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A resolvent approach to metastability

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Abstract. We provide a necessary and sufficient condition for the metastability of a Markov chain, expressed in terms of a property of the solutions of the resolvent equation. As an application of this result, we prove the metastability of reversible, critical zero-range processes starting from a configuration.

Keywords. Metastability, resolvent equation, zero-range process, condensation

1. Introduction

Metastability is a physical phenomenon ubiquitous in first order phase transitions. A tentative of a precise description can be traced back, at least, to Maxwell [51].

In the mid-1980s, Cassandro, Galves, Olivieri and Vares [20], continuing work of Lebowitz and Penrose [55], proposed a first rigorous method for deducing the metastable behavior of Markov processes, based on the theory of large deviations developed by Freidlin and Wentsel [25]. This method, known as the *pathwise approach to metastability*, was successfully applied to many models in statistical mechanics [54].

In the following years, different approaches were put forward. In the early 20th century, Bovier, Eckhoff, Gayrard and Klein [16–18], replaced the large deviations tools with potential theory to derive sharp estimates for the transition times between wells, the so-called Eyring–Kramers law. We refer to [14] for a comprehensive review of this method, known as the *potential-theoretic approach to metastability*.

More recently, Beltrán and Landim [6, 8] characterized the metastable behavior of a process as the convergence of the order process, a coarse-grained projection of the dynamics, to a Markov chain. Inspired by [17] and based on the martingale characterization of

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Markov processes, they provided different sets of sufficient conditions for metastability. We refer to [35] for a review of the *martingale approach to metastability*.

In this article, we show that the metastable behavior of a sequence of Markov chains can be read off from a property of the solutions of the resolvent equation associated to the generator of the process. It turns out that this property is not only sufficient, but also *necessary* for metastability. This is the content of Theorem 2.3.

As these conditions for metastability do not rely on the explicit knowledge of the stationary state, they can, in principle, be employed to derive the metastable behavior of non-reversible dynamics whose stationary states are not known.

To emphasize the strength of our method, we show that the necessary and sufficient conditions for metastability can be derived from the ones introduced in [6, 8], which have been proved to hold for all models whose metastable behavior has been derived through the potential-theoretic method [17] or the martingale method [6, 8]. Moreover, the recent articles [36, 37, 46] successfully apply the approach introduced here to non-reversible overdamped Langevin dynamics.

We further illustrate the extent of possible applications by proving that the conditions for metastability required in this article hold for a dynamics with poor mixing properties: reversible condensing critical zero-range processes. This is a model which does not satisfy the conditions in [6], and whose metastable behavior could only be derived so far when the process starts from measures spread over a well [41]. This new approach permits us to extend this result to reversible dynamics in which the process starts from a configuration.

We leave for the future the investigation of metastability of critical asymmetric zero-range processes. For this model the mixing condition $\mathfrak{M}$, introduced in Section 6.2, is very delicate in that the mixing time is slightly smaller than the escape time. In the reversible situation considered here, we verify condition $\mathfrak{M}$ through a careful construction of a sub-harmonic function. It seems difficult to extend this construction to the non-reversible case. Apart from condition $\mathfrak{M}$, all other steps are identical to the reversible case.

Recent advancements

Before providing a more detailed statement of the main results, we review recent progress in the theory of metastability.

Markov Chain Monte Carlo algorithms have been widely used in order to sample from a given Gibbs measure. Their efficiency is expressed by the speed of convergence to equilibrium. It has been shown in several different contexts that non-reversible dynamics converge faster to equilibrium than their reversible counterpart. This is derived by Kaiser, Jack and Zimmer [30] for the large deviations from the hydrodynamic limit of interacting particle systems described by the Macroscopic Fluctuation Theory. Bouchet and Reygner [13] show that the transition time between two wells in overdamped Langevin dynamics is faster in the non-reversible case. A similar result appears in [44] for random walks in potential fields.

These results raise the problem of finding the non-reversible perturbation of a reversible dynamics that does not alter the invariant distribution and optimizes the rate of convergence. Lelièvre, Nier and Pavliotis [48] solve this problem for overdamped Langevin equations with quadratic potential. Guillin and Monmarché [28] show that the asymptotic rate of convergence of generalized Ornstein–Uhlenbeck processes is maximized by non-reversible hypoelliptic ones.

There are only a few other results on metastability for non-reversible dynamics. Le Peutrec and Michel [49] obtain by semiclassical analysis the Eyring–Kramers formula for the exponentially small eigenvalues of the generator of a non-reversible overdamped Langevin dynamics associated to a potential which is a Morse function satisfying additional regularity properties.

In the last years, the close connection between quasi-stationary states and exponential exit laws have been exploited in many different directions. Bianchi, Gaudillière and Milanesi [11,12] expressed the mean transition time in terms of soft capacities and derived sufficient conditions for metastability in terms of local and global mixing characteristics of the dynamics. Miclo [52] provided an estimate on the distance between the exit time of a set and an exponential law. Di Gesù, Lelièvre, Le Peutrec and Nectoux [22,47] investigated the distribution of the exit point from a domain. Berglund [9] reviews analytical methods to derive metastability. Di Gesú [21] derived, recently, the Eyring–Kramers formula for the exponentially small eigenvalues of the generator of reversible discrete diffusions with semiclassical analysis, an expansion obtained in [42,44] by stochastic methods.

We turn to a precise description of the results.

The model. Consider a sequence of countable sets $\mathcal{H}_N$, $N \in \mathbb{N}$, and a collection of $\mathcal{H}_N$-valued, irreducible, continuous-time Markov chains $(\xi_N(t) : t \geq 0)$. To fix ideas, one may think that the sets $\mathcal{H}_N$ are finite with cardinality increasing to infinity, but this is not necessary.

Let S be a fixed finite set and $\Psi_N : \mathcal{H}_N \to S$ a projection in the sense that the cardinality of S is much smaller than that of $\mathcal{H}_N$. Elements of $\mathcal{H}_N$ are represented by Greek letters η, ξ, while the ones of S by x, y. The problem we address is under what conditions the *order process* $(Y_N(t) : t \geq 0)$, defined by $Y_N(t) = \Psi_N(\xi_N(t))$, is close to a Markovian dynamics which mimics the dynamics of $\xi_N(t)$.

Denote by $\mathcal{E}_N^x$ the inverse image of $x \in S$ by Ψ_N, $\mathcal{E}_N^x = \Psi_N^{-1}(x)$, and by $\mathcal{L}_N$ the generator of the Markov chain $\xi_N(t)$. The sets $\mathcal{E}_N^x$ are called *wells*. The following condition plays a central role in the article.

Resolvent condition. Fix a function $g : S \to \mathbb{R}$, and denote by $G_N : \mathcal{H}_N \to \mathbb{R}$ its lifting: $G_N(\eta) = \sum_{x \in S} g(x) \chi_{\mathcal{E}_N^x}(\eta)$, where $\chi_{\mathcal{A}}$, $A \subset \mathcal{H}_N$, stands for the indicator of the set $\mathcal{A}$. For $\lambda > 0$, denote by F_N the solution of the resolvent equation

$$(\lambda - \mathcal{L}_N) F_N = G_N . \tag{1.1}$$

Assume that for each $\lambda > 0$, F_N is asymptotically constant on each set $\mathcal{E}_N^x$: there exists a function $f : S \to \mathbb{R}$ such that

$$\lim_{N\to\infty} \max_{x \in S} \sup_{\eta \in \mathcal{E}_N^x} |F_N(\eta) - f(x)| = 0. \tag{1.2}$$

Of course, f depends on λ and on g.

Assume, furthermore, that there exists a generator $\mathcal{L}$ of an S-valued continuous-time Markov chain such that

$$(\lambda - \mathcal{L}) f = g \tag{1.3}$$

for all $\lambda > 0$ and $g : S \to \mathbb{R}$.

We claim that, under the resolvent conditions (1.2), (1.3), any limit point of the sequence of processes $Y_N(\cdot) = \Psi_N(\xi_N(\cdot))$ is the law of the continuous-time Markov chain whose generator is $\mathcal{L}$. The proof of this claim is so simple that we present it below. It relies on the martingale characterization of Markovian dynamics.

Assume that the sequence $Y_N(\cdot)$ converges in law. Fix $\lambda > 0$ and a function $f : S \to \mathbb{R}$. Denote by F_N the solution of the resolvent equation (1.1) with $g = (\lambda - \mathcal{L}) f$. Since $\xi_N(\cdot)$ is a Markov process,

$$M_N(t) = e^{-\lambda t} F_N(\xi_N(t)) - F_N(\xi_N(0)) + \int_0^t e^{-\lambda r} [(\lambda - \mathcal{L}_N) F_N](\xi_N(r))\, dr \tag{1.4}$$

is a martingale. As F_N solves the resolvent equation (1.1), $\lambda F_N - \mathcal{L}_N F_N = G_N$. By (1.2), $F_N(\eta)$ is close to $f(\Psi_N(\eta))$. Hence, since $\Psi_N(\xi_N(t)) = Y_N(t)$ and $G_N(\eta) = g(\Psi_N(\eta))$, we can write

$$M_N(t) = e^{-\lambda t} f(Y_N(t)) - f(Y_N(0)) + \int_0^t e^{-\lambda r} g(Y_N(r))\, dr + o_N(1),$$

where $o_N(1)$ is a small error which vanishes uniformly as $N \to \infty$. As $g = (\lambda - \mathcal{L}) f$,

$$M_N(t) = e^{-\lambda t} f(Y_N(t)) - f(Y_N(0)) + \int_0^t e^{-\lambda r} [(\lambda - \mathcal{L}) f](Y_N(r))\, dr + o_N(1).$$

Passing to the limit shows that any limit point solves the martingale problem associated to the generator $\mathcal{L}$. To complete the argument it remains to recall the uniqueness of solutions of martingale problems in finite state spaces.

The resolvent condition is also necessary. The previous approach provides a general method to describe a complex system, a Markovian dynamics evolving in a large space $\mathcal{H}_N$, in terms of a much simpler one, an S-valued Markov chain. This abridgement has been named *Markov chain model reduction* or *metastability*; see [35] and references therein.

The point here is that the existence of this synthetic description of the dynamics can be read off from a simple property of the generator. It is in force if the resolvent operator $\mathfrak{U}_{\lambda,N} := (\lambda - \mathcal{L}_N)^{-1}$ sends functions which are constant on the sets $\mathcal{E}_N^x$ to functions which are asymptotically constant.

The second main point of the article is that conditions (1.2), (1.3) are not only sufficient for the convergence of the order process $Y_N(\cdot)$, but also necessary.

Applications. The last claim of the article is that this method to derive the metastable behavior, in the sense of the model reduction described above, of a sequence of Markov processes can be applied to a wide range of dynamics. We support this assertion by providing sufficient conditions for assumptions (1.2), (1.3) to hold. These conditions rely on mixing properties of the dynamics and have been derived in several different contexts in previous papers. Furthermore, in the last part of the article, we show that these conditions are in force for reversible, critical zero-range dynamics. In particular, we are able to extend the results presented in [41] to the case in which the process starts from a configuration instead of a measure spread over a well $\mathcal{E}_N^x$.

Comments. In concrete examples, one has first to find the time-scale θ_N at which a metastable behavior is observed. Then, one speeds up the evolution by this quantity and proves all properties of the dynamics in this new time-scale. Speeding up the process by θ_N corresponds to multiplying the generator by the time-scale θ_N. In the previous discussion we started from a generator which has already been speeded up, which means that the metastable behavior is observed in the time-scale $\theta_N = 1$.

This approach to metastability, inspired from techniques in PDE to study the asymptotic behavior of solutions of reaction-diffusion equations [23,59], appeared in the context of Markov processes in [45,53,56]. In these articles, for different models, it is proved that the solutions of the Poisson equation $\mathcal{L}_N F_N = G_N$ are asymptotically constant in each well.

Replacing the Poisson equation with resolvent equations has a significant advantage, as the solutions of the later equation are bounded. It permits, in particular, to prove L^∞ estimates instead of the L^2 estimates derived in [41]. This, in turn, allows to start the process from a fixed configuration instead of a measure spread over the sets $\mathcal{E}_N^x$.

The existing methods to derive the metastable behavior of a Markov processes rely on explicit computations involving the stationary state [14, 35]. In contrast, as already pointed out at the beginning of this introduction, the deduction of (1.2) and (1.3) does not appeal to the stationary state.

Introducing a transition region. Condition (1.2) is expected to hold only in very special cases, where the jump rates between configurations belonging to different sets $\mathcal{E}_N^x$ vanish asymptotically. Only in such a case, one can hope for a discontinuity of the solution of the resolvent equation (1.1) at the boundary of the set $\mathcal{E}_N^x$ [an aftermath of condition (1.3)].

To surmount this problem, we introduce a transition set Δ_N which separates the wells $\mathcal{E}_N^x$. The set Δ_N has to be sufficiently large to isolate the wells, but small enough to be irrelevant from the point of view of the dynamics.

In this new set-up, Δ_N, $\mathcal{E}_N$ forms a partition of the state space $\mathcal{H}_N$, where $\mathcal{E}_N = \bigcup_{x\in S} \mathcal{E}_N^x$. To bypass the set Δ_N, we focus our attention on the trace of the process $\xi_N(\cdot)$ on $\mathcal{E}_N$ and provide sufficient conditions for the projection of the trace process to converge to a Markovian dynamics. This result requires conditions (1.2), (1.3) to hold only on the set $\mathcal{E}_N$, as stated in the first equation.

Furthermore, in this framework, Theorem 2.3 asserts that the resolvent conditions (1.2), (1.3) hold if, and only if, (a) the order process converges to the S-valued Markov chain whose generator is $\mathscr{L}$ and (b) the process $\xi_N(\cdot)$ spends only a negligible amount of time outside the wells $\mathcal{E}_N^x$.

Critical zero-range processes. As mentioned above, this approach is applied to a special class of zero-range processes. This Markovian dynamics describes the evolution of particles on a finite set S. Denote by $N \ge 1$ the total number of particles and by $\eta = (\eta_x : x \in S)$ a configuration of particles. Here, η_x represents the number of particles at site x for the configuration η. Let $\mathcal{H}_N = \{\eta \in \mathbb{N}^S : \sum_{x\in S} \eta_x = N\}$ be the state space.

Particles jump on S according to some rates which will be specified in the next section. It has been shown that a condensation phenomenon occurs for this family of rates. The precise statement requires some notation. Fix a sequence $(\ell_N : N \ge 1)$ such that $\ell_N \to \infty$, $\ell_N/N \to 0$. Denote by $\mathcal{E}_N^x$, $x \in S$, the set of configurations given by

$$\mathcal{E}_N^x = \{\eta \in \mathcal{H}_N : \eta_x \ge N - \ell_N\}.$$

For the models alluded to above, $\mu_N(\mathcal{E}_N^x) \to 1/|S|$, where μ_N represents the stationary state of the dynamics [1,3,4,7,24,27,29].

This means that under the stationary state, essentially all particles sit on a single site. In consequence, in terms of the dynamics, one expects the zero-range process to evolve as follows. When it reaches a set $\mathcal{E}_N^x$, it remains there a very long time, performing short excursions in Δ_N. Its sojourn at $\mathcal{E}_N^x$ before it hits a new well $\mathcal{E}_N^y$, $y \ne x$, is long enough for the process to equilibrate inside the well $\mathcal{E}_N^x$. The transition from $\mathcal{E}_N^x$ to a new well $\mathcal{E}_N^y$ is abrupt in the sense that its duration is much shorter compared to the total time the process stayed in $\mathcal{E}_N^x$.

We apply the method presented at the beginning of this introduction to derive the asymptotic evolution of the position of the condensate (the site x where almost all particles sit) for critical, reversible zero-range dynamics.

The metastable behavior of condensing zero-range processes has a long history [2,7,34,41,53,58]. The critical case, examined here and in [41], presents a major difference with respect to the supercritical case considered before. While in the supercritical case, when entering a well, the process visits all its configurations before visiting a new well, this is no longer true in the critical case. This difference prevents the use of the martingale approach, proposed in [6,8], to prove the metastable behavior of a sequence of Markov chains.

To overcome this problem, we show that in the critical case, when entering a well, the process hits the bottom of this well before reaching another well. The proof of this result relies on the superharmonic functions constructed in [41] and on mixing properties of the process reflected at the boundary of the wells. The fact that the process visits one specific configuration inside the well permits us to prove its metastable behavior starting from any configuration inside a well.

Combining the property that the process hits quickly the bottom of a well and that it mixes inside the well before it reaches its boundary permits us to prove that the solution of the resolvent equation fulfills (1.2). The proof of property (1.3) also relies on a computation of capacities between wells. The details are given in Sections 8–12.

To our knowledge, this is the first model which does not visit points and for which one can prove metastability starting from points and derive explicit formulae for the time-scale at which metastability occurs and for the generator $\mathcal{L}$ of the asymptotic dynamics.

Along the same lines, Schlichting and Slowik [57] extended the investigation of metastability to continuous-time Markov chains which do not hit single points. They derived asymptotic sharp estimates for mean hitting times by generalizing the potential-theoretic approach to deal with metastable sets, instead of just metastable points. This technique has been applied by Bovier, den Hollander, Marello, Pulvirenti and Slowik [15] to inhomogeneous mean-field models.

Directions for future research. As observed above, it is conceivable to derive properties (1.2), (1.3) without turning to the stationary state. In particular, this approach might permit one to deduce the metastable behavior of non-reversible dynamics for which the stationary measure is not known explicitly (say, non-reversible diffusions [13]). Furthermore, proving properties (1.2) and (1.3) for a generator $\mathcal{L}_N$ becomes an interesting problem since they yield (modulo a third property) the metastable behavior of the associated Markovian dynamics.

2. A resolvent approach to metastability

In this section, we provide a set of sufficient conditions for a sequence of continuous-time Markov chains to exhibit a metastable behavior. If the framework below seems too abstract, the reader may read this section together with the next, where we apply these results to a concrete example, the critical zero-range process.

We start by introducing the general framework proposed in [6, 8] to describe the metastable behavior of a Markovian dynamics as a Markov chain model reduction. Let $(\mathcal{H}_N : N \geq 1)$ be a collection of finite sets. Elements of the set $\mathcal{H}_N$ are designated by the letters η, ξ, and ζ.

Consider a sequence $(\xi_N(t) : t \geq 0)$ of $\mathcal{H}_N$-valued, irreducible, continuous-time Markov chains, whose generator is represented by $\mathcal{L}_N$. Then, for every function $f : \mathcal{H}_N \to \mathbb{R}$,

$$(\mathcal{L}_N f)(\eta) = \sum_{\xi \in \mathcal{H}_N} R_N(\eta, \xi)[f(\xi) - f(\eta)],$$

where $R_N(\eta, \xi)$ stands for the jump rates. Denote by $\lambda_N(\eta)$ the holding times of the Markov chain, $\lambda_N(\eta) = \sum_{\xi \neq \eta} R_N(\eta, \xi)$, and by μ_N the unique stationary state.

Denote by $D(\mathbb{R}_+, \mathcal{H}_N)$ the space of right-continuous functions $\mathbf{x} : \mathbb{R}_+ \to \mathcal{H}_N$ with left limits, endowed with the Skorokhod topology and its associated Borel σ-field. Let $\mathbf{P}^N_\eta$, $\eta \in \mathcal{H}_N$, be the probability measure on $D(\mathbb{R}_+, \mathcal{H}_N)$ induced by the process $\xi_N(\cdot)$ starting from $\eta \in \mathcal{H}_N$. Expectation with respect to $\mathbf{P}^N_\eta$ is represented by $\mathbf{E}^N_\eta$.

Fix a finite set S, and denote by $\mathcal{E}^x_N$, $x \in S$, a family of disjoint subsets of $\mathcal{H}_N$. Let

$$\mathcal{E}_N = \bigcup_{x \in S} \mathcal{E}^x_N \quad \text{and} \quad \Delta_N = \mathcal{H}_N \setminus \bigcup_{x \in S} \mathcal{E}^x_N.$$

The sets $\mathcal{E}^x_N$, $x \in S$, represent the metastable sets of the dynamics $\xi_N(\cdot)$, in the sense that, as soon as the process $\xi_N(\cdot)$ enters one of these sets, say $\mathcal{E}^x_N$, it equilibrates in $\mathcal{E}^x_N$

before hitting a new set $\mathcal{E}_N^y$, $y \neq x$. The goal of the theory is to describe the evolution between these sets. To this end, we introduce the order process.

For $\mathcal{A} \subset \mathcal{H}_N$, denote by $T^{\mathcal{A}}(t)$ the total time the process $\xi_N(\cdot)$ spends in $\mathcal{A}$ in the time interval $[0, t]$:

$$T^{\mathcal{A}}(t) = \int_0^t \chi_{\mathcal{A}}(\xi_N(s))\, ds,$$

where $\chi_{\mathcal{A}}$ represents the characteristic function of the set $\mathcal{A}$. Denote by $S^{\mathcal{A}}(t)$ the generalized inverse of $T^{\mathcal{A}}(t)$:

$$S^{\mathcal{A}}(t) = \sup\{s \geq 0 : T^{\mathcal{A}}(s) \leq t\}. \tag{2.1}$$

The trace of $\xi_N(\cdot)$ on $\mathcal{A}$, denoted by $(\xi_N^{\mathcal{A}}(t) : t \geq 0)$, is defined by

$$\xi_N^{\mathcal{A}}(t) = \xi_N(S^{\mathcal{A}}(t)), \quad t \geq 0. \tag{2.2}$$

It is an $\mathcal{A}$-valued, continuous-time Markov chain, obtained by turning off the clock when the process $\xi_N(\cdot)$ visits the set $\mathcal{A}^c$, that is, by deleting all excursions to $\mathcal{A}^c$. For this reason, it is called the *trace process* of $\xi_N(\cdot)$ on $\mathcal{A}$.

Let $\Psi_N : \mathcal{E}_N \to S$ be the projection given by

$$\Psi_N(\eta) = \sum_{x \in S} x \cdot \chi_{\mathcal{E}_N^x}(\eta).$$

The order process $(Y_N(t) : t \geq 0)$ is defined as

$$Y_N(t) = \Psi_N(\xi_N^{\mathcal{E}_N}(t)), \quad t \geq 0. \tag{2.3}$$

Denote by $\mathbb{Q}_\eta^N$, $\eta \in \mathcal{E}_N$, the probability measure on $D(\mathbb{R}_+, S)$ induced by the measure $\mathbf{P}_\eta^N$ and the order process Y_N.

The definition of metastability relies on two conditions. Let $\mathcal{L}$ be a generator of an S-valued, continuous-time Markov chain. Denote by $\mathbb{Q}_x^{\mathcal{L}}$, $x \in S$, the probability measure on $D(\mathbb{R}_+, S)$ induced by the Markov chain whose generator is $\mathcal{L}$ and which starts from x.

Condition $\mathfrak{C}_{\mathcal{L}}$. For all $x \in S$ and sequences $(\eta^N)_{N \in \mathbb{N}}$ such that $\eta^N \in \mathcal{E}_N^x$ for all $N \in \mathbb{N}$, the sequence $(\mathbb{Q}_{\eta^N}^N)_{N \in \mathbb{N}}$ of laws converges to $\mathbb{Q}_x^{\mathcal{L}}$ as $N \to \infty$.

