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0-1 Laws of a Probability Measure on a Locally Convex Space

By

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Abstract

We introduce several 0-1 laws for a cylindrical probability measure μ on a locally convex Hausdorff space E and examine the equivalence of them. We show that the following 0-1 laws are equivalent: (a) *for every* x'_n *in* E', $\mu(x; (x'_n(x)) \in c_0) = 0$ *or* 1, (b) *for every* x'_n *in* E', $\mu(x; (x'_n(x)) \in c_1) = 0$ *or* 1, *and* (c) *for every* x'_n *in* E'. $\mu(x; (x'_n(x)) \in l_\infty) = 0$ *or* 1. We also show that each of (a), (b) and (c) implies: (d) *for every* x'_n *in* E', $\mu(x; (x'_n(x)) \in l_p) = 0$ *or* 1. If μ is a Radon probability measure, then (a), (b) and (c) are equivalent to: (e) *for every lower semi-continuous semi-norm* N, $\mu(x; N(x) < \infty) = 0$ *or* 1.

§1. 0-1 Laws

In this section, we present several 0-1 laws which appear in the probability theory.

Let E be a locally convex Hausdorff space, $C(E, E')$ be the cylindrical σ-algebra generated by the topological dual E' and $\mathscr{B}(E)$ be the Borel σ-algebra generated by all open subsets. Let μ be a probability measure on $C(E, E')$ or on $\mathscr{B}(E)$. The measure μ on $\mathscr{B}(E)$ is called a Radon measure if it holds $\mu(A) = \sup\{\mu(K); K \subset A$ and K is compact$\}$ for every $A \in \mathscr{B}(E)$. The Radon measure μ is called a convex Radon measure if it satisfies that $1 = \sup\{\mu(K); K$ is compact and convex$\}$. If E is quasi-complete, then every Radon measure is convex Radon since the closed convex hull of each compact subset is again compact.

The weakest notion of the 0-1 law is the following.

(0) *For every* $x' \in E'$, $\mu(x; x'(x) = 0) = 0$ *or* 1.

The strongest 0-1 law is the following.

(1) *For every* μ*-measurable linear subspace* $F \subset E$, $\mu(F) = 0$ *or* 1.

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For example, convex measures, semi-stable measures satisfy the 0-1 law (1), see Borell [1], Dudley and Kanter [2], Krakowiak [4] and Louie, Rajput and Tortrat [5].

We introduce other intermediate 0-1 laws between (0) and (1). Let $l_\infty=\{(t_n)\in \boldsymbol{R}^\infty;\ \sup_n|t_n|<\infty\}$, $l_p=\{(t_n)\in\boldsymbol{R}^\infty;\ \sum_n|t_n|^p<\infty\}$ $(1\leqq p<\infty)$, $c_1=\{(t_n)\in\boldsymbol{R}^\infty;\ \lim_n t_n$ exists$\}$ and $c_0=\{(t_n)\in\boldsymbol{R}^\infty;\ t_n\to 0\}$ be the usual Banach spaces, where $\boldsymbol{R}^\infty$ be the countable product of the real numbers $\boldsymbol{R}$ with the product topology. A linear subspace F of $\boldsymbol{R}^\infty$ is called a convex Lusin subspace if F is a Borel subset of $\boldsymbol{R}^\infty$ and, for every probability measure ν on $\boldsymbol{R}^\infty$, it holds that $\nu(F)=\sup\{\nu(K);\ K\subset F,\ K$ is compact convex and balanced in $\boldsymbol{R}^\infty\}$. For example, l_∞, l_p $(1\leqq p<\infty)$, c_1 and c_0 are convex Lusin subspaces of $\boldsymbol{R}^\infty$. Every separable Banach subspace is convex Lusin.

(2) *For every sequence* $x'_n\in E'$ *and every convex Lusin subspace* F *of* $\boldsymbol{R}^\infty$, $\mu(x;(x'_n(x))\in F)=0$ *or* 1.

(3) *For every sequence* $x'_n\in E'$, $\mu(x;(x'_n(x))\in l_\infty)=0$ *or* 1.

(4) *For every sequence* $x'_n\in E'$, $\mu(x;(x'_n(x))\in c_1)=0$ *or* 1.

(5) *For every sequence* $x'_n\in E'$, $\mu(x;(x'_n(x))\in c_0)=0$ *or* 1.

(6) *There exist no sequence* $x'_n\in E'$ *such that* $\mu(x;(x'_n(x))\in c_0)>0$ *and that* $\mu(x;(x'_n(x))\notin l_\infty)>0$.

