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The Chern Character on the Odd Spinor Groups

By

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§ 1. Introduction

This paper is a continuation of [10] and contains a calculation of the Chern character ([2, §1])

$$ch : K^*(G) \longrightarrow H^*(G ; Q)$$

for $G=Spin(2l+1)$ with $l\geq 5$. For $G=Spin(n)$ with $7\leq n\leq 9$ it has been done in [10].

Let λ_1' denote the composition

$$Spin(2l+1) \xrightarrow{\pi} SO(2l+1) \xrightarrow{k} SU(2l+1) \xrightarrow{j} U(2l+1)$$

where π is the double covering, k is the usual inclusion and j is the natural inclusion. It gives an element of the representation ring $R(Spin(2l+1))$, which we also denote by λ_1'. Let

$$\beta : R(Spin(2l+1)) \longrightarrow K^{-1}(Spin(2l+1))$$

be the map introduced in [4]; then the image of β generates $K^*(Spin(2l+1))$ multiplicatively. Since the λ-ring ([5, Chapter 12]) structure of $R(Spin(2l+1))$ is convenient for our work, if the value of ch on $\beta(\lambda_1')$ is known, then that on any other element of $K^*(Spin(2l+1))$ can be calculated (see §3). For this reason the result below is the core of this paper. To state it we need some notations. For each integer n, let $s(n)$ be the integer such that

$$2^{s(n)-1} < n \leq 2^{s(n)}.$$

Put

$$\mathscr{P} = \{2^i \mid i\geq 0\}.$$

Recall that $H^*(Spin(2l+1);Q)$ is an exterior algebra on generators

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of degree $4j-1$, $1 \leq j \leq l$. From this fact and the Poincaré duality we infer that there exist elements

$$x_{4j-1} \in H^{4j-1}(Spin(2l+1); Z), \ 1 \leq j \leq l,$$

such that (they are not divisible and) the product $x_3 x_7 \ldots x_{4l-1}$ gives a generator of $H^{2l^2+l}(Spin(2l+1); Z) = Z$.

Theorem 1. *With the above notations, we have*

$$ch\beta(\lambda_1') = \sum_{j=1}^{l} \frac{(-1)^{j-1} 2^{q(l,j)}}{(2j-1)!} \cdot x_{4j-1}$$

where

$$q(l, j) = \begin{cases} 2 & \text{if } l \notin \mathscr{P} \text{ and } j = 2^{s(l)-1} \text{ or} \\ & \quad l \in \mathscr{P} \text{ and } j = 2^{s(l)} \\ 1 & \text{otherwise.} \end{cases}$$

As an application of this theorem, we obtain a precise description of the image of x_{4j-1} under the transgression in the Serre spectral sequence of the fibration $Spin(2l+1) \longrightarrow Spin(2l+1)/T \longrightarrow BT$, where T is a maximal torus of $Spin(2l+1)$.

The paper is organized as follows. In §2 we determine the value of $k_l^*: H^i(Spin(2l+1); Z) \longrightarrow H^i(Spin(2l-1); Z)$ on x_{4j-1}, where k_l: $Spin(2l-1) \longrightarrow Spin(2l+1)$ is the natural inclusion. From the λ-ring structure of $R(Spin(2l+1))$ and a result of Atiyah [1], in §3 we derive a relation among the coefficients of x_{4j-1}'s in $ch\beta(\lambda_1')$. In §4 we calculate the determinant of a certain matrix and then prove Theorem 1. Its consequences will be discussed in §5.

§ 2. The Cohomology of $Spin(2l+1)$

Let us begin by recalling some basic results on the cohomology of $Spin(n)$ (for a reference see [3]). First,

(2.1) *$Spin(n)$ has 2-torsion if and only if $n \geq 7$, and all of its torsion is of order* 2.

Secondly,

(2.2) *If p is an odd prime, then $H^*(Spin(2l+1); Z/p)$ is an exterior* *algebra on generators* x_{4j-1}, $1\leq j\leq l$. *Each* x_{4j-1} *is universally transgressive.*

Let $\Delta(\ )$ denote a $Z/2$-algebra having a simple system of generators. The following result is due to [7] (cf. [6]).