The next condition asserts that the process $\xi_N(\cdot)$ spends a negligible amount of time on Δ_N on each finite time interval. It ensures that the trace process does not differ much from the original one when starting from a well.

Condition $\mathfrak{D}$. For all $t > 0$,

$$\lim_{N \to \infty} \max_{x \in S} \sup_{\eta \in \mathcal{E}_N^x} \mathbf{E}_\eta^N \Big[\int_0^t \chi_{\Delta_N}(\xi_N(s))\, ds \Big] = 0.$$

The next definition is taken from [6].

Definition 2.1. The process $\xi_N(\cdot)$ is said to be $\mathcal{L}$-*metastable* if conditions $\mathfrak{C}_{\mathcal{L}}$ and $\mathfrak{D}$ hold.

The first main result of this article provides sufficient conditions, expressed in terms of properties of the solutions of resolvent equations, for condition $\mathfrak{C}_{\mathcal{L}}$ to hold. The second one asserts that these sufficient conditions are also necessary.

Fix a function $g : S \to \mathbb{R}$, and let $G_N : \mathcal{H}_N \to \mathbb{R}$ be its lifting to $\mathcal{H}_N$ given by

$$G_N(\eta) = \sum_{x \in S} g(x) \chi_{\mathcal{E}_N^x}(\eta). \tag{2.4}$$

Note that the function G_N is constant on each well $\mathcal{E}_N^x$ and vanishes on Δ_N. For $\lambda > 0$, denote by $F_N = F_N^{\lambda,g}$ the unique solution of the resolvent equation

$$(\lambda - \mathcal{L}_N) F_N = G_N. \tag{2.5}$$

Condition $\mathfrak{R}_{\mathcal{L}}$. For all $\lambda > 0$ and $g : S \to \mathbb{R}$, the unique solution F_N of the resolvent equation (2.5) is asymptotically constant in each set $\mathcal{E}_N^x$:

$$\lim_{N\to\infty} \sup_{\eta \in \mathcal{E}_N^x} |F_N(\eta) - f(x)| = 0, \quad x \in S, \tag{2.6}$$

where $f : S \to \mathbb{R}$ is the unique solution of the reduced resolvent equation

$$(\lambda - \mathfrak{L}) f = g. \tag{2.7}$$

Remark 2.2. Condition $\mathfrak{R}_{\mathcal{L}}$ is usually proved in two steps. One first shows that for every $\lambda > 0$ and $g : S \to \mathbb{R}$ the solution F_N of the resolvent equation (2.5) is asymptotically constant on each well. In other words, that (2.6) holds for some f. Then, one proves $(\lambda - \mathfrak{L}) f = g$ for some generator $\mathfrak{L}$.

The first main result of the article reads as follows.

Theorem 2.3. *The process $\xi_N(\cdot)$ is $\mathcal{L}$-metastable if, and only if, condition $\mathfrak{R}_{\mathcal{L}}$ is fulfilled. In other words, Conditions $\mathfrak{D}$ and $\mathfrak{C}_{\mathcal{L}}$ hold if, and only if, condition $\mathfrak{R}_{\mathcal{L}}$ is in force.*

Remark 2.4. This result provides a new tool to prove metastability. The existing methods rely on explicit computations involving the stationary state. In particular, they cannot be applied to non-reversible dynamics whose stationary states are not known explicitly, for example, to small perturbations of dynamical systems or to the superposition of Glauber and Kawasaki dynamics. That the solution of a resolvent equation is constant on the wells might be proven without turning to the stationary state.

Remark 2.5. The introduction of the set Δ_N which separates the wells makes condition $\mathfrak{R}_{\mathcal{L}}$ plausible. The challenge is to tune Δ_N correctly: sufficiently large to prove $\mathfrak{R}_{\mathcal{L}}$, but small enough for $\mathfrak{D}$ to hold.

Remark 2.6. Solving the martingale problem through a resolvent equation, instead of a Poisson equation, considerably simplifies the proof of the metastable behavior of the process. As the solutions of the resolvent equations are bounded (see (4.2)), one can hope to obtain bounds and convergence in L^∞, as we do here, instead of L^2. Moreover, many L^1-estimates simplify substantially due to the L^∞-bound on the solution of the resolvent equation.

Remark 2.7. Condition $\mathfrak{R}_{\mathcal{L}}$ being necessary and sufficient for metastability implies that it holds for all models whose metastable behavior has been derived so far. The reader will find in [14, 35] a list of such dynamics.

2.1. Applications

To convince the skeptical reader that condition $\mathfrak{R}_{\mathcal{L}}$ is not too stringent, besides the fact, mentioned before, that it is also necessary for metastability, we provide two frameworks where this condition can be proven. First, Theorem 2.8 states that condition $\mathfrak{R}_{\mathcal{L}}$ follows from properties (H0) and (H1), introduced in [6, 8]. Then, in the next section, to illustrate how to prove condition $\mathfrak{R}_{\mathcal{L}}$ when assumption (H1) is violated, we prove that it holds for critical condensing zero-range processes.

The statement of Theorem 2.8 requires some notation. Denote by $r_N(x, y)$ the mean-jump rate between the sets $\mathcal{E}_N^x$ and $\mathcal{E}_N^y$:

$$r_N(x, y) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta)\lambda_N(\eta)\mathbf{P}_\eta^N[\tau_{\mathcal{E}_N^y} < \tau^+_{\breve{\mathcal{E}}_N^y}]. \tag{2.8}$$

In this formula, $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{A}}^+$, for $\mathcal{A} \subset \mathcal{H}_N$, stand for the hitting and return time of the set $\mathcal{A}$, respectively:

$$\begin{aligned} \tau_{\mathcal{A}} &= \inf\{t > 0 : \xi_N(t) \in \mathcal{A}\}. \\ \tau_{\mathcal{A}}^+ &= \inf\{t \geq \sigma_1 : \xi_N(t) \in \mathcal{A}\}, \quad \text{where} \quad \sigma_1 = \inf\{t \geq 0 : \xi_N(t) \neq \xi_N(0)\}, \end{aligned} \tag{2.9}$$

and $\breve{\mathcal{E}}_N^y = \bigcup_{x \in S\setminus\{y\}} \mathcal{E}_N^x$ for $y \in S$.

Condition (H0). For all $x \neq y \in S$, the sequence $r_N(x, y)$ converges. Denote its limit by $r(x, y)$:

$$r(x, y) = \lim_{N\to\infty} r_N(x, y).$$

Let $\mathcal{D}_N(F)$ be the Dirichlet form of a function $F : \mathcal{H}_N \to \mathbb{R}$ with respect to the generator $\mathcal{L}_N$:

$$\mathcal{D}_N(F) = \langle F, (-\mathcal{L}_N F)\rangle_{\mu_N}.$$

Summation by parts yields

$$\mathcal{D}_N(F) = \frac{1}{2} \sum_{\eta,\eta' \in \mathcal{H}_N} \mu_N(\eta)R_N(\eta, \eta')[F(\eta') - F(\eta)]^2.$$

Fix two disjoint non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}_N$. The *equilibrium potential* between $\mathcal{A}$ and $\mathcal{B}$ with respect to the process $\xi_N(\cdot)$ is denoted by $h_{\mathcal{A},\mathcal{B}}$ and is given by

$$h_{\mathcal{A},\mathcal{B}}(\eta) = \mathbf{P}^N_\eta[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}], \quad \eta \in \mathcal{H}_N. \tag{2.10}$$

The *capacity* between $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\mathrm{cap}_N(\mathcal{A}, \mathcal{B}) = \mathcal{D}_N(h_{\mathcal{A},\mathcal{B}}).$$

Condition (H1). For each $x \in S$, there exists a sequence $(\xi^x_N : N \ge 1)$ of configurations such that $\xi^x_N \in \mathcal{E}^x_N$ for all $N \ge 1$ and

$$\lim_{N\to\infty} \max_{\eta\in\mathcal{E}^x_N} \frac{\mathrm{cap}_N(\mathcal{E}^x_N, \breve{\mathcal{E}}^x_N)}{\mathrm{cap}_N(\xi^x_N, \eta)} = 0.$$

Theorem 2.8. *Assume that conditions* (H0) *and* (H1) *are in force. Then the solution F_N of the resolvent equation* (2.5) *is asymptotically constant on each well $\mathcal{E}^x_N$ in the sense that*

$$\lim_{N\to\infty} \max_{x\in S} \max_{\eta,\zeta\in\mathcal{E}^x_N} |F_N(\eta) - F_N(\zeta)| = 0.$$

Furthermore, let $f_N : S \to \mathbb{R}$ be the function given by

$$f_N(x) = \frac{1}{\mu_N(\mathcal{E}^x_N)} \sum_{\eta\in\mathcal{E}^x_N} F_N(\eta)\mu_N(\eta), \quad x \in S,$$

and let f be a limit point of the sequence f_N. Then

$$[(\lambda - \mathcal{L}_Y) f](y) = g(y) \tag{2.11}$$

for all $y \in S$ such that $\mu_N(\Delta_N)/\mu_N(\mathcal{E}^y_N) \to 0$, where g is the function in (2.4)*. In this formula, $\mathcal{L}_Y$ is the generator of the continuous-time Markov process whose jump rates are given by $r(x, y)$, introduced in* (H0)*.*

Remark 2.9. Under assumptions (H0), (H1) and $\mathfrak{D}$, condition $\mathfrak{R}_{\mathcal{L}_Y}$ is in force. Indeed, by [8, Theorem 2.1], $\mathfrak{C}_{\mathcal{L}_Y}$ holds. Hence, by Theorem 2.3, $\mathfrak{R}_{\mathcal{L}_Y}$ is fulfilled. A direct proof is also possible.

Remark 2.10. Conditions (H0) and (H1) have been proved in many different contexts including condensing zero-range models [2,7,34,41,58], inclusion processes [10,31,32], Ising, Potts and Blume–Capel models at low temperature [33,38,39,43], random walks and diffusions in potential fields [42,44,45,56] and many others [35,40].

Remark 2.11. Lemma 7.4 provides a sufficient condition for the identity (2.11) to hold in the case where $\mu_N(\Delta_N)/\mu_N(\mathcal{E}^y_N)$ does not vanish asymptotically.

The rest of the article is organized as follows. In Section 3, we introduce the critical zero-range process and state, in Theorem 3.2, that it fulfills conditions $\mathfrak{R}_{\mathcal{L}}$ and $\mathfrak{D}$. In

Section 4, we prove Theorem 2.3. In Sections 5–7, we prove Theorem 2.8 and provide further different sets of sufficient conditions, namely, conditions $\mathfrak{V}$ and $\mathfrak{M}$, for $\mathfrak{D}$ or $\mathfrak{R}_{\mathcal{L}}$ to hold. These families of sufficient conditions were designed to encompass most of the dynamics whose metastable behavior have been derived so far. Sections 8–12 are devoted to the proof of Theorem 3.2.

3. Critical zero-range dynamics

In this section, we introduce the critical condensing zero-range process to which we apply the resolvent approach described in the previous section. Fix a finite set S with $|S| = \kappa \geq 2$ elements, and consider a continuous-time Markov chain on S with generator L_X acting on functions $f : S \to \mathbb{R}$ as

$$(L_X f)(x) = \sum_{y \in S} r(x, y)[f(y) - f(x)]$$

for some jump rate $r : S \times S \to \mathbb{R}_+$ assumed to be symmetric $[r(x, y) = r(y, x)$ for all $x, y \in S]$. Set $r(x, x) = 0$ for all $x \in S$ for convenience. Denote by $(X(t))_{t \geq 0}$ the Markov chain generated by L_X and assume that this chain is irreducible. Note that the process $X(\cdot)$ is reversible with respect to the uniform measure $m(\cdot)$ on S $[m(x) = 1/\kappa$ for all $x \in S]$.

The zero-range process describes the evolution of particles on S. A configuration $\eta \in \mathbb{N}^S$ of particles is written as $\eta = (\eta_x)_{x \in S}$ where η_x represents the number of particle at x under the configuration η. For $N \in \mathbb{N}$ and $S_0 \subset S$, denote by $\mathcal{H}_{N,S_0} \subset \mathbb{N}^{S_0}$ the subset of configurations on S_0 with exactly N particles:

$$\mathcal{H}_{N,S_0} = \Big\{\eta \in \mathbb{N}^{S_0} : \sum_{x \in S_0} \eta_x = N\Big\}. \tag{3.1}$$

Let $\mathcal{H}_N = \mathcal{H}_{N,S}$. The *critical zero-range process* is the continuous-time Markov chain $\{\eta_N(t)\}_{t \geq 0}$ on $\mathcal{H}_N$ with generator acting on functions $F : \mathcal{H}_N \to \mathbb{R}$ as

$$(\mathcal{L}_N F)(\eta) = \sum_{x,y \in S} g(\eta_x) r(x, y)[F(\sigma^{x,y}\eta) - F(\eta)], \quad \eta \in \mathcal{H}_N, \tag{3.2}$$

where

$$g(0) = 0, \quad g(1) = 1, \quad \text{and} \quad g(n) = \frac{n}{n-1} \text{ for } n \geq 2.$$

In this equation, $\sigma^{x,y}\eta$, $x, y \in S$, stands for the configuration obtained from η by moving a particle from x to y, when there is at least one particle at x:

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_x - 1 & \text{if } z = x, \\ \eta_y + 1 & \text{if } z = y, \\ \eta_z & \text{otherwise,} \end{cases}$$

if $\eta_x \geq 1$. Otherwise, $\sigma^{x,y}\eta = \eta$.

3.1. Condensation of particles

It is elementary to check that the unique invariant measure for the irreducible Markov chain $\eta_N(\cdot)$ is given by

$$\mu_N(\eta) = \frac{N}{Z_{N,\kappa}(\log N)^{\kappa-1}} \frac{1}{\mathbf{a}(\eta)},$$

where

$$\mathbf{a}(\eta) = \prod_{x\in S} a(\eta_x) \quad \text{with} \quad a(n) = \max\{n, 1\} \text{ for } n \geq 0,$$

and where the partition function $Z_{N,\kappa}$ is defined by

$$Z_{N,\kappa} = \frac{N}{(\log N)^{\kappa-1}} \sum_{\eta\in\mathcal{H}_N} \frac{1}{\mathbf{a}(\eta)}. \tag{3.3}$$

The factor $N/(\log N)^{\kappa-1}$ was introduced so that $Z_{N,\kappa}$ has a non-degenerate limit when N tends to infinity: By [41, Proposition 4.1], $\lim_{N\to\infty} Z_{N,\kappa} = \kappa$. Furthermore, the zero-range process $\eta_N(\cdot)$ is reversible with respect to $\mu_N(\cdot)$.

Define the *metastable well* as

$$\mathcal{E}_N^x = \{\eta \in \mathcal{H}_N : \eta_x \geq N - \ell_N\}, \quad x \in S,$$

where ℓ_N is any sequence satisfying

$$\lim_{N\to\infty} \frac{\ell_N}{N} = 0 \quad \text{and} \quad \lim_{N\to\infty} \frac{\log \ell_N}{\log N} = 1.$$

Assume that $\ell_N = N/\log N$ for simplicity. The set $\mathcal{E}_N^x$ can be regarded as a collection of configurations in which almost all particles are sitting at site x. As defined previously, let

$$\mathcal{E}_N = \bigcup_{x\in S} \mathcal{E}_N^x \quad \text{and} \quad \Delta_N = \mathcal{H}_N \setminus \bigcup_{x\in S} \mathcal{E}_N^x,$$

so that $\mathcal{H}_N = \mathcal{E}_N \cup \Delta_N$ gives a partition of $\mathcal{H}_N$. The following result is [41, Theorem 2.3].

Theorem 3.1. *For all $x \in S$, $\lim_{N\to\infty} \mu_N(\mathcal{E}_N^x) = 1/\kappa$. In particular,*

$$\lim_{N\to\infty} \mu_N(\mathcal{E}_N) = 1 \quad \textit{and} \quad \lim_{N\to\infty} \mu_N(\Delta_N) = 0.$$

Hence, as $N \to \infty$, under the invariant measure, almost all particles are condensed at a single site. In this sense, the critical zero-range process $\eta_N(\cdot)$ condensates. The main result of the article describes the evolution of the condensate.

This model is said to be "critical" for the following reason. Suppose that we replace $g(\eta_x)$ in (3.2) by $[g(\eta_x)]^\alpha$ for some $\alpha > 0$. It is known that the condensation phenomenon occurs if $\alpha \geq 1$, while a diffusive behavior without condensation is observed if $\alpha < 1$. For this reason the zero-range process $\eta_N(\cdot)$ is said to be critical at $\alpha = 1$.

3.2. Order process

Let $\theta_N = N^2 \log N$ be the time-scale at which the condensate moves, and denote by $\xi_N(\cdot)$ the process obtained by speeding up the zero-range process $\eta_N(\cdot)$ by θ_N, i.e., $\xi_N(t) = \eta_N(t\theta_N)$ for all $t \ge 0$. Note that the process $\xi_N(\cdot)$ is the $\mathcal{H}_N$-valued, continuous-time Markov chain whose generator is given by $\mathcal{L}_N^{\xi} = \theta_N \mathcal{L}_N$.

Denote by $\mathbf{P}_\eta^N$ the probability measure on $D(\mathbb{R}_+, \mathcal{H}_N)$ induced by the process $\xi_N(\cdot)$ starting from $\eta \in \mathcal{H}_N$, and by $\mathbf{E}_\eta^N$ the expectation with respect to $\mathbf{P}_\eta^N$.

Recall from (2.2), (2.3) the definition of the trace process $(\xi_N^{\mathcal{E}_N}(t))_{t\ge 0}$, of the projection $\Psi_N : \mathcal{E}_N \to S$, and of the order process $(Y_N(t))_{t\ge 0}$. For critical zero-range processes, the order process $Y_N(\cdot)$ specifies the position of the condensate for the trace process $\xi_N^{\mathcal{E}_N}(t)$. Recall that $\mathbb{Q}_\eta^N$, $\eta \in \mathcal{H}_N$, denotes the probability law on $D(\mathbb{R}_+, S)$ induced by the order process $Y_N(\cdot)$ when the underlying zero-range process $\xi_N(\cdot)$ starts from η.

3.3. Main result

We first introduce the S-valued Markov chain $(Y(t))_{t\ge 0}$ describing the evolution of the condensate. Denote by τ_C^X, $C \subset S$, the hitting time of the set C with respect to the random walk $X(\cdot)$ introduced above:

$$\tau_C^X = \inf\{t > 0 : X(t) \in C\}.$$

Let $\mathbb{Q}_x^X$, $x \in S$, be the law of the process $X(\cdot)$ starting at x. For two non-empty disjoint subsets A, B of S, the *equilibrium potential* between A and B with respect to the process $X(\cdot)$ is the function $h_{A,B}^X : S \to \mathbb{R}$ defined by

$$h_{A,B}^X(x) = \mathbb{Q}_x^X[\tau_A^X < \tau_B^X], \quad x \in S. \tag{3.4}$$

The *capacity* between A and B is given by

$$\mathrm{cap}_X(A, B) = D_X(h_{A,B}^X), \tag{3.5}$$

where $D_X(\cdot)$ stands for the Dirichlet form associated to the process $X(\cdot)$, which can be written as

$$D_X(f) = \frac{1}{2} \sum_{x,y\in S} m(x) r(x,y) [f(y) - f(x)]^2 \tag{3.6}$$

for $f : S \to \mathbb{R}$. If the sets A, B are singletons, then we write $\mathrm{cap}_X(x, y)$ instead of $\mathrm{cap}_X(\{x\}, \{y\})$.

Denote by $(Y(t))_{t\ge 0}$ the S-valued, continuous-time Markov chain associated to the generator L_Y acting on $f : S \to \mathbb{R}$ as

$$(L_Y f)(x) = 6\kappa \sum_{y\in S} \mathrm{cap}_X(x, y) [f(y) - f(x)]. \tag{3.7}$$

Recall from the previous section that we denote by $\mathbb{Q}_x^{L_Y}$, $x \in S$, the probability measure on $D(\mathbb{R}_+, S)$ induced by the Markov chain $Y(\cdot)$ starting from x. Sometimes, we represent $\mathbb{Q}_x^{L_Y}$ by $\mathbb{Q}_x^{Y}$.

The next theorem is the third main result of the article.

Theorem 3.2. *Conditions $\mathfrak{R}_{L_Y}$ and $\mathfrak{D}$ hold for the critical zero-range process, where L_Y is given by* (3.7)*. In particular, the critical zero-range process is L_Y-metastable.*

In view of Theorem 2.3, this result establishes that, in the time-scale $\theta_N = N^2 \log N$, the condensate evolves as the Markov chain $Y(\cdot)$ and outside some time intervals whose total length is negligible, almost all particles sit on a single site.

Remark 3.3. The so-called martingale approach developed in [6, 8] to derive the metastable behavior of a Markov process, based on potential theory, does not apply here because the process does not visit all points of a well before jumping to a new one, and condition (H1) of [6] is violated. This characteristic is the main difference between the critical zero-range process and the supercritical ones.

Remark 3.4. By using the so-called Poisson equation approach developed in [45,53,56], we proved in [41] a weaker version of Theorem 3.2. Denote by $\mu_N^x(\cdot)$, $x \in S$, the measure on $\mathcal{E}_N^x$ obtained by conditioning μ_N on $\mathcal{E}_N^x$:

$$\mu_N^x(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{E}_N^x)}, \quad \eta \in \mathcal{E}_N^x.$$

We assumed in [41] that the initial distribution is a measure ν_N concentrated on a set $\mathcal{E}_N^x$ for some x, and satisfying the following L^2-condition: there exists a finite constant C_0 such that

$$E_{\mu_N^x}\Big[\Big(\frac{d\nu_N}{d\mu_N^x}\Big)^2\Big] = \sum_{\eta \in \mathcal{E}_N^x} \frac{\nu_N(\eta)^2}{\mu_N^x(\eta)} \le C_0 \quad \text{for all } N \in \mathbb{N}. \tag{3.8}$$

The main novelty of Theorem 3.2 is that it removes assumption (3.8) and allows the process to start from a fixed configuration inside some well.

Remark 3.5. The proof of Theorem 3.2 relies on many estimates obtained in [41], in particular, on the construction of a superharmonic function inside the wells.

Remark 3.6. The equilibration inside the well, or the loss of memory, is obtained in two different manners. First, we derive a sharp bound on the relaxation time of the process reflected at the boundary of a well. This relaxation time is shown to be much smaller than the metastable time-scale θ_N.

Then, we show that the process visits the bottom of the well before visiting a new well. This crucial property is derived with the help of the superharmonic function alluded to above. Thus, although the process does not visit all configurations in a well before reaching a new one, it visits a specific configuration.

Remark 3.7. The symmetry of the jump rates r of the chain X is used in the construction of the superharmonic function. Theorem 3.2 should still hold without this assumption, but a proof is missing.

