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Regularity of stable solutions of p-Laplace equations through geometric Sobolev type inequalities

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Abstract. We prove a Sobolev and a Morrey type inequality involving the mean curvature and the tangential gradient with respect to the level sets of the function that appears in the inequalities. Then, as an application, we establish *a priori* estimates for semistable solutions of $-\Delta_p u = g(u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. In particular, we obtain new L^r and $W^{1,r}$ bounds for the extremal solution $u^\star$ when the domain is strictly convex. More precisely, we prove that $u^\star \in L^\infty(\Omega)$ if $n \le p+2$ and $u^\star \in L^{\frac{np}{n-p-2}}(\Omega) \cap W_0^{1,p}(\Omega)$ if $n > p+2$.

Keywords. Geometric inequalities, mean curvature of level sets, Schwarz symmetrization, p-Laplace equations, regularity of stable solutions

1. Introduction

The aim of this paper is to obtain *a priori* estimates for semistable solutions of p-Laplace equations. We will accomplish this by proving some geometric inequalities involving the functionals

$$I_{p,q}(v;\Omega) := \left(\int_\Omega \left\{ \left(\frac{1}{p'}\left|\nabla_{T,v}|\nabla v|^{p/q}\right|\right)^q + |H_v|^q |\nabla v|^p \right\} dx\right)^{1/p}, \qquad p,q \ge 1, \quad (1.1)$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ with $n \ge 2$ and $v \in C_0^\infty(\overline{\Omega})$. Here, and in the rest of the paper, $H_v(x)$ denotes the mean curvature at x of the hypersurface $\{y \in \Omega : |v(y)| = |v(x)|\}$ (which is smooth at points $x \in \Omega$ satisfying $\nabla v(x) \neq 0$), and $\nabla_{T,v}$ is the tangential gradient along a level set of $|v|$. We will prove a Morrey type inequality when $n < p+q$ and a Sobolev type inequality when $n > p+q$ (see Theorem 1.3 below).

Then, as an application of these inequalities, we establish L^r and $W^{1,r}$ *a priori* estimates for semistable solutions of the reaction-diffusion problem

$$\begin{cases} -\Delta_p u = g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

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Here, the diffusion is modeled by the p-Laplace operator Δ_p (recall that $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$) with $p > 1$, while the reaction term is driven by any positive C^1 nonlinearity g.

As we will see, these estimates will lead to new L^r and $W^{1,r}$ bounds for the extremal solution $u^\star$ of (1.2) when $g(u) = \lambda f(u)$ and the domain Ω is strictly convex. More precisely, we prove that $u^\star \in L^\infty(\Omega)$ if $n \le p+2$ and $u^\star \in L^{\frac{np}{n-p-2}}(\Omega) \cap W_0^{1,p}(\Omega)$ if $n > p+2$.

1.1. Geometric Sobolev inequalities

In the last decades symmetrizations and rearrangements have been useful tools to prove Sobolev inequalities as well as to obtain *a priori* estimates for solutions of elliptic and parabolic equations.

Let v be a Lipschitz continuous function in $\overline{\Omega}$ vanishing at the boundary and denote by $|E|$ the n-dimensional Hausdorff measure of any subset E of $\mathbb{R}^n$. The *Schwarz symmetrization* (or *spherically symmetric rearrangement*) v^* of v is a spherically symmetric function, defined in the ball B_R centered at the origin with the same measure as Ω, which is decreasing with respect to $|x|$ and satisfies $|\{x \in \Omega : |v(x)| > t\}| = |\{x \in B_R : v^*(x) > t\}|$ for every $t \in (0, \|v\|_{L^\infty(\Omega)})$ (see Definition 2.1). Since v and v^* are equidistributed, we have

$$\left(\int_{B_R} |v^*|^r \, dx\right)^{1/r} = \left(\int_\Omega |v|^r \, dx\right)^{1/r} \quad \text{for all } r \in [1, \infty]. \tag{1.3}$$

Another important property of this symmetrization is the Pólya–Szegö principle (see [32]). It establishes that the Dirichlet integral decreases under Schwarz symmetrization:

$$\int_{B_R} |\nabla v^*|^r \, dx \le \int_\Omega |\nabla v|^r \, dx \quad \text{for all } r \in [1, \infty). \tag{1.4}$$

See monographs by Bandle [5] and Kawohl [27] for this and other rearrangements and symmetrizations, their main properties, and several applications.

In the seventies, Talenti [34] proved the Pólya–Szegö principle using the isoperimetric inequality and, as a consequence, he obtained the optimal constant in the classical Sobolev inequality. He also obtained in [35] *a priori* estimates for solutions of some nonlinear elliptic equations with the aid of the Fleming–Rishel formula [24]. Since then several authors have used the ideas and techniques behind [34, 35] to prove comparison results for solutions of elliptic and parabolic equations (see for instance [2, 3, 4]) or Hessian equations [8], to obtain the sharp extinction time in the flow by mean curvature [9], etc.

We proceed in the spirit of Talenti's works to prove our geometric inequalities. We first establish that the functional $I_{p,q}$ defined in (1.1) decreases (up to a universal multiplicative constant) under Schwarz symmetrization, i.e., $I_{p,q}(v^*; B_R) \le C I_{p,q}(v; \Omega)$ for some universal constant C depending only on n, p, and q. This property is analogous to the Pólya–Szegö principle (1.4) for the Dirichlet integral.

Theorem 1.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 2$ and B_R the ball centered at the origin and with radius $R = (|\Omega|/|B_1|)^{1/n}$. Let $v \in C_0^\infty(\overline{\Omega})$ and v^* its Schwarz symmetrization. Let $I_{p,q}$ be the functional defined in (1.1) with $p, q \geq 1$. If $n > q+1$ then there exists a universal constant C, depending only on n, p, and q, such that*

$$\left(\int_{B_R} \frac{1}{|x|^q}|\nabla v^*|^p\,dx\right)^{1/p} = I_{p,q}(v^*; B_R) \leq C I_{p,q}(v;\Omega). \tag{1.5}$$

Remark 1.2. (i) In the proof of Theorem 1.1 we obtain the explicit admissible constant $C = A^{q/p}\mathcal{H}_{n-1}(\partial B_1)^{q/((n-1)p)}$ in (1.5), where A is the universal constant appearing in the geometric Sobolev inequality (1.8) below and $\mathcal{H}_{n-1}(\partial B_1)$ denotes the $(n-1)$-dimensional Hausdorff measure of ∂B_1. The best constant A in (1.8) is unknown: this is the reason why we do not obtain $C = 1$ in (1.5) as in the Pólya–Szegö principle. However, Theorem 1.1 will be enough to prove our geometric Sobolev inequalities.

(ii) Note that the Schwarz symmetrization of v is a spherically symmetric function, i.e., its level sets are spheres. In particular, the mean curvature $H_{v^*}(x)$ equals $1/|x|$ and the tangential gradient $\nabla_{T,v^*}|\nabla v^*|^{p/q}$ is 0. This explains the equality in (1.5).

A related result was proved by Trudinger [38] when $q = 1$ for the class of mean convex functions (i.e., functions for which the mean curvature of level sets is nonnegative). More precisely, he proved Theorem 1.1 replacing the functional $I_{p,q}$ by

$$\tilde{I}_{p,q}(v;\Omega) := \left(\int_\Omega |H_v|^q |\nabla v|^p\,dx\right)^{1/p} \tag{1.6}$$

and considering the Schwarz symmetrization of v with respect to the perimeter instead of the classical one like us. In order to define this symmetrization (with respect to the perimeter) it is essential to know that the mean curvature H_v of the level sets of $|v|$ is nonnegative. Then using an Aleksandrov–Fenchel inequality for mean convex hypersurfaces (see [37]) Trudinger proved Theorem 1.1 for this class of functions when $q = 1$.

We prove Theorem 1.1 using two ingredients. The first one is the classical isoperimetric inequality

$$n|B_1|^{1/n}|D|^{(n-1)/n} \leq \mathcal{H}_{n-1}(\partial D) \tag{1.7}$$

for any smooth bounded domain D of $\mathbb{R}^n$. The second one is a geometric Sobolev inequality, due to Michael and Simon [29] and to Allard [1], on compact $(n-1)$-hypersurfaces M without boundary which involves the mean curvature H of M: for every $q \in [1, n-1)$, there exists a constant A depending only on n and q such that

$$\left(\int_M |\phi|^{q^\star}\,d\sigma\right)^{1/q^\star} \leq A\left(\int_M (|\nabla\phi|^q + |H\phi|^q)\,d\sigma\right)^{1/q} \tag{1.8}$$

for every $\phi \in C^\infty(M)$, where $q^\star = (n-1)q/(n-1-q)$ and $d\sigma$ denotes the area element in M. Using the classical isoperimetric inequality (1.7) and the geometric Sobolev inequality (1.8) with $M = \{x \in \Omega : |v(x)| = t\}$ and $\phi = |\nabla v|^{(p-1)/q}$ we will prove Theorem 1.1.

From Theorem 1.1 and well known 1-dimensional weighted Sobolev inequalities [6] it is standard to prove Morrey and Sobolev geometric inequalities involving the functional $I_{p,q}$. Indeed, by Theorem 1.1, and since Schwarz symmetrization preserves the L^r norm, it is sufficient to prove the existence of a positive constant $\overline{C}$ independent of v^* such that

$$\|v^*\|_{L^r(B_R)} \le \overline{C} I_{p,q}(v^*; B_R).$$

Using this argument we prove the following geometric inequalities.

Theorem 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \ge 2$ and $v \in C_0^\infty(\overline{\Omega})$. Let $I_{p,q}$ be the functional defined in (1.1) with $p, q \ge 1$ and*

$$p_q^\star := \frac{np}{n-(p+q)} \quad \textit{for } n > p+q.$$

Assume $n > q+1$. The following assertions hold:

(a) *If $n < p+q$ then*

$$\|v\|_{L^\infty(\Omega)} \le C_1 |\Omega|^{\frac{p+q-n}{np}} I_{p,q}(v; \Omega) \tag{1.9}$$

for some constant C_1 depending only on n, p, and q.

(b) *If $n > p+q$ then*

$$\|v\|_{L^r(\Omega)} \le C_2 |\Omega|^{1/r - 1/p_q^\star} I_{p,q}(v; \Omega) \quad \textit{for every } 1 \le r \le p_q^\star, \tag{1.10}$$

where C_2 is a constant depending only on n, p, q, and r.

