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A Torelli-like theorem for higher-dimensional function fields

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Abstract. We prove a Torelli-like theorem for higher-dimensional function fields, from the point of view of "almost-abelian" anabelian geometry.

Keywords. Function fields, 1-motives, mixed Hodge structures, anabelian geometry

1. Introduction

The classical *Torelli Theorem*, in its cohomological form, can be stated as follows:

Theorem. *Let X be a smooth compact complex curve. Then the isomorphism type of X is determined by the singular cohomology group $\mathrm{H}^1(X,\mathbb{Z})$, endowed with its canonical polarized Hodge structure.*

In this paper, we develop and prove a Torelli-like theorem for function fields of transcendence degree ≥ 2 over algebraically closed subfields of $\mathbb{C}$. As expected, one must include not only H^1 (with its *mixed* Hodge structure), but also some additional data arising from the cup product, akin to the polarization in the theorem above. In our context, the *two-step nilpotent* information, encoded as the kernel of the cup-product, turns out to be sufficient. While the bound on the transcendence degree is necessary in our context (see Section 1.3) our result works even with *rational* coefficients, in contrast to the theorem above.

1.1. Main result

Let k be an algebraically closed field and $\sigma : k \hookrightarrow \mathbb{C}$ an embedding into the complex numbers. Let Λ be any subring of $\mathbb{Q}$. For a k-variety X, consider $X^{\mathrm{an}} := X(\mathbb{C})$ (computed

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via σ) endowed with the complex topology, and define the *Betti cohomology* of X in the usual way as

$$\mathrm{H}^i(X,\Lambda) := \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}},\Lambda).$$

Following Deligne [14,15], one can endow $\mathrm{H}^i(X,\Lambda)$ with a canonical *mixed Hodge structure* over Λ. We write $\Lambda(j)$ for the j-th Tate–Hodge structure over Λ, which is the unique pure Hodge structure over Λ with underlying module $(2\pi i)^j \cdot \Lambda$ which is of Hodge type $(-j,-j)$. As usual, we write $\mathrm{H}^i(X,\Lambda(j)) := \mathrm{H}^i(X,\Lambda) \otimes \Lambda(j)$.

Let $K|k$ be a function field, and X a *model* of $K|k$, i.e. X is an integral k-variety whose function field is K. Define

$$\mathrm{H}^i(K|k,\Lambda(j)) := \varinjlim_U \mathrm{H}^i(U,\Lambda(j)),$$

where U varies over the nonempty open k-subvarieties of X. We consider $\mathrm{H}^i(K|k,\Lambda(j))$ as a mixed Hodge structure of possibly infinite rank. This construction does not depend on the choice of model X of $K|k$.

The cup-product in singular cohomology yields a well-defined cup-product on

$$\mathrm{H}^*(K|k,\Lambda(*)) := \bigoplus_{i\geq 0} \mathrm{H}^i(K|k,\Lambda(i)),$$

making it into a graded-commutative ring. We write

$$\mathcal{R}(K|k,\Lambda) := \{(x,y) \mid x,y \in \mathrm{H}^1(K|k,\Lambda(1)),\ x \cup y = 0\}$$

for the set of pairs of elements of $\mathrm{H}^1(K|k,\Lambda(1))$ whose cup-product vanishes. With this notation and terminology, we may state our main result as follows (see Theorem 6.1 for a precise formulation).

Theorem 1.1. *In the above context, assume furthermore that* $\mathrm{trdeg}(K|k) \geq 2$. *Then the isomorphism type of* $K|k$*, as fields, is determined by the following data:*

(1) *The mixed Hodge structure* $\mathrm{H}^1(K|k,\Lambda(1))$.

(2) *The subset* $\mathcal{R}(K|k,\Lambda) \subset \mathrm{H}^1(K|k,\Lambda(1)) \times \mathrm{H}^1(K|k,\Lambda(1))$.

1.2. A comment about the proof

The proof of Theorem 1.1 combines results concerning 1-motives and their Hodge realizations with arguments from birational anabelian geometry. The key ingredients include the following:

(1) The comparison of a 1-motive with its Hodge realization, due to Deligne [15]. We will only need a simple case of this comparison which is summarized in Lemma 4.1.

(2) The construction of the Picard 1-motive of a smooth quasi-projective variety and the calculation of its Hodge realization. This is due essentially to Serre [27], and/or the works of Barbieri-Viale and Srinivas [6] and Ramachandran [25].

(3) Methods for reconstructing function fields over algebraically closed fields in birational anabelian geometry, which are similar to those developed by Bogomolov [7], Bogomolov–Tschinkel [9, 10], Pop [21, 23, 24], and the author [29–31].

In addition to the above points, there are several nontrivial hurdles one must overcome, specifically in the case where $\Lambda = \mathbb{Q}$, where the known "global" anabelian techniques (e.g. from Pop [23, 24] and/or Bogomolov–Tschinkel [9, 10]) break down, as one can no longer distinguish between the "divisible" and "nondivisible." We overcome these difficulties by relying on arguments surrounding the connection between algebraic dependence and the cup-product, which refine some of the key ideas from [29, 31]. The local theory developed for the usual Galois-theoretical contexts in [8, 22, 28, 30] must also be modified appropriately, and this is handled in Appendix A.

As we see it, the primary novelty of this work comes from the fact that it applies *anabelian* techniques in a purely *motivic* setting, a combination that seems to be new. In fact, the object $\mathrm{H}^*(K|k, \Lambda(1))$ behaves quite similarly to the Galois cohomology of K in several ways, and this point of view plays a motivating role throughout the paper. For example:

(1) There is a *Kummer map* $K^\times \to \mathrm{H}^1(K|k, \Lambda(1))$ whose kernel is $k^\times$.

(2) There are *residue maps* associated to divisorial valuations of $K|k$.

(3) The cohomological dimension agrees with the transcendence degree of $K|k$.

These properties, among others, are discussed in Sections 2 and 3. The interaction between Kummer theory and the mixed Hodge structure of $\mathrm{H}^1(K|k, \Lambda(1))$ also plays a crucial role in the proof of our main result, as discussed in Sections 6.1 and 6.2.

1.3. A comment about the one-dimensional case

The assumption that $\mathrm{trdeg}(K|k) \geq 2$ in Theorem 1.1 arises from a similar assumption appearing in Theorem 5.1, which is the primary anabelian result used in the proof of our main theorem. There are a few key points in the proof of Theorem 5.1 where this assumption is used in a fundamental way:

(1) In the *local theory*, this assumption is used in the process of detecting valuations from the given data. See Proposition A.6 in particular.

(2) The proof has two key *synchronization steps*, both of which rely on this assumption. The first occurs in Proposition 5.7 whose proof relies on the local theory. And the second occurs in Proposition 5.8, which uses this assumption along with a *birational Bertini result* (Fact 5.4) to obtain an abundance of so-called *general elements*.

(3) This assumption is used in the final *collineation step* of the proof of Theorem 5.1 before applying the fundamental theorem of projective geometry [4, 18]. See Proposition 5.9.

If $\mathrm{trdeg}(K|k) = 1$ and X denotes the smooth proper model of $K|k$, then the set $\mathcal{R}(K|k, \Lambda)$ appearing in Theorem 1.1 is simply $\mathrm{H}^1(K|k, \Lambda(1))^2$, due to cohomological dimension reasons (see Fact 3.1). Thus, the data appearing in Theorem 1.1 reduces to the mixed Hodge structure $\mathrm{H}^1(K|k, \Lambda(1))$ alone. In this case, the genus g of X can be easily recovered from $\mathrm{H}^1(K|k, \Lambda(1))$, for example, since its smallest weight portion has rank $2g$.

In the case where $g \geq 2$, one can use the work of Zilber [33] along with some of the arguments below to see that the isomorphism type of X is determined by $\mathrm{H}^1(K|k, \mathbb{Z}(1))$ along with the collection of kernels of the residue maps $\partial_v : \mathrm{H}^1(K|k, \mathbb{Z}(1)) \to \mathbb{Z}$ associated to the divisorial valuations v of $K|k$. However, it is currently unclear whether the same holds with more general coefficient rings. In any case, such a datum is best viewed within the framework of *cycle modules* in the sense of Rost [26]. The implications of Zilber's work [33] and the results of the present paper in such a context will be investigated in future work.

2. Cohomology

Throughout the paper, we work with a fixed algebraically closed field k endowed with an embedding $\sigma : k \hookrightarrow \mathbb{C}$, and a subring Λ of $\mathbb{Q}$ which will be fixed as our coefficient ring. By a *k-variety* we mean a separated scheme of finite type over k, and *morphisms* of k-varieties are morphisms over $\operatorname{Spec} k$.