(2.3) (i) $H^*(Spin(n);Z/2)=\Delta(u_i,\ u\,|\,0<i<n,\ i\notin\mathscr{P})$ *where* $\deg u_i=i$ *and* $\deg u=2^{s(n)}-1$. u_i *is universally transgressive, and* u *is universally transgressive if and only if* $n\leq 9$.

(ii) $Sq^j(u_i)=\binom{i}{j}u_{i+j}$; *in particular*, $u_i^2=u_{2i}$ (*where we use the convention that* $u_i=0$ *if* $i\geq n$ *or* $i\in\mathscr{P}$). $Sq^1(u)=0$ *and* $u^2=0$.

(iii) *If* $j_n: Spin(n)\longrightarrow Spin(n+1)$ *is the natural inclusion, then* $j_n^*(u_i)=u_i$, $j_n^*(u)=u$ *if* $n\notin\mathscr{P}$, *and* $j_n^*(u)=0$ *if* $n\in\mathscr{P}$.

Then we have

Lemma 2. *Let* ρ: $H^i(Spin(2l+1);Z)\longrightarrow H^i(Spin(2l+1);Z/2)$ *be the* mod 2 *reduction. Then for* $1\leq j\leq l$,

$$\rho(x_{4j-1})=\begin{cases}u_{4j-1} & \text{if } j\in\mathscr{P} \text{ and } j<2^{s(2l+1)-2}\\ u & \text{if } j=2^{s(2l+1)-2}\\ u_{2j-1}u_{2j}+u_{4j-1} & \text{if } j\notin\mathscr{P}\end{cases}$$

(*where* $\deg u=2^{s(2l+1)}-1$).

Proof. Let $\{E_r\}$ be the mod 2 cohomology Bockstein spectral sequence of $Spin(2l+1)$. Since $E_2=E_\infty$ by (2.1), to prove this lemma, it suffices to compute $E_2=\mathrm{Ker}\,Sq^1/\mathrm{Im}\,Sq^1$. But it was done in Corollary 5 of [7] (by using (2.3)(ii)).

Consider the natural inclusion

$$k_l=j_{2l}j_{2l-1}: Spin(2l-1)\longrightarrow Spin(2l+1)$$

and put

$$V_l=Spin(2l+1)/Spin(2l-1).$$

Then there is a fibration

$$S^{2l-1}\longrightarrow V_l\longrightarrow S^{2l}.$$

From the last statement of the theorem in [9, §23.4] it follows that

(2.4)
$$\tilde{H}^i(V_l;\, Z) = \begin{cases} Z & \textit{if } i=4l-1 \\ Z/2 & \textit{if } i=2l \\ 0 & \textit{otherwise.} \end{cases}$$

Consider the homomorphism

$$k_l^*\colon H^i(Spin(2l+1)\,;\, Z) \longrightarrow H^i(Spin(2l-1)\,;\, Z).$$

Clearly $k_l^*(x_{4l-1})=0$. We define integers $b_j(l)$ by the equations

$$k_l^*(x_{4j-1}) = b_j(l)\cdot x_{4j-1}$$

where $1\leq j\leq l-1$. Note that

$$s(2l+1) = \begin{cases} s(l)+1 & \text{if } l\notin\mathscr{P} \\ s(l)+2 & \text{if } l\in\mathscr{P}. \end{cases}$$

Lemma 3. *We have, up to sign,*

$$b_j(l) = \begin{cases} 2 & \textit{if } l\in\mathscr{P} \textit{ and } j=2^{s(l)-1} \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Let p be an odd prime, and consider the Serre spectral sequence for the mod p cohomology of the fibration

$$Spin(2l-1) \xrightarrow{k_l} Spin(2l+1) \longrightarrow V_l.$$

In view of (2.2) and (2.4), we see that it collapses. In other words,

$$k_l^*\colon H^i(Spin(2l+1)\,;\, Z/p) \longrightarrow H^i(Spin(2l-1)\,;\, Z/p)$$

