1, the leading term η_0 given in (1.13) satisfies

$$-\eta_0(x_1) \geq \frac{1}{C}\delta^{-1/2}e^{-2/\delta} \quad \text{for } |x_1| < C^{-1}$$

for some $C > 0$ independent of δ, while its tail is much smaller. Therefore the concentrated vorticity ω creates a surface depression in the near field with rapid decay as $|x_1| \to \infty$; see Figure 1.

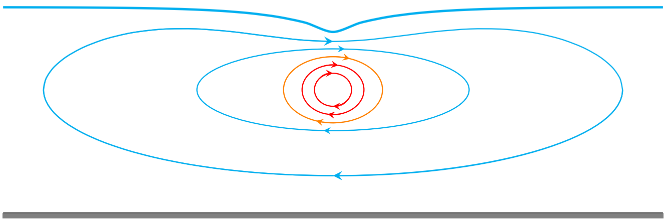


Fig. 1. Schematic representation of the streamline pattern and free surface. Blue lines indicate positive vorticity, red is negative, and orange is zero. Note that there is a critical layer and all streamlines are closed except the boundary components. For stationary waves, the fluid particles will move exactly along these streamlines. The above configuration therefore differs rather dramatically from that of a typical irrotational wave: in the absence of vorticity and surface tension, it is known that *none* of the particle paths can be closed [10].

1.2. History and relation to our construction

Rotational steady water waves have been a very active area of research for nearly two decades, beginning with the construction of large-amplitude periodic gravity waves by Constantin and Strauss [11]. These authors used bifurcation theory starting from a fixed shear flow, and their methodology has since been adapted and expanded upon in many ways (see [9]). It is important to note that, while there do exist explicit rotational water waves (for example, [6, 15, 21, 23, 25]), they are exceedingly rare. From that perspective, the main contribution of [11] was its systematic treatment of a broad class of vorticity distributions. However, Constantin–Strauss – and most of the works that followed them – require both that ω is nonlocalized and that there are no interior stagnation points. In particular, smooth perturbation of a shear flow could never yield decaying vorticity. Interior stagnation and critical layers (regions of closed streamlines), however, can be constructed using variants of this approach. Early papers of Simmem and Saffman [37] and da Silva and Peregrine [39] considered this regime through formal asymptotic analysis and numerical bifurcation theory. In [17], it was rigorously shown that the nonlinear particle paths in the linearized system can have closed orbits, and the behavior of small waves with constant vorticity was studied. Based on it, Wahlén [44] constructed exact periodic waves with one critical layer, and similar waves were subsequently constructed using a harmonic-functions approach, globally, in [13]. These works all treat constant vorticity and the situation where the linearized problem at the shear flow has a one-dimensional kernel. One can also find steady waves with critical layers bifurcating from two-dimensional [16], three-dimensional [18], and even arbitrarily high dimensional kernels [26] of affine or near-affine vorticity functions, as well as from one-dimensional kernels of constant vorticity with one discontinuity [31]. Very recently, a global theory for analytic vorticity functions allowing for several critical layers has been presented in [43] (one might note that even affine vorticity can yield arbitrarily many vertically aligned stagnation points.) The waves built in this paper have vorticity functions of the next order in this development, as Assumption (A) implies that γ is nonlinear with leading-order linear term, although the method of proof is very different.

The first rigorous construction of traveling capillary-gravity waves with localized vorticity in infinite depth is due to Shatah, Walsh, and Zeng [36]. In that paper, two classes of compactly supported vorticity were studied: solitary and periodic waves with a submerged point vortex, and solitary waves with a vortex patch. In the former case, ω is a Dirac measure supported in the interior of Ω. This can be viewed as a solution to a suitably weakened version of the Euler equations. The proof in [36] was based on a splitting of the velocity field into a rotational and irrotational component, followed by a bifurcation argument beginning at the trivial solution $(\Psi, \eta) = (0, 0)$ with the total vorticity $\int \omega \, dx$ serving as the parameter. While the vortex patch solutions were small amplitude, the authors obtained a global curve of periodic traveling waves with a point vortex. The vortex patches have finite energy and the corresponding vorticity is $C^{0,1}(\Omega)$ and smooth on its support. Later, Varholm [42] extended the ideas in [36] to the finite-depth case with arbitrarily many point vortices, and Le [28] studied the existence and orbital stability of finite dipoles inside an infinite-depth capillary-gravity wave. Earlier work in the 50s and 60s that treated point vortices carried by gravity waves in finite depth include [19, 20, 40]. A vortex patch situated near a shoreline and such that the velocity vanishes completely outside a ball has also been constructed in [8], using dynamical systems tools.

The capillary-gravity waves in the current work can be said to live between the above-mentioned types. They can be viewed, for $0 < \delta \ll 1$, as smoothed vortex patches or as the limit, as the period tends to infinity, of steady periodic waves with critical layers. We note that in [36], (i) ω is single-signed and either a Dirac measure or in $C^{0,1}(\Omega)$; and (ii) the measure $\omega \, dx$ vanishes absolutely as one approaches the point of bifurcation. By contrast, in the present paper, the vorticity changes sign and is smooth *throughout* Ω. Moreover, $\omega \, dx$ converges to 0 weakly as $\delta \searrow 0$, while the L^1 norm of ω and the kinetic energy both remain order $O(1)$. This surprising feature results from the fact that we do not perturb from a shear flow, but singularly from U of (1.11) which has fixed, positive energy. In all these respects, the vortex spikes constructed in Theorem 1.1 contrast starkly with the literature described above.

When ω is not compactly supported, it is of little help to decompose the velocity field into rotational and irrotational parts. We are also barred from using shear flows as a model for the stream function. The main new idea is to instead look to the theory of spike and spike-layer solutions to singular perturbations of semilinear elliptic PDE. These equations typically have the form

$$\delta^2 \Delta u = u - u^p \quad \text{in } D, \tag{1.14}$$

where $D \subset \mathbb{R}^n$ is a smooth bounded domain, $p > 1$, and Dirichlet or Neumann conditions are prescribed on ∂D. Beginning in the late 80s, versions of (1.14) were investigated intensively by the elliptic PDE community resulting in a vast literature; see, for example, [30, 33–35].

Drawing inspiration from these works, we model our stream function as a rescaled and translated $U(\frac{\cdot - x_*}{\delta})$ on the unknown fluid domain represented by a conformal mapping Γ. The translation invariance of the problem leads to a degeneracy – as can be seen in Assumption (B) – which is resolved through a Lyapunov–Schmidt reduction. We outline heuristically how to solve the resulting highly degenerate bifurcation equation for x_* in the next subsection.

To the best of our knowledge, ours is the first work exploring singularly perturbed elliptic equations in the hydrodynamical context. The method bears certain similarities to Li and Nirenberg's treatment of (1.14) in [30], in particular, the use of a Lyapunov–Schmidt reduction, bundle coordinates in a tubular neighborhood of a family of translates, and boundary correction projections. However, we stress that the steady water wave problem presents substantial new difficulties: the upper boundary is free, the Bernoulli condition (1.8) imposed there is completely nonlinear, and the domain Ω is horizontally unbounded.

1.3. Heuristic discussions

In this subsection, we discuss several issues related to the *finite energy/spatial decaying* assumptions on *smooth* steady (stationary or traveling) solutions on fluid domains extending to horizontal infinity. We first observe that the support of the vorticity of such solutions should be the whole of Ω. Otherwise, one expects that the vorticity will not be smooth over the boundary of its support, as a consequence of the Hopf lemma for the elliptic equation (1.10).

Traveling waves. While we focus on stationary capillary-gravity waves in the current paper, by shifting to a moving reference frame, Theorem 1.1 immediately furnishes families of *traveling* capillary-gravity waves with exponentially localized vorticity. The velocity field for these waves will be an H^{k_0-1} perturbation of a fixed uniform background current $ce_1 \neq 0$, and the vorticity will be spiked in the same sense as before.

On the other hand, smooth *finite*-energy waves with a nonzero wavespeed are unlikely to exist. In fact, the vorticity level curves for such waves would be closed loops $C_a = \{\omega = a\}$, which are transported by the velocity field $v = (v_1, v_2)$. Therefore

$$v \cdot \nu = ce_1 \cdot \nu = c\nu_1 \quad \text{along } C_a,$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector of C_a. This implies that $|v| \geq \frac{1}{2}|c|$ if $|\nu_1| > \frac{1}{2}$, which usually happens on an $O(1)$ proportion of most level curves. Consequently, $|v|$ is likely to be bounded from below on a set with infinite measure, which is prevented by the finite energy assumption.

Fluid depth and the boundary condition of the stream function. In [36], traveling capillary-gravity waves with compact vortex patches were constructed in fluids of infinite depth. Slightly modifying the formula of the rotational part of the velocity fields, actually the same construction should also work with finite depth. However, we do *not* expect smooth spatially localized stationary waves to exist in infinite depth unless the free surface is overturned.

In fact, let us temporarily not preclude the possibility of Ω with infinite depth. Let a solution Ψ of (1.10) be given satisfying $v = \nabla^\perp \Psi \in H^1(\Omega)$ and $(v \cdot N)|_{\partial\Omega} = 0$ with $N = (N_1, N_2)$ the outward unit normal to Ω. The latter condition implies that Ψ is locally constant on $\partial\Omega$. Fix $\Psi = 0$ on $S = \operatorname{graph}(1 + \eta)$.

Much as in the proof of Proposition 3.1, Ψ and its derivatives decay exponentially as $|x| \to \infty$. Let Γ be the antiderivative of γ with $\Gamma(0) = 0$. We multiply (1.10) by $\partial_{x_2}\Psi$ and integrate to find

$$
\begin{aligned}
\frac{1}{\delta^2}\int_{\partial\Omega} \Gamma(\Psi) N_2 \, \mathrm{d}S = \frac{1}{\delta^2}\int_{\Omega} \partial_{x_2}\Gamma(\Psi)\, \mathrm{d}x &= \int_{\Omega} \partial_{x_2}\Psi \Delta\Psi \, \mathrm{d}x \\
&= -\int_{\Omega} \nabla\partial_{x_2}\Psi \cdot \nabla\Psi \, \mathrm{d}x + \int_{\partial\Omega} \partial_{x_2}\Psi \nabla\Psi \cdot N \, \mathrm{d}S \\
&= -\frac{1}{2}\int_{\partial\Omega} |\nabla\Psi|^2 N_2 \, \mathrm{d}S + \int_{\partial\Omega} \partial_{x_2}\Psi N \cdot \nabla\Psi \, \mathrm{d}S \\
&= \frac{1}{2}\int_{\partial\Omega} |\nabla\Psi|^2 N_2 \, \mathrm{d}S, \qquad (1.15)
\end{aligned}
$$

where in the last step above we used the fact that Ψ is locally constant on $\partial\Omega$ and thus $\nabla\Psi = (N \cdot \nabla\Psi)N$ holds there.

The first implication of this equality is that if Ψ is nontrivial, then $S \subsetneq \partial\Omega$. Otherwise we would have $\int_S |\nabla\Psi|^2 N_2 \, \mathrm{d}S = 0$ with $N_2 > 0$, which is impossible. This argument does not rely on anything but the regularity of γ, in particular, we do not need the full strength of Assumption (A) or (B). Nonexistence of deep water solitary waves in the presence of algebraically localized vorticity has been more thoroughly investigated in the recent paper [4]; see also [14, 24, 38, 45] for results on the irrotational case.

Now suppose instead that the domain is finite depth, and set $\partial\Omega = S \cup B$, with $B = \{x_2 = -1\}$ denoting the flat rigid lower boundary. Suppose also that $S \cap B = \emptyset$. The properties (i) $|\eta(x_1)| \to 0$ as $|x_1| \to \infty$; (ii) $\nabla\Psi \in L^2(\Omega)$; and (iii) Ψ is locally constant on $\partial\Omega$, together imply that $\Psi|_B = \Psi|_S = 0$ based on a simple Hölder estimate on Ψ along vertical lines. Therefore, from (1.15), we infer that

$$
\frac{1}{2}\int_{\partial\Omega} |\nabla\Psi|^2 N_2 \, \mathrm{d}S = 0. \qquad (1.16)
$$

The reduced (degenerate) equation from the Lyapunov–Schmidt reduction. Equation (1.16) is the key to the proof of our main theorem. As mentioned above, we first carry out a Lyapunov–Schmidt reduction argument to reduce the problem to a highly degenerate one-dimensional "bifurcation" equation with the parameter τ as in Theorem 1.1. One of the usual techniques to handle those somewhat degenerate bifurcation equations is to first use a blow-up argument to search for a nondegenerate direction of the linearized problem, and then employ the implicit function theorem. Even though Proposition 4.5 does imply such linear invertibility of the bifurcation equation, the nondegeneracy we find is far too weak for an (obvious) application of the implicit function theorem to be effective.

Instead, in Section 5 we show that the bifurcation equation is equivalent to (1.16) above. Now, on the free surface, $N_2 = (1 + (\eta')^2)^{-1/2} > 0$, while $N_2 = -1$ on the flat bed B. If $\nabla\Psi$ is highly localized close to the surface, then the integral there should dominate so that the left-hand side of (1.16) would be positive, and conversely for $\nabla\Psi$ concentrated near the bed, it would be negative. This mandates a *balancing* between the contributions on the surface and bed. That observation is at the heart of the analysis in the last part of Section 5. It also reveals the importance of the translation parameter τ.

Nonflat bottom and more. With some modifications, the approach of the current paper should also apply when the bed has nontrivial topography.[1] Indeed, suppose $\partial\Omega = S \cup B$, where B is now a horizontally asymptotically flat rigid bottom, for simplicity taken even in x_1. Thus we expect the vorticity to be localized at $\tau e_2 = (0, \tau) \in \Omega$ for some τ. We can parametrize the unknown Ω by a conformal mapping defined on a fixed domain above B and below $\{x_2 = 1\}$. Based on Proposition 3.1, one may adjust the basic estimates in Sections 3 and 4 accordingly to carry out the Lyapunov–Schmidt reduction and arrive at a highly degenerate one-dimensional reduced bifurcation equation that would still turn out to be equivalent to (1.16). As in the current paper, the distance from τe_2 would again play a crucial role. Let

$$d(\tau) = \operatorname{dist}(\tau e_2, \partial\Omega).$$

Much as in [30], stationary solutions are expected to exist with a localized vorticity concentrated near strict local maximums of $d(\tau)$. However, when multiple localized vorticity locations are considered or when B is not necessarily even in x_1, a sphere packing problem arises. See, for example, [22].

Lastly, we remark that it would be very interesting, though quite difficult, to study gravity water waves with a spike vortex. Surface tension allows us to treat the Bernoulli condition (1.8) essentially as an elliptic problem on the boundary. In fact, the linear part is invertible, which greatly simplifies the analysis; see the proof of Lemma 5.2. Perhaps with much more careful estimates it would be possible to allow for $\sigma = 0$.

1.4. Plan

We begin, in Section 2, by rewriting the stationary water wave problem into an analytically more tractable form. Using a conformal mapping Γ, the fluid domain is pulled back to a fixed slab $\mathcal{S}_\delta$ of width $2/\delta$; this mapping Γ becomes one of the unknowns, taking the place of η. We impose the desired ansatz (1.9) on the stream function, thereby reformulating the problem in terms of the deviation of Ψ from a translated and rescaled solution U to (1.11).

In Section 3, we obtain leading-order approximations of U and a boundary correction operator as well as rather precise exponentially small bounds on the remainders.

Section 4 is devoted to the study of the linearized problem at an approximate solution. Specifically, we prove that there is a small simple eigenvalue $l = l(\delta) = O(e^{(2-|\tau|)/\delta})$

[1] This question was also raised by Shuangjie Peng and Shusen Yan during a talk given by the third author.

related to the direction of $\partial_{x_2} U$. The linearized problem is uniformly nondegenerate in the complementary codimension-1 directions.

All of these tools are used in Section 5 to prove Theorem 1.1. Adopting bundle-type coordinates over $\tau \in (-\frac{1}{3}, \frac{1}{3})$, we carry out a Lyapunov–Schmidt reduction in the non-degenerate codimension-1 directions to reduce the problem to a one-dimensional bifurcation equation. As mentioned above, this bifurcation equation is equivalent to (1.16) and the proof is completed by invoking the intermediate value theorem. Here the idea of balancing the two surface integrals is made rigorous through careful estimates of all the quantities involved. Indeed, while this analysis is quite delicate, the simple identity (1.16) is the key to the argument.

Notation

Throughout the paper $\lesssim$, $\gtrsim$ and $\eqsim$ indicate relations that are valid up to a positive factor which can be chosen uniformly in δ small enough and $\tau \in [-\frac{1}{3}, \frac{1}{3}]$. Complex scalars are sometimes viewed as 2-d real vectors, hence "$\cdot$" between complex quantities denotes their dot product. For a given L^2 function $f \neq 0$ defined on a certain domain, we often use $f^\perp$ to denote the L^2-orthogonal complement of the one-dimensional subspace spanned by f. We also use $\mathrm{D} = \partial_{x_1}$.

2. Reformulation

As the first step toward proving Theorem 1.1, the stationary water wave problem (1.10), (1.7), and (1.8) will be reformulated on a fixed domain, and we will build in the spike ansatz for the stream function mentioned in (1.9). The final product of these efforts is an equivalent transformed problem (2.24) that is posed on an infinite strip.

2.1. Rescaling and parametrization

We start by introducing new coordinates that eliminate the free boundary. As mentioned in the introduction, this can be achieved at little cost in the irrotational regime; see, for example, [7,12]. With vorticity, however, one expect to pay a price in the form of increased complexity of the equations. Given that the highest-order operator in the semilinear equation (1.10) is the Laplacian, it is natural to work with conformal mappings. With that in mind, define the *reference domain* to be $\mathbb{R} \times (-1, 1)$, which we identify with the complex strip

$$\mathbb{C}_{|z_2|<1} = \{z = z_1 + \mathrm{i}z_2 \in \mathbb{C} : |z_2| < 1\}.$$

We will look for fluid domains Ω that are expressed as the image of the reference domain under a near-identity holomorphic mapping. Specifically, let $\Gamma = \Gamma_1 + \mathrm{i}\Gamma_2 \colon \mathbb{C}_{|z_2|<1} \to \mathbb{C}$ be holomorphic and satisfy

$$|\Gamma|_{H^{5/2}} \ll 1, \quad \Gamma(-\overline{z}) = -\overline{\Gamma(z)}, \quad \Gamma_2|_{\{z_2=-1\}} = 0. \tag{2.1}$$

Note this implies that $z_1 \mapsto \Gamma_1$ is odd and $z_1 \mapsto \Gamma_2$ is even. Such a conformal mapping is uniquely determined by $\Gamma_2|_{x_2=1}$. In fact, since Γ_2 is harmonic,

$$(\partial_{x_2}^2 - \xi_1^2)\, \mathscr{F}_{x_1}\, \Gamma_2 = 0,$$

where $\mathscr{F}_{x_1}$ denotes the Fourier transform in the x_1 variable. By construction, Γ_2 vanishes on the bottom of the domain, and hence it depends analytically upon its trace on the upper boundary, $\Gamma_2(\cdot\,, 1)$. Explicitly,

$$(\mathscr{F}_{x_1}\, \Gamma_2)(\xi_1, x_2) = \frac{\sinh(|\xi_1|(x_2+1))}{\sinh(2|\xi_1|)} (\mathscr{F}_{x_1}\, \Gamma_2)(\xi_1, 1) \quad \text{for all } \xi_1 \in \mathbb{R},\ |x_2| < 1,$$

so we have

$$\Gamma_2 = \frac{\sinh(|\partial_{x_1}|(x_2+1))}{\sinh(2|\partial_{x_1}|)} \Gamma_2(\cdot\,, 1) \quad \text{in } \mathbb{C}_{|z_2|<1}, \tag{2.2}$$

and

$$\partial_{x_2} \Gamma_2 = |\partial_{x_1}| \coth(2|\partial_{x_1}|) \Gamma_2 \quad \text{on } \{x_2 = 1\}. \tag{2.3}$$

Observe that $|\partial_{x_1}| \coth(2|\partial_{x_1}|)$ above is the Dirichlet–Neumann operator on the strip $\{|x_2| < 1\}$ with a homogeneous Dirichlet condition imposed on the lower boundary. The real part Γ_1 is a harmonic conjugate of Γ_2 whose one degree of freedom is fixed by the symmetry in x_1.