(c) *If $n = p+q$ then*

$$\int_\Omega \exp\left\{\left(\frac{|v|}{C_3 I_{p,q}(v; \Omega)}\right)^{p'}\right\} dx \le \frac{n}{n-1}|\Omega|, \quad \textit{where } p' = p/(p-1), \tag{1.11}$$

for some positive constant C_3 depending only on n and p.

Cabré and the second author [15] proved recently Theorem 1.3 under the assumption $q \ge p$ using a different method (without the use of Schwarz symmetrization). More precisely, they proved the theorem replacing the functional $I_{p,q}(v; \Omega)$ by the one defined in (1.6), $\tilde{I}_{p,q}(v; \Omega)$. Therefore, our geometric inequalities are new only in the range $1 \le q < p$.

Open Problem 1. Is Theorem 1.3 true for the range $1 \le q < p$ and with the functional $I_{p,q}(v; \Omega)$ replaced by the one defined in (1.6), $\tilde{I}_{p,q}(v; \Omega)$?

This question has a positive answer for the class of mean convex functions. Trudinger [38] proved this result for this class of functions when $q = 1$ and can be easily extended for every $q \ge 1$. However, to our knowledge, for general functions (without mean convex level sets) it is an open problem.

1.2. Regularity of semistable solutions

The second part of the paper deals with *a priori* estimates for semistable solutions of problem (1.2). Remember that a regular solution $u \in C^1_0(\overline{\Omega})$ of (1.2) is said to be *semistable* if the second variation of the associated energy functional at u is nonnegative definite, i.e.,

$$\int_\Omega \left(|\nabla u|^{p-2} \left\{ |\nabla \phi|^2 + (p-2) \left(\nabla \phi \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right\} - g'(u)\phi^2 \right) dx \geq 0 \tag{1.12}$$

for every $\phi \in H_0$, where H_0 denotes the space of admissible functions (see Definition 4.1 below). The class of semistable solutions includes local minimizers of the energy functional as well as minimal and extremal solutions of (1.2) when $g(u) = \lambda f(u)$.

Using an appropriate test function in (1.12) we prove the following *a priori* estimates for semistable solutions. This result extends the ones in [12] and [15] for the Laplacian case ($p = 2$) due to Cabré and the second author.

Theorem 1.4. *Let g be any C^∞ function and $\Omega \subset \mathbb{R}^n$ any smooth bounded domain. Let $u \in C^1_0(\overline{\Omega})$ be a semistable solution of (1.2), i.e., a solution satisfying (1.12). The following assertions hold:*

(a) *If $n \leq p+2$ then there exists a constant C depending only on n and p such that*

$$\|u\|_{L^\infty(\Omega)} \leq s + \frac{C}{s^{2/p}} |\Omega|^{\frac{p+2-n}{np}} \left(\int_{\{u \leq s\}} |\nabla u|^{p+2}\, dx \right)^{1/p} \quad \text{for all } s > 0. \tag{1.13}$$

(b) *If $n > p+2$ then there exists a constant C depending only on n and p such that*

$$\left(\int_{\{u>s\}} (u-s)^{\frac{np}{n-(p+2)}}\, dx \right)^{\frac{n-(p+2)}{np}} \leq \frac{C}{s^{2/p}} \left(\int_{\{u \leq s\}} |\nabla u|^{p+2}\, dx \right)^{1/p} \tag{1.14}$$

for all $s > 0$. Moreover, there exists a constant C depending only on n, p, and r such that

$$\int_\Omega |\nabla u|^r\, dx \leq C \left(|\Omega| + \int_\Omega |u|^{\frac{np}{n-(p+2)}}\, dx + \|g(u)\|_{L^1(\Omega)} \right) \tag{1.15}$$

for all $1 \leq r < r_1 := \frac{np^2}{(1+p)n-p-2}$.

To prove (1.13) and (1.14) we use the semistability condition (1.12) with the test function $\phi = |\nabla u|\eta$ to obtain

$$\int_\Omega \left(\frac{4}{p^2} |\nabla_{T,u} |\nabla u|^{p/2}|^2 + \frac{n-1}{p-1} H_u^2 |\nabla u|^p \right) \eta^2\, dx \leq \int_\Omega |\nabla u|^p |\nabla \eta|^2\, dx \tag{1.16}$$

for every Lipschitz function η in $\overline{\Omega}$ with $\eta|_{\partial\Omega} = 0$. Then, taking $\eta = T_s u = \min\{s, u\}$, we obtain (1.13) and (1.14) when $n \neq p+2$ by using the Morrey and Sobolev inequalities established in Theorem 1.3 with $q = 2$. The critical case $n = p+2$ is more involved. In order to get (1.13) in this case, we take another explicit test function $\eta = \eta(u)$ in (1.16) and use the geometric Sobolev inequality (1.8). The gradient estimate established in (1.15) will follow by using a technique introduced by Bénilan et al. [7] to get the regularity of entropy solutions for p-Laplace equations with L^1 data (see Proposition 4.2).

The rest of the introduction deals with the regularity of extremal solutions. Let us recall the problem and some known results in this topic. Consider

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.17$_\lambda$}$$

where λ is a positive parameter and f is a C^1 positive increasing function satisfying

$$\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = \infty. \tag{1.18}$$

For the Laplacian case ($p = 2$), Crandall and Rabinowitz [17] proved, using the Implicit Function Theorem, the existence of an extremal parameter $\lambda^\star \in (0, \infty)$ such that problem $(1.17)_\lambda$ admits a minimal classical solution $u_\lambda \in C_0^2(\overline{\Omega})$ for $\lambda \in (0, \lambda^\star)$ and admits no classical solution for $\lambda > \lambda^\star$. Brezis et al. [10] proved that the limit of minimal solutions

$$u^\star := \lim_{\lambda\uparrow\lambda^\star} u_\lambda$$

is a weak solution of $(1.17)_{\lambda^\star}$. This solution is known as the *extremal solution* of $(1.17)_\lambda$. Moreover, they proved that problem $(1.17)_\lambda$ admits no weak solution for $\lambda > \lambda^\star$. Brezis and Vázquez [11] gave (among other results) a characterization for singular extremal solutions lying in the energy class $H_0^1(\Omega)$ when the nonlinearity f is convex and raised the question of determining the regularity of the extremal solution $u^\star$ depending on the dimension n of the domain. After this important work the study of the regularity of $u^\star$ started to increase dramatically. However, there are few results for general reaction terms f and general domains Ω. Nedev [30, 31] proved, in the case of convex nonlinearities, that $u^\star \in L^\infty(\Omega)$ if $n \le 3$, $u^\star \in L^r(\Omega)$ for all $1 \le r < n/(n-4)$ if $n \ge 4$, and $u^\star \in H_0^1(\Omega)$ independently of the dimension n of Ω if in addition Ω is convex. Recently, Cabré [12] and Cabré and the second author [15] proved, in the case of convex domains and general nonlinearities, that $u^\star \in L^\infty(\Omega)$ if $n \le 4$ and $u^\star \in L^{2n/(n-4)}(\Omega)$ if $n \ge 5$. For more results and bibliography in this topic for $p = 2$ see the recent monograph by Dupaigne [20].

Many of the above results for the Laplacian case have been extended to problem $(1.17)_\lambda$ for arbitrary $p > 1$. García-Azorero, Peral, and Puel [25, 26] considered the exponential nonlinearity $f(u) = e^u$. They proved the existence of minimal solutions $u_\lambda \in C_0^1(\overline{\Omega})$ for $\lambda \in (0, \lambda^\star)$, and that problem $(1.17)_\lambda$ admits no regular solution for $\lambda > \lambda^\star$, using a monotone iteration method instead of the Implicit Function Theorem as in [17]. Cabré and the second author [14] extended this result to positive increasing nonlinearities f satisfying (1.18). Moreover, they proved that every minimal solution u_λ is semistable for $\lambda \in (0, \lambda^\star)$. In [25, 26] and [14] it is proved, for exponential and power-type nonlinearities, that the limit of minimal solutions $u^\star$ is a solution lying in the energy class $W_0^{1,p}(\Omega)$ independently of the dimension n and $p > 1$.

For $p \ne 2$, and without additional assumptions on f, it is unknown if $u^\star$ is a (weak or entropy) solution of $(1.17)_{\lambda^\star}$. In the affirmative case, it is called the *extremal solution* *of* $(1.17)_{\lambda^\star}$. However, in [33] it is proved that $u^\star$ is a weak solution (in the distributional sense) of $(1.17)_{\lambda^\star}$ whenever $p \geq 2$ and f satisfies the additional condition:

$$\text{there exists } T \geq 0 \text{ such that } (f(t) - f(0))^{1/(p-1)} \text{ is convex for all } t \geq T. \tag{1.19}$$

Moreover,

$$u^\star \in L^\infty(\Omega) \quad \text{if } n < p + p'$$

and

$$u^\star \in L^r(\Omega) \quad \text{for all } r < \tilde{r}_0 := (p-1)\frac{n}{n-(p+p')} \quad \text{if } n \geq p + p'.$$

This extends previous results of Nedev [30] for the Laplacian case ($p = 2$) and convex nonlinearities.

Our next result improves the L^q estimate in [30, 33] for strictly convex domains. We also prove that $u^\star$ belongs to the energy class $W_0^{1,p}(\Omega)$ independently of the dimension extending an unpublished result of Nedev [31] for $p = 2$ to every $p \geq 2$ (see also [15]).

Theorem 1.5. *Let f be an increasing positive C^1 function satisfying* (1.18). *Assume that Ω is a smooth bounded and strictly convex domain of $\mathbb{R}^n$. Let $u_\lambda \in C_0^1(\overline{\Omega})$ be the minimal solution of* $(1.17)_\lambda$. *There exists a constant C independent of λ such that:*

(a) *If $n \leq p + 2$ then $\|u_\lambda\|_{L^\infty(\Omega)} \leq C\|f(u_\lambda)\|_{L^1(\Omega)}^{1/(p-1)}$.*

(b) *If $n > p + 2$ then $\|u_\lambda\|_{L^{\frac{np}{n-p-2}}(\Omega)} \leq C\|f(u_\lambda)\|_{L^1(\Omega)}^{1/(p-1)}$. Moreover $\|u_\lambda\|_{W_0^{1,p}(\Omega)} \leq C'$ where C' is a constant depending only on n, p, Ω, f, and $\|f(u_\lambda)\|_{L^1(\Omega)}$.*

Assume, in addition, that $p \geq 2$ and that (1.19) *holds. Then*

(i) *If $n \leq p + 2$ then $u^\star \in L^\infty(\Omega)$. In particular, $u^\star \in C_0^1(\overline{\Omega})$.*

(ii) *If $n > p + 2$ then $u^\star \in L^{\frac{np}{n-p-2}}(\Omega) \cap W_0^{1,p}(\Omega)$.*

Remark 1.6. If $f(u_\lambda)$ is bounded in $L^1(\Omega)$ by a constant independent of λ, then parts (a) and (b) will lead automatically to assertions (i) and (ii) (without the requirement that $p \geq 2$ and (1.19) hold true). However, as we said before, the estimate $f(u^\star) \in L^1(\Omega)$ is unknown in the general case, i.e., for arbitrary positive and increasing nonlinearities f satisfying (1.18) and arbitrary $p > 1$.