The 0-1 law (3) was considered by Sato [6]. It is clear that the 0-1 law (2) is stronger than (3), (4) and (5). It is also clear that each of (3), (4) and (5) is stronger than (6).

The following 0-1 law (7) is weaker than (5).

(7) *For every sequence* $x'_n\in E'$, *if there exists* $(a_n)\in c_0$ *such that* $\mu(x;(a_n^{-1}x'_n(x))\in l_\infty)>0$, *then* $\mu(x;(x'_n(x))\in c_0)=1$.

In fact, if $(a_n^{-1}x'_n(x))\in l_\infty$ for $(a_n)\in c_0$, then $(x'_n(x))\in c_0$. Hence (5)$\Rightarrow$(7) follows.

(8) *For every sequence* $x'_n\in E'$, *if there exists* $(a_n)\in c_0$ *such that* $\mu(x;(a_n^{-1}x'_n(x))\in l_\infty)>0$, *then* $\mu(x;(x'_n(x)\in l_\infty)=1$.

(9) *For every sequence* $x'_n\in E'$, *if there exists* $(a_n)\in c_0$ *such that* $\mu(x;(a_n^{-1}x'_n(x))\in c_1)>0$, *then* $\mu(x;(x'_n(x))\in c_1)=1$.

(10) *For every sequence* $x'_n\in E'$, *if there exists* $(a_n)\in c_0$ *such that* $\mu(x;(a_n^{-1}x'_n(x))\in c_0)>0$, *then* $\mu(x;(x'_n(x))\in c_0)=1$.

(11) *For every sequence* $x'_n \in E'$, *if there exists* $(a_n) \in c_0$ *such that* $\mu(x; (a_n^{-1} x'_n(x)) \in c_0) > 0$, *then* $\mu(x; (x'_n(x)) \in l_\infty) = 1$.

Since $c_0 \subset c_1 \subset l_\infty$, it is easily seen that the 0-1 law (7) is stronger than (8), (9) and (10). Also each of (8), (9) and (10) is stronger than (11). It is clear that the 0-1 law (6) implies (8) and (11), since if $\mu(x; (a_n^{-1} x'_n(x)) \in l_\infty) > 0$, then $\mu(x; (x'_n(x)) \in c_0) \geqq \mu(x; (a_n^{-1} x'_n(x)) \in l_\infty) > 0$ and hence by (6), $\mu(x; (x'_n(x)) \in l_\infty) = 1$. We shall show that the 0-1 law (11) implies (2). Thus we obtain the main result (Theorem 1):

The 0-1 *laws* (2)~(11) *are all equivalent.*

Since l_p $(1 \leqq p < \infty)$ is a convex Lusin subspace of R^∞, each of (2)~(11) implies the following 0-1 law.

(12) *For every sequence* $x'_n \in E'$, $\mu(x; (x'_n(x)) \in l_p) = 0$ *or* 1 $(1 \leqq p < \infty)$.

The above 0-1 laws (2)~(12) are described in terms of the sequences in E', so it is sufficient to suppose that the measure μ is defined only on $C(E, E')$. We state another 0-1 laws for a Radon probability measure μ on $\mathscr{B}(E)$.

(13) *For every closed convex balanced subset* B, $\mu\left(\bigcup_{n=1}^{\infty} nB\right) = 0$ *or* 1.

(14) *For every lower semi-continuous semi-norm* $N(x)$ *on* E *(admitting the value* ∞*)*, $\mu(x; N(x) < \infty) = 0$ *or* 1.

(15) *For every compact convex balanced subset* K, $\mu\left(\bigcup_{n=1}^{\infty} nK\right) = 0$ *or* 1.

For example, the countable product of non-atomic probability measures on R^∞ satisfies the 0-1 law (15), see Hoffmann-Jørgensen [3] and Zinn [8]. Obviously (13) and (14) are equivalent. We show that if μ is Radon, then (2)~(11), (13) and (14) are equivalent (Theorem 2). In the case where μ is convex Radon, (13) and (15) are equivalent, hence the 0-1 laws (2)~(11), (13) (14) and (15) are all equivalent (Theorem 3). The implication (3)⇒(15) for a convex Radon μ was proved by Sato [6] and (15)⇒(3) was remarked by Takahashi [7].