The corresponding fluid domain is taken to be

$$\Omega := (\mathrm{id} + \Gamma)(\{|x_2| < 1\}) = \{(x_1 + \Gamma_1(x),\ x_2 + \Gamma_2(x)) : |x_2| < 1\}.$$

It follows that the free surface is parameterized by $x_1 \mapsto (x_1, 1) + \Gamma(x_1, 1)$, for x_1 ranging over $\mathbb{R}$. This curve can also be written as the graph

$$x_2 = 1 + \eta(x_1), \qquad \eta = \Gamma_2 \circ (\mathrm{id} + \Gamma_1(\cdot\,, 1))^{-1}. \tag{2.4}$$

The stream function can be pulled back,

$$\Phi = \Psi \circ (\mathrm{id} + \Gamma) \colon \{|x_2| < 1\} \to \mathbb{R}, \tag{2.5}$$

yielding a new unknown defined on the fixed reference domain. Then the water wave problem (1.10), (1.7), and (1.8) become

$$\begin{cases} \delta^2 \Delta \Phi = |1 + \Gamma'|^2 \gamma(\Phi) & \text{in } \{|x_2| < 1\}, \\ \Phi = 0 & \text{on } \{|x_2| = 1\}, \end{cases}$$

together with the transformed Bernoulli condition

$$\frac{1}{2} \frac{(\partial_{x_2} \Phi)^2}{|1 + \Gamma'|^2} - \alpha^2 \frac{\operatorname{Im}(\Gamma''(1 + \overline{\Gamma'}))}{|1 + \Gamma'|^3} + g\Gamma_2 = 0 \quad \text{on } \{x_2 = 1\}, \tag{2.6}$$

where we used the fact that $|\nabla \Phi| = |\partial_{x_2} \Phi|$ along $x_2 = 1$ due to the boundary condition on Φ. Here $\Gamma' = \partial_z \Gamma = \partial_{x_1} \Gamma$ in view of the fact that Γ is holomorphic, and

$$\kappa = -\frac{\operatorname{Im}(\Gamma''(1+\overline{\Gamma'}))}{|1+\Gamma'|^3}$$

is the signed curvature of the interface.

Recalling the scaling in (1.9), we define

$$\varphi = \Phi(\delta\,\cdot), \tag{2.7}$$

which then solves a nondimensionalized version of the problem for Φ set on the slab

$$\mathcal{S}_\delta = \left\{x \in \mathbb{R}^2 : |x_2| < \tfrac{1}{\delta}\right\}. \tag{2.8}$$

It is important to realize that this domain is decreasing in δ, so that in particular $\mathcal{S}_{2\delta} \subset \mathcal{S}_\delta$. Now it is easy to compute that φ satisfies the elliptic equation

$$\begin{cases} \Delta\varphi = |1+\Gamma'(\delta\,\cdot)|^2 \gamma(\varphi) & \text{in } \mathcal{S}_\delta, \\ \varphi = 0 & \text{on } \partial\mathcal{S}_\delta, \end{cases} \tag{2.9}$$

and the Bernoulli condition translates to

$$\frac{1}{2\delta^2}\frac{(\partial_{x_2}\varphi(\cdot\,,\frac{1}{\delta}))^2}{|1+\Gamma'(\delta\,\cdot\,,1)|^2} - \alpha^2 \frac{\operatorname{Im}(\Gamma''(\delta\,\cdot\,,1)(1+\overline{\Gamma'(\delta\,\cdot\,,1)}))}{|1+\Gamma'(\delta\,\cdot\,,1)|^3} + g\Gamma_2(\delta\,\cdot\,,1) = 0. \tag{2.10}$$

2.2. *Boundary correction*

Our overarching strategy is to model Ψ, and by extension φ, on a rescaled U of (1.11). However, while U is exponentially localized, it does not satisfy the homogeneous boundary conditions in (2.9). We therefore perform a *boundary correction*, modeled on Assumption (A), subtracting a function from U that shares its trace but is exceedingly smaller in the interior.

For any real-valued function f in a reasonable Sobolev space (see below) defined on $\partial\mathcal{S}_\delta$, we introduce the extension operator

$$\mathrm{bc}\colon f \mapsto f_{\mathrm{bc}},$$

defined uniquely by

$$(\mathcal{F}_{x_1} f_{\mathrm{bc}})(\xi_1, x_2) = \sum_{\pm} \pm \frac{\sinh(\langle\xi_1\rangle(x_2 \pm \frac{1}{\delta}))(\mathcal{F}_{x_1} f_\pm)(\xi_1)}{\sinh(\frac{2\langle\xi_1\rangle}{\delta})}, \tag{2.11}$$

where $f_\pm$ is the restriction of f on $\{x_2 = \pm\frac{1}{\delta}\}$, and we are using the Japanese bracket notation $\langle\xi_1\rangle = (1+|\xi_1|^2)^{1/2}$. Provided that $f_\pm \in H^s(\mathbb{R})$, $s > 0$, the function f_{bc} is an

$H^{s+1/2}(\mathcal{S}_\delta)$-solution[2] of

$$\begin{cases} (1-\Delta) f_{\mathrm{bc}} = 0 & \text{in } \mathcal{S}_\delta, \\ f_{\mathrm{bc}} = f & \text{on } \partial\mathcal{S}_\delta. \end{cases} \tag{2.12}$$

Observe that, due to the localization of U and the assumption $\gamma'(0) = 1$, the above problem closely resembles the linearized operator $\gamma'(U) - \Delta$ away from the origin. It is worth noting that, for any $s > 0$,

$$|f_{\mathrm{bc}}|_{H^{s+1/2}(\mathcal{S}_\delta)} \lesssim |f|_{H^s(\partial\mathcal{S}_\delta)}.$$

2.3. *The perturbed problem*

Let U be given by Assumption (A). As discussed in Section 1, it will be important to consider vertical translates of this function. For each $\tau \in [-1/3, 1/3]$, let

$$U(\cdot\,,\tau) = U\big(\cdot - \tfrac{\tau}{\delta} e_2\big). \tag{2.13}$$

With a slight abuse of notation we shall still write U to denote the function $U(\cdot\,,\tau)$, except when the precise value of τ becomes important. At other times, it will be more convenient to use the notation $U(\tau)(\cdot)$ rather than $U(\cdot\,,\tau)$. The value $\frac{1}{3}$ is unimportant; we will find waves for $|\tau|$ exceedingly much smaller. What is important is that the center of vorticity remains closer to the origin than to the boundary of the reference domain, but $\frac{1}{3}$ has no special significance.

We proceed with the *ansatz*

$$\varphi = u + U - U_{\mathrm{bc}}, \tag{2.14}$$

where bc denotes the boundary correction from (2.12). Thus u measures the deviation of φ from the rescaled, translated, and boundary corrected U. Inserting this into (2.9), we see that it solves the following elliptic PDE set on $\mathcal{S}_\delta$:

$$\begin{aligned} \Delta u &= |1 + \Gamma'(\delta\,\cdot)|^2 \gamma(u + U - U_{\mathrm{bc}}) - \gamma(U) + U_{\mathrm{bc}} \\ &= \gamma'(U)u + |1 + \Gamma'(\delta\,\cdot)|^2 \gamma(u + U - U_{\mathrm{bc}}) \\ &\quad - \gamma(U) - \gamma'(U)u + U_{\mathrm{bc}}. \end{aligned} \tag{2.15}$$

Here, we have made use of the facts that $\Delta U = \gamma(U)$ and $\Delta U_{\mathrm{bc}} = U_{\mathrm{bc}}$. Similarly, the kinematic condition in (2.9) takes the form

$$u = 0 \quad \text{on } \partial\mathcal{S}_\delta, \tag{2.16}$$

since $U = U_{\mathrm{bc}}$ there.

[2] In general the solution of (2.12) need not be unique, as $\mathcal{S}_\delta$ is an infinite slab.

Consider next the Bernoulli condition (2.10). Direct substitution yields

$$0 = \frac{1}{2\delta^2}\frac{(\partial_{x_2}(u+U-U_{\mathrm{bc}})(\cdot\,,\frac{1}{\delta}))^2}{|1+\Gamma'(\delta\,\cdot\,,1)|^2} - \alpha^2\frac{\operatorname{Im}(\Gamma''(\delta\,\cdot\,,1)(1+\overline{\Gamma'(\delta\,\cdot\,,1)}))}{|1+\Gamma'(\delta\,\cdot\,,1)|^3} + g\Gamma_2(\delta\,\cdot\,,1). \tag{2.17}$$

From the Cauchy–Riemann equations and

$$\Gamma' = \partial_{x_1}\Gamma = \partial_{x_2}\Gamma_2 + \mathrm{i}\partial_{x_1}\Gamma_2,$$

any derivatives involving Γ can be expressed in terms of derivatives of Γ_2. Making this replacement in (2.17) yields

$$0 = \frac{1}{2\delta^2}\frac{(\partial_{x_2}(u+U-U_{\mathrm{bc}})(\frac{\cdot}{\delta},\frac{1}{\delta}))^2}{(1+\partial_{x_2}\Gamma_2)^2+(\partial_{x_1}\Gamma_2)^2} - \alpha^2\frac{(1+\partial_{x_2}\Gamma_2)\partial_{x_1}^2\Gamma_2 - \partial_{x_1}\Gamma_2\partial_{x_1x_2}\Gamma_2}{((1+\partial_{x_2}\Gamma_2)^2+(\partial_{x_1}\Gamma_2)^2)^{3/2}} + g\Gamma_2. \tag{2.18}$$

Here, all terms involving Γ_2 are evaluated at $(x_1, 1)$. The idea is that, to the leading order in terms of Γ_2, the right-hand side of (2.18) is determined by the operator $g-\alpha^2\partial_{x_1}^2$ acting on Γ_2, which is invertible $H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ for all $s \in \mathbb{R}$. To make this rigorous, let

$$\Gamma_{\mathrm{s}} = \Gamma_2(\cdot\,,1) \tag{2.19}$$

be the trace of Γ_2 on the top of the reference domain $\mathcal{S}_\delta$. From (2.18) and (2.3) we have

$$\frac{1}{2\delta^2}\frac{(\partial_{x_2}(u+U-U_{\mathrm{bc}})(\frac{\cdot}{\delta},\frac{1}{\delta}))^2}{(1+m(\mathrm{D})\Gamma_{\mathrm{s}})^2+\Gamma_{\mathrm{s}}'^2} - \alpha^2\frac{(1+m(\mathrm{D})\Gamma_{\mathrm{s}})\Gamma_{\mathrm{s}}'' - \Gamma_{\mathrm{s}}'m(\mathrm{D})\Gamma_{\mathrm{s}}'}{((1+m(\mathrm{D})\Gamma_{\mathrm{s}})^2+\Gamma_{\mathrm{s}}'^2)^{3/2}} + g\Gamma_{\mathrm{s}} = 0, \tag{2.20}$$

where $\mathrm{D} = \partial_{x_1}$, $m(\mathrm{D}) = |\mathrm{D}|\coth(2|\mathrm{D}|)$, and $\Gamma_{\mathrm{s}}' = \partial_{x_1}\Gamma_{\mathrm{s}}$. Let $A(\Gamma_{\mathrm{s}})$ be a linear operator depending on Γ_{s} acting on $v\colon \mathbb{R} \to \mathbb{R}$ as

$$A(\Gamma_{\mathrm{s}}) := \Big(g - \alpha^2((1+m(\mathrm{D})\Gamma_{\mathrm{s}})^2+\Gamma_{\mathrm{s}}'^2)^{-3/2}\big((1+m(\mathrm{D})\Gamma_{\mathrm{s}})\mathrm{D}^2 - \Gamma_{\mathrm{s}}'m(\mathrm{D})\mathrm{D}\big)\Big)(g-\alpha^2\mathrm{D}^2)^{-1}.$$

Notice also that $|\mathrm{D}|$ preserves the even-odd parity. For a given smooth Γ_{s}, A is a zero-order operator on any Sobolev space $H^s_{\mathrm{e}}(\mathbb{R})$, $s \in \mathbb{R}$; here and elsewhere the subscript "e" indicates that the functions are even in x_1. More precisely, if $\Gamma_{\mathrm{s}} \in H^s_{\mathrm{e}}(\mathbb{R})$ for $s > 3/2$, then the map[3]

$$H^s_{\mathrm{e}}(\mathbb{R}) \ni \Gamma_{\mathrm{s}} \mapsto A(\Gamma_{\mathrm{s}}) \in \mathcal{L}(H^{s'}_{\mathrm{e}}(\mathbb{R})) \text{ is analytic, } s' \in [1-s, s-1]. \tag{2.21}$$

[3] Note here that the lower right s used in Γ_{s} stands for "surface", while the italic s is a (general) regularity index.

Recall here that $H^{-s}(\mathbb{R})$ is the continuous dual of $H^s(\mathbb{R})$, whence the lower bound $1-s$ is needed to ensure that products can be made sense of when applying $A(\Gamma_s)$ to $H^{s'}(\mathbb{R})$. Now $A(0) = \mathrm{id}$ and we have the bound

$$|A(\Gamma_s) - \mathrm{id}|_{\mathcal{L}(H^{s'})} \lesssim |\Gamma_s|_{H^s}.$$

Thus $A(\Gamma_s) \in \mathcal{L}(H^{s'})$ is invertible for $|\Gamma_s|_{H^s} \ll 1$. We can now isolate the leading-order terms in (2.20) by applying $A(\Gamma_s)^{-1}$ to it:

$$0 = (g - \alpha^2 \mathrm{D}^2)\Gamma_s + \frac{1}{2\delta^2} A(\Gamma_s)^{-1}\left[\frac{(\partial_{x_2}(u + U - U_{\mathrm{bc}})(\frac{\cdot}{\delta}, \frac{1}{\delta}))^2}{(1 + m(\mathrm{D})\Gamma_s)^2 + \Gamma_s'^2}\right], \tag{2.22}$$

which is valid as long as $|\Gamma_s|_{H^s} \ll 1$.

Now we are roughly in a position to make a rigorous statement of our problem. For any $\delta > 0$, we define

$$X_\delta^k = H_{\mathrm{e}}^k(\mathcal{S}_\delta) \cap H_0^1(\mathcal{S}_\delta), \quad k \geq 1, \quad X_\delta^0 = L_{\mathrm{e}}^2(\mathcal{S}_\delta). \tag{2.23}$$

Summarizing the analysis of this section, we see that if u, Γ_s, and τ satisfy

$$(-\Delta + \gamma'(U))u + F(\tau, u, \Gamma_s) = 0 \quad \text{in } \mathcal{S}_\delta, \tag{2.24a}$$

$$(g - \alpha^2 \mathrm{D}^2)\Gamma_s + G(\tau, u, \Gamma_s) = 0 \quad \text{on } \mathbb{R}, \tag{2.24b}$$

where $F\colon [-\frac{1}{3}, \frac{1}{3}] \times X_\delta^k \times H_{\mathrm{e}}^k(\mathbb{R}) \to X_\delta^{k-2}$ and $G\colon [-\frac{1}{3}, \frac{1}{3}] \times X_\delta^k \times H_{\mathrm{e}}^k(\mathbb{R}) \to H_{\mathrm{e}}^{k-2}(\mathbb{R})$ are the mappings

$$F(\tau, \cdot)\colon (u, \Gamma_s) \mapsto |1 + \Gamma'(\delta\,\cdot)|^2 \gamma(u + U - U_{\mathrm{bc}}) - \gamma(U) - \gamma'(U)u + U_{\mathrm{bc}}, \tag{2.25}$$

and

$$G(\tau, \cdot)\colon (u, \Gamma_s) \mapsto \frac{1}{2\delta^2} A(\Gamma_s)^{-1}\left[\frac{(\partial_{x_2}(u + U - U_{\mathrm{bc}})(\frac{\cdot}{\delta}, \frac{1}{\delta}))^2}{(1 + |\mathrm{D}|\coth(2|\mathrm{D}|)\Gamma_s)^2 + \Gamma_s'^2}\right], \tag{2.26}$$

then (Ψ, η), reconstructed via (2.5), (2.7), (2.14), (2.1), (2.19), and the Cauchy–Riemann equations, will solve the stationary water wave problem (1.10), (1.7), and (1.8). Recall here that U is a shorthand for $U(\cdot\,, \tau) = U(\cdot - \frac{\tau}{\delta} e_2)$. For the class of γ satisfying Assumptions (A) and (B), the mappings F and G are well defined and continuously differentiable given some basic estimates on U and U_{bc} that are derived in the next section. For that reason, we postpone making a precise statement, or offering a proof, until Lemma 3.8.

3. Estimates of U and its boundary corrections

This section is devoted to the estimates of $U(\cdot\,, \tau)$, its derivatives, and boundary corrections of the same functions, assuming that Assumptions (A) and (B) from Section 1 hold. Finally, we give some estimates of the nonlinearities F and G, defined in (2.25) and (2.26), in the elliptic system (2.24), which is equivalent to the original problem (1.10), (1.7), and (1.8) of finding stationary water waves.

Estimates for U, $(\partial_{x_2} U)_{\mathrm{bc}}$, and $\partial_{x_2}(U_{\mathrm{bc}})$

We start with some basic estimates on U. Recall from Assumption (A) that $\gamma \in C^{k_0}$, for a fixed integer $k_0 \geq 2$.

Proposition 3.1. *Under Assumptions (A) and (B), there exists $\lambda \neq 0$ such that*

$$\lim_{r\to\infty} (-1)^k r^{1/2} \mathrm{e}^r \partial_r^k U(r) = \lambda \quad \textit{for all } \ 0 \leq k \leq k_0 + 2. \tag{3.1}$$

Remark 3.2. As

$$\nabla = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix},$$

and

$$\partial_{x_2}^2 = \sin^2\theta\, \partial_r^2 + \frac{2\cos\theta\sin\theta}{r}\partial_{r\theta} + \frac{\cos^2\theta}{r^2}\partial_\theta^2 + \frac{\cos^2\theta}{r}\partial_r - \frac{\cos\theta\sin\theta}{r^2}\partial_\theta,$$

when applied to radial functions, we have

$$\partial_{x_2} U = \sin\theta\, U_r, \quad \partial_{x_2}^2 U = \sin^2\theta\, U_{rr} + \frac{\cos^2\theta}{r} U_r. \tag{3.2}$$

Proposition 3.1 readily induces signs on the Cartesian derivatives of these functions. In particular, $\operatorname{sgn} \partial_{x_2} U = -\operatorname{sgn}(\lambda x_2)$ globally with

$$\partial_{x_2} U \eqsim -\lambda x_2 r^{-3/2} e^{-r}, \quad \partial_{x_2}^2 U \eqsim \lambda x_2^2 r^{-5/2} e^{-r}$$

when $r \gg 1$. Note also that

$$|1 - \gamma'(U)| \lesssim r^{-1/2} e^{-r}, \quad |\gamma(U) - U| \lesssim r^{-1} e^{-2r}, \quad r \gg 1.$$

Remark 3.3. For $|\tau| < \frac{1}{3}$, the function $U(\,\cdot\,, \tau)$ from (2.13) is just a translation of the center and global maximum of the radial function U from the origin to $(0, \frac{\tau}{\delta})$. It follows that Proposition 3.1 applies to $U(\,\cdot\,, \tau)$ with r changed accordingly.