is surjective for all i. On the other hand, by (2.3) (iii) we find that

$$k_l^*\colon H^i(Spin(2l+1)\,;\, Z/2) \longrightarrow H^i(Spin(2l-1)\,;\, Z/2)$$

is surjective if $l\notin\mathscr{P}$ (or $l\in\mathscr{P}$ and $i\neq 2^{s(l)+1}-1$). Therefore

$$k_l^*\colon H^i(Spin(2l+1)\,;\, Z) \longrightarrow H^i(Spin(2l-1)\,;\, Z)$$

is surjective if $l\notin\mathscr{P}$ (or $l\in\mathscr{P}$ and $i\neq 2^{s(l)+1}-1$). Hence $b_j(l)=1$ if $l\notin\mathscr{P}$ (or $l\in\mathscr{P}$ and $j\neq 2^{s(l)-1}$).

It remains to consider the case where $l\in\mathscr{P}$, i.e., $l=2^{s(l)}$. In this case, by (2.3) (iii) we find that

$$u\in H^{2l-1}(Spin(2l-1)\,;\, Z/2)$$

is the only element which does not belong to the image of k_l^*. This implies that, in the integral cohomology spectral sequence of the above fibration, the only non-zero differential is

$$d_{2l}(1\otimes u)=v\otimes 1$$

where v is a generator of $H^{2l}(V_l;\ Z)=Z/2$. By Lemma 2, this reveals that $k_l^*(x_{2l-1})=2\cdot x_{2l-1}$. That is, $b_{2^{s(l)}-1}(2^{s(l)})=2$.

§ 3. The Representation Ring of $Spin(2l+1)$

For further details of the following result, see [5] and [11].

(3.1) (i) *There exist representations* $\lambda_1', \lambda_2', \dots, \lambda_{l-1}', \Delta_{2l+1}$ *of* $Spin(2l+1)$ *such that*

$$R(Spin(2l+1))=Z[\lambda_1', \dots, \lambda_{l-1}', \Delta_{2l+1}]$$

where $\dim \lambda_k'=\binom{2l+1}{k}$, $\dim \Delta_{2l+1}=2^l$ *and relations*

(a) $$\Lambda^k\lambda_1'=\lambda_k',$$

(b) $$\sum_{k=0}^{l}\lambda_k'=\Delta_{2l+1}^2$$

hold (where Λ^k *denotes the k-th exterior power and we use the convention that* $\lambda_0'=1$ *and* $\lambda_l'=\Lambda^l\lambda_1'$*).*

(ii) *The homomorphism*

$$k_l^*:\ R(Spin(2l+1))\longrightarrow R(Spin(2l-1))$$

sends λ_1' *to* $\lambda_1'+2$.

Consider now the composite

$$R(Spin(2l+1))\xrightarrow{\beta}K^{-1}(Spin(2l+1))\xrightarrow{ch}H^*(Spin(2l+1);\ Q).$$

Since the cohomology suspension $\sigma^*:\ H^i(BSpin(2l+1);\ Q)\longrightarrow H^{i-1}(Spin(2l+1);\ Q)$ has image contained in the module of primitives

$$PH^*(Spin(2l+1);\ Q)=Q\{x_{4j-1}\,|\,1\le j\le l\},$$

it follows from the argument in [10, §1] that, for any $\lambda\in R(Spin(2l+1))$, $ch\beta(\lambda)$ can be written in the form

(3.2) $$ch\beta(\lambda)=\sum_{j=1}^{l}a(\lambda,j)\cdot x_{4j-1}\ \text{with}\ a(\lambda,j)\in Q.$$

For clarity we sometimes write $\lambda_k'(l)$ for $\lambda_k'\in R(Spin(2l+1))$.