Open Problem 2. Is it true that $f(u^\star) \in L^1(\Omega)$ for arbitrary positive and increasing nonlinearities f satisfying (1.18)?

Under the assumptions $p \geq 2$ and (1.19) it is proved in [33] that $f(u^\star) \in L^r(\Omega)$ for all $1 \leq r < n/(n-p')$ when $n \geq p'$ and $f(u^\star) \in L^\infty(\Omega)$ if $n < p'$. In particular, one has $f(u^\star) \in L^1(\Omega)$ independently of the dimension n and of the parameter $p > 1$. As a consequence, assertions (i) and (ii) follow immediately from parts (a) and (b) of the theorem.

The proof of the L^r *a priori* estimates stated in (a) and (b) is accomplished in three steps. First, we use the strict convexity of the domain Ω to prove that

$$\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\} \subset \{x \in \Omega : u_\lambda(x) < s\}$$

for a suitable s. This is done using a moving plane procedure for p-Laplace equations (see Proposition 3.1 below). Then, we prove that the Morrey and Sobolev type inequalities, stated in Theorem 1.3 for smooth functions, also hold for regular solutions of (1.2) when $1 \le q \le 2$. Finally, taking a test function η related to $\operatorname{dist}(\cdot, \partial\Omega)$ in (1.16) and proceeding as in the proof of Theorem 1.4 we will obtain the L^r *a priori* estimates established in the theorem.

The energy estimate established in parts (b) and (ii) of Theorem 1.5 follows by extending the arguments of Nedev [31] for the Laplacian case (see also [15, Theorem 2.9]). First, using a Pokhozhaev identity we obtain

$$\int_\Omega |\nabla u_\lambda|^p\, dx \le \frac{1}{p'} \int_{\partial\Omega} |\nabla u_\lambda|^p\, x \cdot \nu\, d\sigma \quad \text{for all } p > 1 \text{ and } \lambda \in (0, \lambda^\star), \tag{1.20}$$

where $d\sigma$ denotes the area element in $\partial\Omega$ and ν is the outward unit normal to Ω. Then, using the strict convexity of the domain (as in the L^r estimates) and standard regularity estimates for $-\Delta_p u = \lambda f(u_\lambda(x))$ in a neighborhood of the boundary, we are able to control the right hand side of (1.20) by a constant whose dependence on λ is given by a function of $\|f(u_\lambda)\|_{L^1(\Omega)}$.

Remark 1.7. Let us compare our regularity results with the sharp ones proved by Cabré, Capella, and the second author in [13] when Ω is the unit ball B_1 of $\mathbb{R}^n$. In the radial case, the extremal solution $u^\star$ of $(1.17)_{\lambda^\star}$ is bounded if the dimension $n < p + 4p/(p-1)$. Moreover, if $n \ge p + 4p/(p-1)$ then $u^\star \in W_0^{1,r}(B_1)$ for all $1 \le r < \bar{r}_1$, where

$$\bar{r}_1 := \frac{np}{n - 2\sqrt{\frac{n-1}{p-1}} - 2}.$$

In particular, $u^\star \in L^r(B_1)$ for all $1 \le r < \bar{r}_0$, where

$$\bar{r}_0 := \frac{np}{n - 2\sqrt{\frac{n-1}{p-1}} - p - 2}.$$

It can be shown that these regularity results are sharp by taking the exponential and power nonlinearities (see [14, 25, 26]).

Note that the $L^r(\Omega)$-estimate established in Theorem 1.5 differs with the sharp exponent $\bar{r}_0$ defined above by the term $2\sqrt{(n-1)/(p-1)}$. Moreover, observe that $\bar{r}_1$ is larger than p and tends to it as n goes to infinity. In particular, the best expected regularity independent of the dimension n for the extremal solution $u^\star$ is $W_0^{1,p}(\Omega)$, which is the one we obtain in Theorem 1.5.

1.3. Outline of the paper

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and the geometric type inequalities stated in Theorem 1.3. In Section 3 we prove that Theorem 1.3 holds for solutions of (1.2) when $1 \le q \le 2$. Moreover we establish boundary estimates when the domain is strictly convex. In Section 4, we present the semistability condition (1.12) and the space of admissible functions H_0. The rest of the section deals with the regularity of semistable solutions, proving Theorems 1.4 and 1.5.

2. Geometric Hardy–Sobolev type inequalities

In this section we prove Theorems 1.1 and 1.3. As we said in the introduction, the geometric inequalities established in Theorem 1.3 are new for the range $1 \le q < p$ since the case $q \ge p$ was proved in [15]. However, we will give the proof in all cases using Schwarz symmetrization, giving an alternative proof for the known range of parameters $q \ge p$.

We start by recalling the definition of Schwarz symmetrization of a compact set and of a Lipschitz continuous function.

Definition 2.1. We define the *Schwarz symmetrization of a compact set* $D \subset \mathbb{R}^n$ as

$$D^* := \begin{cases} B_R(0) \text{ with } R = (|D|/|B_1|)^{1/n} & \text{if } D \neq \emptyset, \\ \emptyset & \text{if } D = \emptyset. \end{cases}$$

Let v be a Lipschitz continuous function in $\overline{\Omega}$ vanishing at the boundary $\partial\Omega$, and set $\Omega_t := \{x \in \Omega : |v(x)| > t\}$. We define the *Schwarz symmetrization of* v as

$$v^*(x) := \sup\{t \in \mathbb{R} : x \in \Omega_t^*\}.$$

Equivalently, we can define the Schwarz symmetrization of v as

$$v^*(x) = \inf\{t \ge 0 : V(t) < |B_1|\,|x|^n\},$$

where $V(t) := |\Omega_t| = |\{x \in \Omega : |v(x)| > t\}|$ denotes the distribution function of v.

The Schwarz symmetrization v^* of v is a spherically symmetric and Lipschitz continuous function (see [27]). Moreover, it is equidistributed with v, i.e.,

$$|\{x \in \Omega : |v(x)| > t\}| = |\{x \in B_R : v^*(x) > t\}| \quad \text{for every } t \in (0, \|v\|_{L^\infty(\Omega)}).$$

As a consequence, the L^r norm of v is preserved under this symmetrization for all $r \in [1, \infty]$ (i.e., identity (1.3) holds).

Talenti [34] proved the Pólya–Szegö principle (1.4) using the isoperimetric inequality. From these properties, the classical Sobolev inequality reduces to the existence of a constant C depending only on n and q such that

$$\left(\int_{B_R} |v^*|^{\frac{nq}{n-q}}\, dx\right)^{\frac{n-q}{nq}} \le C\left(\int_{B_R} |\nabla v^*|^q\, dx\right)^{1/q}$$

when $n > q$. At that point, noting that v^* is a spherically symmetric function, the result follows from the Bliss inequalities [6].

To prove our Theorems 1.1 and 1.3 we proceed in the same way. The first ingredient in the proof of Theorem 1.1 is the isoperimetric inequality (1.7). More precisely, let $v \in C_0^\infty(\overline{\Omega})$ and note that $\partial\{x \in \Omega : |v(x)| > t\} = \{x \in \Omega : |v(x)| = t\}$ is a C^∞ immersed $(n-1)$-dimensional compact hypersurface of $\mathbb{R}^n$ without boundary for every regular value

$t \in (0, \|v\|_{L^\infty(\Omega)})$. Since, by Sard's theorem almost every $t \in (0, \|v\|_{L^\infty(\Omega)})$ is a regular value of $|v|$, we can apply (1.7) to $\{x \in \Omega : |v(x)| > t\}$ to obtain

$$n|B_1|^{1/n} V(t)^{(n-1)/n} \le P(t) := \mathcal{H}_{n-1}(\{x \in \Omega : |v(x)| = t\}) \quad \text{for a.e. } t > 0. \tag{2.1}$$

The second ingredient is the following Sobolev inequality on compact hypersurfaces without boundary due to Michael and Simon [29] and to Allard [1].

Theorem 2.2 ([1, 29]). *Let $M \subset \mathbb{R}^n$ be a C^∞ immersed $(n-1)$-dimensional compact hypersurface without boundary and $\phi \in C^\infty(M)$. If $q \in [1, n-1)$, then there exists a constant A depending only on n and q such that*

$$\left(\int_M |\phi|^{q^\star} d\sigma\right)^{1/q^\star} \le A\left(\int_M (|\nabla\phi|^q + |H\phi|^q)\, d\sigma\right)^{1/q}, \tag{2.2}$$

where H is the mean curvature of M, $d\sigma$ denotes the area element in M, and $q^\star = \frac{(n-1)q}{n-1-q}$.

Let us prove that Schwarz symmetrization reduces (up to a multiplicative constant) the functional $I_{p,q}$ defined in (1.1) using the isoperimetric inequality (2.1) and the geometric inequality (2.2) applied to $M = M_t = \{x \in \Omega : |v(x)| = t\}$ and $\phi = |\nabla v|^{(p-1)/q}$.