Given a k-variety X, write $X^{\mathrm{an}} := X(\mathbb{C})$, computed via σ, and endowed with the complex topology. We will work with *Betti cohomology* (with respect to σ) defined in the usual way as the singular cohomology of X^{an}:

$$\mathrm{H}^i(X, \Lambda) := \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}}, \Lambda).$$

This is a Λ-module which is canonically endowed with a mixed Hodge structure [14, 15]. We write $\mathrm{H}^i(X, \Lambda(j)) := \mathrm{H}^i(X, \Lambda) \otimes \Lambda(j)$ for its j-th Tate twist. Of course, the construction of $\mathrm{H}^i(X, \Lambda(j))$ depends on the choice of σ, but we will exclude it from the notation while ensuring it is understood from context.

2.1. Models

Let K be a function field over k. By a *model* of $K|k$, we mean an integral quasi-projective k-variety whose function field is K. Given such a model, we define

$$\mathrm{H}^i(K|k, \Lambda(j)) := \varinjlim_U \mathrm{H}^i(U, \Lambda(j)),$$

where U varies over the nonempty open k-subvarieties of X, considered as a (possibly infinite-rank) mixed Hodge structure over Λ. The cup-product in singular cohomology yields a natural *cup-product*:

$$\cup : \mathrm{H}^i(K|k, \Lambda(j)) \otimes_\Lambda \mathrm{H}^{i'}(K|k, \Lambda(j')) \to \mathrm{H}^{i+i'}(K|k, \Lambda(j + j'))$$

which makes $\mathrm{H}^*(K|k, \Lambda(*)) := \bigoplus_{i \geq 0} \mathrm{H}^i(K|k, \Lambda(i))$ into a graded-commutative Λ-algebra.

For a *smooth* model X of $K|k$ and nonempty open k-subvariety U of X, the morphism

$$\mathrm{H}^1(X, \Lambda(1)) \to \mathrm{H}^1(U, \Lambda(1)),$$

induced by the inclusion $U \hookrightarrow X$, is known to be injective. Thus the structure map $\mathrm{H}^1(X, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1))$ is injective as well. In other words, the underlying Λ-module of $\mathrm{H}^1(K|k, \Lambda(1))$ can be considered as an *inductive union* of $\mathrm{H}^1(U, \Lambda(1))$ as U varies over the smooth models of $K|k$. We will tacitly identify $\mathrm{H}^1(U, \Lambda(1))$ with its image in $\mathrm{H}^1(K|k, \Lambda(1))$ whenever U is such a smooth model of $K|k$.

2.2. *Functoriality*

Let $\iota : L \hookrightarrow K$ be a k-embedding of function fields over k. By a *model* of ι, we mean a dominant morphism $f : X \to Y$ where X is a model of $K|k$ and Y is a model of $L|k$ which induces ι on the level of function fields. Given such a model $f : X \to Y$ of ι, we obtain a canonical map

$$\iota_* : \mathrm{H}^i(L|k, \Lambda(j)) = \varinjlim_U \mathrm{H}^i(U, \Lambda(j)) \xrightarrow{f^*} \varinjlim_U \mathrm{H}^i(f^{-1}(U), \Lambda(j)) \to \mathrm{H}^i(K|k, \Lambda(j)),$$

where U varies over the nonempty open k-subvarieties of Y. This morphism does not depend on the choice of model f, and this construction makes $\mathrm{H}^i(K|k, \Lambda(j))$ (covariantly) functorial in K with respect to k-embeddings.

2.3. *Kummer theory*

It is well-known that $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \Lambda(1)) = \Lambda(0)$ as mixed Hodge structures. Let $K|k$ be a function field and $f \in K^\times$ be given. Let X be a model of $K|k$ and U an open k-subvariety of X such that $f \in \mathcal{O}^\times(U)$. Then f corresponds to a morphism $f : U \to \mathbb{G}_\mathrm{m}$, and hence induces a canonical map of Λ-modules

$$\Lambda = \mathrm{H}^1(\mathbb{G}_\mathrm{m}, \Lambda(1)) \xrightarrow{f^*} \mathrm{H}^1(U, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1)).$$

We write $\kappa_U(f)$ for the image of $1 \in \Lambda$ in $\mathrm{H}^1(U, \Lambda(1))$ with respect to this map.

It is a straightforward consequence of the Künneth formula that the corresponding map

$$\kappa_U : \mathcal{O}^\times(U) \to \mathrm{H}^1(U, \Lambda(1))$$

is a homomorphism of abelian groups. Indeed, suppose we are given a pair of functions $f, g \in \mathcal{O}^\times(U)$. The morphism $U \to \mathbb{G}_\mathrm{m}$ associated to the product $f \cdot g$ factors as

$$U \xrightarrow{(f,g)} \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \xrightarrow{\mu} \mathbb{G}_\mathrm{m},$$

where μ is multiplication in $\mathbb{G}_\mathrm{m}$. Thus, the corresponding map on $\mathrm{H}^1(-, \Lambda(1))$ factors as follows:

$$\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \Lambda(1)) \to \mathrm{H}^1(\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}, \Lambda(1)) \to \mathrm{H}^1(U, \Lambda(1)).$$

On the other hand, the Künneth formula yields an isomorphism

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \Lambda(1)) \oplus \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \Lambda(1)) \cong \mathrm{H}^1(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}, \Lambda(1)),$$

which is the sum of the maps associated to the two projections $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$. From this it follows that the induced map

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \Lambda(1)) \oplus \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \Lambda(1)) \cong \mathrm{H}^1(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}, \Lambda(1)) \to \mathrm{H}^1(U, \Lambda(1))$$

is the sum of the maps $\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \Lambda(1)) \to \mathrm{H}^1(U, \Lambda(1))$ associated to f and g. Tracing through all the definitions, it follows that indeed $\kappa_U(f \cdot g) = \kappa_U(f) + \kappa_U(g)$.

As any element of $K^\times$ is contained in $\mathcal{O}^\times(U)$ for any sufficiently small U as above, we obtain a map

$$\kappa_K : K^\times \to \mathrm{H}^1(K|k, \Lambda(1))$$

by taking the colimit of the maps κ_U. This map κ_K is therefore also a morphism of abelian groups. It is a straightforward consequence of the definitions that $k^\times$ is contained in the kernels of κ_U and κ_K.

Throughout the paper we will write $\mathscr{K}_\Lambda(K|k) := (K^\times/k^\times) \otimes_{\mathbb{Z}} \Lambda$. For $t \in K^\times$, we write $\bar{t}$ for the image of t in $\mathscr{K}_\Lambda(K|k)$. We will always use additive notation for the Λ-module $\mathscr{K}_\Lambda(K|k)$, while $K^\times/k^\times$ will be written multiplicatively. The assignment $K \mapsto \mathscr{K}_\Lambda(K|k)$ is clearly functorial in K with respect to k-embeddings. For a k-embedding $\iota : L \hookrightarrow K$, we will write $\iota_* : \mathscr{K}_\Lambda(L|k) \to \mathscr{K}_\Lambda(K|k)$ for the corresponding morphism of Λ-modules. The map κ_K defined above induces a morphism of Λ-modules

$$\kappa_K^\Lambda : \mathscr{K}_\Lambda(K|k) \to \mathrm{H}^1(K|k, \Lambda(1)),$$

which is natural in K with respect to k-embeddings.

The maps κ_U and κ_K may be seen as analogues of the usual Kummer map in étale cohomology. To see this, suppose that $f \in \mathcal{O}^\times(U)$ is given, considered also as a morphism $f : U \to \mathbb{G}_{\mathrm{m}}$. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^\times(\mathbb{G}_{\mathrm{m}}) & = & \mathrm{H}^0_{\text{ét}}(\mathbb{G}_{\mathrm{m}}, \mathcal{O}^\times) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{\mathrm{m}}, \mu_n) \\ \downarrow & & & & \downarrow \\ \mathcal{O}^\times(U) & = & \mathrm{H}^0_{\text{ét}}(U, \mathcal{O}^\times) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(U, \mu_n) \end{array}$$

The horizontal maps are the usual Kummer maps in étale cohomology, obtained from the exact sequence of étale sheaves

$$1 \to \mu_n \to \mathcal{O}^\times \xrightarrow{z \mapsto z^n} \mathcal{O}^\times \to 1,$$

while the vertical maps are induced by $f : U \to \mathbb{G}_{\mathrm{m}}$. If we write $\mathbb{G}_{\mathrm{m}} = \operatorname{Spec} k[t^{\pm 1}]$, then the image of $t \in \mathcal{O}^\times(\mathbb{G}_{\mathrm{m}})$ in $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_{\mathrm{m}}, \mu_n) = \mathbb{Z}/n$ agrees with the generator $1 \in \mathbb{Z}/n$, while its image in $\mathcal{O}^\times(U)$ agrees with f.

