J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1400

Published by EMS Press

Hongjie Dong · Hanye Zhu

Gradient estimates for singular p-Laplace type equations with measure data

Received February 17, 2021; revised September 9, 2022

Abstract. We are concerned with interior and global gradient estimates for solutions to a class of singular quasilinear elliptic equations with measure data, whose prototype is given by the p-Laplace equation $-\Delta_p u = \mu$ with $p \in (1,2)$. The cases when $p \in (2 - \frac{1}{n}, 2)$ and $p \in (\frac{3n-2}{2n-1}, 2 - \frac{1}{n}]$ were studied in Duzaar and Mingione [J. Funct. Anal. 259 (2010), 379–418] and Nguyen and Phuc [J. Funct. Anal. 278 (2020), art. 108391] respectively. In this paper, we improve the results of Nguyen and Phuc and address the open case when $p \in (1, \frac{3n-2}{2n-1}]$. Interior and global modulus of continuity estimates of the gradients of solutions are also established.

Keywords. p-Laplace type equations, gradient estimates, measure data, Dini continuity

1. Introduction

In this paper, we consider the quasilinear elliptic equation with measure data

$$-\mathrm{div}(A(x, \nabla u)) = \mu \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. Here μ is a locally finite signed Radon measure in Ω, namely, $|\mu|(B_R(x) \cap \Omega) < \infty$ for any open ball $B_R(x) \subset \mathbb{R}^n$. By setting $|\mu|(\mathbb{R}^n \setminus \Omega) = 0$, we will always assume that μ is defined in the whole space $\mathbb{R}^n$. The vector field $A = (A_1, \dots, A_n) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy the following growth, ellipticity, and continuity conditions: there exist constants $\lambda \geq 1$, $s \geq 0$, and $p > 1$ such that

$$|A(x,\xi)| \leq \lambda(s^2 + |\xi|^2)^{(p-1)/2}, \quad |D_\xi A(x,\xi)| \leq \lambda(s^2 + |\xi|^2)^{(p-2)/2}, \tag{1.2}$$

$$\langle D_\xi A(x,\xi)\eta, \eta\rangle \geq \lambda^{-1}(s^2 + |\xi|^2)^{(p-2)/2}|\eta|^2, \tag{1.3}$$

Hongjie Dong: Division of Applied Mathematics, Brown University, Providence, RI 02912-9056, USA; hongjie_dong@brown.edu

Hanye Zhu: Division of Applied Mathematics, Brown University, Providence, RI 02912-9056, USA; hanye_zhu@brown.edu

Mathematics Subject Classification (2020): Primary 35J62; Secondary 35J75, 31C45, 35B65, 35R06

and

$$|A(x,\xi)-A(x_0,\xi)| \le \lambda\omega(|x-x_0|)(s^2+|\xi|^2)^{(p-1)/2} \tag{1.4}$$

for all $x, x_0 \in \Omega$ and every $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$, and $\omega : [0,\infty) \to [0,1]$ is a concave nondecreasing function satisfying

$$\lim_{r\to 0^+} \omega(r) = \omega(0) = 0$$

and the Dini condition

$$\int_0^1 \omega(r)\,\frac{dr}{r} < \infty. \tag{1.5}$$

A typical model equation is given by the (possibly nondegenerate) p-Laplace equation with measure data and $s \ge 0$:

$$-\mathrm{div}\big(a(x)(|\nabla u|^2+s^2)^{(p-2)/2}\nabla u\big) = \mu \quad \text{in } \Omega, \tag{1.6}$$

where $a(\cdot)$ is a Dini continuous function in Ω, satisfying

$$0 < \lambda^{-1} \le a(x) \le \lambda \tag{1.7}$$

and

$$|a(x)-a(x_0)| \le \lambda\omega(|x-x_0|) \tag{1.8}$$

for all $x, x_0 \in \Omega$.

By a (*weak*) *solution* to (1.1), we mean a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that the distributional relation

$$\int_\Omega \langle A(x,\nabla u), D\varphi\rangle\,dx = \int_\Omega \varphi\,d\mu$$

holds whenever $\varphi \in C_0^\infty(\Omega)$ has compact support in Ω. We denote

$$B_R = B_R(0), \qquad \Omega_R(x) = \Omega \cap B_R(x).$$

The gradient estimates for the superquadratic case when $p \ge 2$ have been well studied in the literature; see [8, 10, 13, 19, 20]. However, the corresponding results for the singular case when $p \in (1,2)$ are far from complete.

In this paper, we are only concerned with the singular case when $p \in (1,2)$.

For singular quasilinear equations, the case when $p \in (2-\frac{1}{n}, 2)$ was considered in the pioneering work [9], in which the authors proved that under conditions (1.2)–(1.5), if $u \in C^1(\Omega)$ solves (1.1), then

$$|\nabla u(x)| \le C[\mathbf{I}_1^R(|\mu|)(x)]^{\frac{1}{p-1}} + C \fint_{B_R(x)} (|\nabla u(y)|+s)\,dy$$

for every ball $B_R(x) \subset \Omega$ with $R \in (0,1]$, where $C = C(n,p,\lambda,\omega)$. Here $\fint_E$ stands for the integral average over a measurable set E, and

$$\mathbf{I}_1^R(|\mu|)(x) := \int_0^R \frac{|\mu|(B_t(x))}{t^{n-1}}\,\frac{dt}{t} \tag{1.9}$$

is the truncated first-order *Riesz potential*. Later, the case when $p \in (\frac{3n-2}{2n-1}, 2 - \frac{1}{n}]$ was treated in [22], where the authors obtained a pointwise gradient bound involving the Wolff potential under stronger assumptions on A and ω. Namely, under the conditions (1.2)–(1.4) and further assuming that

$$|D_\xi A(x,\xi) - D_\xi A(x,\eta)| \\ \leq \lambda (s^2 + |\xi|^2)^{(p-2)/2}(s^2 + |\eta|^2)^{(p-2)/2}(s^2 + |\xi|^2 + |\eta|^2)^{(2-p-\alpha_0)/2}|\xi - \eta|^{\alpha_0} \tag{1.10}$$

and

$$\int_0^1 \omega(r)^\gamma \, \frac{dr}{r} < \infty$$

for some $\alpha_0 \in (0, 2-p)$ and $\gamma \in (\frac{n}{2n-1}, \frac{n(p-1)}{n-1}) \subset (0,1)$, if $u \in C^1(\Omega)$ is a solution to (1.1), then

$$|\nabla u(x)| \leq C[\mathbf{P}_\gamma^R(|\mu|)(x)]^{\frac{1}{\gamma(p-1)}} + C\left(\fint_{B_R(x)} (|\nabla u(y)| + s)^\gamma \, dy\right)^{1/\gamma},$$

where $C = C(n, p, \lambda, \alpha_0, \omega, \gamma)$ and

$$\mathbf{P}_\gamma^R(|\mu|)(x) := \int_0^R \left(\frac{|\mu|(B_t(x))}{t^{n-1}}\right)^\gamma \frac{dt}{t}$$

is a truncated nonlinear Wolff potential. We recall that in general, the truncated *Wolff potential* is defined as

$$\mathbf{W}_{\beta,p}^R(|\mu|)(x) := \int_0^R \left(\frac{|\mu|(B_t(x))}{t^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}, \quad \beta \in (0, n/p]. \tag{1.11}$$

Our first main result is stated as follows.

Theorem 1.1 (Interior pointwise gradient estimate). *Let $p \in (\frac{3n-2}{2n-1}, 2)$ and suppose that $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a solution to (1.1). Then under the assumptions (1.2)–(1.5), there exists a constant $C = C(n, p, \lambda, \omega)$ such that the estimate*

$$|\nabla u(x)| \leq C[\mathbf{I}_1^R(|\mu|)(x)]^{\frac{1}{p-1}} + C\left(\fint_{B_R(x)} (|\nabla u(y)| + s)^{2-p} \, dy\right)^{\frac{1}{2-p}} \tag{1.12}$$

holds for any Lebesgue point x of the vector-valued function ∇u and any $R \in (0,1]$ with $B_R(x) \subset \Omega$.

Remark 1.2. Our pointwise bound in Theorem 1.1 using the Riesz potential $\mathbf{I}_1^R(|\mu|)$ is an improvement of the bound in [22, Theorem 1.1] which contains the Wolff potential $\mathbf{P}_\gamma^R(|\mu|)$, since

$$\mathbf{I}_1^R(|\mu|) \leq C\mathbf{P}_\gamma^{2R}(|\mu|)^{1/\gamma} \quad \forall \gamma < 1.$$

The conditions on ω and A in Theorem 1.1 are also weaker. In particular, (1.10) is not assumed.

For the more singular case when $p \in (1, \frac{3n-2}{2n-1}]$, which has been open, we obtain the following Lipschitz estimate.

Theorem 1.3 (Interior Lipschitz estimate). *Let $p \in (1,2)$ and suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to (1.1). Then under the assumptions (1.2)–(1.5), there exists a constant $C = C(n, p, \lambda, \omega)$ such that the estimate*

$$\|\nabla u\|_{L^\infty(B_{R/2}(x))} \le C\,\|\mathbf{I}^R_1(|\mu|)\|^{\frac{1}{p-1}}_{L^\infty(B_R(x))} + CR^{-\frac{n}{2-p}}\,\big\||\nabla u| + s\big\|_{L^{2-p}(B_R(x))} \tag{1.13}$$

holds for any $R \in (0, 1]$ with $B_R(x) \subset \Omega$.

We also obtain a modulus of continuity estimate of ∇u in Theorem 4.3, which directly implies the following sufficient condition for the continuity of ∇u.

Theorem 1.4 (Gradient continuity via Riesz potential). *Let $p \in (1,2)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to (1.1). Assume that (1.2)–(1.5) are satisfied and the functions*

$$x \mapsto \mathbf{I}^R_1(|\mu|)(x) \text{ converge locally uniformly to zero in } \Omega \text{ as } R \to 0. \tag{1.14}$$

Then ∇u is continuous in Ω.

Recall the Lorentz space $L^{n,1}$ is the collection of measurable functions f such that

$$\int_0^\infty |\{x : |f(x)| \ge t\}|^{1/n}\, dt < \infty.$$

Theorem 1.4 has the following corollary.

Corollary 1.5 (Gradient continuity via Lorentz spaces). *Let $p \in (1,2)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to (1.1). Assume that (1.2)–(1.5) are satisfied and*

$$\mu \in L^{n,1} \quad \text{locally in } \Omega. \tag{1.15}$$

Then ∇u is continuous in Ω.

We remark that the Lorentz-space result above was proved in [14] for a p-Laplacian system similar to (1.6) when $p \in (1,\infty)$.

A further, actually immediate, corollary of Theorem 1.4 concerns measures with certain density properties.

Corollary 1.6 (Gradient continuity via density). *Let $p \in (1,2)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to (1.1). Assume that (1.2)–(1.5) are satisfied and μ satisfies*

$$|\mu|(B_\rho(x)) \le C\rho^{n-1}h(\rho) \tag{1.16}$$

for every ball $B_\rho(x) \subset\subset \Omega$, where C is a positive constant and $h : [0,\infty) \to [0,\infty)$ is a function satisfying the Dini condition

$$\int_0^R h(r)\,\frac{dr}{r} < \infty \quad \text{for some } R > 0. \tag{1.17}$$

Then ∇u is continuous in Ω.

We remark that Theorem 1.4 and Corollaries 1.5 and 1.6 above are indeed the subquadratic ($p \in (1,2)$) counterparts of [13, Theorem 1.5 and Corollaries 1.6, 1.7]. See also [8, Theorems 1, 3, and 4] and [20, Theorems 5.5 and 5.6]. We refer the reader to [8,13,20] for a discussion of the borderline nature of the assumptions in these results.

Another interesting consequence of Theorem 4.3 is the following gradient Hölder continuity result.

Corollary 1.7 (Gradient Hölder continuity via density). *Let $p \in (1,2)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to (1.1). Then under the assumptions (1.2)–(1.5), there exists a constant $\alpha \in (0,1)$, depending only on n, p, and λ, such that if $\omega(r) \le C r^\beta$ whenever $r > 0$ and $|\mu|(B_\rho(x)) \le C\rho^{n-1+\beta}$ whenever $B_\rho(x) \subset\subset \Omega$, for some constants $C > 0$ and $\beta \in (0,\alpha)$, then $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$.*

Remark 1.8. We should stress that the constant α in Corollary 1.7 is the natural Hölder exponent of the gradients of solutions to corresponding homogeneous equations with x-independent nonlinearities (cf. Lemma 2.2). Therefore, our result in Corollary 1.7 provides the best possible Hölder exponent for the gradient of the solution. The previous corollary is an improvement of the gradient Hölder regularity result by Lieberman [18, Theorem 5.3], who proved $u \in C^{1,\beta_1}_{\mathrm{loc}}$ for some $\beta_1 = \beta_1(n,p,\lambda,\beta) \in (0,1)$ under the same assumptions.

We also obtain up-to-boundary gradient estimates for the p-Laplace equations with measure data in domains with $C^{1,\mathrm{Dini}}$ boundaries.

Definition 1.9. Let Ω be a domain in $\mathbb{R}^n$. We say that Ω has $C^{1,\mathrm{Dini}}$ *boundary* if there exists a constant $R_0 \in (0,1]$ and a nondecreasing function $\omega_0 : [0,1] \to [0,1]$ satisfying the Dini condition

$$\int_0^1 \omega_0(r)\,\frac{dr}{r} < \infty,$$

such that the following holds: for any $x_0 = (x_{01}, x_0') \in \partial\Omega$, there exists a $C^{1,\mathrm{Dini}}$ function (i.e., C^1 function whose first derivatives are uniformly Dini continuous) $\chi : \mathbb{R}^{n-1} \to \mathbb{R}$ and a coordinate system depending on x_0 such that

$$\sup_{|x_1'-x_2'|\le r} |\nabla_{x'}\chi(x_1') - \nabla_{x'}\chi(x_2')| \le \omega_0(r), \quad \forall r \in (0,R_0),$$

and in the new coordinate system, we have

$$|\nabla_{x'}\chi(x_0')| = 0, \quad \Omega_{R_0}(x_0) = \{x \in B_{R_0}(x_0) : x_1 > \chi(x')\}.$$

Our global pointwise gradient estimate and Lipschitz estimate are stated as follows.

Theorem 1.10 (Boundary pointwise gradient estimate). *Let $p \in (\frac{3n-2}{2n-1}, 2)$ and suppose that $u \in W^{1,p}_0(\Omega)$ is a solution to (1.6) with Dirichlet boundary data $u = 0$ on $\partial\Omega$. Assuming that (1.5), (1.7), and (1.8) are satisfied and Ω has a $C^{1,\mathrm{Dini}}$ boundary characterized* *by R_0 and ω_0 as in Definition 1.9, there exists a constant $C = C(n, p, \lambda, \omega, R_0, \omega_0)$ such that the estimate*

$$|\nabla u(x)| \le C[\mathbf{I}_1^R(|\mu|)(x)]^{\frac{1}{p-1}} + C\left(\fint_{\Omega_R(x)} (|\nabla u(y)| + s)^{2-p}\, dy\right)^{\frac{1}{2-p}} \tag{1.18}$$

holds for any Lebesgue point $x \in \Omega$ of the vector-valued function ∇u and for all $R \in (0, 1]$. Moreover, if $u \in C^1(\bar{\Omega})$, then (1.18) holds for any $x \in \bar{\Omega}$.

Theorem 1.11 (Boundary Lipschitz estimate). *Let $p \in (1, 2)$ and suppose that $u \in W_0^{1,p}(\Omega)$ is a solution to (1.6) with Dirichlet boundary data $u = 0$ on $\partial\Omega$. Assuming that (1.5), (1.7), and (1.8) are satisfied and Ω has a $C^{1,\mathrm{Dini}}$ boundary characterized by R_0 and ω_0 as in Definition 1.9, there exists a constant $C = C(n, p, \lambda, \omega, R_0, \omega_0)$ such that the estimate*

$$\|\nabla u\|_{L^\infty(\Omega_{R/2}(x))} \le C\|\mathbf{I}_1^R(|\mu|)\|_{L^\infty(\Omega_R(x))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\big\||\nabla u| + s\big\|_{L^{2-p}(\Omega_R(x))} \tag{1.19}$$

holds for any $x \in \bar{\Omega}$ and $R \in (0, 1]$.

As a corollary, we also obtain the global Lipschitz estimate when Ω is bounded.

Corollary 1.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Under the assumptions of Theorem 1.11, there exists a constant $C = C(n, p, \lambda, \omega, R_0, \omega_0, \mathrm{diam}(\Omega))$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\|\mathbf{I}_1^1(|\mu|)\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} + Cs.$$

A global modulus of continuity estimate is established in Theorem 5.10 under the assumptions of Theorem 1.11. One may also deduce corresponding up-to-boundary gradient continuity results from Theorem 5.10 similar to Theorem 1.4 and Corollaries 1.5–1.7 from Theorem 4.3.

Let us give a brief description of the proofs. We first apply an iteration argument to get an L^{γ_0}-mean oscillation estimate of the gradients of solutions to the homogeneous equation with x-independent nonlinearities

$$-\mathrm{div}(A_0(\nabla v)) = 0$$

in Section 2, where $\gamma_0 \in (0, 1)$. Our proofs of the interior gradient estimates are then based on a comparison estimate between the original solution u of (1.1) and the solution to the homogeneous equation $-\mathrm{div}(A(x, \nabla w)) = 0$ in a ball B_R with the boundary condition $u = w$ on ∂B_R. The outcome is the inequality

$$\left(\fint_{B_R} |\nabla u - \nabla w|^{\gamma_0}\, dx\right)^{1/\gamma_0} \le C\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{p-1}} + C\frac{|\mu|(B_R)}{R^{n-1}}\fint_{B_R} (|\nabla u| + s)^{2-p}\, dx, \tag{1.20}$$

which holds for some constant $\gamma_0 \in (0,1)$. The details can be found in Lemma 3.2. For the case when $p \in (\frac{3n-2}{2n-1}, 2)$, we can choose $\gamma_0 = 2-p$, the same integral exponent as on the right-hand side. We then borrow an idea in [6] by estimating the L^{γ_0}-mean oscillation to adapt the iteration scheme used, for instance, in [9]. However, for the case when $p \in (1, \frac{3n-2}{2n-1}]$, we are only able to prove the comparison estimate (1.20) for some $\gamma_0 < 2-p$ and that is the reason why we only obtain Lipschitz estimates instead of pointwise gradient estimates in this case.

For the gradient estimates up to the boundary, we use the technique of flattening the boundary and generalize the interior oscillation estimates to half-balls. We adapt an idea in [2] to establish the global L^{γ_0}-mean oscillation estimates by a delicate combination of the interior estimates and the estimates near a flat boundary. To this end, we also apply an odd extension argument to derive an L^{γ_0}-mean oscillation estimate on half-balls for homogeneous equations with x-independent nonlinearities. This argument only works for equations in diagonalized form, such as the p-Laplace equation, so the global estimates for general equations remain open. As a partial result in this direction, we refer the reader to [22] for a weighted pointwise boundary estimate under the condition that $\partial\Omega$ is sufficiently flat in the sense of Reifenberg. We also refer the reader to [1, 16] for boundary regularity results for quasilinear equations with sufficiently regular right-hand side.

The rest of the paper is organized as follows. In the next section, we derive an L^{γ_0}-mean oscillation estimate of solutions to the homogeneous equation with x-independent nonlinearities. In Section 3, we give the proof of Theorem 1.1. Section 4 is devoted to the Lipschitz estimate and the interior modulus of continuity estimate of the gradient of solutions as well as some corollaries. Finally, in Section 5 we consider the corresponding boundary estimates.