Proof of Theorem 1.1. Let $v \in C_0^\infty(\overline{\Omega})$, $p \ge 1$, and $1 \le q < n-1$. Applying inequality (2.2) to $M = M_t = \{x \in \Omega : |v(x)| = t\}$ and $\phi = |\nabla v|^{(p-1)/q}$ we obtain

$$\left(\int_{M_t} |\nabla v|^{(p-1)q^\star/q}\, d\sigma\right)^{q/q^\star} \le A^q \int_{M_t} \left(\left|\nabla_{T,v} |\nabla v|^{(p-1)/q}\right|^q + |H_v|^q |\nabla v|^{p-1}\right) d\sigma \tag{2.3}$$

for a.e. $t \in (0, \|v\|_{L^\infty(\Omega)})$, where H_v denotes the mean curvature of M_t, $d\sigma$ is the area element in M_t, A is the constant in (2.2) which depends only on n and q, and

$$q^\star := \frac{(n-1)q}{n-1-q}.$$

Recall that the distribution function $V(t)$, being a nonincreasing function, is differentiable almost everywhere and, thanks to the coarea formula (see for instance [21]) and the fact that almost every $t \in (0, \|v\|_{L^\infty(\Omega)})$ is a regular value of $|v|$, we have

$$-V'(t) = \int_{M_t} \frac{1}{|\nabla v|}\, d\sigma \quad \text{and} \quad P(t) = \int_{M_t} d\sigma \quad \text{for a.e. } t \in (0, \|v\|_{L^\infty(\Omega)}).$$

Therefore applying the Jensen inequality and then using the isoperimetric inequality (2.1), we obtain

$$\left(\int_{M_t} |\nabla v|^{(p-1)q^\star/q+1} \frac{d\sigma}{|\nabla v|}\right)^{q/q^\star} \ge \frac{P(t)^{p-1+q/q^\star}}{(-V'(t))^{p-1}} \ge \frac{(A_1 V(t)^{(n-1)/n})^{p-1+q/q^\star}}{(-V'(t))^{p-1}} \tag{2.4}$$

for a.e. $t \in (0, \|v\|_{L^\infty(\Omega)})$, where $A_1 := n|B_1|^{1/n}$. Integrating the previous inequality with respect to t in $(0, \|v\|_{L^\infty(\Omega)})$ and using (2.3) and the coarea formula, we obtain

$$\int_0^{\|v\|_{L^\infty(\Omega)}} \frac{(A_1 V(t)^{(n-1)/n})^{p-1+q/q^\star}}{(-V'(t))^{p-1}}\,dt \\ \le A^q \int_\Omega \left\{ \left(\frac{1}{p'} \left| \nabla_{T,v} |\nabla v|^{p/q} \right| \right)^q + |H_v|^q |\nabla v|^p \right\} dx. \qquad (2.5)$$

At this point, we proceed as in [34, proof of Lemma 1]. Since the Schwarz symmetrization v^* of v is spherically symmetric, is equidistributed with v, and is Lipschitz continuous (see for instance [34]), the inequalities in (2.4) are equalities when we replace v by v^*, i.e.,

$$\int_0^{\|v\|_{L^\infty(\Omega)}} \frac{(A_1 V(t)^{n-1/n})^{p-1+q/q^\star}}{(-V'(t))^{p-1}}\,dt \\ = \int_0^{\|v\|_{L^\infty(\Omega)}} \left(\int_{\{v^*=t\}} |\nabla v^*|^{(p-1)q^\star/q}\,d\sigma \right)^{q/q^\star} dt \\ = \mathcal{H}_{n-1}(\partial B_1)^{-q/(n-1)} \int_0^{\|v\|_{L^\infty(\Omega)}} \int_{\{v^*=t\}} |H_{v^*}|^q |\nabla v^*|^{p-1}\,d\sigma\,dt.$$

Combining the previous identity with inequality (2.5) and using the coarea formula, we obtain inequality (1.5) with the explicit constant $C = A^{q/p}\mathcal{H}_{n-1}(\partial B_1)^{q/((n-1)p)}$. □

Now, to prove Theorem 1.3 we proceed as in the proof of the classical Sobolev inequality. From (1.3) and Theorem 1.1 it will be enough to prove the result for spherically symmetric functions v^* in a ball B_R. Then using the Bliss inequalities [6] we conclude the proof. While the method is standard, we have decided to include the proof for convenience to the reader.

Proof of Theorem 1.3. Let $v \in C_0^\infty(\overline{\Omega})$ and v^* its Schwarz symmetrization. Recall that v^* is defined in B_R with $R = (|\Omega|/|B_1|)^{1/n}$.

(a) Assume $1 + q < n < p + q$. Using the Hölder inequality we obtain

$$v^*(s) = \int_s^R |(v^*)'(\tau)|\,d\tau \\ \le \left(\int_0^R |(v^*)'(\tau)|^p \tau^{-q} \tau^{n-1}\,d\tau \right)^{1/p} \left(\int_s^R \tau^{\frac{1+q-n}{p-1}}\,d\tau \right)^{1/p'} \qquad (2.6)$$

for a.e. $s \in (0, R)$. In particular,

$$v^*(s) \le \mathcal{H}_{n-1}(\partial B_1)^{-1/p} \left(\frac{p-1}{p+q-n} \right)^{1/p'} \left(\frac{|\Omega|}{|B_1|} \right)^{\frac{p+q-n}{np}} I_{p,q}(v^*; B_R)$$

for a.e. $s \in (0, R)$. We conclude this case, by Theorem 1.1, noting that $\|v\|_{L^\infty(\Omega)} = v^*(0)$.

(b) Assume $n > p + q$. We use the following 1-dimensional weighted Sobolev inequality:

$$\left(\int_0^R |\varphi(s)|^{p_q^\star} s^{n-1}\,ds\right)^{1/p_q^\star} \le C(n,p,q)\left(\int_0^R s^{-q}|\varphi'(s)|^p s^{n-1}\,ds\right)^{1/p} \tag{2.7}$$

with optimal constant

$$C(n,p,q) := \left(\frac{p-1}{n-(p+q)}\right)^{1/p'} n^{-1/p_q^\star}\left[\frac{\Gamma\left(\frac{np}{p+q}\right)}{\Gamma\left(\frac{n}{p+q}\right)\Gamma\left(1+\frac{n(p-1)}{p+q}\right)}\right]^{\frac{p+q}{np}} \tag{2.8}$$

proved by Bliss [6] in 1930 (see also [38]). Applying inequality (2.7) to $\varphi = v^*$, we obtain

$$\begin{aligned}\mathcal{H}_{n-1}(\partial B_1)^{-1/p_q^\star}&\left(\int_\Omega |v|^{p_q^\star}\,dx\right)^{1/p_q^\star}\\ &\le C(n,p,q)\mathcal{H}_{n-1}(\partial B_1)^{-1/p}\left(\int_{B_R}|x|^{-q}|\nabla v^*|^p\,dx\right)^{1/p}.\end{aligned}$$

Using Theorem 1.1 again and noting that the $L^{p_q^\star}$ norm is preserved under Schwarz symmetrization, we prove (1.10) for $r = p_q^\star$. The remaining cases, $1 \le r < p_q^\star$, now follow easily from the Hölder inequality.

(c) Assume $n = p + q$. From (2.6) and Theorem 1.1 we obtain

$$\begin{aligned}v^*(s) &\le \left(\int_0^R |(v^*)'(\tau)|^p \tau^{-q}\tau^{n-1}\,d\tau\right)^{1/p}\left(\int_s^R \tau^{-1}\,d\tau\right)^{1/p'}\\ &\le \mathcal{H}_{n-1}(\partial B_1)^{-1/p} C I_{p,q}(v;\Omega)\left(\ln\left(\frac{R}{s}\right)\right)^{1/p'}\end{aligned}$$

for a.e. $s \in (0,R)$. Equivalently

$$\exp\left\{\left(\frac{v^*(s)}{\mathcal{H}_{n-1}(\partial B_1)^{-1/p} C I_{p,q}(v;\Omega)}\right)^{p'}\right\}\mathcal{H}_{n-1}(\partial B_1)s^{n-1} \le \frac{R}{s}\mathcal{H}_{n-1}(\partial B_1)s^{n-1}$$

for a.e. $s \in (0,R)$. Integrating the previous inequality with respect to s in $(0,R)$ we obtain

$$\int_{B_R}\exp\left\{\left(\frac{v^*}{\mathcal{H}_{n-1}(\partial B_1)^{-1/p} C I_{p,q}(v;\Omega)}\right)^{p'}\right\}dx \le \mathcal{H}_{n-1}(\partial B_1)\frac{R^n}{n-1} = \frac{n}{n-1}|\Omega|.$$

We conclude the proof noting that the integral in inequality (1.11) is preserved under Schwarz symmetrization. □

Remark 2.3. Note that we have obtained explicit admissible constants C_1, C_2, and C_3 in inequalities of Theorem 1.3. More precisely, we obtained

$$
\begin{aligned}
C_1 &= \mathcal{H}_{n-1}(\partial B_1)^{-1/p}\left(\frac{p-1}{p+q-n}\right)^{1/p'}\left(\frac{|\Omega|}{|B_1|}\right)^{\frac{p+q-n}{np}} A^{q/p}\mathcal{H}_{n-1}(\partial B_1)^{\frac{q}{(n-1)p}},\\
C_2 &= C(n,p,q)\mathcal{H}_{n-1}(\partial B_1)^{1/p_q^\star-1/p}A^{q/p}\mathcal{H}_{n-1}(\partial B_1)^{\frac{q}{(n-1)p}},\\
C_3 &= \mathcal{H}_{n-1}(\partial B_1)^{-1/p}A^{n-p/p}\mathcal{H}_{n-1}(\partial B_1)^{\frac{n-p}{(n-1)p}},
\end{aligned}
$$

where A is the universal constant appearing in (2.2) and $C(n,p,q)$ is defined in (2.8).

All the constants C_i depend only on n, p, and q. However, the best constant A in (2.2) is unknown (even for mean convex hypersurfaces). Behind the Sobolev inequality (2.2) there is the following geometric isoperimetric inequality:

$$
|M|^{\frac{n-2}{n-1}} \le A_2 \int_M |H(x)|\,d\sigma. \tag{2.9}
$$

Here, $M \subset \mathbb{R}^n$ is a C^∞ immersed $(n-1)$-dimensional compact hypersurface without boundary and H is the mean curvature of M as in Theorem 2.2. The best constant in (2.9) is also unknown, even for mean convex hypersurfaces.

3. Properties of solutions of p-Laplace equations

In this section, we first establish an *a priori* L^∞ estimate in a neighborhood of the boundary $\partial\Omega$ for any regular solution u of (1.2) when the domain Ω is strictly convex. More precisely, we prove that there exist positive constants ε and γ, depending only on the domain Ω, such that

$$
\|u\|_{L^\infty(\Omega_\varepsilon)} \le \frac{1}{\gamma}\|u\|_{L^1(\Omega)}, \quad \text{where} \quad \Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \varepsilon\}. \tag{3.1}
$$

Then we establish that the geometric inequalities of Theorem 1.3 still hold for solutions of (1.2) in the smaller range $1 \le q \le 2$. In the next section, these two ingredients will allow us to obtain *a priori* estimates for semistable solutions.