The relationship between κ_U and κ_K and the usual Kummer maps in étale cohomology arises from Artin's comparison isomorphism between singular and étale cohomology for smooth k-varieties [5, exp. XI]. Consider, for U a smooth open k-subvariety of X, the morphism $\mathrm{H}^1(U, \mathbb{Z}(1)) \to \mathrm{H}^1_{\text{ét}}(U, \mu_n)$ which is obtained by composing the natural map $\mathrm{H}^1(U, \mathbb{Z}(1)) \to \mathrm{H}^1_{\text{sing}}(U^{\text{an}}, \mu_n)$ with the comparison isomorphism $\mathrm{H}^1_{\text{sing}}(U^{\text{an}}, \mu_n) \cong \mathrm{H}^1_{\text{ét}}(U, \mu_n)$. This map is natural in U, and thus, by passing to the colimit, we also obtain a natural map $\mathrm{H}^1(K|k, \mathbb{Z}(1)) \to \mathrm{H}^1_{\text{ét}}(K, \mu_n)$. The discussion above shows that the maps κ_U and κ_K are compatible with the usual Kummer maps to étale cohomology with respect to these morphisms, in the sense that the following two diagrams commute:

$$
\begin{array}{ccc}
\mathcal{O}^\times(U) & \xrightarrow{\kappa_U} & \mathrm{H}^1(U, \mathbb{Z}(1)) \\
& \searrow^{\text{Kummer}} & \downarrow{\scriptstyle\text{Artin}} \\
& & \mathrm{H}^1_{\text{ét}}(U, \mu_n)
\end{array}
\qquad
\begin{array}{ccc}
K^\times & \xrightarrow{\kappa_K} & \mathrm{H}^1(K|k, \mathbb{Z}(1)) \\
& \searrow^{\text{Kummer}} & \downarrow{\scriptstyle\text{Artin}} \\
& & \mathrm{H}^1_{\text{ét}}(K, \mu_n)
\end{array}
$$

Here the maps labeled "Kummer" are the usual Kummer maps, and those labeled "Artin" are the ones obtained from Artin's comparison isomorphism as described above. In any case, we are thus justified in henceforth using the name "Kummer map" for both κ_U and κ_K.

2.4. *Milnor K-theory*

The *Milnor K-ring* of K is the graded-commutative ring which is denoted and defined as follows:

$$\mathrm{K}^{\mathrm{M}}_*(K) := \frac{\mathrm{T}_*(K^\times)}{\langle x \otimes (1-x) \mid x \in K \smallsetminus \{0,1\} \rangle}.$$

Here $\mathrm{T}_*(K^\times)$ denotes the (graded) tensor algebra of $K^\times$, considered as an abelian group. It is customary to write $\{f_1, \dots, f_n\} \in \mathrm{K}^{\mathrm{M}}_n(K)$ for the product of $f_1, \dots, f_n \in K^\times = \mathrm{K}^{\mathrm{M}}_1(K)$ in this ring.

Since $\mathrm{H}^2(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, \Lambda(2)) = 0$, by functoriality we have $\kappa_K(t) \cup \kappa_K(1-t) = 0$ in $\mathrm{H}^2(K|k, \Lambda(2))$ for all $t \in K \smallsetminus \{0, 1\}$. Hence the universal property of $\mathrm{K}^{\mathrm{M}}_*(K)$ shows that κ_K extends to a morphism of graded-commutative rings

$$\kappa_K^* : \mathrm{K}^{\mathrm{M}}_*(K) \to \mathrm{H}^*(K|k, \Lambda(*)).$$

The r-th component of this map, denoted $\kappa_K^r : \mathrm{K}^{\mathrm{M}}_r(K) \to \mathrm{H}^r(K|k, \Lambda(r))$, is uniquely determined by the rule $\kappa_K^r\{f_1, \dots, f_r\} := \kappa_K(f_1) \cup \dots \cup \kappa_K(f_r)$ for $f_1, \dots, f_r \in K^\times$.

2.5. *Residues*

Suppose that X is a smooth k-variety and Z is a smooth closed k-subvariety of X which is pure of codimension 1. Put $U := X \smallsetminus Z$. Recall that one has a so-called *residue morphism* associated to (X, Z):

$$\partial_{X,Z} : \mathrm{H}^{i+1}(U, \Lambda(j+1)) \to \mathrm{H}^i(Z, \Lambda(j)).$$

The following fact seems to be well-known. For a purely algebraic proof of this assertion, which works with any suitable cohomology theory, we refer the reader to [13, Proposition 2.6.5] and the surrounding discussion. In the statement of Fact 2.1 and the rest of the paper, the word “fibre” will refer to the scheme-theoretic fibre.

Fact 2.1. *In the above context, assume furthermore that* $f : X \to \mathbb{A}^1$ *is a morphism of* k*-varieties such that* Z *is the fibre of* f *above* 0*. Consider its restriction* $f : U \to \mathbb{G}_{\mathrm{m}}$ *and the associated function* $f \in \mathcal{O}^\times(U)$*. Let* $\alpha \in \mathrm{H}^i(X, \Lambda(j))$ *be given, and write* $\alpha|_U$ *for its image in* $\mathrm{H}^i(U, \Lambda(j))$ *and* $\alpha|_Z$ *for its image in* $\mathrm{H}^i(Z, \Lambda(j))$*. Then one has* $\partial_{X,Z}(\kappa_U(f) \cup \alpha|_U) = \alpha|_Z$*.*

2.6. Divisorial valuations

Suppose that v is a *divisorial valuation* of $K|k$, i.e. v arises from a prime divisor on some model of $K|k$. Equivalently, v satisfies the following properties:

(1) The value group vK of v is isomorphic (as an ordered abelian group) to $\mathbb{Z}$. Since k is algebraically closed, this automatically implies that v is trivial on k.

(2) The residue field Kv of v is finitely-generated over k, and it has transcendence degree $\mathrm{trdeg}(K|k) - 1$ over k.

In addition to the notations vK for the value group and Kv for the residue field, we will write $\mathcal{O}_v$ for the valuation ring, $\mathfrak{m}_v$ for the valuation ideal, $\mathrm{U}_v := \mathcal{O}_v^\times$ for the v-units and $\mathrm{U}_v^1 := 1 + \mathfrak{m}_v$ for the principal v-units.

Let X be a model of $K|k$. We say that X is a v-*model* provided that the following conditions hold true:

(1) The valuation v has a (necessarily unique) centre ξ_v on X.

(2) The centre ξ_v is a regular codimension 1 point in X.

Given a v-model X of $K|k$ with v-centre ξ_v, we write X_v for the closure of ξ_v in X. An open k-subvariety of X will be called v-*open* provided that $\xi_v \in U$, or equivalently that $U \cap X_v$ is dense in X_v. Note that any v-open k-subvariety U of a v-model X of $K|k$ is again a v-model, and one has $U \cap X_v = U_v$. If X is any v-model of $K|k$, we define

$$\mathrm{H}^i(\mathcal{O}_v|k, \Lambda(j)) := \varinjlim_U \mathrm{H}^i(U, \Lambda(j)),$$

where U varies over the v-open k-subvarieties of X. This clearly does not depend on the choice of v-model X.

2.7. Residues for divisorial valuations

Let v be a divisorial valuation of $K|k$ and X a v-model of $K|k$. The restriction maps

$$\mathrm{H}^i(X, \Lambda(j)) \to \mathrm{H}^i(X \smallsetminus X_v, \Lambda(j)),\ \mathrm{H}^i(X, \Lambda(j)) \to \mathrm{H}^i(X_v, \Lambda(j))$$

are compatible with passage to v-open k-subvarieties of X. Passing to the colimit over the v-open k-subvarieties of X, we obtain canonical maps

$$\mathfrak{u}_v : \mathrm{H}^i(\mathcal{O}_v|k, \Lambda(j)) \to \mathrm{H}^i(K|k, \Lambda(j)), \quad \mathfrak{s}_v : \mathrm{H}^i(\mathcal{O}_v|k, \Lambda(j)) \to \mathrm{H}^i(Kv|k, \Lambda(j)).$$

Similarly, the residue maps ∂_{X,X_v} are compatible with passage to v-open k-subvarieties of X, and, passing to the colimit, we thereby obtain a *residue map* associated to v:

$$\partial_v : \mathrm{H}^{i+1}(K|k, \Lambda(j+1)) \to \mathrm{H}^i(Kv|k, \Lambda(j)).$$

Fact 2.1 then translates, in this birational context, to the following.