2. An oscillation estimate

This section is devoted to the proof of the following interior oscillation estimate for solutions to the homogeneous equation

$$-\operatorname{div}(A_0(\nabla v)) = 0 \quad \text{in } \Omega, \tag{2.1}$$

where $A_0 = A_0(\xi)$ is a vector field independent of x satisfying conditions (1.2) and (1.3) for some $s \geq 0$, $\lambda \geq 1$, and $p > 1$. In this section, we denote the integral average over $B_R(x)$ by $(\cdot)_{B_R(x)}$.

Theorem 2.1. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to (2.1) and $\gamma_0 \in (0,1)$. Then there exist constants $\alpha \in (0,1)$ depending on n, p, and λ, and $C > 1$ depending on n, p, λ, and γ_0, such that for every $B_R(x_0) \subset \Omega$ and $\rho \in (0,R)$, we have*

$$\inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_\rho(x_0)} |\nabla v - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0} \leq C \left(\frac{\rho}{R} \right)^{\alpha} \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_R(x_0)} |\nabla v - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}. \tag{2.2}$$

To prove the above theorem, we first recall a classical oscillation estimate. Estimates of this type, with different exponents involved, were developed in [4, 9, 17].

Lemma 2.2. *Let $v \in W^{1,p}_{\rm loc}(\Omega)$ be a solution to (2.1). There exist constants $C > 1$ and $\alpha \in (0, 1)$, depending only on n, p, and λ, such that $v \in C^{1,\alpha}_{\rm loc}(\Omega)$ and for every $B_R(x_0) \subset \Omega$ and $r \in (0, R)$, we have*

$$\fint_{B_r(x_0)} |\nabla v - (\nabla v)_{B_r(x_0)}|^p \, dx \le C \left(\frac{r}{R}\right)^{\alpha p} \fint_{B_R(x_0)} |\nabla v - (\nabla v)_{B_R(x_0)}|^p \, dx.$$

The lemma above directly implies the following corollary.

Corollary 2.3. *Under the conditions of Lemma 2.2, there exist constants $C > 1$ and $\alpha \in (0, 1)$, depending only on n, p, and λ, such that for every $B_R(x_0) \subset \Omega$, we have*

$$R^\alpha [\nabla v]_{C^\alpha(B_{R/2}(x_0))} \le C \left(\fint_{B_R(x_0)} |\nabla v - (\nabla v)_{B_R(x_0)}|^p \right)^{1/p}, \tag{2.3}$$

and for any $r \in [R/2, R)$,

$$[\nabla v]_{C^\alpha(B_r(x_0))} \le C \frac{R^{n/p+1-\alpha}}{(R-r)^{n/p+1}} \left(\fint_{B_R(x_0)} |\nabla v - (\nabla v)_{B_R(x_0)}|^p \right)^{1/p}. \tag{2.4}$$

Proof. Without loss of generality, we assume $x_0 = 0$. For any $x \in B_{R/2}$ and $r \le R/2$, by Lemma 2.2 we have

$$\begin{aligned} \fint_{B_r(x)} |\nabla v - (\nabla v)_{B_r(x)}|^p &\le C \left(\frac{r}{R}\right)^{\alpha p} \fint_{B_{R/2}(x)} |\nabla v - (\nabla v)_{B_{R/2}(x)}|^p \\ &\le C \left(\frac{r}{R}\right)^{\alpha p} \fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^p. \end{aligned}$$

By Campanato's characterization of Hölder continuous functions, we obtain (2.3).

Now for any $r > R/2$ and $z \in B_r$, using (2.3) and the triangle inequality, we have

$$\begin{aligned} [\nabla v]_{C^\alpha(B_{(R-r)/2}(z))} &\le C(R-r)^{-\alpha} \left(\fint_{B_{R-r}(z)} |\nabla v - (\nabla v)_{B_{R-r}(z)}|^p \right)^{1/p} \\ &\le C \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^p \right)^{1/p}. \end{aligned} \tag{2.5}$$

Thus for any $x, y \in B_r$, let

$$N = \min \left\{ m \in \mathbb{Z} : m > \frac{2|x-y|}{R-r} \right\}.$$

We can divide the line segment connecting x and y into N equal segments using $x_1, \ldots, x_{N-1}$, $x_0 = x$, and $x_N = y$, so that

$$|x_k - x_{k+1}| = \frac{|x-y|}{N} < \frac{R-r}{2}.$$

Then by the triangle inequality and (2.5), we have

$$\begin{aligned}
|\nabla v(x)-\nabla v(y)| &\le \sum_{k=0}^{N-1} |\nabla v(x_k)-\nabla v(x_{k+1})| \\
&\le C\sum_{k=0}^{N-1} \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\int_{B_R} |\nabla v-(\nabla v)_{B_R}|^p\right)^{1/p} \left|\frac{x-y}{N}\right|^{\alpha} \\
&\le CN^{1-\alpha} \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\int_{B_R} |\nabla v-(\nabla v)_{B_R}|^p\right)^{1/p} |x-y|^{\alpha} \\
&\le C\left(\frac{R}{R-r}\right)^{1-\alpha} \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\int_{B_R} |\nabla v-(\nabla v)_{B_R}|^p\right)^{1/p} |x-y|^{\alpha},
\end{aligned}$$

which directly implies (2.4). ∎

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. As before, without loss of generality, we assume $x_0 = 0$. Clearly for any $B_\rho = B_\rho(0) \subset \Omega$, there exists $\mathbf{q}_\rho = (q_\rho^{(1)}, \dots, q_\rho^{(n)}) \in \mathbb{R}^n$ such that

$$\left(\fint_{B_\rho} |\nabla v - \mathbf{q}_\rho|^{\gamma_0}\right)^{1/\gamma_0} = \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_\rho} |\nabla v - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}.$$

Also, it is easily seen that

$$q_\rho^{(i)} \in \mathrm{Range}(D_i v|_{B_\rho}). \tag{2.6}$$

We claim that there exists a constant C, depending only on n, p, λ, and γ_0, such that

$$\|\nabla v - \mathbf{q}_{\rho/2}\|_{L^\infty(B_{\rho/2})} \le C\left(\fint_{B_\rho} |\nabla v - \mathbf{q}_\rho|^{\gamma_0}\right)^{1/\gamma_0} \tag{2.7}$$

for any $B_\rho(x_0) \subset \Omega$.

We prove the claim by using Corollary 2.3 and iteration.

For any $R/2 < r < R \le \mathrm{dist}(x_0, \partial\Omega)$, using (2.4) and the triangle inequality, we get

$$\begin{aligned}
\|\nabla v - \mathbf{q}_r\|_{L^\infty(B_r)} &\le r^\alpha [\nabla v]_{C^\alpha(B_r)} \\
&\le C\frac{R^{n/p+1}}{(R-r)^{n/p+1}} \left(\fint_{B_R} |\nabla v-(\nabla v)_{B_R}|^p\right)^{1/p} \\
&\le C\frac{R^{n/p+1}}{(R-r)^{n/p+1}} \left(\fint_{B_R} |\nabla v-\mathbf{q}_R|^p\right)^{1/p} \\
&\le C\frac{R^{n/p+1}}{(R-r)^{n/p+1}} \|\nabla v-\mathbf{q}_R\|_{L^\infty(B_R)}^{\frac{p-\gamma_0}{p}} \left(\fint_{B_R} |\nabla v-\mathbf{q}_R|^{\gamma_0}\right)^{1/p} \\
&\le \varepsilon\|\nabla v-\mathbf{q}_R\|_{L^\infty(B_R)} + C_\varepsilon\left(\frac{R^{n/p+1}}{(R-r)^{n/p+1}}\right)^{p/\gamma_0} \left(\fint_{B_R} |\nabla v-\mathbf{q}_R|^{\gamma_0}\right)^{1/\gamma_0},
\end{aligned}$$

where we have used Young's inequality with exponents p/γ_0 and $p/(p-\gamma_0)$ in the last line.

Now taking $r_k = (1-2^{-k})\rho$, $r = r_k$, and $R = r_{k+1}$, we have

$$\begin{aligned}
&\|\nabla v - \mathbf{q}_{r_k}\|_{L^\infty(B_{r_k})} \\
&\quad \le \varepsilon \|\nabla v - \mathbf{q}_{r_{k+1}}\|_{L^\infty(B_{r_{k+1}})} + C_\varepsilon 2^{k\beta} \left(\fint_{B_{r_{k+1}}} |\nabla v - \mathbf{q}_{r_{k+1}}|^{\gamma_0} \right)^{1/\gamma_0} \\
&\quad \le \varepsilon \|\nabla v - \mathbf{q}_{r_{k+1}}\|_{L^\infty(B_{r_{k+1}})} + C_\varepsilon 2^{k\beta + n/\gamma_0} \left(\fint_{B_\rho} |\nabla v - \mathbf{q}_\rho|^{\gamma_0} \right)^{1/\gamma_0},
\end{aligned}$$

where $\beta = (n+p)/\gamma_0$. Taking $\varepsilon = 3^{-\beta}$, multiplying both sides by ε^k and summing in k, we get

$$\begin{aligned}
\sum_{k=1}^{\infty} \varepsilon^k \|\nabla v - \mathbf{q}_{r_k}\|_{L^\infty(B_{r_k})} &\le \sum_{k=1}^{\infty} \varepsilon^{k+1} \|\nabla v - \mathbf{q}_{r_{k+1}}\|_{L^\infty(B_{r_{k+1}})} \\
&\quad + C \left(\fint_{B_\rho} |\nabla v - \mathbf{q}_\rho|^{\gamma_0} \right)^{1/\gamma_0},
\end{aligned}$$

where the summations are finite and $C = C(n, p, \lambda, \gamma_0)$. By subtracting

$$\sum_{k=2}^{\infty} \varepsilon^k \|\nabla v - \mathbf{q}_{r_k}\|_{L^\infty(B_{r_k})}$$

from both sides of the above inequality, we obtain (2.7). The claim is proved.

Now we are ready to prove (2.2). If $r \le R/4$, by (2.3), (2.6), and (2.7) we get

$$\begin{aligned}
&\left(\fint_{B_r} |\nabla v - \mathbf{q}_r|^{\gamma_0} \right)^{1/\gamma_0} \le C r^\alpha [\nabla v]_{C^\alpha(B_{R/4})} \\
&\quad \le C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_{R/2}} |\nabla v - (\nabla v)_{B_{R/2}}|^p \right)^{1/p} \le C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_{R/2}} |\nabla v - \mathbf{q}_{R/2}|^p \right)^{1/p} \\
&\quad \le C \left(\frac{r}{R} \right)^\alpha \|\nabla v - \mathbf{q}_{R/2}\|_{L^\infty(B_{R/2})} \le C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_R} |\nabla v - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0}.
\end{aligned}$$

If $r > R/4$, we have

$$\begin{aligned}
\left(\fint_{B_r} |\nabla v - \mathbf{q}_r|^{\gamma_0} \right)^{1/\gamma_0} &\le \left(\fint_{B_r} |\nabla v - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0} \\
&\le C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_R} |\nabla v - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0}.
\end{aligned}$$

The theorem is proved. ■

3. Interior pointwise gradient estimates

In order to prove the interior pointwise gradient estimates, we follow the outline of arguments given in [22] while replacing their oscillation estimates with our new oscillation estimate in Section 2. We also borrow an idea from [6] by estimating the L^{γ_0}-mean oscillations of solutions, where $\gamma_0 \in (0,1)$.

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution to (1.1) and $B_{2r}(x_0) \subset\subset \Omega$. We consider the unique solution $w \in u + W^{1,p}_0(B_{2r}(x_0))$ to the equation

$$\begin{cases} -\operatorname{div}(A(x,\nabla w)) = 0 & \text{in } B_{2r}(x_0), \\ w = u & \text{on } \partial B_{2r}(x_0). \end{cases} \tag{3.1}$$

We first recall an interior reverse Hölder inequality [11, Theorem 6.7]. See also [22, Lemma 3.1].

Lemma 3.1. *Let w be a solution to (3.1). There exists a constant $\theta_1 > p$ depending only on n, p, and λ such that for any $t > 0$, the estimate*

$$\left(\fint_{B_{\rho/2}(y)} (|\nabla w| + s)^{\theta_1} \, dx \right)^{1/\theta_1} \le C \left(\fint_{B_\rho(y)} (|\nabla w| + s)^t \, dx \right)^{1/t} \tag{3.2}$$

holds for all $B_\rho(y) \subset B_{2r}(x_0)$, where $C = C(n,p,\lambda,t) > 0$.

We also have the following comparison result, which generalizes and refines similar results in [9, 21, 23].

Lemma 3.2. *Let w be a solution to (3.1) and assume that $p \in (1,2)$. Then for any $\gamma_0 \in (0, 2-p]$ when $p \in (\frac{3n-2}{2n-1}, 2)$, or $\gamma_0 \in (0, \frac{(p-1)n}{n-1})$ when $p \in (1, \frac{3n-2}{2n-1}]$, one has*

$$\begin{aligned} &\left(\fint_{B_{2r}(x_0)} |\nabla u - \nabla w|^{\gamma_0} \, dx \right)^{1/\gamma_0} \\ &\qquad \le C \left[\frac{|\mu|(B_{2r}(x_0))}{r^{n-1}} \right]^{\frac{1}{p-1}} + C \frac{|\mu|(B_{2r}(x_0))}{r^{n-1}} \fint_{B_{2r}(x_0)} (|\nabla u| + s)^{2-p} \, dx, \end{aligned}$$

where C is a constant depending only on n, p, λ, and γ_0.

Proof. The case when $p \in (1, \frac{3n-2}{2n-1}]$ and $s = 0$ was proved in [23, Lemma 2.1] and their proof also works for $p \in (1, \frac{3n-2}{2n-1}]$ and $s > 0$. Therefore, we focus on the case when $p \in (\frac{3n-2}{2n-1}, 2)$ and $s \ge 0$. By scaling invariance (see [9, Remark 4.1] for example), we may assume that $B_{2r}(x_0) = B_2$ and $|\mu|(B_2) = 1$. For $k > 0$, using

$$\varphi_1 = T_{2k}(u-w) := \max\left\{ \min\{u-w, 2k\}, -2k \right\}$$

as a test function in (1.1) and (3.1) and recalling (1.3), we have

$$\int_{B_2 \cap \{|u-w|<2k\}} g^s(u,w) \le Ck \quad \text{with} \quad g^s(u,w) = \frac{|\nabla(u-w)|^2}{(|\nabla w| + |\nabla u| + s)^{2-p}}. \tag{3.3}$$

By the triangle inequality, we have

$$\begin{aligned}|\nabla(u-w)| &= g^s(u,w)^{1/2}(|\nabla w|+|\nabla u|+s)^{\frac{2-p}{2}}\\ &\le g^s(u,w)^{1/2}(|\nabla(u-w)|+2|\nabla u|+s)^{\frac{2-p}{2}}\\ &\le Cg^s(u,w)^{1/2}|\nabla(u-w)|^{\frac{2-p}{2}}+Cg^s(u,w)^{1/2}(|\nabla u|+s)^{\frac{2-p}{2}}.\end{aligned}$$

Using Young's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we obtain

$$|\nabla(u-w)| \le Cg^s(u,w)^{1/p}+Cg^s(u,w)^{1/2}(|\nabla u|+s)^{\frac{2-p}{2}}. \tag{3.4}$$

Now we set

$$E_k = B_2\cap\{k<|u-w|<2k\} \quad \text{and} \quad F_k = B_2\cap\{|u-w|>k\}.$$

Using the Sobolev inequality, Hölder's inequality, and (3.4), we obtain

$$\begin{aligned}&k|\{x:|u(x)-w(x)|>2k\}\cap B_2|^{\frac{n-1}{n}}\\ &\le C\left(\int_{B_2}|T_{2k}(u-w)-T_k(u-w)|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\\ &\le C\int_{E_k}|\nabla(u-w)|\\ &\le C\int_{E_k}\left(g^s(u,w)^{1/p}+g^s(u,w)^{1/2}(|\nabla u|+s)^{\frac{2-p}{2}}\right)\\ &\le C|E_k|^{\frac{p-1}{p}}\left(\int_{E_k}g^s(u,w)\right)^{1/p}+C\left(\int_{E_k}g^s(u,w)\right)^{1/2}\left(\int_{E_k}(|\nabla u|+s)^{2-p}\right)^{1/2}.\end{aligned} \tag{3.5}$$

From (3.3) and (3.5), we get

$$k^{1/2}|F_{2k}|^{\frac{n-1}{n}} \le Ck^{-1/2+1/p}|F_k|^{\frac{p-1}{p}}+CQ_1^{\frac{2-p}{2}},$$

where $Q_1 := \big\||\nabla u|+s\big\|_{L^{2-p}(B_2)}$. Therefore, by taking the sup over $k\in(0,\infty)$, we obtain

$$\|u-w\|^{1/2}_{L^{\frac{n}{2(n-1)},\infty}(B_2)} \le C\|u-w\|^{1/p-1/2}_{L^{\frac{2-p}{2(p-1)},\infty}(B_2)}+CQ_1^{\frac{2-p}{2}}.$$

Since $\frac{3n-2}{2n-1}<p<2$, we have

$$\frac{n}{2(n-1)}>\frac{2-p}{2(p-1)},$$

which implies

$$\|u-w\|^{1/2}_{L^{\frac{n}{2(n-1)},\infty}(B_2)} \le C\|u-w\|^{1/p-1/2}_{L^{\frac{n}{2(n-1)},\infty}(B_2)}+CQ_1^{\frac{2-p}{2}}.$$

Thus, by Young's inequality, we obtain

$$\|u-w\|_{L^{\frac{n}{2(n-1)},\infty}(B_2)} \le C + CQ_1^{\frac{2-p}{2}}. \tag{3.6}$$

Let $k, l > 0$ and $q = \frac{n}{2(n-1)}$. By the Chebyshev inequality and (3.3), we have

$$\begin{aligned} &|\{x : g^s(u,w) > l\} \cap B_2| \\ &\quad \le |\{x : |u-w| > k\} \cap B_2| + |\{x : |u-w| \le k, g^s(u,w) > l\} \cap B_2| \\ &\quad \le Ck^{-q}\|u-w\|^q_{L^{q,\infty}(B_2)} + \frac{1}{l}\int_{B_2\cap\{x:|u-w|\le k\}} g^s(u,w)\,dx \\ &\quad \le Ck^{-q}\|u-w\|^q_{L^{q,\infty}(B_2)} + Ck/l. \end{aligned}$$

By choosing

$$k = [l\|u-w\|^q_{L^{q,\infty}(B_2)}]^{\frac{1}{1+q}},$$

we get

$$l^{\frac{q}{1+q}}|\{x : g^s(u,w) > l\} \cap B_2| \le C\|u-w\|^{\frac{q}{1+q}}_{L^{q,\infty}(B_2)}.$$

Therefore, by taking the sup over $l \in (0,\infty)$, we obtain

$$\|g^s(u,w)\|_{L^{\frac{q}{1+q},\infty}(B_2)} \le C\|u-w\|_{L^{q,\infty}(B_2)}. \tag{3.7}$$

Let $\gamma_0 \in (0, 2-p]$. By (3.4) and Hölder's inequality with exponents $2/p$ and $2/(2-p)$, we get

$$\begin{aligned} \int_{B_2} |\nabla(u-w)|^{\gamma_0} &\le C\int_{B_2} \left(g^s(u,w)^{\gamma_0/p} + g^s(u,w)^{\gamma_0/2}(|\nabla u|+s)^{\gamma_0(2-p)/2}\right) \\ &\le C\left(\int_{B_2} g^s(u,w)^{\gamma_0/p}\right) + C\left(\int_{B_2} g^s(u,w)^{\gamma_0/p}\right)^{p/2}\left(\int_{B_2}(|\nabla u|+s)^{\gamma_0}\right)^{\frac{2-p}{2}} \\ &\le C\|g^s(u,w)\|^{\gamma_0/p}_{L^{\frac{q}{1+q},\infty}(B_2)} + C\|g^s(u,w)\|^{\gamma_0/2}_{L^{\frac{q}{1+q},\infty}(B_2)} Q_1^{\gamma_0(2-p)/2}. \end{aligned} \tag{3.8}$$