Remark 3.4. The solution U to (1.11) is often obtained through a variational approach carried out in H^1 space. In fact, for any $\gamma \in C^1(\mathbb{R}^2)$ satisfying $\gamma(0) = 0$, any radial solution $U \in C^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ automatically satisfies the decay assumption $\lim_{r\to\infty} U(r) = \lim_{r\to\infty} U'(r) = 0$ in Assumption (A). This is due to the inequality, for $r_2 > r_1 > 0$,

$$\begin{aligned} |U(r_2)^2 - U(r_1)^2| &= 2\left|\int_{r_1}^{r_2} U'(r)U(r)\,\mathrm{d}r\right| \leq \frac{2}{r_1}\left|\int_{r_1}^{r_2} U'(r)U(r) r\,\mathrm{d}r\right| \\ &\leq \frac{2}{r_1} |U|_{L^2(\mathbb{R}^2)} |U'|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

which implies $\lim_{r\to\infty} U(r) = 0$, and hence $U \in L^\infty$. The boundedness of U yields $\Delta U = \gamma(U) \in L^2(\mathbb{R}^2)$, and thus $U \in H^2(\mathbb{R}^2)$. The fact that $\lim_{r\to\infty} U'(r) = 0$ follows from the same argument. Obviously one may also replace $U \in C^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ by $U \in C^2(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Proof of Proposition 3.1. The decay rate (3.1) is stated by Li and Nirenberg [30] for the case $\gamma(t) = t - t^p$, with a reference to an earlier paper of Berestycki and P.-L. Lions [3]. However, while that work could be extended to our setting, as written it does not contain the same sharp result and it is restricted to three or higher dimensions. Here we provide a sketch of a proof that does cover the case of interest; it is based on invariant manifold methods rather than variational techniques. For a reference, see for example [5].

In polar coordinates the semilinear problem for U is

$$\partial_r^2 U = -\frac{1}{r}\partial_r U + \gamma(U) \tag{3.3}$$

and thus it suffices to obtain the estimate for $k = 0, 1$, due to the fact that $\gamma'(0) = 1$. Letting

$$w_1 = \frac{1}{2}(U + \partial_r U), \quad w_2 = \frac{1}{2}(U - \partial_r U), \quad s = \frac{1}{r}, \quad \gamma_1(U) = \gamma(U) - U = O(U^2),$$

we rewrite (3.3) as

$$\begin{cases} \partial_r w_1 = \left(1 - \frac{s}{2}\right)w_1 + \frac{s}{2}w_2 + \frac{1}{2}\gamma_1(w_1 + w_2), \\ \partial_r w_2 = \frac{s}{2}w_1 - \left(1 + \frac{s}{2}\right)w_2 - \frac{1}{2}\gamma_1(w_1 + w_2), \\ \partial_r s = -s^2. \end{cases} \tag{3.4}$$

Clearly $(0, 0, 0)$ is an unstable equilibrium of the ODE system with w_1, w_2, and s being in the unstable, stable, and the center directions, respectively. Therefore there exists a C^{k_0} center-stable manifold W^{cs} in a neighborhood of $(0, 0, 0)$ given by a graph

$$w_1 = \phi(w_2, s) \quad \text{with } \phi \in C^{k_0} \text{ and } \phi(0, 0) = 0,\ \nabla\phi(0, 0) = 0.$$

Even though the center-stable manifold W^{cs} is usually not unique, the subset $W^{\mathrm{cs}} \cap \{s \geq 0\}$ is indeed unique because of its positive invariance under the ODE flow. Due to Assumption (A), both the orbit corresponding to U and the trivial state $(0, 0, s = \frac{1}{r})$ converge to $(0, 0, 0)$ as $r \to +\infty$. Hence they both belong to W^{cs}. This implies

$$\phi(0, s) = 0, \ \text{ and thus } w_1 = \phi(w_2, s) = O(|w_2|(|s| + |w_2|)), \quad |w_2|,\ |s| \ll 1.$$

Therefore the only orbit on W^{cs} intersecting $\{w_2 = 0\}$ is the one corresponding to the trivial solution. On W^{cs}, the w_2 equation in (3.4) and the above properties of ϕ yield

$$\left|\partial_r w_2 + \left(1 + \frac{s}{2} - \frac{s}{2}\phi_{w_2}(0, s)\right)w_2\right| = O(w_2^2), \quad |w_2|, |s| \ll 1.$$

Using this, we first calculate that

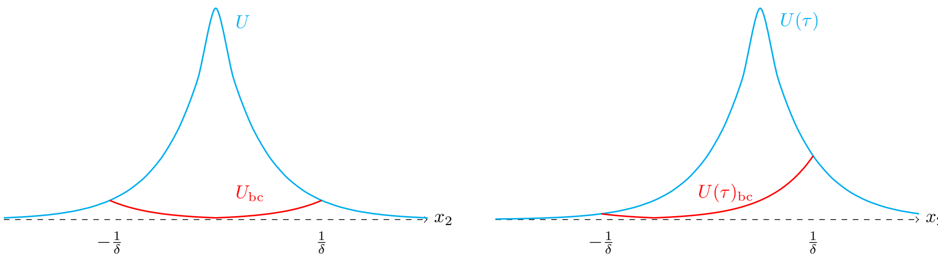
$$\frac{d}{dr}(e^r w_2^2) = \left(1 - 2\left(1 + \frac{s}{2} - \frac{s}{2}\phi_{w_2}(0, s)\right) + O(|w_2|)\right)e^r w_2^2 \leq 0, \quad |w_2|,\ |s| \ll 1.$$

Therefore $e^{r/2}|w_2|$ is decreasing in r for $1 \ll r$ and thus $\int_{r_0}^{\infty} w_2 \, dr$ converges absolutely. Moreover, the above estimate of $\partial_r w_2$ on W^{cs} further implies

$$w_2(r) = \left(\frac{r_0}{r}\right)^{1/2} e^{r_0 - r} e^{\tilde{w}(r)} w_2(r_0), \quad \tilde{w}(r) = \int_{r_0}^{r} \left[\frac{1}{2r'} \phi_{w_2}\left(0, \frac{1}{r'}\right) + O(|w_2(r')|)\right] dr'.$$

Since $\phi_{w_2}(0, s) = O(s)$, the above estimate implies that $\lim_{r \to +\infty} r^{1/2} e^r w_2(r)$ exists and belongs to $(0, \infty)$, which along with the fact that $w_1 = O(|w_2|(|\frac{1}{r}| + |w_2|))$ yields (3.1) for $k = 0, 1$. ■

The following corollary will be used to analyze the boundary correction operator. For this and the coming results, especially Corollary 3.7, it can be good to consult Figure 2. Note, in particular, that the estimate below essentially concerns the behavior of U *outside* of the slab $\mathcal{S}_\delta$ (on the slab reflected over its own boundaries, modulo the translation τ).



(a) $\tau = 0$ **(b)** $\tau \in (0, \frac{1}{3})$

Fig. 2. Graphs of U and $U_{\rm bc}$ along the line $x_1 = 0$. On the left, $U(x)$ is centered at the origin; on the right, it is shifted closer to the free surface. See also Corollary 3.7.

Corollary 3.5. *For any* $\tau \in [-\frac{1}{3}, \frac{1}{3}]$, $0 \le k \le k_0 + 1$, *and* $0 \le k' \le k_0 + 2$, $U(\tau)$ *and* $\partial_{x_2} U(\tau)$ *satisfy*

$$\left|U\left(\cdot, \pm\tfrac{2}{\delta} - \cdot, \tau\right)\right|_{H^{k'}(\mathcal{S}_\delta)}, \left|\partial_{x_2} U\left(\cdot, \pm\tfrac{2}{\delta} - \cdot, \tau\right)\right|_{H^k(\mathcal{S}_\delta)} \eqsim \delta^{1/4} e^{-\frac{1 \mp \tau}{\delta}}.$$

Proof. We shall focus on $\partial_{x_2} U(\cdot, \frac{2}{\delta} - \cdot, \tau)$ as the others can be handled similarly. Consider the following subset S of $\mathcal{S}_\delta$:

$$S = \left\{x \in \mathcal{S}_\delta : x_1^2 + \left(\tfrac{2-\tau}{\delta} - x_2\right)^2 < \left(\tfrac{3-\tau}{\delta}\right)^2\right\}. \tag{3.5}$$

Let (ρ, β) be the polar coordinates of $(x_1, \frac{2-\tau}{\delta} - x_2)$ so that

$$S = \left\{(\rho, \beta) : \rho \in \left(\tfrac{1-\tau}{\delta}, \tfrac{3-\tau}{\delta}\right), \, \beta \in (\beta_0(\rho), \pi - \beta_0(\rho))\right\}, \quad \beta_0(\rho) = \sin^{-1}\left(\tfrac{1-\tau}{\delta\rho}\right).$$

Since $|\tau| \le \frac{1}{3}$, we have $\sin\beta \eqsim 1$ in S, and

$$\frac{\pi}{2} - \beta_0(\rho) = \sin^{-1}\left(1 - \left(\tfrac{1-\tau}{\delta\rho}\right)^2\right)^{1/2} \eqsim (\delta\rho - 1 + \tau)^{1/2} \quad \text{for all } \rho \in \left(\tfrac{1-\tau}{\delta}, \tfrac{3-\tau}{\delta}\right). \tag{3.6}$$

Along with Proposition 3.1, this implies

$$I + II := \left(\int_S + \int_{\mathcal{S}_\delta \setminus S}\right)\left(x_1^2 + \left(\tfrac{2-\tau}{\delta} - x_2\right)^2\right)^{-1/2} e^{-2(x_1^2+(\frac{2-\tau}{\delta}-x_2)^2)^{1/2}}\, \mathrm{d}x$$
$$\gtrsim \left|\partial_{x_2} U\left(\cdot, \tfrac{2}{\delta} - \cdot, \tau\right)\right|^2_{H^{k_0+1}(\mathcal{S}_\delta)} \gtrsim \left|\partial_{x_2} U\left(\cdot, \tfrac{2}{\delta} - \cdot, \tau\right)\right|^2_{L^2(\mathcal{S}_\delta)} \gtrsim I.$$

Again, it follows from Proposition 3.1 and (3.6) that

$$I \eqsim \int_{\frac{1-\tau}{\delta}}^{\frac{3-\tau}{\delta}} \int_{\beta_0(\rho)}^{\pi-\beta_0(\rho)} e^{-2\rho}\, \mathrm{d}\beta\, \mathrm{d}\rho \eqsim \int_0^{2/\delta} (\delta\rho')^{1/2} e^{-2(1-\tau)/\delta - 2\rho'}\, \mathrm{d}\rho' \eqsim \delta^{1/2} e^{-2(1-\tau)/\delta},$$

while

$$II \lesssim \int_{|x| \geq \frac{3-\tau}{\delta}} |x|^{-1} e^{-2|x|}\, \mathrm{d}x \lesssim \int_{\frac{3-\tau}{\delta}}^{\infty} e^{-2\rho}\, \mathrm{d}\rho \eqsim e^{-2(3-\tau)/\delta}.$$

This completes the proof of the corollary. ∎

In order to estimate the boundary correction operator defined in (2.12), we will need the following auxiliary lemma.

Lemma 3.6. *Suppose $k \geq 2$ is an integer, $|\tau| \leq \frac{1}{3}$, and $h \in C^k(\mathbb{R}^2, \mathbb{R})$ satisfies*

$$|\partial^j h(x)| \lesssim \left(1 + \left|x - \tfrac{\tau}{\delta} e_2\right|\right)^{-1/2} e^{-|x - \frac{\tau}{\delta} e_2|} \quad \textit{for } 0 \leq j \leq k,$$

and

$$|\partial^j (1-\Delta) h(x)| \lesssim \left(1 + \left|x - \tfrac{\tau}{\delta} e_2\right|\right)^{-1} e^{-2|x - \frac{\tau}{\delta} e_2|} \quad \textit{for } 0 \leq j \leq k-2.$$

Then

$$v(x_1, x_2) := (h|_{\partial \mathcal{S}_\delta})_{\mathrm{bc}}(x_1, x_2) - h\left(x_1, \tfrac{2}{\delta} - x_2\right) - h\left(x_1, -\tfrac{2}{\delta} - x_2\right)$$

satisfies

$$|v|_{H^k(\mathcal{S}_\delta)} \lesssim \delta^{3/4} e^{-2(1-|\tau|)/\delta}.$$

Intuitively, this says that the boundary correction of h is, to leading order, found by subtracting the reflections of h over the top and bottom boundaries of the slab.

Proof. From the definition of bc in (2.12), we see that v satisfies

$$\begin{cases} (1-\Delta) v(x_1, x_2) = -(1-\Delta) h\left(x_1, \tfrac{2}{\delta} - x_2\right) - (1-\Delta) h\left(x_1, -\tfrac{2}{\delta} - x_2\right) & \text{in } \mathcal{S}_\delta, \\ v|_{x_2 = \pm 1/\delta} = -h\left(x_1, \mp \tfrac{3}{\delta}\right). \end{cases}$$

One can immediately deduce the energy estimate

$$|v|_{H^k(\mathcal{S}_\delta)} \lesssim \sum_{\pm} \left|(1-\Delta) h\left(\cdot, \pm \tfrac{2}{\delta} - \cdot\right)\right|_{H^{k-2}(\mathcal{S}_\delta)} + |h|_{H^{k-1/2}(\{|x_2| = 3/\delta\})}.$$

An upper bound of the first term on the right-hand side above can be obtained much as in the proof of Corollary 3.5, and so we only provide a sketch and focus on the "+" case. Let

S be given as in (3.5), and split the slab $\mathcal{S}_\delta = S \cup (\mathcal{S}_\delta \setminus S)$. From the properties assumed on h, we see that

$$\begin{aligned}
&\big|(1-\Delta)h\big(\cdot, \tfrac{2}{\delta} - \cdot\big)\big|^2_{H^{k-2}(\mathcal{S}_\delta)} \\
&\quad \lesssim \Big(\int_S + \int_{\mathcal{S}_\delta \setminus S}\Big)\big(x_1^2 + \big(\tfrac{2-\tau}{\delta} - x_2\big)^2\big)^{-1} e^{-4(x_1^2 + (\frac{2-\tau}{\delta} - x_2)^2)^{1/2}} \, \mathrm{d}x \\
&\quad \lesssim \int_{\frac{1-\tau}{\delta}}^{\frac{3-\tau}{\delta}} \int_{\beta_0(\rho)}^{\pi - \beta_0(\rho)} \rho^{-1} e^{-4\rho} \, \mathrm{d}\beta \, \mathrm{d}\rho + \int_{|x| \geq \frac{3-\tau}{\delta}} |x|^{-2} e^{-4|x|} \, \mathrm{d}x \\
&\quad \lesssim \delta^{1/2} e^{-\frac{4(1-\tau)}{\delta}} \int_0^{2/\delta} (\rho')^{1/2} \big(\tfrac{1-\tau}{\delta} + \rho'\big)^{-1} e^{-4\rho'} \, \mathrm{d}\rho' + \int_{\frac{3-\tau}{\delta}}^{\infty} \rho^{-1} e^{-4\rho} \, \mathrm{d}\rho \\
&\quad \lesssim \delta^{3/2} e^{-4(1-\tau)/\delta}.
\end{aligned}$$

The $H^{k-1/2}(\partial \mathcal{S}_{3/\delta})$ norm can be estimated by interpolating it between H^k and H^{k-1} and then appealing to the assumptions on h:

$$\begin{aligned}
|h|^2_{H^{k-1/2}(\{|x_2| = 3/\delta\})} &\lesssim \Big(\int_0^{\delta^{-1/2}} + \int_{\delta^{-\frac{1}{2}}}^{\infty}\Big)\big(x_1^2 + \tfrac{(3-|\tau|)^2}{\delta^2}\big)^{-1/2} e^{-2(x_1^2 + \frac{(3-|\tau|)^2}{\delta^2})^{1/2}} \, \mathrm{d}x_1 \\
&\lesssim \delta^{1/2} e^{-2(3-|\tau|)/\delta} + \int_{(\frac{\delta + (3-|\tau|)^2}{\delta^2})^{1/2}}^{\infty} x_1(s)^{-1} e^{-2s} \, \mathrm{d}s \lesssim \delta^{1/2} e^{-2(3-|\tau|)/\delta}, \qquad (3.7)
\end{aligned}$$

where the substitution $x_1(s) = (s^2 - \frac{(3-|\tau|)^2}{\delta^2})^{1/2}$ was used to evaluate the integral on $[\delta^{-1/2}, \infty)$. Combining the above inequalities concludes the proof of the lemma. ∎

Lemma 3.6 is mainly applied to U_{bc} and $(\partial_{x_2} U)_{\mathrm{bc}}$ for $|\tau| \leq \frac{1}{3}$. In fact, (1.11) yields

$$(1-\Delta)\partial_{x_2} U = (1 - \gamma'(U))\partial_{x_2} U,$$

and so the assumption that $\gamma'(0) = 1$ together with Proposition 3.1 ensures that U and $\partial_{x_2} U$ satisfy the hypotheses of Lemma 3.6. Therefore, in addition to Corollary 3.5 we obtain the following estimates, which will be essential to us later.