Let $u \in W_0^{1,p}(\Omega)$ be a weak solution (i.e., a solution in the distributional sense) of the problem

$$
\begin{cases}
-\Delta_p u = g(u) & \text{in } \Omega,\\
u > 0 & \text{in } \Omega,\\
u = 0 & \text{on } \partial\Omega,
\end{cases} \tag{3.2}
$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, with $n \ge 2$, and g is any positive smooth nonlinearity.

We say that u is a *regular solution* of (3.2) if it satisfies the equation in the distributional sense and $g(u) \in L^\infty(\Omega)$. By well known regularity results for degenerate elliptic equations, every regular solution u belongs to $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1]$ (see

[18, 36]). Moreover, $u \in C^1(\overline{\Omega})$ (see [28]). This is the best regularity that one can expect for solutions of p-Laplace equations. Therefore, equation (3.2) is always meant in the distributional sense.

We prove the boundary *a priori* estimate (3.1) through a moving plane procedure for the p-Laplacian which is developed in [19].

Proposition 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ and g any positive smooth function. Let u be any positive regular solution of (3.2). If Ω is strictly convex, then there exist positive constants ε and γ depending only on the domain Ω such that for every $x \in \Omega$ with* $\operatorname{dist}(x, \partial\Omega) < \varepsilon$, *there exists a set $I_x \subset \Omega$ with the following properties:*

$$|I_x| \geq \gamma \quad \text{and} \quad u(x) \leq u(y) \quad \text{for all } y \in I_x.$$

As a consequence,

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{1}{\gamma}\|u\|_{L^1(\Omega)}, \quad \text{where} \quad \Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}. \tag{3.3}$$

Proof. First let us observe that by the regularity of the solution u up to the boundary $\partial\Omega$ and the fact that $\Delta_p u \leq 0$, we can apply the generalized Hopf boundary lemma [39] to deduce that the normal derivative $\frac{\partial u}{\partial \nu}$ is negative on $\partial\Omega$. Thus, if we let $Z_u := \{x \in \Omega : \nabla u(x) = 0\}$ be the critical set of u, then $Z_u \cap \partial\Omega = \emptyset$. By the compactness of both sets, there exists $\varepsilon_0 > 0$ such that $Z_u \cap \Omega_\varepsilon = \emptyset$ for any $\varepsilon \leq \varepsilon_0$.

We will now prove that this neighborhood of the boundary is in fact independent of the solution u. In order to begin a moving plane argument, we need some notation. Let $e \in S^{n-1}$ be any direction and for $\lambda \in \mathbb{R}$ let us consider the hyperplane

$$T = T_{\lambda,e} = \{x \in \mathbb{R}^n : x \cdot e = \lambda\}$$

and the corresponding cap

$$\Sigma = \Sigma_{\lambda,e} = \{x \in \Omega : x \cdot e < \lambda\}.$$

Set

$$a(e) = \inf_{x \in \Omega} x \cdot e$$

and, for any $x \in \Omega$, let $x' = x_{\lambda,e}$ be its reflection with respect to the hyperplane T, i.e.,

$$x' = x + 2(\lambda - x \cdot e)e.$$

For any $\lambda > a(e)$ the cap

$$\Sigma' = \{x \in \Omega : x' \in \Sigma\}$$

is the (nonempty) reflected cap of Σ with respect to T.

Furthermore, consider the function $v(x) = u(x') = u(x_{\lambda,e})$, which is just the reflection of u with respect to the same hyperplane. By the boundedness of Ω, for $\lambda - a(e)$ small, the corresponding reflected cap Σ' is contained in Ω. Moreover, by the strict convexity of Ω, there exists $\lambda_0 = \lambda_0(\Omega)$ (independent of e) such that Σ' remains in Ω for any $\lambda \leq \lambda_0$.

Let us now compare u and its reflection v for such values of λ in the cap Σ. First of all, both functions solve the same equation since Δ_p is invariant under reflection; secondly, on the hyperplane T the functions coincide, whereas for any $x \in \partial\Sigma \cap \partial\Omega$ we have $u(x) = 0$ and $v(x) = u(x') > 0$, since $x' \in \Omega$. Hence

$$\Delta_p(u) + f(u) = \Delta_p(v) + f(v) \quad \text{in } \Sigma, \qquad u \le v \quad \text{on } \partial\Sigma.$$

Again by the boundedness of Ω, if $\lambda - a(e)$ is small, the measure of the cap Σ will be small. Therefore, from the Comparison Principle in small domains (see [19]) we have $u \le v$ in Σ. Moreover, by the Strong Comparison Principle and Hopf Lemma, we see that $u \le v$ in $\Sigma_{\lambda,e}$ for any $a(e) < \lambda \le \lambda_0$. In particular, this spells that $u(x)$ is nondecreasing in the e direction for all $x \in \Sigma$.

Now, fix $x_0 \in \partial\Omega$ and let $e = \nu(x_0)$ be the outward unit normal to $\partial\Omega$ at x_0. By the convexity assumption, $T_{a(\nu(x_0)),\nu(x_0)} \cap \partial\Omega = \{x_0\}$. If we let $\theta \in S^{n-1}$ be another direction close to the outer normal $\nu(x_0)$, the reflection of the cap $\Sigma_{\lambda,\theta}$ with respect to the hyperplane $T_{\lambda,\theta}$ (which is close to the tangent one) would still be contained in Ω thanks to its strict convexity. So the above argument could also be applied to the new direction θ. In particular, we see that we can get a neighborhood Θ of $\nu(x_0)$ in S^{n-1} such that $u(x)$ is nondecreasing in every direction $\theta \in \Theta$ and for any x such that $x \cdot \theta < \lambda_0/2$.

By eventually taking a smaller neighborhood Θ, we may assume that

$$|x \cdot (\theta - \nu(x_0))| < \lambda_0/8$$

for any $x \in \Sigma_{\lambda_0,\theta}$ and $\theta \in \Theta$. Moreover, noticing that

$$x \cdot \theta = x \cdot (\theta - \nu(x_0)) + x \cdot \nu(x_0)$$

and

$$\lambda_0/2 = \lambda_0/8 + 3\lambda_0/8 > x \cdot \theta > \lambda_0/8 - \lambda_0/8 = 0,$$

it is then easy to see that u is nondecreasing in any direction $\theta \in \Theta$ on $\Sigma_0 = \{x \in \Omega : \lambda_0/8 < x \cdot \nu(x_0) < 3\lambda_0/8\}$.

Finally, let us choose $\varepsilon = \lambda_0/8$. Fix any point $x \in \Omega_\varepsilon$ and let x_0 be its projection onto $\partial\Omega$. From the above arguments we see that

$$u(x) \le u(x_0 - \varepsilon\nu(x_0)) \le u(y)$$

for any $y \in I_x$, where $I_x \subset \Sigma_0$ is a truncated cone with vertex at x_0, opening angle Θ, and height $\lambda_0/4$. Hence, there exists a positive constant $\gamma = \gamma(\Omega, \varepsilon)$ such that $|I_x| \ge \gamma$ and $u(x) \le u(y)$ for any $y \in I_x$. Finally, choosing x_ε as the maximum point of u in Ω_ε, we get

$$\|u\|_{L^\infty(\Omega_\varepsilon)} = u_\varepsilon(x_\varepsilon) \le \frac{1}{\gamma}\int_{I_{x_\varepsilon}} u(y)\,dy \le \frac{1}{\gamma}\|u\|_{L^1(\Omega)},$$

which proves (3.3). □

We will now prove that the inequalities in Theorem 1.3 are also valid for a positive solution u of (3.2) in the smaller range $1 \le q \le 2$. To do this, we will construct an approximation of u by smooth functions and see that, thanks to strong uniform estimates on this approximation, we can pass to the limit in all of the inequalities.

Proposition 3.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ and g any positive smooth function. Let u be any positive regular solution of (3.2). If $1 \le q \le 2$, then inequalities in Theorem 1.3 hold for $v = u$. Given $s > 0$, the same holds true also for $v = u - s$ and Ω replaced by $\Omega_s := \{x \in \Omega : u > s\}$.*

Proof. Let $Z_u = \{x \in \Omega : \nabla u(x) = 0\}$. Recall that by standard elliptic regularity $u \in C^\infty(\Omega \setminus Z_u)$ and that $|Z_u| = 0$ by [19]. Therefore, u is smooth almost everywhere in Ω. Let $x \in \Omega \setminus Z_u$ and observe that for the mean curvature H_u of the level set passing through x we have the explicit expression

$$-(n-1)H_u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{\Delta u}{|\nabla u|} - \frac{\langle D^2 u \nabla u, \nabla u\rangle}{|\nabla u|^3}, \tag{3.4}$$

whereas for the tangential gradient term we have

$$\nabla_{T,u}|\nabla u| = \frac{D^2 u \nabla u}{|\nabla u|} - \frac{\langle D^2 u \nabla u, \nabla u\rangle \nabla u}{|\nabla u|^3}, \tag{3.5}$$

where all the terms in these expressions are evaluated at x. Hence, there exists a positive constant $C = C(n, p, q)$ such that

$$\left(\frac{1}{p'}\left|\nabla_{T,u}|\nabla u|^{p/q}\right|\right)^q + |H_u|^q |\nabla u|^p \le C|D^2 u|^q |\nabla u|^{p-q} \quad \text{for a.e. } x \in \Omega. \tag{3.6}$$

From [19] we recall the following important estimate: for any $1 \le q \le 2$,

$$\int_\Omega |D^2 u|^q |\nabla u|^{p-q}\, dx < \infty. \tag{3.7}$$

Thanks to (3.6) and (3.7), all of the integrals in the geometric Hardy–Sobolev inequalities are well defined for any $1 \le q \le 2$.

However, since the solution u is not smooth around Z_u, we need to regularize it in order to apply the inequalities of Theorem 1.3. We will now describe an approximation argument due to Canino, Le, and Sciunzi [16] for the $p(\cdot)$-Laplacian (in our case $p(x) \equiv p$ constant).