In the last inequality, we have used the fact that

$$\frac{\gamma_0}{p} < \frac{q}{1+q}, \quad \gamma_0 \le 2-p.$$

Combining (3.6)–(3.8), we have

$$\int_{B_2} |\nabla(u-w)|^{\gamma_0} \le C + CQ_1^{\gamma_0(2-p)},$$

which implies the desired result. ∎

We now let $v \in w + W_0^{1,p}(B_r(x_0))$ be the unique solution to

$$\begin{cases} -\mathrm{div}(A(x_0,\nabla v)) = 0 & \text{in } B_r(x_0), \\ v = w & \text{on } \partial B_r(x_0). \end{cases} \tag{3.9}$$

By testing (3.1) and (3.9) with $v - w$, we obtain an estimate for the difference $\nabla v - \nabla w$:

$$\fint_{B_r(x_0)} |\nabla v - \nabla w|^p \, dx \le C\omega(r)^p \fint_{B_r(x_0)} (|\nabla w| + s)^p \, dx. \tag{3.10}$$

A detailed proof of this result can be found in [9, (4.35)]. Thus by (3.2) and Hölder's inequality, we get

$$\fint_{B_r(x_0)} |\nabla v - \nabla w|^{\gamma_0} \, dx \le C\omega(r)^{\gamma_0} \fint_{B_{2r}(x_0)} (|\nabla w| + s)^{\gamma_0} \, dx. \tag{3.11}$$

For a ball $B_\rho(x) \subset\subset \Omega$ and a function $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$, there exists $\mathbf{q}_{x,\rho}(f) \in \mathbb{R}^n$ such that

$$\left(\fint_{B_\rho(x)} |\nabla f - \mathbf{q}_{x,\rho}(f)|^{\gamma_0}\right)^{1/\gamma_0} = \inf_{\mathbf{q} \in \mathbb{R}^n} \left(\fint_{B_\rho(x)} |\nabla f - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}.$$

We denote $\mathbf{q}_{x,\rho} = \mathbf{q}_{x,\rho}(u)$ and

$$\phi(x,\rho) = \inf_{\mathbf{q} \in \mathbb{R}^n} \left(\fint_{B_\rho(x)} |\nabla u - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}.$$

Since

$$|\mathbf{q}_{x,\rho} - \nabla u(x)|^{\gamma_0} \le |\mathbf{q}_{x,\rho} - \nabla u(z)|^{\gamma_0} + |\nabla u(z) - \nabla u(x)|^{\gamma_0},$$

by taking the average over $z \in B_\rho(x)$ and then taking the γ_0-th root, we obtain

$$|\mathbf{q}_{x,\rho} - \nabla u(x)| \le C\phi(x,\rho) + C\left(\fint_{B_\rho(x)} |\nabla u(z) - \nabla u(x)|^{\gamma_0} \, dz\right)^{1/\gamma_0}.$$

Therefore, from the definition of ϕ and the fact that $0 < \gamma_0 < 1$, we obtain

$$\lim_{\rho \to 0} \mathbf{q}_{x,\rho} = \nabla u(x) \tag{3.12}$$

for any Lebesgue point $x \in \Omega$ of the vector-valued function ∇u.

Proposition 3.3. *Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to (1.1). Then for any $\varepsilon \in (0,1)$ and $B_{2r}(x_0) \subset\subset \Omega$, we have*

$$\begin{aligned} \phi(x_0, \varepsilon r) \le C\varepsilon^\alpha \phi(x_0, r) &+ C_\varepsilon \left(\frac{|\mu|(B_{2r}(x_0))}{r^{n-1}}\right)^{\frac{1}{p-1}} \\ &+ C_\varepsilon \frac{|\mu|(B_{2r}(x_0))}{r^{n-1}} \fint_{B_{2r}(x_0)} (|\nabla u| + s)^{2-p} \\ &+ C_\varepsilon \omega(r) \left(\fint_{B_{2r}(x_0)} (|\nabla u| + s)^{2-p}\right)^{\frac{1}{2-p}}, \end{aligned} \tag{3.13}$$

where $\alpha \in (0,1)$ is the constant in Theorem 2.1, γ_0 is the same constant as in Lemma 3.2, C_ε is a constant depending on ε, n, p, λ, and γ_0, and C is a constant depending on n, p, λ, and γ_0.

Proof. By Theorem 2.1 and the definition of $\mathbf{q}_{x,\rho}(\cdot)$, we have

$$
\begin{aligned}
\left(\fint_{B_{\varepsilon r}(x_0)} |\nabla u - \mathbf{q}_{x_0,\varepsilon r}(u)|^{\gamma_0}\right)^{1/\gamma_0} &\le \left(\fint_{B_{\varepsilon r}(x_0)} |\nabla u - \mathbf{q}_{x_0,\varepsilon r}(v)|^{\gamma_0}\right)^{1/\gamma_0} \\
&\le C\left(\fint_{B_{\varepsilon r}(x_0)} |\nabla v - \mathbf{q}_{x_0,\varepsilon r}(v)|^{\gamma_0}\right)^{1/\gamma_0} + C\left(\fint_{B_{\varepsilon r}(x_0)} |\nabla u - \nabla v|^{\gamma_0}\right)^{1/\gamma_0} \\
&\le C\varepsilon^{\alpha}\left(\fint_{B_r(x_0)} |\nabla v - \mathbf{q}_{x_0,r}(v)|^{\gamma_0}\right)^{1/\gamma_0} + C\varepsilon^{-n/\gamma_0}\left(\fint_{B_r(x_0)} |\nabla u - \nabla v|^{\gamma_0}\right)^{1/\gamma_0} \\
&\le C\varepsilon^{\alpha}\left(\fint_{B_r(x_0)} |\nabla v - \mathbf{q}_{x_0,r}(u)|^{\gamma_0}\right)^{1/\gamma_0} + C\varepsilon^{-n/\gamma_0}\left(\fint_{B_r(x_0)} |\nabla u - \nabla v|^{\gamma_0}\right)^{1/\gamma_0} \\
&\le C\varepsilon^{\alpha}\left(\fint_{B_r(x_0)} |\nabla u - \mathbf{q}_{x_0,r}(u)|^{\gamma_0}\right)^{1/\gamma_0} + C\varepsilon^{-n/\gamma_0}\left(\fint_{B_r(x_0)} |\nabla u - \nabla v|^{\gamma_0}\right)^{1/\gamma_0}.
\end{aligned}
\tag{3.14}
$$

Moreover, by (3.11) and the fact that $|\omega(r)| \le 1$, one has

$$
\begin{aligned}
\fint_{B_r(x_0)} |\nabla u - \nabla v|^{\gamma_0} &\le \fint_{B_r(x_0)} |\nabla u - \nabla w|^{\gamma_0} + \fint_{B_r(x_0)} |\nabla w - \nabla v|^{\gamma_0} \\
&\le C\fint_{B_{2r}(x_0)} |\nabla u - \nabla w|^{\gamma_0} + C\omega(r)^{\gamma_0}\fint_{B_{2r}(x_0)} (|\nabla w| + s)^{\gamma_0} \\
&\le C\fint_{B_{2r}(x_0)} |\nabla u - \nabla w|^{\gamma_0} + C\omega(r)^{\gamma_0}\fint_{B_{2r}(x_0)} (|\nabla u| + s)^{\gamma_0}.
\end{aligned}
\tag{3.15}
$$

Thus from (3.14) and (3.15), we have

$$
\begin{aligned}
\phi(x_0, \varepsilon r) \le C\varepsilon^{\alpha}\phi(x_0, r) &+ C_{\varepsilon}\left(\fint_{B_{2r}(x_0)} |\nabla u - \nabla w|^{\gamma_0}\right)^{1/\gamma_0} \\
&+ C_{\varepsilon}\omega(r)\left(\fint_{B_{2r}(x_0)} (|\nabla u| + s)^{\gamma_0}\right)^{1/\gamma_0}.
\end{aligned}
\tag{3.16}
$$

Now we can apply Lemma 3.2 to bound the second term on the right-hand side of (3.16) to conclude the proof. ∎

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We prove the theorem at a Lebesgue point $x = x_0$ of the vector-valued function ∇u, assuming that $B_R(x_0) \subset \Omega$. Since $p \in (\frac{3n-2}{2n-1}, 2)$, we choose $\gamma_0 = 2 - p$ in Lemma 3.2. Choose $\varepsilon = \varepsilon(n, p, \lambda, \alpha) \in (0, 1/4)$ so small that $C\varepsilon^{\alpha} \le 1/4$, where C is the constant in (3.13).

For an integer $j \ge 0$, set $r_j = \varepsilon^j R$, $B^j = B_{2r_j}(x_0)$, and

$$
T_j = \left(\fint_{B^j} (|\nabla u| + s)^{2-p}\, dx\right)^{\frac{1}{2-p}}, \quad \phi_j = \phi(x_0, r_j), \quad \mathbf{q}_j = \mathbf{q}_{x_0, r_j}.
$$

Applying (3.13) yields

$$\phi_{j+1} \leq \frac{1}{4}\phi_j + C\left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} + C\frac{|\mu|(B^j)}{r_j^{n-1}}T_j^{2-p} + C\omega(r_j)T_j.$$

Let j_0 and m be positive integers to be specified later such that $j_0 \leq m$. Summing the above inequality over $j = j_0, j_0+1, \dots, m$, we obtain

$$\begin{aligned}\sum_{j=j_0}^{m+1}\phi_j &\leq C\phi_{j_0} + C\sum_{j=j_0}^{m}\left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} \\ &\quad + C\sum_{j=j_0}^{m}\frac{|\mu|(B^j)}{r_j^{n-1}}T_j^{2-p} + C\sum_{j=j_0}^{m}\omega(r_j)T_j. \end{aligned} \tag{3.17}$$

Since

$$|\mathbf{q}_{j+1} - \mathbf{q}_j|^{\gamma_0} \leq |\mathbf{q}_{j+1} - \nabla u(x)|^{\gamma_0} + |\nabla u(x) - \mathbf{q}_j|^{\gamma_0},$$

by taking the average over $x \in B_{r_{j+1}}(x_0)$ and then taking the γ_0-th root we obtain

$$|\mathbf{q}_{j+1} - \mathbf{q}_j| \leq C\phi_j + C\phi_{j+1}.$$

Then, by iterating, we get

$$|\mathbf{q}_{m+1} - \mathbf{q}_{j_0}| \leq C\sum_{j=j_0}^{m+1}\phi_j,$$

which together with (3.17) implies

$$\begin{aligned}|\mathbf{q}_{m+1}| + \sum_{j=j_0}^{m+1}\phi_j &\leq C\phi_{j_0} + |\mathbf{q}_{j_0}| + C\sum_{j=j_0}^{m}\left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} \\ &\quad + C\sum_{j=j_0}^{m}\frac{|\mu|(B^j)}{r_j^{n-1}}T_j^{2-p} + C\sum_{j=j_0}^{m}\omega(r_j)T_j. \end{aligned} \tag{3.18}$$

By the definition of ϕ_{j_0}, we have

$$\phi_{j_0} \leq C\left(\fint_{B^{j_0}}|\nabla u|^{\gamma_0}\,dx\right)^{1/\gamma_0} \leq CT_{j_0}.$$

Since

$$|\mathbf{q}_{j_0}|^{\gamma_0} \leq |\nabla u(x) - \mathbf{q}_{j_0}|^{\gamma_0} + |\nabla u(x)|^{\gamma_0},$$

by taking the average over $x \in B_{r_{j_0}}(x_0)$ and taking the γ_0-th root we obtain

$$|\mathbf{q}_{j_0}| \leq C\phi_{j_0} + C\left(\fint_{B^{j_0}}|\nabla u|^{\gamma_0}\,dx\right)^{1/\gamma_0} \leq CT_{j_0}.$$

Therefore, (3.18) implies that

$$|\mathbf{q}_{m+1}| + \sum_{j=j_0}^{m+1} \phi_j \le C T_{j_0} + C \sum_{j=j_0}^{m} \left(\frac{|\mu|(B^j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C \sum_{j=j_0}^{m} \frac{|\mu|(B^j)}{r_j^{n-1}} T_j^{2-p} + C \sum_{j=j_0}^{m} \omega(r_j) T_j. \tag{3.19}$$

By (1.5) and the comparison principle for Riemann integrals, there exists $j_0 = j_0(n, p, \varepsilon, C, \omega) > 1$ so large that

$$4^{1/\gamma_0}(2\varepsilon)^{-n/\gamma_0} C \sum_{j=j_0}^{\infty} \omega(r_j) \le \frac{1}{10}, \tag{3.20}$$

where C is the constant in (3.19).

Note that by the comparison principle for Riemann integrals,

$$\sum_{j=j_0}^{m} \frac{|\mu|(B^j)}{r_j^{n-1}} \le C \int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho}, \tag{3.21}$$

and since $p < 2$ we also have

$$\sum_{j=j_0}^{m} \left(\frac{|\mu|(B^j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \le C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}}. \tag{3.22}$$

To prove (1.12) at $x = x_0$, it is sufficient to show that

$$|\nabla u(x_0)| \le C T_{j_0} + C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}}. \tag{3.23}$$

To this end, we consider the following possibilities.

Case 1: If $|\nabla u(x_0)| \le T_{j_0}$, then (3.23) easily follows.

Case 2: If $T_j < |\nabla u(x_0)|$ for all $j_0 \le j \le j_1$, and $|\nabla u(x_0)| \le T_{j_1+1}$, then since $\gamma_0 = 2 - p < 1$, we have

$$\begin{aligned}
|\nabla u(x_0)| &\le \left(\fint_{B^{j_1+1}} (|\nabla u| + s)^{\gamma_0} \, dx \right)^{1/\gamma_0} \\
&\le 2^{1/\gamma_0} \left(\fint_{B^{j_1+1}} |\nabla u|^{\gamma_0} \, dx \right)^{1/\gamma_0} + 2^{1/\gamma_0} s \\
&\le 2^{1/\gamma_0} (2\varepsilon)^{-n/\gamma_0} \left(\fint_{B_{r_{j_1}}(x_0)} |\nabla u|^{\gamma_0} \, dx \right)^{1/\gamma_0} + 2^{1/\gamma_0} s \\
&\le 4^{1/\gamma_0} (2\varepsilon)^{-n/\gamma_0} (\phi_{j_1} + |\mathbf{q}_{j_1}|) + 2^{1/\gamma_0} s,
\end{aligned} \tag{3.24}$$

where the last inequality follows from the definitions of ϕ_{j_1} and $\mathbf{q}_{j_1}$. Now applying (3.19) with $m = j_1 - 1$ and using (3.21) and (3.22), from (3.24) we get

$$
\begin{aligned}
|\nabla u(x_0)| \le C' T_{j_0} &+ C''\Big(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\Big)^{\frac{1}{p-1}}\\
&+ C''\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\cdot|\nabla u(x_0)|^{2-p}\\
&+ C'\sum_{j=j_0}^{m}\omega(r_j)|\nabla u(x_0)| + 2^{1/\gamma_0}s,
\end{aligned}
$$

where $C' = 4^{1/\gamma_0}(2\varepsilon)^{-n/\gamma_0}C$, C is the constant in (3.19), and C'' is a constant depending on n, p, and λ. Hence using (3.20) and Young's inequality, we find

$$
|\nabla u(x_0)| \le C T_{j_0} + C\Big(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\Big)^{\frac{1}{p-1}} + \frac{1}{5}|\nabla u(x_0)| + Cs.
$$

This implies (3.23) as desired.

Case 3: If $T_j < |\nabla u(x_0)|$ for any $j \ge j_0$, then from (3.19), (3.21), and (3.22) we have, for any $m > j_0$,

$$
\begin{aligned}
|\mathbf{q}_{m+1}| &\le C T_{j_0} + C\Big(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\Big)^{\frac{1}{p-1}}\\
&\quad + C\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\cdot|\nabla u(x_0)|^{2-p} + C\sum_{j=j_0}^{m}\omega(r_j)|\nabla u(x_0)|\\
&\le C T_{j_0} + C\Big(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\Big)^{\frac{1}{p-1}}\\
&\quad + C\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\cdot|\nabla u(x_0)|^{2-p} + \frac{1}{10}|\nabla u(x_0)|.
\end{aligned}
$$

Here we have used (3.20) in the last inequality. Letting $m \to \infty$ and using (3.12), we get

$$
\begin{aligned}
|\nabla u(x_0)| \le C T_{j_0} &+ C\Big(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\Big)^{\frac{1}{p-1}}\\
&+ C\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\,\frac{d\rho}{\rho}\cdot|\nabla u(x_0)|^{2-p} + \frac{1}{10}|\nabla u(x_0)|.
\end{aligned}
$$

Then using Young's inequality, we deduce (3.23). ∎

4. Interior Lipschitz estimate and modulus of continuity estimate of the gradient

In this section, we give the proof of the Lipschitz estimate in Theorem 1.3 and derive an interior modulus of continuity estimate of ∇u under the same conditions. We first adapt the argument in [6] to obtain some decay estimates from Proposition 3.3. Let $\alpha \in (0,1)$ be the same constant as in Theorem 2.1, $\alpha_1 \in (0,\alpha)$, $R \in (0,1]$ and $B_R(x_0) \subset \Omega$. Choose $\varepsilon = \varepsilon(n,p,\lambda,\gamma_0,\alpha,\alpha_1) > 0$ sufficiently small such that

$$C\varepsilon^{\alpha-\alpha_1} < 1 \quad \text{and} \quad \varepsilon^{\alpha_1} < 1/4,$$

where C is the constant in (3.13).

Proposition 3.3 implies that for any $B_{2r}(x) \subset\subset B_R(x_0)$,

$$\begin{aligned}\phi(x,\varepsilon r) \le \varepsilon^{\alpha_1}\phi(x,r) &+ C\left(\frac{|\mu|(B_{2r}(x))}{r^{n-1}}\right)^{\frac{1}{p-1}} \\ &+ C\frac{|\mu|(B_{2r}(x))}{r^{n-1}}(\|\nabla u\|_{L^\infty(B_{2r}(x))}+s)^{2-p} \\ &+ C\omega(r)(\|\nabla u\|_{L^\infty(B_{2r}(x))}+s). \end{aligned} \tag{4.1}$$

Denote

$$g(x,r) = \frac{|\mu|(B_r(x))}{r^{n-1}}, \quad h(x,r) = g(x,r)^{\frac{1}{p-1}}. \tag{4.2}$$

By iteration, from (4.1) we get

$$\begin{aligned}\phi(x,\varepsilon^j r) \le \varepsilon^{\alpha_1 j}\phi(x,r) &+ C\sum_{i=1}^{j}\varepsilon^{\alpha_1(i-1)}h(x,2\varepsilon^{j-i}r) \\ &+ C\sum_{i=1}^{j}\varepsilon^{\alpha_1(i-1)}g(x,2\varepsilon^{j-i}r)(\|\nabla u\|_{L^\infty(B_{2r}(x))}+s)^{2-p} \\ &+ C\sum_{i=1}^{j}\varepsilon^{\alpha_1(i-1)}\omega(\varepsilon^{j-i}r)(\|\nabla u\|_{L^\infty(B_{2r}(x))}+s)\end{aligned}$$

for any $B_{2r}(x) \subset\subset B_R(x_0)$ with $r \in (0,R/4)$. Thus,

$$\begin{aligned}\phi(x,\varepsilon^j r) \le \varepsilon^{\alpha_1 j}\phi(x,r) + C\tilde{h}(x,2\varepsilon^j r) + C\tilde{g}(x,2\varepsilon^j r)(\|\nabla u\|_{L^\infty(B_{2r}(x))}+s)^{2-p} \\ + C\tilde{\omega}(\varepsilon^j r)(\|\nabla u\|_{L^\infty(B_{2r}(x))}+s),\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\tilde{h}(x,t) &:= \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}\big(h(x,\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + h(x,R/2)[\varepsilon^{-i}t > R/2]\big), \\ \tilde{g}(x,t) &:= \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}\big(g(x,\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + g(x,R/2)[\varepsilon^{-i}t > R/2]\big), \\ \tilde{\omega}(t) &:= \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}\big(\omega(\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + \omega(R/2)[\varepsilon^{-i}t > R/2]\big).\end{aligned} \tag{4.4}$$

Here and throughout the paper, we use the Iverson bracket notation, i.e., $[P] = 1$ if P is true and $[P] = 0$ if P is false. We obtain the following lemma from (4.3).