Corollary 3.7. *For any $\tau \in [-\frac{1}{3}, \frac{1}{3}]$, $U(\tau)_{\mathrm{bc}}$ and $(\partial_{x_2} U)(\tau)_{\mathrm{bc}}$ satisfy*

$$\begin{aligned}
&\big|U(\tau)_{\mathrm{bc}} - U\big(\cdot, \tfrac{2}{\delta} - \cdot, \tau\big) - U\big(\cdot, -\tfrac{2}{\delta} - \cdot, \tau\big)\big|_{H^{k_0+2}(\mathcal{S}_\delta)} \lesssim \delta^{3/4} e^{-2(1-|\tau|)/\delta}, \\
&\big|(\partial_{x_2} U)(\tau)_{\mathrm{bc}} - (\partial_{x_2} U)\big(\cdot, \tfrac{2}{\delta} - \cdot, \tau\big) - (\partial_{x_2} U)\big(\cdot, -\tfrac{2}{\delta} - \cdot, \tau\big)\big|_{H^{k_0+1}(\mathcal{S}_\delta)} \lesssim \delta^{3/4} e^{-2(1-|\tau|)/\delta}, \\
&|U(\tau)_{\mathrm{bc}}|_{H^{k_0+2}(\mathcal{S}_\delta)}, \ |(\partial_{x_2} U)(\tau)_{\mathrm{bc}}|_{H^{k_0+1}(\mathcal{S}_\delta)} \eqsim \delta^{1/4} e^{-1-|\tau|/\delta}.
\end{aligned}$$

Proof. The first two inequalities follow directly from Proposition 3.1 and Lemma 3.6. To obtained the estimate on $U(\tau)_{\mathrm{bc}}$ based on the first inequality and Corollary 3.5, we only need to show the almost orthogonality

$$\big|\big(U\big(\cdot, \tfrac{2}{\delta} - \cdot, \tau\big), U\big(\cdot, -\tfrac{2}{\delta} - \cdot, \tau\big)\big)_{H^{k_0+2}(\mathcal{S}_\delta)}\big| \ll \delta^{1/2} e^{-2(1-|\tau|)/\delta}. \qquad (3.8)$$

In fact, consider

$$b_1, b_2 \in \mathbb{R}, \ |b_1| - 1 \in \left[\tfrac{1}{3}, 3\right], \quad x \in \mathcal{S}_\delta, \ |x_1| \le \tfrac{4}{\delta};$$

we have $|x_1|\,|\frac{b_1}{\delta} - x_2|^{-1} \le 12$. Observing that $\frac{d^2}{dt^2}\langle t\rangle = \langle t\rangle^{-3} > 0$, it follows that $t \mapsto \langle t\rangle$ is convex and

$$\left(1 + \left|\tfrac{b_1}{\delta} - x_2\right|^{-2} x_1^2\right)^{1/2} \ge 1 + 5\sigma\left|\tfrac{b_1}{\delta} - x_2\right|^{-2} x_1^2, \quad \text{where} \quad \sigma = \tfrac{1}{10}\left(\tfrac{1}{145}\right)^{3/2}.$$

Therefore, for small $\delta > 0$,

$$\begin{aligned} \left|\tfrac{b_1}{\delta} e_2 - x\right| + \left|x - \tfrac{b_2}{\delta} e_2\right| &\ge \left|x_2 - \tfrac{b_2}{\delta}\right| + \left|\tfrac{b_1}{\delta} - x_2\right|\left(1 + \left|\tfrac{b_1}{\delta} - x_2\right|^{-2} x_1^2\right)^{1/2} \\ &\ge \left|x_2 - \tfrac{b_2}{\delta}\right| + \left|\tfrac{b_1}{\delta} - x_2\right| + 5\sigma\left|\tfrac{b_1}{\delta} - x_2\right|^{-1} x_1^2 \ge \tfrac{|b_1 - b_2|}{\delta} + \sigma\delta x_1^2. \end{aligned} \tag{3.9}$$

It is also clear that

$$1 + \left|\tfrac{b_1}{\delta} e_2 - x\right| \eqsim \delta^{-1}, \quad 1 + \left|x - \tfrac{b_2}{\delta} e_2\right| \ge 1 + \left|x_2 - \tfrac{b_2}{\delta}\right|. \tag{3.10}$$

Applying these inequalities to $b_1 = 2 - \tau$ and $b_2 = -(2 + \tau)$ we obtain

$$\begin{aligned} &\int_{\mathcal{S}_\delta} \left(1 + \left|x - \tfrac{2-\tau}{\delta} e_2\right|\right)^{-1/2} \left(1 + \left|x + \tfrac{2+\tau}{\delta} e_2\right|\right)^{-1/2} e^{-|x + \frac{2+\tau}{\delta} e_2| - |x - \frac{2-\tau}{\delta} e_2|}\, dx \\ &\qquad \lesssim \delta \left(\int_{|x_1| \le 4/\delta} + \int_{|x_1| \ge 4/\delta}\right) \int_{-1/\delta}^{1/\delta} e^{-|x + \frac{2+\tau}{\delta} e_2| - |\frac{2-\tau}{\delta} e_2 - x|}\, dx_2\, dx_1 \\ &\qquad \lesssim \int_{|x_1| \le 4/\delta} e^{-\frac{4 - 2|\tau|}{\delta} - \sigma\delta x_1^2}\, dx_1 + \int_{4/\delta}^{\infty} e^{-2x_1}\, dx_1 \lesssim e^{-3/\delta}. \end{aligned}$$

Together with Proposition 3.1, this immediately implies (3.8) and completes the proof of the corollary. ∎

Estimating the nonlinearity

Finally, we give some estimates of the nonlinearities F and G occurring in the reformulated water wave problem (2.24).

Lemma 3.8. *For γ as in Assumptions (A) and (B) and any integer $2 \le k \le k_0$, there exists $\sigma \in (0,1)$, depending only on g and α, such that the operators F and G given in (2.25) and (2.26) satisfy*

$$\begin{aligned} &F \colon \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times H_{\mathrm{e}}^k(\mathcal{S}_\delta) \times H_{\mathrm{e}}^k(\mathbb{R}) \to H_{\mathrm{e}}^{k-1}(\mathcal{S}_\delta) \text{ is } C^{k_0 - k + 1} \text{ in } u, \Gamma_{\mathrm{s}}, \text{ and } C^{k_0 - k} \text{ in } \tau; \\ &G \colon \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times H_{\mathrm{e}}^k(\mathcal{S}_\delta) \times B_\sigma(H_{\mathrm{e}}^k(\mathbb{R})) \to H_{\mathrm{e}}^{k'}(\mathbb{R}) \text{ is } C^\infty \text{ in } u, \Gamma_{\mathrm{s}}, \text{ and } C^{k_0 - k' + 1} \text{ in } \tau, \end{aligned}$$

where $B_\sigma(H_{\mathrm{e}}^k(\mathbb{R}))$ is the ball in $H_{\mathrm{e}}^k(\mathbb{R})$ centered at 0 with radius σ and $k' = k - \frac{3}{2}$ if $k > 2$ and k' can be any number smaller than $k - \frac{3}{2}$ if $k = 2$. Moreover, for any $\sigma_u \in (0,1)$,

$\sigma_\Gamma \in (0,\sigma)$, $\tau \in (-\frac{1}{3},\frac{1}{3})$, $u \in B_{\sigma_u}(H^k_{\mathrm{e}}(\mathcal{S}_\delta))$ and $\Gamma_{\mathrm{s}} \in B_{\sigma_\Gamma}(H^k_{\mathrm{e}}(\mathbb{R}))$, we have

$$\begin{aligned}
|D_u F|_{\mathcal{L}(H^k_{\mathrm{e}}(\mathcal{S}_\delta),H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta))} &\lesssim \sigma_u + \delta^{-1}\sigma_\Gamma + \delta^{1/4}e^{-(1-|\tau|)/\delta},\\
|D_{\Gamma_{\mathrm{s}}} F|_{\mathcal{L}(H^k_{\mathrm{e}}(\mathbb{R}),H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta))} &\lesssim \delta^{-1},\\
|F(\tau,0,0)|_{H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta)} &\lesssim \delta^{1/4}|\log\delta|^{1/2}e^{-2(1-|\tau|)/\delta},
\end{aligned}$$

and

$$\begin{aligned}
|D_u G|_{\mathcal{L}(H^k_{\mathrm{e}}(\mathcal{S}_\delta),H^{k-2}_{\mathrm{e}}(\mathbb{R}))} &\lesssim \delta^{1/2-k}\sigma_u + \delta^{3/4-k}e^{-(1-\tau)/\delta},\\
|D_{\Gamma_{\mathrm{s}}} G|_{\mathcal{L}(H^k_{\mathrm{e}}(\mathbb{R}),H^{k-2}_{\mathrm{e}}(\mathbb{R}))} &\lesssim \delta^{1/2-k}(\sigma_u^2 + \delta^{1/2}e^{-2(1-\tau)/\delta}),\\
|G(\tau,0,0)|_{H^{k-2}_{\mathrm{e}}(\mathbb{R})} &\lesssim \delta^{1-k}e^{-2(1-\tau)/\delta}.
\end{aligned}$$

Proof. Verifying the smoothness of F and G is tedious but straightforward. The argument is based on (i) standard regularity results on products in Sobolev spaces, properties of the harmonic extension, the trace theorem, and (ii) the $C^{k_0-l'}$ smoothness of the mapping $H^l \ni u \mapsto \gamma \circ u \in H^{l'}$ for a given $\gamma \in C^{k_0}$, which holds for $l' \le l$ and $l > \frac{n}{2}+1$ in n dimensions. The limitation on the smoothness of F and G with respect to τ is only due to the C^{k_0+2} dependence of U_{bc} in τ. The small $\sigma > 0$ is chosen such that the denominator in the definition of G is bounded away from zero and $A(\Gamma_{\mathrm{s}})$ has a bounded inverse, which can be done independent of $|\tau| \le \frac{1}{3}$ and small $\delta > 0$. We omit the details and focus on the quantitative estimates related to F and G. In what follows, let

$$\Gamma = \Gamma_1 + \mathrm{i}\Gamma_2 \in H^{k+1/2}_{\mathrm{e}}(\mathcal{S}_\delta)$$

be the conformal mapping determined by Γ_{s} through (2.1) and (2.19). Note that this involves just the harmonic extension (2.2) and harmonic conjugate operators.

From the definition of F,

$$\begin{aligned}
F(\tau,0,0) &= \gamma(U-U_{\mathrm{bc}}) - \gamma(U) + U_{\mathrm{bc}}\\
&= \int_0^1\int_0^1 \gamma''(s_2U - s_1s_2U_{\mathrm{bc}})(s_1U_{\mathrm{bc}} - U)U_{\mathrm{bc}}\,\mathrm{d}s_2\,\mathrm{d}s_1,
\end{aligned}$$

which, along with Corollary 3.7, implies that

$$\begin{aligned}
|F(\tau,0,0)|_{H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta)} &\lesssim |UU_{\mathrm{bc}}|_{H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta)} + |U^2_{\mathrm{bc}}|_{H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta)}\\
&\lesssim \Big|\sum_{\pm} U\big(\cdot,\pm\tfrac{2}{\delta}-\cdot\big)U\Big|_{H^{k-2}_{\mathrm{e}}(\mathcal{S}_\delta)} + \delta^{1/2}e^{-2(1-|\tau|)/\delta}.
\end{aligned} \tag{3.11}$$

Without loss of generality, we only need to consider the "+" term in the summation. According to Assumption (A) and Proposition 3.1, for any $0 \le j \le k-2$ and $x \in \mathcal{S}_\delta$,

$$\begin{aligned}
&\big|\partial^j\big(U(x)U\big(x_1,\tfrac{2}{\delta}-x_2\big)\big)\big|\\
&\qquad\lesssim \big(1+\big|x-\tfrac{\tau}{\delta}e_2\big|\big)^{-1/2}\big|\tfrac{2-\tau}{\delta}e_2 - x\big|^{-1/2}e^{-(|x-\frac{\tau}{\delta}e_2|+|\frac{2-\tau}{\delta}e_2-x|)}
\end{aligned}$$

which can be estimated much as (3.8). Applying (3.9) and (3.10) to $b_1 = 2-\tau$ and $b_2 = \tau$, we have

$$\begin{aligned}
&\int_{\mathcal{S}_\delta} \big(1+\big|x-\tfrac{\tau}{\delta}e_2\big|\big)^{-1}\big|\tfrac{2-\tau}{\delta}e_2 - x\big|^{-1} e^{-2(|x-\frac{\tau}{\delta}e_2|+|\frac{2-\tau}{\delta}e_2-x|)}\,\mathrm{d}x \\
&\qquad \lesssim \left(\int_{|x_1|\le 4/\delta} + \int_{|x_1|\ge 4/\delta}\right)\int_{-1/\delta}^{1/\delta} \big(1+\big|x-\tfrac{\tau}{\delta}e_2\big|\big)^{-1}\big|\tfrac{2-\tau}{\delta}e_2 - x\big|^{-1} \\
&\qquad\qquad \times e^{-2(|x-\frac{\tau}{\delta}e_2|+|\frac{2-\tau}{\delta}e_2-x|)}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&\qquad \lesssim \int_{|x_1|\le 4/\delta}\int_0^{\frac{1+|\tau|}{\delta}} \delta(1+s)^{-1}e^{-2(\frac{2(1-|\tau|)}{\delta}+\sigma\delta x_1^2)}\,\mathrm{d}s\,\mathrm{d}x_1 + \int_{|x|\ge 4/\delta}\delta^2 e^{-2|x|}\,\mathrm{d}x,
\end{aligned}$$

and so

$$\begin{aligned}
\int_{\mathcal{S}_\delta} \big(1+\big|x-\tfrac{\tau}{\delta}e_2\big|\big)^{-1}\big|\tfrac{2-\tau}{\delta}e_2 - x\big|^{-1}|e^{-2(|x-\frac{\tau}{\delta}e_2|+|\frac{2-\tau}{\delta}e_2-x|)}\,\mathrm{d}x \\
\lesssim \delta^{1/2}|\log\delta|e^{-4(1-|\tau|)/\delta}. \qquad (3.12)
\end{aligned}$$

This further implies

$$\big|U\big(\cdot,\tfrac{2}{\delta}-\cdot\big)U\big|^2_{H^{k-2}(\mathcal{S}_\delta)} \lesssim \delta^{1/2}|\log\delta|e^{-4(1-|\tau|)/\delta},$$

which, with (3.11), furnishes the desired estimate of $F(\tau,0,0)$.

Next, observe that, for any $\tilde{u}\in H^k_{\mathrm{e}}(\mathcal{S}_\delta)$ with $k\ge 2$,

$$\begin{aligned}
&D_uF(\tau,u,\Gamma_{\mathrm{s}})\tilde{u} = \big(|1+\Gamma'(\delta\,\cdot)|^2\gamma'(u+U-U_{\mathrm{bc}})-\gamma'(U)\big)\tilde{u} \\
&= \left(\big(2\Gamma_1'(\delta\,\cdot)+|\Gamma'(\delta\,\cdot)|^2\big)\gamma'(u+U-U_{\mathrm{bc}})+(u-U_{\mathrm{bc}})\int_0^1\gamma''(U+s(u-U_{\mathrm{bc}}))\,\mathrm{d}s\right)\tilde{u}. \qquad (3.13)
\end{aligned}$$

Now, for any s we have the the scaling identity

$$|f(\delta\,\cdot)|_{\dot{H}^s(\mathcal{S}_\delta)} = \delta^{s-1}|f|_{\dot{H}^s(\mathcal{S}_1)},$$

and, for $k-\frac{1}{2}\ge\frac{3}{2}$ and $|\Gamma_{\mathrm{s}}|_{H^k}<1$,

$$\big|2\Gamma_1'(\delta\,\cdot)+|\Gamma'(\delta\,\cdot)|^2\big|_{H^{k-1/2}(\mathcal{S}_\delta)} \lesssim \delta^{-1}|\Gamma_{\mathrm{s}}|_{H^k(\mathbb{R})}.$$

Thus, the $H^{k-2}(\mathcal{S}_\delta)$ norm of the last line of (3.13) has the upper bound

$$O(\delta^{-1}|\Gamma_{\mathrm{s}}|_{H^k(\mathbb{R})}+|u|_{H^k(\mathcal{S}_\delta)}+|U_{\mathrm{bc}}|_{H^k(\mathcal{S}_\delta)})|\tilde{u}|_{H^{k-2}(\mathcal{S}_\delta)}.$$

Corollary 3.7 therefore gives the claimed estimate of D_uF.

Regarding $D_{\Gamma_{\mathrm{s}}}F$, we have, for any $\tilde{\Gamma}_{\mathrm{s}}\in H^k_{\mathrm{e}}(\mathbb{R})$,

$$D_{\Gamma_{\mathrm{s}}}F(\tau,u,\Gamma_{\mathrm{s}})\tilde{\Gamma}_{\mathrm{s}} = 2\big(\tilde{\Gamma}_1'(\delta\,\cdot)+\Gamma'(\delta\,\cdot)\cdot\tilde{\Gamma}'(\delta\,\cdot)\big)\gamma(u+U-U_{\mathrm{bc}}),$$

where $\tilde{\Gamma} = \tilde{\Gamma}_1 + \mathrm{i}\tilde{\Gamma}_2$ is the complex holomorphic function determined by $\tilde{\Gamma}_{\mathrm{s}}$. The estimate on $D_{\Gamma_{\mathrm{s}}} F$ follows from this expression and the scaling of the Sobolev norms explained above.

From the definition of G, one can directly compute

$$G(\tau, 0, 0) = \frac{1}{2\delta^2}(\partial_{x_2}(U - U_{\mathrm{bc}}))\big(\tfrac{\cdot}{\delta}, \tfrac{1}{\delta}\big)^2. \tag{3.14}$$

Recall the one-dimensional scaling property,

$$|f(\delta^{-1}\,\cdot)|_{\dot{H}^s(\mathbb{R})} = \delta^{-s+1/2}|f|_{\dot{H}^s(\mathbb{R})},$$

which holds for all $s \in \mathbb{R}$. The term $(\partial_{x_2}(U - U_{\mathrm{bc}}))\big(\cdot, \frac{1}{\delta}\big)$ can be estimated by approximating $\partial_{x_2} U_{\mathrm{bc}}(\cdot, \frac{1}{\delta})$ by $-\partial_{x_2} U(\cdot, \frac{1}{\delta})$. From Corollary 3.7 and the trace theorem, we then have, for any $k_0 + \frac{1}{2} \geq k \geq 0$,

$$\big|\partial_{x_2}(U - U_{\mathrm{bc}})\big(\cdot, \tfrac{1}{\delta}\big) - 2(\partial_{x_2} U)\big(\cdot, \tfrac{1}{\delta}\big)\big|_{H^k(\mathbb{R})} \lesssim \big|(\partial_{x_2} U)\big(\cdot, -\tfrac{3}{\delta}\big)\big|_{H^k(\mathbb{R})} + \delta^{3/4} e^{-2(1-|\tau|)/\delta}. \tag{3.15}$$

Using Proposition 3.1 and the change of variables $x_1(\rho) = (\rho^2 - (\frac{1-\tau}{\delta})^2)^{1/2}$, we can estimate $(\partial_{x_2} U)(\cdot, \frac{1}{\delta})$ while the terms on the right-hand side are obviously much smaller,

$$\begin{aligned}\big|(\partial_{x_2} U)\big(\cdot, \tfrac{1}{\delta}\big)\big|^2_{H^k(\mathbb{R})} &\lesssim \bigg(\int_0^{\delta^{-1/2}} + \int_{\delta^{-1/2}}^{\infty}\bigg)\Big(x_1^2 + \big(\tfrac{1-\tau}{\delta}\big)^2\Big)^{-1/2} e^{-2(x_1^2 + (\frac{1-\tau}{\delta})^2)^{1/2}}\,\mathrm{d}x_1 \\ &\lesssim \delta^{1/2} e^{-2(1-\tau)/\delta} + \int_{\frac{((1-\tau)^2+\delta)^{1/2}}{\delta}}^{\infty} \frac{\mathrm{d}\rho}{x_1(\rho) e^{2\rho}} \lesssim \delta^{1/2} e^{-2(1-\tau)/\delta}.\end{aligned} \tag{3.16}$$

This implies that

$$\big|(\partial_{x_2}(U - U_{\mathrm{bc}}))\big(\cdot, \tfrac{1}{\delta}\big)\big|_{H^k(\mathbb{R})} \lesssim \delta^{1/4} e^{-1-\tau/\delta}. \tag{3.17}$$

We therefore obtain the estimate on $G(\tau, 0, 0)$ from the scaling property as

$$|G(\tau,0,0)|_{H^{k-2}(\mathbb{R})} \lesssim \delta^{1/2-(k-2)-2}\big|\big((\partial_{x_2}(U - U_{\mathrm{bc}}))\big(\cdot, \tfrac{1}{\delta}\big)\big)^2\big|_{H^{k-2}(\mathbb{R})} \lesssim \delta^{1-k} e^{-2(1-\tau)/\delta}.$$

Consider next the bound on $D_u G$. It is easy to see from the definition of G that

$$D_u G(\tau, u, \Gamma_{\mathrm{s}})\tilde{u} = \frac{1}{\delta^2} A(\Gamma_{\mathrm{s}})^{-1}\bigg[\frac{(\partial_{x_2}(u + U - U_{\mathrm{bc}})\partial_{x_2}\tilde{u})(\frac{\cdot}{\delta}, \frac{1}{\delta})}{(1 + |\mathrm{D}|\coth(2|\mathrm{D}|)\Gamma_{\mathrm{s}})^2 + \Gamma_{\mathrm{s}}'^2}\bigg].$$

By the trace theorem and (3.17), we have

$$\begin{aligned}&\big|(\partial_{x_2}(u + U - U_{\mathrm{bc}})\partial_{x_2}\tilde{u})\big(\cdot, \tfrac{1}{\delta}\big)\big|_{H^{k-2}(\mathbb{R})} \\ &\qquad \lesssim \big|\partial_{x_2}(u + U - U_{\mathrm{bc}})\big(\cdot, \tfrac{1}{\delta}\big)\big|_{H^{k-3/2}(\mathbb{R})}\big|\partial_{x_2}\tilde{u}\big(\cdot, \tfrac{1}{\delta}\big)\big|_{H^{k-3/2}(\mathbb{R})} \\ &\qquad \lesssim (\sigma_u + \delta^{1/4} e^{-1-\tau/\delta})|\tilde{u}|_{H^k(\mathcal{S}_\delta)}.\end{aligned}$$

The desired bound on $D_u G$ then follows from the scaling property.