The proof of Theorem 3.2 relies on Theorem 2.3. The strategy is presented in Section 8 and the details in Sections 9–12.

4. Proof of Theorem 2.3

In the first part of this section, we show that condition $\mathfrak{R}_{\mathcal{L}}$ implies conditions $\mathfrak{C}_{\mathcal{L}}$ and $\mathfrak{D}$. In the second part, we prove the converse.

4.1. *Condition $\mathfrak{R}_{\mathcal{L}}$ entails $\mathfrak{C}_{\mathcal{L}}$ and $\mathfrak{D}$*

We first show that the solution of the resolvent equation is bounded. Fix a function $g : S \to \mathbb{R}$ and $\lambda > 0$. It is well known that the solution of the resolvent equation (2.5) can be represented as

$$F_N(\eta) = \mathbf{E}_\eta^N \Big[\int_0^\infty e^{-\lambda s} G_N(\xi_N(s))\, ds \Big]. \tag{4.1}$$

In particular, there exists a finite constant $C_0 = C_0(\lambda, g)$ such that

$$\max_{\eta \in \mathcal{H}_N} |F_N(\eta)| \le C_0. \tag{4.2}$$

The next result asserts that condition $\mathfrak{R}_{\mathcal{L}}$ implies condition $\mathfrak{D}$.

Lemma 4.1. *Assume that condition $\mathfrak{R}_{\mathcal{L}}$ holds. Then condition $\mathfrak{D}$ is in force.*

Proof. We first claim that for all $\lambda > 0$,

$$\lim_{N\to\infty} \max_{\eta \in \mathcal{E}_N} \mathbf{E}_\eta^N \Big[\int_0^\infty e^{-\lambda s} \chi_{\Delta_N}(\xi_N(s))\, ds \Big] = 0. \tag{4.3}$$

Indeed, fix $\lambda > 0$ and $g : S \to \mathbb{R}$ given by $g(x) = 1$ for all $x \in S$. Let G_N, F_N be given by (2.4) and (2.5), respectively. By (4.1) and since $G_N = \chi_{\mathcal{E}_N}$, for all $\eta \in \mathcal{H}_N$,

$$F_N(\eta) - \frac{1}{\lambda} = \mathbf{E}_\eta^N \Big[\int_0^\infty e^{-\lambda s} [G_N(\xi_N(s)) - 1]\, ds \Big] = -\mathbf{E}_\eta^N \Big[\int_0^\infty e^{-\lambda s} \chi_{\Delta_N}(\xi_N(s))\, ds \Big].$$

Since the solution f of the reduced resolvent equation $(\lambda - \mathfrak{L}) f = g$ is $f(x) = 1/\lambda$ for all $x \in S$, claim (4.3) follows from (2.6).

Fix $t, \lambda > 0$, and observe that

$$\begin{aligned} \mathbf{E}_\eta^N \Big[\int_0^t \chi_{\Delta_N}(\xi_N(s))\, ds \Big] &\le e^{\lambda t} \mathbf{E}_\eta^N \Big[\int_0^t e^{-\lambda s} \chi_{\Delta_N}(\xi_N(s))\, ds \Big] \\ &\le e^{\lambda t} \mathbf{E}_\eta^N \Big[\int_0^\infty e^{-\lambda s} \chi_{\Delta_N}(\xi_N(s))\, ds \Big] \end{aligned}$$

for all $\eta \in \mathcal{H}_N$. Hence, condition $\mathfrak{D}$ follows from (4.3). ∎

We prove some consequences of condition $\mathfrak{R}_{\mathcal{L}}$. The next result asserts that the process $\xi_N(\cdot)$ cannot jump from one well to another quickly. The proof of this result is similar to the one of [56, Proposition 5.2]. Recall from (2.9) that we denote by $\tau_{\mathcal{A}}$, $\mathcal{A} \subset \mathcal{H}_N$, the hitting time of the set $\mathcal{A}$. Let

$$\breve{\mathcal{E}}_N^x = \bigcup_{y \in S \setminus \{x\}} \mathcal{E}_N^y, \quad x \in S. \tag{4.4}$$

Lemma 4.2. *Assume that condition $\mathfrak{R}_{\mathcal{L}}$ holds. Then, for all $x \in S$,*

$$\limsup_{t \to 0} \limsup_{N \to \infty} \sup_{\eta \in \mathcal{E}_N^x} \mathbf{P}_\eta^N [\tau_{\breve{\mathcal{E}}_N^x} < t] = 0. \tag{4.5}$$

Proof. Fix $\lambda > 0$, $x \in S$ and $\eta^N \in \mathcal{E}_N^x$. Let $f : S \to \mathbb{R}$ be the function given by $f(y) = 1 - \delta_{x,y}$. Set $g = (\lambda - \mathcal{L}) f$, and denote by F_N the solution of the resolvent equation (2.5). Let $M_N(t)$ be the martingale defined by

$$M_N(t) = F_N(\xi_N(t)) - F_N(\xi_N(0)) - \int_0^t (\mathcal{L}_N F_N)(\xi_N(r))\, dr.$$

As $\mathcal{L}_N F_N = \lambda F_N - G_N$, for every $t > 0$,

$$\mathbf{E}_\eta^N [F_N(\xi_N(t \wedge \tau))] = F_N(\eta^N) + \mathbf{E}_\eta^N \Big[\int_0^{t \wedge \tau} (\lambda F_N - G_N)(\xi_N(r))\, dr \Big], \tag{4.6}$$

where $\tau = \tau_{\breve{\mathcal{E}}_N^x}$.

By condition $\mathfrak{R}_{\mathcal{L}}$ and the definition of f, $\lim_{N\to\infty} F_N(\eta^N) = 0$. By (4.2) and by definition of G_N, $\lambda F_N - G_N$ is bounded. The right-hand side of (4.6) is thus bounded by $a_N + C_0 t$ for some finite constant C_0 and a sequence a_N such that $a_N \to 0$.

We turn to the left-hand side of (4.6). Since $f \ge 0$, by condition $\mathfrak{R}_{\mathcal{L}}$ there exists a constant $c_N \ge 0$ such that $c_N \to 0$ and $\widetilde{F}_N(\zeta) = F_N(\zeta) + c_N \ge 0$ for all $\zeta \in \mathcal{E}_N$.

Claim A. $\widetilde{F}_N(\zeta) \ge 0$ *for all* $\zeta \in \mathcal{H}_N$.

To prove this claim, let ζ be a configuration at which $\widetilde{F}_N$ achieves its minimum value so that $(\mathcal{L}_N F_N)(\zeta) = (\mathcal{L}_N \widetilde{F}_N)(\zeta) \ge 0$. If $\zeta \in \mathcal{E}_N$, there is nothing to prove. If $\zeta \in \Delta_N$, then since G_N vanishes on Δ_N and $(\lambda - \mathcal{L}_N) F_N = G_N$, it follows that $F_N(\zeta) = \lambda^{-1} (\mathcal{L}_N F_N)(\zeta) \ge 0$ so that $\widetilde{F}_N(\zeta) = F_N(\zeta) + c_N \ge 0$, as claimed.

The left-hand side of (4.6) is equal to $\mathbf{E}_\eta^N [\widetilde{F}_N(\xi_N(t \wedge \tau))] - c_N$. By $\mathfrak{R}_{\mathcal{L}}$, for N sufficiently large, $\widetilde{F}_N(\zeta) \ge 1/2$ on $\breve{\mathcal{E}}_N^x$. Hence, the left-hand side of (4.6) is bounded below by $(1/2)\mathbf{P}_\eta^N [\xi_N(t \wedge \tau) \in \breve{\mathcal{E}}_N^x] - c_N$.

Putting together the previous estimates yields

$$\mathbf{P}_\eta^N [\xi_N(t \wedge \tau) \in \breve{\mathcal{E}}_N^x] \le 2(c_N + a_N + C_0 t).$$

To complete the proof of the lemma, it remains to remark that

$$\mathbf{P}_\eta^N [\tau_{\breve{\mathcal{E}}_N^x} < t] \le \mathbf{P}_\eta^N [\xi_N(t \wedge \tau) \in \breve{\mathcal{E}}_N^x]. \qquad \blacksquare$$

The next result states that the sequence $(\mathbb{Q}^N_{\eta^N})_{N\in\mathbb{N}}$ is tight.

Proposition 4.3. *Assume that condition $\mathfrak{D}$ and (4.5) hold. Then the sequence $(\mathbb{Q}^N_{\eta^N})_{N\in\mathbb{N}}$ is tight, and any limit point $\mathbb{Q}^*$ of this sequence is such that*

$$\mathbb{Q}^*[Y(t) \neq Y(t-)] = 0 \quad \textit{for all } t > 0. \tag{4.7}$$

Proof. This result follows from conditions $\mathfrak{D}$, (4.5) and Aldous' criterion. We refer to [41, Theorem 5.4] for a proof. ∎

Recall from (2.1) the definition of the time-change $S^{\mathcal{A}}(t)$, $\mathcal{A} \subset \mathcal{H}_N$. Clearly, for all $t \geq 0$,

$$T^{\mathcal{A}}(S^{\mathcal{A}}(t)) = t. \tag{4.8}$$

In contrast, we only have $S^{\mathcal{A}}(T^{\mathcal{A}}(t)) \geq t$ and a strict inequality may occur. Furthermore, for all $t > 0$ and $\varepsilon > 0$,

$$\{S^{\mathcal{A}}(t) - t \geq \varepsilon\} \subset \left\{ \int_0^{t+\varepsilon} \chi_{\mathcal{A}^c}(\xi_N(s))\, ds \geq \varepsilon \right\}. \tag{4.9}$$

Indeed, if $S^{\mathcal{A}}(t) \geq t + \varepsilon$, applying $T^{\mathcal{A}}$ on both sides of this inequality, as $T^{\mathcal{A}}$ is an increasing function, by (4.8),

$$t \geq T^{\mathcal{A}}(t+\varepsilon) \quad \text{so that} \quad t + \varepsilon - T^{\mathcal{A}}(t+\varepsilon) \geq \varepsilon.$$

This last relation corresponds exactly to the right-hand side of (4.9).

Denote by $\{\mathcal{F}^0_t\}_{t\geq 0}$ the natural filtration of $D(\mathbb{R}_+, \mathcal{H}_N)$ generated by the process $\xi_N(\cdot)$, $\mathcal{F}^0_t = \sigma(\xi_N(s) : s \in [0,t])$, and by $\{\mathcal{F}_t\}_{t\geq 0}$ its usual augmentation. Let $\mathcal{G}^N_t$ be the filtrations defined by

$$\mathcal{G}^N_t := \mathcal{F}_{S^{\mathcal{E}_N}(t)} \quad \text{for } t > 0. \tag{4.10}$$

Lemma 4.4. *Assume that condition $\mathfrak{D}$ is in force. Then, for all $\lambda > 0$ and $t > 0$,*

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N} \mathbf{E}^N_\eta [e^{-\lambda t} - e^{-\lambda S^{\mathcal{E}_N}(t)}] = 0,$$

and

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N} \mathbf{E}^N_\eta \left[\int_0^t \{e^{-\lambda r} - e^{-\lambda S^{\mathcal{E}_N}(r)}\}\, dr \right] = 0.$$

Note that these expressions are positive since $S^{\mathcal{E}_N}(r) \geq r$ for all $r \geq 0$.

Proof. To prove the first assertion, note that the expectation is bounded by

$$\mathbf{E}^N_\eta [K_\lambda(S^{\mathcal{E}_N}(t) - t)],$$

where $K_\lambda(a) = 1 - e^{-\lambda a}$.

Fix $\varepsilon > 0$. As K_λ is continuous, there exists $\delta > 0$ such that $K_\lambda(a) \leq \varepsilon$ for all $0 \leq a \leq \delta$. Since K_λ is bounded by 1, the previous expectation is less than or equal to

$$\varepsilon + \mathbf{P}^N_\eta [S^{\mathcal{E}_N}(t) - t > \delta].$$

By (4.9) and Chebyshev's inequality, this expression is bounded by

$$\varepsilon + \frac{1}{\delta}\mathbf{E}^N_\eta\Big[\int_0^{t+\delta} \chi_{\Delta_N}(\xi_N(s))\, ds\Big].$$

At this point, the first claim of the lemma follows from condition $\mathfrak{D}$ by taking the limit $N \to \infty$ and then $\varepsilon \to 0$.

The proof of the second assertion is similar. The expectation is equal to

$$\mathbf{E}^N_\eta\Big[\int_0^t e^{-\lambda r} K_\lambda(S^{\mathcal{E}_N}(r) - r)\, dr\Big].$$

As $r \mapsto S^{\mathcal{E}_N}(r) - r$ and K_λ are increasing maps, this expectation is bounded by

$$\frac{1}{\lambda}\mathbf{E}^N_\eta[K_\lambda(S^{\mathcal{E}_N}(t) - t)].$$

At this point, the second assertion of the lemma follows from the first one. ■

The next result establishes the uniqueness of limit points of the sequence $\mathbb{Q}^N_{\eta^N}$.

Proposition 4.5. *Assume condition $\mathfrak{R}_{\mathcal{L}}$ is in force. Fix $x \in S$ and a sequence $(\eta^N)_{N\in\mathbb{N}}$ such that $\eta^N \in \mathcal{E}^x_N$ for all $N \in \mathbb{N}$. Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^N_{\eta^N}$ which satisfies (4.7). Then $\mathbb{Q}^* = \mathbb{Q}^{\mathcal{L}}_x$.*

Proof. Fix $\lambda > 0$ and a function $f : S \to \mathbb{R}$, and let $g = (\lambda - \mathcal{L})f$. Denote by F_N the solution of (2.5). Under the measure $\mathbf{P}^N_{\eta^N}$, the process $M_N(t)$ given by

$$M_N(t) = e^{-\lambda t} F_N(\xi_N(t)) - F_N(\xi_N(0)) + \int_0^t e^{-\lambda r}[(\lambda - \mathcal{L}_N)F_N](\xi_N(r))\, dr$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ defined above (4.10). By (2.5), we may replace $(\lambda - \mathcal{L}_N)F_N$ by G_N. Thus, since G_N vanishes on Δ_N,

$$M_N(t) = e^{-\lambda t} F_N(\xi_N(t)) - F_N(\xi_N(0)) + \int_0^t e^{-\lambda r} G_N(\xi_N(r))\chi_{\mathcal{E}_N}(\xi_N(r))\, dr.$$

Recall the definition of the filtration $\{\mathcal{G}^N_t\}_{t\ge 0}$ from (4.10). Since $S^{\mathcal{E}_N}(t)$ is a stopping time with respect to $\mathcal{F}_t$, the process $\widehat{M}_N(t) = M(S^{\mathcal{E}_N}(t))$ is a martingale with respect to the filtration $\{\mathcal{G}^N_t\}_{t\ge 0}$:

$$\begin{aligned}\widehat{M}_N(t) &= e^{-\lambda S^{\mathcal{E}_N}(t)} F_N(\xi^{\mathcal{E}_N}_N(t)) - F_N(\xi^{\mathcal{E}_N}_N(0)) \\ &\quad + \int_0^{S^{\mathcal{E}_N}(t)} e^{-\lambda r} G_N(\xi_N(r))\chi_{\mathcal{E}_N}(\xi_N(r))\, dr.\end{aligned}$$

The presence of the indicator of the set $\mathcal{E}_N$ in the integral permits us to perform the change of variables $r' = T^{\mathcal{E}_N}(r)$. Hence, by (4.8),

$$\widehat{M}_N(t) = e^{-\lambda S^{\mathcal{E}_N}(t)} F_N(\xi^{\mathcal{E}_N}_N(t)) - F_N(\xi^{\mathcal{E}_N}_N(0)) + \int_0^t e^{-\lambda S^{\mathcal{E}_N}(r')} G_N(\xi^{\mathcal{E}_N}(r'))\, dr'.$$

By definitions of G_N, $Y_N(\cdot)$, by condition $\mathfrak{R}_{\mathcal{L}}$ and by Lemmata 4.1 and 4.4,

$$\widehat{M}_N(t) = e^{-\lambda t} f(Y_N(t)) - f(Y_N(0)) - \int_0^t e^{-\lambda r'} g(Y_N(r'))\, dr' + R_N(t),$$

where, for all $t > 0$,

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N^x} \mathbf{E}_\eta^N[R_N(t)] = 0. \tag{4.11}$$

Fix $0 \le s < t$, $p \ge 1$, $0 \le s_1 < s_2 < \cdots < s_p \le s$ and a bounded measurable function $h : S^p \to \mathbb{R}$. Let

$$\mathfrak{M}_f^{s,t}(Y(\cdot)) := e^{-\lambda t} f(Y(t)) - e^{-\lambda s} f(Y(s)) + \int_s^t e^{-\lambda r}[(\lambda - \mathfrak{L}_Y) f](Y(r))\, dr,$$
$$\mathfrak{H}(Y(\cdot)) := h(Y(s_1), \dots, Y(s_p)),$$

and let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^N_{\eta^N}$ satisfying the hypothesis of the proposition. As $\widehat{M}_N(t)$ is a martingale and $\eta^N \in \mathcal{E}_N^x$, by (4.11),

$$E_{\mathbb{Q}^*}[\mathfrak{M}_f^{s,t}(Y(\cdot))\mathfrak{H}(Y(\cdot))] = \lim_{N\to\infty} \mathbf{E}^N_{\eta^N}[\mathfrak{M}_f^{s,t}(Y_N(\cdot))\mathfrak{H}(Y_N(\cdot))] = 0.$$

To complete the proof, it remains to appeal to the uniqueness of solutions of martingale problems in finite state spaces. ■

We are now in a position to prove that condition $\mathfrak{R}_{\mathcal{L}}$ entails $\mathfrak{C}_{\mathcal{L}}$ and $\mathfrak{D}$.

Proof. The statement follows from Lemma 4.1 and Propositions 4.3 and 4.5. ■

4.2. *Conditions $\mathfrak{C}_{\mathcal{L}}$ and $\mathfrak{D}$ imply $\mathfrak{R}_{\mathcal{L}}$*

Recall equation (4.1) for F_N. Since G_N vanishes on Δ_N, we may rewrite this identity as

$$F_N(\eta) = \mathbf{E}_\eta^N\Big[\int_0^\infty e^{-\lambda t} G_N(\xi(t))\chi_{\mathcal{E}_N}(\xi(t))\, dt\Big].$$

As the chain $\xi_N(t)$ is irreducible, $\lim_{t\to\infty} T^{\mathcal{E}}(t) = \infty$. Hence, by the change of variables $t' = T^{\mathcal{E}}(t)$,

$$F_N(\eta) = \mathbf{E}_\eta^N\Big[\int_0^\infty e^{-\lambda S^{\mathcal{E}}(t)} G_N(\xi^{\mathcal{E}}(t))\, dt\Big] = \mathbf{E}_\eta^N\Big[\int_0^\infty e^{-\lambda S^{\mathcal{E}}(t)} g(Y_N(t))\, dt\Big]$$

because $G_N(\xi^{\mathcal{E}}(t)) = g(Y_N(t))$. Therefore,

$$F_N(\eta) = \mathbf{E}_\eta^N\Big[\int_0^\infty e^{-\lambda t} g(Y_N(t))\, dt\Big] + R_N^{(1)}(\eta),$$

where the absolute value of the remainder $R_N^{(1)}(\eta)$ is bounded by

$$\|g\|_\infty \mathbf{E}_\eta^N\Big[\int_0^\infty \{e^{-\lambda t} - e^{-\lambda S^{\mathcal{E}}(t)}\}\, dt\Big]$$

because $S^{\mathcal{E}}(t) \ge t$ for all $t \ge 0$.

By Condition $\mathfrak{C}_{\mathcal{L}}$, for all $x \in S$,

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N^x} \Big| \mathbf{E}_\eta^N \Big[\int_0^\infty e^{-\lambda t} g(Y_N(t))\, dt \Big] - \mathbb{Q}_x^{\mathcal{L}} \Big[\int_0^\infty e^{-\lambda t} g(Y(t))\, dt \Big] \Big| = 0.$$

Note that the convergence is uniform in $\mathcal{E}_N^x$ because we may consider a subsequence $\eta^N \in \mathcal{E}_N^x$ of initial conditions which attains the maximum and apply condition $\mathfrak{C}_{\mathcal{L}}$ to this sequence. By (4.1), the second term in the previous formula is $f(x)$, where f is the solution of (2.7).

To complete the proof of the theorem, it remains to show that the remainder $R_N^{(1)}(\eta)$ converges uniformly to 0. This is a consequence of the second assertion of Lemma 4.4.

5. Potential theory

We review below some results of potential theory used in the next three sections. The notation is the one introduced in Section 2. Recall that we represent by $R_N : \mathcal{H}_N \times \mathcal{H}_N \to [0,\infty)$ the jump rates of the process $\xi_N(\cdot)$, and by $\lambda_N(\eta) = \sum_{\zeta\neq\eta} R_N(\eta,\zeta)$ the holding times. We adopt the convention that the jump rates vanish on the diagonal: $R_N(\eta,\eta) = 0$ for all $\eta \in \mathcal{H}_N$. Denote the jump probabilities by $p_N(\eta,\zeta) = R_N(\eta,\zeta)/\lambda_N(\eta)$.

We represent by $\langle\cdot,\cdot\rangle_{\mu_N}$ the scalar product in $L^2(\mu_N)$: for $F, G : \mathcal{H}_N \to \mathbb{R}$,

$$\langle F, G\rangle_{\mu_N} = \sum_{\eta\in\mathcal{H}_N} F(\eta) G(\eta) \mu_N(\eta).$$

Denote by $\mathcal{L}_N^\dagger$ the adjoint of the generator $\mathcal{L}_N$ in $L^2(\mu_N)$. It is well known that $\mathcal{L}_N^\dagger$ is the generator of an $\mathcal{H}_N$-valued, continuous-time Markov chain, represented by $\xi_N^\dagger(\cdot)$. The jump rates, holding times and jump probabilities of this process are denoted by $R_N^\dagger(\eta,\zeta)$, $\lambda_N^\dagger(\eta)$ and $p_N^\dagger(\eta,\zeta)$, respectively. For a probability measure ν on $\mathcal{H}_N$, we denote by $\mathbf{P}_\nu^{\dagger,N}$ the measure on $D(\mathbb{R}_+, \mathcal{H}_N)$ induced by $\xi_N^\dagger(\cdot)$ starting from ν. Expectation with respect to $\mathbf{P}_\nu^{\dagger,N}$ is represented by $\mathbf{E}_\nu^{\dagger,N}$.

Fix two disjoint non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}_N$. The equilibrium potential between $\mathcal{A}$ and $\mathcal{B}$ with respect to the process $\xi_N(\cdot)$ has been introduced in (2.10). The one for the adjoint process $\xi_N^\dagger(\cdot)$ is denoted by $h_{\mathcal{A},\mathcal{B}}^\dagger : \mathcal{H}_N \to [0,1]$ and is given by

$$h_{\mathcal{A},\mathcal{B}}^\dagger(\eta) = \mathbf{P}_\eta^{\dagger,N}[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}], \quad \eta \in \mathcal{H}_N. \tag{5.1}$$

Recall from (2.8) the definition of the mean-jump rates between $\mathcal{E}_N^x$ and $\mathcal{E}_N^y$ for the process $\xi_N(\cdot)$. The ones for the adjoint process $\xi_N^\dagger(\cdot)$, represented by $r_N^\dagger(x,y)$, are defined analogously. Since the holding times of the adjoint process coincide with the original ones, $\lambda_N^\dagger(\eta) = \lambda_N(\eta)$, $r_N^\dagger(x,y)$ is equal to the right-hand side of (2.8) with $\mathbf{P}_\eta^N$ replaced by $\mathbf{P}_\eta^{\dagger,N}$.