Lemma 4.1. *Let $B_{2r}(x) \subset\subset B_R(x_0) \subset \Omega$ with $r \le R/4$. There exists a constant C, depending only on ε, n, p, λ, γ_0, and α_1, such that for any $\rho \in (0, r]$, we have*

$$\phi(x,\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\tilde{h}(x,2\rho) + C\tilde{g}(x,2\rho)(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s)^{2-p} + C\tilde{\omega}(\rho)(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s), \tag{4.5}$$

and

$$\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt + C(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s)^{2-p}\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt + C(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s)\int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt. \tag{4.6}$$

To prove Lemma 4.1, we need the following technical lemma.

Lemma 4.2. *Let $B_R(x_0) \subset \Omega$. Then there exist constants c_1 and c_2, depending on ε, n, p, and α_1, such that for any fixed $x \in B_R(x_0)$ and any $f \in \{\tilde{\omega}, \tilde{g}(x,\cdot), \tilde{h}(x,\cdot)\}$, one has $c_1 f(t) \le f(s) \le c_2 f(t)$ whenever $0 < \varepsilon t \le s \le t$.*

Proof. We will only give the proof for g since the other cases are similar. For fixed $x \in B_{R/4}(x_0)$, we set

$$G(x,r) := \begin{cases} g(x,r) & \text{if } 0 < r \le R/2, \\ g(x,R/2) & \text{if } r > R/2, \end{cases}$$

and observe that by (4.4),

$$\tilde{g}(x,r) = \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}G(x,\varepsilon^{-i}r).$$

Suppose that $0 < \varepsilon t \le s \le t$. It is easy to see from the definitions of g and G that $G(x,s) \le \varepsilon^{1-n}G(x,t)$ and therefore $\tilde{g}(x,s) \le \varepsilon^{1-n}\tilde{g}(x,t)$. Also the fact that $0 < \varepsilon s \le \varepsilon t \le s$ implies that $\tilde{g}(x,\varepsilon t) \le \varepsilon^{1-n}\tilde{g}(x,s)$. On the other hand,

$$\tilde{g}(x,t) = \varepsilon^{-\alpha_1}\sum_{i=1}^{\infty}\varepsilon^{\alpha_1(i+1)}G(x,\varepsilon^{-(i+1)}\varepsilon t) \le \varepsilon^{-\alpha_1}\tilde{g}(x,\varepsilon t).$$

The lemma is proved. ∎

Now we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. We first prove (i). For given $\rho \in (0, r]$, let j be the integer such that $\varepsilon^{j+1} < \rho/r \le \varepsilon^j$. Then by (4.3) with $\varepsilon^{-j}\rho$ in place of r, we get

$$\begin{aligned}\phi(x,\rho) &\le \varepsilon^{\alpha_1 j}\phi(x,\varepsilon^{-j}\rho) + C\tilde{h}(x,2\rho) + C\tilde{g}(x,2\rho)(\|\nabla u\|_{L^\infty(B_{2\varepsilon^{-j}\rho}(x))} + s)^{2-p} \\ &\quad + C\tilde{\omega}(\rho)(\|\nabla u\|_{L^\infty(B_{2\varepsilon^{-j}\rho}(x))} + s) \\ &\le C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\tilde{h}(x,2\rho) + C\tilde{g}(x,2\rho)(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s)^{2-p} \\ &\quad + C\tilde{\omega}(\rho)(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s).\end{aligned}$$

Therefore, (i) holds. Now applying (4.5) with $\varepsilon^j\rho$ in place of ρ and summing in j, we get

$$\begin{aligned}\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) &\le C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\sum_{j=1}^{\infty}\tilde{h}(x,2\varepsilon^j\rho) \\ &\quad + C(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s)^{2-p}\sum_{j=1}^{\infty}\tilde{g}(x,2\varepsilon^j\rho) \\ &\quad + C(\|\nabla u\|_{L^\infty(B_{2r}(x))} + s)\sum_{j=1}^{\infty}\tilde{\omega}(\varepsilon^j\rho).\end{aligned}$$

Hence, by using Lemma 4.2 and the comparison principle for Riemann integrals, we can easily get (4.6). The lemma is proved. ∎

Recall the definition of $\mathbf{q}_{x,\rho}$ from Section 3. Since

$$|\mathbf{q}_{x,\varepsilon\rho} - \mathbf{q}_{x,\rho}|^{\gamma_0} \le |\nabla u(z) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\nabla u(z) - \mathbf{q}_{x,\varepsilon\rho}|^{\gamma_0},$$

by taking the average over $z \in B_{\varepsilon\rho}(x)$ and then taking the γ_0-th root we obtain

$$|\mathbf{q}_{x,\varepsilon\rho} - \mathbf{q}_{x,\rho}| \le C\phi(x,\varepsilon\rho) + C\phi(x,\rho).$$

Then, by iterating, we get

$$|\mathbf{q}_{x,\varepsilon^j\rho} - \mathbf{q}_{x,\rho}| \le C\sum_{i=0}^{j}\phi(x,\varepsilon^i\rho).$$

Therefore, by using (3.12), we obtain

$$|\nabla u(x) - \mathbf{q}_{x,\rho}| \le C\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) \tag{4.7}$$

for any Lebesgue point $x \in \Omega$ of the vector-valued function ∇u.

Now we are ready to prove the interior Lipschitz estimate.

Proof of Theorem 1.3. We prove the theorem around a given point $x = x_0$ assuming that $B_R(x_0) \subset \Omega$ with $R \in (0,1]$ and

$$\|\mathbf{I}_1^R(|\mu|)\|_{L^\infty(B_R(x_0))} < \infty. \tag{4.8}$$

We first derive an a priori estimate for the case when $u \in C^1$ and then use approximation to prove the general case.

Step 1: The case when $u \in C^1(\overline{B_R(x_0)})$. Using (4.6) with $\rho = r$ and (4.7), we obtain

$$
\begin{aligned}
|\nabla u(x) - \mathbf{q}_{x,\rho}| \le C\phi(x,\rho) + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\
+ C(\|\nabla u\|_{L^\infty(B_{2\rho}(x))} + s)^{2-p}\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt \\
+ C(\|\nabla u\|_{L^\infty(B_{2\rho}(x))} + s)\int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt
\end{aligned}
$$

for any $B_{2\rho}(x) \subset\subset B_R(x_0)$ with $\rho \in (0, R/4]$.

Note that

$$
|\mathbf{q}_{x,\rho}| \le C\phi(x,\rho) + C\rho^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(B_\rho(x))} \le C\rho^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(B_\rho(x))}.
$$

Combining the above two inequalities, we have

$$
\begin{aligned}
|\nabla u(x)| \le C\rho^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(B_\rho(x))} + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\
+ C(\|\nabla u\|_{L^\infty(B_{2\rho}(x))} + s)^{2-p}\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt \\
+ C(\|\nabla u\|_{L^\infty(B_{2\rho}(x))} + s)\int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt.
\end{aligned} \tag{4.9}
$$

Note that $\tilde{\omega}$ also satisfies the Dini condition (1.5); see [5, Lemma 1]. Thus we can take $\rho_0 = \rho_0(n, p, \lambda, \omega, \alpha_1, \gamma_0, R) \in (0, R/4]$ sufficiently small such that

$$
C\int_0^{\rho_0} \frac{\tilde{\omega}(t)}{t}\,dt \le 3^{-1-n/\gamma_0},
$$

where C is the constant in (4.9). Then for any $B_{2\rho}(x) \subset\subset B_R(x_0)$ with $0 < \rho \le \rho_0$, by (4.9) and Young's inequality, we have

$$
\begin{aligned}
|\nabla u(x)| \le C\rho^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(B_\rho(x))} + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\
+ C\left(\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt\right)^{\frac{1}{p-1}} + 3^{-n/\gamma_0}(\|\nabla u\|_{L^\infty(B_{2\rho}(x))} + s).
\end{aligned} \tag{4.10}
$$

For $k \ge 1$, we denote $\rho_k = (1 - 2^{-k})R$. Since $\rho_{k+1} - \rho_k = 2^{-k-1}R$, we have $B_{2\rho}(x) \subset B_{\rho_{k+1}}(x_0)$ for any $x \in B_{\rho_k}(x_0)$ and $\rho = 2^{-k-2}R$. We take k_0 sufficiently large such that $2^{-k_0-2} \le \rho_0$. Then by (4.10) with $\rho = 2^{-k-2}R$ we have, for any $k \ge k_0$,

$$
\begin{aligned}
&\|\nabla u\|_{L^\infty(B_{\rho_k}(x_0))} + s \\
&\le 3^{-n/\gamma_0}(\|\nabla u\|_{L^\infty(B_{\rho_{k+1}}(x_0))} + s) + C\left(\frac{2^{k+2}}{R}\right)^{n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(B_{\rho_{k+1}}(x_0))} \\
&\quad + C\sup_{x\in B_{\rho_k}(x_0)}\int_0^{R/2} \frac{\tilde{h}(x,t)}{t}\,dt + C\sup_{x\in B_{\rho_k}(x_0)}\left(\int_0^{R/2} \frac{\tilde{g}(x,t)}{t}\,dt\right)^{\frac{1}{p-1}} + s.
\end{aligned}
$$

Multiplying the above inequality by $3^{-nk/\gamma_0}$, and summing the terms with respect to $k = k_0, k_0+1, \dots,$ we obtain

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} 3^{-nk/\gamma_0}(\|\nabla u\|_{L^\infty(B_{\rho_k}(x_0))} + s) \\
&\quad \le \sum_{k=k_0+1}^{\infty} 3^{-nk/\gamma_0}(\|\nabla u\|_{L^\infty(B_{\rho_k}(x_0))} + s) + CR^{-n/\gamma_0}(\|\nabla u\|_{L^{\gamma_0}(B_R(x_0))} + s) \\
&\qquad + C \sup_{x\in B_R(x_0)} \int_0^{R/2} \frac{\tilde{h}(x,t)}{t}\,dt + C \sup_{x\in B_R(x_0)} \left(\int_0^{R/2} \frac{\tilde{g}(x,t)}{t}\,dt\right)^{\frac{1}{p-1}} + Cs,
\end{aligned}$$

where each summation is finite. By subtracting

$$\sum_{k=k_0+1}^{\infty} 3^{-nk/\gamma_0}(\|\nabla u\|_{L^\infty(B_{\rho_k}(x_0))} + s)$$

from both sides of the above inequality, we get the following L^∞-estimate for ∇u:

$$\begin{aligned}
\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + s &\le CR^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(B_R(x_0))} \\
&\quad + C \sup_{x\in B_R(x_0)} \int_0^{R/2} \frac{\tilde{h}(x,t)}{t}\,dt \\
&\quad + C \sup_{x\in B_R(x_0)} \left(\int_0^{R/2} \frac{\tilde{g}(x,t)}{t}\,dt\right)^{\frac{1}{p-1}} + Cs. \qquad (4.11)
\end{aligned}$$

We can simplify the terms in (4.11) to get

$$\begin{aligned}
&\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} \\
&\quad \le C\|\mathbf{I}_1^R(|\mu|)\|_{L^\infty(B_R(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\big\||\nabla u| + s\big\|_{L^{2-p}(B_R(x_0))}. \qquad (4.12)
\end{aligned}$$

Indeed, by the definition of $\tilde{g}$ in (4.4), we have

$$\begin{aligned}
\int_0^{R/2} \frac{\tilde{g}(x,t)}{t}\,dt &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^{R/2} \frac{g(x,\varepsilon^{-i}t)}{t}[\varepsilon^{-i}t \le R/2]\,dt \\
&\quad + \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^{R/2} \frac{g(x,R/2)}{t}[\varepsilon^{-i}t > R/2]\,dt.
\end{aligned}$$

The first term above is equal to

$$\begin{aligned}
\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^{\varepsilon^i R/2} \frac{g(x,\varepsilon^{-i}t)}{t}\,dt &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^{R/2} \frac{g(x,t)}{t}\,dt \\
&\le C\mathbf{I}_1^R(|\mu|)(x).
\end{aligned}$$

The second term is equal to

$$\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \ln(\varepsilon^{-i}) g(x, R/2) \le C g(x, R/2) \le C \mathbf{I}_1^R(|\mu|)(x).$$

Therefore,

$$\int_0^{R/2} \frac{\tilde{g}(x,t)}{t}\, dt \le C \mathbf{I}_1^R(|\mu|)(x). \tag{4.13}$$

We can similarly get

$$\begin{aligned}\int_0^{R/2} \frac{\tilde{h}(x,t)}{t}\, dt &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^{R/2} \frac{h(x,t)}{t}\, dt + \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \ln(\varepsilon^{-i}) h(x, R/2) \\ &\le C\left(\int_0^R \frac{g(x,t)}{t}\, dt\right)^{\frac{1}{p-1}} + C\left(\mathbf{I}_1^R(|\mu|)(x)\right)^{\frac{1}{p-1}} \le C\left(\mathbf{I}_1^R(|\mu|)(x)\right)^{\frac{1}{p-1}}.\end{aligned} \tag{4.14}$$

Recalling the fact that $\gamma_0 \le 2 - p$ (cf. Lemma 3.2), using (4.13), (4.14), and Hölder's inequality, from (4.11) we obtain (4.12).

Step 2: The general case. We take $r_1 \in (0, R)$, $r_2 = (R + r_1)/2$, and a sequence $\{\varphi_k\}$ of standard mollifiers such that for any positive integer k,

$$\varphi_k \in C_0^\infty(B_{1/k}(0)), \quad \varphi_k \ge 0, \quad \text{and} \quad \fint_{B_{1/k}(0)} \varphi_k = 1.$$

Then we mollify μ and A by setting

$$\mu_k(x) = (\mu * \varphi_k)(x), \quad A_k(x, \xi) = (A(\cdot, \xi) * \varphi_k)(x), \quad x \in B_{r_2}(x_0).$$

We note that A_k is well defined and satisfies the growth, ellipticity and continuity assumptions (1.2)–(1.4) in $B_{r_2}(x_0)$ for $k > 1/(R - r_2)$. By the corollary after [12, Theorem 1], (4.8) implies $\mu \in W^{-1,p'}(B_{r_2}(x_0))$, where $p' = p/(p-1)$, and therefore

$$\|\mu_k - \mu\|_{W^{-1,p'}(B_{r_2}(x_0))} \to 0. \tag{4.15}$$

Next we let $u_k \in u + W_0^{1,p}(B_{r_2}(x_0))$ be the unique solution to

$$\begin{cases} -\operatorname{div}(A_k(x, \nabla u_k)) = \mu_k & \text{in } B_{r_2}(x_0), \\ u_k = u & \text{on } \partial B_{r_2}(x_0). \end{cases} \tag{4.16}$$

Choosing $u_k - u$ as a test function in (4.16), we obtain

$$\begin{aligned}&\int_{B_{r_2}(x_0)} \langle A(x, \nabla u_k), \nabla u_k \rangle\, dx \\ &\qquad = \int_{B_{r_2}(x_0)} \langle A(x, \nabla u_k), \nabla u \rangle\, dx + \int_{B_{r_2}(x_0)} (u_k - u)\, d\mu_k.\end{aligned} \tag{4.17}$$

Using the fundamental theorem of calculus, (1.2), (1.3), and Young's inequality with exponents p and $p/(p-1)$, we have

$$\begin{aligned}&\int_{B_{r_2}(x_0)} \langle A(x,\nabla u_k),\nabla u_k\rangle\,dx\\&\qquad = \int_{B_{r_2}(x_0)}\left(\int_0^1 \langle D_\xi A(x,t\nabla u_k)\nabla u_k,\nabla u_k\rangle\,dt + \langle A(x,0),\nabla u_k\rangle\right)dx\\&\qquad \ge \int_{B_{r_2}(x_0)}\left(\int_0^1 \lambda^{-1}(s^2+|\nabla u_k|^2t^2)^{\frac{p-2}{2}}|\nabla u_k|^2\,dt - \lambda s^{p-1}|\nabla u_k|\right)dx\\&\qquad \ge \lambda^{-1}\int_{B_{r_2}(x_0)}(s^2+|\nabla u_k|^2)^{\frac{p-2}{2}}|\nabla u_k|^2\,dx - \int_{B_{r_2}(x_0)}\lambda s^{p-1}|\nabla u_k|\,dx\\&\qquad \ge c(\lambda,p)\int_{B_{r_2}(x_0)}|\nabla u_k|^p\,dx - C'(\lambda,p)\int_{B_{r_2}(x_0)} s^p\,dx.\end{aligned}$$

On the other hand, using (1.2) and Young's inequality, we obtain

$$\begin{aligned}&\int_{B_{r_2}(x_0)}\langle A(x,\nabla u_k),\nabla u\rangle\,dx + \int_{B_{r_2}(x_0)}(u_k-u)\,d\mu_k\\&\quad\le \lambda\int_{B_{r_2}(x_0)}(s^2+|\nabla u_k|^2)^{\frac{p-1}{2}}|\nabla u|\,dx + \|u_k-u\|_{W_0^{1,p}(B_{r_2}(x_0))}\|\mu_k\|_{W^{-1,p'}(B_{r_2}(x_0))}\\&\quad\le \tfrac14 c(\lambda,p)\int_{B_{r_2}(x_0)}(|\nabla u_k|+s)^p\,dx + C''(\lambda,p)\int_{B_{r_2}(x_0)}|\nabla u|^p\,dx\\&\qquad + \tfrac18 c(\lambda,p)\|\nabla u_k-\nabla u\|^p_{L^p(B_{r_2}(x_0))} + C''(\lambda,p)\|\mu_k\|^{p'}_{W^{-1,p'}(B_{r_2}(x_0))}.\end{aligned}$$

Therefore, (4.17) implies that

$$\|\nabla u_k\|^p_{L^p(B_{r_2}(x_0))} \le C\,\big\||\nabla u|+s\big\|^p_{L^p(B_{r_2}(x_0))} + C\,\|\mu_k\|^{p'}_{W^{-1,p'}(B_{r_2}(x_0))}, \tag{4.18}$$

where C is a constant not depending on k.