Lemma 4. *For* $1\le j\le l-1$,

$$a(\lambda_1'(l),j)=\begin{cases}2^{-1}a(\lambda_1'(l-1),j) & \text{if } l\in\mathscr{P} \text{ and } j=2^{s(l)-1}\\ a(\lambda_1'(l-1),j) & \text{otherwise.}\end{cases}$$

Proof. By (3.2),

$$ch\beta(\lambda_1'(l))=\sum_{j=1}^{l}a(\lambda_1'(l),j)x_{4j-1}.$$

Apply k_l^* to this equation. Then, by (3.1)(ii) and the fact that $\beta(2)=0$([4, Lemma 4.1]), the left hand side becomes

$$\begin{aligned}k_l^*ch\beta(\lambda_1'(l))&=ch\beta k_l^*(\lambda_1'(l))\\&=ch\beta(\lambda_1'(l-1))\\&=\sum_{j=1}^{l-1}a(\lambda_1'(l-1),j)x_{4j-1}\end{aligned}$$

and the right hand side becomes

$$\begin{aligned}k_l^*(\sum_{j=1}^{l}a(\lambda_1'(l),j)x_{4j-1})&=\sum_{j=1}^{l}a(\lambda_1'(l),j)k_l^*(x_{4j-1})\\&=\sum_{j=1}^{l-1}a(\lambda_1'(l),j)b_j(l)x_{4j-1}.\end{aligned}$$

Hence

$$a(\lambda_1'(l-1),j)=a(\lambda_1'(l),j)\cdot b_j(l)$$

for $1\leq j\leq l-1$. So the result follows from Lemma 3.

Let us review a result of Atiyah [1]. By (3.1)(i), the set $\{\lambda_1',\dots,\lambda_{l-1}',\Delta_{2l+1}\}$ is a system of generators of the ring $R(Spin(2l+1))$. Via (3.2), with such a system there is associated an $l\times l$-matrix

$$\Phi(l)=\begin{pmatrix}a(\lambda_1',1)&\dots&a(\lambda_1',l)\\ \vdots&&\vdots\\ a(\lambda_{l-1}',1)&\dots&a(\lambda_{l-1}',l)\\ a(\Delta_{2l+1},1)&\dots&a(\Delta_{2l+1},l)\end{pmatrix}.$$

Then his result (Proposition 1 of [1]) can be reformulated as det $\Phi(l)=\pm1$ (see [10, §0]). Here we adopt

(3.3) $$\det\Phi(l)=1$$

only, because the problem of sign (e.g., the sign of x_{4j-1} and the arrangement of x_{4j-1}'s) has no essential influence on our argument.

In what follows we express $a(\lambda_k',j)$ and $a(\Delta_{2l+1},j)$ in terms of $a(\lambda_1',j)$. First, by Lemma 1 of [10], we deduce from the relation (a) in (3.1)(i) that

$$a(\lambda_k',j)=\varphi(2l+1,k,2j)\cdot a(\lambda_1',j)$$

for all $k\geq1$ where

$$\varphi(2l+1, k, 2j) = \sum_{i=1}^{k} (-1)^{i-1} \binom{2l+1}{k-i} i_{2j-1}.$$

Consider next the relation (b) in (3.1)(i). Since β is additive and $\beta(1)=0$, we have

$$\begin{aligned} ch\beta(\sum_{k=0}^{l} \lambda'_k) &= \sum_{k=0}^{l} ch\beta(\lambda'_k) = \sum_{k=1}^{l} ch\beta(\lambda'_k) \\ &= \sum_{k=1}^{l} \sum_{j=1}^{l} \varphi(2l+1, k, 2j)\, a(\lambda'_1, j)\, x_{4j-1}. \end{aligned}$$

Using the formula (2) of [4, p. 8], we have

$$\begin{aligned} ch\beta(\Delta_{2l+1}^2) &= ch\{2^l\beta(\Delta_{2l+1}) + 2^l\beta(\Delta_{2l+1})\} \\ &= 2^{l+1} ch\beta(\Delta_{2l+1}). \end{aligned}$$

Therefore

$$a(\Delta_{2l+1}, j) = 2^{-(l+1)} \{\sum_{k=1}^{l} \varphi(2l+1, k, 2j)\} \cdot a(\lambda'_1, j).$$