The first result of this section establishes an elementary identity between mean jump rates of the process and its adjoint. Recall that $\breve{\mathcal{E}}_N^x$ has been introduced in (4.4).

Lemma 5.1. *For all $x \neq y \in S$,*

$$\mu_N(\mathcal{E}_N^x) r_N^\dagger(x,y) = \mu_N(\mathcal{E}_N^y) r_N(y,x),$$

and

$$\sum_{z\neq x} r_N^\dagger(x,z) = \sum_{z\neq x} r_N(x,z) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \operatorname{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x).$$

Proof. By the definition (2.8) of the jump rates $r_N(x,y)$,

$$\mu_N(\mathcal{E}_N^x) r_N(x,y) = \sum_{\eta\in\mathcal{E}_N^x} \mu_N(\eta)\lambda_N(\eta)\mathbf{P}_\eta^N[\tau_{\mathcal{E}_N^y} < \tau_{\breve{\mathcal{E}}_N^y}^+].$$

Let $M_N(\eta) = \mu_N(\eta)\lambda_N(\eta)$. This measure is invariant for the embedded, discrete-time Markov chain. With this notation, the right-hand side can be written as

$$\sum_{\eta\in\mathcal{E}_N^x} \sum_{\zeta\in\mathcal{E}_N^y} M_N(\eta)\mathbf{P}_\eta^N[\tau_\zeta = \tau_{\breve{\mathcal{E}}_N}^+].$$

Reversing the trajectory, this sum is seen to be equal to

$$\sum_{\eta\in\mathcal{E}_N^x} \sum_{\zeta\in\mathcal{E}_N^y} M_N(\zeta)\mathbf{P}_\zeta^{\dagger,N}[\tau_\eta = \tau_{\breve{\mathcal{E}}_N}^+] = \mu_N(\mathcal{E}_N^y) r_N^\dagger(y,x),$$

which proves the first assertion of the lemma.

To prove the second one, note that

$$\mu_N(\mathcal{E}_N^x) \sum_{z\neq x} r_N(x,z) = \sum_{\eta\in\mathcal{E}_N^x} M_N(\eta)\mathbf{P}_\eta^N[\tau_{\breve{\mathcal{E}}_N^x} < \tau_{\mathcal{E}_N^x}^+] = \operatorname{cap}_N(\breve{\mathcal{E}}_N^x, \mathcal{E}_N^x).$$

By [26, (2.4) and Lemma 2.3], $\operatorname{cap}_N(\breve{\mathcal{E}}_N^x, \mathcal{E}_N^x) = \operatorname{cap}_N^\dagger(\breve{\mathcal{E}}_N^x, \mathcal{E}_N^x)$, where the last expression represents the capacity with respect to the adjoint process. To conclude the proof, it remains to rewrite the same two identities for the adjoint process. ∎

Conditions (H0), (H1). Recall from Section 2 the statement of these conditions. We present below some consequences of them. The next result is [6, Proposition 5.10], which essentially asserts that the process hits every configuration inside a metastable set before arriving at another metastable set.

Lemma 5.2. *Assume that condition* (H1) *is in force. Fix $x \in S$ and a sequence $(\zeta^N : N \geq 1)$ such that $\zeta^N \in \mathcal{E}_N^x$ for all $N \geq 1$. Then*

$$\limsup_{N\to\infty} \max_{\eta\in\mathcal{E}_N^x} \mathbf{P}_\eta[\tau_{\zeta^N} > \tau_{\breve{\mathcal{E}}_N^x}] = 0.$$

The next result asserts that, starting from a well $\mathcal{E}_N^x$, the process $\xi_N(\cdot)$ visits any point in $\mathcal{E}_N^x$ quickly.

Lemma 5.3. *Assume that conditions* (H0) *and* (H1) *are in force. Fix $x \in S$, and let $(\zeta^N : N \ge 1)$ be a sequence of configurations such that $\zeta^N \in \mathcal{E}_N^x$ for all $N \ge 1$. Then, for all $\delta > 0$,*

$$\limsup_{N\to\infty} \max_{\eta\in\mathcal{E}_N^x} \mathbf{P}_\eta[\tau_{\zeta^N} > \delta] = 0.$$

Proof. Fix a sequence $(\eta^N : N \ge 1)$ such that $\eta^N \in \mathcal{E}_N^x$ for all $N \ge 1$. By [8, Theorem 2.1], the process $Y_N(\cdot)$ converges to $Y(\cdot)$. The assertion of the lemma follows from this fact, Lemma 5.2 and [6, Lemma 3.1]. ■

In the reversible case, the mean jump rate $r_N(\cdot,\cdot)$ can be expressed in terms of capacities: By [6, Lemma 6.8], $r_N(x, y)$ is equal to

$$\frac{1}{\mu_N(\mathcal{E}_N^x)}\big[\mathrm{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x) + \mathrm{cap}_N(\mathcal{E}_N^y, \breve{\mathcal{E}}_N^y) - \mathrm{cap}_N(\mathcal{E}_N^x \cup \mathcal{E}_N^x, \mathcal{E}_N(S\setminus\{x,y\}))\big], \quad (5.2)$$

where $\mathcal{E}_N(S\setminus\{x,y\}) = \bigcup_{z\in S\setminus\{x,y\}} \mathcal{E}_N^z$. Hence, estimating the mean-jump rates boils down to estimating the capacity between metastable wells, which can be achieved by using the variational characterizations of capacities, known as the Dirichlet and the Thomson principles [35]. In the non-reversible case, a robust strategy of estimating mean-jump rates via capacities between wells has also been developed in [8, 34, 44].

We complete this section with a formula for the average of equilibrium potentials. Fix two disjoint non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}_N$. According to [8, Proposition A.2],

$$\sum_{\eta\notin\mathcal{A}\cup\mathcal{B}} \mu_N(\eta) h^\dagger_{\mathcal{A},\mathcal{B}}(\eta) = \mathrm{cap}_N(\mathcal{A},\mathcal{B})\, \mathbf{E}^N_{\nu^\dagger_{\mathcal{A},\mathcal{B}}}\Big[\int_0^{\tau_\mathcal{B}} \chi_{[\mathcal{A}\cup\mathcal{B}]^c}(\xi_N(s))\, ds\Big], \quad (5.3)$$

where $\nu^\dagger_{\mathcal{A},\mathcal{B}}$ is the equilibrium measure between $\mathcal{A}$ and $\mathcal{B}$:

$$\nu^\dagger_{\mathcal{A},\mathcal{B}}(\zeta) = \frac{1}{\mathrm{cap}_N(\mathcal{A},\mathcal{B})} \mu_N(\zeta)\lambda_N(\zeta)\, \mathbf{P}^{\dagger,N}_\zeta[\tau_\mathcal{B} < \tau^+_\mathcal{A}], \quad \zeta\in\mathcal{A}. \quad (5.4)$$

6. The solutions of the resolvent equation

Theorem 2.3 asserts that a sequence of Markov processes is metastable if conditions $\mathfrak{R}_\mathcal{L}$ and $\mathfrak{D}$ are fulfilled. In this section and in the next, we present sufficient conditions for $\mathfrak{R}_\mathcal{L}$ and $\mathfrak{D}$ to hold. We start by dividing condition $\mathfrak{R}_\mathcal{L}$ into two subconditions, $\mathfrak{R}^{(1)}$ and $\mathfrak{R}^{(2)}_\mathcal{L}$.

In this section, we present two mixing properties, assumptions $\mathfrak{V}$ and $\mathfrak{M}$, which imply condition $\mathfrak{R}^{(1)}$. As a by-product, we show that condition $\mathfrak{M}$ implies condition $\mathfrak{D}$ if $\mu_N(\Delta_N)/\mu_N(\mathcal{E}_N^x) \to 0$ for all $x\in S$. We leave condition $\mathfrak{R}^{(2)}_\mathcal{L}$ to the next section.

Condition $\mathfrak{R}^{(1)}$. The solution F_N of the resolvent equation (2.5) is asymptotically constant on each well $\mathcal{E}_N^x$ in the sense that

$$\lim_{N\to\infty} \max_{x\in S} \max_{\eta,\zeta\in\mathcal{E}_N^x} |F_N(\eta) - F_N(\zeta)| = 0.$$

Remark 6.1. Clearly, condition $\mathfrak{R}^{(1)}$ is satisfied if the wells $\mathcal{E}_N^x$ are singletons as in the Ising model under Glauber dynamics [19] or in the simple inclusion process [10, 32].

6.1. Visiting condition $\mathfrak{V}$

The first condition is built upon the existence in each well of a configuration which is visited in a time-scale much shorter than the metastable one.

Condition $\mathfrak{V}$. There exist configurations $\zeta_N^x \in \mathcal{E}_N^x$, $x \in S$, such that

$$\lim_{N\to\infty} \max_{\eta\in\mathcal{E}_N^y} \mathbf{P}_\eta^N[\tau_{\zeta_N^y} \geq s] = 0 \tag{6.1}$$

for all $s > 0$ and $y \in S$.

The next result asserts that this property is sufficient for $\mathfrak{R}^{(1)}$ to hold. The proof is postponed to the end of the subsection.

Proposition 6.2. *Condition $\mathfrak{V}$ implies condition $\mathfrak{R}^{(1)}$.*

Remark 6.3. Condition (6.1) requires the process to visit the bottom of the well quickly. It is weaker than (H1), which implies that the process visits all configurations in a well before jumping to a new one. Actually, Proposition 10.1 below asserts that a stronger version of condition (6.1) holds for reversible, critical zero-range processes, a model which does not satisfy condition (H1).

Corollary 6.4. *Assume that conditions* (H0)*,* (H1) *are in force. Then $\mathfrak{R}^{(1)}$ holds.*

Proof. By Lemma 5.3, condition (6.1) holds under the assumptions (H0) and (H1). The assertion of the corollary follows, therefore, from Proposition 6.2. ∎

Remark 6.5. Conditions (H0) and (H1) have been derived for supercritical condensing zero-range processes in [7, 34, 58] and for many other dynamics. These results support the introduction of condition (6.1).

We turn to the proof of Proposition 6.2. We start by showing that we may mollify the solution with the semigroup $(\mathcal{P}_N(t) : t \geq 0)$ associated to the generator $\mathcal{L}_N$.

Lemma 6.6. *For all $T > 0$,*

$$\sup_{0\leq t\leq T} \max_{\eta\in\mathcal{H}_N} |F_N(\eta) - (\mathcal{P}_N(t)F_N)(\eta)| \leq 2T\|G_N\|_\infty .$$

Proof. Fix $T > 0$ and $0 < t \leq T$. By the representation (4.1) of F_N,

$$(\mathcal{P}_N(t)F_N)(\eta) = \mathbf{E}_\eta^N\Big[\int_0^\infty e^{-\lambda s} G_N(\xi_N(s+t))\, ds\Big].$$

By a change of variables, the right-hand side can be rewritten as

$$\mathbf{E}_\eta^N\Big[\int_t^\infty e^{-\lambda s} G_N(\xi_N(s))\, ds\Big] + \mathbf{E}_\eta^N\Big[\int_t^\infty \{e^{-\lambda(s-t)} - e^{-\lambda s}\} G_N(\xi_N(s))\, ds\Big].$$

The first term is equal to $F_N(\eta) + R_N$, where the absolute value of R_N is bounded by $t\|G_N\|_\infty$. As $1 - e^{-a} \le a$ for $a \ge 0$, the second term is bounded by $t\|G_N\|_\infty$. ■

Proof of Proposition 6.2. Fix $x \in S$, $\eta \in \mathcal{E}_N^x$ and $s > 0$, and write $(\mathcal{P}_N(s)F_N)(\eta)$ as

$$\mathbf{E}_\eta^N[F_N(\xi_N(s)),\ \tau_{\zeta_N^x} \le s] + R_N^{(1)},$$

where the remainder $R_N^{(1)}$ is bounded by $\max_{\zeta\in\mathcal{E}_N^x} \mathbf{P}_\zeta^N[\tau_{\zeta_N^x} > s]\|F_N\|_\infty$. By the strong Markov property, the previous expression is equal to

$$\mathbf{E}_\eta^N\big[[\mathcal{P}_N(s-\tau_{\zeta_N^x})F_N](\zeta_N^x),\ \tau_{\zeta_N^x} \le s\big] + R_N^{(1)}.$$

By Lemma 6.6, this expression is equal to $F_N(\xi_N^x) + R_N^{(2)}$, where

$$|R_N^{(2)}| \le 2s\|G_N\|_\infty + 2\max_{\zeta\in\mathcal{E}_N^x} \mathbf{P}_\zeta^N[\tau_{\zeta_N^x} > s]\|F_N\|_\infty.$$

Hence, by Lemma 6.6 once more,

$$\max_{\eta\in\mathcal{E}_N^x} |F_N(\eta) - F_N(\zeta_N^x)| \le 4s\|G_N\|_\infty + 2\max_{\zeta\in\mathcal{E}_N^x} \mathbf{P}_\zeta^N[\tau_{\zeta_N^x} > s]\|F_N\|_\infty.$$

By (4.2), the sequence F_N is uniformly bounded. The same property holds for the sequence G_N by definition. To complete the proof of the assertion, it remains to let $N \to \infty$ and then $s \to 0$ and to recall the hypothesis (6.1). ■

6.2. *Mixing condition* $\mathfrak{M}$

The second set of assumptions requires the mixing time of the reflected process on a well to be much smaller than the hitting time of the boundary.

Denote by $\mathcal{V}_N^x$, $x \in S$, a set of large wells which contain the wells $\mathcal{E}_N^x$: $\mathcal{E}_N^x \subset \mathcal{V}_N^x$. Let $(\xi_N^{R,x}(t) : t \ge 0)$ be the continuous-time Markov chain on $\mathcal{V}_N^x$ obtained by reflecting the process $\xi_N(\cdot)$ at the boundary of this set. In other words, in the discrete setting, the process $\xi_N^{R,x}(\cdot)$ behaves as the original process inside the well $\mathcal{V}_N^x$, but its jumps to the set $(\mathcal{V}_N^x)^c$ are suppressed.

Denote by $d_{\rm TV}^x(\mu,\nu) = d_{\rm TV}^{x,N}(\mu,\nu)$ the total variation distance between two probability measures μ, ν on $\mathcal{V}_N^x$:

$$d_{\rm TV}^x(\mu,\nu) = \frac{1}{2}\sup_J \left|\int J(\eta)\,\mu(d\eta) - \int J(\eta)\,\nu(d\eta)\right|, \tag{6.2}$$

where the supremum is taken over all measurable functions $J : \mathcal{V}_N^x \to \mathbb{R}$ bounded by 1, $\sup_{\xi\in\mathcal{V}_N^x} |J(\xi)| \le 1$.

Assume that the reflected process $\xi_N^{R,x}(\cdot)$ is ergodic. Denote by $(\mathcal{P}_N^{R,x}(t) : t \ge 0)$ its semigroup, by $\pi^{R,x} = \pi_N^{R,x}$ its stationary state, and by $t_{\rm mix}^x(\varepsilon) = t_{{\rm mix},N}^x(\varepsilon)$, $0 < \varepsilon < 1$, its mixing time:

$$t^x_{\rm mix}(\varepsilon) = \inf\Big\{t > 0 : \sup_{\eta\in\mathcal{E}^x_N} d^x_{\rm TV}(\delta_\eta \mathcal{P}^{R,x}_N(t), \pi^{R,x}) \le \varepsilon\Big\}.$$

The next result asserts that the following mixing properties entail condition $\mathfrak{R}^{(1)}$.

Condition $\mathfrak{M}$. The process $\xi_N(\cdot)$ starting from a well $\mathcal{E}^x_N$ cannot escape from the well $\mathcal{V}^x_N$ within a time-scale $\mathbf{h}_N \ll 1$: For all $x \in S$,

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}^x_N} \mathbf{P}^N_\eta[\tau_{(\mathcal{V}^x_N)^c} \le \mathbf{h}_N] = 0. \tag{6.3}$$

Furthermore, for every $x \in S$, the reflected process $\xi^{R,x}_N(\cdot)$ is ergodic, and for all $\varepsilon > 0$,

$$t^x_{\rm mix}(\varepsilon) \le \mathbf{h}_N \tag{6.4}$$

for all N sufficiently large.

Proposition 6.7. *If the mixing property $\mathfrak{M}$ is satisfied, then condition $\mathfrak{R}^{(1)}$ holds.*

Remark 6.8. Barrera and Jara [5] proved that the mixing time of small random perturbations of dynamical systems satisfying certain regularity assumptions is of polynomial order. Since the hitting time of the boundary is exponentially large [25], the previous result applies to this setting.

The proof of Proposition 6.7 relies on a simple estimate between the semigroup of the original process and the semigroup of the reflected one.

Lemma 6.9. *For each $x \in S$, $\eta \in \mathcal{E}^x_N$ and $t > 0$,*

$$|(\mathcal{P}_N(t)F_N)(\eta) - (\mathcal{P}^{R,x}_N(t)F_N)(\eta)| \le 2\|F_N\|_\infty \mathbf{P}^N_\eta[\tau_{(\mathcal{V}^x_N)^c} \le t].$$

Proof. Fix x in S, η in $\mathcal{E}^x_N$, and write $(\mathcal{P}_N(t)F_N)(\eta)$ as

$$\mathbf{E}^N_\eta[F_N(\xi_N(t)),\ \tau_{(\mathcal{V}^x_N)^c} > t] + \mathbf{E}^N_\eta[F_N(\xi_N(t)),\ \tau_{(\mathcal{V}^x_N)^c} \le t].$$

In the first term, we may replace the process $\xi_N(\cdot)$ by the reflected one since the process remained in the set $\mathcal{V}^x_N$ in the time interval $[0,t]$. The second term is bounded by $\mathbf{P}^N_\eta[\tau_{(\mathcal{V}^x_N)^c} \le t]\|F_N\|_\infty$. Writing the indicator function of the set $\{\tau_{(\mathcal{V}^x_N)^c} > t\}$ as 1 minus the indicator of the complement, we conclude the proof. ∎

Proof of Proposition 6.7. By Lemmata 6.6 and 6.9 with $T = t = \mathbf{h}_N$, and hypothesis (6.3),

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}^x_N} |F_N(\eta) - (\mathcal{P}^{R,x}_N(\mathbf{h}_N)F_N)(\eta)| = 0.$$

Fix $x \in S$, $\eta \in \mathcal{E}^x_N$ and $\varepsilon > 0$. By definition of the total variation distance,

$$|(\mathcal{P}^{R,x}_N(\mathbf{h}_N)F_N)(\eta) - E_{\pi^{R,x}}[F_N]| \le 2\|F_N\|_\infty d^x_{\rm TV}\big(\delta_\eta\mathcal{P}^{R,x}_N(\mathbf{h}_N), \pi^{R,x}\big). \tag{6.5}$$

By the contracting property of the semigroup, the distance

$$d^x_{\mathrm{TV}}(\delta_\eta \mathcal{P}^{R,x}_N(t), \pi^{R,x}) = d^x_{\mathrm{TV}}\big(\delta_\eta \mathcal{P}^{R,x}_N(t), \pi^{R,x}\mathcal{P}^{R,x}_N(t)\big)$$

is decreasing in t, and thus by (6.4), the right-hand side of (6.5) is bounded from above by

$$2\|F_N\|_\infty \sup_{\xi\in\mathcal{E}^x_N} d^x_{\mathrm{TV}}\big(\delta_\xi \mathcal{P}^{R,x}_N(t^x_{\mathrm{mix}}(\varepsilon)), \pi^{R,x}\big) = 2\|F_N\|_\infty \varepsilon$$

by definition of the mixing time. This completes the proof of the proposition because the sequence (F_N) is uniformly bounded in N. ∎

6.3. *Local equilibration and condition $\mathfrak{D}$*

The same argument shows that condition $\mathfrak{M}$ guarantees a fast local equilibration inside each well. In particular, condition $\mathfrak{D}$ results from assumption $\mathfrak{M}$ and the property that $\mu_N(\Delta_N)/\mu_N(\mathcal{E}^x_N) \to 0$ for all $x \in S$.

Consider a uniformly bounded sequence $(Q_N)_{N\in\mathbb{N}}$ of functions $Q_N : \mathcal{H}_N \to \mathbb{R}$: There exists a finite constant $M > 0$ such that

$$\sup_{\eta\in\mathcal{H}_N} |Q_N(\eta)| \le M \quad \text{for all } N \in \mathbb{N}. \tag{6.6}$$

Recall the definition of the probability measure μ^x_N, introduced in Remark 3.4.

Proposition 6.10. *Assume that condition $\mathfrak{M}$ is in force. Then, for all $x \in S$ and $T > 0$,*

$$\sup_{\eta\in\mathcal{E}^x_N} \Big| \mathbf{E}^N_\eta \Big[\int_0^T Q_N(\xi_N(t))\, dt \Big] - \mathbf{E}^N_{\mu^x_N} \Big[\int_0^T Q_N(\xi_N(t))\, dt \Big] \Big| \le 6M(T+1) o_N(1),$$

where the error term $o_N(1)$ on the right-hand side is uniform in N, M and T.

Proof. Fix $x \in S$ and $\eta \in \mathcal{E}^x_N$, and let

$$q_N(\eta) = \mathbf{E}^N_\eta \Big[\int_0^T Q_N(\xi_N(s))\, ds \Big].$$

Note that

$$|q_N(\zeta)| \le TM \quad \text{for all } N \in \mathbb{N} \text{ and } \zeta \in \mathcal{H}_N. \tag{6.7}$$

By (6.4), there exists a sequence $(\varepsilon_N : N \ge 1)$ such that $\lim_N \varepsilon_N = 0$ and $t^x_{\mathrm{mix}}(\varepsilon_N) \le \mathbf{h}_N$ for all $N \ge 1$. Let $\mathbf{s}_N = t^x_{\mathrm{mix}}(\varepsilon_N)$. Since Q_N is uniformly bounded by M,

$$q_N(\eta) = \mathbf{E}^N_\eta \Big[\int_{\mathbf{h}_N}^{T+\mathbf{h}_N} Q_N(\xi_N(s))\, ds \Big] + M O(\mathbf{h}_N).$$

By (6.3), this expectation is equal to

$$\mathbf{E}^N_\eta \Big[\int_{\mathbf{h}_N}^{T+\mathbf{h}_N} Q_N(\xi_N(s))\, ds,\ \tau_{(\mathcal{V}^x_N)^c} > \mathbf{h}_N \Big] + M T o_N(1).$$

By the Markov property and the definition of q_N, we may write the previous expectation as

$$\mathbf{E}_\eta^N [q_N(\xi_N(\mathbf{h}_N)),\ \tau_{(\mathcal{V}_N^x)^c} > \mathbf{h}_N].$$

Recall that we denote by $\xi_N^{R,x}(\cdot)$ the reflected process at the boundary of $\mathcal{V}_N^x$. Denote by $\mathbf{P}_\eta^{R,x}$ the law of the reflected process $\xi_N^{R,x}(\cdot)$, and by $\mathbf{E}_\eta^{R,x}$ the expectation with respect to $\mathbf{P}_\eta^{R,x}$.

