Publ. RIMS, Kyoto Univ.
22 (1986), 543-549

Cohomology Mod 2 of the Classifying Space of $Spin^c(n)$

By

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In this paper we determine the mod 2 cohomology ring and the integral cohomology ring of the classifying space of the compact, connected Lie group $Spin^c(n)$, which is a subgroup of the group of units in the complex Clifford algebra $C_n \otimes C$ (see [1]). The group $Spin^c(n)$ is very important for the orientations in the KO-theory. We also determine (the mod 2 reduction of) the Chern classes of the complex spin representations and the Hopf algebra structure of the mod 2 cohomology ring of $Spin^c(n)$.

The first section is devoted to studying an ideal of a polynomial ring over F_2 which is associated to a symplectic bilinear form on a F_2 vector space and whose variety of geometric points is the union of the maximal isotropic subspaces rational over F_2. We show that the generators of the ideal form a regular sequence and we determine the decomposition of the ideal into prime ideals. These algebraic-geometric results are applied in the second and third sections to compute the mod 2 and integral cohomology ring of $BSpin^c(n)$ and determine the Chern classes of the spin representation of $Spin^c(n)$. In the last section we compute the Steenrod operations and the coproducts of the mod 2 cohomology of $Spin^c(n)$.

Throughout the paper $H^*(X)$ denotes the mod 2 cohomology ring.

1. Let V be an n-dimensional vector space over F_2, V^* its dual, $S(V^*)$ the symmetric algebra over V^* and B a symplectic bilinear form on V. Let h' be the codimension of a B-isotropic subspace of maximum dimension. Consider the following sequence of homogeneous

Communicated by N. Shimada, October 25, 1985.
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elements of length h' in $S(V^*)$:

(1.1) $$B(x, x^2), \ldots, B(x, x^{2^{h'}}).$$

Let Ω be a universal field of $\boldsymbol{F}_2$, $V_\Omega = V \otimes \Omega$, J' the ideal of $S(V^*)$ generated by (1.1) and $\mathrm{Var}\, J'$ the variety of zeros in V_Ω. First we prove the following:

Theorem 1.2. $\mathrm{Var}\, J' = \cup W_\Omega$ *where* W *ranges over the maximal* B*-isotropic subspaces of* V.

Proof. Using the identity

$$B(x^{2^i}, x^{2^j}) = B(x, x^{2^{j-i}})^{2^i} \quad (i \leq j),$$

one see for an element $x \in V_\Omega$, that $x \in \mathrm{Var}\, J'$ if and only if the Ω-subspace

$$N_x = \Omega x + \Omega x^2 + \cdots + \Omega x^{2^{h'}}$$

of V_Ω is B-isotropic. To prove the theorem we must therefore show that $x \in \mathrm{Var}\, J'$ if and only if N_x is stable under the Frobenius, which is shown by computing the dimension of maximal B-isotropic subspaces of V_Ω as was done in the proof of Theorem 2.4 of [5].

Corollary 1.3. *The sequence* (1.1) *is a regular sequence.*

Remark 1.4. All maximal B-isotropic subspaces of V are of the same dimension $n - h'$.

Counting the number of the maximal B-isotropic subspaces, we can prove the following by Bezout's theorem (see Section 3 of [5]):

Theorem 1.5. *The ideal* J' *has a prime decomposition* $J' = \cap \mathfrak{p}_W$, *where* W *ranges over all maximal* B*-isotropic subspaces and* $\mathfrak{p}_W = \mathrm{Ker}\{S(V^*) \to S(W^*)\}$.

Let Q be a quadratic form on V. Then $B(x, y) = Q(x+y) + Q(x) + Q(y)$ is a symplectic bilinear form. Let h be the codimension of a Q-isotropic subspace of maximum dimension. Then we can easily get the following:

Lemma 1.6. *$h=h'+\varepsilon$ where $\varepsilon=0$, 1, or 1 depending on Q is real, complex, or quaternion respectively.*

See Section 3 of [5].