Fact 2.2. *Let v be a divisorial valuation of $K|k$ and $\varpi \in K^\times$ a uniformizer of v. Let $\alpha \in \mathrm{H}^i(\mathcal{O}_v|k, \Lambda(j))$ be given. Then*

$$\partial_v(\kappa_K(\varpi) \cup \mathfrak{u}_v\alpha) = \mathfrak{s}_v\alpha \quad \textit{as elements of } \mathrm{H}^i(Kv|k, \Lambda(j)).$$

Proof. This follows from Fact 2.1 since we can find some v-model X of $K|k$ such that $\varpi \in \mathcal{O}(X)$ and X_v is the zero-locus of ϖ. ∎

To put Fact 2.2 in the right perspective, recall the existence of the *tame symbol* in Milnor K-theory associated to a divisorial valuation v of $K|k$. This is a morphism

$$\partial_v : \mathrm{K}^{\mathrm{M}}_{r+1}(K) \to \mathrm{K}^{\mathrm{M}}_r(Kv)$$

which is uniquely characterized by the formula

$$\partial_v\{\varpi, u_1, \ldots, u_r\} = \{\bar{u}_1, \ldots, \bar{u}_r\},$$

where ϖ is a uniformizer of v, $u_1, \ldots, u_r \in \mathrm{U}_v$, and $\bar{u}_i$ denotes the image of u_i in $Kv^\times$. The residue maps described above are compatible with these tame symbols in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{M}}_{r+1}(K) & \xrightarrow{\partial_v} & \mathrm{K}^{\mathrm{M}}_r(Kv) \\ {\scriptstyle \kappa_K^{r+1}}\downarrow & & \downarrow{\scriptstyle \kappa_{Kv}^r} \\ \mathrm{H}^{r+1}(K|k, \Lambda(r+1)) & \xrightarrow{\partial_v} & \mathrm{H}^r(Kv|k, \Lambda(r)) \end{array} \tag{2.1}$$

3. Algebraic dependence

In this section we discuss the relationship between the cohomological structures described above and algebraic (in)dependence in function fields. Throughout this section we work with a fixed function field $K|k$.

3.1. Cohomological dimension

Recall that the *Andreotti–Frankel Theorem* [2] combined with the universal coefficient theorem shows that $\mathrm{H}^i(X, \Lambda(j)) = 0$ whenever X is a smooth affine k-variety of dimension smaller than i. We immediately obtain the following consequence.

Fact 3.1. *One has* $\mathrm{H}^i(K|k, \Lambda(j)) = 0$ *for all* $i > \mathrm{trdeg}(K|k)$.

This bound on the cohomological dimension of $K|k$ is sharp, as the following lemma shows.

Lemma 3.2. *Let $f_1, \dots, f_r \in K^\times$ be given. The following are equivalent:*

(1) *The element $\kappa_K^r\{f_1, \dots, f_r\} \in \mathrm{H}^r(K|k, \Lambda(r))$ vanishes.*

(2) *The element $\kappa_K^r\{f_1, \dots, f_r\} \in \mathrm{H}^r(K|k, \Lambda(r))$ is Λ-torsion.*

(3) *The elements $f_1, \dots, f_r \in K^\times$ are algebraically dependent over k.*

Proof. The implication (3)⇒(1) follows from Fact 3.1, while (1)⇒(2) is tautological. To conclude, assume that $f_1, \dots, f_r \in K^\times$ are algebraically independent over k. We will show that $\kappa_K^r\{f_1, \dots, f_r\}$ is non-Λ-torsion in $\mathrm{H}^r(K|k, \Lambda(r))$. We proceed by induction on r with $r = 0$ being trivial. For the inductive case, choose a divisorial valuation v of $K|k$ which has the following properties:

(1) $v(f_1) \neq 0 = v(f_2) = \cdots = v(f_r)$.

(2) Letting $\overline{f_i}$, $i = 2, \dots, r$, denote the image of f_i in Kv, the elements $\overline{f_2}, \dots, \overline{f_r}$ are algebraically independent in $Kv|k$.

We then have

$$\partial_v(\kappa_K^r\{f_1, \dots, f_r\}) = v(f_1) \cdot \kappa_{Kv}^{r-1}\{\overline{f_2}, \dots, \overline{f_r}\}.$$

Since Λ is a subring of $\mathbb{Q}$, this element is non-Λ-torsion by our inductive hypothesis, and thus the same holds for $\kappa_K^r\{f_1, \dots, f_r\}$. ∎

3.2. Good models

Let $K|k$ be a function field and L a subextension of $K|k$ which is relatively algebraically closed in K. We say that a model $X \to B$ of $\iota : L \hookrightarrow K$ is *good* provided that:

(1) The k-varieties X, B and the morphism $X \to B$ are all smooth.

(2) The fibres of $X \to B$ are all geometrically integral.

(3) The induced map $X^{\mathrm{an}} \to B^{\mathrm{an}}$ on topological spaces is a fibre bundle.

Clearly, if $f : X \to B$ is a good model, and U is a nonempty k-open subvariety in B, then the restriction $f^{-1}(U) \to U$ is again good.

The good models of $\iota : L \hookrightarrow K$ are cofinal among all models. Indeed, if $f : X \to B$ is any model of ι, then f has generically geometrically integral fibres because L was assumed to be relatively algebraically closed in K. Hence the models of ι satisfying (1) and (2) above are cofinal among all models. Moreover, if $f : X \to B$ is any model of ι satisfying (1) and (2), then there exists a nonempty open k-subvariety U of B such that the restriction $f^{-1}(U) \to U$ is a good model (see [32, Corollaire 5.1]).

3.3. Geometric submodules

Let L be a subextension of $K|k$ which is algebraically closed in K. In this subsection we study the map induced by the inclusion $L \hookrightarrow K$ on H^1.

Lemma 3.3. *Let L be a subextension of $K|k$ which is relatively algebraically closed in K. Then the canonical map*

$$\mathrm{H}^1(L|k, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1))$$

is injective.

Proof. Suppose α is in the kernel of this map, and let $X \to B$ be a good model of $L \hookrightarrow K$ such that $\alpha \in \mathrm{H}^1(B, \Lambda(1))$. Since $X^{\mathrm{an}} \to B^{\mathrm{an}}$ is a fibre bundle, the map

$$\mathrm{H}^1(B, \Lambda(1)) \to \mathrm{H}^1(X, \Lambda(1))$$

is injective. Since X is smooth, the map $\mathrm{H}^1(X, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1))$ is injective as well. Hence $\alpha = 0$. ■

Proposition 3.4. *Let L be a subextension of $K|k$ which is relatively algebraically closed in K and let $\alpha \in \mathrm{H}^1(K|k, \Lambda(1))$ be given. Assume that α is not contained in the image of the injective map*

$$\mathrm{H}^1(L|k, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1)).$$

Then there exists a smooth model B of $L|k$ such that for all closed points $b \in B$ and all systems of regular parameters $(f_1, \ldots, f_r)$ of $\mathcal{O}_{B,b}$, the element $\kappa_K^r\{f_1, \ldots, f_r\} \cup \alpha$ is non-Λ-torsion (in particular, nontrivial) in $\mathrm{H}^{r+1}(K|k, \Lambda(r+1))$.

Proof. Choose a good model $f : X \to B$ of $L \hookrightarrow K$ with $\alpha \in \mathrm{H}^1(X, \Lambda(1))$. Since L is assumed to be a subfield of K, we will consider elements of L both as rational functions on B and as rational functions on X.

Let $b \in B$ be a closed point and $f_1, \ldots, f_r$ a system of regular parameters at $b \in B$. Let ξ denote the generic point of $Z := f^{-1}(b)$, and note that $f_1, \ldots, f_r$, when considered as rational functions on X, are also a system of regular parameters at $\xi \in X$. Replacing X resp. B with a sufficiently small neighborhood of ξ resp. b, we may also assume that the following conditions hold true:

(1) One has $f_1, \ldots, f_r \in \mathcal{O}(B)$, $\{b\}$ is the zero-locus of $(f_1, \ldots, f_r)$ in B, and Z is the zero-locus of $(f_1, \ldots, f_r)$ in X. Let W_i denote the zero-locus of $(f_1, \ldots, f_i)$ in B and Z_i the zero-locus of $(f_1, \ldots, f_i)$ in X. Put $W_0 := B$ and $Z_0 := X$.