Lemma 3.3 ([16]). *Let D be a smooth bounded domain of $\mathbb{R}^n$, $1 \le q \le 2$, and $\varepsilon \in (0, 1)$. Let $u \in C^1(\overline{\Omega})$ be a local solution of $-\Delta_p u = g(u)$ where g is any positive smooth nonlinearity. Set $h := g(u)$ and let $h_\varepsilon \in C^\infty(\overline{D})$ be any sequence converging to h in $C^1(\overline{D})$ as $\varepsilon \downarrow 0$. Then there exists a unique solution $u_\varepsilon \in W^{1,p}(D)$ of the regularized problem*

$$\begin{cases} -\operatorname{div}\left((\varepsilon^2 + |\nabla u_\varepsilon|^2)^{(p-2)/2} \nabla u_\varepsilon\right) = h_\varepsilon(x) & \text{in } D, \\ u_\varepsilon = u & \text{on } \partial D, \end{cases} \tag{3.8}$$

Moreover, u_ε is smooth, $u_\varepsilon \to u$ strongly in $W^{1,p}(D)$ and there exists a constant C independent of ε such that

$$\int_D |D^2 u_\varepsilon|^q (\varepsilon^2 + |\nabla u_\varepsilon|^2)^{(p-q)/2}\, dx \le C$$

and

$$\lim_{\varepsilon \to 0} \int_D |D^2 u_\varepsilon|^q (\varepsilon^2 + |\nabla u_\varepsilon|^2)^{(p-q)/2}\, dx = \int_D |D^2 u|^q |\nabla u|^{p-q}\, dx. \tag{3.9}$$

Applying Lemma 3.3 with $D = \Omega$ we obtain a smooth regularization u_ε of u. We can then apply Theorem 1.3 to any u_ε to get the appropriate inequality (a), (b), or (c). From [18, 28] we know that the regularization u_ε will converge to u, as $\varepsilon \downarrow 0$, in $C^1(\overline{\Omega})$. Hence we can easily pass to the limit as $\varepsilon \downarrow 0$ in the left hand side of (1.9) and (1.10).

We will now see that also the remaining terms $I_{p,q}(u_\varepsilon; \Omega)$, which involve tangential gradient and mean curvature, behave well under this approximation. By the uniform convergence of $|\nabla u_\varepsilon|$ to $|\nabla u|$ as $\varepsilon \to 0$, we can choose an $\varepsilon_0 > 0$ such that

$$\frac{1}{2} \leq \frac{|\nabla u_\varepsilon|}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \leq 2$$

for any $\varepsilon \leq \varepsilon_0$ and for all $x \in \overline{\Omega}$. In particular we see that

$$|\nabla u_\varepsilon|^{p-q} \leq \max\{2^{p-q}, 2^{q-p}\}(\varepsilon^2 + |\nabla u_\varepsilon|^2)^{(p-q)/2}. \tag{3.10}$$

Hence from (3.6) and (3.10) we see that for a sufficiently small $\varepsilon_0 > 0$ there exists a constant $K = K(n, p, q, \varepsilon_0) > 0$ such that for any $\varepsilon \leq \varepsilon_0$ we have

$$\left(\frac{1}{p'}\left|\nabla_{T,u_\varepsilon}|\nabla u_\varepsilon|^{p/q}\right|\right)^q + |H_{u_\varepsilon}|^q|\nabla u_\varepsilon|^p \leq K|D^2 u_\varepsilon|^q(\varepsilon^2 + |\nabla u_\varepsilon|^2)^{(p-q)/2}. \tag{3.11}$$

Moreover, by the fact that $u_\varepsilon \to u$ in $C^2(\Omega \setminus Z_u)$ and $|Z_u| = 0$, almost everywhere in Ω we have

$$\lim_{\varepsilon\to 0}\left\{\left(\frac{1}{p'}\left|\nabla_{T,u_\varepsilon}|\nabla u_\varepsilon|^{p/q}\right|\right)^q + |H_{u_\varepsilon}|^q|\nabla u_\varepsilon|^p\right\} = \left(\frac{1}{p'}\left|\nabla_{T,u}|\nabla u|^{p/q}\right|\right)^q + |H_u|^q|\nabla u|^p. \tag{3.12}$$

Now, thanks to (3.9), (3.11), and (3.12), by dominated convergence we see that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega&\left\{\left(\frac{1}{p'}\left|\nabla_{T,u_\varepsilon}|\nabla u_\varepsilon|^{p/q}\right|\right)^q + |H_{u_\varepsilon}|^q|\nabla u_\varepsilon|^p\right\}dx\\ &= \int_\Omega\left\{\left(\frac{1}{p'}\left|\nabla_{T,u}|\nabla u|^{p/q}\right|\right)^q + |H_u|^q|\nabla u|^p\right\}dx.\end{aligned}$$

Thus, the assertions of Theorem 1.3 hold for $v = u$.

To conclude the proof let us fix any $s > 0$ and consider $v = u - s$ on $\Omega_s = \{x \in \Omega : u > s\}$. It is clear that the integrands in the inequalities remain unchanged in this case, so the only problem comes from the fact Ω_s might not be smooth. If this is the case, let us consider two sequences $\varepsilon_n \to 0$ and $s_n \to s$, with the corresponding regularizations of v given by $v_n := v_{\varepsilon_n} = u_{\varepsilon_n} - s_n$. Thanks to the smoothness of any v_n and the Sard Lemma, we can choose each s_n to be a regular value of v_n, so that the level set $\{v_n > 0\} = \{u_n > s_n\}$ is smooth. Moreover, from the C^1 convergence, it is clear that for the characteristic functions we have $\chi_{\{u_n > s_n\}} \to \chi_{\{u>s\}}$. Hence we can conclude the proof using the same dominated convergence argument as above. □

4. Regularity of stable solutions. Proof of Theorems 1.4 and 1.5

We are now ready to establish L^r and $W^{1,r}$ *a priori* estimates of semistable solutions to p-Laplace equations proving Theorems 1.4 and 1.5.

Before the proof of our regularity results let us recall some known facts on the linearized operator associated to (1.2) and on semistable solutions.

4.1. Linearized operator and semistable solutions

This subsection deals with the linearized operator at any regular semistable solution $u \in C_0^1(\overline{\Omega})$ of

$$\begin{cases} -\Delta_p u = g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, with $n \geq 2$, and g is any positive C^1 nonlinearity.

The linearized operator L_u associated to (4.1) at u is defined by duality as

$$L_u(v,\phi) := \int_\Omega |\nabla u|^{p-2}\left\{\nabla v \cdot \nabla \phi + (p-2)\left(\nabla v \cdot \frac{\nabla u}{|\nabla u|}\right)\left(\nabla \phi \cdot \frac{\nabla u}{|\nabla u|}\right)\right\} dx - \int_\Omega g'(u) v\phi \, dx$$

for all $(v,\phi) \in H_0 \times H_0$, where the Hilbert space H_0 is defined according to [19] as follows.

Definition 4.1. Let $u \in C_0^1(\overline{\Omega})$ be a regular semistable solution of (4.1). We introduce the following weighted L^2 norm of the gradient:

$$|\phi| := \left(\int_\Omega \rho |\nabla \phi|^2 \, dx\right)^{1/2} \quad \text{where} \quad \rho := |\nabla u|^{p-2}.$$

According to [19], the space

$$H^1_\rho(\Omega) := \{\phi \in L^2(\Omega) \text{ weakly differentiable} : |\phi| < \infty\}$$

is a Hilbert space and is the completion of $C^\infty(\Omega)$ with respect to the $|\cdot|$-norm.

We define the Hilbert space H_0 of admissible test functions as

$$H_0 := \begin{cases} \{\phi \in H_0^1(\Omega) : |\phi| < \infty\} & \text{if } 1 < p \leq 2, \\ \text{the closure of } C_0^\infty(\Omega) \text{ in } H^1_\rho(\Omega) & \text{if } p > 2. \end{cases}$$

Note that for $1 < p \leq 2$, H_0 is a subspace of $H_0^1(\Omega)$ and since

$$\int_\Omega |\nabla \phi|^2 \leq \|\nabla u\|^{2-p}_{L^\infty(\Omega)} |\phi|^2,$$

we see that $(H_0, |\cdot|)$ is a Hilbert space. For $p > 2$, the weight $\rho = |\nabla u|^{p-2}$ is in $L^\infty(\Omega)$ and satisfies $\rho^{-1} \in L^1(\Omega)$, as shown in [19].

Now, thanks to the above definition, the operator L_u is well defined for $\phi \in H_0$, and therefore the semistability condition for the solution u reads

$$L_u(\phi,\phi) = \int_\Omega \left[|\nabla u|^{p-2} \left\{ |\nabla \phi|^2 + (p-2)\left(\nabla\phi \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right\} - g'(u)\phi^2 \right] dx \geq 0 \qquad (4.2)$$

for every $\phi \in H_0$.

On the one hand, considering $\phi = |\nabla u|\eta$ as a test function in the semistability condition (4.2) for u, we obtain

$$\int_\Omega \left[(p-1)|\nabla u|^{p-2} \left| \nabla_{T,u} |\nabla u| \right|^2 + B_u^2 |\nabla u|^p \right] \eta^2 \, dx \leq (p-1) \int_\Omega |\nabla u|^p |\nabla \eta|^2 \, dx \qquad (4.3)$$

for any Lipschitz continuous function η with compact support. Here, B_u^2 denotes the L^2 norm of the second fundamental form of the level set of $|u|$ through x (i.e., the sum of the squares of its principal curvatures). The fact that $\phi = \eta|\nabla u|$ is an admissible test function derives from the estimate (3.7), whereas the computations behind (4.3) are done in [22] (see also [23, Theorem 1]).

On the other hand, noting that $(n-1)H_u^2 \leq B_u^2$ and

$$|\nabla u|^{p-2} \left| \nabla_{T,u} |\nabla u| \right|^2 = \frac{4}{p^2} \left| \nabla_{T,u} |\nabla u|^{p/2} \right|^2,$$

we obtain the key inequality to prove our regularity results for semistable solutions:

$$\int_\Omega \left(\frac{4}{p^2} \left| \nabla_{T,u} |\nabla u|^{p/2} \right|^2 + \frac{n-1}{p-1} H_u^2 |\nabla u|^p \right) \eta^2 \, dx \leq \int_\Omega |\nabla u|^p |\nabla \eta|^2 \, dx \qquad (4.4)$$

for any Lipschitz continuous function η with compact support.

4.2. A priori estimates of stable solutions. Proof of Theorem 1.4

In order to prove the gradient estimate (1.15) stated in Theorem 1.4(b) we will use the following result. Its proof is based on a technique introduced by Bénilan et al. [7] to obtain the regularity of entropy solutions for p-Laplace equations with L^1 data.