Finally, for any $\tilde{\Gamma}_s \in H^k_e(\mathbb{R})$,

$$2\delta^2 D_{\Gamma_s} G(\tau, u, \Gamma_s)\tilde{\Gamma}_s = \big(D_{\Gamma_s}(A(\Gamma_s)^{-1})\tilde{\Gamma}_s\big)\left[\frac{(\partial_{x_2}(u + U - U_{\mathrm{bc}}))(\frac{\cdot}{\delta}, \frac{1}{\delta})^2}{(1 + |\mathrm{D}|\coth(2|\mathrm{D}|)\Gamma_s)^2 + \Gamma_s'^2}\right] - 2A(\Gamma_s)^{-1}\left[\frac{(\partial_{x_2}(u + U - U_{\mathrm{bc}}))(\frac{\cdot}{\delta}, \frac{1}{\delta})^2}{((1 + |\mathrm{D}|\coth(2|\mathrm{D}|)\Gamma_s)^2 + \Gamma_s'^2)^2} \times ((1 + |\mathrm{D}|\coth(2|\mathrm{D}|)\Gamma_s)|\mathrm{D}|\coth(2|\mathrm{D}|)\tilde{\Gamma}_s + \Gamma_s'\tilde{\Gamma}_s')\right].$$

Since $|u|_{H^k(\mathcal{S}_\delta)} < \sigma_u$ and $|\Gamma_s|_{H^k(\mathbb{R})} < \sigma_\Gamma < \sigma < 1$ with $k \geq 2$, straightforwardly we obtain

$$|D_{\Gamma_s} G(\tau, u, \Gamma_s)\tilde{\Gamma}_s|_{H^{k-2}(\mathbb{R})} \lesssim \delta^{-2}\big|(\partial_{x_2}(u + U - U_{\mathrm{bc}}))\big(\tfrac{\cdot}{\delta}, \tfrac{1}{\delta}\big)^2\big|_{H^{k-2}(\mathbb{R})}|\tilde{\Gamma}_s|_{H^k(\mathbb{R})} \lesssim \delta^{1/2-k}\big(|u|^2_{H^k(\mathcal{S}_\delta)} + \big|(\partial_{x_2}(U - U_{\mathrm{bc}}))\big(\cdot, \tfrac{1}{\delta}\big)\big|^2_{H^{k-1}(\mathbb{R})}\big)|\tilde{\Gamma}_s|_{H^k(\mathbb{R})},$$

where the scaling property and the trace theorem have been used. The estimate on $D_{\Gamma_s} G$ now follows immediately from (3.17). ■

One notices that $|D_{\Gamma_s} F(\tau, u, \Gamma_s)|$ is not small no matter how small u and Γ_s are. Fortunately this is an “off-diagonal term” in the linearization, which will be handled by a simple rescaling argument in Section 5.

4. Spectral properties

Having the necessary estimates on U and the boundary correction operator bc now in hand, we next consider the linear operator

$$L_\tau = -\Delta + \gamma'(U(\tau)) \colon X^2_\delta \to X^0_\delta,$$

in the elliptic equation (2.24a). Recall that the Dirichlet boundary conditions on $\partial\mathcal{S}_\delta$ are encoded in the definition of the space X^2_δ, and that we usually suppress the dependence on the translation parameter τ in the notation for $U = U(\cdot, \tau) = U(\tau)(\cdot)$.

The inherent difficulty here is that equation (1.11) implies

$$\Delta\partial_{x_2} U = \gamma'(U)\partial_{x_2} U,$$

so that $\partial_{x_2} U$ is in the kernel of $-\Delta + \gamma'(U)$ viewed as an operator with domain $H^2_e(\mathbb{R}^2)$. Working in the strip $\mathcal{S}_\delta$ breaks the vertical translation symmetry and eliminates this kernel direction. It is therefore expected that L_τ will be invertible $X^2_\delta \to X^0_\delta$, and, indeed, this is proved in Lemma 4.6. However, as $\delta \searrow 0$, heuristically the strip approximates $\mathbb{R}^2$, and so we cannot hope to obtain bounds for $L_\tau^{-1} \colon X^0_\delta \to X^2_\delta$ that are *uniform* in δ. Another way to see this is to note that

$$L_\tau \partial_{x_2} U = 0 \quad \text{in } \mathcal{S}_\delta, \qquad \partial_{x_2} U \eqsim \delta^{1/2} e^{-(1-|\tau|)/\delta} \quad \text{on } \partial\mathcal{S}_\delta,$$

as U and its derivatives decay exponentially. Thus, L_τ is nearly degenerate in the direction close to $\partial_{x_2} U$.

In order to proceed, it is therefore necessary to have detailed information about the behavior of L_τ as $\delta \searrow 0$. We will prove that there is a positive (simple) eigenvalue $l = l(\delta, \tau)$ that is exponentially small in δ for fixed τ, and whose eigenfunction U_0 approaches $\partial_{x_2} U$ as $\delta \searrow 0$. In the orthogonal complement of U_0, the inverse of L_τ is bounded uniformly in δ. In the next section, we will make use of this fact to perform a Lyapunov–Schmidt type reduction to (2.24a), first solving the problem on a codimension-1 subspace where L_τ is well-behaved, and then studying the reduced equation on the near-degenerate direction.

4.1. *An approximate eigenfunction*

As a preparation for proving the existence of l and U_0, we first study the function

$$U_2(\,\cdot\,,\tau) = \partial_{x_2} U(\,\cdot\,,\tau) - (\partial_{x_2} U)(\,\cdot\,,\tau)_{\mathrm{bc}} \in X_\delta^2, \tag{4.1}$$

which results from taking $\partial_{x_2} U$ and perturbing it slightly so that the homogeneous boundary condition on $\partial \mathcal{S}_\delta$ is satisfied (see Figure 2). In what follows, the dependence of U_2 on τ will be suppressed when there is no risk of confusion. While U_2 is not likely to be an eigenfunction itself, we will see that it does help in identifying the asymptotically degenerate direction. Observe that it solves the elliptic problem

$$\begin{cases} (-\Delta + \gamma'(U))U_2 = (1-\gamma'(U))(\partial_{x_2} U)_{\mathrm{bc}} & \text{in } \mathcal{S}_\delta, \\ U_2 = 0 & \text{on } \partial\mathcal{S}_\delta, \end{cases} \tag{4.2}$$

as $(-\Delta + \gamma'(U))\partial_{x_2} U = 0$ and $\Delta(\partial_{x_2} U)_{\mathrm{bc}} = (\partial_{x_2} U)_{\mathrm{bc}}$ by the definition of the boundary correction operator. Recall also that $\gamma \in C^{k_0}$ according to Assumption (A).

Lemma 4.1. *For $|\tau| \le \frac{1}{3}$, we have $|U_2|_{H^{k_0+1}(\mathcal{S}_\delta)} \eqsim 1$ and*

$$|L_\tau U_2|_{H^{k_0-1}(\mathcal{S}_\delta)} \lesssim \delta^{1/4} |\log \delta|^{1/2} e^{-2(1-|\tau|)/\delta},$$
$$0 < (U_2, L_\tau U_2)_{L^2(\mathcal{S}_\delta)} \eqsim \delta^{1/2} e^{-2(1-|\tau|)/\delta}.$$

Proof. Simply from the exponential decay of $U(x)$ as $|x| \to \infty$, Proposition 3.1, Corollary 3.7, and its definition, it is clear that $U_2 = O(1)$ in H^{k_0+1} for $|\tau| \le \frac{1}{3}$. On the other hand, from (4.2) and Corollary 3.7, we obtain the estimate

$$\Big| L_\tau U_2 - (1-\gamma'(U)) \sum_{\pm} \partial_{x_2} U\big(\,\cdot\,, \pm\tfrac{2}{\delta} - \cdot\,\big) \Big|_{H^{k_0-1}(\mathcal{S}_\delta)} \lesssim \delta^{3/4} e^{-2(1-|\tau|)/\delta}. \tag{4.3}$$

Without loss of generality, we only need to consider the "+" case. According to Assumption (A) and Proposition 3.1, for any $0 \le k \le k_0 - 1$ and $(x_1, x_2) \in \mathcal{S}_\delta$,

$$\big|\partial^k\big((1-\gamma'(U(x)))\partial_{x_2} U\big(x_1, \tfrac{2}{\delta} - x_2\big)\big)\big| \lesssim \frac{e^{-(|x-\frac{\tau}{\delta}e_2| + |\frac{2-\tau}{\delta}e_2 - x|)}}{(1+|x-\frac{\tau}{\delta}e_2|)^{1/2} |\frac{2-\tau}{\delta}e_2 - x|^{1/2}}$$

and thus the claimed bound on $L_\tau U_2$ follows from (3.12). Likewise, using the above estimate on $L_\tau U_2$ in conjunction with Corollary 3.7, (4.1), and (4.2), one can estimate

$$\begin{aligned}(U_2,\, L_\tau U_2)_{L^2(\mathcal{S}_\delta)} &= \big(\partial_{x_2} U - (\partial_{x_2} U)_{\mathrm{bc}}, L_\tau U_2\big)_{L^2(\mathcal{S}_\delta)} \\ &= \big((1-\gamma'(U))\partial_{x_2} U, (\partial_{x_2} U)_{\mathrm{bc}}\big)_{L^2(\mathcal{S}_\delta)} + O(e^{-3(1-|\tau|)/\delta}).\end{aligned}$$

We concentrate on the first term on the right-hand side, as it will clearly dominate as $\delta \searrow 0$. Using the identities

$$(1-\gamma'(U))\partial_{x_2} U = (1-\Delta)\partial_{x_2} U, \quad (1-\Delta)(\partial_{x_2} U)_{\mathrm{bc}} = 0,$$

and integrating by parts twice yields

$$\begin{aligned}\int_{\mathcal{S}_\delta} [(1-\gamma'(U))\partial_{x_2} U](\partial_{x_2} U)_{\mathrm{bc}}\, \mathrm{d}x &= \int_{\mathcal{S}_\delta} [(1-\Delta)\partial_{x_2} U](\partial_{x_2} U)_{\mathrm{bc}}\, \mathrm{d}x \\ &= \int_{\partial\mathcal{S}_\delta} \big(-[\partial_{x_2}^2 U](\partial_{x_2} U)_{\mathrm{bc}} + \partial_{x_2} U\, \partial_{x_2}(\partial_{x_2} U)_{\mathrm{bc}}\big) N_2\, \mathrm{d}x_1 \\ &= \int_{\partial\mathcal{S}_\delta} \big((\partial_{x_2} U)_{\mathrm{bc}}\partial_{x_2}(\partial_{x_2} U)_{\mathrm{bc}} - [\partial_{x_2}^2 U]\partial_{x_2} U\big) N_2\, \mathrm{d}x_1.\end{aligned}$$

The first of the boundary integrals we treat by integrating back to the interior domain $\mathcal{S}_\delta$ and using the definition of the boundary correction operator:

$$\begin{aligned}\int_{\partial\mathcal{S}_\delta} (\partial_{x_2} U)_{\mathrm{bc}}\partial_{x_2}(\partial_{x_2} U)_{\mathrm{bc}} N_2\, \mathrm{d}x_1 &= \int_{\mathcal{S}_\delta} \big(|\nabla(\partial_{x_2} U)_{\mathrm{bc}}|^2 + (\partial_{x_2} U)_{\mathrm{bc}}^2\big)\, \mathrm{d}x \\ &\eqsim \delta^{1/2} e^{-2(1-|\tau|)/\delta},\end{aligned}$$

where Corollary 3.5 and 3.7 are used in the last step above. The second boundary integral is instead estimated by integrating into the *outer* domain $\mathcal{S}_\delta^c$, where U and its derivatives are well defined and exponentially decaying in all radial directions. In analogy to the above, we use the elliptic equation that $\partial_{x_2} U$ satisfies to find

$$\begin{aligned}-\int_{\partial\mathcal{S}_\delta} (\partial_{x_2}^2 U)(\partial_{x_2} U) N_2\, \mathrm{d}x_1 &= \int_{\mathcal{S}_\delta^c} \big(|\nabla(\partial_{x_2} U)|^2 + (\partial_{x_2} U)\Delta(\partial_{x_2} U)\big)\, \mathrm{d}x \\ &= \int_{\mathcal{S}_\delta^c} \big(|\nabla(\partial_{x_2} U)|^2 + \gamma'(U)(\partial_{x_2} U)^2\big)\, \mathrm{d}x \\ &= \int_{|x_2|\in(1/\delta,3/\delta)} \big(|\nabla(\partial_{x_2} U)|^2 + (\partial_{x_2} U)^2\big)\, \mathrm{d}x + O(e^{-3(1-|\tau|)/\delta}) \eqsim \delta^{1/2} e^{-2(1-|\tau|)/\delta},\end{aligned}$$

where the last bound is from Corollary 3.5. Observe also that both boundary integrals are positive. This implies the positivity of $(U_2, L_\tau U_2)_{L^2(\mathcal{S}_\delta)}$ for δ small enough, and so the proof is complete. ∎

We wish to show that $L_\tau\colon X^2_\delta \to X^0_\delta$ is well behaved as $\delta \searrow 0$ except in a one-dimensional near-degenerate direction that anticipates the kernel of $-\Delta + \gamma'(U)$ on $\mathbb{R}^2$. To be more precise, define the function spaces

$$X = H^2_{\mathrm{e}}(\mathbb{R}^2), \quad Y = L^2_{\mathrm{e}}(\mathbb{R}^2).$$

Then under Assumption (B) we see that $-\Delta + \gamma'(U)\colon X \to Y$ has a one-dimensional kernel spanned by $\partial_{x_2} U$. Note that here the even symmetry restriction eliminates the kernel direction generated by $\partial_{x_1} U$. Let $P = P(\tau)$ denote the orthogonal projection of Y onto span $\{\partial_{x_2} U\}$; abusing notation somewhat, we use the same symbol for the induced projection $X \to$ span $\{\partial_{x_2} U\}$.

The following nondegeneracy result is a direct consequence of Assumption (B).

Lemma 4.2 (Nondegeneracy in $\mathbb{R}^2$). *The operator $-\Delta + \gamma'(U)\colon (I-P)X \to (I-P)Y$ is an isomorphism with bounds uniform in $|\tau| \le \frac{1}{3}$.*

Next, we establish an estimate for $L_\tau\colon X^2_\delta \to X^0_\delta$. Let $\varrho \in C^\infty(\mathbb{R}, [0,1])$ be a smooth cut-off function with

$$\varrho(r) = \begin{cases} 1 & \text{for } r < 1/3, \\ 0 & \text{for } r > 1/2. \end{cases}$$

Given a function $h\colon \overline{\mathcal{S}_\delta} \to \mathbb{R}$ we define its (odd) extension $Eh\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$(Eh)(x) := \begin{cases} h(x) & \text{for } x \in \mathcal{S}_\delta, \\ -h\big(x_1, \pm\frac{2}{\delta} - x_2\big)\varrho\big(\big(|x_2| - \frac{1}{\delta}\big)\delta\big) & \text{for } \pm x_2 \ge 1/\delta. \end{cases} \tag{4.4}$$

Notice that

$$|h|_{L^2(\mathcal{S}_\delta)} \le |Eh|_{L^2(\mathbb{R}^2)} \le 3|h|_{L^2(\mathcal{S}_\delta)}, \tag{4.5}$$

and hence $h \in L^2(\mathcal{S}_\delta)$ if and only if $Eh \in L^2(\mathbb{R}^2)$. By a standard property of odd extensions, we find in fact that $h \in X^2_\delta$ if and only if $Eh \in X$.

Now, let

$$U_2^\perp := \{u \in X^0_\delta : (u, U_2)_{L^2(\mathcal{S}_\delta)} = 0\}. \tag{4.6}$$

For any $|\tau| \le \frac{1}{3}$ and $u \in U_2^\perp$, we see that $Eu \in Y$ and

$$\begin{aligned} \int_{\mathbb{R}^2} (Eu)\partial_{x_2} U \,\mathrm{d}x &= \int_{\mathcal{S}_\delta^c} (Eu)\partial_{x_2} U \,\mathrm{d}x + \int_{\mathcal{S}_\delta} u(U_2 + (\partial_{x_2} U)_{\mathrm{bc}})\,\mathrm{d}x \\ &= -\sum_{\pm} \int_{\{\frac{1}{\delta} < \pm x_2 < \frac{3}{2\delta}\}} u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big)\varrho(\delta|x_2| - 1)\partial_{x_2} U \,\mathrm{d}x \\ &\quad + \int_{\mathcal{S}_\delta} u(\partial_{x_2} U)_{\mathrm{bc}}\,\mathrm{d}x. \end{aligned}$$

Using the L^2 bounds of $\partial_{x_2} U$ and $(\partial_{x_2} U)_{\mathrm{bc}}$ given in Corollaries 3.5 and 3.7, we estimate that

$$\left| \int_{\mathbb{R}^2} (Eu)\partial_{x_2} U \,\mathrm{d}x \right| \lesssim \delta^{1/4} e^{-(1-|\tau|)/\delta} |u|_{L^2(\mathcal{S}_\delta)} \quad \text{for all } u \in U_2^\perp, \tag{4.7}$$

uniformly for $|\tau| \le \frac{1}{3}$ and all small values of δ. In other words, the extension Eu is nearly orthogonal to $\partial_{x_2} U$ in $L^2_{\mathrm{e}}(\mathbb{R}^2)$. Combining these observations leads to the bound

$$|PEu|_{L^2(\mathbb{R}^2)} = \frac{|\int_{\mathbb{R}^2}(Eu)\partial_{x_2}U\,\mathrm{d}x|}{|\partial_{x_2}U|_{L^2(\mathbb{R}^2)}} \lesssim \delta^{1/4}e^{-(1-|\tau|)/\delta}|u|_{L^2(\mathcal{S}_\delta)}, \tag{4.8}$$

which holds for all $u \in U_2^\perp$.