Due to the presence of the indicator of the set $\{\tau_{(\mathcal{V}_N^x)^c} > \mathbf{h}_N\}$, we may replace in the previous expectation $\xi_N(\mathbf{h}_N)$ by $\xi_N^{R,x}(\mathbf{h}_N)$ and then remove the indicator of that set. After these modifications the previous expression becomes

$$\mathbf{E}_\eta^{R,x}[q_N(\xi_N(\mathbf{h}_N))] + MTo_N(1).$$

By definition of $\mathbf{s}_N$ and since $\mathbf{s}_N \le \mathbf{h}_N$, the expectation is equal to

$$E_{\pi^{R,x}}[q_N] + MTo_N(1).$$

We have just proved that

$$\sup_{\eta,\zeta\in\mathcal{E}_N^x} |q_N(\eta) - q_N(\zeta)| \le 6M(T+1)o_N(1).$$

The assertion of the proposition follows from this bound by averaging ζ according to μ_N^x. ■

Corollary 6.11. *Assume that condition $\mathfrak{M}$ is in force. Then, for all $x \in S$ and $T > 0$,*

$$\sup_{\eta\in\mathcal{E}_N^x} \mathbf{E}_\eta^N \Big[\int_0^T \chi_{\Delta_N}(\xi_N(t)))\, dt \Big] \le \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} T + 6(T+1)o_N(1).$$

In particular, if $\mu_N(\Delta_N)/\mu_N(\mathcal{E}_N^x) \to 0$ for all $x \in S$, then condition $\mathfrak{D}$ holds.

Proof. By the proposition, for every $x \in S$, $\eta \in \mathcal{E}_N^x$ and $T > 0$,

$$\mathbf{E}_\eta^N \Big[\int_0^T \chi_{\Delta_N}(\xi_N(t))\, dt \Big] \le \mathbf{E}_{\mu_N^x}^N \Big[\int_0^T \chi_{\Delta_N}(\xi_N(t))\, dt \Big] + 6(T+1)o_N(1).$$

The expectation is bounded by

$$\frac{1}{\mu_N(\mathcal{E}_N^x)} \mathbf{E}_{\mu_N}^N \Big[\int_0^T \chi_{\Delta_N}(\xi_N(t)))\, dt \Big] = \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} T,$$

where the last identity follows from the fact that μ_N is the stationary state. ■

7. Proof of Theorem 2.8

In this section, we examine the possible limits of the average of the solutions of the resolvent equation (2.5) in each well and prove Theorem 2.8. Most of the notation is borrowed from Section 5.

Recall from the statement of Theorem 2.8 the definition of the function f_N. Note that condition $\mathfrak{R}^{(1)}$ holds if and only if

$$\lim_{N\to\infty} \max_{z\in S} \max_{\eta\in\mathcal{E}_N^z} |F_N(\eta) - f_N(z)| = 0.$$

Condition $\mathfrak{R}^{(2)}_{\mathcal{L}}$. Let $\mathcal{L}$ be the generator of an S-valued, continuous-time Markov chain. For all $x \in S$,

$$\lim_{N\to\infty} f_N(x) = f(x),$$

where $f : S \to \mathbb{R}$ is the solution of the reduced resolvent equation

$$(\lambda - \mathcal{L}) f = g.$$

Remark 7.1. It is clear that $\mathfrak{R}^{(1)}$ and $\mathfrak{R}^{(2)}_{\mathcal{L}}$ together imply condition $\mathfrak{R}_{\mathcal{L}}$.

By (4.2) and the definition of f_N, there exists a finite constant $C_0 = C_0(\lambda, g)$ such that

$$\sup_{N\ge 1} \max_{x\in S} |f_N(x)| \le C_0.$$

Let $\mathcal{L}$ be the generator of the S-valued Markov chain induced by the rates r introduced in condition (H0):

$$(\mathcal{L} f)(x) = \sum_{y\in S} r(x,y)[f(y) - f(x)],$$

and let $\nu_y^\dagger = \nu^\dagger_{\mathcal{E}_N^y, \breve{\mathcal{E}}_N^y}$, $y \in S$, be the equilibrium measure between $\mathcal{E}_N^y$ and $\breve{\mathcal{E}}_N^y$, as defined in (5.4).

Proposition 7.2. *Assume that conditions* (H0) *and* $\mathfrak{R}^{(1)}$ *are in force. Let f be a limit point of the sequence f_N. Then*

$$[(\lambda - \mathcal{L}) f](y) = g(y)$$

for all $y \in S$ such that

$$\lim_{N\to\infty} \Big(\sum_{z\neq y} r_N(y,z)\Big) \mathbf{E}^N_{\nu_y^\dagger}\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}} \chi_{\Delta_N}(\xi_N(s))\, ds\Big] = 0. \tag{7.1}$$

Proof. Fix $y \in S$, and denote by $h_y^\dagger = h^\dagger_{\mathcal{E}_N^y, \breve{\mathcal{E}}_N^y}$ the equilibrium potential between $\mathcal{E}_N^y$ and $\breve{\mathcal{E}}_N^y$ for the adjoint process, as defined in (2.10). Multiply the resolvent equation (2.5) by $h_y^\dagger$ and integrate with respect to the stationary measure μ_N to get

$$\lambda\langle F_N, h_y^\dagger\rangle_{\mu_N} - \langle \mathcal{L}_N F_N, h_y^\dagger\rangle_{\mu_N} = \langle G_N, h_y^\dagger\rangle_{\mu_N}. \tag{7.2}$$

Consider the right-hand side of this equation. Since G_N vanishes on Δ_N and is equal to $g(z)$ on $\mathcal{E}_N^z$, $z \in S$, and since on the set $\mathcal{E}_N$, $h_y^\dagger$ is equal to the indicator of the set $\mathcal{E}_N^y$,

$$\langle G_N, h_y^\dagger\rangle_{\mu_N} = g(y)\mu_N(\mathcal{E}_N^y).$$

We turn to the first term on the left-hand side of (7.2). For similar reasons, it is equal to

$$\lambda \sum_{\eta \in \mathcal{E}_N^y} \mu_N(\eta) F_N(\eta) + \sum_{\eta \in \Delta_N} \mu_N(\eta) F_N(\eta) h_y^\dagger(\eta).$$

By (4.2), the sequence F_N is uniformly bounded. As $h_y^\dagger$ is bounded by 1, the second term is bounded by $C_0(\lambda, g) \sum_{\eta \in \Delta_N} \mu_N(\eta) h_y^\dagger(\eta)$. On the other hand, by definition of f_N, the first term is equal to $\lambda \mu_N(\mathcal{E}_N^y) f_N(y)$.

We turn to the second term on the left-hand side of (7.2). Since $\mathcal{L}_N^\dagger h_y^\dagger = 0$ on Δ_N,

$$\langle \mathcal{L}_N F_N, h_y^\dagger \rangle_{\mu_N} = \langle F_N, \mathcal{L}_N^\dagger h_y^\dagger \rangle_{\mu_N} = \sum_{x \in S} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) F_N(\eta) (\mathcal{L}_N^\dagger h_y^\dagger)(\eta).$$

Since the equilibrium potential $h_y^\dagger$ vanishes on $\breve{\mathcal{E}}_N^y$ and is equal to 1 on $\mathcal{E}_N^y$, for $\eta \in \mathcal{E}_N^x$, $x \neq y$, as $\lambda_N^\dagger(\eta) = \lambda_N(\eta)$,

$$\begin{aligned}
(\mathcal{L}_N^\dagger h_y^\dagger)(\eta) &= \sum_{\zeta \in \mathcal{H}_N} R_N^\dagger(\eta, \zeta)[h_y^\dagger(\zeta) - h_y^\dagger(\eta)] \\
&= \lambda_N(\eta) \sum_{\zeta \in \mathcal{H}_N} p_N^\dagger(\eta, \zeta) \mathbf{P}_\zeta^{N,\dagger}[\tau_{\mathcal{E}_N^y} < \tau_{\breve{\mathcal{E}}_N^y}] = \lambda_N(\eta) \mathbf{P}_\eta^{N,\dagger}[\tau_{\mathcal{E}_N^y} < \tau_{\breve{\mathcal{E}}_N^y}^+].
\end{aligned}$$

Similarly, as $h_y^\dagger(\zeta) - 1 = -\mathbf{P}_\zeta^{N,\dagger}[\tau_{\breve{\mathcal{E}}_N^y} < \tau_{\mathcal{E}_N^y}]$, for $\eta \in \mathcal{E}_N^y$,

$$(\mathcal{L}_N^\dagger h_y^\dagger)(\eta) = -\lambda_N(\eta) \mathbf{P}_\eta^{N,\dagger}[\tau_{\breve{\mathcal{E}}_N^y} < \tau_{\mathcal{E}_N^y}^+].$$

Therefore,

$$\begin{aligned}
\langle \mathcal{L}_N F_N, h_y^\dagger \rangle_{\mu_N} &= \sum_{x \neq y} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) \lambda_N(\eta) F_N(\eta) \mathbf{P}_\eta^{N,\dagger}[\tau_{\mathcal{E}_N^y} < \tau_{\breve{\mathcal{E}}_N^y}^+] \\
&\quad - \sum_{\eta \in \mathcal{E}_N^y} \mu_N(\eta) \lambda_N(\eta) F_N(\eta) \mathbf{P}_\eta^{N,\dagger}[\tau_{\breve{\mathcal{E}}_N^y} < \tau_{\mathcal{E}_N^y}^+].
\end{aligned}$$

Recall from (2.8) the definition of $r_N^\dagger(z, z')$. Add and subtract f_N to rewrite the right-hand side as

$$\sum_{x \neq y} \mu_N(\mathcal{E}_N^x) f_N(x) r_N^\dagger(x, y) - \mu_N(\mathcal{E}_N^y) f_N(y) \sum_{x \neq y} r_N^\dagger(y, x) + R_N, \tag{7.3}$$

where the absolute value of the remainder R_N is bounded by

$$\max_{z \in S} \max_{\eta \in \mathcal{E}_N^z} |F_N(\eta) - f_N(z)| \Big\{ \sum_{x \neq y} \mu_N(\mathcal{E}_N^x) r_N^\dagger(x, y) + \mu_N(\mathcal{E}_N^y) \sum_{x \neq y} r_N^\dagger(y, x) \Big\}.$$

By Lemma 5.1, this expression can be rewritten as

$$2\mu_N(\mathcal{E}_N^y) \max_{z \in S} \max_{\eta \in \mathcal{E}_N^z} |F_N(\eta) - f_N(z)| \sum_{x \neq y} r_N(y, x).$$

For the same reasons, the sum of the first two terms in (7.3) is equal to

$$\mu_N(\mathcal{E}_N^y)\sum_{x\neq y} r_N(y,x)[f_N(x)-f_N(y)].$$

Recollecting all previous calculations and dividing by $\mu_N(\mathcal{E}_N^y)$ permits us to rewrite (7.2) as

$$\lambda f_N(y)-\sum_{x\neq y} r_N(y,x)[f_N(x)-f_N(y)] = g(y)+R_N^{(2)},$$

where the absolute value of $R_N^{(2)}$ is bounded by

$$\frac{C_0}{\mu_N(\mathcal{E}_N^y)}\sum_{\eta\in\Delta_N}\mu_N(\eta)h_y^\dagger(\eta)+2\max_{z\in S}\max_{\eta\in\mathcal{E}_N^z}|F_N(\eta)-f_N(z)|\sum_{x\neq y} r_N(y,x)$$

for some finite constant $C_0=C_0(\lambda,g)$. By (5.3), with $\mathcal{A}=\mathcal{E}_N^y$, $\mathcal{B}=\breve{\mathcal{E}}_N^y$, and the second assertion of Lemma 5.1, this expression can be rewritten as

$$C_0\sum_{x\neq y} r_N(y,x)\Big\{\mathbf{E}^N_{\nu_y^\dagger}\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}}\chi_{\Delta_N}(\xi_N(s))\,ds\Big]+\max_{z\in S}\max_{\eta\in\mathcal{E}_N^z}|F_N(\eta)-f_N(z)|\Big\}$$

for a possibly different constant C_0. To conclude the proof, it remains to recall the statement of conditions (H0), $\mathfrak{R}^{(1)}$, and the hypotheses of the proposition. ■

In the previous proof we used the identity

$$\Big(\sum_{z\neq y} r_N(y,z)\Big)\mathbf{E}^N_{\nu_y^\dagger}\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}}\chi_{\Delta_N}(\xi_N(s))\,ds\Big]=\frac{1}{\mu_N(\mathcal{E}_N^y)}\sum_{\eta\in\Delta_N}\mu_N(\eta)h_y^\dagger(\eta). \tag{7.4}$$

In particular, (7.1) holds for $y\in S$ if and only if the right-hand side vanishes as $N\to\infty$.

Corollary 7.3. *Assume that conditions* (H0) *and* $\mathfrak{R}^{(1)}$ *are in force. Let f be a limit point of the sequence f_N. Then*

$$[(\lambda-\mathcal{L}_Y)f](y)=g(y)$$

for all $y\in S$ such that $\mu_N(\Delta_N)/\mu_N(\mathcal{E}_N^y)\to 0$. In this formula, $\mathcal{L}_Y$ is the generator of the continuous-time Markov process whose jump rates are given by $r(x,y)$, introduced in (H0).

Proof. The right-hand side of (7.4) is bounded by $\mu_N(\Delta_N)/\mu_N(\mathcal{E}_N^y)$. Thus, the assertion follows from Proposition 7.2. ■

Proof of Theorem 2.8. Theorem 2.8 follows from Corollaries 6.4 and 7.3. ■

We complete this section with a method to prove condition (7.1) when the hypotheses of Corollary 7.3 are not satisfied. The idea behind the decomposition below is that $\mathcal{A}_N$ is contained in the basin of attraction of $\breve{\mathcal{E}}_N^y$. In particular, starting from a configuration in $\mathcal{A}_N$ the set $\breve{\mathcal{E}}_N^y$ is reached quickly. We refer to Figure 1 for an example of illustration of the set $\mathcal{A}_N$.

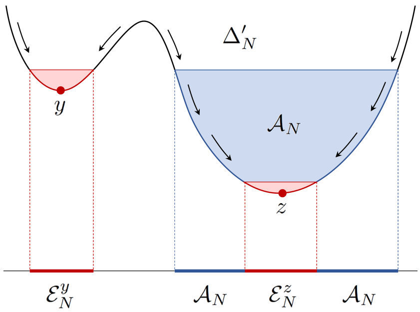

Fig. 1. This picture illustrates the idea behind the statement of Lemma 7.4. To simplify, we argue in a continuous setting, but the same idea applies to the discrete setting. Consider a diffusion on the potential field appearing in the picture. The valley $\mathcal{E}_N^y$ is a metastable set and $\mathcal{E}_N^z$ a stable one. As $\mu_N(\Delta_N)/\mu_N(\mathcal{E}_N^y)$ does not converge to 0, we decompose Δ_N as $\Delta'_N \cup \mathcal{A}_N$, so that $\mu_N(\Delta'_N)/\mu_N(\mathcal{E}_N^y) \to 0$. On the other hand, as $\mathcal{A}_N$ is a subset of the domain of attraction of the valley $\mathcal{E}_N^z$, we can expect (7.5) to hold.

Lemma 7.4. *Fix $y \in S$, and suppose that Δ_N may be decomposed as $\Delta_N = \Delta'_N \cup \mathcal{A}_N$, $\Delta'_N \cap \mathcal{A}_N = \varnothing$, where $\mu_N(\Delta'_N)/\mu_N(\mathcal{E}_N^y) \to 0$, and*

$$\lim_{N\to\infty} \sup_{\xi\in\mathcal{A}_N} \mathbf{E}_\xi^N\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}} \chi_{\mathcal{A}_N}(\xi_N(s))\, ds\Big] = 0. \tag{7.5}$$

Then (7.1) holds for y.

Proof. In (7.1), write χ_{Δ_N} as $\chi_{\mathcal{A}_N} + \chi_{\Delta'_N}$. We estimate the two pieces separately. By (7.4) with Δ'_N instead of Δ_N,

$$\Big(\sum_{x\neq y} r_N(y,x)\Big)\mathbf{E}_{\nu_y^\dagger}^N\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}} \chi_{\Delta'_N}(\xi_N(s))\, ds\Big] \le \frac{\mu_N(\Delta'_N)}{\mu_N(\mathcal{E}_N^y)}.$$

By hypothesis, this expression vanishes as $N \to \infty$.

On the other hand, starting the integral from the hitting time of $\mathcal{A}_N$ and applying the strong Markov property yields

$$\begin{aligned}\mathbf{E}_{\nu_y^\dagger}^N\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}} \chi_{\mathcal{A}_N}(\xi_N(s))\, ds\Big] &= \mathbf{E}_{\nu_y^\dagger}^N\Big[\int_{\tau_{\mathcal{A}_N}}^{\tau_{\breve{\mathcal{E}}_N^y}} \chi_{\mathcal{A}_N}(\xi_N(s))\, ds\Big] \\ &\le \sup_{\xi\in\mathcal{A}_N} \mathbf{E}_\xi^N\Big[\int_0^{\tau_{\breve{\mathcal{E}}_N^y}} \chi_{\mathcal{A}_N}(\xi_N(s))\, ds\Big].\end{aligned}$$

By assumption this expression vanishes as $N \to \infty$, which completes the proof. ∎

8. Proof of Theorem 3.2

In view of Theorem 2.3, to prove Theorem 3.2 we have to show that conditions $\mathfrak{D}$ and $\mathfrak{R}_\mathcal{L}$ hold. The proof is based on the theory developed in the previous sections. We proceed as follows.

Condition $\mathfrak{R}_{\mathcal{L}}$. In Proposition 10.1, we show that condition $\mathfrak{V}$ is fulfilled. Hence, by Proposition 6.2, property $\mathfrak{R}^{(1)}$ holds.

In Corollary 12.2 we show that condition (H0) holds. Since we have already proved that condition $\mathfrak{R}^{(1)}$ is fulfilled, and since, by Theorem 3.1, $\mu_N(\Delta_N)/\mu_N(\mathcal{E}_N^x) \to 0$ for all $x \in S$, by Corollary 7.3, property $\mathfrak{R}_{L_Y}^{(2)}$ is in force, where L_Y is the generator introduced in (3.7).

Condition $\mathfrak{D}$. Recall the assumptions (6.3) and (6.4) of condition $\mathfrak{M}$. In Corollary 9.2, we show that condition (6.3) holds for some enlarged wells $\mathcal{V}_N^x$ and a time-scale $\mathbf{h}_N \ll 1$. Then, in Proposition 11.1, we prove that, for every $\varepsilon > 0$, the mixing time $t_{\text{mix}}^x(\varepsilon)$ of the zero-range process reflected at the boundary of $\mathcal{V}_N^x$ is bounded by a sequence $\mathbf{s}_N \ll \mathbf{h}_N$. This property implies condition (6.4). These two results yield condition $\mathfrak{M}$, which is the assertion of Corollary 11.2. Thus, by Theorem 3.1 and Corollary 6.11, property $\mathfrak{D}$ is fulfilled.

Remark 8.1. To deduce property $\mathfrak{R}^{(1)}$ one could also invoke Corollary 11.2 and Proposition 6.7. On the other hand, condition $\mathfrak{R}_{L_Y}^{(2)}$ has been proven in an alternative way in [41, Section 7].

9. Escape from large wells

In this section, we prove that condition (6.3) holds for the critical zero-range process for a sequence $(\mathbf{h}_N)_{N\in\mathbb{N}}$, $\mathbf{h}_N \to 0$, and enlarged wells $(\mathcal{V}_N^x, x \in S)_{N\in\mathbb{N}}$, $\mathcal{V}_N^x \supset \mathcal{E}_N^x$.

For $N \in \mathbb{N}$, set

$$m_N = N/(\log N)^\delta \quad \text{with} \quad \delta \in (0,1)$$

and let $\mathbf{h}_N$ be the macroscopic time-scales given by

$$\mathbf{h}_N := \frac{m_N^2 (\log N)^{1/2}}{\theta_N} = \frac{1}{(\log N)^{1/2+2\delta}}. \tag{9.1}$$

For $x \in S$, define a larger well by

$$\mathcal{V}_N^x = \big\{\eta \in \mathcal{H}_N : \eta_y \le m_N \text{ for all } y \in S \setminus \{x\}\big\}. \tag{9.2}$$

As in [41], denote by $\mathcal{W}_N^x$, $\mathcal{D}_N^x$, $x \in S$, the wells given by

$$\mathcal{W}_N^x = \{\eta \in \mathcal{H}_N : \eta_x \ge N - m_N\}, \quad \mathcal{D}_N^x = \{\eta \in \mathcal{H}_N : \eta_x \ge N - N^\gamma\}. \tag{9.3}$$

In this formula, $\gamma \in (0, 2/\kappa)$ is a fixed constant. The sets $\mathcal{D}_N^x$ are called *deep wells* and the sets $\mathcal{W}_N^x$ *shallow wells*.

Denote by $\zeta_N^x \in \mathcal{H}_N$ the configuration such that all N particles are located at site x, so that

$$\zeta_N^x \in \mathcal{D}_N^x \subset \mathcal{E}_N^x \subset \mathcal{W}_N^x \subset \mathcal{V}_N^x. \tag{9.4}$$

The main result of this section asserts that the process $\xi_N(\cdot)$ starting from a well $\mathcal{E}_N^x$ cannot escape from the well $\mathcal{W}_N^x$ within the time-scale $\mathbf{h}_N$.

Proposition 9.1. *For all $x \in S$,*

$$\lim_{N\to\infty} \sup_{\eta \in \mathcal{E}_N^x} \mathbf{P}_\eta^N [\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N] = 0.$$

By the last inclusion of (9.4), the next result is a straightforward consequence of Proposition 9.1.

Corollary 9.2. *For all $x \in S$,*

$$\lim_{N\to\infty} \sup_{\eta \in \mathcal{E}_N^x} \mathbf{P}_\eta^N [\tau_{(\mathcal{V}_N^x)^c} \le \mathbf{h}_N] = 0.$$

9.1. Estimates based on capacity

In this subsection, we state several estimates based on the following bound of the capacity between $\mathcal{E}_N^x$ and $(\mathcal{W}_N^x)^c$ with respect to the critical zero-range processes.

Lemma 9.3. *There exists a finite constant C such that*

$$\mathrm{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c) \le \frac{C\theta_N}{m_N^2 \log N}$$

for all $x \in S$ and $N \ge 1$.

