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On the Cohomology of the 2-connected Cover of the Loop Space of Simple Lie Groups

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

By

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§ 1. Introduction

Let G be a compact, connected, simply connected, simple, Lie group, p a prime and $t \in H^2(\Omega G; \boldsymbol{F}_p)$ a generator. Let $x_3 \in H^3(G; \boldsymbol{F}_p)$ be a generator satisfying $\sigma(x_3) = t$, where σ is the cohomology suspension. Since G is compact and $H^*(\Omega G; \boldsymbol{F}_p)$ is a Hopf algebra, there exists an integer $d = d(G, p)$ satisfying

$$t^{p^d - 1} \neq 0 \text{ and } t^{p^d} = 0.$$

Let $G\langle 3\rangle$ be the 3-connected cover of G. For a graded module $A = \oplus A_i$ of finite type over $\boldsymbol{F}_p$, we define $P(A, q) = \sum (\dim A_i) q^i$. The purpose of this note is to determine $P(H^*(\Omega G\langle 3\rangle; \boldsymbol{F}_p), q)$. There is a fibering (*) $S^1 \to \Omega G\langle 3\rangle \to \Omega G$. Since $H^*(\Omega G; \boldsymbol{F}_p)$ is a Hopf algebra, there exists a graded algebra $A(G, p)$ satisfying

$$H^*(\Omega G; \boldsymbol{F}_p) = \boldsymbol{F}_p[t]/(t^{p^d}) \otimes A(G, p).$$

On the other hand $H^*(\Omega G; \boldsymbol{Z})$ is torsion free (cf. Bott [8]) and so

$$P(A(G, p), q)^{-1} = \left(\prod_{j=2}^{l} (1 - q^{2m(j)})\right) \circ (1 - q^{2a(G, p)})$$

where $a(G, p) = p^d$ and $m(1) = 1 < m(2) \leq \cdots \leq m(l)$ are the exponent of the weyl group of G. Using the Gysin exact sequence for (*) we have

Theorem 1. $P(H^*(\Omega G\langle 3\rangle; \boldsymbol{F}_p), q) = P(A(G, q), q) \circ (1 + q^{2a(G, p) - 1})$.

On the other hand the number $d(G, p)$ is given by

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Theorem 2. (1) *For a classical type G,*

$$d(G,p)=\begin{cases} r(n,p) & \text{if } G=SU(n),\\ r([n/2],2) & \text{if } G=Spin(n) \text{ and } p=2,\\ r(2n,p) & \text{if } G=Spin(2n+1),\ Spin(2n) \text{ or}\\ & Sp(n) \text{ and } p \text{ is an odd prime,}\end{cases}$$

where $p^{r(n,p)-1}<n\leq p^{r(n,p)}$ *and* $d(G,p)=1$ *if* $G=Sp(n)$ *and* $p=2$.
(2) $d(G,2)=2$ *if* $G=G_2$ *or* F_4 *and* $d(G,2)=4$ if $G=E_6$, E_7 *or* E_8.
(3) *For an odd prime p,* $d(G,p)$ *is given by the following table*:

G	G_2		F_4, E_6		E_7			E_8		
p	5	$\neq 5$	≤ 11	>11	3	$5\leq p\leq 17$	>17	3	$5\leq p\leq 29$	>29
$d(G,p)$	2	1	2	1	3	2	1	3	2	1

§ 2. Proof of Theorem 2

In this section $P(k)=\mathcal{P}^{p^{k-1}}\dots\mathcal{P}^1$ ($\mathcal{P}^i=Sq^{2i}$ if $p=2$). Put $E=\{G_2, F_4, E_6, E_7, E_8\}$ and $PT=\{(G,2);\ G\in E\}\cup\{(G,3);\ G\in E,\ G\neq G_2\}\cup\{(E_8, 5)\}\cup\{(Spin(n),2);\ n\geq 7\}$. For a compact 1-connected simple Lie group G, $H^*(G;Z)$ is p-torsion free if and only if $(G,p)\notin PT$.

Lemma 1. *If* $P(k)x_3$ *is decomposable, then* $t^{p^k}=0$.

Proof. $t^{p^k}=P(k)t=P(k)\sigma(x_3)=\sigma(P(k)x_3)=0$.

Lemma 2. *For any G,* $P(d(G,p))x_3$ *is decomposable.*

Proof. Since deg $P(d)x_3=2a+1$, we need only show $H^{2a+1}(G;\boldsymbol{F}_p)$ is decomposable, which is shown in [4] and [7] for $G\notin E$ and in [1], [2], [3] and [6] for $G\in E$.