2. First consider the central extension

$$0 \longrightarrow S^1 \xrightarrow{i} \tilde{V}^c \xrightarrow{\pi} V \longrightarrow 1 \tag{2.1}$$

where V is an elementary abelian 2-group. As is well known that (2.1) is classified by an element $b \in H^3(BV; \boldsymbol{Z})$. Let ρ be the mod 2 reduction and $B'=\rho(b)$. Since $\operatorname{Im}\rho=\operatorname{Im} Sq^1$, there is an element $Q \in H^2(BV)$ such that $B'=Sq^1Q$. Note that $H^*(BV)$ is isomorphic to $S(V^*)$ and so Q is a quadratic form and $B'=B(x, x^2)$. Let W be a maximal B-isotropic subspace. Then $\tilde{W}^c=\pi^{-1}(W)=W\times S^1$ since ρ is a monomorphism. Let $\chi: \tilde{W}^c \to S^1$ be a complex representation of $\tilde{W}^c$ whose restriction to S^1 is the standerd representation ι and let Δ be the representation of $\tilde{V}^c$ obtained by inducing χ from $\tilde{W}^c$ to $\tilde{V}^c$. Then Δ has dimension $2^{h'}$ and $i^*(\Delta)=2^{h'}\iota$. Now we can prove the following:

Theorem 2.2. *As an algebra $H^*(B\tilde{V}^c)$ is isomorphic to $S(V^*)/J' \otimes F_2[e]$, where J' is the ideal generated by $B(x, x^2), \dots, B(x, x^{2^{h'}})$ and $e \in H^{2^{h'+1}}(BV^c)$ is the Euler class of Δ.*

Proof. Consider the Serre spectral sequence for the fibering $BS^1 \xrightarrow{i} B\tilde{V}^c \xrightarrow{\pi} BV$

$$E_2^{p,q}=H^p(BV; H^q(BS^1)) \Rightarrow E_\infty=\operatorname{Gr}(H^*(B\tilde{V}^c)).$$

Let z be a generator of $H^2(BS^1)$ so that $H^*(BS^1)=F_2[z]$. The element z is transgressive with $\tau(z)=B'=B(x, x^2)$. Therefore

$$\tau(z^{2^k})=\tau(S_k z)=S_k B(x, x^2)=B(x, x^{2^{k+1}})$$

where $S_k=Sq^{2^k}\dots Sq^2$. Since $B(x, x^2), \dots, B(x, x^{2^{h'}})$ is a regular sequence by Corollary 1.3, we can easily get

$$E_{2^{h'}+2}=S(V^*)/J' \otimes F_2[z^{2^{h'}}].$$

On the other hand $i^*(e)=z^{2^{h'}}$ since $i^*(\Delta)=2^{h'}\iota$. Hence $E_\infty=E_{2^{h'}+2}$ and we get the theorem.

Theorem 2.3. *The homomorphism $H^*(B\tilde{V}^c) \to \prod_W H^*(B\tilde{W}^c)$ is injec-*

tive, where the product is taken over all maximal B-isotropic subspaces of V.

Proof. Recall the fact that Ker $\{H^*(B\tilde{V}^c)\to H^*(B\tilde{W}^c)\}$ is equal to $(p_W/J')\otimes F_2[e]$. Therefore Theorem 2.3 follows from Theorem 1.5.

Remark 2.4. We can determine the Chern classes of Δ using Theorem 2.3.