Combining the above, we see that

$$\det \Phi(l) = 2^{-(l+1)} \cdot \det \begin{pmatrix} \Phi_1(l) \\ \vdots \\ \Phi_{l-1}(l) \\ \sum_{k=1}^{l} \Phi_k(l) \end{pmatrix} \cdot \prod_{j=1}^{l} a(\lambda'_1, j)$$

where $\Phi_k(l)$ is the $1 \times l$-matrix $(\varphi(2l+1, k, 2j))_{1 \le j \le l}$. By elementary properties of determinant,

$$\det \begin{pmatrix} \Phi_1(l) \\ \vdots \\ \Phi_{l-1}(l) \\ \sum_{k=1}^{l} \Phi_k(l) \end{pmatrix} = \sum_{k=1}^{l} \det \begin{pmatrix} \Phi_1(l) \\ \vdots \\ \Phi_{l-1}(l) \\ \Phi_k(l) \end{pmatrix} = \det \begin{pmatrix} \Phi_1(l) \\ \vdots \\ \Phi_{l-1}(l) \\ \Phi_l(l) \end{pmatrix}.$$

Therefore

$$2^{l+1} \cdot \det \Phi(l) = \det(\varphi(2l+1, i, 2j))_{1 \le i, j \le l} \cdot \prod_{j=1}^{l} a(\lambda'_1, j).$$

Substituting (3.3) to this, we get

$$\prod_{j=1}^{l} a(\lambda'_1, j) = 2^{l+1} / \det(\varphi(2l+1, i, 2j))_{1 \le i, j \le l}. \tag{3.4}$$

§4. Proof of Theorem 1

The determinant in (3.4) is given by

Lemma 5. $\det(\varphi(2l+1, i, 2j))_{1\le i,j\le l} = (-1)^{[l/2]}\prod_{k=1}^{l}(2k-1)!$ *where* $[l/2]$ *denotes the greatest integer less than or equal to* $l/2$.

Proof. For $k-1\le i\le l$ and $k\le j\le l$, let

$$\varphi_{i,j}^{(k)} = \sum_{m=k}^{i}(-1)^{m-1}\binom{2l+1}{i-m}(m^2-1)(m^2-2^2)\dots(m^2-(k-1)^2)m^{2j-2k+1}$$

Note that when $k=1$, $\varphi_{i,j}^{(1)}=\varphi(2l+1, i, 2j)$. As is easily checked, these integers satisfy:

$$\varphi_{k-1,j}^{(k)}=0 \quad \text{for } k\le j\le l;$$
$$\varphi_{k,k}^{(k)}=(-1)^{k-1}(2k-1)!;$$
$$\varphi_{k,j}^{(k)}=k^2\varphi_{k,j-1}^{(k)} \quad \text{for } k+1\le j\le l.$$

Furthermore, if $k\le i\le l$ and $k+1\le j\le l$, then

$$\begin{aligned}
&\varphi_{i,j}^{(k)}-k^2\varphi_{i,j-1}^{(k)}\\
&=\sum_{m=k}^{i}(-1)^{m-1}\binom{2l+1}{i-m}(m^2-1)\dots(m^2-(k-1)^2)(m^{2j-2k+1}-k^2m^{2j-2k-1})\\
&=\sum_{m=k+1}^{i}(-1)^{m-1}\binom{2l+1}{i-m}(m^2-1)\dots(m^2-(k-1)^2)(m^2-k^2)m^{2j-2(k+1)+1}\\
&=\varphi_{i,j}^{(k+1)}.
\end{aligned}$$

For each k with $1\le k\le l$, we consider the $(l-k+1)\times(l-k+1)$-matrix $(\varphi_{i,j}^{(k)})_{k\le i,j\le l}$. Transform this matrix by adding $(-k^2)$-times the $(l-k)$-th column to the $(l-k+1)$-th column. Transform next the resulting matrix by adding $(-k^2)$-times the $(l-k-1)$-th column to the $(l-k)$-th column. Iterate such elementary transformations. This procedure, which ends in the first column, yields the following first equality:

$$\det(\varphi_{i,j}^{(k)}) = \det\begin{pmatrix}\varphi_{k,k}^{(k)} & 0\\ \varphi_{i,k}^{(k)} & \varphi_{i,j}^{(k+1)}\end{pmatrix} = \varphi_{k,k}^{(k)}\det(\varphi_{i,j}^{(k+1)}).$$