Now we recall a well-known inequality

$$\begin{aligned}c^{-1}(s^2+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}} &\le \frac{|V(\xi_2)-V(\xi_1)|^2}{|\xi_2-\xi_1|^2}\\&\le c(s^2+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}},\end{aligned}\tag{4.19}$$

where $c = c(n,p) > 1$ is a positive constant and the mapping $V(\cdot)$ is defined as

$$V(\xi) = (|\xi|^2+s^2)^{\frac{p-2}{4}}\xi, \quad \xi\in\mathbb{R}^n.$$

Combining (1.3) and (4.19) yields

$$c_0^{-1}|V(\xi_2)-V(\xi_1)|^2 \le \langle A(x,\xi_2)-A(x,\xi_1),\xi_2-\xi_1\rangle,$$

for some positive constant $c_0 = c_0(n, p, \lambda)$. We note that the above inequality also holds for A_k. Then choosing $(u_k - u)1_{B_{r_2}(x_0)}$ as a test function in (1.1) and (4.16), we have

$$\begin{aligned}
c_0^{-1} \int_{B_{r_2}(x_0)} &|V(\nabla u_k) - V(\nabla u)|^2 \, dx \\
&\le \int_{B_{r_2}(x_0)} \langle A_k(x, \nabla u_k) - A_k(x, \nabla u), \nabla u_k - \nabla u \rangle \, dx \\
&= \int_{B_{r_2}(x_0)} \langle A(x, \nabla u) - A_k(x, \nabla u), \nabla u_k - \nabla u \rangle \, dx + \int_{B_{r_2}(x_0)} (u_k - u) \, d(\mu_k - \mu) \\
&\le \| A(\cdot, \nabla u) - A_k(\cdot, \nabla u) \|_{L^{p/(p-1)}(B_{r_2}(x_0))} \cdot \| u_k - u \|_{W_0^{1,p}(B_{r_2}(x_0))} \\
&\quad + \| u_k - u \|_{W_0^{1,p}(B_{r_2}(x_0))} \cdot \| \mu_k - \mu \|_{W^{-1,p'}(B_{r_2}(x_0))}.
\end{aligned}$$

From the definition of A_k, by using the Minkowski inequality and (1.4), we obtain

$$\| A(\cdot, \nabla u) - A_k(\cdot, \nabla u) \|_{L^{p/(p-1)}(B_{r_2}(x_0))} \le \lambda \omega(1/k) \Big(\int_{B_{r_2}(x_0)} (s^2 + |\nabla u|^2)^{p/2} \, dx \Big)^{\frac{p-1}{p}},$$

which together with (4.15) and (4.18), yields

$$\int_{B_{r_2}(x_0)} |V(\nabla u_k) - V(\nabla u)|^2 \, dx \to 0.$$

By (4.19), we have

$$|\nabla u_k - \nabla u|^p \le c |V(\nabla u_k) - V(\nabla u)|^p (|\nabla u_k|^2 + |\nabla u|^2 + s^2)^{p(2-p)/4}$$

and therefore using Hölder's inequality with exponents $2/p$ and $2/(2-p)$, we obtain

$$\begin{aligned}
&\int_{B_{r_2}(x_0)} |\nabla u_k - \nabla u|^p \, dx \\
&\le c \Big(\int_{B_{r_2}(x_0)} |V(\nabla u_k) - V(\nabla u)|^2 \, dx \Big)^{p/2} \Big(\int_{B_{r_2}(x_0)} (|\nabla u_k|^2 + |\nabla u|^2 + s^2)^{p/2} \, dx \Big)^{\frac{2-p}{2}},
\end{aligned}$$

which implies that

$$\nabla u_k \to \nabla u \quad \text{strongly in } L^p(B_{r_2}(x_0)).$$

Thus there exists a subsequence $\{k_j\}$ such that $\nabla u_{k_j} \to \nabla u$ almost everywhere in $B_{r_2}(x_0)$.

Since A_k and μ_k are smooth in x, by the classical regularity theory (see, for instance, [3, 24]), we know that $u_k \in C^{1,\alpha}_{\mathrm{loc}}(B_{r_2}(x_0))$. Therefore the Lipschitz estimate (4.12) from Step 1 holds for u_k in $B_{r_1/2}(x_0)$. Namely,

$$\| \nabla u_k \|_{L^\infty(B_{r_1/2}(x_0))} \le C \| \mathbf{I}_1^{r_1}(|\mu_k|) \|_{L^\infty(B_{r_1}(x_0))}^{\frac{1}{p-1}} + C r_1^{\frac{n}{2-p}} \big\| |\nabla u_k| + s \big\|_{L^{2-p}(B_{r_1}(x_0))}.$$

Note that by direct computation, for any $t > 0$, it follows that

$$\mu_k(B_t(x)) = \int_{\mathbb{R}^n} \mu(B_t(x-y)) \varphi_k(y) \, dy.$$

Therefore, for sufficiently large k, by the Fubini–Tonelli theorem we have

$$\|\mathbf{I}_1^{r_1}(|\mu_k|)\|_{L^\infty(B_{r_1}(x_0))} \le \|\mathbf{I}_1^{r_1}(|\mu|)\|_{L^\infty(B_{r_2}(x_0))}.$$

Thus by taking $k = k_j \nearrow \infty$ and then $r_1 \nearrow R$, we obtain the Lipschitz estimate (1.13) around $x = x_0$. ■

In the rest of the section, we derive an interior modulus of continuity estimate of ∇u under the conditions of Theorem 1.3. Recall that we fixed an $\varepsilon \in (0, 1/4)$ sufficiently small such that

$$C\varepsilon^{\alpha-\alpha_1} < 1 \quad \text{and} \quad \varepsilon^{\alpha_1} < 1/4,$$

where C is the constant in (3.13), $\alpha \in (0, 1)$ is the same constant as in Theorem 2.1 and $\alpha_1 \in (0, \alpha)$. We also took a ball $B_R(x_0) \subset \Omega$ with $R \in (0, 1]$. Similar to (4.4), we define

$$\begin{aligned}
\tilde{\omega}(t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\omega(\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + \omega(R/2)[\varepsilon^{-i}t > R/2]\big),\\
\tilde{\mathbf{I}}_1^{\rho}(|\mu|)(x) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{I}_1^{R/2}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2]\big),\\
\tilde{\mathbf{W}}_{1/p,p}^{\rho}&(|\mu|)(x)\\
&= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\mathbf{W}_{1/p,p}^{\varepsilon^{-i}\rho}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{W}_{1/p,p}^{R/2}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2]\big),
\end{aligned} \tag{4.20}$$

where $\mathbf{I}_1$ and $\mathbf{W}_{1/p,p}$ are the Riesz and Wolff potentials defined in (1.9) and (1.11), respectively. We note that since $1/(p-1) > 1$, we have

$$\mathbf{W}_{1/p,p}^{\rho}(|\mu|)(x) \le C\big(\mathbf{I}_1^{2\rho}(|\mu|)(x)\big)^{\frac{1}{p-1}}, \tag{4.21}$$

so that $\tilde{\mathbf{W}}_{1/p,p}^{\rho}(|\mu|)(x)$ and $\tilde{\mathbf{I}}_1^{\rho}(|\mu|)(x)$ are bounded and converge to zero as $\rho \to 0$ as long as $\mathbf{I}_1^{R}(|\mu|)(x)$ is finite.

Our interior modulus of continuity estimate is stated as follows.

Theorem 4.3. *Assume the conditions of Theorem 1.3 and $\alpha_1 \in (0, \alpha)$, where α is the constant in Theorem 2.1. Then there exist a constant $C = C(n, p, \lambda, \alpha_1, \omega)$ such that for any $R \in (0, 1]$ with $B_R(x_0) \subset \Omega$, and $x, y \in B_{R/4}(x_0)$ that are Lebesgue points of the vector-valued function ∇u, we have*

$$\begin{aligned}
|\nabla u(x) - \nabla u(y)|&\\
\le C\mathbf{M}&\left[\left(\frac{\rho}{R}\right)^{\alpha_1} + \int_0^{\rho} \frac{\tilde{\omega}(t)}{t}\,dt\right] + C\|\tilde{\mathbf{W}}_{1/p,p}^{\rho}(|\mu|)\|_{L^\infty(B_{R/4}(x_0))}\\
&+ C\mathbf{M}^{2-p}\|\tilde{\mathbf{I}}_1^{\rho}(|\mu|)\|_{L^\infty(B_{R/4}(x_0))},
\end{aligned} \tag{4.22}$$

where $\rho = |x-y|$, $\tilde{\omega}$, $\tilde{\mathbf{W}}_{1/p,p}$, and $\tilde{\mathbf{I}}_1$ are defined in (4.20), and

$$\mathbf{M} := R^{-\frac{n}{2-p}}\big\||\nabla u| + s\big\|_{L^{2-p}(B_R(x_0))} + \|\mathbf{I}_1^{R}(|\mu|)\|_{L^\infty(B_R(x_0))}^{\frac{1}{p-1}}.$$

Note that due to (4.21), the term $\|\tilde{\mathbf{W}}^{\rho}_{1/p,p}(|\mu|)\|_{L^\infty(B_{R/4}(x_0))}$ in (4.22) can be replaced with the sup norm of a summation of the truncated Riesz potentials similar to (4.20).

Proof of Theorem 4.3. For any $x, y \in B_{R/4}(x_0)$ that are Lebesgue points of ∇u, by the triangle inequality we have

$$
\begin{aligned}
&|\nabla u(x) - \nabla u(y)|^{\gamma_0} \\
&\quad \le |\nabla u(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\mathbf{q}_{x,\rho} - \mathbf{q}_{y,\rho}|^{\gamma_0} + |\nabla u(y) - \mathbf{q}_{y,\rho}|^{\gamma_0} \\
&\quad \le 2 \sup_{y_0 \in B_{R/4}(x_0)} |\nabla u(y_0) - \mathbf{q}_{y_0,\rho}|^{\gamma_0} + |\nabla u(z) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\nabla u(z) - \mathbf{q}_{y,\rho}|^{\gamma_0}.
\end{aligned}
$$

We set $\rho = |x - y|$, take the average over $z \in B(x,\rho) \cap B(y,\rho)$, and then take the γ_0-th root to get

$$
\begin{aligned}
|\nabla u(x) - \nabla u(y)| &\le C \sup_{y_0 \in B_{R/4}(x_0)} |\nabla u(y_0) - \mathbf{q}_{y_0,\rho}| + C\phi(x,\rho) + C\phi(y,\rho) \\
&\le C \sup_{y_0 \in B_{R/4}(x_0)} \sum_{j=0}^{\infty} \phi(y_0, \varepsilon^j \rho) + C \sup_{y_0 \in B_{R/4}(x_0)} \phi(y_0,\rho) \\
&\le C \sup_{y_0 \in B_{R/4}(x_0)} \sum_{j=0}^{\infty} \phi(y_0, \varepsilon^j \rho). \qquad (4.23)
\end{aligned}
$$

Here we have used (4.7) in the second inequality.

If $\rho < R/8$, by using (4.23), (4.6) with $R/8$ in place of r, and the fact that

$$
B_{R/4}(y_0) \subset B_{R/2}(x_0) \quad \forall y_0 \in B_{R/4}(x_0),
$$

we obtain

$$
\begin{aligned}
&|\nabla u(x) - \nabla u(y)| \\
&\quad \le C \left(\frac{\rho}{R}\right)^{\alpha_1} \|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + C \sup_{y_0 \in B_{R/4}(x_0)} \int_0^\rho \frac{\tilde{h}(y_0,t)}{t}\,dt \\
&\qquad + C(\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + s)^{2-p} \sup_{y_0 \in B_{R/4}(x_0)} \int_0^\rho \frac{\tilde{g}(y_0,t)}{t}\,dt \\
&\qquad + C(\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + s) \int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt. \qquad (4.24)
\end{aligned}
$$

Clearly, (4.24) still holds when $\rho \ge R/8$.

We can simplify the terms in (4.24) as follows. For any $y_0 \in B_{R/4}(x_0)$ and $\rho \in (0, R/2)$, by the definition of $\tilde{g}$ in (4.4), we have

$$
\begin{aligned}
\int_0^\rho \frac{\tilde{g}(y_0,t)}{t}\,dt &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^\rho \frac{g(y_0, \varepsilon^{-i} t)}{t} [\varepsilon^{-i} t \le R/2]\,dt \\
&\quad + \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^\rho \frac{g(y_0, R/2)}{t} [\varepsilon^{-i} t > R/2]\,dt.
\end{aligned}
$$

Recalling the definition of $\tilde{\mathbf{I}}_1$ from (4.20), the first term above is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\Big(\int_0^{\rho} \frac{g(y_0, \varepsilon^{-i}t)}{t}\,dt\,[\varepsilon^{-i}\rho \le R/2] + \int_0^{\varepsilon^i R/2} \frac{g(y_0, \varepsilon^{-i}t)}{t}\,dt\,[\varepsilon^{-i}\rho > R/2]\Big)\\
&\quad = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\Big(\int_0^{\varepsilon^{-i}\rho} \frac{g(y_0, t)}{t}\,dt\,[\varepsilon^{-i}\rho \le R/2] + \int_0^{R/2} \frac{g(y_0, t)}{t}\,dt\,[\varepsilon^{-i}\rho > R/2]\Big)\\
&\quad = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)(y_0)[\varepsilon^{-i}\rho \le R/2] + \mathbf{I}_1^{R/2}(|\mu|)(y_0)[\varepsilon^{-i}\rho > R/2]\big)\\
&\quad = \tilde{\mathbf{I}}_1^{\rho}(|\mu|)(y_0).
\end{aligned}$$

The second term is equal to

$$\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}[\varepsilon^{-i}\rho > R/2]\ln(2\varepsilon^{-i}\rho/R)g(y_0, R/2).$$

Let K be the positive integer such that $\varepsilon^{-K}\rho > R/2$ and $\varepsilon^{-(K-1)}\rho \le R/2$. Then

$$\begin{aligned}
&\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}[\varepsilon^{-i}\rho > R/2]\ln(2\varepsilon^{-i}\rho/R)\\
&\qquad = \sum_{i=K}^{\infty} \varepsilon^{\alpha_1 i}\ln(2\varepsilon^{-i}\rho/R)\\
&\qquad = \varepsilon^{\alpha_1 K}\sum_{i=K}^{\infty} \varepsilon^{\alpha_1(i-K)}\big(\ln(2\varepsilon^{-K}\rho/R) + (i-K)\ln(\varepsilon^{-1})\big)\\
&\qquad \le \Big(\frac{2\rho}{R}\Big)^{\alpha_1}\sum_{i=K}^{\infty} \varepsilon^{\alpha_1(i-K)}(i-K+1)\ln(\varepsilon^{-1}) \le C\Big(\frac{\rho}{R}\Big)^{\alpha_1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{\rho} \frac{\tilde{g}(y_0, t)}{t}\,dt &\le \tilde{\mathbf{I}}_1^{\rho}(|\mu|)(y_0) + C(\rho/R)^{\alpha_1} g(y_0, R/2)\\
&\le \tilde{\mathbf{I}}_1^{\rho}(|\mu|)(y_0) + C(\rho/R)^{\alpha_1}\mathbf{I}_1^{R}(|\mu|)(y_0). \qquad (4.25)
\end{aligned}$$

We can similarly get the following estimate:

$$\begin{aligned}
\int_0^{\rho} \frac{\tilde{h}(y_0, t)}{t}\,dt &\le \sum_{i=1}^{\infty}\Big(\varepsilon^{\alpha_1 i}\int_0^{\varepsilon^{-i}\rho} h(y_0, t)\,\frac{dt}{t}\,[\varepsilon^{-i}\rho \le R/2]\\
&\qquad + \int_0^{R/2} h(y_0, t)\,\frac{dt}{t}\,[\varepsilon^{-i}\rho > R/2]\Big) + C\Big(\frac{\rho}{R}\Big)^{\alpha_1} h(y_0, R/2)\\
&\le \tilde{\mathbf{W}}_{1/p,p}^{\rho}(|\mu|)(y_0) + C(\rho/R)^{\alpha_1}(\mathbf{I}_1^{R}(|\mu|)(y_0))^{1/(p-1)}. \qquad (4.26)
\end{aligned}$$

Using (1.13), (4.25), and (4.26), from (4.24) we obtain (4.22). ■

Proof of Theorem 1.4. Since the set of Lebesgue points of ∇u is dense in Ω, it suffices to show the right-hand side of (4.22) converges to zero when $\rho \to 0$. In fact, we have

$$\|\tilde{\mathbf{I}}_1^{\rho}(|\mu|)\|_{L^\infty(B_{R/4}(x_0))} \le \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\|\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)\|_{L^\infty(B_{R/4}(x_0))}[\varepsilon^{-i}\rho \le R/2] + \|\mathbf{I}_1^{R/2}(|\mu|)\|_{L^\infty(B_{R/4}(x_0))}[\varepsilon^{-i}\rho > R/2]\big),$$

which must converge to 0 by using (1.14) and the dominated convergence theorem. We can similarly prove the convergence of other terms in (4.22). ∎

Proof of Corollary 1.5. By [8, Lemma 3] with $p = 2$ and $k = 1$, the assumption (1.15) implies (1.14). Therefore Corollary 1.5 follows from Theorem 1.4. ∎

Proof of Corollary 1.6. This is an immediate consequence of Theorem 1.4 since the assumption (1.14) is verified by (1.16) and (1.17). ∎

Proof of Corollary 1.7. We choose $\alpha_1 \in (\beta, \alpha)$ in Theorem 4.3. Then Corollary 1.7 follows by a direct computation using (4.22). ∎

5. Global gradient estimates for the p-Laplacian equations

This section is devoted to the proof of the global pointwise gradient estimate in Theorem 1.10, the Lipschitz estimate in Theorem 1.11, Corollary 1.12, as well as the derivation of a global modulus of continuity estimate of ∇u stated in Theorem 5.10 for the following (possibly nondegenerate) p-Laplace equation with Dirichlet boundary condition:

$$\begin{cases} -\operatorname{div}\big(a(x)(|\nabla u|^2 + s^2)^{\frac{p-2}{2}}\nabla u\big) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $a(\cdot)$ satisfies (1.5), (1.7), and (1.8), and Ω has a $C^{1,\text{Dini}}$ boundary characterized by R_0 and ω_0 as in Definition 1.9.

First, we derive a gradient estimate around any point $x_0 \in \partial\Omega$. Without loss of generality, we assume that $x_0 = 0 \in \partial\Omega$. Then we can choose a local coordinate around $x_0 = 0$ and a function χ as in Definition 1.9 such that $\chi(0') = 0$. Let

$$\Gamma(y) = (y_1 + \chi(y'), y') \quad \text{and} \quad \Lambda(x) = \Gamma^{-1}(x) = (x_1 - \chi(x'), x').$$

Note that the determinants of the Jacobian of $\Gamma(\cdot)$ and $\Lambda(\cdot)$ are equal to 1. Since Ω has $C^{1,\text{Dini}}$ boundary, from the proof of [2, Lemma 2.2], there exists $R_1 = R_1(\omega_0, R_0) \in (0, R_0)$ such that

$$|\nabla_{x'}\chi(x')| \le 1/2 \quad \text{if } |x'| \le R_1, \tag{5.2}$$

$$\Omega_{r/2} \subset \Gamma(B_r^+) \subset \Omega_{2r} \quad \forall r \in (0, R_1/2]. \tag{5.3}$$

Therefore, there exist constants $c_1(n)$ and $c_2(n)$, depending only on n, such that for any $x \in \overline{\Omega}$ and $0 < r \leq R_1$,

$$c_1(n) r^n \leq |\Omega_r(x)| \leq c_2(n) r^n. \tag{5.4}$$

Now we use the technique of flattening the boundary. We denote $u_1(y) = u(\Gamma(y))$, $a_1(y) = a(\Gamma(y))$, and $\mu_1(A) = \mu(\Gamma(A))$ for any Borel set $A \subset \mathbb{R}^n$. Then u_1 satisfies

$$\begin{cases} -\mathrm{div}_y\big(a_1(y)(|(D\Lambda)^T D_y u_1|^2 + s^2)^{\frac{p-2}{2}} D\Lambda (D\Lambda)^T D_y u_1\big) = \mu_1 & \text{in } B^+_{R_1/2}, \\ u_1 = 0 & \text{on } B_{R_1/2} \cap \partial\mathbb{R}^n_+. \end{cases} \tag{5.5}$$

We set

$$A_1(y,\xi) = a_1(y)(|(D\Lambda)^T \xi|^2 + s^2)^{\frac{p-2}{2}} D\Lambda (D\Lambda)^T \xi.$$

By direct computations with (5.2) in hand, A_1 satisfies the following conditions with $\omega_1 = \omega + \omega_0$ and some constant $\lambda_1 = \lambda_1(n,p,\lambda)$:

$$|A_1(y,\xi)| \leq \lambda_1 (s^2 + |\xi|^2)^{(p-1)/2}, \quad |D_\xi A_1(y,\xi)| \leq \lambda_1 (s^2 + |\xi|^2)^{(p-2)/2}, \tag{5.6}$$

$$\langle D_\xi A_1(y,\xi)\eta, \eta\rangle \geq \lambda_1^{-1} (s^2 + |\xi|^2)^{(p-2)/2} |\eta|^2, \tag{5.7}$$

$$|A_1(y,\xi) - A_1(y_0,\xi)| \leq \lambda_1 \omega_1(|y - y_0|)(s^2 + |\xi|^2)^{(p-1)/2} \tag{5.8}$$

for all $y, y_0 \in B^+_{R_1/2}$ and $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$.