Proposition 4.2. *Assume $n \geq 3$ and $h \in L^1(\Omega)$. Let u be the entropy solution of*

$$\begin{cases} -\Delta_p u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (4.5)$$

Let $r_0 \geq (p-1)n/(n-p)$. If $\int_\Omega |u|^{r_0} \, dx < \infty$, then the following a priori estimate holds:

$$\int_\Omega |\nabla u|^r \, dx \leq r|\Omega| + \left(\frac{r_1}{r} - 1 \right)^{-1} \left(\int_\Omega |u|^{r_0} \, dx + \|h\|_{L^1(\Omega)} \right)$$

for all $r < r_1 := pr_0/(r_0+1)$.

Remark 4.3. Bénilan et al. [7] proved the existence and uniqueness of entropy solutions to problem (4.5). Moreover, they proved that $|\nabla u|^{p-1} \in L^r(\Omega)$ for all $1 \le r < n/(n-1)$ and $|u|^{p-1} \in L^r(\Omega)$ for all $1 \le r < n/(n-p)$. Proposition 4.2 establishes an improvement of the previous gradient estimate if we know an *a priori* estimate of $\int_\Omega |u|^{r_0}\,dx$ for some $r_0 > (p-1)n/(n-p)$.

Proof of Proposition 4.2. Multiplying (4.5) by $T_s u = \max\{-s, \min\{s, u\}\}$ we obtain

$$\int_{\{|u|\le s\}} |\nabla u|^p\,dx = \int_\Omega h(x) T_s u\,dx \le s\|h\|_{L^1(\Omega)}.$$

Let $t = s^{(r_0+1)/p}$. From the previous inequality, recalling that $V(s) = |\{x \in \Omega : |u| > s\}|$, we deduce

$$s^{r_0}|\{|\nabla u| > t\}| \le s^{r_0} \int_{\{|\nabla u|>t\}\cap\{|u|\le s\}} (|\nabla u|/t)^p\,dx + s^{r_0}\int_{\{|u|>s\}} dx$$

$$\le \|h\|_{L^1(\Omega)} + s^{r_0} V(s) \qquad \text{for a.e. } s > 0.$$

In particular

$$t^{\frac{pr_0}{r_0+1}} |\{|\nabla u| > t\}| \le \|h\|_{L^1(\Omega)} + \sup_{\tau>0} \tau^{r_0} V(\tau) \qquad \text{for a.e. } t > 0. \tag{4.6}$$

Moreover, since

$$\tau^{r_0} V(\tau) \le \tau^{r_0} \int_{\{|u|>\tau\}} (|u|/\tau)^{r_0}\,dx \le \int_\Omega |u|^{r_0}\,dx \qquad \text{for a.e. } \tau > 0,$$

we have $\sup_{\tau>0} \tau^{r_0} V(\tau) \le \int_\Omega |u|^{r_0}\,dx$.

Let $r < r_1 := pr_0/(r_0+1)$. From (4.6) and the previous inequality, we have

$$\int_\Omega |\nabla u|^r\,dx = r\int_0^\infty t^{r-1}|\{|\nabla u| > t\}|\,dt$$
$$\le r|\Omega| + r\left(\int_\Omega |u|^{r_0}\,dx + \|h\|_{L^1(\Omega)}\right)\int_1^\infty t^{r-1} t^{-\frac{pr_0}{r_0+1}}\,dt,$$

proving the proposition. □

Now, we have all the ingredients to prove the *a priori* estimates stated in Theorem 1.4 for semistable solutions. These will follow from Theorem 1.3 and Propositions 3.2 and 4.2 by choosing suitable test functions in the semistability condition (4.4).

First, we prove Theorem 1.4 when $n \ne p+2$. We will take $\eta = T_s u = \min\{s, u\}$ as a test function in (4.4) and then, thanks to Proposition 3.2, we apply our Morrey and Sobolev inequalities (1.9) and (1.10) with $q = 2$.

Proof of Theorem 1.4 for $n \neq p+2$. Assume $n \neq p+2$. Let $u \in C_0^1(\overline{\Omega})$ be a semistable solution of (1.2). By taking $\eta = T_s u = \min\{s, u\}$ in the semistability condition (4.4) we obtain

$$\int_{\{u>s\}} \left(\frac{4}{p^2} \left| \nabla_{T,u} |\nabla u|^{p/2} \right|^2 + \frac{n-1}{p-1} H_u^2 |\nabla u|^p \right) dx \leq \frac{1}{s^2} \int_{\{u<s\}} |\nabla u|^{p+2}\, dx$$

for a.e. $s > 0$. In particular,

$$\min\left(\frac{4}{(n-1)p}, 1 \right) I_{p,2}(u-s; \{x \in \Omega : u > s\})^p \leq \frac{p-1}{(n-1)s^2} \int_{\{u<s\}} |\nabla u|^{p+2}\, dx$$

for a.e. $s > 0$, where $I_{p,2}$ is the functional defined in (1.1) with $q = 2$. By Proposition 3.2 we can apply Theorem 1.3 with Ω replaced by $\{x \in \Omega : u > s\}$, $v = u - s$, and $q = 2$. Then, the L^r estimates established in parts (a) and (b) follow directly from the Morrey and Sobolev type inequalities (1.9) and (1.10).

Finally, the gradient estimate (1.15) follows directly from Proposition 4.2 with $r_0 = np/(n-p-2)$. □

Now, we deal with the proof of Theorem 1.4(a) when $n = p+2$. This critical case follows from Theorem 2.2 and the semistability condition (4.4) with the test function $\eta = \eta(u)$ defined in (4.11) and (4.10) below.

Proof of Theorem 1.4 when $n = p+2$. Assume $n = p+2$ (and hence $n > 3$). Taking a Lipschitz function $\eta = \eta(u)$ (to be chosen later) in (4.4) and using the coarea formula we obtain

$$\begin{aligned} C \int_0^\infty \int_{\{u=t\}} \left\{ \left| \nabla_{T,u} |\nabla u|^{(p-1)/2} \right|^2 + \left| H_u |\nabla u|^{(p-1)/2} \right|^2 \right\} \eta(t)^2\, d\sigma\, dt \\ \leq \int_0^\infty \int_{\{u=t\}} |\nabla u|^{p+1} \dot{\eta}(t)^2\, d\sigma\, dt, \end{aligned} \tag{4.7}$$

where $d\sigma$ denotes the area element in $\{u = t\}$ and C, here and in the rest of the proof, is a constant depending only on p.

To apply the Sobolev inequality (2.2) in the left hand side of the previous inequality we need to make an approximation argument. Consider the sequence u_k of smooth regularizations of u introduced in the proof of Proposition 3.2 and note that $\{u_k = t\}$ is a smooth hypersurface for a.e. $t \geq 0$. Then, from the Sobolev inequality (2.2) with $\phi = |\nabla u_k|^{(p-1)/2}$, $q = 2$, and $M = \{u_k = t\}$, and noting that

$$(p-1)\frac{n-1}{n-3} = p+1 \qquad \text{when } n = p+2,$$

we obtain

$$\begin{aligned} C \int_0^\infty \left(\int_{\{u_k=t\}} |\nabla u_k|^{p+1} \right)^{\frac{n-3}{n-1}} \eta(t)^2\, d\sigma\, dt \\ \leq \int_0^\infty \int_{\{u_k=t\}} \left\{ \left| \nabla_{T,u_k} |\nabla u_k|^{(p-1)/2} \right|^2 + \left| H_{u_k} |\nabla u_k|^{(p-1)/2} \right|^2 \right\} \eta(t)^2\, d\sigma\, dt. \end{aligned} \tag{4.8}$$

Now, we will pass to the limit in the previous inequality. Note that, if η is bounded, through a dominated convergence argument as in Proposition 3.2 we have

$$\lim_{k\to\infty}\int_0^\infty\int_{\{u_k=t\}}\left\{\left|\nabla_{T,u_k}|\nabla u_k|^{(p-1)/2}\right|^2+\left|H_{u_k}|\nabla u_k|^{(p-1)/2}\right|^2\right\}\eta(t)^2\,d\sigma\,dt$$
$$=\int_0^\infty\int_{\{u=t\}}\left\{\left|\nabla_{T,u}|\nabla u|^{(p-1)/2}\right|^2+\left|H_u|\nabla u|^{(p-1)/2}\right|^2\right\}\eta(t)^2\,d\sigma\,dt.$$

Moreover, from the C^1 convergence of u_k to u we obtain

$$\lim_{k\to\infty}\int_0^\infty\left(\int_{\{u_k=t\}}|\nabla u_k|^{p+1}\right)^{\frac{n-3}{n-1}}\eta(t)^2\,d\sigma\,dt=\int_0^\infty\left(\int_{\{u=t\}}|\nabla u|^{p+1}\right)^{\frac{n-3}{n-1}}\eta(t)^2\,d\sigma\,dt.$$

Therefore, taking the limit as k goes to infinity in (4.8) and using (4.7), we get

$$C\int_0^\infty\psi(t)^{\frac{n-3}{n-1}}\eta(t)^2\,dt\le\int_0^\infty\psi(t)\dot\eta(t)^2\,dt=\int_0^\infty\int_{\{u=t\}}|\nabla u|^{p+1}\,d\sigma\,\dot\eta(t)^2\,dt,\tag{4.9}$$

where

$$\psi(t):=\int_{\{u=t\}}|\nabla u|^{p+1}\,d\sigma.\tag{4.10}$$

Now, let $\bar M:=\|u\|_{L^\infty(\Omega)}$. Given $s>0$, choose

$$\eta(t)=\eta_s(t):=\begin{cases}t/s & \text{if } 0\le t\le s,\\ \exp\left(\frac{1}{\sqrt2}\int_s^t\left(\frac{C\psi(\tau)^{\frac{n-3}{n-1}}}{\psi(\tau)}\right)^{1/2}d\tau\right) & \text{if } s<t\le\bar M\\ \eta(\bar M) & \text{if } t>\bar M.\end{cases}\tag{4.11}$$

It is then clear that

$$\int_0^{\bar M}\int_{\{u=t\}}|\nabla u|^{p+1}\,d\sigma\,\dot\eta_s(t)^2\,dt=\frac{1}{s^2}\int_{\{u\le s\}}|\nabla u|^{p+2}\,dx+\frac{C}{2}\int_s^{\bar M}\psi(t)^{\frac{n-3}{n-1}}\eta_s(t)^2\,dt.$$