Lemma 4.3 (Nondegeneracy in $\mathcal{S}_\delta$). (a) *There exist $\delta_0 > 0$ and $\lambda_0 > 0$ such that, for all $\delta \in (0, \delta_0)$ and $|\tau| \le \frac{1}{3}$,*

$$|L_\tau u|_{L^2(\mathcal{S}_\delta)} \ge \lambda_0 |u|_{L^2(\mathcal{S}_\delta)} \quad \textit{for all } u \in X^2_\delta \cap U_2^\perp. \tag{4.9}$$

(b) *For every $\theta \in (0, 1)$, there exist $\delta_0 = \delta_0(\theta) > 0$ and $\mu_0 = \mu_0(\theta) > 0$ such that, for all $\delta \in (0, \delta_0)$, $|\tau| \le \frac{1}{3}$, and $u \in X^2_\delta$ satisfying*

$$|PEu|_{L^2(\mathbb{R}^2)} \le \theta |u|_{L^2(\mathcal{S}_\delta)}, \tag{4.10}$$

we have

$$|L_\tau u|_{L^2(\mathcal{S}_\delta)} \ge \mu_0 |u|_{L^2(\mathcal{S}_\delta)}. \tag{4.11}$$

Proof. First observe that in light of (4.8), for any fixed $\theta \in (0, 1)$, any element $u \in U_2^\perp$ will satisfy (4.10) for δ sufficiently small. It therefore suffices to prove part (b). Fix $\theta \in (0, 1)$ as above and let u satisfy the near-orthogonality condition (4.10). By linearity, we can assume $|u|_{L^2(\mathcal{S}_\delta)} \le 1$.

From the definition (4.4) of the extension E,

$$[-\Delta + \gamma'(U), E]u = 0 \quad \text{on } \mathcal{S}_\delta \cup (\mathcal{S}_{3\delta/2})^c.$$

We compute that, for $\pm x_2 > 1/\delta$, one has

$$\begin{aligned} &((-\Delta + \gamma'(U))Eu)(x) \\ &\quad = \Delta u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big)\varrho(\pm\delta x_2 - 1) \mp 2\delta\varrho'(\pm\delta x_2 - 1)\partial_{x_2}u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big) \\ &\qquad + \delta^2 u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big)\varrho''(\pm\delta x_2 - 1) - \gamma'(U)u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big)\varrho(\pm\delta x_2 - 1). \end{aligned}$$

This leads directly to the following expression for the commutator on the set $\{\pm x_2 > 1/\delta\}$:

$$\begin{aligned} ([-\Delta + \gamma'(U),\ E]u)(x) &= ((-\Delta + \gamma'(U))Eu)(x) - (E(-\Delta + \gamma'(U))u)(x) \\ &= \mp 2\delta\varrho'(\pm\delta x_2 - 1)\partial_{x_2}u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big) + \delta^2 u\big(x_1, \pm\tfrac{2}{\delta} - x_2\big)\varrho''(\pm\delta x_2 - 1) \\ &\quad - \big(\gamma'(U) - \gamma'\big(U\big(x_1, \pm\tfrac{2}{\delta} - x_2\big)\big)\big)u(x_1, \pm\tfrac{2}{\delta} - x_2)\varrho(\pm\delta x_2 - 1). \end{aligned} \tag{4.12}$$

Now, measuring the left- and right-hand sides of (4.12) in $L^2(\mathbb{R}^2)$, taking into account the estimates of $\gamma'(U) = 1 + O(U)$ for small U and Proposition 3.1, we find that

$$|[-\Delta + \gamma'(U),\ E]u|_{L^2(\mathbb{R}^2)} \lesssim \delta|\partial_{x_2}u|_{L^2(\mathcal{S}_\delta)} + \delta^2|u|_{L^2(\mathcal{S}_\delta)}.$$

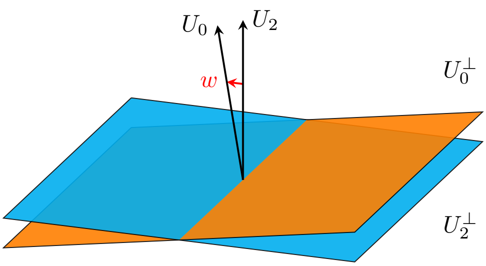


Fig. 3. We construct an eigenfunction U_0 that is almost parallel to U_2, and thus it will take the form $U_0 \approx U_2 + w$ for some $w \in U_2^\perp$ with $|w| \ll 1$.

The $\partial_{x_2} u$ term above can be eliminated via interpolation:

$$\begin{aligned}|\partial_{x_2} u|^2_{L^2(\mathcal{S}_\delta)} &\lesssim |\Delta u|_{L^2(\mathcal{S}_\delta)} |u|_{L^2(\mathcal{S}_\delta)} \lesssim |L_\tau u|_{L^2(\mathcal{S}_\delta)} |u|_{L^2(\mathcal{S}_\delta)} + |u|^2_{L^2(\mathcal{S}_\delta)} \\ &\lesssim |L_\tau u|^2_{L^2(\mathcal{S}_\delta)} + |u|^2_{L^2(\mathcal{S}_\delta)}.\end{aligned}$$

Inserting this inequality into (4.12), we arrive at the commutator bound

$$|[-\Delta + \gamma'(U),\ E]u|_{L^2(\mathbb{R}^2)} \lesssim \delta(|L_\tau u|_{L^2(\mathcal{S}_\delta)} + |u|_{L^2(\mathcal{S}_\delta)}), \tag{4.13}$$

independent of δ, τ, and u.

We are now prepared to prove the estimate (4.11). From Lemma 4.2 we have

$$\begin{aligned}|(-\Delta + \gamma'(U))Eu|^2_{L^2(\mathbb{R}^2)} &= |(-\Delta + \gamma'(U))(1 - P)Eu|^2_{L^2(\mathbb{R}^2)} \\ &\gtrsim |(1 - P)Eu|^2_{L^2(\mathbb{R}^2)} = |Eu|^2_{L^2(\mathbb{R}^2)} - |PEu|^2_{L^2(\mathbb{R}^2)} \\ &\geq (1 - \theta^2)|u|^2_{L^2(\mathcal{S}_\delta)},\end{aligned} \tag{4.14}$$

where the last inequality follows from hypothesis (4.10) and (4.5). On the other hand, together (4.5) and the commutator estimate (4.13) reveal that

$$\begin{aligned}|(-\Delta + \gamma'(U))Eu|_{L^2(\mathbb{R}^2)} &\lesssim |EL_\tau u|_{L^2(\mathbb{R}^2)} + \delta(|L_\tau u|_{L^2(\mathcal{S}_\delta)} + |u|_{L^2(\mathcal{S}_\delta)}) \\ &\lesssim |L_\tau u|_{L^2(\mathcal{S}_\delta)} + \delta |u|_{L^2(\mathcal{S}_\delta)}.\end{aligned} \tag{4.15}$$

Combined, (4.14) and (4.15) imply that (4.11) holds when δ is taken sufficiently small, which completes the proof. ∎

4.2. *Construction of a near-degenerate eigenfunction*

In Section 4.1, it was shown that the function U_2 roughly aligns with the near-degenerate direction of L_τ in the sense that the restriction $L_\tau \colon X^2_\delta \cap U_2^\perp \to X^0_\delta$ is uniformly positive according to (4.9), where $U_2^\perp$ is defined in (4.6). We now refine our analysis to find a (very small) eigenvalue l and corresponding eigenfunction U_0 near U_2 that limits to $\partial_{x_2} U$ in some sense as $\delta \searrow 0$. Similar to P above, denote by P_2 the $L^2(\mathcal{S}_\delta)$ orthogonal projection $X^0_\delta \to \operatorname{span}\{U_2\}$ and also the projection it induces from X^2_δ to $\operatorname{span}\{U_2\}$.

Lemma 4.4. *Consider the operator*

$$\tilde{L}_\tau : D(\tilde{L}_\tau) = U_2^\perp \cap X_\delta^2 = (1 - P_2)X_\delta^2 \to U_2^\perp = (1 - P_2)X_\delta^0$$

defined by

$$\tilde{L}_\tau u = (1 - P_2)L_\tau u \quad \textit{for all } u \in (1 - P_2)X_\delta^2.$$

There exists $\delta_0 > 0$ such that, for all $|\tau| \le \frac{1}{3}$ and $\delta \in (0, \delta_0)$, $\tilde{L}_\tau$ is an isomorphism and is self-adjoint as an unbounded and densely defined operator on $U_2^\perp$ with $|\tilde{L}_\tau^{-1}| \approx 1$.

Proof. Throughout the proof, all norms and inner products are evaluated on $\mathcal{S}_\delta$. By definition, $u \in D(\tilde{L}_\tau^*) \subset U_2^\perp$ and $\tilde{u} = \tilde{L}_\tau^* u \in U_2^\perp$ if and only if

$$(\tilde{u}, v)_{L^2} - (u, \tilde{L}_\tau v)_{L^2} = 0 \quad \text{for all } v \in U_2^\perp \cap X_\delta^2,$$

which holds if and only if

$$\left(\tilde{u} + \frac{(u,\, L_\tau U_2)_{L^2}}{|U_2|_{L^2}^2} U_2,\, aU_2 + v\right)_{L^2} - (u,\, L_\tau(aU_2 + v))_{L^2} = 0$$

for all $a \in \mathbb{R}$ and $v \in U_2^\perp \cap X_\delta^2$. Since $L_\tau^* = L_\tau$ on X_δ^0, we see that $u \in D(\tilde{L}_\tau^*)$ and $\tilde{u} = \tilde{L}_\tau^* u$ if and only if

$$u \in U_2^\perp \cap X_\delta^2 \quad \text{and} \quad \tilde{u} + \frac{(u,\, L_\tau U_2)_{L^2}}{|U_2|_{L^2}^2} U_2 = L_\tau u.$$

Thus $D(\tilde{L}_\tau^*) = U_2^\perp \cap X_\delta^2$, and $\tilde{u} = (I - P_2)L_\tau u = \tilde{L}_\tau u$. This implies that $\tilde{L}_\tau$ is indeed self-adjoint on its domain in $U_2^\perp$.

Next, we slightly improve the bound of $\tilde{L}_\tau$ in (4.9): observe that, for all u in $(1 - P_2)X_\delta^2$,

$$|u|_{H^2(\mathcal{S}_\delta)} \lesssim |u|_{L^2} + |\Delta u|_{L^2} \lesssim |u|_{L^2} + |L_\tau u|_{L^2} \lesssim |L_\tau u|_{L^2} \lesssim |\tilde{L}_\tau u|_{L^2} + |P_2 L_\tau u|_{L^2}.$$

But, due to equation (4.2) satisfied by U_2 and Lemma 4.1, we know that

$$|P_2 L_\tau u|_{L^2(\mathcal{S}_\delta)} \approx |(L_\tau u,\, U_2)_{L^2(\mathcal{S}_\delta)}| = |(u,\, L_\tau U_2)_{L^2(\mathcal{S}_\delta)}| \lesssim e^{-2(1-|\tau|)/\delta} |u|_{L^2(\mathcal{S}_\delta)},$$

and thus

$$|u|_{H^2(\mathcal{S}_\delta)} \lesssim |\tilde{L}_\tau u|_{L^2(\mathcal{S}_\delta)}. \tag{4.16}$$

This implies that $\tilde{L}_\tau$ is an isomorphism from $U_2^\perp \cap X_\delta^2$ to its range – a closed subspace of $U_2^\perp$. It follows from the self-adjointness of $\tilde{L}_\tau$ on $U_2^\perp$ that it is an isomorphism from $U_2^\perp \cap X_\delta^2$ to $U_2^\perp$. ∎

Proposition 4.5 (Existence of U_0). *For each $|\tau| \le \frac{1}{3}$ and $\delta > 0$ sufficiently small, there exists an eigenfunction*

$$U_0 = a_0(w + U_2) \in X_\delta^{k_0+1}, \quad w \in (I - P_2)X_\delta^{k_0+1}, \; |U_0|_{L^2(\mathcal{S}_\delta)} = 1, \tag{4.17}$$

of L_τ with a real eigenvalue $l = l(\delta, \tau)$:

$$L_\tau U_0 = l U_0 \quad \textit{in } \mathcal{S}_\delta.$$

They obey the estimates

$$|w|_{H^{k_0+1}(\mathcal{S}_\delta)} \lesssim \delta^{1/4} |\log \delta|^{1/2} e^{-2(1-|\tau|)/\delta}, \quad 0 < l(\delta,\tau) \eqsim \delta^{1/2} e^{-2(1-|\tau|)/\delta}, \quad 0 < a_0 \eqsim 1.$$

Moreover, for fixed δ, l is C^{k_0-1} in τ and $U_0 \in X_\delta^{k+2}$ is C^{k_0-k-1} in τ for $0 \le k \le k_0 - 1$, respectively.

Here a_0 is simply a normalizing constant so that $|U_0|_{L^2(\mathcal{S}_\delta)} = 1$.

Proof. From Lemma 4.4, we know that there exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, $\tilde{L}_\tau$ is an isomorphism from $(I - P_2)X_\delta^2$ to $(I - P_2)X_\delta^0$. By (4.16), its inverse satisfies

$$|\tilde{L}_\tau^{-1}|_{\mathcal{L}((I-P_2)X_\delta^0;(I-P_2)X_\delta^2)} \le \lambda_0^{-1} \quad \text{for all } |\tau| \le \tfrac{1}{3}, \tag{4.18}$$

for some $\lambda_0 > 0$ independent of $|\tau| \le \frac{1}{3}$ and small $\delta > 0$. A function $U_2 + w$ with w in $(I - P_2)X_\delta^2$ is an eigenfunction corresponding to l if

$$L_\tau(U_2 + w) = l(U_2 + w).$$

Taking the inner product of the above equation with U_2 yields

$$l = \frac{(w + U_2,\ L_\tau U_2)_{L^2(\mathcal{S}_\delta)}}{|U_2|^2_{L^2(\mathcal{S}_\delta)}}.$$

On the other hand, applying $I - P_2$ to the eigenfunction equation and recalling that $P_2 w = 0$, we see that

$$\tilde{L}_\tau w = l w - (I - P_2) L_\tau U_2. \tag{4.19}$$

This motivates us to consider the mapping $\Lambda \colon U_2^\perp \to U_2^\perp$ defined by

$$\Lambda(w) = \ell(w) \tilde{L}_\tau^{-1} w - \tilde{L}_\tau^{-1} (1 - P_2) L_\tau U_2, \tag{4.20}$$

where

$$\ell(w) = \frac{(w + U_2,\ L_\tau U_2)_{L^2(\mathcal{S}_\delta)}}{|U_2|^2_{L^2(\mathcal{S}_\delta)}} \tag{4.21}$$

is the presumptive eigenvalue. Clearly a small fixed point $w \in U_2^\perp$ of Λ yields an eigenfunction $w + U_2$ of L_τ close to U_2 associated to the eigenvalue $\ell(w)$.

It is straightforward to estimate

$$|\ell(w)| \lesssim |L_\tau U_2|_{L^2(\mathcal{S}_\delta)} |w|_{L^2(\mathcal{S}_\delta)} + (U_2, L_\tau U_2)_{L^2(\mathcal{S}_\delta)}, \tag{4.22}$$

and

$$|\ell(w_1) - \ell(w_2)| \le |U_2|^{-2}_{L^2(\mathcal{S}_\delta)} |L_\tau U_2|_{L^2(\mathcal{S}_\delta)} |w_1 - w_2|_{L^2(\mathcal{S}_\delta)}.$$

Likewise, we have

$$\begin{aligned}|\Lambda(w_1)-\Lambda(w_2)|_{L^2(\mathcal{S}_\delta)}\\ \leq |\ell(w_1)|\,|\tilde{L}_\tau^{-1}(w_1-w_2)|_{L^2(\mathcal{S}_\delta)}+|\ell(w_1)-\ell(w_2)|\,|\tilde{L}_\tau^{-1}w_2|_{L^2(\mathcal{S}_\delta)}\end{aligned}$$

and

$$|\Lambda(0)|_{L^2(\mathcal{S}_\delta)}\lesssim |L_\tau U_2|_{L^2(\mathcal{S}_\delta)},$$

where all the above inequalities are uniform in $|\tau|\leq\frac{1}{3}$ and small δ. Consequently, Lemma 4.1 and (4.18) imply Λ is a contraction map that sends B_1, the closed unit ball centered at the origin in X_δ^0, to itself. It therefore has a unique fixed point $w^*=w^*(\delta,\tau)$ in $(1-P_2)X_\delta^2\cap B_1$. This yields the eigenvalue $l=\ell(w^*)$ defined by (4.21) and the corresponding eigenfunction $w+U_2$ whose higher Sobolev regularity is due to the ellipticity in (4.19). The normalizing constant $a_0>0$ is chosen such that $|U_0|_{L^2(\mathcal{S}_\delta)}=1$. Since $\gamma\in C^{k_0}$ and $U\in C^{k_0+2}$ with exponential decay, it is easy to see that $L_\tau\colon X_\delta^{k+2}\to X_\delta^k$ is C^{k_0-k-1} in τ for $k\geq 0$. From standard spectral theory, the simple eigenvalue ℓ is C^{k_0-1} in τ and the *unit* eigenfunction $U_0\in X_\delta^{k+2}$ of L_τ is C^{k_0-k-1} in τ for $0\leq k\leq k_0-1$.

As w^* is a fixed point of the contraction Λ, its definition (4.20) and Lemma 4.1 imply

$$|w^*|_{L^2(\mathcal{S}_\delta)}\lesssim|\Lambda(0)|_{L^2(\mathcal{S}_\delta)}\lesssim\delta^{1/4}|\log\delta|^{1/2}e^{-2(1-|\tau|)/\delta}.$$

The higher Sobolev norms satisfy similar estimates due to the elliptic regularity given in (4.19). Finally, we conclude from (4.21), the above inequality, and Lemma 4.1, that

$$\left|\ell(w^*)-\frac{(U_2,\,L_\tau U_2)_{L^2(\mathcal{S}_\delta)}}{|U_2|^2_{L^2(\mathcal{S}_\delta)}}\right|\leq e^{-4(1-|\tau|)/\delta}.$$

Along with Lemma 4.1, this yields the desired estimate on ℓ. The positivity of ℓ is a consequence of the sign of $(U_2,\,L_\tau U_2)_{L^2(\mathcal{S}_\delta)}$ proved in Lemma 4.1. ■

Using the estimates just obtained, we can now confirm that L_τ is invertible (with near-degeneracy in the U_0 direction) and, more important, that the inverse of its restriction to the orthogonal complement of the one-dimensional subspace spanned by U_0 is *bounded independently* of δ. That said, let U_0 be given as in Proposition 4.5 and denote the orthogonal complement in X_δ^0 of span $\{U_0\}$ by $U_0^\perp$.