Suppose that $4r \leq R_1$. We now consider the unique solution $w \in u_1 + W_0^{1,p}(B^+_{2r})$ to the equation

$$\begin{cases} -\mathrm{div}_y(A_1(y, \nabla_y w)) = 0 & \text{in } B^+_{2r}, \\ w = u_1 & \text{on } \partial B^+_{2r}. \end{cases} \tag{5.9}$$

We first derive a boundary version of the reverse Hölder inequality.

Lemma 5.1. *Let w be a solution to (5.9). There exists a constant $\theta_1 > p$, depending only on n, p, and λ, such that for any $t > 0$, the estimate*

$$\left(\fint_{B^+_{\rho/2}(y_0)} (|\nabla_y w| + s)^{\theta_1}\, dx\right)^{1/\theta_1} \leq C \left(\fint_{B^+_{\rho}(y_0)} (|\nabla_y w| + s)^{t}\, dx\right)^{1/t} \tag{5.10}$$

holds for all $B^+_\rho(y_0) \subset B^+_{2r}$, where $C = C(n,p,\lambda,t) > 0$.

Proof. For simplicity, we still denote $\nabla = \nabla_y$ through this proof. First we prove a Caccioppoli type inequality in half-balls. Suppose that $y_0 \in B_{2r} \cap \partial\mathbb{R}^n_+$ and $B_{2\rho}(y_0) \subset\subset B_{2r}$. Let ζ be a nonnegative smooth function satisfying $\zeta = 1$ in $B_\rho(y_0)$, $|\nabla\zeta| \leq 2\rho^{-1}$, and $\zeta = 0$ outside $B_{2\rho}(y_0)$. Using $\zeta^p w$ as a test function in (5.9), we get

$$0 = \int_{B^+_{2r}} \langle A_1(y,\nabla w), \zeta^p \nabla w\rangle\, dy + p \int_{B^+_{2r}} \langle A_1(y,\nabla w), \zeta^{p-1} w \nabla\zeta\rangle\, dy =: \mathrm{I} + \mathrm{II}. \tag{5.11}$$

Using the fundamental theorem of calculus, (5.6), (5.7), and Young's inequality with exponents p and $p/(p-1)$, we have

$$\begin{aligned}
\mathrm{I} &= \int_{B_{2r}^+} \zeta^p \Big(\int_0^1 \langle D_\xi A_1(y, t\nabla w)\nabla w, \nabla w\rangle \, dt + A_1(y,0)\nabla w \Big)\, dy \\
&\ge \int_{B_{2r}^+} \zeta^p \Big(\int_0^1 \lambda_1^{-1}(s^2 + |\nabla w|^2 t^2)^{\frac{p-2}{2}} |\nabla w|^2 \, dt - \lambda_1 s^{p-1}|\nabla w| \Big)\, dy \\
&\ge c(\lambda_1, p) \int_{B_{2r}^+} \zeta^p |\nabla w|^p \, dy - C'(\lambda_1, p) \int_{B_{2r}^+} \zeta^p s^p \, dy.
\end{aligned}$$

On the other hand, using (5.6) and Young's inequality, we have

$$\begin{aligned}
|\mathrm{II}| &\le p\lambda_1 \int_{B_{2\rho}^+(y_0)} (s^2 + |\nabla w|^2)^{\frac{p-1}{2}} \zeta^{p-1} |w \nabla \zeta| \, dy \\
&\le \frac{1}{2} c(\lambda_1, p) \int_{B_{2r}^+} \zeta^p (s^2 + |\nabla w|^2)^{p/2} \, dy + C''(\lambda_1, p) \int_{B_{2\rho}^+(y_0)} |w\nabla\zeta|^p \, dy \\
&\le \frac{1}{2} c(\lambda_1, p) \int_{B_{2r}^+} \zeta^p (s^p + |\nabla w|^p) \, dy + C''(\lambda_1, p) \int_{B_{2\rho}^+(y_0)} |w\nabla\zeta|^p \, dy.
\end{aligned}$$

Therefore, (5.11) implies the following Caccioppoli type inequality:

$$\int_{B_\rho^+(y_0)} |\nabla w|^p \, dx \le C\rho^n s^p + C\rho^{-p} \int_{B_{2\rho}^+(y_0)} |w|^p \, dx. \tag{5.12}$$

Since $w = u = 0$ on $B_{2r} \cap \partial\mathbb{R}^n_+$, by the Sobolev–Poincaré inequality,

$$\Big(\int_{B_{2\rho}^+(y_0)} |w|^p \, dx \Big)^{1/p} \le C\rho^{1+n/p-n/q} \Big(\int_{B_{2\rho}^+(y_0)} |\nabla w|^q \, dx \Big)^{1/q} \tag{5.13}$$

for any q such that $\max\left\{1, \frac{np}{n+p}\right\} \le q < p$. Thus by combining (5.12) and (5.13), we have

$$\Big(\fint_{B_\rho^+(y_0)} (|\nabla w| + s)^p \, dx \Big)^{1/p} \le C \Big(\fint_{B_{2\rho}^+(y_0)} (|\nabla w| + s)^q \, dx \Big)^{1/q}.$$

Similarly, the interior version

$$\Big(\fint_{B_\rho(y_0)} (|\nabla w| + s)^p \, dx \Big)^{1/p} \le C \Big(\fint_{B_{2\rho}(y_0)} (|\nabla w| + s)^q \, dx \Big)^{1/q}$$

holds for all $B_{2\rho}(y_0) \subset\subset B_{2r}^+$. Therefore, by a standard covering argument and Gehring's lemma, we get (5.10). ∎

We also have a boundary comparison result analogous to Lemma 3.2 by following almost the same proof.

Lemma 5.2. *Let w be a solution to (5.9) and assume that $p \in (1,2)$. Then for any $\gamma_0 \in (0, 2-p]$ when $p \in (\frac{3n-2}{2n-1}, 2)$ and $\gamma_0 \in (0, \frac{(p-1)n}{n-1})$ when $p \in (1, \frac{3n-2}{2n-1}]$, we have*

$$\left(\fint_{B_{2r}^+} |\nabla_y u_1 - \nabla_y w|^{\gamma_0}\,dx\right)^{1/\gamma_0} \le C\left[\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}} \fint_{B_{2r}^+} (|\nabla_y u_1| + s)^{2-p}\,dx,$$

where C is a constant depending only on n, p, λ, and γ_0.

We now let $v \in w + W_0^{1,p}(B_r^+)$ be the unique solution to

$$\begin{cases} -\mathrm{div}_y(A_1(0, \nabla_y v)) = 0 & \text{in } B_r^+, \\ v = w & \text{on } \partial B_r^+. \end{cases} \tag{5.14}$$

We also have an estimate for the difference $\nabla v - \nabla w$ analogous to (3.10) by following almost the same proof as that of [9, (4.35)]:

$$\fint_{B_r^+} |\nabla_y v - \nabla_y w|^p\,dx \le C\omega_0(r)^p \fint_{B_r^+} (|\nabla_y w| + s)^p\,dx.$$

Thus by (5.10) and Hölder's inequality, we get

$$\fint_{B_r^+} |\nabla_y v - \nabla_y w|^{\gamma_0}\,dx \le C\omega_0(r)^{\gamma_0} \fint_{B_{2r}^+} (|\nabla_y w| + s)^{\gamma_0}\,dx.$$

Next we prove an oscillation estimate for v.

Lemma 5.3. *Let v be a solution to (5.14). There exists a constant $C > 1$, depending only on n, p, λ, and γ_0, such that for any half-ball $B_\rho^+ \subset B_R^+ \subset B_r^+$, we have*

$$\inf_{\theta\in\mathbb{R}}\left(\fint_{B_\rho^+} (|D_{y_1} v - \theta|^{\gamma_0} + |D_{y'} v|^{\gamma_0})\right)^{1/\gamma_0} \le C\left(\frac{\rho}{R}\right)^{\alpha} \inf_{\theta\in\mathbb{R}}\left(\fint_{B_R^+} (|D_{y_1} v - \theta|^{\gamma_0} + |D_{y'} v|^{\gamma_0})\right)^{1/\gamma_0}, \tag{5.15}$$

where $\alpha \in (0,1)$ is the same constant as in Theorem 2.1.

Proof. Let $\bar{v}$ be an odd extension of v in B_r, namely,

$$\bar{v}(y) = \begin{cases} v(y) & \text{if } y_1 \ge 0, \\ -v(-y_1, y') & \text{if } y_1 < 0. \end{cases}$$

Then since

$$A_1(0,\xi) = a_1(0)(|\xi|^2 + s^2)^{\frac{p-2}{2}}\xi,$$

$\bar{v} \in W^{1,p}(B_r)$ is a solution to the equation

$$-\mathrm{div}_y\big(a_1(0)(|\nabla_y \bar{v}|^2 + s^2)^{\frac{p-2}{2}} \nabla_y \bar{v}\big) = 0 \quad \text{in } B_r.$$

Thus we can apply Theorem 2.1 to $\bar{v}$ to get

$$\inf_{\mathbf{q}\in\mathbb{R}^n}\left(\fint_{B_\rho} |\nabla_y \bar{v} - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0} \le C\left(\frac{\rho}{R}\right)^\alpha \inf_{\mathbf{q}\in\mathbb{R}^n}\left(\fint_{B_R} |\nabla_y \bar{v} - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}.$$

Since $\bar{v}$ is an odd function in y_1, by the triangle inequality there exists $\theta_\rho \in \mathbb{R}$ such that

$$\left(\fint_{B_\rho^+} (|D_{y_1} v - \theta_\rho|^{\gamma_0} + |D_{y'} v|^{\gamma_0})\right)^{1/\gamma_0} \le C \inf_{\mathbf{q}\in\mathbb{R}^n}\left(\fint_{B_\rho} |\nabla_y \bar{v} - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}.$$

By the triangle inequality again, it is easily seen that

$$\inf_{\mathbf{q}\in\mathbb{R}^n}\left(\fint_{B_R} |\nabla_y \bar{v} - \mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0} \le C \inf_{\theta\in\mathbb{R}}\left(\fint_{B_R^+} (|D_{y_1} v - \theta|^{\gamma_0} + |D_{y'} v|^{\gamma_0})\right)^{1/\gamma_0}.$$

Then (5.15) is a direct consequence of the three inequalities above. ∎

Lemma 5.4. *Suppose that $u_1 \in W^{1,p}(B_{R_1}^+)$ is a solution to (5.5). Then for any $\varepsilon \in (0,1)$ and $r \in (0, R_1/4]$, we have*

$$\begin{aligned}
&\inf_{\theta\in\mathbb{R}}\left(\fint_{B_{\varepsilon r}^+} (|D_{y_1} u_1 - \theta|^{\gamma_0} + |D_{y'} u_1|^{\gamma_0})\right)^{1/\gamma_0} \\
&\quad \le C\varepsilon^\alpha \inf_{\theta\in\mathbb{R}}\left(\fint_{B_r^+} (|D_{y_1} u_1 - \theta|^{\gamma_0} + |D_{y'} u_1|^{\gamma_0})\right)^{1/\gamma_0} \\
&\qquad + C_\varepsilon \left(\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}}\right)^{\frac{1}{p-1}} + C_\varepsilon \omega_1(r)\left(\fint_{B_{2r}^+} (|\nabla_y u_1| + s)^{2-p}\right)^{\frac{1}{2-p}} \\
&\qquad + C_\varepsilon \frac{|\mu_1|(B_{2r}^+)}{r^{n-1}} \fint_{B_{2r}^+} (|\nabla_y u_1| + s)^{2-p},
\end{aligned} \tag{5.16}$$

where α and γ_0 are the same constants as in Proposition 3.3, C_ε is a constant depending on ε, n, p, λ, and γ_0, and C is a constant depending on n, p, λ, and γ_0.

Proof. By using Lemmas 5.1, 5.2, and 5.3, the proof is almost identical to that of Proposition 3.3, so we omit it. ∎

We now define

$$\psi(x_0, r) = \inf_{\theta\in\mathbb{R}}\left(\fint_{\Omega_r(x_0)} (|D_1 u - \theta|^{\gamma_0} + |D_{x'} u|^{\gamma_0})\right)^{1/\gamma_0}.$$

Let $\varepsilon \in (0,1)$ and $r \in (0, R_1/4]$. By using change of variables, (5.3), (5.4), and the triangle inequality, we have

$$\begin{aligned}
&\inf_{\theta\in\mathbb{R}}\left(\fint_{B^+_{\varepsilon r}}(|D_{y_1}u_1-\theta|^{\gamma_0}+|D_{y'}u_1|^{\gamma_0})\right)^{1/\gamma_0}\\
&=\inf_{\theta\in\mathbb{R}}\left(\fint_{\Gamma(B^+_{\varepsilon r})}(|D_1u-\theta|^{\gamma_0}+|D_1u\,D_{x'}\chi+D_{x'}u|^{\gamma_0})\right)^{1/\gamma_0}\\
&\geq C\inf_{\theta\in\mathbb{R}}\left(\fint_{\Omega_{\varepsilon r/2}}(|D_1u-\theta|^{\gamma_0}+|D_{x'}u|^{\gamma_0})\right)^{1/\gamma_0}-C'\left(\fint_{\Omega_{\varepsilon r/2}}|D_1u\,D_{x'}\chi|^{\gamma_0}\right)^{1/\gamma_0}\\
&\geq C\psi(0,\varepsilon r/2)-C'\omega_1(\varepsilon r/2)\left(\fint_{\Omega_{\varepsilon r/2}}|\nabla u|^{\gamma_0}\right)^{1/\gamma_0}, \qquad (5.17)
\end{aligned}$$

where C and C' are positive constants depending only on n and γ_0. Similarly,

$$\begin{aligned}
&\inf_{\theta\in\mathbb{R}}\left(\fint_{B^+_{r}}(|D_{y_1}u_1-\theta|^{\gamma_0}+|D_{y'}u_1|^{\gamma_0})\right)^{1/\gamma_0}\\
&=\inf_{\theta\in\mathbb{R}}\left(\fint_{\Gamma(B^+_{r})}(|D_1u-\theta|^{\gamma_0}+|D_1u\,D_{x'}\chi+D_{x'}u|^{\gamma_0})\right)^{1/\gamma_0}\\
&\leq C''\inf_{\theta\in\mathbb{R}}\left(\fint_{\Omega_{2r}}(|D_1u-\theta|^{\gamma_0}+|D_{x'}u|^{\gamma_0})\right)^{1/\gamma_0}+C''\left(\fint_{\Omega_{2r}}|D_1u\,D_{x'}\chi|^{\gamma_0}\right)^{1/\gamma_0}\\
&\leq C''\psi(0,2r)+C''\omega_1(2r)\left(\fint_{\Omega_{2r}}|\nabla u|^{\gamma_0}\right)^{1/\gamma_0}, \qquad (5.18)
\end{aligned}$$

where C'' is a positive constant depending only on n and γ_0. Therefore, by using (5.17), (5.18), (5.2), and (5.4), (5.16) implies that

$$\begin{aligned}
\psi(0,\varepsilon r/2) &\leq C\varepsilon^{\alpha}\psi(0,2r)+C_\varepsilon\left(\frac{|\mu|(\Omega_{4r})}{r^{n-1}}\right)^{\frac{1}{p-1}}\\
&\quad+C_\varepsilon\omega_1(2r)\left(\fint_{\Omega_{4r}}(|\nabla u|+s)^{2-p}\right)^{\frac{1}{2-p}}+C_\varepsilon\frac{|\mu|(\Omega_{4r})}{r^{n-1}}\fint_{\Omega_{4r}}(|\nabla u|+s)^{2-p}.
\end{aligned}$$

By replacing $\varepsilon/4$ and $2r$ with ε and r respectively, we obtain

Corollary 5.5. *Suppose that $u \in W^{1,p}_0(\Omega)$ is a solution to (5.1) and $x_0 \in \partial\Omega$. Then for $\varepsilon \in (0, 1/4)$, $r \leq R_1/2$, and α, C, C_ε as above, we have*

$$\begin{aligned}
\psi(x_0,\varepsilon r) &\leq C\varepsilon^{\alpha}\psi(x_0,r)+C_\varepsilon\left(\frac{|\mu|(\Omega_{2r}(x_0))}{r^{n-1}}\right)^{\frac{1}{p-1}}\\
&\quad+C_\varepsilon\omega_1(r)\left(\fint_{\Omega_{2r}(x_0)}(|\nabla u|+s)^{2-p}\right)^{\frac{1}{2-p}}+C_\varepsilon\frac{|\mu|(\Omega_{2r}(x_0))}{r^{n-1}}\fint_{\Omega_{2r}(x_0)}(|\nabla u|+s)^{2-p}.
\end{aligned} \qquad (5.19)$$

As in Section 3, for any $x \in \bar{\Omega}$, we define

$$\phi(x,\rho) = \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{\Omega_\rho(x)} |\nabla u - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}$$

and choose $\mathbf{q}_{x,r} \in \mathbb{R}^n$ such that

$$\left(\fint_{\Omega_r(x)} |\nabla u - \mathbf{q}_{x,r}|^{\gamma_0} \right)^{1/\gamma_0} = \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{\Omega_r(x)} |\nabla u - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}. \tag{5.20}$$

We remark that our definition of ϕ is invariant under any orthogonal change of coordinates as in Definition 1.9 and that (3.12) still holds for any Lebesgue point $x \in \Omega$ of the vector-valued function ∇u from the same argument as in Section 3. Moreover, if we assume $u \in C^1(\bar{\Omega})$, then (3.12) actually holds for any $x \in \bar{\Omega}$.

5.1. Global pointwise gradient estimates

To prove the pointwise gradient estimate for $p \in (\frac{3n-2}{2n-1}, 2)$, we choose $\gamma_0 = 2 - p$ and $\varepsilon = \varepsilon(n, p, \lambda, \alpha) \in (0, 1/4)$ sufficiently small such that $C\varepsilon^\alpha \le 1/4$ for both constants C in (3.13) and (5.19). Fix $x_0 \in \partial\Omega$ and $R \le R_1/2$. For $j \ge 0$, set $r_j = \varepsilon^j R$, $\Omega_j = \Omega_{2r_j}(x_0)$,

$$T_j = \left(\fint_{\Omega_j} (|\nabla u| + s)^{2-p}\, dx \right)^{\frac{1}{2-p}}, \quad \phi_j = \phi(x_0, r_j), \quad \text{and} \quad \psi_j = \psi(x_0, r_j).$$

Applying (5.19) yields

$$\psi_{j+1} \le \frac{1}{4}\psi_j + C\left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C\frac{|\mu|(\Omega_j)}{r_j^{n-1}} T_j^{2-p} + C\omega_1(r_j)T_j.$$

Let j_0 and m be positive integers such that $j_0 \le m$. Summing the above inequality over $j \in \{j_0, j_0+1, \ldots, m\}$ and noting that $\phi_j \le \psi_j \le CT_j$, we obtain

$$\begin{aligned} \sum_{j=j_0}^{m+1} \phi_j \le \sum_{j=j_0}^{m+1} \psi_j \le CT_{j_0} + C \sum_{j=j_0}^{m} \left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\ + C \sum_{j=j_0}^{m} \frac{|\mu|(\Omega_j)}{r_j^{n-1}} T_j^{2-p} + C \sum_{j=j_0}^{m} \omega_1(r_j)T_j \end{aligned} \tag{5.21}$$

for any $x_0 \in \partial\Omega$ and $R \le R_1/2$.