So the $(l-k)\times(l-k)$-matrix $(\varphi_{i,j}^{(k+1)})_{k+1\le i,j\le l}$ is left. From this recursive process we deduce that

$$\begin{aligned}
\det(\varphi_{i,j}^{(1)})_{1\le i,j\le l} &= \prod_{k=1}^{l}\varphi_{k,k}^{(k)}\\
&=\prod_{k=1}^{l}(-1)^{k-1}(2k-1)!\\
&=(-1)^{[l/2]}\prod_{k=1}^{l}(2k-1)!.
\end{aligned}$$

By this lemma and (3.4),

$$\prod_{j=1}^{l} a(\lambda_1',j) = (-1)^{[l/2]}2^{l+1}/\prod_{k=1}^{l}(2k-1)!. \tag{4.1}$$

The following is a restatement of Theorem 1.

Theorem 6. *For* $1 \leq j \leq l$,

$$a(\lambda_1'(l),j) = \begin{cases} -2^2/(2j-1)! & \text{if } l \notin \mathscr{P} \text{ and } j=2^{s(l)-1} \text{ or} \\ & l \in \mathscr{P} \text{ and } j=2^{s(l)} \\ (-1)^{j-1}2/(2j-1)! & \text{otherwise.} \end{cases}$$

Proof. We prove this by induction on l. The result for $l=3$, 4 was shown in [10, §3]. Assume that it is true for $l=m$ and consider the case $l=m+1$. Our argument is divided into two cases. (Recall that $2^{s(m)-1} < m \leq 2^{s(m)}$.)

First suppose that $m+1 \notin \mathscr{P}$. By Lemma 4 and the inductive hypothesis, we have, for $1 \leq j \leq m$,

$$a(\lambda_1'(m+1),j) = a(\lambda_1'(m),j)$$
$$= \begin{cases} -2^2/(2j-1)! & \text{if } m \notin \mathscr{P} \text{ and } j=2^{s(m)-1} \text{ or} \\ & m \in \mathscr{P} \text{ and } j=2^{s(m)} \\ (-1)^{j-1}2/(2j-1)! & \text{otherwise.} \end{cases}$$

If $m \notin \mathscr{P}$, then $s(m+1)=s(m)$. If $m \in \mathscr{P}$, then $m=2^{s(m)}$ and so $s(m+1) = s(m)+1$. Thus the desired result is obtained for $j \neq m+1$. Using the above and (4.1), we have

$$\begin{aligned} a(\lambda_1'(m+1),m+1) &= \{\prod_{j=1}^{m+1} a(\lambda_1'(m+1),j)\}/\{\prod_{j=1}^{m} a(\lambda_1'(m+1),j)\} \\ &= \{\prod_{j=1}^{m+1} a(\lambda_1'(m+1),j)\}/\{\prod_{j=1}^{m} a(\lambda_1'(m),j)\} \\ &= \frac{\{(-1)^{[(m+1)/2]}2^{m+2}/\prod_{j=1}^{m+1}(2j-1)!\}}{\{(-1)^{[m/2]}2^{m+1}/\prod_{j=1}^{m}(2j-1)!\}} \\ &= (-1)^{[(m+1)/2]-[m/2]}2/(2m+1)! \\ &= (-1)^m 2/(2m+1)!. \end{aligned}$$

Finally suppose that $m+1 \in \mathscr{P}$. Then $m+1=2^{s(m)}$ and $s(m+1)= s(m)$. By Lemma 4 and the inductive hypothesis, we have, for $1 \leq j \leq m$,

$$a(\lambda_1'(m+1),j)=\begin{cases}2^{-1}a(\lambda_1'(m),j) & \text{if } j=2^{s(m+1)-1}\\ a(\lambda_1'(m),j) & \text{otherwise}\end{cases}$$
$$=\begin{cases}2^{-1}(-2^2)/(2j-1)\,! & \text{if } j=2^{s(m)-1}\\ (-1)^{j-1}2/(2j-1)\,! & \text{otherwise}\end{cases}$$
$$=(-1)^{j-1}2/(2j-1)\,!.$$