Therefore, from (4.9) we obtain

$$\frac{C}{2}\int_s^{\bar M}\psi(t)^{\frac{n-3}{n-1}}\eta_s(t)^2\,dt\le\frac{1}{s^2}\int_{\{u\le s\}}|\nabla u|^{p+2}\,dx.\tag{4.12}$$

Let us choose $\alpha=\frac{2}{n-2}$, $\beta=\frac{n-3}{(n-2)(n-1)}$, and $m=n-2$. Note that $\alpha,\beta>0$, $m>1$, and $\beta m'=1/(n-1)$. Moreover, using the definition of η_s we have

$$\frac{1}{\psi(t)^{\beta m'}\eta_s(t)^{\alpha m'}}=\sqrt{\frac{2}{C}}\,\frac{\dot\eta_s(t)}{\eta_s(t)^{\alpha m'+1}}\tag{4.13}$$

for all $t > s$. By (4.13), the Hölder inequality, and (4.12), we see that

$$\begin{aligned}
\bar{M} - s &= \int_s^{\bar{M}} \frac{\psi(t)^\beta \eta_s(t)^\alpha}{\psi(t)^\beta \eta_s(t)^\alpha}\, dt \\
&\le \left(\int_s^{\bar{M}} \psi(t)^{\beta m} \eta_s(t)^{\alpha m}\, dt\right)^{1/m} \left(\int_s^{\bar{M}} \frac{dt}{\psi(t)^{\beta m'} \eta_s(t)^{\alpha m'}}\right)^{1/m'} \\
&\le \left(\int_s^{\bar{M}} \psi(t)^{\frac{n-3}{n-1}} \eta_s(t)^2\, dt\right)^{\frac{1}{n-2}} \left(\sqrt{\frac{2}{C}} \int_s^{\bar{M}} \frac{\dot{\eta}_s(t)}{\eta_s(t)^{m'\alpha+1}}\, dt\right)^{\frac{n-3}{n-2}} \\
&\le \left(\frac{2}{Cs^2} \int_{\{u \le s\}} |\nabla u|^{p+2}\, dx\right)^{\frac{1}{n-2}} \left(\sqrt{\frac{2}{C}}\, \frac{n-3}{2}\right)^{\frac{n-3}{n-2}},
\end{aligned}$$

which is exactly (1.13) (note that $n - 2 = p$ and $\eta(\bar{M}) \ge 1$). □

4.3. Regularity of the extremal solution. Proof of Theorem 1.5

In this subsection we will prove the *a priori* estimates for minimal and extremal solutions of $(1.17)_\lambda$ stated in Theorem 1.5. Let us remark that in the proof of Theorem 1.5 we will assume the nonlinearity f to be smooth. However, if it is only C^1 we can proceed with an approximation argument as in [12, proof of Theorem 1.2].

The $W^{1,p}$ estimate established in Theorem 1.5 has as main ingredient the following result.

Lemma 4.4. *Let f be an increasing positive C^1 function satisfying (1.18) and $\lambda \in (0, \lambda^\star)$. Let $u = u_\lambda \in C_0^1(\overline{\Omega})$ be the minimal solution of $(1.17)_\lambda$. Then*

$$\int_\Omega |\nabla u|^p\, dx \le \left(\max_{x \in \overline{\Omega}} |x|\right) \frac{1}{p'} \int_{\partial\Omega} |\nabla u|^p\, d\sigma. \tag{4.14}$$

Proof. Let $G'(t) = g(t) = \lambda f(t)$. First, we note that

$$x \cdot \nabla u g(u) = x \cdot \nabla G(u) = \operatorname{div}(G(u)x) - nG(u)$$

and that almost everywhere on Ω we can evaluate

$$\begin{aligned}
x \cdot \nabla u \Delta_p u - \operatorname{div}(x \cdot \nabla u\, |\nabla u|^{p-2} \nabla u) &= -|\nabla u|^{p-2} \nabla u \cdot \nabla(x \cdot \nabla u) \\
&= -|\nabla u|^p - \frac{1}{p} \nabla |\nabla u|^p \cdot x \\
&= \frac{n-p}{p} |\nabla u|^p - \frac{1}{p} \operatorname{div}(|\nabla u|^p x).
\end{aligned}$$

As a consequence, multiplying $(1.17)_\lambda$ by $x \cdot \nabla u$ and integrating on Ω, we have

$$n \int_\Omega G(u)\, dx - \frac{n-p}{p} \int_\Omega |\nabla u|^p\, dx = \frac{1}{p'} \int_{\partial\Omega} |\nabla u|^p\, x \cdot \nu\, d\sigma, \tag{4.15}$$

where ν is the outward unit normal to Ω.

Noting that u is an absolute minimizer of the energy functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega G(u)\,dx$$

in the convex set $\{v \in W_0^{1,p}(\Omega) : 0 \le v \le u\}$ (see [14]), we see that $J(u) \le J(0) = 0$. Therefore, from (4.15) we obtain

$$\int_\Omega |\nabla u|^p\,dx = nJ(u) + \frac{1}{p'}\int_{\partial\Omega} |\nabla u|^p\, x\cdot\nu\,d\sigma \le \Big(\max_{x\in\overline{\Omega}}|x|\Big)\frac{1}{p'}\int_{\partial\Omega}|\nabla u|^p\,d\sigma,$$

proving the lemma. □

Finally, we prove Theorem 1.5 (using the semistability condition (4.4) with an appropriate test function), Theorem 1.3, and Lemma 4.4.

Proof of Theorem 1.5. Let u_λ be the minimal solution of $(1.17)_\lambda$ for $\lambda \in (0, \lambda^\star)$. From [14] we know that minimal solutions are semistable. In particular, u_λ satisfies the semistability condition (4.4) for all $\lambda \in (0, \lambda^\star)$.

Assume that Ω is strictly convex. Let $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ be the distance to the boundary and $\Omega_\varepsilon := \{x \in \Omega : \delta(x) < \varepsilon\}$. By Proposition 3.1 there exist positive constants ε and γ such that for every $x_0 \in \Omega_\varepsilon$ there exists a set $I_{x_0} \subset \Omega$ satisfying $|I_{x_0}| > \gamma$ and

$$u_\lambda(x_0)^{p-1} \le u_\lambda(y)^{p-1} \quad \text{for all } y \in I_{x_0}. \tag{4.16}$$

Let $x_\varepsilon \in \overline{\Omega}_\varepsilon$ be such that $u_\lambda(x_\varepsilon) = \|u_\lambda\|_{L^\infty(\Omega_\varepsilon)}$. Integrating inequality (4.16) with respect to y in I_{x_ε} and using (1.18), we obtain

$$\|u_\lambda\|^{p-1}_{L^\infty(\Omega_\varepsilon)} \le \frac{1}{\gamma}\int_{I_{x_\varepsilon}} u_\lambda^{p-1}\,dy \le \frac{1}{\gamma}\int_\Omega u_\lambda^{p-1}\,dy \le \frac{C}{\gamma}\|f(u_\lambda)\|_{L^1(\Omega)}, \tag{4.17}$$

where C, here and in the rest of the proof, is a constant independent of λ. Letting $s = ((C/\gamma)\|f(u_\lambda)\|_{L^1(\Omega)})^{1/(p-1)}$, we deduce

$$\Omega_\varepsilon \subset \{x \in \Omega : u_\lambda(x) \le s\}. \tag{4.18}$$

Now, choose

$$\eta(x) := \begin{cases} \delta(x) & \text{if } \delta(x) < \varepsilon, \\ \varepsilon & \text{if } \delta(x) \ge \varepsilon, \end{cases}$$

as a test function in (4.4) and use (4.18) to obtain

$$\varepsilon^2 \int_{\{u_\lambda > s\}} \left(\frac{4}{p^2}\big|\nabla_{T,u_\lambda}|\nabla u_\lambda|^{p/2}\big|^2 + \frac{n-1}{p-1}H^2_{u_\lambda}|\nabla u_\lambda|^p\right)dx \le \int_{\{u_\lambda \le s\}} |\nabla u_\lambda|^p\,dx.$$

Multiplying equation $(1.17)_\lambda$ by $T_s u_\lambda = \min\{s, u_\lambda\}$ we have

$$\int_{\{u_\lambda \le s\}} |\nabla u_\lambda|^p\,dx = \lambda\int_\Omega f(u_\lambda)T_s u\,dx \le \lambda^\star s\|f(u_\lambda)\|_{L^1(\Omega)} = C\|f(u_\lambda)\|^{p'}_{L^1(\Omega)}.$$

Combining the previous two inequalities we obtain

$$\int_{\{u_\lambda>s\}}\left(\frac{4}{p^2}\big|\nabla_{T,u_\lambda}|\nabla u_\lambda|^{p/2}\big|^2+\frac{n-1}{p-1}H_{u_\lambda}^2|\nabla u_\lambda|^p\right)dx\le C\|f(u_\lambda)\|_{L^1(\Omega)}^{p'}.$$

At this point, proceeding exactly as in the proof of Theorem 1.4, we deduce the L^r estimates stated in parts (a) and (b).

In order to prove the $W^{1,p}$ estimate of (b), recall that by (4.14) we have

$$\int_\Omega|\nabla u_\lambda|^p\,dx\le C\int_{\partial\Omega}|\nabla u_\lambda|^p\,d\sigma.$$

Therefore, we need to control the right hand side of the previous inequality. Since the nonlinearity f is increasing by hypothesis, we obtain

$$f(u_\lambda)\le f\big(C\|f(u_\lambda)\|_{L^1(\Omega)}^{1/(p-1)}\big)\quad\text{ in }\Omega_\varepsilon$$

by (4.17), where C is a constant independent of λ.

Now, since $-\Delta_p u_\lambda=\lambda f(u_\lambda)\in L^\infty(\Omega_\varepsilon)$ in Ω_ε, we have

$$\|u_\lambda\|_{C^{1,\beta}(\overline{\Omega}_\varepsilon)}\le C'$$

for some $\beta\in(0,1)$ by [28], where C' is a constant depending only on n, p, Ω, f, and $\|f(u_\lambda)\|_{L^1(\Omega)}$, proving the assertion.

Finally, assume that $p\ge 2$ and (1.19) holds. From [33] we know that $f(u^\star)\in L^r(\Omega)$ for all $1\le r<n/(n-p')$. In particular, $f(u^\star)\in L^1(\Omega)$. Therefore, parts (i) and (ii) follow directly from (a) and (b). □

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