On the other hand, according to (3.17),

$$\begin{aligned} \sum_{j=j_0}^{m+1} \phi_j \le C\phi_{j_0} + C \sum_{j=j_0}^{m} \left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\ + C \sum_{j=j_0}^{m} \frac{|\mu|(\Omega_j)}{r_j^{n-1}} T_j^{2-p} + C \sum_{j=j_0}^{m} \omega(r_j)T_j \end{aligned} \tag{5.22}$$

for any $x_0 \in \Omega$ and $R > 0$ such that $r_{j_0} = \varepsilon^{j_0} R < \operatorname{dist}(x_0, \partial\Omega)/2$.

We now define

$$\Omega'_j = \Omega_{8r_j}(x_0), \quad Z_j = \left(\fint_{\Omega'_j} (|\nabla u| + s)^{2-p} \, dx \right)^{\frac{1}{2-p}}.$$

Then we can obtain the following lemma.

Lemma 5.6. *Suppose that $u \in W_0^{1,p}(\Omega)$ is a solution to (5.1), $x_0 \in \bar{\Omega}$, and $R \le R_1/6$. Then*

$$\sum_{j=j_0}^{m+1} \phi_j \le C Z_{j_0} + C \sum_{j=j_0}^{m} \left(\frac{|\mu|(\Omega'_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C \sum_{j=j_0}^{m} \frac{|\mu|(\Omega'_j)}{r_j^{n-1}} Z_j^{2-p} + C \sum_{j=j_0}^{m+1} \omega_1(r_j) Z_j, \tag{5.23}$$

where C is a constant depending only on n, p, λ, and γ_0.

Proof. First when $x_0 \in \partial\Omega$, since $\Omega_j \subset \Omega'_j$, we have $T_j \le C Z_j$. Thus (5.21) directly implies (5.23). It remains to prove the lemma for $x_0 \in \Omega$. Since (5.22) holds when $r_{j_0} < \operatorname{dist}(x_0, \partial\Omega)/2$, we only need to show that (5.23) holds when $r_{j_0} \ge \operatorname{dist}(x_0, \partial\Omega)/2$. Now assume $r_{j_1} \ge \operatorname{dist}(x_0, \partial\Omega)/2$ and $r_{j_1+1} < \operatorname{dist}(x_0, \partial\Omega)/2$. By (5.22), we have

$$\sum_{j=j_1+1}^{m+1} \phi_j \le C\phi_{j_1+1} + C \sum_{j=j_1+1}^{m} \left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C \sum_{j=j_1+1}^{m} \frac{|\mu|(\Omega_j)}{r_j^{n-1}} T_j^{2-p} + C \sum_{j=j_1+1}^{m} \omega(r_j) T_j. \tag{5.24}$$

By (5.4), we also have

$$\phi_{j_1+1} \le \left(\fint_{\Omega_{r_{j_1+1}}(x_0)} |\nabla u - \mathbf{q}_{x_0, r_{j_1}}|^{\gamma_0} \right)^{1/\gamma_0} \le C\phi_{j_1}. \tag{5.25}$$

Now for any $j \in \{j_0, j_0+1, \ldots, j_1\}$, $r_j \ge \operatorname{dist}(x_0, \partial\Omega)/2$. Choose $y_0 \in \partial\Omega$ such that $d := \operatorname{dist}(x_0, \partial\Omega) = |y_0 - x_0|$, so that $\Omega_{r_j}(x_0) \subset \Omega_{3r_j}(y_0)$ and $\Omega_{6r_j}(y_0) \subset \Omega_{8r_j}(x_0)$. Thus by using (5.21) at $y_0 \in \partial\Omega$, we have

$$\begin{aligned}
\sum_{j=j_0}^{j_1} \phi_j &\le C \sum_{j=j_0}^{j_1} \phi(y_0, 3r_j) \\
&\le C Y_{j_0} + C \sum_{j=j_0}^{j_1} \left(\frac{|\mu|(\Omega_{6r_j}(y_0))}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\
&\quad + C \sum_{j=j_0}^{j_1} \frac{|\mu|(\Omega_{6r_j}(y_0))}{r_j^{n-1}} Y_j^{2-p} + C \sum_{j=j_0}^{j_1+1} \omega_1(3r_j) Y_j \\
&\le C Z_{j_0} + C \sum_{j=j_0}^{j_1} \left(\frac{|\mu|(\Omega'_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C \sum_{j=j_0}^{j_1} \frac{|\mu|(\Omega'_j)}{r_j^{n-1}} Z_j^{2-p} + C \sum_{j=j_0}^{j_1+1} \omega_1(r_j) Z_j,
\end{aligned} \tag{5.26}$$

where

$$Y_j := \left(\fint_{\Omega_{6r_j}(y_0)} (|\nabla u| + s)^{2-p} \, dx \right)^{\frac{1}{2-p}}.$$

Recall that $\omega_1 = \omega + \omega_0$. Combining (5.24)–(5.26), we obtain (5.23). ■

Proof of Theorem 1.10. With (5.23) in place of (3.17), we can easily get the global pointwise gradient estimate (1.18) using the same ideas as in the proof of Theorem 1.1. ■

5.2. *Global Lipschitz estimates and modulus of continuity estimates of the gradient*

Let $x_0 \in \overline{\Omega}$ and $0 < R \le R_1$. For any fixed $\alpha_1 \in (0, \alpha)$, let $\alpha_2 = (\alpha_1 + \alpha)/2$, and choose $\varepsilon = \varepsilon(n, p, \lambda, \gamma_0, \alpha, \alpha_1) \in (0, 1/4)$ sufficiently small such that $\varepsilon^{\alpha_2} < 1/4$ and $C\varepsilon^{\alpha - \alpha_2} < 1$ for both constants C in (3.13) and (5.19). Next we define

$$g_1(x, r) = \frac{|\mu|(B_r(x) \cap B_{R/2}(x_0))}{r^{n-1}}, \quad h_1(x, r) = g_1(x, r)^{\frac{1}{p-1}},$$
$$\omega_1^*(r) = \omega_1(r)[r \le R/2] + \omega_1(R/2)[r > R/2],$$

and

$$\hat{g}_1(x, t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_2 i} g_1(x, \varepsilon^{-i} t), \quad \breve{g}_1(x, t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} g_1(x, \varepsilon^{-i} t),$$
$$\hat{h}_1(x, t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_2 i} h_1(x, \varepsilon^{-i} t), \quad \breve{h}_1(x, t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} h_1(x, \varepsilon^{-i} t),$$
$$\hat{\omega}_1(t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_2 i} \omega_1^*(\varepsilon^{-i} t), \qquad \breve{\omega}_1(t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \omega_1^*(\varepsilon^{-i} t),$$

Indeed, we have

$$\begin{aligned}
\breve{\omega}_1(t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \big(\omega_1(\varepsilon^{-i} t)[\varepsilon^{-i} t \le R/2] + \omega_1(R/2)[\varepsilon^{-i} t > R/2] \big) =: \tilde{\omega}_1(t), \\
\breve{g}_1(x, t) &\le \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \big(g(x, \varepsilon^{-i} t)[\varepsilon^{-i} t \le R/2] + g(x_0, R/2)[\varepsilon^{-i} t > R/2] \big), \\
\breve{h}_1(x, t) &\le \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \big(h(x, \varepsilon^{-i} t)[\varepsilon^{-i} t \le R/2] + h(x_0, R/2)[\varepsilon^{-i} t > R/2] \big),
\end{aligned} \tag{5.27}$$

where the functions g and h are defined in (4.2).

Using the same iteration technique as in Lemma 4.1, we can deduce from (5.19) that

$$\begin{aligned}
\psi(x, \rho) \le C \left(\frac{\rho}{r} \right)^{\alpha_2} \psi(x, r) &+ C \hat{h}_1(x, 2\rho) \\
&+ C \hat{g}_1(x, 2\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))} + s)^{2-p} \\
&+ C \hat{\omega}_1(\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))} + s)
\end{aligned} \tag{5.28}$$

for any $x \in \partial\Omega$, $B_{2r}(x) \subset B_{R/2}(x_0)$, and $0 < \rho \le r$.

Similarly, from (4.1) and the fact that $\omega \le \omega_1$, we have

$$\begin{aligned}\phi(x,\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_2}\phi(x,r) + C\hat{h}_1(x,2\rho) \\ + C\hat{g}_1(x,2\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))}+s)^{2-p} \\ + C\hat{\omega}_1(\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))}+s)\end{aligned} \tag{5.29}$$

for any $B_{2r}(x) \subset\subset \Omega$, $B_{2r}(x) \subset B_{R/2}(x_0)$, and $0 < \rho \le r$.

By combining (5.28) and (5.29), we will show the following estimates.

Lemma 5.7. *Let $x \in \overline{\Omega}$ and $B_{2r}(x) \subset B_{R/2}(x_0)$. There exists a constant C, depending only on ε, n, p, λ, γ_0, and α_1, such that, for any $0 < \rho \le r \le R_1$, we have*

$$\begin{aligned}\phi(x,\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_2} r^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(\Omega_r(x))} + C\check{h}_1(x,\rho) \\ + C\check{g}_1(x,\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))}+s)^{2-p} \\ + C\check{\omega}_1(\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))}+s),\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_2} r^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(\Omega_r(x))} + C\int_0^\rho \frac{\check{h}_1(x,t)}{t}\,dt \\ + C(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))}+s)^{2-p}\int_0^\rho \frac{\check{g}_1(x,t)}{t}\,dt \\ + C(\|\nabla u\|_{L^\infty(\Omega_{2r}(x))}+s)\int_0^\rho \frac{\check{\omega}_1(t)}{t}\,dt.\end{aligned} \tag{5.31}$$

To prove Lemma 5.7, we also need the following technical lemma.

Lemma 5.8. *Let $x, y \in \overline{\Omega}$ and $p \in (1,2)$. Then for $g_1, h_1, \hat{g}_1, \hat{h}_1, \check{g}_1, \check{h}_1, \hat{\omega}_1, \check{\omega}_1$ defined as above, we have the following:*

(i) *There exist constants $C_1, C_2 > 0$, depending on ε, n, p, α and α_1, such that for any fixed $x \in \overline{\Omega}$, and any $f \in \{\hat{g}_1(x,\cdot), \hat{h}_1(x,\cdot), \hat{\omega}_1, \check{g}_1(x,\cdot), \check{h}_1(x,\cdot), \check{\omega}_1\}$, we have*

$$C_1 f(t) \le f(s) \le C_2 f(t) \quad \text{whenever} \quad 0 < \varepsilon t \le s \le t.$$

(ii) *There exists a constant $C > 0$, depending on ε, n, p, α and α_1, such that for any $0 < \varepsilon r \le \rho \le r$ with $\Omega_\rho(x) \subset \Omega_r(y)$, and any $F \in \{\hat{g}_1, \hat{h}_1, \check{g}_1, \check{h}_1\}$, we have*

$$F(x,\rho) \le CF(y,r).$$

(iii) *For any $0 < \rho \le r$, there exists a constant $C > 0$, depending on ε, n, p, α and α_1, such that*

$$\left(\frac{\rho}{r}\right)^{\alpha_2}\hat{\omega}_1(r) \le C\check{\omega}_1(\rho),$$

$$\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{g}_1(x,r) \le C\breve{g}_1(x,\rho),$$
$$\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{h}_1(x,r) \le C\breve{h}_1(x,\rho).$$

Proof. We will only give the proof for g since the other cases are similar. Noting that

$$g_1(x,s) \le \varepsilon^{1-n} g_1(x,t), \quad g_1(x,\varepsilon t) \le \varepsilon^{1-n} g_1(x,s)$$

whenever $\varepsilon t \le s \le t$ and $\hat{g}_1(x,t) \le \varepsilon^{-\alpha_2}\hat{g}_1(x,\varepsilon t)$, assertion (i) follows. Assertion (ii) follows similarly by observing that $\Omega_{\varepsilon^{-i}\rho}(x) \subset \Omega_{\varepsilon^{-i}r}(y)$ when $i \ge 0$ since $\Omega_\rho(x) \subset \Omega_r(y)$. It remains to prove assertion (iii). Since $0 < \rho \le r$, there exists an integer $j \ge 0$ such that $\varepsilon^{-j}\rho \le r < \varepsilon^{-j-1}\rho$. Therefore, by part (i),

$$\begin{aligned}\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{g}_1(x,r) &\le C\varepsilon^{\alpha_2 j}\hat{g}_1(x,\varepsilon^{-j}\rho)\\ &\le C\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}\varepsilon^{\alpha_2(i+j)} g_1(x,\varepsilon^{-i-j}\rho)\\ &= C\sum_{k=1}^{\infty} k\varepsilon^{\alpha_2 k} g_1(x,\varepsilon^{-k}\rho) \le C\breve{g}_1(x,\rho),\end{aligned}$$

where we have used the fact that $k\varepsilon^{\alpha_2 k} \le C\varepsilon^{\alpha_1 k}$ in the last inequality, since $\alpha_1 < \alpha_2$. ■

Now we are ready to prove Lemma 5.7.

Proof of Lemma 5.7. Without loss of generality, we may assume $x = 0$. Note that if $r/16 \le \rho \le r$, then (5.30) follows from the definition of ϕ. Hence we only need to consider the case when $0 < \rho < r/16$. We consider the following three cases:

$$r/4 \le \operatorname{dist}(0,\partial\Omega), \quad \operatorname{dist}(0,\partial\Omega) \le 4\rho, \quad 4\rho < \operatorname{dist}(0,\partial\Omega) < r/4.$$

Case 1: $r/4 \le \operatorname{dist}(0,\partial\Omega)$. Set $r_1 = r/16$. Since $B_{4r_1} \subset \Omega$, from (5.29) we have

$$\begin{aligned}\phi(0,\rho) \le C\left(\frac{\rho}{r_1}\right)^{\alpha_2}\phi(0,r_1) + C\hat{h}_1(0,2\rho)&\\ + C\hat{g}_1(0,2\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r_1})} + s)^{2-p} + C\hat{\omega}_1(\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r_1})} + s).&\end{aligned}$$

Thus we can easily get (5.30) from Lemma 5.8 and the fact that

$$\phi(0,r_1) \le Cr^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(\Omega_r)}.$$

Case 2: $\operatorname{dist}(0,\partial\Omega) \le 4\rho$. Choose $y_0 \in \partial\Omega$ such that $\operatorname{dist}(0,\partial\Omega) = |y_0|$. Then $B_{10\rho}(y_0) \subset B_{14\rho} \subset B_r$, and from (5.28) we have

$$\begin{aligned}\phi(0,\rho) &\le C\psi(y_0,5\rho)\\ &\le C\left(\frac{\rho}{r}\right)^{\alpha_2}\psi(y_0,r/2) + C\hat{h}_1(y_0,10\rho)\\ &\quad + C\hat{g}_1(y_0,10\rho)(\|\nabla u\|_{L^\infty(\Omega_r(y_0))} + s)^{2-p} + C\hat{\omega}_1(5\rho)(\|\nabla u\|_{L^\infty(\Omega_r(y_0))} + s).\end{aligned}$$

Thus from the fact that $\Omega_r(y_0) \subset \Omega_{2r}$, $\Omega_{r/2}(y_0) \subset \Omega_r$, and $\Omega_{10\rho}(y_0) \subset \Omega_{14\rho}$, we get (5.30) by using Lemma 5.8.

Case 3: $4\rho < \operatorname{dist}(0, \partial\Omega) < r/4$. Set $r_1 = \operatorname{dist}(0, \partial\Omega)/4 > \rho$. Using (5.29), we obtain

$$\begin{aligned}\phi(0,\rho) \le C\left(\frac{\rho}{r_1}\right)^{\alpha_2}\phi(0,r_1) + C\hat{h}_1(0,2\rho) \\ + C\hat{g}_1(0,2\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r_1})} + s)^{2-p} + C\hat{\omega}_1(\rho)(\|\nabla u\|_{L^\infty(\Omega_{2r_1})} + s).\end{aligned}$$

On the other hand, choose $y_0 \in \partial\Omega$ such that $\operatorname{dist}(0, \partial\Omega) = |y_0|$. Then $B_{r_1} \subset B_{5r_1}(y_0)$, $B_r(y_0) \subset B_{2r}$ and from (5.28) we have

$$\begin{aligned}\phi(0,r_1) &\le C\psi(y_0, 5r_1) \\ &\le C\left(\frac{r_1}{r}\right)^{\alpha_2}\psi(y_0, r/2) + C\hat{h}_1(y_0, 10r_1) \\ &\quad + C\hat{g}_1(y_0, 10r_1)(\|\nabla u\|_{L^\infty(\Omega_r(y_0))} + s)^{2-p} + C\hat{\omega}_1(5r_1)(\|\nabla u\|_{L^\infty(\Omega_r(y_0))} + s).\end{aligned}$$

Noting that $\Omega_r(y_0) \subset \Omega_{2r}$, $\Omega_{r/2}(y_0) \subset \Omega_r$, and $\Omega_{10r_1}(y_0) \subset \Omega_{14r_1}$, we get (5.30) by combining the last two estimates and applying Lemma 5.8.

Finally, replacing ρ with $\varepsilon^j\rho$ and summing in j, we get (5.31) by using Lemma 5.8 and the comparison principle of Riemann integrals. ■

Remark 5.9. We emphasize that Lemma 5.7 has a local nature. Indeed, it can be seen from the proof that we only need the Dirichlet boundary condition $u = 0$ on $\partial\Omega \cap B_{R/2}(x_0)$ and $C^{1,\mathrm{Dini}}$ regularity of $\partial\Omega \cap B_{R/2}(x_0)$ for these estimates to hold. Therefore, our Lipschitz estimates and modulus of continuity estimates, which will be deduced from Lemma 5.7, also have a local nature.

Recall the definition of $\mathbf{q}_{x,r}$ from (5.20) and keep (5.4) in mind. By following almost the same proof of (4.7), we find that for any Lebesgue point $x \in \Omega$ of the vector-valued function ∇u and $\rho \in (0, R_1]$,

$$|\nabla u(x) - \mathbf{q}_{x,\rho}| \le C\sum_{j=0}^{\infty}\phi(x, \varepsilon^j\rho), \tag{5.32}$$

where C is a constant depending only on n and γ_0.

Proof of Theorem 1.11. We will prove a boundary Lipschitz estimate

$$\|\nabla u\|_{L^\infty(\Omega_{R/16}(x_0))} \le C\|\mathbf{I}_1^R(|\mu|)\|_{L^\infty(\Omega_R(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\big\||\nabla u| + s\big\|_{L^{2-p}(\Omega_R(x_0))} \tag{5.33}$$

for any $x_0 \in \partial\Omega$ and $R \le R_1$, assuming that

$$\|\mathbf{I}_1^R(|\mu|)\|_{L^\infty(B_R(x_0))} < \infty. \tag{5.34}$$

Then (1.19) follows by a standard covering argument using (1.13) and (5.33). The proof of (5.33) is similar to that of Theorem 1.3 so we will only focus on the differences.