Lemma 4.6 (Invertibility of L_τ). *There exists $\delta_0>0$ such that, for all $\delta\in(0,\delta_0)$ and $|\tau|\leq\frac{1}{3}$,*

$$L_\tau\colon X_\delta^2\to X_\delta^0\quad\text{is invertible.}\tag{4.23}$$

Moreover, there exists $\mu_0=\mu_0(\delta_0)>0$ such that

$$|L_\tau u|_{L^2(\mathcal{S}_\delta)}\geq\mu_0|u|_{L^2(\mathcal{S}_\delta)}\quad\text{for all } u\in U_0^\perp\cap X_\delta^2.$$

Proof. Since L_τ is self-adjoint and U_0 is an eigenfunction with eigenvalue l, it is standard that $U_0^\perp$ is invariant under L_τ in the sense that

$$L_\tau(U_0^\perp\cap X_\delta^2)\subset U_0^\perp.$$

Because of $l > 0$ and again the self-adjointness of L_τ, it suffices to prove that $L_\tau|_{U_0^\perp \cap X_\delta^2} : U_0^\perp \cap X_\delta^2 \to U_0^\perp$ has a lower bound independent of $|\tau| \le \frac{1}{3}$ and small $\delta > 0$. In fact, recall

$$U_0 = a_0(U_2 + w) \quad \text{with} \quad w \in U_2^\perp,\ |w|_{L^2(\mathcal{S}_\delta)} \lesssim e^{-2(1-|\tau|)/\delta}.$$

Any $v \in U_0^\perp$ can be written as

$$v = v_1 + bU_2, \quad \text{where} \quad v_1 = (I - P_2)v \in U_2^\perp.$$

We have

$$(v_1, w)_{L^2(\mathcal{S}_\delta)} = (v, w)_{L^2(\mathcal{S}_\delta)} = \left(v, \frac{U_0}{a_0} - U_2\right)_{L^2(\mathcal{S}_\delta)} = -(v, U_2)_{L^2(\mathcal{S}_\delta)} = -b|U_2|^2_{L^2(\mathcal{S}_\delta)}$$

and thus

$$b = -\frac{(v_1, w)_{L^2(\mathcal{S}_\delta)}}{|U_2|^2_{L^2(\mathcal{S}_\delta)}} = -\frac{(v, w)_{L^2(\mathcal{S}_\delta)}}{|U_2|^2_{L^2(\mathcal{S}_\delta)}}.$$

It implies that $U_0^\perp$ and $U_2^\perp$ are isomorphic through

$$v_1 = (I - P_2)v \quad \text{with} \quad |v - v_1|_{L^2(\mathcal{S}_\delta)} \eqsim |b| \lesssim e^{-2(1-|\tau|)/\delta}|v|_{L^2(\mathcal{S}_\delta)},$$

where $a_0 \eqsim 1$ was also used. Together with Lemmas 4.1 and 4.3 we obtain

$$|L_\tau v|_{L^2(\mathcal{S}_\delta)} \ge |L_\tau v_1|_{L^2(\mathcal{S}_\delta)} - |b|\,|L_\tau U_2|_{L^2(\mathcal{S}_\delta)} \ge \frac{\lambda_0}{2}|v_1|_{L^2(\mathcal{S}_\delta)} \ge \frac{\lambda_0}{4}|v|_{L^2(\mathcal{S}_\delta)},$$

which completes the proof. ■

The invertibility of L_τ also holds in higher Sobolev spaces due to elliptic theory.

Corollary 4.7. *There exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, $|\tau| \le \frac{1}{3}$, and $0 \le k \le k_0 - 1$,*

$$L_\tau \colon X_\delta^{k+2} \to X_\delta^k \quad \textit{is invertible}. \tag{4.24}$$

Moreover, there exists $\mu_0 = \mu_0(\delta_0) > 0$ such that

$$|L_\tau u|_{H^k(\mathcal{S}_\delta)} \ge \mu_0 |u|_{H^{k+2}(\mathcal{S}_\delta)} \quad \textit{for all } u \in U_0^\perp \cap X_\delta^{k+2}. \tag{4.25}$$

5. Proof of the main result

In this section we complete the argument leading to the proof of Theorem 1.1.

5.1. Normal bundle coordinates

Recall from Section 2 that the waves we study are represented by two quantities: the boundary value Γ_s of a conformal mapping, that determines the fluid domain, and a (re-scaled) stationary stream function φ that gives the velocity field. Our basic approach is to construct waves for which $|\Gamma_{\mathrm{s}}|_{H^{k_0}} \ll 1$ and φ is a perturbation of $U(\tau) - U(\tau)_{\mathrm{bc}}$, where the parameter $\tau \in (-\frac{1}{3}, \frac{1}{3})$ selects the approximate altitude of the center of vorticity.

At this stage, we have obtained detailed information regarding the spectrum of the linearized operator

$$L_\tau = -\Delta + \gamma'(U(\tau)) \colon X_\delta^{k+2} \to X_\delta^k$$

and its dependence on τ and δ. In particular, we proved in Proposition 4.5 that there exists a unique simple eigenvalue $l = l(\delta, \tau)$, associated to an eigenfunction $U_0(\tau)$, that converges to 0 exponentially fast as $\delta \searrow 0$. This presents an obvious obstruction to a naïve fixed point scheme. We will see that τ is the key to ameliorating the issue.

To see the connection, observe that the family

$$\mathscr{C} := \left\{ U(\tau) - U(\tau)_{\mathrm{bc}} : \tau \in \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \right\}$$

can be viewed as a C^{k_0+2-k} curve in the ambient space X_δ^k. At a fixed τ, the tangent vector to $\mathscr{C}$ is

$$\begin{aligned} T_\tau \mathscr{C} &= \partial_\tau (U(\tau) - U(\tau)_{\mathrm{bc}}) \\ &= -\delta^{-1} (\partial_{x_2} U(\tau) - (\partial_{x_2} U(\tau))_{\mathrm{bc}}) \\ &= -\delta^{-1} U_2(\tau) \sim -\delta^{-1} U_0(\tau), \end{aligned}$$

where the second equality follows from the linearity of the boundary correction operator. The above calculation shows that the tangent direction along the curve $\mathscr{C}$ is almost parallel to the near-degenerate subspace.

Therefore, our strategy is to seek a (rescaled) stationary stream function of the form

$$\varphi = U(\tau) - U(\tau)_{\mathrm{bc}} + v, \tag{5.1}$$

with the unknowns

$$(\tau, v) \in \mathcal{X}_{\delta,k} := \left\{ (\tau, v) : \tau \in \left(-\tfrac{1}{3}, \tfrac{1}{3}\right),\ v \in X_\delta^k \cap U_0(\tau)^\perp \right\}, \quad k \geq 2. \tag{5.2}$$

This ensures that v avoids the near-degenerate direction of L_τ, while the linear part $g - \alpha^2 \mathrm{D}^2$ of the Bernoulli boundary condition (2.24b) is already invertible. We may then perform a Lyapunov–Schmidt reduction: for each fixed τ, we solve for v and Γ_{s}, leaving a one-dimensional problem of the form $b(\tau) = 0$ for a certain bifurcation function b. Finally, we will appeal to an intermediate value theorem argument to infer the existence of solutions to this reduced problem, as anticipated by the model calculation carried out in Section 1.

It is therefore imperative that the Lypanuov–Schmidt reduction be performed in such a way that $b(\tau)$ is *continuous* (or even smooth). Because the near-degenerate and nondegenerate subspaces vary as we change τ, it is natural to view $\mathcal{X}_{\delta,k}$ as a smooth vector bundle over the base $(-\frac{1}{3}, \frac{1}{3})$, with the fibers being the nondegenerate subspaces

$$\mathcal{X}_{\delta,k}^\tau := X_\delta^k \cap U_0(\tau)^\perp$$

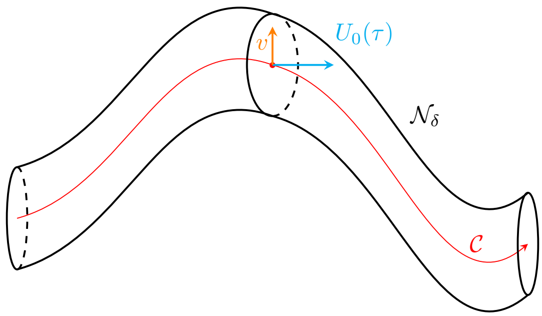


Fig. 4. Schematic of the tubular neighborhood $\mathcal{N}_\delta$ of the curve $\mathcal{C}_\delta$. At each $\tau \in (-\frac{1}{3}, \frac{1}{3})$, we locally decompose the space X^k_δ into a component v in the nondegenerate direction $U_0(\tau)^\perp$, and a component in the near-degenerate direction $U_0(\tau)$.

(see also Figure 4). According to Proposition 4.5, the C^{k_0}-regularity of γ ensures that $\tau \mapsto U_0(\tau) \in X^{k+2}_\delta$ is C^{k_0-k-1} for $0 \le k \le k_0 - 1$, and hence the orthogonal projection $P_0(\tau)$ onto span $U_0(\tau)$ enjoys the same regularity with respect to τ. It then follows that each $\tau_0 \in (-\frac{1}{3}, \frac{1}{3})$ is contained in a neighborhood $\mathcal{I}_0$ such that the mapping

$$\mathcal{I}_0 \times \mathcal{X}^{\tau_0}_{\delta,k} \ni (\tau, v) \mapsto (\tau, (I - P_0(\tau))v) \in \mathcal{X}_{\delta,k} \quad \text{for } 2 \le k \le k_0 + 1$$

is a C^{k_0-k+1} local trivialization of $\mathcal{X}_{\delta,k}$. Note that here and in what follows, we reserve calligraphic script for bundles. In the Lyapunov–Schmidt reduction, we fix τ, while tracking the continuous dependence on it.

Remark 5.1. While continuity in τ is sufficient for our purpose, in differential geometry, there are standard notions of smoothness of mappings related to vector bundles based on the smoothness of the trivializations, which allow implicit function theorem type arguments to be carried out as on flat spaces or manifolds. Moreover, it is standard to prove that

$$\chi\colon \mathcal{X}_{\delta,k} \ni (\tau, v) \mapsto v + U(\tau) - U(\tau)_{\mathrm{bc}} \in X^k_\delta \quad \text{for } 2 \le k \le k_0$$

defines a C^{k_0-k+1} local coordinate map (usually referred to as the transversal bundle coordinates) near $\mathcal{C}$.

To simplify notation, we introduce the set

$$\mathcal{W}_{\delta,k} := \mathcal{X}_{\delta,k} \times H^k_{\mathrm{e}}(\mathbb{R})$$

and endow it with the structure of a vector bundle over $(-\frac{1}{3}, \frac{1}{3})$ having fibers

$$\mathcal{W}^\tau_{\delta,k} := \mathcal{X}^\tau_{\delta,k} \times H^k_{\mathrm{e}}(\mathbb{R}),$$

and locally trivialized in the obvious way.

5.2. Lyapunov–Schmidt reduction

Let us now reconsider the elliptic system (2.24),

$$\begin{cases} (-\Delta + \gamma'(U))v + F(\tau, v, \Gamma_s) = 0 & \text{in } \mathcal{S}_\delta, \\ (g - \alpha^2 \mathrm{D}^2)\Gamma_s + G(\tau, v, \Gamma_s) = 0 & \text{on } \mathbb{R}, \end{cases}$$

from this geometrical standpoint. In the previous subsection, we argued that this system is equivalent to finding Γ_s together with a scaled stream function having the ansatz

$$\varphi = v + U(\tau) - U(\tau)_{\mathrm{bc}}, \quad (\tau, v) \in \mathcal{X}_{\delta,k}.$$

As before, we suppress the dependence of U and U_{bc} on τ whenever there is no risk of confusion. With a slight abuse of notation we also as above view

$$F(\tau, v, \Gamma_s) = |1 + \Gamma'(\delta\,\cdot)|^2 \gamma(v + U - U_{\mathrm{bc}}) - \gamma(U) - \gamma'(U)v + U_{\mathrm{bc}}, \tag{5.3}$$

$$G(\tau, v, \Gamma_s) = \frac{1}{2\delta^2} A(\Gamma_s)^{-1} \left[\frac{(\partial_{x_2}(v + U - U_{\mathrm{bc}})(\,\cdot\,, \frac{1}{\delta}))^2}{(1 + |\mathrm{D}| \coth(2|\mathrm{D}|)\Gamma_s)^2 + \Gamma_s'^2} \right], \tag{5.4}$$

from (2.25) and (2.26) to be the bundle map from a subset (with Γ_s small) of $\mathcal{W}_{\delta,k}$ to $\mathcal{W}_{\delta,k-2}$. It is easily seen that the slightly reinterpreted (F, G) enjoy the same regularity as in Lemma 3.8.

Projecting the semilinear elliptic problem into the near-degenerate and nondegenerate subspaces (which are invariant under L_τ), we can reconfigure the governing equations as the following system:

$$P_0 F(\tau, v, \Gamma_s) = 0 \quad \text{in } \mathcal{S}_\delta, \tag{5.5a}$$

$$(-\Delta + \gamma'(U))v + (I - P_0)F(\tau, v, \Gamma_s) = 0 \quad \text{in } \mathcal{S}_\delta, \tag{5.5b}$$

$$(g - \alpha^2 \mathrm{D}^2)\Gamma_s + G(\tau, v, \Gamma_s) = 0 \quad \text{on } \mathbb{R}. \tag{5.5c}$$

Notice that for a fixed τ, (5.5b)–(5.5c) are solved on the fiber $\mathcal{W}^\tau_{\delta,k}$. In the next lemma, we prove that one can always do this, and the solution depends smoothly on τ. We therefore reduce the system to the one-dimensional equation (5.5a) related to the near-degenerate subspace.

Lemma 5.2 (Lyapunov–Schmidt reduction). *There exist $C, \delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$ and $\tau \in (-\frac{1}{3}, \frac{1}{3})$, there exists a solution $(\tilde{v}(\tau), \tilde{\Gamma}_s(\tau)) \in \mathcal{W}^\tau_{\delta,k_0}$ to (5.5b)–(5.5c) which is unique in the set*

$$\{(v, \Gamma_s) \in \mathcal{W}^\tau_{\delta,k_0} : |v|_{H^{k_0}(\mathcal{S}_\delta)} + C\delta^{-1}|\Gamma_s|_{H^{k_0}(\mathbb{R})} \le \delta^{k_0+1}\},$$

and satisfies

$$|\tilde{v}|_{H^{k_0}(\mathcal{S}_\delta)} + C\delta^{-1}|\tilde{\Gamma}_s|_{H^{k_0}(\mathbb{R})} \lesssim C\delta^{-k_0} e^{-2(1-|\tau|)/\delta},$$
$$|\tilde{\Gamma}_s - \eta_0|_{H^{k_0}(\mathbb{R})} \lesssim C\delta^{3/4-2k_0} e^{-3(1-|\tau|)/\delta},$$

where

$$\eta_0 = -2\delta^{-2}(g - \alpha^2 \mathrm{D}^2)^{-1}\big((\partial_{x_2} U\big(\tfrac{\cdot}{\delta}, \tfrac{1}{\delta}\big))^2\big).$$

Moreover, $(\tilde{v}, \tilde{\Gamma}_{\mathrm{s}}) \in H^{k_0}(\mathcal{S}_\delta) \times H^{k_0}(\mathbb{R})$ *depends continuously on* τ.

Remark 5.3. As a consequence, the system (5.5) is locally equivalent to the one-dimensional problem

$$0 = b(\tau) := \big(U_0(\tau), F(\tau, \tilde{v}(\tau), \tilde{\Gamma}_{\mathrm{s}}(\tau))\big)_{L^2(\mathcal{S}_\delta)} = (U_0, L_\tau \tilde{v} + F(\tau, \tilde{v}, \tilde{\Gamma}_{\mathrm{s}}))_{L^2(\mathcal{S}_\delta)}. \quad (5.6)$$

Also, it is worth noting that, since $\gamma \in C^k$ for any $2 \le k \le k_0$, the above lemma holds for all such k. The uniqueness property of $(\tilde{v}(\tau), \tilde{\Gamma}_{\mathrm{s}}(\tau))$ implies that it is independent of k.

Proof of Lemma 5.2. Let $\delta \in (0, \delta_0)$ be given, where δ_0 will be determined over the course of the proof, which is largely based on the estimates given in Lemma 3.8. To tame the singular bound δ^{-1} of $D_{\Gamma_{\mathrm{s}}} F$, we introduce the rescaled variable

$$\check{\Gamma}_{\mathrm{s}} := \frac{C}{\delta} \Gamma_{\mathrm{s}},$$

where $C > 0$ will be determined independent of τ and δ, and the corresponding scaling of the nonlinearities

$$\check{F}(\tau, v, \check{\Gamma}_{\mathrm{s}}) := F\bigg(\tau, v, \frac{\delta}{C}\check{\Gamma}_{\mathrm{s}}\bigg), \quad \check{G}(\tau, v, \check{\Gamma}_{\mathrm{s}}) := \frac{C}{\delta} G\bigg(\tau, v, \frac{\delta}{C}\check{\Gamma}_{\mathrm{s}}\bigg).$$

Denote

$$\begin{aligned} L_1^{-1}(\tau) &= \big((-\Delta + \gamma'(U(\tau)))|_{\mathcal{X}^\tau_{\delta,k_0}}\big)^{-1} \colon \mathcal{X}^\tau_{\delta,k_0-2} \to \mathcal{X}^\tau_{\delta,k_0}, \\ L_2^{-1} &= (g - \alpha^2 \mathrm{D}^2)^{-1} \colon H^{k_0-2}(\mathbb{R}) \to H^{k_0}(\mathbb{R}), \end{aligned}$$

where we recall that the existence and boundedness of $L_1^{-1}(\tau)$ were established in Corollary 4.7. In particular, notice that, because $-\Delta + \gamma'(U) \colon \mathcal{X}^\tau_{\delta,k_0} \to \mathcal{X}^\tau_{\delta,k_0-2}$ is self-adjoint with respect to the $L^2(\mathcal{S}_\delta)$ inner product and U_0 is an eigenfunction, the range of $L_1^{-1}(\tau)$ is contained in $U_0(\tau)^\perp$. Then we see that $(\tau, v, \Gamma_{\mathrm{s}})$ solves (5.5b) and (5.5c) if and only if $(\tau, v, \check{\Gamma}_{\mathrm{s}})$ is a fixed point of the mapping

$$\Lambda^\tau = \big(\Lambda_1^\tau(v, \check{\Gamma}_{\mathrm{s}}), \Lambda_2^\tau(v, \check{\Gamma}_{\mathrm{s}})\big) \colon B \to \mathcal{W}^\tau_{\delta,k_0}$$

given by

$$\begin{aligned} \Lambda_1^\tau(v, \check{\Gamma}_{\mathrm{s}}) &= -L_1^{-1}(\tau)(I - P_0(\tau))\check{F}(\tau, v, \check{\Gamma}_{\mathrm{s}}), \\ \Lambda_2^\tau(v, \check{\Gamma}_{\mathrm{s}}) &= -L_2^{-1}\check{G}(\tau, v, \check{\Gamma}_{\mathrm{s}}) \end{aligned} \quad (5.7)$$

on the set

$$B = \{(v, \check{\Gamma}_{\mathrm{s}}) \in \mathcal{W}^\tau_{\delta,k_0} : |v|_{H^{k_0}(\mathcal{S}_\delta)} + |\check{\Gamma}_{\mathrm{s}}|_{H^{k_0}(\mathbb{R})} \le \delta^{k_0+1}\}.$$

From Lemma 3.8, we have

$$|\Lambda^\tau(0,0)|_{H^{k_0}(\mathcal{S}_\delta)\times H^{k_0}(\mathbb{R})} \lesssim C\delta^{-k_0}e^{-2(1-|\tau|)/\delta}, \quad |D\Lambda^\tau|_{C^0(B,\mathcal{L}(\mathcal{W}^\tau_{\delta,k_0}))} \lesssim C^{-1}.$$

Therefore, for a sufficiently large $C > 0$, which can be chosen independently of δ and τ, Λ^τ is a contraction on B, and so it possesses a unique fixed point

$$(\tilde{v}(\tau), \check{\Gamma}_{\mathrm{s}}(\tau)) = (\tilde{v}(\tau), C\delta^{-1}\tilde{\Gamma}_{\mathrm{s}}(\tau)) \in B.$$

Moreover, we have the estimate

$$|\tilde{v}(\tau)|_{H^{k_0}(\mathcal{S}_\delta)} + |\check{\Gamma}_{\mathrm{s}}(\tau)|_{H^{k_0}(\mathbb{R})} \lesssim |\Lambda^\tau(0,0)|_{H^{k_0}(\mathcal{S}_\delta)\times H^{k_0}(\mathbb{R})} \lesssim C\delta^{-k_0}e^{-2(1-|\tau|)/\delta}.$$

The continuity of $\tilde{v}(\tau)$ and $\tilde{\Gamma}_{\mathrm{s}}(\tau)$ follows from the continuity of Λ^τ in τ, where we can view it as a mapping defined on a smooth bundle.