(2) $B = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_r = \{b\}$ is a flag of smooth integral closed subvarieties with W_{i+1} having codimension 1 in W_i.

(3) $X = Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r = Z$ is a flag of smooth integral closed subvarieties with Z_{i+1} having codimension 1 in Z_i.

(4) For all $i = 0, \ldots, r-1$, the function f_{i+1} is a regular parameter for the generic point of W_{i+1} in W_i, and similarly for the generic point of Z_{i+1} in Z_i.

Put $Z_{r+1} = \varnothing$ and, for $i = 1, \ldots, r$, let ∂_i denote the composition

$$\mathrm{H}^{s+1}(Z_{i-1} \smallsetminus Z_i, \Lambda(s+1)) \xrightarrow{\partial_{Z_{i-1}, Z_i}} \mathrm{H}^s(Z_i, \Lambda(s)) \xrightarrow{\text{restriction}} \mathrm{H}^s(Z_i \smallsetminus Z_{i+1}, \Lambda(s)),$$

where $s = 1 + r - i$. Applying Fact 2.1 successively r times, we find

$$(\partial_r \circ \partial_{r-1} \circ \cdots \circ \partial_1)(\kappa_{Z_0 \smallsetminus Z_1}(f_1) \cup \cdots \cup \kappa_{Z_{r-1} \smallsetminus Z_r}(f_r) \cup \alpha) = \beta,$$

where β denotes the image of α in $\mathrm{H}^1(Z, \Lambda(1))$. This map to $\mathrm{H}^1(Z, \Lambda(1))$ fits in an exact sequence of the form

$$0 \to \mathrm{H}^1(B, \Lambda(1)) \to \mathrm{H}^1(X, \Lambda(1)) \to \mathrm{H}^1(Z, \Lambda(1))$$

since $X^{\mathrm{an}} \to B^{\mathrm{an}}$ is a fibre bundle. Passing to the colimit over good models of $L \hookrightarrow K$, we obtain a similar exact sequence

$$0 \to \mathrm{H}^1(L|k, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1)) \to \mathrm{H}^1(k(Z)|k, \Lambda(1)).$$

Writing v_i for the divisorial valuation on $k(Z_{i-1})$ associated to Z_i, the above calculation shows that

$$(\partial_{v_r} \circ \cdots \circ \partial_{v_1})(\kappa_K^r\{f_1, \ldots, f_r\} \cup \alpha)$$

agrees with the image of α in $\mathrm{H}^1(k(Z)|k, \Lambda(1))$. The assertion follows since this element is nontrivial, and $\mathrm{H}^1(k(Z)|k, \Lambda(1))$ is torsion-free. ∎

4. Picard 1-motives

Recall that a 1-*motive* over k consists of the following data:

(1) A semiabelian k-variety $\mathbf{G}$.

(2) A finitely-generated free abelian group $\mathbf{L}$.

(3) A morphism $\mathbf{L} \to \mathbf{G}(k)$.

This data is summarized as a complex $[\mathbf{L} \to \mathbf{G}]$ of group schemes over k where $\mathbf{L}$ is placed in degree 0 and $\mathbf{G}$ in degree 1. A morphism of 1-motives is simply a morphism of such complexes. The set of morphisms $\mathrm{Hom}_k(\mathbf{M}_1, \mathbf{M}_2)$ between two 1-motives over k naturally forms an abelian group.

4.1. Hodge realizations

Let $\mathbf{M} = [\mathbf{L} \to \mathbf{G}]$ be a 1-motive over k. The *Hodge realization* of $\mathbf{M}$, denoted $\mathrm{H}(\mathbf{M})$, is constructed as follows (see [15, Section 10]). First, consider the exponential sequence

$$0 \to \mathrm{H}_1(\mathbf{G}^{\mathrm{an}}, \mathbb{Z}) \to \mathrm{Lie}\, \mathbf{G}^{\mathrm{an}} \to \mathbf{G}^{\mathrm{an}} \to 0.$$

Pull back this sequence with respect to the map $\mathbf{L} \to \mathbf{G}(k) \hookrightarrow \mathbf{G}^{\mathrm{an}}$ to obtain $\mathrm{H}(\mathbf{M})$ which then fits in an exact sequence of the form

$$0 \to \mathrm{H}_1(\mathbf{G}^{\mathrm{an}}, \mathbb{Z}) \to \mathrm{H}(\mathbf{M}) \to \mathbf{L} \to 0. \tag{4.1}$$

The mixed Hodge structure of $\mathrm{H}(\mathbf{M})$ is the unique one making the exact sequence above compatible with the usual mixed Hodge structure on $\mathrm{H}_1(\mathbf{G}^{\mathrm{an}}, \mathbb{Z})$, the trivial Hodge structure on $\mathbf{L}$, where $\mathrm{F}^0(\mathrm{H}(\mathbf{M}) \otimes \mathbb{C})$ is the kernel of the induced map

$$\mathrm{H}(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{Lie}\, \mathbf{G}^{\mathrm{an}}$$

obtained from the construction of $\mathrm{H}(\mathbf{M})$.

Given any 1-motive $\mathbf{M} = [\mathbf{L} \to \mathbf{G}]$, we may consider $\mathbf{L}$ as the 1-motive $[\mathbf{L} \to 0]$ and $\mathbf{G}$ as the 1-motive $[0 \to \mathbf{G}]$. These fit into an exact sequence (of complexes) of the form

$$0 \to \mathbf{G} \to \mathbf{M} \to \mathbf{L} \to 0. \tag{4.2}$$

Furthermore, one has $\mathbf{L} = \mathrm{H}(\mathbf{L})$ and $\mathrm{H}_1(\mathbf{G}^{\mathrm{an}}, \mathbb{Z}) = \mathrm{H}(\mathbf{G})$ with their natural mixed Hodge structures, while (4.1) can thus be considered as the exact sequence of mixed Hodge structures

$$0 \to \mathrm{H}(\mathbf{G}) \to \mathrm{H}(\mathbf{M}) \to \mathrm{H}(\mathbf{L}) \to 0 \tag{4.3}$$

obtained from (4.2) by applying the Hodge realization functor $\mathrm{H}(-)$.

We need the following lemma, which is the simplest case of Deligne's much more general result [15, Section 10.1.3].

Lemma 4.1. *Let $\mathbf{M} = [\mathbf{L} \to \mathbf{A}]$ be a 1-motive over k with $\mathbf{A}$ an abelian variety. Consider the 1-motive $\mathbb{Z} := [\mathbb{Z} \to 0]$. Then the trivial Hodge structure $\mathbb{Z}$ is isomorphic to $\mathrm{H}(\mathbb{Z})$, and $\mathrm{H}(-)$ induces an isomorphism $\mathrm{Hom}_k(\mathbb{Z}, \mathbf{M}) \cong \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{M}))$.*

Proof. The fact that $\mathbb{Z}$ (as a mixed Hodge structure) agrees with $\mathrm{H}(\mathbb{Z})$ follows directly from the definitions. As above, we view $\mathbf{A}$ and $\mathbf{L}$ as the 1-motives $[0 \to \mathbf{A}]$ and $[\mathbf{L} \to 0]$ respectively. We have an exact sequence of 1-motives,

$$0 \to \mathbf{A} \to \mathbf{M} \to \mathbf{L} \to 0,$$

and a corresponding exact sequence of Hodge realizations:

$$0 \to \mathrm{H}(\mathbf{A}) \to \mathrm{H}(\mathbf{M}) \to \mathrm{H}(\mathbf{L}) \to 0.$$

Note $\mathrm{H}(\mathbf{A}) = \mathrm{H}_1(\mathbf{A}^{\mathrm{an}}, \mathbb{Z})$ and $\mathrm{H}(\mathbf{L}) = \mathbf{L}$.

The lemma is easily verified in the case where $\mathbf{M} = \mathbf{A}$ or $\mathbf{M} = \mathbf{L}$. Indeed, the theory of weights ensures $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{A})) = 0$, while $\mathrm{Hom}_k(\mathbb{Z}, \mathbf{A}) = 0$ by definition. On the other hand, $\mathrm{Hom}_k(\mathbb{Z}, \mathbf{L}) = \mathbf{L}$ while $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathbf{L}) = \mathbf{L}$ and the map in question is the obvious isomorphism.