Step 1: The case when $u \in C^1(\overline{\Omega_{R/2}(x_0)})$. With (5.31) in place of (4.6), using the same iteration technique as in the proof of Theorem 1.3, we get the following estimate:

$$\|\nabla u\|_{L^\infty(\Omega_{R/4}(x_0))} + s \le CR^{-n/\gamma_0}\|\nabla u\|_{L^{\gamma_0}(\Omega_{R/2}(x_0))} + C \sup_{x\in\Omega_{R/2}(x_0)} \int_0^{R/2} \frac{\breve{h}_1(x,t)}{t}\,dt + C \sup_{x\in\Omega_{R/2}(x_0)} \Big(\int_0^{R/2} \frac{\breve{g}_1(x,t)}{t}\,dt\Big)^{\frac{1}{p-1}} + Cs. \tag{5.35}$$

Using (5.27) and direct computations, we have

$$\int_0^{R/2} \frac{\breve{g}_1(x,t)}{t}\,dt \le C\mathbf{I}_1^R(|\mu|)(x) + C\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}},$$
$$\int_0^{R/2} \frac{\tilde{h}_1(x,t)}{t}\,dt \le C(\mathbf{I}_1^R(|\mu|)(x))^{\frac{1}{p-1}} + C\Big(\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}}\Big)^{\frac{1}{p-1}}.$$

Therefore, from (5.35) and the fact that $\gamma_0 \le 2-p$ (cf. Lemma 5.2), we obtain

$$\|\nabla u\|_{L^\infty(\Omega_{R/4}(x_0))} \le C\|\mathbf{I}_1^R(|\mu|)\|_{L^\infty(\Omega_R(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\big\||\nabla u| + s\big\|_{L^{2-p}(\Omega_R(x_0))}. \tag{5.36}$$

Step 2: The general case. We use an approximation argument with the aid of the regularized distance introduced by Lieberman [15]. Here we refer to a modified version in [7]. Let $d(\cdot)$ be the regularized distance defined in [7, Lemma 5.1] ($\psi(\cdot)$ in that paper) and $\Omega^k = \{x \in \Omega : d(x) > 1/k\}$. Then from [7, Lemma 5.1], we know that Ω^k has a smooth boundary and the $C^{1,\text{Dini}}$-properties of $\partial\Omega_k$ are the same as those of $\partial\Omega$ up to some constant independent of k. We take a sequence $\{\varphi_k\}$ of standard mollifiers and mollify μ and a by setting

$$\mu_k(x) = (\mu * \varphi_k)(x), \quad x \in \Omega; \qquad a^k(x) = (a * \varphi_k)(x), \quad x \in \Omega^k.$$

We know that $\mu \in W^{-1,p'}(\Omega_R(x_0))$ and therefore

$$\|\mu_k - \mu\|_{W^{-1,p'}(\Omega_R(x_0))} \to 0.$$

Recalling that we have a $C^{1,\text{Dini}}$ coordinate in $\Omega_R(x_0)$ since $R \le R_1$, we can take a sequence of cut-off functions $\zeta_k \in C^\infty(\mathbb{R}^n)$ satisfying $\zeta_k = 1$ in $\Omega^{k/4} \cap B_R(x_0)$, $\zeta_k = 0$ in $(\Omega \setminus \Omega^{k/2}) \cap B_R(x_0)$, and $\|\nabla\zeta_k\|_{L^\infty} \le 16k$.

Next we let $u_k \in u\zeta_k + W_0^{1,p}(\Omega^k \cap B_R(x_0))$ be the unique solution to

$$\begin{cases} -\operatorname{div}\big(a^k(x)(|\nabla u_k|^2 + s^2)^{\frac{p-2}{2}}\nabla u_k\big) = \mu_k & \text{in } \Omega^k \cap B_R(x_0), \\ u_k = u\zeta_k & \text{on } \partial(\Omega^k \cap B_R(x_0)). \end{cases} \tag{5.37}$$

Since $u_k = u\zeta_k = 0$ on $B_R(x_0) \cap \partial\Omega^k$, we can always assume $u_k \in u\zeta_k + W_0^{1,p}(\Omega_R(x_0))$ by taking the zero extension of u_k in $(\Omega \setminus \Omega^k) \cap B_R(x_0)$. Since $u = 0$ on $B_R(x_0) \cap \partial\Omega$, by Hardy's inequality we have

$$\begin{aligned}\|u\nabla\zeta_k\|_{L^p(\Omega_R(x_0))} &\le 16\|ku\|_{L^p((\Omega\setminus\Omega^{k/4})\cap B_R(x_0))}\\ &\le C\|u/d\|_{L^p((\Omega\setminus\Omega^{k/4})\cap B_R(x_0))} \le C\|\nabla u\|_{L^p((\Omega\setminus\Omega^{k/4})\cap B_R(x_0))} \to 0\end{aligned}$$

as $k \to \infty$. Therefore, we know that

$$\|u - u\zeta_k\|_{W^{1,p}(\Omega_R(x_0))} \to 0. \tag{5.38}$$

Thus by choosing $u_k - u\zeta_k$ as a test function in (5.37), following the proof of (4.18), and using (5.38), we can show that $\|\nabla u_k\|_{L^p(\Omega_R(x_0))}$ is uniformly bounded in k. Also, choosing $(u_k - u\zeta_k)1_{\Omega_R(x_0)}$ as a test function in (5.1) and (5.37), similarly we obtain

$$\int_{\Omega_R(x_0)} |V(\nabla u_k) - V(\nabla u)|^2\,dx \to 0 \quad \text{as } k \to \infty,$$

which again implies

$$\nabla u_k \to \nabla u \quad \text{strongly in } L^p(\Omega_R(x_0)).$$

By the classical boundary regularity theory (see, for instance, [16]), we have

$$u_k \in C^1(\overline{\Omega^k \cap B_{R/2}(x_0)}).$$

Note that for sufficiently large k, there exists $x_k \in B_R(x_0) \cap \partial\Omega^k$ such that $|x_k - x_0| \le R/16$. Thus $B_{R/16}(x_0) \subset B_{R/8}(x_k)$, $B_{R/4}(x_k) \subset B_{R/2}(x_0)$ and $B_{R/2}(x_k) \subset B_{3R/4}(x_0)$. Therefore, using (5.36) in Step 1 and Remark 5.9, we get

$$\begin{aligned}\|\nabla u_k\|_{L^\infty(\Omega^k\cap B_{R/16}(x_0))} &\le \|\nabla u_k\|_{L^\infty(\Omega^k\cap B_{R/8}(x_k))}\\ &\le C\|\mathbf{I}_1^{R/2}(|\mu_k|)\|_{L^\infty(\Omega^k\cap B_{R/2}(x_k))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\big\||\nabla u_k| + s\big\|_{L^{2-p}(\Omega^k\cap B_{R/2}(x_k))}\\ &\le C\|\mathbf{I}_1^{R/2}(|\mu_k|)\|_{L^\infty(\Omega^k\cap B_{3R/4}(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\big\||\nabla u_k| + s\big\|_{L^{2-p}(\Omega^k\cap B_{3R/4}(x_0))}.\end{aligned}$$

By extracting a subsequence and taking the limit as $k \to \infty$, we obtain (5.33). ■

Proof of Corollary 1.12. By testing (5.1) with u, following the proof of (4.18) we obtain

$$\|\nabla u\|_{L^p(\Omega)} \le C\|\mu\|_{W^{-1,p'}(\mathbb{R}^n)}^{\frac{1}{p-1}} + Cs,$$

where $p' = p/(p-1)$. From [12, Theorem 1], we also have

$$\|\mu\|_{W^{-1,p'}(\mathbb{R}^n)} \le C\left(\int_{\mathbb{R}^n} \mathbf{W}_{1,p}^1(|\mu|)\,d|\mu|\right)^{\frac{p-1}{p}} \le C\|\mathbf{I}_1^1(|\mu|)\|_{L^\infty(\Omega)}.$$

Therefore, Corollary 1.12 follows by combining (1.19), Hölder's inequality, and the last two inequalities. ■

Now we turn to global modulus of continuity estimates of the gradient. Recall that we fixed an $\varepsilon \in (0, 1/4)$ sufficiently small such that

$$C\varepsilon^{\alpha-\alpha_2} < 1 \quad \text{and} \quad \varepsilon^{\alpha_2} < 1/4$$

for both constants C in (3.13) and (5.19), where $\alpha \in (0,1)$ is the same constant as in Theorem 2.1, $\alpha_1 \in (0, \alpha)$, and $\alpha_2 = (\alpha_1 + \alpha)/2$. We also took $R \in (0, R_1]$ and defined

$$\begin{aligned}
\tilde{\omega}_1(t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\omega_1(\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + \omega_1(R/2)[\varepsilon^{-i}t > R/2]\big),\\
\tilde{\mathbf{I}}_1^{\rho}(|\mu|)(x) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{I}_1^{R/2}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2]\big),\\
&\tilde{\mathbf{W}}^{\rho}_{1/p,p}(|\mu|)(x)\\
&= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\big(\mathbf{W}^{\varepsilon^{-i}\rho}_{1/p,p}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{W}^{R/2}_{1/p,p}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2]\big),
\end{aligned} \tag{5.39}$$

where $\mathbf{I}_1$ and $\mathbf{W}_{1/p,p}$ are the Riesz and Wolff potentials defined in (1.9) and (1.11), respectively.

Our global modulus of continuity estimate of the gradient is as follows.

Theorem 5.10. *Assume the conditions of Theorem 1.11 and $\alpha_1 \in (0, \alpha)$, where α is the constant in Theorem 2.1. Then there exist constants $R_1 = R_1(R_0, \omega_0) \in (0, R_0)$ and $C = C(n, p, \lambda, \alpha_1, \omega, R_0, \omega_0)$ such that for any $x_0 \in \overline{\Omega}$, any $R \in (0, R_1]$, and any $x, y \in \Omega_{R/4}(x_0)$ that are Lebesgue points of the vector-valued function ∇u, we have*

$$\begin{aligned}
&|\nabla u(x) - \nabla u(y)|\\
&\quad \le C\mathbf{M}_1\left[\left(\frac{\rho}{R}\right)^{\alpha_1} + \int_0^{\rho} \frac{\tilde{\omega}_1(t)}{t}\,dt\right] + C\|\tilde{\mathbf{W}}^{\rho}_{1/p,p}(|\mu|)\|_{L^\infty(\Omega_{R/4}(x_0))}\\
&\qquad + C\mathbf{M}_1^{2-p}\|\tilde{\mathbf{I}}_1^{\rho}(|\mu|)\|_{L^\infty(\Omega_{R/4}(x_0))},
\end{aligned} \tag{5.40}$$

where $\rho = |x - y|$, $\omega_1 = \omega + \omega_0$, $\tilde{\omega}_1$, $\tilde{\mathbf{W}}_{1/p,p}$, and $\tilde{\mathbf{I}}_1$ are defined in (5.39), and

$$\mathbf{M}_1 := R^{-\frac{n}{2-p}}\big\||\nabla u| + s\big\|_{L^{2-p}(\Omega_R(x_0))} + \|\mathbf{I}_1^R(|\mu|)\|^{\frac{1}{p-1}}_{L^\infty(\Omega_R(x_0))}.$$

Proof. For any $x, y \in \Omega_{R/4}(x_0)$ that are Lebesgue points of ∇u and $\rho > 0$,

$$\begin{aligned}
&|\nabla u(x) - \nabla u(y)|^{\gamma_0}\\
&\quad \le |\nabla u(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\nabla u(y) - \mathbf{q}_{y,2\rho}|^{\gamma_0} + |\mathbf{q}_{x,\rho} - \mathbf{q}_{y,2\rho}|^{\gamma_0}\\
&\quad \le |\nabla u(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\nabla u(y) - \mathbf{q}_{y,2\rho}|^{\gamma_0}\\
&\qquad + |\nabla u(z) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\nabla u(z) - \mathbf{q}_{y,2\rho}|^{\gamma_0}.
\end{aligned}$$

We set $\rho = |x - y|$, take the average over $z \in \Omega_\rho(x)$, and then take the γ_0-th root to get

$$
\begin{aligned}
&|\nabla u(x) - \nabla u(y)| \\
&\quad \le C|\nabla u(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + C|\nabla u(y) - \mathbf{q}_{y,2\rho}|^{\gamma_0} + C\phi(x,\rho) + C\phi(y,2\rho) \\
&\quad \le C\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) + C\sum_{j=0}^{\infty}\phi(y,2\varepsilon^j\rho) + C\phi(x,\rho) + C\phi(y,2\rho) \\
&\quad \le C\sup_{y_0\in\Omega_{R/4}(x_0)}\sum_{j=0}^{\infty}\phi(y_0,2\varepsilon^j\rho),
\end{aligned}
$$

where we have used the fact that $\Omega_\rho(x) \subset \Omega_{2\rho}(y)$ in the first inequality and (5.32) in the second inequality.

If $\rho < R/16$, by using (5.31) with $R/8$ in place of r and the fact that

$$
\Omega_{R/4}(y_0) \subset \Omega_{R/2}(x_0) \quad \forall y_0 \in \Omega_{R/4}(x_0),
$$

we obtain

$$
\begin{aligned}
&|\nabla u(x) - \nabla u(y)| \\
&\quad \le C\Big(\frac{\rho}{R}\Big)^{\alpha_2}\|\nabla u\|_{L^\infty(\Omega_{R/2}(x_0))} + C\sup_{y_0\in\Omega_{R/4}(x_0)}\int_0^\rho \frac{\breve{h}_1(y_0,t)}{t}\,dt \\
&\qquad + C(\|\nabla u\|_{L^\infty(\Omega_{R/2}(x_0))} + s)^{2-p}\sup_{y_0\in\Omega_{R/4}(x_0)}\int_0^\rho \frac{\breve{g}_1(y_0,t)}{t}\,dt \\
&\qquad + C(\|\nabla u\|_{L^\infty(\Omega_{R/2}(x_0))} + s)\int_0^\rho \frac{\breve{\omega}_1(t)}{t}\,dt.
\end{aligned} \tag{5.41}
$$

Clearly, (5.41) still holds when $\rho \ge R/16$. Using (5.27) and similar calculations to those in the proof of Theorem 4.3, for any $y_0 \in \Omega_{R/4}(x_0)$ and $\rho \in (0, R/2)$ we have

$$
\int_0^\rho \frac{\breve{g}_1(y_0,t)}{t}\,dt \le \tilde{\mathbf{I}}_1^\rho(|\mu|)(y_0) + C\Big(\frac{\rho}{R}\Big)^{\alpha_1}\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}}, \tag{5.42}
$$

$$
\int_0^\rho \frac{\breve{h}_1(y_0,t)}{t}\,dt \le \tilde{\mathbf{W}}_{1/p,p}^\rho(|\mu|)(y_0) + C\Big(\frac{\rho}{R}\Big)^{\alpha_1}\Big(\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}}\Big)^{\frac{1}{p-1}}. \tag{5.43}
$$

Using (1.19), (5.42), and (5.43), the estimate (5.41) implies (5.40). ∎

Acknowledgments. We would like to thank Quoc-Hung Nguyen for helpful comments and pointing out a gap in an earlier version of the manuscript. We also thank the referees for their careful reading and very useful suggestions.

Funding. H. Dong was partially supported by the Simons Foundation, grant # 709545, a Simons fellowship, grant no. 007638, and the NSF under agreement DMS-2055244. H. Zhu was partially supported by the NSF under agreement DMS-2055244.

References

[1] Breit, D., Cianchi, A., Diening, L., Schwarzacher, S.: Global Schauder estimates for the p-Laplace system. Arch. Ration. Mech. Anal. **243**, 201–255 (2022) Zbl 1481.35235 MR 4359452

[2] Choi, J., Dong, H.: Gradient estimates for Stokes systems in domains. Dynam. Partial Differential Equations **16**, 1–24 (2019) Zbl 1404.35162 MR 3890946

[3] DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. **7**, 827–850 (1983) Zbl 0539.35027 MR 709038

[4] DiBenedetto, E., Manfredi, J.: On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. Amer. J. Math. **115**, 1107–1134 (1993) Zbl 0805.35037 MR 1246185

[5] Dong, H.: Gradient estimates for parabolic and elliptic systems from linear laminates. Arch. Ration. Mech. Anal. **205**, 119–149 (2012) Zbl 1258.35040 MR 2927619

[6] Dong, H., Kim, S.: On C^1, C^2, and weak type-$(1,1)$ estimates for linear elliptic operators. Comm. Partial Differential Equations **42**, 417–435 (2017) Zbl 1373.35117 MR 3620893

[7] Dong, H., Lee, J., Kim, S.: On conormal and oblique derivative problem for elliptic equations with Dini mean oscillation coefficients. Indiana Univ. Math. J. **69**, 1815–1853 (2020) Zbl 1459.35119 MR 4170081

[8] Duzaar, F., Mingione, G.: Gradient continuity estimates. Calc. Var. Partial Differential Equations **39**, 379–418 (2010) Zbl 1204.35100 MR 2729305

[9] Duzaar, F., Mingione, G.: Gradient estimates via linear and nonlinear potentials. J. Funct. Anal. **259**, 2961–2998 (2010) Zbl 1200.35313 MR 2719282

[10] Duzaar, F., Mingione, G.: Gradient estimates via non-linear potentials. Amer. J. Math. **133**, 1093–1149 (2011) Zbl 1230.35028 MR 2823872

[11] Giusti, E.: Direct methods in the calculus of variations. World Scientific, River Edge, NJ (2003) Zbl 1028.49001 MR 1962933

[12] Hedberg, L. I., Wolff, T. H.: Thin sets in nonlinear potential theory. Ann. Inst. Fourier (Grenoble) **33**, 161–187 (1983) Zbl 0508.31008 MR 727526

[13] Kuusi, T., Mingione, G.: Linear potentials in nonlinear potential theory. Arch. Ration. Mech. Anal. **207**, 215–246 (2013) Zbl 1266.31011 MR 3004772

[14] Kuusi, T., Mingione, G.: A nonlinear Stein theorem. Calc. Var. Partial Differential Equations **51**, 45–86 (2014) Zbl 1316.35132 MR 3247381

[15] Lieberman, G. M.: Regularized distance and its applications. Pacific J. Math. **117**, 329–352 (1985) Zbl 0535.35028 MR 779924

[16] Lieberman, G. M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988) Zbl 0675.35042 MR 969499

[17] Lieberman, G. M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. Comm. Partial Differential Equations **16**, 311–361 (1991) Zbl 0742.35028 MR 1104103

[18] Lieberman, G. M.: Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures. Comm. Partial Differential Equations **18**, 1191–1212 (1993) Zbl 0802.35041 MR 1233190

[19] Mingione, G.: Gradient potential estimates. J. Eur. Math. Soc. **13**, 459–486 (2011) Zbl 1217.35077 MR 2746772

[20] Mingione, G.: Nonlinear measure data problems. Milan J. Math. **79**, 429–496 (2011) Zbl 1236.35068 MR 2862024

[21] Nguyen, Q.-H., Phuc, N. C.: Good-λ and Muckenhoupt–Wheeden type bounds in quasilinear measure datum problems, with applications. Math. Ann. **374**, 67–98 (2019) Zbl 1432.35089 MR 3961305

[22] Nguyen, Q.-H., Phuc, N. C.: Pointwise gradient estimates for a class of singular quasilinear equations with measure data. J. Funct. Anal. **278**, art. 108391, 35 pp. (2020) Zbl 1437.35123 MR 4046205

[23] Nguyen, Q.-H., Phuc, N. C.: Existence and regularity estimates for quasilinear equations with measure data: the case $1 < p \leq (3n-2)/(2n-1)$. Anal. PDE **15**, 1879–1895 (2022) Zbl 1512.35304 MR 4546498

[24] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations **51**, 126–150 (1984) Zbl 0488.35017 MR 727034