Finally, we identify the leading-order term of $\tilde{\Gamma}_{\mathrm{s}}(\tau)$. Due to the fixed point property, we have

$$(g - \alpha^2\mathrm{D}^2)\tilde{\Gamma}_{\mathrm{s}}(\tau) = -G(\tau, \tilde{v}, \tilde{\Gamma}_{\mathrm{s}}).$$

Lemma 3.8 and the above upper bounds of $(\tilde{v}, \check{\Gamma}_{\mathrm{s}})$ imply

$$|G(\tau, \tilde{v}, \tilde{\Gamma}_{\mathrm{s}}) - G(\tau, 0, 0)|_{H^{k_0-2}(\mathbb{R})} \lesssim C\delta^{3/4-2k_0}e^{-3(1-|\tau|)/\delta}.$$

From (3.14)–(3.16) and the scaling property, we have

$$\left|G(\tau,0,0) - 2\delta^{-2}\left(\partial_{x_2}U\left(\tfrac{\cdot}{\delta}, \tfrac{1}{\delta}\right)\right)^2\right|_{H^{k_0-2}(\mathbb{R})} \lesssim \delta^{-1/2-k_0}e^{-3(1-|\tau|)/\delta},$$

which along with the above inequality yields the desire estimate on $\tilde{\Gamma}_{\mathrm{s}}(\tau)$. ∎

5.3. *Proof of the main result*

Proof of Theorem 1.1. The Lyapunov–Schmidt reduction carried out in Lemma 5.2 shows that it suffices to find $\tau \in (-\frac{1}{3}, \frac{1}{3})$ with $b(\tau) = 0$, where $b(\tau)$ is defined in (5.6). Our strategy will be to relate the bifurcation equation to the model calculation (1.16).

With that in mind, fix $\tau \in (-\frac{1}{3}, \frac{1}{3})$ and recall

$$(\tilde{v}, \tilde{\Gamma}_{\mathrm{s}}) = (\tilde{v}(\tau), \tilde{\Gamma}_{\mathrm{s}}(\tau)), \quad U = U(\tau), \quad U_0(\tau) = a_0(U_2 + w), \quad \varphi = U - U_{\mathrm{bc}} + \tilde{v},$$

remembering that U_0, U_2, a_0, and w were obtained in Section 4. In particular, $1 \eqsim a_0 = a_0(\tau) > 0$ is a normalizing constant introduced to ensure that $|U_0|_{L^2} = 1$.

Since $(\tilde{v}, \tilde{\Gamma}_{\mathrm{s}})$ solves (5.5b), we have

$$L_\tau\tilde{v} + F(\tau, \tilde{v}, \tilde{\Gamma}_{\mathrm{s}}) = b(\tau)U_0(\tau). \tag{5.8}$$

Now, let

$$\psi(\tau) = \varphi(\tau) \circ (\mathrm{id} + \delta^{-1}\tilde{\Gamma}(\delta\,\cdot))^{-1},$$

where $\tilde{\Gamma} = \tilde{\Gamma}_1 + i\tilde{\Gamma}_2$ is the holomorphic function constructed from $\tilde{\Gamma}_s$ through (2.19). According to Lemma 5.2, the domain of ψ is the (slightly) perturbed strip

$$\tilde{\Omega}(\tau) = (\mathrm{id} + \delta^{-1}\tilde{\Gamma}(\delta\,\cdot))(\mathcal{S}_\delta) \sim \mathcal{S}_\delta.$$

For clarity, we use $y = (y_1, y_2)$ as the coordinate variable in $\tilde{\Omega}(\tau)$. It is easy to compute

$$\partial_{y_2}\psi = \left(\frac{i}{1+\tilde{\Gamma}'(\delta\,\cdot)}\cdot\nabla\varphi\right)\circ(\mathrm{id}+\delta^{-1}\tilde{\Gamma}(\delta\,\cdot))^{-1},$$

where the complex number $i(1+\tilde{\Gamma}'(\delta\,\cdot))^{-1}$ is understood as a two-dimensional vector. Corollary 3.7, Proposition 4.5, and Lemma 5.2 together imply that

$$\left|U_0(\tau) - a_0(\tau)\frac{i}{1+\tilde{\Gamma}'(\delta\,\cdot)}\cdot\nabla\varphi\right|_{L^2(\mathcal{S}_\delta)} \ll 1 = |U_0|_{L^2(\mathcal{S}_\delta)}.$$

In view of (5.8), we see that (5.6) holds for $(\tau, \tilde{v}, \tilde{\Gamma}_s)$ if and only if

$$\tilde{b}(\tau) := \left(\frac{i}{1+\tilde{\Gamma}'(\delta\,\cdot)}\cdot\nabla\varphi, L_\tau\tilde{v} + F(\tau,\tilde{v},\tilde{\Gamma}_s)\right)_{L^2(\mathcal{S}_\delta)} = 0.$$

By the definitions of F and the boundary correction operator,

$$L_\tau\tilde{v} + F(\tau,\tilde{v},\tilde{\Gamma}_s) = -\Delta\varphi(\tau) + |1+\tilde{\Gamma}'(\delta\,\cdot)|^2\gamma(\varphi),$$

which, along with the coordinate change $y = x + \delta^{-1}\tilde{\Gamma}(\delta x)$, gives

$$\tilde{b}(\tau) = \int_{\tilde{\Omega}(\tau)} (-\Delta\psi + \gamma(\psi))\partial_{y_2}\psi\,\mathrm{d}y.$$

Following the same calculation leading to (1.16), we then find that

$$\tilde{b}(\tau) = -\frac{1}{2}\int_{\partial\tilde{\Omega}(\tau)} |\nabla\psi|^2 N_2\,\mathrm{d}S_y$$

where

$$N = (N_1, N_2) = \pm\left(\frac{i + i\tilde{\Gamma}'(\delta\,\cdot)}{|1+\tilde{\Gamma}'(\delta\,\cdot)|}\right)\circ(\mathrm{id}+\delta^{-1}\tilde{\Gamma}(\delta\,\cdot))^{-1}$$

is the outward unit normal vector on the upper/lower component of $\partial\tilde{\Omega}(\tau)$, and

$$\mathrm{d}S_y = |1+\tilde{\Gamma}'(\delta\,\cdot)|\circ(\mathrm{id}+\delta^{-1}\tilde{\Gamma}(\delta\,\cdot))^{-1}\,\mathrm{d}x_1$$

is the length element along $\partial\tilde{\Omega}(\tau)$. We can rewrite $\tilde{b}(\tau)$ as an integral on $\mathcal{S}_\delta$ by reversing the coordinate change:

$$\begin{aligned}\tilde{b}(\tau) = &-\frac{1}{2}\int_{\mathbb{R}} \frac{1+\partial_{x_1}\tilde{\Gamma}_1(\delta x_1, 1)}{|1+\tilde{\Gamma}'(\delta x_1, 1)|^2}\left|\partial_{x_2}\varphi\big(x_1, \tfrac{1}{\delta}\big)\right|^2\mathrm{d}x_1\\ &+\frac{1}{2}\int_{\mathbb{R}} \frac{1+\partial_{x_1}\tilde{\Gamma}_1(\delta x_1, -1)}{|1+\tilde{\Gamma}'(\delta x_1, -1)|^2}\left|\partial_{x_2}\varphi\big(x_1, -\tfrac{1}{\delta}\big)\right|^2\mathrm{d}x_1.\end{aligned} \tag{5.9}$$

Notice that tangential derivatives do not appear because $\varphi|_{\partial \mathcal{S}_\delta} = 0$. Without loss of generality, we just consider the first term. From the definition of φ, (3.15), (3.16), Lemma 5.2 (taking $k_0 = 2$), and the trace theorem, we obtain

$$\left|\partial_{x_2}\varphi\left(\cdot, \tfrac{1}{\delta}\right) - 2\partial_{x_2}U\left(\cdot, \tfrac{1-\tau}{\delta}\right)\right|_{L^2(\mathbb{R})} \lesssim |\tilde{v}|_{H^2(\mathcal{S}_\delta)} + \delta^{3/4}e^{-2(1-|\tau|)/\delta} \lesssim \delta^{-2}e^{-2(1-|\tau|)/\delta},$$

and

$$\left|\partial_{x_2}\varphi\left(\cdot, \tfrac{1}{\delta}\right)\right|_{L^2(\mathbb{R})} \lesssim \delta^{1/4}e^{-(1-|\tau|)/\delta}.$$

Therefore, again Lemma 5.2 implies

$$|\tilde{b}(\tau) - \tilde{b}_1(\tau)| \lesssim \delta^{-7/4}e^{-3(1-|\tau|)/\delta},$$

where

$$\tilde{b}_1(\tau) := -2\int_{\mathbb{R}} \left[\left(\partial_{x_2}U\left(x_1, \tfrac{1-\tau}{\delta}\right)\right)^2 - \left(\partial_{x_2}U\left(x_1, \tfrac{1+\tau}{\delta}\right)\right)^2\right] \mathrm{d}x_1.$$

Here we have used the radial symmetry of U to slightly simplify the expression. Clearly, it also implies that $\tilde{b}_1$ is odd.

Due to the exponential localization, $\tilde{b}_1$ can be effectively determined by integrating only over a δ-dependent but compact interval. Indeed, from Proposition 3.1, it is easy to see

$$\left|\tilde{b}(\tau) + \tilde{b}_2(\tau)\right| \lesssim \delta^{-7/4}e^{-3(1-|\tau|)/\delta}, \tag{5.10}$$

where

$$\tilde{b}_2(\tau) := 2\int_{-5/\delta}^{5/\delta} \left[\left(\partial_{x_2}U\left(x_1, \tfrac{1-\tau}{\delta}\right)\right)^2 - \left(\partial_{x_2}U\left(x_1, \tfrac{1+\tau}{\delta}\right)\right)^2\right] \mathrm{d}x_1.$$

Since $\tilde{b}_2$ is also odd, we consider $\tau \in (0, \frac{1}{3})$. Using Proposition 3.1 once more, along with (3.2), we compute that

$$\begin{aligned}
\tilde{b}_2(\tau) &= -4\int_{-5/\delta}^{5/\delta}\int_{\frac{1-\tau}{\delta}}^{\frac{1+\tau}{\delta}} (\partial_{x_2}U\,\partial_{x_2}^2 U)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= -4\int_{-5/\delta}^{5/\delta}\int_{\frac{1-\tau}{\delta}}^{\frac{1+\tau}{\delta}} \sin\theta\, U_r\left(\sin^2\theta\, U_{rr} + \frac{\cos^2\theta}{r}U_r\right)\mathrm{d}x_2\,\mathrm{d}x_1 \\
&\eqsim \int_{-5/\delta}^{5/\delta}\int_{\frac{1-\tau}{\delta}}^{\frac{1+\tau}{\delta}} r^{-1}e^{-2r}\,\mathrm{d}x_2\,\mathrm{d}x_1,
\end{aligned}$$

where we used the fact that $0 < \sin\theta \eqsim 1$ in this integral region. Let

$$S = \left\{x : |x_1| < \tfrac{5}{\delta},\ \left|x_2 - \tfrac{1}{\delta}\right| < \tfrac{\tau}{\delta},\ |x| < \tfrac{1+\tau}{\delta}\right\},$$

which has the polar coordinates representation

$$S = \left\{(r, \theta) : r \in \left(\tfrac{1-\tau}{\delta}, \tfrac{1+\tau}{\delta}\right),\ \theta \in (\beta(r), \pi - \beta(r))\right\},$$

where, because we are restricting to $\tau \in (0, \frac{1}{3})$,

$$\beta(r) = \arcsin\left(\tfrac{1-\tau}{\delta r}\right), \quad \pi/2 - \beta(r) \eqsim (\delta r - 1 + \tau)^{1/2}.$$

Therefore we have

$$\tilde{b}_2(\tau) \gtrsim \int_S r^{-1} e^{-2r}\, \mathrm{d}x_2\, \mathrm{d}x_1 = \int_{\frac{1-\tau}{\delta}}^{\frac{1+\tau}{\delta}} \int_{\beta(r)}^{\pi-\beta(r)} e^{-2r}\, \mathrm{d}\theta\, \mathrm{d}r = \int_{\frac{1-\tau}{\delta}}^{\frac{1+\tau}{\delta}} (\pi - 2\beta(r)) e^{-2r}\, \mathrm{d}r$$

$$\eqsim \int_{\frac{1-\tau}{\delta}}^{\frac{1+\tau}{\delta}} (\delta r - (1-\tau))^{1/2} e^{-2r}\, \mathrm{d}r = \delta^{1/2} e^{-2(1-\tau)/\delta} \int_0^{\frac{2\tau}{\delta}} (r')^{1/2} e^{-2r'}\, \mathrm{d}r'.$$

This implies that, for $\frac{\tau}{\delta} \le 1$,

$$\tilde{b}_2(\tau) \gtrsim \tau^{1/2} e^{-2/\delta},$$

and thus we deduce from (5.10) that there exists $C > 0$ independent of $\delta > 0$ such that

$$\tilde{b}(\tau_0) < 0, \quad \tau_0 = C\delta^{-7/2} e^{-2/\delta}.$$

From the oddness of $\tilde{b}_2$, we can then conclude that there exists $\tilde{\tau}$ with $|\tilde{\tau}| \lesssim \delta^{-7/2} e^{-2/\delta}$ and such that $(\tilde{\tau}, \tilde{v}(\tilde{\tau}), \tilde{\Gamma}_{\mathrm{s}}(\tilde{\tau}))$ is a solution to (5.5), and thus corresponds to a solution to the stationary capillary-gravity wave problem. The stream function is given by

$$\Psi = \big(U\big(\cdot - \tfrac{\tilde{\tau}}{\delta} e_2\big) - U\big(\cdot - \tfrac{\tilde{\tau}}{\delta} e_2\big)_{\mathrm{bc}} + \tilde{v}(\tilde{\tau})\big) \circ \big(\tfrac{1}{\delta}(\mathrm{id} + \tilde{\Gamma}(\tilde{\tau}))^{-1}\big), \tag{5.11}$$

defined on

$$\Omega = (\mathrm{id} + \tilde{\Gamma}(\tilde{\tau}))(\{|x_2| < 1\}).$$

From the estimate $|\tilde{\tau}| \lesssim \delta^{-7/2} e^{-2/\delta}$, Corollary 3.7, and Lemma 5.2, we have

$$\big|\tilde{v}(\tilde{\tau}) \circ \big(\tfrac{1}{\delta}(\mathrm{id} + \tilde{\Gamma}(\tilde{\tau}))^{-1}\big)\big|_{H^{k_0}(\Omega)} \lesssim \delta^{1-k_0} |\tilde{v}(\tilde{\tau})|_{H^{k_0}(\mathcal{S}_\delta)} \lesssim \delta^{1-2k_0} e^{-2/\delta},$$

and

$$\begin{aligned}
&\big|\big(U\big(\cdot - \tfrac{\tilde{\tau}}{\delta} e_2\big) - U\big(\cdot - \tfrac{\tilde{\tau}}{\delta} e_2\big)_{\mathrm{bc}}\big) \circ \big(\tfrac{1}{\delta}(\mathrm{id} + \tilde{\Gamma}(\tilde{\tau}))^{-1}\big) - \tilde{\Psi}_0\big(\tfrac{\cdot}{\delta}\big)\big|_{H^{k_0}(\Omega)} \\
&\quad \le \big|\big(U\big(\cdot - \tfrac{\tilde{\tau}}{\delta} e_2\big) - U\big(\cdot - \tfrac{\tilde{\tau}}{\delta} e_2\big)_{\mathrm{bc}} - \tilde{\Psi}_0\big) \circ \big(\tfrac{1}{\delta}(\mathrm{id} + \tilde{\Gamma}(\tilde{\tau}))^{-1}\big)\big|_{H^{k_0}(\Omega)} \\
&\qquad + \big|\tilde{\Psi}_0 \circ \big(\tfrac{1}{\delta}(\mathrm{id} + \tilde{\Gamma}(\tilde{\tau}))^{-1}\big) - \tilde{\Psi}_0\big(\tfrac{\cdot}{\delta}\big)\big|_{H^{k_0}(\Omega)} \\
&\quad \lesssim \delta^{7/4-k_0} e^{-2/\delta} + \big|\tilde{\Psi}_0\big(\tfrac{\cdot}{\delta}\big)\big|_{H^{k_0+1}(\mathbb{R}^2)} |\tilde{\Gamma}_{\mathrm{s}}(\tilde{\tau})|_{H^{k_0-1/2}(\mathbb{R})} \lesssim \delta^{1-2k_0} e^{-2/\delta},
\end{aligned}$$

where

$$\tilde{\Psi}_0(x) = U\big(x - \tfrac{\tilde{\tau}}{\delta} e_2\big) - U\big(x_1, \tfrac{2-\tilde{\tau}}{\delta} - x_2\big) - U\big(x_1, -\tfrac{2+\tilde{\tau}}{\delta} - x_2\big).$$

The desired estimate on Ψ in Theorem 1.1 follows immediately.

Finally, the corresponding free surface profile η is given by

$$\eta = \tilde{\Gamma}_{\mathrm{s}}(\tilde{\tau}) \circ (\mathrm{id} + \tilde{\Gamma}_1(\tilde{\tau}, \cdot, 1))^{-1},$$

which clearly satisfies

$$|\eta|_{H^{k_0}(\mathbb{R})} \lesssim \delta^{1-k_0} e^{-2/\delta}.$$

Using (3.16) and Lemma 5.2, it is straightforward to identify the leading-order term of η coinciding with that of $\tilde{\Gamma}_{\mathrm{s}}(\tilde{\tau})$ and to obtain the same remainder estimate much as in the above procedure for Ψ. This completes the proof of the main theorem. ■

Acknowledgements. The authors wish to thank one of the referees whose extraordinarily thorough reading and many suggestions led to substantial improvements.

Funding. ME was supported by grant nos. 231668 and 250070 from the Research Council of Norway.

SW was supported by the National Science Foundation through the awards DMS-1514910 and DMS-1812436.

CZ was supported in part by the National Science Foundation through the award DMS-1900083.

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