Consider the following commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_k(\mathbb{Z}, \mathbf{M}) & \longrightarrow & \mathrm{Hom}_k(\mathbb{Z}, \mathbf{L}) & \xrightarrow{\delta_k} & \mathrm{Ext}_k(\mathbb{Z}, \mathbf{A}) \\
 & & \downarrow & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{M})) & \longrightarrow & \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{L})) & \xrightarrow{\delta_{\mathrm{MHS}}} & \mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{A}))
\end{array}$$

Exactness on the left follows from the above observation that the groups $\mathrm{Hom}_k(\mathbb{Z}, \mathbf{A})$ and $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{A}))$ are both trivial. The group $\mathrm{Ext}_k(\mathbb{Z}, \mathbf{A})$ at the top-right is the abelian group of extensions of $\mathbb{Z}$ by $\mathbf{A}$, with respect to the Baer sum, in the category of 1-motives over k. Similarly, $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{A}))$ is the group of extensions of $\mathbb{Z}$ by $\mathrm{H}(\mathbf{A})$, again with respect to the Baer sum, in the category of mixed Hodge structures. The vertical maps are all induced by the functor H. The map δ_k sends a morphism $f : \mathbb{Z} \to \mathbf{L}$ to the pullback of

$$0 \to \mathbf{A} \to \mathbf{M} \to \mathbf{L} \to 0$$

with respect to f. The map δ_{MHS} can be defined similarly.

As noted in the diagram above, the map $\mathrm{Hom}_k(\mathbb{Z}, \mathbf{L}) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{L}))$ is an isomorphism. A simple diagram chase shows it is enough to prove that the map

$$\mathrm{Ext}_k(\mathbb{Z}, \mathbf{A}) \to \mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{A}))$$

induced by $\mathrm{H}(-)$ is injective. Note that $\mathrm{Ext}_k(\mathbb{Z}, \mathbf{A}) = \mathbf{A}(k)$, merely as a consequence of the definitions. Also, it is well-known that $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{H}(\mathbf{A})) = \mathbf{A}^{\mathrm{an}}$ (see [12]), and the corresponding map from $\mathbf{A}(k)$ to $\mathbf{A}^{\mathrm{an}} = \mathbf{A}(\mathbb{C})$ is the usual (injective) inclusion. This concludes the proof of the lemma. ■

4.2. Picard 1-motives

Let X be a smooth projective integral k-variety, and U a nonempty open k-subvariety of X. Put $Z := X \smallsetminus U$. Consider $\mathrm{Div}^0(X)$, the group of algebraically trivial Weil divisors on X, as well as the subgroup $\mathrm{Div}^0_Z(X)$ of algebraically trivial Weil divisors on X which are supported on Z. Note $\mathrm{Div}^0_Z(X)$ is a finitely-generated free abelian group.

Next, consider the Picard variety $\mathbf{Pic}^0_X$ of X. This is an abelian k-variety, and we have a canonical morphism

$$\mathrm{Div}^0_Z(X) \hookrightarrow \mathrm{Div}^0(X) \to \mathrm{Pic}^0(X) = \mathbf{Pic}^0_X(k),$$

mapping a Weil divisor to its associated line bundle. We thereby obtain the so-called *Picard* 1-*motive* of U (associated to the inclusion $U \hookrightarrow X$):

$$\mathbf{M}^{1,1}(U) := [\mathrm{Div}^0_Z(X) \to \mathbf{Pic}^0_X].$$

Whenever $V \subset U$ is a nonempty open k-subvariety, we obtain a canonical morphism

$$\mathbf{M}^{1,1}(U) \to \mathbf{M}^{1,1}(V)$$

of 1-motives over k, which simply arises from the inclusion $\mathrm{Div}^0_{X \smallsetminus U}(X) \hookrightarrow \mathrm{Div}^0_{X \smallsetminus V}(X)$.

The following theorem, due to Barbieri-Viale and Srinivas [6] and Ramachandran [25], describes the Hodge realization of such Picard 1-motives.

Theorem 4.2 ([6, Theorem 4.7], [25, Theorem 2.5]). *In the above context, there is a canonical isomorphism of mixed Hodge structures* $\mathrm{H}(\mathbf{M}^{1,1}(U)) \cong \mathrm{H}^1(U, \mathbb{Z}(1))$ *which is functorial with respect to open embeddings* $V \hookrightarrow U$ *of open* k*-subvarieties of* X.

5. An anabelian result

In this section, we discuss an anabelian result from which we will eventually deduce our main theorem. Write $\Lambda_{\neq 0}$ for the set of nonzero elements of Λ. Since Λ is a subring of $\mathbb{Q}$, we see that for every $x \in \mathcal{K}_\Lambda(K|k)$, there exists some $t \in K^\times$ such that $\overline{t} \in \Lambda_{\neq 0} \cdot x$. Given two elements $x, y \in \mathcal{K}_\Lambda(K|k)$ and elements $u, v \in K^\times$ such that $\overline{u} \in \Lambda_{\neq 0} \cdot x$ and $\overline{v} \in \Lambda_{\neq 0} \cdot y$, we say that x, y are *(in)dependent* provided that u, v are algebraically (in)dependent over k. It is easy to see that this definition does not depend on the choice of u, v as above, and that x, y are dependent if and only if they are not independent (this again relies on the assumption that $\Lambda \subset \mathbb{Q}$).

For a subextension M of $K|k$, the canonical map

$$\mathcal{K}_\Lambda(M|k) \to \mathcal{K}_\Lambda(K|k)$$

is injective since Λ is flat over $\mathbb{Z}$. We will tacitly identify $\mathcal{K}_\Lambda(M|k)$ with its image in $\mathcal{K}_\Lambda(K|k)$ with respect to this inclusion. For a subset S of K, we write

$$\mathrm{acl}_K(S) := \overline{k(S)} \cap K$$

for the relative algebraic closure of $k(S)$ in K. We say that a submodule $\mathcal{K}$ of $\mathcal{K}_\Lambda(K|k)$ is *rational* if there exists some $t \in K \smallsetminus k$ such that $\mathrm{acl}_K(t) = k(t)$ and $\mathcal{K} = \mathcal{K}_\Lambda(k(t)|k)$.

Next suppose that $L|l$ is a further function field over an algebraically closed field l of characteristic 0, and let

$$\phi : \mathcal{K}_\Lambda(K|k) \cong \mathcal{K}_\Lambda(L|l)$$

be an isomorphism of Λ-modules. We say that:

(1) *ϕ is compatible with dependence* provided that for all $x, y \in \mathcal{K}_\Lambda(K|k)$, the pair x, y is dependent in $\mathcal{K}_\Lambda(K|k)$ if and only if the pair $\phi x, \phi y$ is dependent in $\mathcal{K}_\Lambda(L|l)$.

(2) *ϕ is compatible with rational submodules* provided that ϕ induces a bijection between the rational submodules of $\mathcal{K}_\Lambda(K|k)$ and the rational submodules of $\mathcal{K}_\Lambda(L|l)$.

The collection of all isomorphisms $\mathcal{K}_\Lambda(K|k) \cong \mathcal{K}_\Lambda(L|l)$ which are compatible with dependence and with rational submodules will be denoted by

$$\mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l)).$$

Note that for any $\phi \in \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$ and $\varepsilon \in \Lambda^\times$, the corresponding isomorphism $\varepsilon \cdot \phi$ is again compatible with dependence and with rational submodules. We thus obtain an action of $\Lambda^\times$ on $\mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$, and we denote its orbits by

$$\underline{\mathrm{Isom}}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l)).$$

Any isomorphism of fields $K \cong L$ restricts to an isomorphism $k \cong l$, hence we obtain a canonical map

$$\mathrm{Isom}(K, L) \to \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l)) \twoheadrightarrow \underline{\mathrm{Isom}}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l)),$$

which is the focus of the following main result of this section.

Theorem 5.1. *In the above context, assume furthermore that* $\mathrm{trdeg}(K|k) \geq 2$*. Then the canonical map*

$$\mathrm{Isom}(K, L) \to \underline{\mathrm{Isom}}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$$

is a bijection.

We stated Theorem 5.1 as a theorem because it may be of independent interest. However, it can be deduced from known results in the literature in certain special cases. For example, if $\Lambda = \mathbb{Z}$, this theorem follows from the main results of Bogomolov–Tschinkel [10] and/or Cadoret–Pirutka [11]. If Λ is a *proper* subring of $\mathbb{Q}$, then this theorem can be deduced from the work of Pop [24]. And finally, if $\mathrm{trdeg}(K|k) \geq 5$, then one can deduce this result from the work of Evans–Hrushovski [16, 17] and Gismatullin [19], along with arguments similar to the ones appearing below. Moreover, in all of these cases the condition of *compatibility with rational submodules* can be removed.