Thus the desired result is obtained for $j\neq m+1$. Using the above and (4. 1), we have

$$a(\lambda_1'(m+1),m+1)=\{\prod_{j=1}^{m+1}a(\lambda_1'(m+1),j)\}/\{\prod_{j=1}^{m}a(\lambda_1'(m+1),j)\}$$
$$=\{\prod_{j=1}^{m+1}a(\lambda_1'(m+1),j)\}/\{\prod_{j=1}^{m}(-1)^{j-1}2/(2j-1)\,!\}$$
$$=\{2^{m+2}/\prod_{j=1}^{m+1}(2j-1)\,!\}/\{-2^m/\prod_{j=1}^{m}(2j-1)\,!\}$$
$$=-2^2/(2m+1)\,!$$

and the proof is completed.

§ 5. Some Consequences

The argument of this section depends on that of [10, §1]; the reader is referred to it.

A maximal torus T of $Spin(2l+1)$ can be chosen so that

$$H^*(BT;Z)=Z[t_1,t_2,\dots,t_l,\gamma]/(c_1-2\gamma)$$

where $\deg t_j=\deg\gamma=2$ and $c_1=t_1+\cdots+t_l$. Moreover, if $W(Spin(2l+1))$ is the corresponding Weyl group, then it acts on $H^*(BT;Z)$ and the $W(Spin(2l+1))$-invariants in $H^*(BT;Q)$ form a polynomial algebra on generators $p_1,p_2,\dots,p_l$ where

$$p_j=\sigma_j(t_1^2,t_2^2,\dots,t_l^2)\in H^{4j}(BT;Z)$$

(σ_j denotes the j-th elementary symmetric function). Let $i\colon T\to Spin(2l+1)$ be the inclusion and $i^*:R(Spin(2l+1))\longrightarrow R(T)$ the homomorphism induced by it. Let $\alpha:R(T)\longrightarrow K^*(BT)$ be the (λ-ring) homomorphism given in [2]. For $k\geq 0$ let $ch^k:K^*(BT)\longrightarrow H^{2k}(BT;Q)$ be the composition $K^*(BT)\xrightarrow{ch}H^*(BT;Q)\longrightarrow H^{2k}(BT;Q)$ where the second map is the projection onto the $2k$-dimensional component. Consider the cohomology transgression

$$\tau':H^{i-1}(Spin(2l+1);Q)\longrightarrow H^i(BT;Q)$$

in the Serre spectral sequence of the fibration

$$Spin(2l+1) \longrightarrow Spin(2l+1)/T \longrightarrow BT.$$

Then we have

Proposition 7. *With the above notations, for* $1 \leq j \leq l$,

$$\tau'(x_{4j-1}) = 2^{-q(l,j)} p_j$$

(*modulo decomposables in* $Q[p_1, p_2, \ldots, p_l]$).

Proof. Since the set of weights of λ_1' is given by $\{\pm t_j, 0 \mid 1 \leq j \leq l\}$ (see [11]), it follows from the same calculation as in [10, §2] that

$$ch^{2j} \alpha i^* (\lambda_1') = \frac{(-1)^{j-1}}{(2j-1)!} p_j$$

modulo decomposables in $Q[p_1, \ldots, p_l]$. But by [10, §1] the left hand side is equal to $a(\lambda_1', j) \cdot \tau'(x_{4j-1})$. Since

$$a(\lambda_1', j) = \frac{(-1)^{j-1}}{(2j-1)!} 2^{q(l,j)}$$

by Theorem 1, the result follows.