Proof. Let $Q(\eta) = q(N - \eta_x)$ for some function $q : \mathbb{Z} \to \mathbb{R}$ such that

$$q(k) = \begin{cases} 0 & \text{if } k \le \ell_N, \\ 1 & \text{if } k \ge m_N. \end{cases}$$

The precise expression for q will be specified below in (9.5). By the Dirichlet principle and since $Q(\sigma^{y,z}\eta) = Q(\eta)$ if $y, z \neq x$ or $\eta \notin \mathcal{W}_N^x \setminus \mathcal{E}_N^x$,

$$\begin{aligned} &\mathrm{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c) \\ &\quad \le \mathcal{D}_N(Q) = \theta_N \sum_{\eta \in \mathcal{W}_N^x \setminus \mathcal{E}_N^x} \sum_{y \in S} \mu_N(\eta) g(\eta_x) r(x,y) [Q(\sigma^{x,y}\eta) - Q(\eta)]^2. \end{aligned}$$

By definition of the jump rates and of Q, this expression is bounded by

$$C\theta_N \sum_{\eta \in \mathcal{W}_N^x \setminus \mathcal{E}_N^x} \mu_N(\eta) [q(N - \eta_x + 1) - q(N - \eta_x)]^2$$

for some finite constant C. Let

$$\mathcal{R}_k = \{\eta \in \mathcal{H}_N : N - \eta_x = k\}$$

so that

$$\mathrm{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c) \le C\theta_N \sum_{k=\ell_N}^{m_N - 1} \mu_N(\mathcal{R}_k) [q(k+1) - q(k)]^2.$$

Define

$$q(k) = \frac{\sum_{i=\ell_N}^{k-1} \mu_N(\mathcal{R}_i)^{-1}}{\sum_{i=\ell_N}^{m_N-1} \mu_N(\mathcal{R}_i)^{-1}}, \quad k \in [\ell_N, m_N]. \tag{9.5}$$

It follows from the penultimate displayed equation that

$$\operatorname{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c) \le \frac{C\theta_N}{\sum_{i=\ell_N}^{m_N-1} \mu_N(\mathcal{R}_i)^{-1}}. \tag{9.6}$$

For $\ell_N \le i < m_N$,

$$\mu_N(\mathcal{R}_i) = \frac{N}{Z_{N,\kappa}(\log N)^{\kappa-1}} \frac{1}{N-i} \frac{Z_{i,\kappa-1}(\log i)^{\kappa-2}}{i} \le \frac{C}{\log N} \frac{1}{i}.$$

Here, we have used the facts that $Z_{N,\kappa}$, $Z_{i,\kappa-1}$ are bounded, that $N/(N-i) \simeq 1$ and $\log i/\log N \simeq 1$ for $i \in [\ell_N, m_N - 1]$. Inserting this into (9.6) yields

$$\operatorname{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c) \le \frac{C\theta_N}{(\log N)\sum_{i=\ell_N}^{m_N-1} i} \le \frac{C\theta_N}{m_N^2 \log N},$$

as claimed. ∎

Based on the previous estimate of capacity, we can refine [41, Propositions 9.4 and 8.6], replacing the set $\breve{\mathcal{E}}_N^x = \mathcal{E}_N \setminus \mathcal{E}_N^x$ by the much larger set $(\mathcal{W}_N^x)^c$.

Lemma 9.4. *For all $x \in S$,*

$$\lim_{N\to\infty} \inf_{\eta\in\mathcal{D}_N^x} \inf_{\zeta\in\mathcal{D}_N^x} \mathbf{P}_\eta^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}] = 1.$$

Proof. By [38, (3.3)] and the monotonicity of capacity,

$$\mathbf{P}_\eta^N[\tau_\zeta > \tau_{(\mathcal{W}_N^x)^c}] \le \frac{\operatorname{cap}_N(\eta, (\mathcal{W}_N^x)^c)}{\operatorname{cap}_N(\eta, \zeta)} \le \frac{\operatorname{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c)}{\operatorname{cap}_N(\eta, \zeta)}.$$

Hence, by [41, Lemma 9.3] and Lemma 9.3,

$$\mathbf{P}_\eta^N[\tau_\zeta > \tau_{(\mathcal{W}_N^x)^c}] \le C\, \frac{N^{\gamma\kappa}(\log N)^{\kappa-1}}{m_N^2 \log N} = o_N(1),$$

where the last equality holds because $\gamma < 2/\kappa$. This completes the proof. ∎

Recall from [41, Proposition 9.1] that the deep wells $\mathcal{D}_N^x$ are attractors, in the sense that

$$\lim_{N\to\infty} \inf_{\eta\in\mathcal{E}_N^x} \mathbf{P}_\eta^N[\tau_{\mathcal{D}_N^x} < \tau_{(\mathcal{W}_N^x)^c}] = 1 \quad \text{for all } x \in S. \tag{9.7}$$

Lemma 9.5. *For all $x \in S$,*

$$\lim_{N\to\infty} \inf_{\zeta\in\mathcal{D}_N^x} \inf_{\eta\in\mathcal{E}_N^x} \mathbf{P}_\eta^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}] = 1.$$

Proof. Fix $x \in S$, $\eta \in \mathcal{E}_N^x$, and $\zeta \in \mathcal{D}_N^x$. Then, by the strong Markov property,

$$\begin{aligned}\mathbf{P}_\eta^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}] &\ge \mathbf{P}_\eta^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c},\ \tau_{\mathcal{D}_N^x} < \tau_{(\mathcal{W}_N^x)^c}] \\ &\ge \mathbf{P}_\eta^N[\tau_{\mathcal{D}_N^x} < \tau_{(\mathcal{W}_N^x)^c}] \inf_{\xi \in \mathcal{D}_N^x} \mathbf{P}_\xi^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\inf_{\zeta \in \mathcal{D}_N^x} \inf_{\eta \in \mathcal{E}_N^x} \mathbf{P}_\eta^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}] \\ &\qquad \ge \inf_{\eta \in \mathcal{E}_N^x} \mathbf{P}_\eta^N[\tau_{\mathcal{D}_N^x} < \tau_{(\mathcal{W}_N^x)^c}] \times \inf_{\zeta \in \mathcal{D}_N^x} \inf_{\xi \in \mathcal{D}_N^x} \mathbf{P}_\xi^N[\tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}].\end{aligned}$$

The first term on the right-hand side is $1 - o_N(1)$ by (9.7), and the second one is $1 - o_N(1)$ by Lemma 9.4. This completes the proof of the lemma. ∎

9.2. *Proof of Proposition 9.1*

The proof of Proposition 9.1 is similar to the one of [41, Theorem 3.2]. First, we establish the following estimate, whose proof is omitted since it is completely identical to the proof of [41, Proposition 8.4]. It suffices to replace $\breve{\mathcal{E}}_N^x$ by $(\mathcal{W}_N^x)^c$ and $1/\gamma_N$ by t_N.

Lemma 9.6. *For all $x \in S$ and every probability measure υ_N concentrated on $\mathcal{E}_N^x$,*

$$(\mathbf{P}_{\upsilon_N}^N[\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N])^2 \le e^2 \mathbf{h}_N \mathbb{E}_{\mu_N^x}\Big[\Big(\frac{\upsilon_N}{\mu_N^x}\Big)^2\Big] \frac{1}{\mu_N(\mathcal{E}_N^x)} \operatorname{cap}_N(\mathcal{E}_N^x, (\mathcal{W}_N^x)^c).$$

By inserting $\upsilon_N = \pi_N^x(\cdot) := \mu_N(\cdot|\mathcal{D}_N^x)$ into the previous equation, we obtain the following estimate. The order of magnitude of $\mathbf{h}_N$ is critically used in the proof of this result.

Lemma 9.7. *For all $x \in S$,*

$$\lim_{N\to\infty} \mathbf{P}_{\pi_N^x}^N[\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N] = 0.$$

Proof. Since

$$\mathbb{E}_{\mu_N^x}\Big[\Big(\frac{\upsilon_N}{\mu_N^x}\Big)\Big]^2 = \frac{\mu_N(\mathcal{E}_N^x)}{\mu_N(\mathcal{D}_N^x)},$$

by Lemmata 9.3 and 9.6 we get

$$(\mathbf{P}_{\upsilon_N}^N[\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N])^2 \le C\mathbf{h}_N \frac{1}{\mu_N(\mathcal{D}_N^x)} \frac{\theta_N}{m_N^2 \log N} = \frac{C}{(\log N)^{1/2}} \frac{1}{\mu_N(\mathcal{D}_N^x)}.$$

By [41, Lemma 4.2], $\mu_N(\mathcal{D}_N^x) \simeq \frac{1}{\kappa}\gamma^{\kappa-1}$, which completes the proof. ∎

Proof of Proposition 9.1. Fix $\eta \in \mathcal{E}_N^x$ and $\zeta \in \mathcal{D}_N^x$. By Lemma 9.5,

$$\begin{aligned}\mathbf{P}_\eta^N[\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N] &\le \mathbf{P}_\eta^N[\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N,\ \tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}] + \mathbf{P}_\eta^N[\tau_\zeta > \tau_{(\mathcal{W}_N^x)^c}] \\ &= \mathbf{P}_\eta^N[\tau_{(\mathcal{W}_N^x)^c} \le \mathbf{h}_N,\ \tau_\zeta < \tau_{(\mathcal{W}_N^x)^c}] + o_N(1).\end{aligned}$$

By the strong Markov property,

$$\mathbf{P}^N_\eta[\tau_{(\mathcal{W}^x_N)^c} \le \mathbf{h}_N,\ \tau_\zeta < \tau_{(\mathcal{W}^x_N)^c}] \le \mathbf{P}^N_\zeta[\tau_{(\mathcal{W}^x_N)^c} \le \mathbf{h}_N]$$

so that

$$\mathbf{P}^N_\eta[\tau_{(\mathcal{W}^x_N)^c} \le \mathbf{h}_N] \le \mathbf{P}^N_\zeta[\tau_{(\mathcal{W}^x_N)^c} \le \mathbf{h}_N] + o_N(1).$$

Multiplying both sides by $\pi^x_N(\zeta)$ and summing over $\zeta \in \mathcal{D}^x_N$, we get

$$\mathbf{P}^N_\eta[\tau_{(\mathcal{W}^x_N)^c} \le \mathbf{h}_N] \le \mathbf{P}^N_{\pi^x_N}[\tau_{(\mathcal{W}^x_N)^c} \le \mathbf{h}_N] + o_N(1).$$

Apply Lemma 9.7 to complete the proof. ■

10. Condition $\mathfrak{V}$ for critical zero-range processes

In this section, we prove that the process $\xi_N(\cdot)$ starting from a well $\mathcal{E}^x_N$ hits quickly the configuration ζ^x_N. For $N \ge 1$, define the time-scale $\mathbf{u}_N$ by

$$\mathbf{u}_N := \frac{m_N^2}{\theta_N} = \frac{1}{(\log N)^{1+2\delta}}.$$

Proposition 10.1. *For all $x \in S$,*

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}^x_N} \mathbf{P}^N_\eta[\tau_{\zeta^x_N} \ge \mathbf{u}_N] = 0.$$

In particular, condition $\mathfrak{V}$ holds for the critical zero-range processes.

This result is crucially used in the next section to verify the requirement (6.4) of condition $\mathfrak{M}$.

10.1. A superharmonic function

We first establish, in Lemma 10.6 below, the estimate stated in Proposition 10.1 for the process which is reflected at the boundary of $\mathcal{W}^x_N$. The proof of Lemma 10.6 is based on the construction, carried out in [41, Section 10], of a function $G^x_N : \mathcal{H}_N \to \mathbb{R}$, for $x \in S$, which is superharmonic on $\mathcal{W}^x_N \setminus \mathcal{D}^x_N$. For the sake of completeness, we recall its definition and main properties below.

Fix $x_0 \in S$, and let $S_0 = S \setminus \{x_0\}$. For a subset $\mathcal{C}$ of $\mathcal{H}_N$, let

$$\begin{aligned}
\operatorname{int}\mathcal{C} &= \{\eta \in \mathcal{C} : \sigma^{x,y}\eta \in \mathcal{C} \text{ for all } x, y \text{ with } r(x,y) > 0\},\\
\partial\mathcal{C} &= \mathcal{C} \setminus \operatorname{int}\mathcal{C},\\
\overline{\mathcal{C}} &= \{\eta \in \mathcal{H}_N : \eta \in \mathcal{C} \text{ or } \sigma^{x,y}\eta \in \mathcal{C} \text{ for some } x, y \text{ with } r(x,y) > 0\}.
\end{aligned}$$

With this notation, let

$$\mathcal{U}^{x_0}_N = \overline{\mathcal{W}^{x_0}_N \setminus \mathcal{D}^{x_0}_N} \quad \text{so that} \quad \operatorname{int}\mathcal{U}^{x_0}_N = \mathcal{W}^{x_0}_N \setminus \mathcal{D}^{x_0}_N.$$

Recall, from (3.5), that $h_{A,B}(\cdot)$ and $\mathrm{cap}_X(\cdot,\cdot)$ represent the equilibrium potential and the capacity, respectively, associated to the underlying random walk X. For each non-empty subset A of S_0, consider the sequence $(b^A_{x,y})_{x,y\in S}$ defined by

$$b^A_{x,y} = \frac{1}{\kappa}\,\frac{h_{x,A^c}(y)}{\mathrm{cap}_X(x,A^c)}, \quad x, y \in A,$$

and $b^A_{x,y} = 0$ otherwise. By elementary properties of the capacity and the equilibrium potential, $b^A_{x,y} = b^A_{y,x}$ for all $x, y \in S$ (see [41, Lemma 10.2]). Moreover, by [41, Lemma 10.3],

$$b^A_{x,y} \le b^B_{x,y} \quad \text{for all } x, y \in S \tag{10.1}$$

if $A \subset B \subset S_0$.

For each non-empty subset A of S_0, define the quadratic function $P^A(\cdot)$ as

$$P^A(\eta) = \frac{1}{2}\sum_{x\in A} b^A_{x,x}\,\eta_x(\eta_x - 1) + \sum_{\{x,y\}\subset A} b^A_{x,y}\,\eta_x\eta_y .$$

By [41, Lemma 10.8],

$$c_1\Big(\sum_{x\in S_0}\eta_x\Big)^2 \le P^{S_0}(\eta) \le c_2\Big(\sum_{x\in S_0}\eta_x\Big)^2 . \tag{10.2}$$

Fix $A \subsetneq S_0$. For each constant $c_A > 0$ and positive integer $\ell \ge 1$, let $P^A_\ell : \mathcal{U}^{x_0}_N \to \mathbb{R}$ be given by

$$P^A_\ell(\eta) = P^A(\eta) - c_A\ell^2 .$$

The dependence of P^A_ℓ on the constant c_A is omitted from the notation. Taking $P^{\varnothing}_\ell(\eta) = 0$ for all $\eta \in \mathcal{U}^{x_0}_N$, define the corrector function $W_\ell : \mathcal{U}^{x_0}_N \to \mathbb{R}$ by

$$W_\ell(\eta) = \min\{P^A_\ell(\eta) : A \subset S_0,\ A \ne S_0\}.$$

By [41, Lemma 10.10], there exists a constant $0 < C < \infty$ such that

$$-C\ell^2 \le W_\ell(\eta) \le 0 .$$

Hence, by (10.2) and the previous bound, $P^{S_0}(\eta) - W_\ell(\eta) > 0$ for all $\eta \in \mathcal{U}^{x_0}_N$.

For each positive integer $m > 2$, define the function $G^{x_0}_N : \mathcal{H}_N \to \mathbb{R}$ by

$$G^{x_0}_N(\eta) = \begin{cases} \displaystyle\sum_{\ell=2}^{m} \frac{1}{\ell}\big[P^{S_0}(\eta) - W_\ell(\eta)\big]^{1/2}, & \eta \in \mathcal{U}^{x_0}_N, \\ 0, & \zeta \in \mathcal{H}_N \setminus \mathcal{U}^{x_0}_N . \end{cases} \tag{10.3}$$

Here, again, the dependence of $G^{x_0}_N$ on m is omitted. The next result is [41, Theorem 9.2]. Recall that $\mathcal{L}^{\xi}_N = \theta_N \mathcal{L}_N$, introduced in Section 3.2, is the generator of the speeded-up process.

Theorem 10.2. *For large enough m and a suitable selection $(c_A)_{A\subsetneq S_0}$ of constants, the function $G_N^{x_0}$ is superharmonic in $\mathcal{W}_N^{x_0} \setminus \mathcal{D}_N^{x_0}$. More precisely, there exists a positive constant $C > 0$ such that*

$$(\mathcal{L}_N^{\xi} G_N^{x_0})(\eta) \le -\frac{C\theta_N}{N-\eta_{x_0}} \quad \textit{for all } \eta \in \mathcal{W}_N^{x_0} \setminus \mathcal{D}_N^{x_0}.$$

Additionally, there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1(N-\eta_{x_0}) \le G_N^{x_0}(\eta) \le c_2(N-\eta_{x_0}) \tag{10.4}$$

for all $\eta \in \mathcal{W}_N^{x_0} \setminus \mathcal{D}_N^{x_0}$.

10.2. Reflected processes

Denote by $(\widehat{\xi}_N^x(t))_{t\ge 0}$ the continuous-time Markov chain on $\mathcal{W}_N^x$ obtained by reflecting the zero-range process $\xi_N(\cdot)$ at the boundary of this set. In other words, the process $\widehat{\xi}_N^x(\cdot)$ behaves as the zero-range process inside the well $\mathcal{W}_N^x$, but its jumps to the set $(\mathcal{W}_N^x)^c$ are suppressed. Denote by $\widehat{\mathbf{P}}_\eta^{N,x}$ the law of the reflected process $\widehat{\xi}_N^x(\cdot)$, and by $\widehat{\mathbf{E}}_\eta^{N,x}$ the expectation with respect to $\widehat{\mathbf{P}}_\eta^{N,x}$.

The next result asserts that the function $G_N^{x_0}$ is also superharmonic in $\mathcal{W}_N^{x_0} \setminus \mathcal{D}_N^{x_0}$ for the reflected process. Denote by $\mathcal{L}_N^{x_0}$ the generator associated to the reflected process $\widehat{\xi}_N^{x_0}(\cdot)$.

Lemma 10.3. *Fix $x_0 \in S$, and let $G_N^{x_0}$ be the function given by (10.3). Then there exists $C > 0$ such that*

$$(\mathcal{L}_N^{x_0} G_N^{x_0})(\eta) \le -\frac{C\theta_N}{N-\eta_{x_0}} \quad \textit{for all } \eta \in \mathcal{W}_N^{x_0} \setminus \mathcal{D}_N^{x_0}.$$

The main difference between this lemma and Theorem 10.2 is the analysis around the boundary of $\mathcal{W}_N^{x_0}$, since the generator $\mathcal{L}_N^{x_0}$ differs from $\mathcal{L}_N$ there, as the jumps to $(\mathcal{W}_N^{x_0})^c$ are excluded.

Proof of Lemma 10.3. Since we possibly have $(\mathcal{L}_N^{x_0} G_N^{x_0})(\eta) \neq (\mathcal{L}_N G_N^{x_0})(\eta)$ only at the boundary $\partial\mathcal{W}_N^{x_0} = \{\eta : \eta_{x_0} = N - m_N\}$, it suffices to show that

$$(\mathcal{L}_N^{x_0} G_N^{x_0})(\eta) \le (\mathcal{L}_N^{\xi} G_N^{x_0})(\eta) \quad \text{for all } \eta \in \partial\mathcal{W}_N^{x_0}.$$

At $\partial\mathcal{W}_N^{x_0}$, the reflected process cannot decrease the number of particles at site x_0. Thus,

$$(\mathcal{L}_N^{x_0} G_N^{x_0})(\eta) = (\mathcal{L}_N^{\xi} G_N^{x_0})(\eta) - \theta_N \sum_{y\in S} g(\eta_{x_0}) r(x_0,y)[G_N^{x_0}(\sigma^{x_0,y}\eta) - G_N^{x_0}(\eta)],$$

and it is enough to show that

$$G_N^{x_0}(\sigma^{x_0,y}\eta) \ge G_N^{x_0}(\eta) \quad \text{ for all } \eta \in \partial\mathcal{W}_N^{x_0}.$$

Actually, by the definition (10.3) of $G_N^{x_0}$, it is enough to show that

$$P^{S_0}(\sigma^{x_0,y}\eta) - W_\ell(\sigma^{x_0,y}\eta) \geq P^{S_0}(\eta) - W_\ell(\eta) \tag{10.5}$$

for all $\ell \geq 2$ and $\eta \in \partial \mathcal{W}_N^{x_0}$.

Fix $A \subsetneq S_0$. By definitions of P^{S_0} and P^A along with the increasing property (10.1),

$$\begin{aligned} P^{S_0}(\sigma^{x_0,y}\eta) - P^{S_0}(\eta) &= \sum_{z\in S_0} b_{y,z}^{S_0}\eta_z \geq \sum_{z\in A} b_{y,z}^{A}\eta_z \\ &= P^A(\sigma^{x_0,y}\eta) - P^A(\eta). \end{aligned} \tag{10.6}$$

Hence, if $W_\ell(\eta) = P_\ell^A(\eta)$ and $W_\ell(\sigma^{x,y}\eta) = P_\ell^A(\sigma^{x,y}\eta)$ for the same set $A \subsetneq S \setminus \{x_0\}$, then (10.5) follows from (10.6).

On the other hand, if $W_\ell(\eta) = P_\ell^A(\eta)$ and $W_\ell(\sigma^{x,y}\eta) = P_\ell^B(\sigma^{x,y}\eta)$ for some $A \neq B$, then by definition of W_ℓ and (10.6),

$$\begin{aligned} W_\ell(\sigma^{x,y}\eta) - W_\ell(\eta) &= P_\ell^B(\sigma^{x,y}\eta) - P_\ell^A(\eta) \\ &\leq P_\ell^A(\sigma^{x,y}\eta) - P_\ell^A(\eta) \leq P(\sigma^{x,y}\eta) - P(\eta). \end{aligned}$$

This completes the proof of (10.5) and of the lemma. ■

10.3. Hitting times of the reflected process

In this subsection, we establish, in Lemma 10.6 below, that the assertion of Proposition 10.1 holds for the reflected process $\widehat{\xi}_N^x(\cdot)$. The first result asserts that the process $\widehat{\xi}_N^x(\cdot)$ hits the set $\mathcal{D}_N^x$ quickly when it starts from a configuration in $\mathcal{E}_N^x$.