In this respect, the most interesting case of Theorem 5.1 is where $\Lambda = \mathbb{Q}$, and where one considers function fields of transcendence degree ≥ 2. In such cases, we do not know of a straightforward way to deduce this result from what has appeared in the literature. Moreover, it is currently unclear whether the condition of compatibility with rational submodules in Theorem 5.1 can be relaxed when $\Lambda = \mathbb{Q}$.

The rest of this section is devoted to proving Theorem 5.1, and the bulk of the proof is devoted to constructing a (functorial) left inverse of the map appearing in its statement. For the rest of this section, we put ourselves in the context of Theorem 5.1 and fix an element $\phi \in \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$.

5.1. Divisorial valuations

For a divisorial valuation v of $K|k$, we write

$$\mathcal{U}_v := \mathrm{image}((\mathrm{U}_v/k^\times) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{K}_\Lambda(K|k)),$$
$$\mathcal{U}_v^1 := \mathrm{image}((\mathrm{U}_v^1 \cdot k^\times/k^\times) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{K}_\Lambda(K|k)).$$

The maps in the formulas above are the ones induced by the inclusions $\mathrm{U}_v \hookrightarrow K^\times$ and $\mathrm{U}_v^1 \hookrightarrow K^\times$, respectively. Note that $\mathcal{U}_v^1 \subset \mathcal{U}_v \subset \mathcal{K}_\Lambda(K|k)$, and that the map $\mathrm{U}_v \to (Kv)^\times$ induces an isomorphism $\mathcal{U}_v/\mathcal{U}_v^1 \cong \mathcal{K}_\Lambda(Kv|k)$.

We will need to use a variant of the *local theory* from almost-abelian anabelian geometry in order to show the compatibility of ϕ with divisorial valuations. Such local theories have been extensively developed; see [8, 22, 28, 30] for instance. However, the precise statement we need in our context has not appeared in the literature. For the sake of completeness, we provide the details for the local theory needed here in an appendix to this paper. We summarize the precise result we need in the following fact, which follows directly from Theorem A.1 in the appendix.

Fact 5.2. *In the above context, for all divisorial valuations* v *of* $K|k$*, there exists a unique divisorial valuation* v^ϕ *of* $L|l$ *such that* $\phi(\mathcal{U}_v) = \mathcal{U}_{v^\phi}$ *and* $\phi(\mathcal{U}_v^1) = \mathcal{U}_{v^\phi}^1$.

5.2. Rational submodules

For $t \in K \smallsetminus k$, put

$$\mathscr{K}_t := \mathscr{K}_\Lambda(\mathrm{acl}_K(t)|k)$$

considered as a submodule of $\mathscr{K}_\Lambda(K|k)$.

Lemma 5.3. *A submodule $\mathscr{K}$ of $\mathscr{K}_\Lambda(K|k)$ has the form $\mathscr{K}_t$ for some $t \in K \smallsetminus k$ if and only if it satisfies the following conditions:*

(1) *The submodule $\mathscr{K}$ is nontrivial.*

(2) *For all nontrivial $\alpha \in \mathscr{K}$ and $\beta \in \mathscr{K}_\Lambda(K|k)$ dependent with α, one has $\beta \in \mathscr{K}$.*

(3) *Any two elements $\alpha, \beta \in \mathscr{K}$ are dependent.*

Proof. It is easy to see that the three conditions hold for $\mathscr{K}_t$ for $t \in K \smallsetminus k$. Conversely, suppose that $\mathscr{K}$ satisfies the three conditions. By condition (1), there exists some $t \in K \smallsetminus k$ such that $\overline{t} \in \mathscr{K}$, and condition (2) implies that $\mathscr{K}_t$ is contained in $\mathscr{K}$. Finally, if $\beta \in \mathscr{K}$ is not contained in $\mathscr{K}_t$, then $\overline{t}$ and β would be independent, contradicting condition (3). Thus $\mathscr{K}_t = \mathscr{K}$, as required. ∎

We say that t is *general in* $K|k$ provided that K is regular over $k(t)$. Note that if t is general in $K|k$ then $\mathscr{K}_t$ is a rational submodule of $\mathscr{K}_\Lambda(K|k)$. We will need the following *birational Bertini* result.

Fact 5.4 (Birational Bertini [20, Ch. VIII, p. 213]). *Let $x, y \in K$ be algebraically independent over k. For all but finitely many $a \in k$, the element $x + a \cdot y$ is general in $K|k$.*

5.3. Divisors on curves

Let $t \in K \smallsetminus k$ be given, and put $\mathscr{K} := \mathscr{K}_t$. Consider the following collection of submodules of $\mathscr{K}$:

$$\mathscr{D}_t = \mathscr{D}_\mathscr{K} := \{\mathscr{U}_v \cap \mathscr{K} \mid \mathscr{K} \not\subset \mathscr{U}_v\},$$

where v varies over the divisorial valuations of $K|k$. Also write $\mathbb{D}_t = \mathbb{D}_{\mathrm{acl}_K(t)}$ for the collection of divisorial valuations of $\mathrm{acl}_K(t)|k$, so that $\mathbb{D}_t$ is in bijection with the closed points of the unique projective normal model of $\mathrm{acl}_K(t)|k$.

Lemma 5.5. *In the above context, the following hold:*

(1) *For all $\mathscr{U} \in \mathscr{D}_t$, the quotient $\mathscr{K}/\mathscr{U}$ is isomorphic to Λ.*

(2) *One has a canonical bijection $\mathbb{D}_t \cong \mathscr{D}_t$ defined by $w \mapsto \mathscr{U}_w$, $w \in \mathbb{D}_t$, with inverse sending $\mathscr{U} = \mathscr{U}_v \cap \mathscr{K}$ to the restriction of v to $\mathrm{acl}_K(t)$.*

Proof. For (1), let v be a divisorial valuation of $K|k$ with $\mathscr{K} \not\subset \mathscr{U}_v$ and put $\mathscr{U} = \mathscr{U}_v \cap \mathscr{K}$. Note that $\mathscr{K}_\Lambda(K|k)/\mathscr{U}_v$ is isomorphic to Λ, hence one has a canonical injective morphism of Λ-modules

$$\mathscr{K}/\mathscr{U} \hookrightarrow \mathscr{K}_\Lambda(K|k)/\mathscr{U}_v \cong \Lambda.$$

The image of this map is nontrivial by assumption and since Λ is a subring of $\mathbb{Q}$, hence a PID, assertion (1) follows.

For (2), put $M := \mathrm{acl}_K(t)$ and suppose first that w is a divisorial valuation of $M|k$. Then there exists a divisorial valuation v of $K|k$ whose restriction to M is w, hence $\mathscr{U}_w \subset \mathscr{U}_v \cap \mathscr{K}$ while $\mathscr{K} \not\subset \mathscr{U}_v$. The exact sequence of Λ-modules

$$0 \to \mathscr{U}_v \cap \mathscr{K}/\mathscr{U}_w \to \mathscr{K}/\mathscr{U}_w \to \mathscr{K}/\mathscr{U}_v \cap \mathscr{K} \to 0$$

splits since $\mathscr{K}/\mathscr{U}_v \cap \mathscr{K}$ is a free Λ-module of rank 1. As $\mathscr{K}/\mathscr{U}_w$ is also a free Λ-module of rank 1, using the fact that Λ is a PID, it follows that $\mathscr{U}_v \cap \mathscr{K}/\mathscr{U}_w = 0$, hence $\mathscr{U}_w = \mathscr{U}_v \cap \mathscr{K} \in \mathscr{D}_t$. The map $\mathbb{D}_t \to \mathscr{D}_t$ given by $w \mapsto \mathscr{U}_w$ is thus well-defined, and, since the valuations in $\mathbb{D}_t$ are all pairwise independent, this map is injective.