3. First recall the fact that the extension $0\to Z/2\to Spin(n)\to SO(n)\to 1$ is classified by $w_2\in H^2(BSO(n))$ and the extension

$$0\longrightarrow S^1\xrightarrow{i} Spin^c(n)\xrightarrow{\pi} SO(n)\longrightarrow 1 \tag{3.1}$$

is classified by $b'\in H^3(BSO(n);Z)$ where $\rho(b')=w_3=Sq^1w_2$ since $Spin^c(n)=Spin(n)\times_{Z/2}S^1$. Let V be the diagonal matrices in $SO(n)$, j: $V\to SO(n)$ the inclusion and $\tilde{V}^c=\pi^{-1}(V)$. Then the extension $0\to S^1\to \tilde{V}^c\to V\to 1$ is classified by $b=j^*(b')$ and so $Q=j^*(w_2)$ and $B'=j^*(w_3)$ in Section 2. Now using Table 6.2 of [5] and Lemma 1.6, we have the following:

Lemma 3.2. *Let h' be the codimension of a B-isotropic subspace of maximum dimension where B is the associated bilinear form of $Q=j^*(w_2)$. Then $h'=\left[\frac{n-1}{2}\right]$ and $2^{h'}$ is equal to the dimension of the complex spin representation of $Spin^c(n)$.*

Consider the following commutative diagram:

$$\begin{array}{ccccc} SO(n)/V & \longrightarrow & B\tilde{V}^c & \xrightarrow{\tilde{j}} & BSpin^c(n) \\ \| & & \downarrow & & \downarrow \\ SO(n)/V & \longrightarrow & BV & \xrightarrow[j]{} & BSO(n) \end{array}$$

where the horizontal lines are fiberings. Since the Serre spectral sequence for the fibering $SO(n)/V\to BV\to BSO(n)$ collapses, the Serre spectral sequence for $SO(n)/V\to B\tilde{V}^c\to BSpin^c(n)$ also collapses. Therefore we have the following:

Lemma 3.3. *$H^*(BV)$ is a free module over $H^*(BSO(n))$ and $H^*(B\tilde{V}^c)$ is a free module over $H^*(BSpin^c(n))$.*

Since $j^*(S_kw_3)=S_kj^*(w_3)=S_kB(x, x^2)=B(x, x^{2^{k+1}})$, we have that

$j^*(w_3)$, ..., $j^*(S_{h'-1}w_3)$ form a regular sequence. We have therefore the following by Lemma 3.3:

Lemma 3.4. *The sequence* w_3, ..., $S_{h'-1}w_3$ *is a regular sequence.*

Put $V_0=\{x\in V; B(x, y)=0$ for all $y\in V\}$. Then $\dim V_0=0$ if n is odd and $\dim V_0=1$ if n is even. There is a unique spin representation Δ_{2m+1} if $n=2m+1$ and there are two spin representations $\Delta^{\pm}_{2m}$ if $n=2m$. Consider the following orthogonal decomposition:

$$V\cong W_1\oplus W_1^*\oplus V_0.$$

Then $W=W_1\oplus V_0$. Put $\tilde{W}^c=\pi^{-1}(W)$ then $\tilde{W}^c=(W_1\oplus V_0)\times S^1$ and

$$\Delta|_{\tilde{W}^c}=(\text{reg } W_1)\otimes\theta\otimes\iota$$

where reg W_1 is the regular representation, $\dim\theta=1$ and θ is trivial (resp. non trivial) if $\Delta=\Delta^+_{2m}$ (resp. if $\Delta=\Delta^-_{2m}$). Therefore $i^*(\Delta)=2^{h'}\iota$. In the Serre spectral sequence for $BS^1\to BSpin^c(n)\to BSO(n)$, z is transgressive with $\tau(z)=w_3$. Now we have the following:

Theorem 3.5. *As an algebra* $H^*(BSpin^c(n))$ *is isomorphic to* $H^*(BSO(n))/J'\otimes F_2[e]$, *where* J' *is the ideal generated by* w_3, S_1w_3, ..., $S_{h'-1}w_3$ *and* e *is the Euler class of the complex spin representation* Δ.

This follows by computing the Serre spectral sequence for $BS^1\to BSpin^c(n)\to BSO(n)$ as was done in the proof of Theorem 2.2.