Lemma 3. *If* $H^*(G;\boldsymbol{Z})$ *is p-torsion free and* $P(k)x_3\neq 0$, *then* $t^{p^k}\neq 0$.

Proof. Since $H^*(G;\boldsymbol{Z})$ is p-torsion free, as an algebra

$$H_*(G;\boldsymbol{F}_p)=\Lambda(y_3, y_{2m(2)+1}, \dots, y_{2m(l)+1})$$

where deg $y_j=j$ and y_3 is the dual of x_3. Since $P(k)x_3\neq 0$ and $P(k)x_3$

is primitive, the dual of $P(k)x_3$ is not decomposable and so there exists $y_u(u=2p^k+1)$ such that $P(k)_* y_u=y_3$ (cf. [17]). On the other hand

$$H_*(\Omega G;F_p)=F_p[s_2, s_{2m(2)}, \dots, s_{2m(l)}]$$

where deg $s_j=j$. $s_j=\tau'(y_{j+1})$ (τ' is the homology transgression) and s_2 is the dual of t. By the naturalities of τ' we have $P(k)_* s_{u-1}=s_2$ mod decomposables and so $P(k)_* s_{u-1}=s_2$ and so $t^{p^k}=P(k)t\neq 0$.

Lemma 4. *If* $(G,p)=(Spin(n), 2)$, *then* $t^{2^{d(G,p)-1}}\neq 0$.

Proof. Since $H^*(G;F_2)$ is a polynomial algebra for $*\leq n-1$ (cf. [4] or Theorem 3.2 of [10]), the cohomology suspension σ induces a monomorphism $\bar{\sigma}: QH^*(G;F_2)\rightarrow H^{*-1}(\Omega G;F_2)$ for $*\leq n-1$, where Q denotes the indecomposable quotient. Since $P(d(G,2)-1)x_3$ is not decomposables by [3], we have

$$t^{2^{d-1}}=P(d-1)t=P(d-1)\bar{\sigma}(x_3)=\bar{\sigma}(P(d-1)x_3)\neq 0.$$

Lemma 5. *For any* $(G,p)\notin PT$, $P(d-1)x_3$ *is not decomposable.*

Proof. See [6] for $G\notin E$ and [18] for $G\in E$.

Lemma 6. *For any* $(G,p)\in PT$, $(G,p)\neq(Spin(n),2)$, $t^{p^{d-1}}\neq 0$.

Proof. If p is an odd prime we can easily show $H^{2k-1}(\Omega G\langle 3\rangle;F_p)=0$ for $k\leq p^{d-1}$ by making use of the structure of $H^*(G\langle 3\rangle;F_p)$ (cf. [11],, [13], [17]). If $p=2$, $H_*(\Omega G\langle 3\rangle;F_2)$ is known (see [15]). (See also [14]).

Proof of Theorem 2. By Lemma 1 and Lemma 2, $t^{p^d}=0$. Therefore we need only show $t^{p^{d-1}}\neq 0$, since $H^*(\Omega G;F_p)$ is a Hopf algebra. It is shown by Lemma 3, Lemma 4, Lemma 5 and Lemma 6.

Remark 7. (1) Since $P(k)x_3$ is primitive and deg $P(k)x_3$ is odd, $P(k)x_3$ is decomposable if and only if $P(k)x_3=0$.
(2) If $H^*(G;Z)$ is p-torsion free, then $t^{p^k}\neq 0$ if and only if $P(k)x_3\neq 0$ by Lemma 1 and Lemma 3. Moreover by Theorem 2, $t^{p^k}\neq 0$ if and

only if $P(k)x_3 \neq 0$ for any G.

Remark 8. If $G=SU(n)$ or $Spin(n)$, we can prove $t^{p^{d-1}} \neq 0$ by making use of the following (cf. Bott [9] 8-10): There exist maps $j: X_n = \boldsymbol{CP}^{n-1} \to \Omega SU(n)$ and $j': X'_n = SO(n+1)/SO(n-1)\times SO(2) \to \Omega Spin(n)$ satisfying $j^*: H^2(\Omega SU(n); \boldsymbol{Z}) \to H^2(X_n; \boldsymbol{Z})$ and $j'^*: H^2(\Omega Spin(n); \boldsymbol{Z}) \to H^2(X'_n; \boldsymbol{Z})$ are isomorphic.

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Theorem 1 is a special case of a general theory of V. G. Kac and D.H. Peterson on the topology of Kac-Moody Lie groups. They also computed $d(G,p)$ in terms of the Affine Weyl group of G and showed that $d(G,p)=1$ if $p>m(l)$. Their work has prompted me to write this paper. (See [19].)

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