Lemma 10.4. *There exists $C > 0$ such that, for all $x \in S$ and $N \geq 1$,*

$$\sup_{\eta\in\mathcal{E}_N^x} \widehat{\mathbf{E}}_\eta^{N,x}[\tau_{\mathcal{D}_N^x}] \leq C\,\frac{m_N \ell_N}{\theta_N}.$$

Proof. By the martingale formulation, for every $t > 0$,

$$\widehat{\mathbf{E}}_\eta^{N,x}[G_N^x(\widehat{\eta}_N^x(\tau_{\mathcal{D}_N^x}\wedge t))] = G_N^x(\eta) + \widehat{\mathbf{E}}_\eta^{N,x}\Big[\int_0^{\tau_{\mathcal{D}_N^x}\wedge t} (\mathcal{L}_N^x G_N^x)(\widehat{\eta}_N^x(s))\,ds\Big].$$

By Lemma 10.3, there exists a positive constant C, whose value may change from line to line, such that

$$(\mathcal{L}_N^x G_N^x)(\eta) \leq -\frac{C\theta_N}{N-\eta_x} \leq -\frac{C\theta_N}{m_N} \quad \text{for } \eta \in \mathcal{W}_N^x.$$

On the other hand, by (10.4), G_N^x is non-negative. Therefore, by the next to last displayed equation,

$$\frac{C\theta_N}{m_N}\widehat{\mathbf{E}}_\eta^{N,x}[\tau_{\mathcal{D}_N^x}\wedge t] \leq G_N^x(\eta).$$

By (10.4), there exists a finite constant C_1 such that $G_N^x(\eta) \le C_1(N - \eta_x)$. Hence, since $N - \eta_x \le \ell_N$ for $\eta \in \mathcal{E}_N^x$,

$$\frac{C\theta_N}{m_N} \widehat{\mathbf{E}}_\eta^{N,x}[\tau_{\mathcal{D}_N^x} \wedge t] \le C_1 \ell_N .$$

To complete the proof of the lemma, it remains to let $t \to \infty$. ∎

The next result asserts that $\widehat{\xi}_N^x(\cdot)$ hits the configuration ζ_N^x quickly when it starts from a configuration in $\mathcal{D}_N^x$.

Lemma 10.5. *There exists a finite constant C such that*

$$\sup_{\eta \in \mathcal{D}_N^x} \widehat{\mathbf{E}}_\eta^{N,x}[\tau_{\zeta_N^x}] \le C \frac{N^{\gamma\kappa}(\log N)^{\kappa-1}}{\theta_N}$$

for all $x \in S$ and $N \ge 1$.

Proof. If $\eta = \zeta_N^x$, there is nothing to prove. For $\eta \ne \zeta_N^x$, we recall the well-known identity (see [6, Proposition 6.10])

$$\widehat{\mathbf{E}}_\eta^{N,x}[\tau_{\zeta_N^x}] = \frac{\mathbb{E}_{\widehat{\mu}_N^x}[\mathfrak{h}_{\eta,\zeta_N^x}^{N,x}]}{\operatorname{cap}_N^x(\eta, \zeta_N^x)}, \tag{10.7}$$

where $\mathfrak{h}_{\eta,\zeta_N^x}^{N,x}$ and $\operatorname{cap}_N^x(\eta, \zeta_N^x)$ denote the equilibrium potential and the capacity between η and ζ_N^x with respect to the reflected process $\widehat{\xi}_N^x(\cdot)$, respectively, and where $\widehat{\mu}_N^x(\cdot)$ denotes the invariant measure conditioned on $\mathcal{W}_N^x$, i.e.,

$$\widehat{\mu}_N^x(\cdot) = \mu_N(\cdot | \mathcal{W}_N^x) = \frac{\mu_N(\cdot)}{\mu_N(\mathcal{W}_N^x)}.$$

Observe that $\widehat{\mu}_N^x(\cdot)$ is the invariant measure of the reflected process $\widehat{\xi}_N^x(\cdot)$.

Applying the trivial bound $\mathfrak{h}_{\eta,\zeta_N^x}^{N,x} \le 1$ to (10.7), we get

$$\widehat{\mathbf{E}}_\eta^{N,x}[\tau_{\zeta_N^x}] \le \frac{1}{\operatorname{cap}_N^x(\eta, \zeta_N^x)}.$$

By [41, Lemma 9.3],

$$\operatorname{cap}_N^x(\eta, \zeta_N^x) \ge \frac{C\theta_N}{N^{\gamma\kappa}(\log N)^{\kappa-1}}.$$

Actually, in [41, Lemma 9.3] this bound is proved for the capacity with respect to the original zero-range process, but the same proof applies to the reflected process. To complete the proof, it remains to combine the previous bounds. ∎

Lemma 10.6. *For all $x \in S$,*

$$\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N^x} \widehat{\mathbf{P}}_\eta^{N,x}[\tau_{\zeta_N^x} \ge \mathbf{u}_N] = 0.$$

Proof. By Lemmata 10.4 and 10.5, and the strong Markov property,

$$\widehat{\mathbf{E}}^{N,x}_{\eta}[\tau_{\xi^x_N}] \le \frac{C}{\theta_N}\{m_N\ell_N + CN^{\gamma\kappa}(\log N)^{\kappa-1}\} \ll \frac{m_N^2}{\theta_N},$$

since we have assumed that $\gamma < 2/\kappa$. The assertion of the lemma follows from the Chebyshev inequality. ■

10.4. Proof of Proposition 10.1

Consider the canonical coupling of the zero-range process $\xi_N(\cdot)$ and the reflected process $\widehat{\xi}^x_N(\cdot)$ starting together at $\eta \in \mathcal{W}^x_N$. The two processes move together until $\xi_N(\cdot)$ hits $(\mathcal{W}^x_N)^c$. From this point on, they move independently according to their respective dynamics. By Proposition 9.1, starting from $\mathcal{E}^x_N$, we can couple the original zero-range process and the reflected process $\widehat{\xi}^x_N(\cdot)$ up to time $\mathbf{h}_N$ with a probability close to 1.

The joint law of $\xi_N(\cdot)$ and $\widehat{\xi}^x_N(\cdot)$ under this canonical coupling is represented by $\widehat{\mathbf{P}}^{N,x}_{\eta}$. Denote by $\tau_{\mathcal{A}}$ and $\widehat{\tau}_{\mathcal{A}}$ the hitting time of a set $\mathcal{A}$ with respect to $\xi_N(\cdot)$ and $\widehat{\xi}^x_N(\cdot)$, respectively.

Proof of Proposition 10.1. Recall the definition of the sequence $\mathbf{u}_N$ introduced at the beginning of Section 10, and the one of $\mathbf{h}_N$ presented in (9.1). Fix $\eta \in \mathcal{E}^x_N$. By Proposition 9.1,

$$\mathbf{P}^N_{\eta}[\tau_{\xi^x_N} \ge \mathbf{u}_N] = \mathbf{P}^N_{\eta}[\tau_{\xi^x_N} \ge \mathbf{u}_N,\, \tau_{(\mathcal{W}^x_N)^c} > \mathbf{h}_N] + o_N(1).$$

Recall the canonical coupling introduced above. On the event $\{\tau_{(\mathcal{W}^x_N)^c} > \mathbf{h}_N\}$, the two processes $\xi_N(t)$ and $\widehat{\xi}^x_N(t)$ move together until $\mathbf{h}_N$. Since $\mathbf{u}_N \ll \mathbf{h}_N$, on the previous event the sets $\{\tau_{\xi^x_N} \ge \mathbf{u}_N\}$ and $\{\widehat{\tau}_{\xi^x_N} \ge \mathbf{u}_N\}$ coincide. Thus,

$$\begin{aligned}\mathbf{P}^N_{\eta}[\tau_{\xi^x_N} \ge \mathbf{u}_N,\, \tau_{(\mathcal{W}^x_N)^c} > \mathbf{h}_N] &= \widehat{\mathbf{P}}^{N,x}_{\eta}[\tau_{\xi^x_N} \ge \mathbf{u}_N,\, \tau_{(\mathcal{W}^x_N)^c} > \mathbf{h}_N] \\ &= \widehat{\mathbf{P}}^{N,x}_{\eta}[\widehat{\tau}_{\xi^x_N} \ge \mathbf{u}_N,\, \tau_{(\mathcal{W}^x_N)^c} > \mathbf{h}_N].\end{aligned}$$

Since

$$\widehat{\mathbf{P}}^{N,x}_{\eta}[\widehat{\tau}_{\xi^x_N} \ge \mathbf{u}_N,\, \tau_{(\mathcal{W}^x_N)^c} > \mathbf{h}_N] \le \widehat{\mathbf{P}}^{N,x}_{\eta}[\widehat{\tau}_{\xi^x_N} \ge \mathbf{u}_N] = \widehat{\mathbf{P}}^{N,x}_{\eta}[\tau_{\xi^x_N} \ge \mathbf{u}_N],$$

by Lemma 10.6 this quantity vanishes as $N \to \infty$. It remains to combine the previous estimates. ■

11. Condition $\mathfrak{M}$ for critical zero-range processes

In this section, we prove condition (6.4) for a time-scale $\mathbf{s}_N \ll \mathbf{h}_N$ and the large wells $\mathcal{V}^x_N$ introduced in (9.2). For $N \ge 1$, define

$$\mathbf{s}_N = (1 + (\log N)^{1/4})\mathbf{u}_N. \tag{11.1}$$

Note that $\mathbf{s}_N \ll \mathbf{h}_N$.

Recall from Section 6.2 and equation (6.2) the definitions of the reflected process $\xi_N^{R,x}(\cdot)$ and of the total variation distance $d_{\mathrm{TV}}^x(\cdot,\cdot)$. For $t \ge 0$, let

$$D_{\mathrm{TV}}^x(t) := \sup_{\eta \in \mathcal{V}_N^x} d_{\mathrm{TV}}^x(\delta_\eta \mathcal{P}_N^{R,x}(t), \pi^{R,x}).$$

Note here that we prove a stronger version of mixing than the one required in condition $\mathfrak{M}$ since the supremum in the definition of $D_{\mathrm{TV}}^x(t)$ is taken over all configurations in $\mathcal{V}_N^x$.

Proposition 11.1. *For all $x \in S$,*

$$\lim_{N\to\infty} D_{\mathrm{TV}}^x(\mathbf{s}_N) = 0.$$

It follows from this result that for all $\varepsilon > 0$ the mixing time $t_{\mathrm{mix}}^x(\varepsilon)$ is bounded by $\mathbf{s}_N$ for N sufficiently large. In particular, condition (6.4) holds because $\mathbf{s}_N \ll \mathbf{h}_N$.

Corollary 11.2. *Condition $\mathfrak{M}$ holds for the critical zero-range processes.*

Proof. This follows from Corollary 9.2, Proposition 11.1, and the fact that $\mathbf{s}_N \ll \mathbf{h}_N$. ■

The proof of Proposition 11.1 is divided into several steps. We first show that the process $\xi_N^{R,x}(\cdot)$ hits the configuration ζ_N^x in the time-scale $\mathbf{u}_N$. The reasoning carried out in the proof of Lemma 10.6 does not apply to the process $\xi_N^{R,x}(\cdot)$ because Lemma 10.3 does not hold for it. We present below an alternative argument, based on Propositions 9.1 and 10.1.

Recall from Section 10.4 the definition of the canonical coupling of the zero-range process $\xi_N(\cdot)$ and the reflected process $\widehat{\xi}_N^x(\cdot)$. The same definition permits one to couple $\xi_N(\cdot)$ and $\xi_N^{R,x}(\cdot)$. Denote by $\widehat{\mathbf{P}}_\eta^{R,x}$ the joint law of $\xi_N(\cdot)$ and $\xi_N^{R,x}(\cdot)$ under the canonical coupling.

Lemma 11.3. *For all $x \in S$,*

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N^x} \mathbf{P}_\eta^{R,x}[\tau_{\xi_N^x} \ge \mathbf{u}_N] = 0.$$

Proof. By Proposition 10.1,

$$o_N(1) = \mathbf{P}_\eta^N[\tau_{\xi_N^x} \ge \mathbf{u}_N] \ge \mathbf{P}_\eta^N[\tau_{\xi_N^x} \ge \mathbf{u}_N,\ \tau_{(\mathcal{V}_N^x)^c} > \mathbf{h}_N].$$

Let $\tau_{\xi_N^x}^R$ stand for the hitting time of the configuration ζ_N^x with respect to the reflected process $\xi_N^{R,x}(\cdot)$. Replace the probability measure on the right-hand side of the previous equation by the coupling measure $\widehat{\mathbf{P}}_\eta^{R,x}$. Since $\mathbf{h}_N \gg \mathbf{u}_N$,

$$\widehat{\mathbf{P}}_\eta^{R,x}[\tau_{\xi_N^x} \ge \mathbf{u}_N,\ \tau_{(\mathcal{V}_N^x)^c} > \mathbf{h}_N] = \widehat{\mathbf{P}}_\eta^{R,x}[\tau_{\xi_N^x}^R \ge \mathbf{u}_N,\ \tau_{(\mathcal{V}_N^x)^c} > \mathbf{h}_N]$$

By Proposition 9.1, the right-hand side is equal to

$$\widehat{\mathbf{P}}_\eta^{R,x}[\tau_{\xi_N^x}^R \ge \mathbf{u}_N] - o_N(1) = \mathbf{P}_\eta^{R,x}[\tau_{\xi_N^x}^R \ge \mathbf{u}_N] - o_N(1),$$

as claimed. ■

We next recall a bound on the spectral gap established in [41].

Theorem 11.4. *There exists a constant* $c_0 > 0$ *such that the spectral gap of the reflected process* $\xi_N^{R,x}(\cdot)$ *on* $\mathcal{V}_N^x$ *is bounded below by* $c_0 \mathbf{s}_N^{-1}$ *for all* $N \geq 1$.

This result is [41, Theorem 6.1]. One just has to replace ℓ_N by m_N in the statement and in the proof of that result.

Recall from Section 6.2 that $\pi^{R,x}$, $x \in S$, represents the stationary state of the reflected process $\xi_N^{R,x}(\cdot)$. Moreover, for $\eta \in \mathcal{V}_N^x$ and $t > 0$, the measure $\delta_\eta \mathcal{P}_N^{R,x}(t)$ on $\mathcal{V}_N^x$ stands for the distribution of the reflected process $\xi_N^{R,x}(t)$ starting at η. Let

$$\pi_N^x(\cdot, t) = \delta_{\zeta_N^x} \mathcal{P}_N^{R,x}(t).$$

The next result asserts that the reflected process $\xi_N^{R,x}(\cdot)$ starting from ζ_N^x mixes in the time-scale $(\log N)^{1/4} \mathbf{u}_N$.

Lemma 11.5. *There exist two constants* $C_1, C_2 > 0$ *such that, for all* $x \in S$ *and* $t \geq (\log N)^{1/4} \mathbf{u}_N$,

$$d_{\mathrm{TV}}^x(\pi_N^x(\cdot, t), \pi^{R,x}) \leq C_1 e^{-C_2 (\log N)^{1/8}}.$$

Proof. By the Cauchy–Schwarz inequality,

$$d_{\mathrm{TV}}^x(\pi_N^x(\cdot, t), \pi^{R,x})^2 \leq \frac{1}{4} \sum_{\eta \in \mathcal{V}_N^x} \left\{ \frac{\pi_N^x(\eta, t)}{\pi^{R,x}(\eta)} - 1 \right\}^2 \pi^{R,x}(\eta).$$

Since the process $\xi_N(\cdot)$ is reversible, the conditioned measure $\mu_N(\cdot | \mathcal{V}_N^x)$ is the stationary measure for the reflected process $\xi_N^{R,x}(\cdot)$. Hence, by the standard L^2-contraction inequality (see [50, Lemma 20.5]) and Theorem 11.4, the summation on the right-hand side, which is actually the square of the L^2-distance between $\pi_N^x(\cdot, t)$ and $\pi^{R,x}(\cdot)$, is less than or equal to

$$e^{-c_0 (t/\mathbf{u}_N)} \sum_{\eta \in \mathcal{V}_N^x} \left\{ \frac{\pi_N^x(\eta, 0)}{\pi^{R,x}(\eta)} - 1 \right\}^2 \pi^{R,x}(\eta)$$

for some constant $c_0 > 0$ independent of N. As $t \geq (\log N)^{1/4} \mathbf{u}_N$ and $\pi_N^x(\eta, 0) = \mathbf{1}\{\eta = \zeta_N^x\}$, this expression is bounded by

$$e^{-c_0 (\log N)^{1/4}} \left(\frac{1}{\pi^{R,x}(\zeta_N^x)} - 1 \right),$$

By the explicit formula for the invariant measure μ_N,

$$\pi^{R,x}(\zeta_N^x) = \frac{1}{\mu_N(\mathcal{V}_N^x)} \frac{N}{Z_{N,\kappa} (\log N)^{\kappa - 1}} \frac{1}{\mathbf{a}(\zeta_N^x)} \geq \frac{c_1}{(\log N)^{\kappa - 1}}$$

for some constant $c_1 > 0$. Putting together the previous estimates yields

$$d_{\mathrm{TV}}^x(\pi_N^x(\cdot, t), \pi^{R,x}) \leq c_1^{-1} e^{-c_0 (\log N)^{1/4}} (\log N)^{\kappa - 1} \ll c_1^{-1} e^{-(c_0/2)(\log N)^{1/4}}.$$

This completes the proof. ∎

Proof of Proposition 11.1. The proof relies on Lemmata 11.3 and 11.5. Fix $x \in S$, $\eta \in \mathcal{E}_N^x$ and $\mathcal{A} \subset \mathcal{V}_N^x$. By Lemma 11.3, we can write

$$\mathbf{P}_\eta^{R,x}[\xi_N^{R,x}(\mathbf{s}_N) \in \mathcal{A}] = \mathbf{P}_\eta^{R,x}[\xi_N^{R,x}(\mathbf{s}_N) \in \mathcal{A} | \tau_{\xi_N^x}^R < \mathbf{u}_N] + o_N(1), \tag{11.2}$$

where the error term $o_N(1)$ on the right-hand side is bounded by $2\mathbf{P}_\eta^{R,x}[\tau_{\xi_N^x}^R \geq \mathbf{u}_N]$ and hence is independent of $\mathcal{A}$.

Denote by $\alpha_N^x(t)\, dt$ the distribution of $\tau_{\xi_N^x}^R$ conditioned on $\tau_{\xi_N^x}^R < \mathbf{u}_N$. Then, by the strong Markov property, we can write the probability on the right-hand side as

$$\int_0^{\mathbf{u}_N} \mathbf{P}_{\xi_N^x}^{R,x}[\xi_N^{R,x}(\mathbf{s}_N - t) \in \mathcal{A}] \alpha_N^x(t)\, dt. \tag{11.3}$$

Since $\mathbf{s}_N - t \geq (\log N)^{1/4} \mathbf{u}_N$ for all $t \in [0, \mathbf{u}_N]$, by definition of $\mathbf{s}_N$, it follows from Lemma 11.5 that

$$\begin{aligned} |\mathbf{P}_{\xi_N^x}^{R,x}[\xi_N^{R,x}(\mathbf{s}_N - t) \in \mathcal{A}] - \pi^{R,x}(\mathcal{A})| &\leq d_{\mathrm{TV}}^x(\pi_N^x(\cdot, \mathbf{s}_N - t), \pi^{R,x}) \\ &\leq C_1 e^{-C_2 (\log N)^{1/8}}, \end{aligned} \tag{11.4}$$

where we have used the fact that

$$d_{\mathrm{TV}}^x(\nu_1, \nu_2) = \sup_{\mathcal{A} \subset \mathcal{V}_N^x} |\nu_1(\mathcal{A}) - \nu_2(\mathcal{A})| \tag{11.5}$$

for any probability measures ν_1 and ν_2 on $\mathcal{H}_N$. By (11.4) and (11.3), we can assert that the right-hand side of (11.2) is $\pi^{R,x}(\mathcal{A}) + o_N(1)$. Thus,

$$\mathbf{P}_\eta^{R,x}[\xi_N^{R,x}(\mathbf{s}_N) \in \mathcal{A}] - \pi^{R,x}(\mathcal{A}) = o_N(1),$$

where the error term is independent of $\mathcal{A}$. Therefore, by (11.5), we can conclude that

$$D_{\mathrm{TV}}^x(\mathbf{s}_N) = \sup_{\mathcal{A} \subset \mathcal{V}_N^x} |\mathbf{P}_\eta^{R,x}[\xi_N^{R,x}(\mathbf{s}_N) \in \mathcal{A}] - \pi^{R,x}(\mathcal{A})| = o_N(1),$$

as claimed. ■

12. Condition (H0) for critical zero-range processes

In this section, we verify condition (H0) by establishing the following proposition. For each $A \subset S$, write

$$\mathcal{E}_N(A) = \bigcup_{x \in A} \mathcal{E}_N^x.$$

Proposition 12.1. *Fix a non-empty subset $S_1 \subsetneq S$, and let $S_2 = S \setminus S_1$. Then*

$$\lim_{N \to \infty} \mathrm{cap}_N(\mathcal{E}_N(S_1), \mathcal{E}_N(S_2)) = 6 \sum_{x \in S^1, y \in S^2} \mathrm{cap}_X(x, y).$$

The proof is similar to that of [7, Theorem 2.2]. As in [7], we prove the lower and upper bounds separately. The respective proofs is given in Sections 12.1 and 12.2. We prove several technical lemmata in Section 12.3.

Note that the zero-range dynamics that we are considering now is reversible, and thus we can express the mean-jump rate $r_N(x, y)$ as in (5.2). Hence, the following is immediate consequence of Theorem 3.1 and Proposition 12.1.

Corollary 12.2. *The critical zero-range processes satisfy condition* (H0) *with*

$$r(x, y) = 6\kappa \mathrm{cap}_X(x, y), \quad x, y \in S.$$

Now we turn to the proof of Proposition 12.1.

12.1. Lower bound

We start with a lower bound whose proof is a modification of [7, Proposition 4.1]. For the proof, we have to introduce a notion of tube along which the metastable transition occurs. For $x, y \in S$, $x \neq y$, define the tube $\mathcal{I}_N^{x,y}$ between $\mathcal{E}_N^x$ and $\mathcal{E}_N^y$ as

$$\mathcal{I}_N^{x,y} = \{\xi \in \mathcal{H}_{N-1} : \xi_x + \xi_y \geq (N-1) - \ell_N/3 \text{ and } \xi_x, \xi_y \leq (N-1) - \ell_N\}.$$

Then we can observe that

$$\mathcal{I}_N^{x,y} = \mathcal{I}_N^{y,x} \quad \text{and} \quad \mathcal{I}_N^{x,y} \cap \mathcal{I}_N^{z,w} = \emptyset \text{ if } \{x, y\} \neq \{z, w\} \tag{12.1}$$

for all large enough N. From now on, all the computations implicitly assume that N is large enough. This is legitimate since we will send N to ∞ in the end. The former condition in (12.1) is immediate from the definition. For the latter, the statement is obvious if $\{x, y\} \cap \{z, w\} = \emptyset$. To check the other case, suppose that $\xi \in \mathcal{I}_N^{x,y} \cap \mathcal{I}_N^{x,w}$ for some $x, y, w \in S$. Then we must have

$$2(N-1-\ell_N/3) \leq \xi_x + (\xi_y + \xi_x + \xi_w) \leq \xi_x + N - 1$$

and hence $\xi_x \geq N - 1 - 2\ell_N/3$. This contradicts the condition $\xi_x \leq N - 1 - \ell_N$ of $\mathcal{I}_N^{x,y}$.