§ 2. Main Results

Theorem 1. *The* 0-1 *laws* (2)~(11) *are all equivalent.*

Proof. As we have remarked in the preceding section, it is sufficient to prove (11)⇒(2). Let $x'_n \in E'$ and F be a convex Lusin subspace of R^∞. Suppose that $\mu(x; (x'_n(x)) \in F) > 0$. Then we must show that $\mu(x : (x'_n(x)) \in F) = 1$. If we set $\Pi: E \to \boldsymbol{R}^{\infty}$ by $\Pi(x)=(x'_n(x))$, then by the definition of the convex Lusin subspace, there exists a compact convex balanced subset K in $\boldsymbol{R}^{\infty}$ such that $K \subset F$ and that $\Pi(\mu)(K)=\mu(\Pi^{-1}(K))>0$, where $\Pi(\mu)$ is the image measure of μ by Π. Let $\bigcup_{n=1}^{\infty} nK$ be the linear subspace of $\boldsymbol{R}^{\infty}$ spanned by K. We show $\Pi(\mu)\left(\bigcup_{n=1}^{\infty} nK\right)=1$.

Suppose that $\Pi(\mu)\left(\bigcup_{n=1}^{\infty} nK\right)<1$. Then there exists a compact subset L in $\boldsymbol{R}^{\infty}$ such that

$$\Pi(\mu)(L)>0 \quad \text{and} \quad L \cap \left(\bigcup_{n=1}^{\infty} nK\right)=\emptyset .$$

For every $x \in L$ and every n, by the Hahn-Banach theorem, there exists $\xi_{n,x} \in (\boldsymbol{R}^{\infty})'$ such that

$$\xi_{n,x}(x)>1\,, \quad \text{and} \quad |\xi_{n,x}(y)| \leqq 1 \qquad \text{on} \quad nK\,,$$

that is, $\xi_{n,x}$ separates nK and x. For every fixed n, the open subsets $U_{n,x}=\{y: \xi_{n,x}(y)>1\}$, $x \in L$, form a covering of L. Take a finite sub-covering U_{n,x_j^n}, $j=1, 2, \cdots, j(n)$. Then for every n and every $x \in L$, there exists some $j \in \{1, 2, \cdots, j(n)\}$ such that

$$\xi_{n,x_j^n}(x)>1\,.$$

Consider the following sequence in $(\boldsymbol{R}^{\infty})'$:

$$(*) \qquad \xi_{1,x_1^1}, \cdots, \xi_{1,x_{j(1)}^1}, \cdots, n^{1/3}\xi_{n,x_1^n}, \cdots, n^{1/3}\xi_{n,x_{j(n)}^n}, (n+1)^{1/3}\xi_{n+1,x_1^{n+1}}, \cdots .$$

Then for every $x \in L$ and every n, there exists suitable j in $\{1, 2, \cdots, j(n)\}$ such that

$$n^{1/3}\xi_{n,x_j^n}(x)>n^{1/3}.$$

On the other hand, for every $x \in mK$ (m be arbitrarily fixed), we have as $n \to \infty$,

$$n^{2/3}\xi_{n,x_j^n}(x)=n^{2/3}\frac{m}{n}\xi_{n,x_j^n}\left(\frac{n}{m}x\right)$$

$$\leqq n^{2/3}\frac{m}{n}=mn^{-1/3} \to 0\,,$$

since $(n/m)x \in nK$. Thus we have

$$n^{2/3}\xi_{n,x_j^n}(x) \to 0 \quad \text{on} \quad \bigcup_{m=1}^{\infty} mK\,,$$

as $n \to \infty$. Denote by (η_k) the sequence (*) and $(a_k) \in c_0$ be the following sequence:

$$1, \cdots, 1, \cdots, n^{1/3}, \cdots, n^{1/3}, (n+1)^{1/3}, \cdots,$$

that is,

$$a_k=n^{1/3} \qquad \text{for} \quad \sum_{i=1}^{n-1} j(i)<k \leqq \sum_{i=1}^{n} j(i)\,.$$

We put $z'_k=\eta_k\circ\Pi$. Then $z'_k\in E'$ and it hold that

$$a_k z'_k(x)\to 0 \quad \text{on} \quad \Pi^{-1}\Big(\bigcup_{m=1}^{\infty} mK\Big), \quad \text{and}$$

$$\sup_k |z'_k(x)|=\infty \quad \text{on} \quad \Pi^{-1}(L),$$

which contradict to (11), since $\mu\Big(\Pi^{-1}\Big(\bigcup_{m=1}^{\infty} mK\Big)\Big)>0$ and $\mu(\Pi^{-1}(L))>0$.

This completes the proof.

Corollary 1. *Each of the* 0-1 *law* (2)~(11) *implies* (12).