Finally we mention a consequence of this result. Let

$$\tau : H^{i-1}(Spin(2l+1); Q) \longrightarrow H^i(BSpin(2l+1); Q)$$

be the transgression in the Serre spectral sequence of the universal bundle

$$(5.1) \qquad Spin(2l+1) \longrightarrow ESpin(2l+1) \longrightarrow BSpin(2l+1).$$

For $1 \leq j \leq l$ let

$$y_{4j} \in H^{4j}(BSpin(2l+1); Z)$$

and

$$f_{4j} \in H^{4j}(BT; Z)$$

be as in [10, §§0 and 1] (in particular, they are not divisible). Then we can define integers $b(2j)$ and $c(2j)$, $1 \leq j \leq l$, by

$$\tau(x_{4j-1}) = \frac{1}{b(2j)} \cdot y_{4j} \quad \text{(modulo decomposables)}$$

and

$$(Bi)^* (y_{4j}) = c(2j) \cdot f_{4j} \quad \text{(modulo decomposables)}$$

respectively. Since $\tau' = (Bi)^* \tau$, it follows from Proposition 7 that

(5. 2) $$\frac{c(2j)}{b(2j)} f_{4j} = 2^{-q(l,j)} p_j \quad \text{(modulo decomposables)}.$$

Thus if f_{4j} is known, then $b(2j)$ determines $c(2j)$ and vice versa.

Let us cite an example. Consider the group $Spin(11)$, i. e., the case $l=5$. In this case, by (2. 3),

$$H^*(Spin(11); Z/2) = \Delta(u_3, u_5, u_6, u_7, u_9, u_{10}, u)$$

where deg $u=15$; $Sq^1(u_5) = u_6$ and $Sq^1(u_9) = u_{10}$. Then by Lemma 2,

$$\rho(x_3) = u_3, \rho(x_7) = u_7, \rho(x_{11}) = u_5u_6, \rho(x_{15}) = u \text{ and } \rho(x_{19}) = u_9u_{10}.$$

On the other hand, it follows from Theorem 6. 5 of [8] that, in degrees ≤ 20,

$$H^*(BSpin(11); Z/2) = Z/2[w_4, w_6, w_7, w_8, w_{10}, w_{11}]/(w_{10}w_7 + w_{11}w_6)$$

where deg $w_i = i$; $Sq^1(w_6) = w_7$ and $Sq^1(w_{10}) = w_{11}$. By computing Ker $Sq^1/\text{Im}\, Sq^1$, we see that

$$\rho(y_4) = w_4, \rho(y_8) = w_8, \rho(y_{12}) = w_6^2, \rho(y_{16}) = w_6w_{10} \text{ and } \rho(y_{20}) = w_{10}^2.$$

In the Serre spectral sequence $\{E_r, d_r\}$ for the mod 2 cohomology of the fibration (5. 1), we find that

$$d_4(1\otimes u_3) = w_4\otimes 1;$$
$$d_8(1\otimes u_7) = w_8\otimes 1;$$
$$d_6(1\otimes u_5u_6) = w_6\otimes u_6, \beta_2^F(w_6\otimes u_5) = w_6\otimes u_6, d_6(w_6\otimes u_5) = w_6^2\otimes 1;$$
$$d_{10}(1\otimes u) = w_{10}\otimes u_6, \beta_2^F(w_{10}\otimes u_5) = w_{10}\otimes u_6, d_6(w_{10}\otimes u_5) = w_6w_{10}\otimes 1;$$
$$d_{10}(1\otimes u_9u_{10}) = w_{10}\otimes u_{10}, \beta_2^F(w_{10}\otimes u_9) = w_{10}\otimes u_{10}, d_{10}(w_{10}\otimes u_9) = w_{10}^2\otimes 1$$

where $\beta_2^F = 1\otimes Sq^1$. These facts, together with (2. 2), imply that

$$b(2) = 1, b(4) = 1, b(6) = 2, b(8) = 2 \text{ and } b(10) = 2.$$

On the other hand, modulo decomposables, $f_{4j} = (1/2)p_j$ if $j = 1, 2, 4$ and $f_{4j} = p_j$ if $j = 3, 5$ (for details see [10, § 3]). From these results, (5. 2) and Theorem 1 it follows that $c(2j) = 1$ for $1 \leq j \leq 5$.

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