Proposition 12.3. *Fix a non-empty subset $S_1 \subsetneq S$, and let $S_2 = S \setminus S_1$. Then*

$$\liminf_{N\to\infty} \mathrm{cap}_N(\mathcal{E}_N(S_1), \mathcal{E}_N(S_2)) \geq 6 \sum_{x \in S^1,\, y \in S^2} \mathrm{cap}_X(x, y).$$

Proof. Write $\mathfrak{h} = h^N_{\mathcal{E}_N(S_1), \mathcal{E}_N(S_2)}$ (see (2.10)) for the equilibrium potential between $\mathcal{E}_N(S_1)$ and $\mathcal{E}_N(S_2)$ so that $\mathrm{cap}_N(\mathcal{E}_N(S_1), \mathcal{E}_N(S_2)) = \mathcal{D}_N(\mathfrak{h})$.

Let $\mathfrak{d}_x \in \mathcal{H}_1$ denote the configuration with one particle at site x and no particles at the other sites. We can write the Dirichlet form as

$$\mathcal{D}_N(\mathfrak{h}) = \frac{N\theta_N}{2(\log N)^{\kappa-1} Z_{N,\kappa}} \sum_{\xi \in \mathcal{H}_{N-1}} \sum_{z,w \in S} \frac{r(z, w)}{\mathbf{a}(\xi)} [\mathfrak{h}(\xi + \mathfrak{d}_w) - \mathfrak{h}(\xi + \mathfrak{d}_z)]^2.$$

Thus, by (12.1), we can bound $\mathcal{D}_N(\mathfrak{h})$ from below by

$$\frac{N\theta_N}{2(\log N)^{\kappa-1}Z_{N,\kappa}}\sum_{x\in S_1,\,y\in S_2}\sum_{\xi\in\mathcal{I}_N^{x,y}}\sum_{z,w\in S}\frac{r(z,w)}{\mathbf{a}(\xi)}[\mathfrak{h}(\xi+\mathfrak{d}_w)-\mathfrak{h}(\xi+\mathfrak{d}_z)]^2. \tag{12.2}$$

For $x\in S_1$ and $y\in S_2$, fix a configuration $\xi\in\mathcal{I}_N^{x,y}$ such that $\mathfrak{h}(\xi+\mathfrak{d}_x)\neq\mathfrak{h}(\xi+\mathfrak{d}_y)$. Define a function $f:S\to\mathbb{R}$ as

$$f(v)=\frac{\mathfrak{h}(\xi+\mathfrak{d}_v)-\mathfrak{h}(\xi+\mathfrak{d}_y)}{\mathfrak{h}(\xi+\mathfrak{d}_x)-\mathfrak{h}(\xi+\mathfrak{d}_y)},\quad v\in S.$$

Since $f(x)=1$ and $f(y)=0$, we can apply the Dirichlet principle for the underlying random walk to get (see (3.5) and (3.6))

$$\begin{aligned}&\frac{1}{2}\sum_{z,w\in S}r(z,w)[\mathfrak{h}(\xi+\mathfrak{d}_w)-\mathfrak{h}(\xi+\mathfrak{d}_z)]^2\\&\qquad=\kappa D_X(f)[\mathfrak{h}(\xi+\mathfrak{d}_x)-h(\xi+\mathfrak{d}_y)]^2\geq\kappa\mathrm{cap}_X(x,y)[\mathfrak{h}(\xi+\mathfrak{d}_x)-\mathfrak{h}(\xi+\mathfrak{d}_y)]^2.\end{aligned}$$

The same inequality obviously holds when $\mathfrak{h}(\xi+\mathfrak{d}_x)=\mathfrak{h}(\xi+\mathfrak{d}_y)$. Hence, we can bound the summation at (12.2) from below by

$$\sum_{x\in S_1,y\in S_2}\Bigg[\kappa\mathrm{cap}_X(x,y)\sum_{\xi\in\mathcal{I}_N^{x,y}}\frac{1}{\mathbf{a}(\xi)}[\mathfrak{h}(\xi+\mathfrak{d}_x)-\mathfrak{h}(\xi+\mathfrak{d}_y)]^2\Bigg]. \tag{12.3}$$

Fix $x_0\in S_1$ and $y_0\in S_2$ and denote $S_0=S\setminus\{x_0,y_0\}$. For each $\zeta\in\mathcal{H}_{k,S_0}$ (see (3.1)) with $k\leq\ell_N/3$, let $G_\zeta:\{0,\dots,N-1-k\}\to\mathbb{R}$ be the function defined by $G_\zeta(i)=\mathfrak{h}(\xi)$ where ξ is the configuration in $\mathcal{H}_N$ given by $\xi_v=\zeta_v$ for $v\in S_0$, $\xi_{x_0}=i$, and $\xi_{y_0}=N-k-i$. Then we can rewrite the second sum of (12.3) as

$$\sum_{k=0}^{\ell_N/3}\sum_{\zeta\in\mathcal{H}_{k,S_0}}\Bigg[\frac{1}{\mathbf{a}(\zeta)}\sum_{i=\ell_N-k}^{N-\ell_N-1}\frac{1}{i(N-k-1-i)}[G_\zeta(i+1)-G_\zeta(i)]^2\Bigg].$$

By the Cauchy–Schwarz inequality, the innermost sum above is bounded from below by

$$\Bigg[\sum_{i=\ell_N-k}^{N-\ell_N-1}i(N-k-1-i)\Bigg]^{-1}[G_\zeta(N-\ell_N)-G_\zeta(\ell_N-k)]^2. \tag{12.4}$$

By an elementary estimate,

$$\lim_{N\to\infty}N^3\Bigg[\sum_{i=\ell_N-k}^{N-\ell_N-1}i(N-k-1-i)\Bigg]^{-1}=\Bigg[\int_0^1u(1-u)\,du\Bigg]^{-1}=6,$$

and by the fact that $G_\zeta(N-\ell_N)=1$ and $G_\zeta(\ell_N-k)=0$, we can assert that (12.4) is

$(6+o_N(1))/N^3$. Summing up, we have shown so far that

$$\mathcal{D}_N(\mathfrak{h}) \ge [1+o_N(1)]\frac{6\kappa}{Z_{N,\kappa}}\sum_{x\in S_1,\,y\in S_2}\mathrm{cap}_X(x,y)\times\Big[\frac{1}{(\log N)^{\kappa-2}}\sum_{k=0}^{\ell_N/3}\sum_{\zeta\in\mathcal{H}_{k,S_0}}\frac{1}{\mathbf{a}(\zeta)}\Big].$$

Now it suffices to apply Lemma 12.7 and [41, Proposition 4.1] to complete the proof. ■

12.2. Upper bound

Now we deduce the upper bound of the capacity.

Proposition 12.4. *Fix a non-empty subset $S_1 \subsetneq S$, and let $S_2 = S\setminus S_1$. Then*

$$\limsup_{N\to\infty}\mathrm{cap}_N(\mathcal{E}_N(S_1),\mathcal{E}_N(S_2)) \le 6\sum_{x\in S_1, y\in S_2}\mathrm{cap}_X(x,y).$$

We remark at this moment that Proposition 12.1 is an immediate consequence of Propositions 12.3 and 12.4.

We fix S_1 and S_2 throughout this subsection. We prove the proposition exactly as in [7, Section 5] where a test function is constructed and then the upper bound is established via the Dirichlet principle. This test function can be constructed as a suitable approximation for the equilibrium potential $\mathfrak{h}_{\mathcal{E}_N(S_1),\mathcal{E}_N(S_2)}$. We repeat the construction of the test function in exactly the same way as in [7, Section 5].

The set $\mathcal{D}\subset\mathbb{R}^S$ is defined as

$$\mathcal{D} = \Big\{u\in[0,1]^S : \sum_{x\in S}u_x = 1\Big\},$$

and for each $x, y\in S$ and $\varepsilon\in(0,1/6)$ we define

$$\begin{aligned}\mathcal{D}_\varepsilon^x &= \{u\in\mathcal{D} : u_x > 1-\varepsilon\},\\ \mathcal{L}_\varepsilon^{x,y} &= \{u\in\mathcal{D} : u_x+u_y \ge 1-\varepsilon\}.\end{aligned}$$

From now on, fix $x\in S$ and $\varepsilon\in(0,1/6)$. Let $\phi:[0,1]\to[0,1]$ be a smooth bijective function such that $\phi(t)+\phi(1-t)=1$ for all $t\in[0,1]$ and $\phi\equiv 0$ on $[0,3\varepsilon]$. Then, define $H:[0,1]\to[0,1]$ as $H(t) = 6\int_0^{\phi(t)}u(1-u)\,du$. For each $y\in S\setminus\{x\}$, we write $h_{x,y} = h^X_{\{x\},\{y\}}$ (see (3.4)) for the equilibrium potential between x and y with respect to the underlying random walk. Then, enumerate the elements of S by $x=z_1, z_2, \dots, z_\kappa = y$ in such a manner that

$$1 = h_{x,y}(z_1) \ge h_{x,y}(z_2) \ge \cdots \ge h_{x,y}(z_\kappa) = 0.$$

Then, for each $y\in S\setminus\{x\}$, define $F^j_{xy}:\mathcal{H}_N\to\mathbb{R}$, $1\le j\le\kappa-1$, as $F^1_{xy}(\eta) = H(\eta_x/N)$ and

$$F^j_{xy}(\eta) = H\bigg(\frac{\eta_x}{N} + \min\Big\{\frac{1}{N}\sum_{i=1}^{j}\eta_{z_i},\varepsilon\Big\}\bigg)\quad\text{for } j\in[2,\kappa-1].$$

Define $F_{xy} : \mathcal{H}_N \to \mathbb{R}$ as

$$F_{xy}(\eta) = \sum_{j=1}^{\kappa-1} [h_{x,y}(z_j) - h_{x,y}(z_{j+1})] F^j_{xy}(\eta).$$

For $y \neq x$, write $\mathcal{K}^x_y = \mathcal{L}^{x,y}_\varepsilon \setminus \mathcal{D}^x_{3\varepsilon}$. We can observe that the $\kappa - 1$ sets $\mathcal{K}^x_y$, $y \neq x$, are pairwise disjoint compact subsets of $\mathcal{D}$. Therefore, there exists a smooth partition of unity $\Theta^x_y : \mathcal{D} \to [0, 1]$, $y \neq x$, such that

$$\sum_{y \in S\setminus\{x\}} \Theta^x_y \equiv 1 \quad \text{on } \mathcal{D} \quad \text{and} \quad \Theta^x_y \equiv 1 \quad \text{on } \mathcal{K}^x_y \text{ for } y \in S \setminus \{x\}.$$

With the constructions above, we define $F_x : \mathcal{H}_N \to \mathbb{R}$ as

$$F_x(\eta) = \sum_{y \in S\setminus\{x\}} \Theta^x_y(\eta/N) F_{xy}(\eta).$$

Finally, the test function $F_{S_1} : \mathcal{H}_N \to \mathbb{R}$ is defined by

$$F_{S_1}(\eta) = \sum_{x \in S_1} F_x(\eta).$$

The main property of this test function is the following lemma.

Lemma 12.5. *We have*

$$\limsup_{N\to\infty} \mathcal{D}_N(F_{S_1}) \le 6(1 + \varepsilon^{1/2})^3 \sum_{x \in S_1,\, y \in S_2} \mathrm{cap}_X(x, y).$$

We omit the proof of this lemma since it is identical to those of [7, (5.11), (5.12), Proposition 5.3]. Even if those results have been proved for $\alpha > 1$, the same argument also holds for $\alpha = 1$. The only different part is [7, Lemma 5.2] which is used in the proof of [7, (5.11)]. We substitute this lemma by the following lemma.

Lemma 12.6. *For $x, y \in S$, define*

$$\mathcal{I}^{xy}_N = \{\eta \in \mathcal{H}_N : \eta_x + \eta_y \ge N - \ell_N\}.$$

and let $\mathcal{I}^x_N = \bigcup_{y\in S\setminus\{x\}} \mathcal{I}^{xy}_N$. Then, for all $x \in S$ and $\varepsilon \in (0, 1/6)$, there exists a constant $C_\varepsilon > 0$ depending only on $\varepsilon > 0$ such that

$$\frac{1}{2} \sum_{\eta \in \mathcal{H}_N \setminus \mathcal{I}^x_N} \sum_{z,w \in S} \mu_N(\eta) \mathbf{g}(\eta_z) r(z, w) [F_x(\sigma^{zw}\eta) - F_x(\eta)]^2 \le \frac{C_\varepsilon (\log\log N)^2}{Z_{N,\kappa} N^2 (\log N)^2}. \tag{12.5}$$

Proof. Basically, we perform the same proof as in [7, Lemma 5.2], but the fact that $\alpha = 1$ makes it slightly different.

Since $F_x(\eta) = 1$ if $\eta_x \ge (1 - 3\varepsilon)N$ and $F_x(\eta) = 0$ if $\eta_x \le 2\varepsilon N$, we can restrict the first sum of (12.5) to configurations $\eta \in \mathcal{H}_N \setminus \mathcal{I}^x_N$ satisfying $\varepsilon N \le \eta_x \le (1 - \varepsilon)N$. Since there exists $C_\varepsilon > 0$ such that

$$\max_{\eta\in\mathcal{H}_N} |F_x(\sigma^{z,w}\eta) - F_x(\eta)| \le \frac{C_\varepsilon}{N},$$

we can bound the left-hand side of (12.5) from above by

$$\begin{aligned}
&\frac{C_\varepsilon}{N^2} \sum_{\eta\in\mathcal{H}_N\setminus\mathcal{I}_N^x,\, \varepsilon N\le\eta_x\le(1-\varepsilon)N} \mu_N(\eta) \\
&\quad\le \frac{C_\varepsilon N}{Z_{N,\kappa}(\log N)^{\kappa-1}N^2} \sum_{i=\varepsilon N}^{(1-\varepsilon)N} \sum_{\substack{\eta:\,\eta_x=i\\ \max\{\eta_y:y\neq x\}\le N-i-\ell_N}} \frac{1}{\mathbf{a}(\eta)} \\
&\quad= \frac{C_\varepsilon}{Z_{N,\kappa}(\log N)^{\kappa-1}N} \sum_{i=\varepsilon N}^{(1-\varepsilon)N} \sum_{\xi\in\mathcal{H}_{N-i,S\setminus\{x\}}(\ell_N)} \frac{1}{i}\,\frac{1}{\mathbf{a}(\xi)},
\end{aligned} \tag{12.6}$$

where, for $S_0 \subset S$, we define

$$\mathcal{H}_{N,S_0}(\ell) = \{\eta\in\mathcal{H}_{N,S_0} : \eta_x \le N-\ell \text{ for all } x\in S_0\}. \tag{12.7}$$

Since $\mathcal{H}_{N-i,S\setminus\{x\}}(\ell_N)$ is a subset of $\mathcal{H}_{N-i,S\setminus\{x\}}(\ell_{N-i})$, we can further bound (12.6) from above by

$$\begin{aligned}
&\frac{C_\varepsilon}{Z_{N,S}(\log N)^{\kappa-1}N} \sum_{i=\varepsilon N}^{(1-\varepsilon)N} \frac{1}{i} \sum_{\xi\in\mathcal{H}_{N-i,S\setminus\{x\}}(\ell_{N-i})} \frac{1}{\mathbf{a}(\xi)} \\
&\quad\le \frac{C_\varepsilon \log\log N}{Z_{N,S}N^2\log N} \left\{ \frac{N}{(\log N)^{\kappa-1}} \sum_{i=\varepsilon N}^{(1-\varepsilon)N} \frac{(\log(N-i))^{\kappa-2}}{i(N-i)} \right\} \\
&\quad\le \frac{C_\varepsilon \log\log N}{Z_{N,S}N^2\log N} \left\{ \frac{N}{\log N} \sum_{i=N/\log N}^{N-N/\log N} \frac{1}{i(N-i)} \right\} \le \frac{C_\varepsilon(\log\log N)^2}{Z_{N,S}N^2(\log N)^2}
\end{aligned}$$

where the first and last inequalities follow from Lemma 12.8. ■

Now we are ready to prove the upper bound.

Proof of Proposition 12.4. It is immediate that

$$F_{S_1}(\eta) = \begin{cases} 1 & \text{if } \eta\in\mathcal{E}_N(S_1), \\ 0 & \text{if } \eta\in\mathcal{E}_N(S_2). \end{cases}$$

Hence, by Lemma 12.5 and the Dirichlet principle, we get

$$\limsup_{N\to\infty} \mathrm{cap}_N(\mathcal{E}_N(S_1), \mathcal{E}_N(S_2)) \le (1+\varepsilon^{1/2})^3 \frac{6}{\kappa} \sum_{x\in S_1, y\in S_2} \mathrm{cap}_X(x,y).$$

Letting $\varepsilon\to 0$ completes the proof. ■

12.3. Auxiliary lemmata

In this subsection we prove two technical lemmata. The first one is used in the proof of Proposition 12.3.

Lemma 12.7. *For all $c > 0$ and $x, y \in S$, we have*

$$\lim_{N\to\infty} \frac{1}{(\log N)^\kappa} \sum_{n=0}^{c\ell_N} \sum_{\xi\in\mathcal{H}_{n,S}} \frac{1}{\mathbf{a}(\xi)} = 1. \tag{12.8}$$

Proof. By the definition (3.3) of the partition function, we can rewrite (12.8) as

$$\lim_{N\to\infty} \frac{1}{(\log N)^\kappa} \sum_{n=1}^{c\ell_N} \frac{(\log n)^{\kappa-1}}{n} Z_{n,\kappa} = 1$$

as the case $n = 0$ is negligible. Since $(Z_{n,\kappa})_{n\in\mathbb{N}}$ is bounded by [41, Proposition 4.1], and since

$$\frac{1}{(\log\log N)^\kappa} \sum_{n=1}^{\log N} \frac{(\log n)^{\kappa-1}}{n} \simeq \frac{1}{\kappa},$$

it suffices to prove that

$$\lim_{N\to\infty} \frac{1}{(\log N)^\kappa} \sum_{n=\log N}^{c\ell_N} \frac{(\log n)^{\kappa-1}}{n} Z_{n,\kappa} = 1.$$

This follows from [41, Proposition 4.1], from the elementary fact that

$$\sum_{n=\log N}^{c\ell_N} \frac{(\log n)^{\kappa-1}}{n} \simeq \frac{(\log(c\ell_N))^\kappa - (\log\log N)^\kappa}{\kappa},$$

and from $\lim_{N\to\infty} \log \ell_N / \log N = 1$. ∎

The next lemma is used in the proof of Proposition 12.4. Recall the definition of $\mathcal{H}_{N,S_0}(\ell)$ from (12.7). We simply write $\mathcal{H}_N(\ell) = \mathcal{H}_{N,S}(\ell)$.

Lemma 12.8. *Define $\ell_N^{(a)} = N/(\log N)^a$. For $\kappa \geq 2$ and $a > 0$, there exists a constant $C = C_{\kappa,a} > 0$ such that*

$$\frac{N}{(\log N)^{\kappa-1}} \sum_{\eta\in\mathcal{H}_N(\ell_N^{(a)})} \frac{1}{\mathbf{a}(\eta)} \leq C \frac{\log\log N}{\log N}. \tag{12.9}$$

Proof. We proceed by induction. For $\kappa = 2$, we can rewrite and bound the left-hand side of (12.9) as

$$\frac{N}{\log N} \sum_{n=\ell_N^{(a)}}^{N-\ell_N^{(a)}} \frac{1}{n(N-n)} \leq \frac{C}{\log N} \sum_{n=\ell_N^{(a)}}^{N/2} \frac{1}{n} \leq C \frac{\log\log N}{\log N}.$$

Now we assume the result holds for all $\kappa \in [2, \kappa_0 - 1]$ and all $a > 0$. Then, we will show that (12.9) holds for $\kappa = \kappa_0$ and $a > 0$. Define, for $n \le \ell_N^{(a+1)}$,

$$\mathcal{A}_{N,n}^{x,a} = \big\{\xi \in \mathcal{H}_{N-n,S\setminus\{x\}} : \xi_y \le N - \ell_N^{(a)} \text{ for all } y \in S \setminus \{x\}\big\}.$$

We first claim that

$$\mathcal{A}_{N,n}^{x,a} \subset \mathcal{H}_{N-n,S\setminus\{x\}}(\ell_{N-n}^{(a+1)}). \tag{12.10}$$

To verify this, it suffices to check that

$$(N-n) - \frac{N-n}{(\log(N-n))^{a+1}} \ge N - \frac{N}{(\log N)^a}.$$

This follows for $n \le \ell_N^{(a+1)}$ from the inequality

$$n + \frac{N-n}{(\log(N-n))^{a+1}} \le \frac{2N}{(\log N)^{a+1}} \le \frac{N}{(\log N)^a}.$$

Observe that we can write

$$\begin{aligned}
&\frac{N}{(\log N)^{\kappa_0-1}} \sum_{\eta \in \mathcal{H}_N(\ell_N^{(a)})} \frac{1}{\mathbf{a}(\eta)} \\
&\quad = \Bigg\{ \sum_{n=0}^{\ell_N^{(a+1)}} + \sum_{n=\ell_N^{(a+1)}+1}^{N-\ell_N^{(a)}} \Bigg\} \Bigg[\frac{N}{\mathbf{a}(n)(\log N)^{\kappa_0-1}} \sum_{\xi \in \mathcal{A}_{N,n}^{x,a}} \frac{1}{\mathbf{a}(\xi)} \Bigg].
\end{aligned} \tag{12.11}$$

By the induction hypothesis and (12.10), the first summation above is bounded by

$$\begin{aligned}
&\frac{N}{(\log N)^{\kappa_0-1}} \sum_{n=0}^{\ell_N^{(a+1)}} \Bigg[\frac{1}{\mathbf{a}(n)} \sum_{\xi \in \mathcal{H}_{N-n,S\setminus\{x\}}(\ell_{N-n}^{(a+1)})} \frac{1}{\mathbf{a}(\xi)} \Bigg] \\
&\quad \le \frac{CN}{(\log N)^{\kappa_0-1}} \sum_{n=0}^{\ell_N^{(a+1)}} \frac{1}{\mathbf{a}(n)} \frac{(\log(N-n))^{\kappa_0-2}}{N-n} \frac{\log\log(N-n)}{\log(N-n)} \\
&\quad \le C \frac{\log\log N}{(\log N)^2} \sum_{n=0}^{\ell_N^{(a+1)}} \frac{1}{\mathbf{a}(n)} \le C \frac{\log\log N}{\log N}.
\end{aligned}$$

On the other hand, for the second summation of (12.11), we can enlarge $\mathcal{A}_{N,n}^{x,a}$ to $\mathcal{H}_{N-n,S\setminus\{x\}}$ so that the summation is bounded by

$$C \frac{N}{(\log N)^{\kappa_0-1}} \sum_{n=\ell_N^{(a+1)}}^{N-\ell_N^{(a+1)}} \frac{Z_{N,\kappa_0-1}(\log(N-n))^{\kappa_0-2}}{(N-n)} \frac{1}{\mathbf{a}(n)} \le C \frac{\log\log N}{\log N}$$

by [41, Proposition 4.1]. ∎

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