Now we determine the Chern classes of Δ. By Theorem 2.3 and Lemma 3.3, we need only determine $c_i(\Delta|_{\tilde{W}^c})$. By a similar method to that of [5], we have the following (cf. Section 5 of [5]):

Theorem 3.6. (1) *The classes* $c_i(\Delta^{\pm}_{2m})$ *for* $i<2^{h'}$ *are independent of* $\pm$.
(2) $c_i(\Delta)=0$ *for* $i\neq 2^{h'}$, $2^{h'}-2^j$, $(j=0, 1, \dots, h')$.
(3) *The sequence* $\{c_i(\Delta); i=2^{h'}, 2^{h'}-2^j, (j=0, 1, \dots, h'-1)\}$ *is a regular sequence in* $H^*(BSpin^c(n))$.

On the other hand for the integral cohomology we can prove the following:

Theorem 3.7. *The torsion elements of* $H^*(BSpin^c(n);\boldsymbol{Z})$ *are of order* 2.

This follows by computing the Sq^1-cohomology of $H^*(BSpin^c(n))$ as was done in [4].

Remark 3.8. The natural map $H^*(BSpin^c(n);\boldsymbol{Z})\to H^*(BSpin^c(n)) \times H^*(BSpin^c(n);\boldsymbol{R})$ is injective (see [4]).

4. Let $s=s(n)$ be the integer given by $2^{s-1}<n\leq 2^s$. Define $x_j\in H^j(Spin^c(n))$ by $\sigma(\pi^*(w_{j+1}))$ where σ is the cohomology suspension. Note that $w_j=0$ if $j>n$. By Theorem 3.5, as an algebra $H^*(BSpin^c(n))$ is isomorphic to

$$\boldsymbol{F}_2[w'_j;2\leq j\leq n,\ j\neq 2^{j'}+1\ \ (j'\geq 1)]/(r)$$

for $*\leq 2^s+1$, where

$$r=\sum_{i=2}^{2^s-1} w'_i w'_{2^s+1-i}+\text{higher} \tag{4.1}$$

and $w'_j=\pi^*(w_j)$ (w'_j is decomposable if $j=2^{j'}+1$). ($S_k w_3=\sum_{i=0}^{2^k} w_i w_{2^k+1-i}$ mod $\tilde{H}^*(BO)^3$ can be shown by induction on k using the Wu's formula (see 15.7 of [2])).

On the other hand by Theorem 1.1 of [3], there exists $a\in H^{2^s-1}(Spin^c(n))$ which is transgressive with respect to $Spin^c(n)\to Spin^c(n)/T\to BT$, where T is a maximal torus, so that

$$H^*(Spin^c(n))=\Delta(x_j;1\leq j<n,\ j\neq 2^{j'}(j'\geq 1))\otimes\Delta(a) \tag{4.2}$$

where $\Delta(\ldots)$ means that $(\ldots)$ is a simple system of generators. Since $\bar{\phi}(x_j)=0$ by definitions where $\bar{\phi}$ denotes the reduced coproduct of $H^*(Spin^c(n))$,

$$\bar{\phi}(a)=\sum_{i+j=2^{s-1}}\alpha_i x_{2i}\otimes x_{2j-1}\qquad(\alpha_i\in\boldsymbol{F}_2) \tag{4.3}$$

by Theorem 2.2 of [3] (see also Lemma 3.4 of [3]). Then by the Rothenberg-Steenrod spectral sequence ([6]) and (4.1), $\alpha_i=1$ for any i. Then we have (see the proof of Theorem 3.2 of [3]):

Theorem 4.4. (1) *In* (4.2) $\bar{\phi}(x_j)=0$ *and* $\bar{\phi}(a)=\sum_{i+j=2^{s-1}} x_{2i}\otimes x_{2j-1}$

$(x_j=0$ *if* $j=2^{j'}(j'\geq 1))$.

(2) $Sq^i x_j=\binom{j}{i}x_{j+i}(x_{2j}=x_j^2)$, $Sq^1 a=\sum_{\substack{i+j=2^s-1\\ i<j}} x_{2i}x_{2j}$ *and* $Sq^i a=0$ *for* $i\geq 2$ $(a^2=0)$.

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