Theorem 2. *Suppose that* μ *is a Radon probability measure. Then the* 0-1 *laws* (2)~(11), (13) *and* (14) *are all equivalent.*

Proof. By Theorem 1, it is sufficient to show (5)$\Leftrightarrow$(13).

(5)$\Rightarrow$(13) Let B be a closed convex balanced subset with $\mu(B)>0$. We show that $\mu\Big(\bigcup_{n=1}^{\infty} nB\Big)=1$. Suppose that $\mu\Big(\bigcup_{n=1}^{\infty} nB\Big)<1$. Then since μ is Radon, there is a compact subset L such that

$$\mu(L)>0 \quad \text{and} \quad L\cap\Big(\bigcup_{n=1}^{\infty} nB\Big)=\emptyset\,.$$

For every n and every $x\in L$, by the Hahn-Banach theorem, there exists $\xi_{n,x}\in E'$ such that

$$\xi_{n,x}(x)>1\,, \quad \text{and} \quad |\xi_{n,x}(y)|\leqq 1 \quad \text{on} \quad nB\,.$$

The open subsets $U_{n,x}=\{y,\ \xi_{n,x}(y)>1\}$, $x\in L$, cover L. Take a finite subcovering U_{n,x_j^n}, $j=1, 2, \cdots, j(n)$ and consider the sequence

(**) $$\xi_{1,x_1^1}, \cdots, \xi_{1,x_{j(1)}^1}, \cdots, \xi_{n,x_1^n}, \cdots, \xi_{n,x_{j(n)}^n}, \cdots.$$

Then we have

$$\sup_n |\xi_{n,x_j^n}(x)|>1 \quad \text{on} \quad L, \quad \text{and} \quad \xi_{n,x_j^n}(x)\to 0 \quad \text{on} \quad \bigcup_{m=1}^{\infty} mB\,,$$

since for every $x\in mB$

$$\xi_{n,x_j^n}(x)=\frac{m}{n}\xi_{n,x_j^n}\Big(\frac{n}{m}x\Big)$$

$$\leqq\frac{m}{n}\to 0 \quad \text{as} \quad n\to\infty\,.$$

These contradict to (5).

(13)$\Rightarrow$(5) Suppose that $\mu(x\,;\,(x'_n(x))\in c_0)>0$, that is, $\mu(x\,;\,x'_n(x)\to 0)>0$. By the Egorov's theorem, there exists $A\subset\{x\,;\,x'_n(x)\to 0\}$ such that $\mu(A)>0$ and $x'_n(x)\to 0$ uniformly on A. Let B be the closed convex balanced hull of A. We shall see $x'_n(x)\to 0$ uniformly also on B. For every $\varepsilon>0$, there is an $N=N(\varepsilon)$

such that

$$\sup_{x\in A}|x'_n(x)|<\varepsilon \qquad \text{for every} \quad n>N,$$

since (x'_n) converges to 0 uniformly on A. The subset $\{x;\ |x'_n(x)|\leqq\varepsilon$ for every $n>N\}$ is closed convex balanced, and contains A so it follows that $B\subset\{x;\ |x'_n(x)|\leqq\varepsilon$ for every $n>N\}$.

Hence we have

$$\sup_{x\in B}|x'_n(x)|\leqq\varepsilon \qquad \text{for every} \quad n>N,$$

which proves the assertion. By the assumption (13), we have

$$\mu(x;\ x'_n(x)\to 0)\geqq\mu\Big(\bigcup_{n=1}^{\infty} nB\Big)=1\,.$$

Thus the 0-1 law (5) is valid.

This completes the proof.

Theorem 3. *If μ is a convex Radon measure, then the* 0-1 *laws* (2)～(11), (13), (14) *and* (15) *are all equivalent.*

Proof. It is sufficient to show (13)$\Leftrightarrow$(15). (13)$\Rightarrow$(15) is obvious.

(15)$\Rightarrow$(13) Let B be a closed convex balanced subset with $\mu(B)>0$. We show that $\mu\Big(\bigcup_{n=1}^{\infty} nB\Big)=1$. Since μ is a convex Radon measure, there exists a compact convex balanced subset K such that $\mu(K\cap B)>0$. Let L be the closed convex balanced hull of $K\cap B$. Since L is a closed subset of the compact set K, L is also compact. By the assumption (15), we have $\mu\Big(\bigcup_{n=1}^{\infty} nB\Big)\geqq\mu\Big(\bigcup_{n=1}^{\infty} nL\Big)=1$.

This completes the proof.

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