On the other hand, if $\mathscr{U}$ is an element of $\mathscr{D}_t$, and v is a divisorial valuation of $K|k$ with $\mathscr{U} = \mathscr{U}_v \cap \mathscr{K}$ and $\mathscr{K} \not\subset \mathscr{U}_v$, then the restriction w of v to M is a divisorial valuation of $M|k$ with $\mathscr{U}_w \subset \mathscr{U}_v \cap \mathscr{K}$. Arguing as above, we find again that $\mathscr{U}_w = \mathscr{U}_v \cap \mathscr{K} = \mathscr{U}$. This shows that the map $\mathbb{D}_t \to \mathscr{D}_t$ is indeed bijective, and that its inverse is as described in the statement of the lemma. ∎

5.4. *Rational-like collections*

Assume now that t is a general element of $K|k$ so that $\mathscr{K} := \mathscr{K}_t$ is a rational submodule of $\mathscr{K}_\Lambda(K|k)$. By Lemma 5.5, we have $\mathscr{K}/\mathscr{U} \cong \Lambda$ for every $\mathscr{U} \in \mathscr{D}_t$. Consider a collection of such isomorphisms:

$$\Phi = (\Phi_{\mathscr{U}} : \mathscr{K}/\mathscr{U} \xrightarrow{\cong} \Lambda)_{\mathscr{U} \in \mathscr{D}_t}.$$

Any element of $\mathscr{K}$ is contained in all but finitely many of the $\mathscr{U} \in \mathscr{D}_t$ by Lemma 5.5, hence Φ induces a canonical map

$$\mathrm{div}_\Phi : \mathscr{K} \to \bigoplus_{\mathscr{U} \in \mathscr{D}_t} \Lambda \cdot [\mathscr{U}], \quad \mathrm{div}_\Phi(x) = \sum_{\mathscr{U} \in \mathscr{D}_t} \Phi_{\mathscr{U}}(x) \cdot [\mathscr{U}].$$

Here $[\mathscr{U}]$, $\mathscr{U} \in \mathscr{D}_t$, denote formal basis elements for the direct sum.

We say that Φ is a *rational-like collection* provided that div_Φ fits in a short exact sequence of the form

$$0 \to \mathscr{K} \xrightarrow{\mathrm{div}_\Phi} \bigoplus_{\mathscr{U} \in \mathscr{D}_t} \Lambda \cdot [\mathscr{U}] \xrightarrow{\mathrm{sum}} \Lambda \to 0.$$

If Φ is a rational-like collection and $\varepsilon \in \Lambda^\times$ is given, then we obtain another rational-like collection

$$\varepsilon \cdot \Phi := (\varepsilon \cdot \Phi_{\mathscr{U}})_{\mathscr{U} \in \mathscr{D}_t}.$$

By Lemma 5.5, there is a *canonical* rational-like collection for $\mathscr{K}$, constructed from the field structure of $M := \mathrm{acl}_K(t) = k(t)$, as follows. For $\mathscr{U} \in \mathscr{D}_t$ and $w \in \mathbb{D}_t$ such that $\mathscr{U} = \mathscr{U}_w$, the isomorphism $\Phi^{\mathrm{can}}_{\mathscr{U}}$ is the unique one making the following diagram commute:

$$\begin{array}{ccccc} \mathscr{K} & = & (M^\times/k^\times) \otimes_{\mathbb{Z}} \Lambda & \xrightarrow{w \otimes \Lambda} & \mathbb{Z} \otimes_{\mathbb{Z}} \Lambda = \Lambda \\ \downarrow\!\!\downarrow & & & & \uparrow \Phi^{\mathrm{can}}_{\mathscr{U}} \\ \mathscr{K}/\mathscr{U} & & = & & \mathscr{K}/\mathscr{U} \end{array}$$

Write $\Phi^{\mathrm{can}}_{\mathscr{K}} := (\Phi^{\mathrm{can}}_{\mathscr{U}})_{\mathscr{U} \in \mathscr{D}_{\mathscr{K}}}$. This is clearly a rational-like collection, which we call *the canonical rational-like collection of* $\mathscr{K}$. We simplify the notation by writing $\mathrm{div}_{\mathrm{can}} := \mathrm{div}_{\Phi^{\mathrm{can}}_{\mathscr{K}}}$ if $\mathscr{K}$ is understood from context.

Lemma 5.6. *In the above context, let* Φ *be a rational-like collection for* $\mathscr{K}$. *Then there exists a unique* $\varepsilon = \varepsilon_{\mathscr{K}} \in \Lambda^\times$ *such that* $\Phi = \varepsilon \cdot \Phi^{\mathrm{can}}_{\mathscr{K}}$.

Proof. By Lemma 5.5, for each $\mathscr{U} \in \mathscr{D}_{\mathscr{K}}$, we may choose an $\varepsilon_{\mathscr{U}} \in \Lambda^\times$ such that

$$\Phi_{\mathscr{U}} = \varepsilon_{\mathscr{U}} \cdot \Phi^{\mathrm{can}}_{\mathscr{U}}.$$

We must show that $\varepsilon_{\mathscr{U}}$ is independent of the choice of $\mathscr{U}$. For two different $\mathscr{U}, \mathscr{V} \in \mathscr{D}_{\mathscr{K}}$, there exists a unique $x \in \mathscr{K}$ such that

$$\mathrm{div}_{\mathrm{can}}(x) = [\mathscr{U}] - [\mathscr{V}].$$

Hence $\mathrm{div}_\Phi(x) = \varepsilon_{\mathscr{U}} \cdot [\mathscr{U}] - \varepsilon_{\mathscr{V}} \cdot [\mathscr{V}]$. The "exactness" in the definition of a rational-like collection (applied to Φ) shows that $\varepsilon_{\mathscr{U}} - \varepsilon_{\mathscr{V}} = 0$. Hence $\varepsilon_{\mathscr{U}}$ does not depend on $\mathscr{U}$. ∎

5.5. Rational synchronization

Let t be a general element of $K|k$. By Lemma 5.5, $\mathscr{D}_t$ is parameterized by $\mathbb{P}^1(k) = k \cup \{\infty\}$ by identifying $a \in k$ resp. ∞ with $\mathscr{U}_w$ with w the divisorial valuation whose centre is $t = a$ resp. $t = \infty$. Write $\mathscr{U}_{t,a}$ for the element of $\mathscr{D}_t$ corresponding to $a \in k \cup \infty$. Note that

$$\mathrm{div}_{\mathrm{can}} \overline{(t-c)} = [\mathscr{U}_{t,c}] - [\mathscr{U}_{t,\infty}]$$

for all $c \in k$. Also, if $\mathscr{U}_1, \mathscr{U}_2 \in \mathscr{D}_t$ are two distinct elements, then there exists a general element x of $K|k$ such that $k(x) = k(t)$ and

$$\mathrm{div}_{\mathrm{can}} \overline{x} = [\mathscr{U}_1] - [\mathscr{U}_2].$$

Proposition 5.7. *Let* $\phi : \mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$ *be an isomorphism of* Λ*-modules which is compatible with dependence and with rational submodules, and let* x *be a general element of* $K|k$. *Then there exists a general element* y *of* $L|l$, *a unit* $\varepsilon \in \Lambda^\times$, *and a bijection* $\eta : k \cong l$ *such that* $\eta 0 = 0$, $\eta 1 = 1$, *and* $\phi\overline{(x-a)} = \varepsilon \cdot \overline{(y - \eta a)}$ *for all* $a \in k$.

Proof. Put $\mathscr{K} := \mathscr{K}_x$. Since ϕ is compatible with rational submodules, we see that $\mathscr{L} := \phi\mathscr{K}$ is a rational submodule of $\mathscr{K}_\Lambda(L|l)$. By Fact 5.2, ϕ induces a bijection

$$\mathscr{U} \mapsto \phi\mathscr{U} \ : \ \mathscr{D}_{\mathscr{K}} \xrightarrow{\cong} \mathscr{D}_{\mathscr{L}}.$$

Consider the canonical rational-like collection $\Phi := \Phi^{\rm can}_{\mathscr{K}}$ on $\mathscr{K}$. Let Ψ denote the rational-like collection on $\mathscr{L}$ induced by Φ and ϕ. Explicitly, $\Psi_{\mathscr{V}} : \mathscr{L}/\mathscr{V} \cong \Lambda$, $\mathscr{V} \in \mathscr{D}_{\mathscr{L}}$, is the isomorphism

$$\mathscr{L}/\mathscr{V} \xrightarrow{\phi^{-1}} \mathscr{K}/\mathscr{U} \xrightarrow{\Phi_{\mathscr{U}}} \Lambda$$

where $\mathscr{U} = \phi^{-1}\mathscr{V}$. By Lemma 5.6 there exists an $\varepsilon \in \Lambda^\times$ such that $\Psi = \varepsilon^{-1} \cdot \Phi^{\rm can}_{\mathscr{L}}$, and by the construction of Ψ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K} & \xrightarrow{\mathrm{div_{can}}} & \bigoplus_{\mathscr{U}\in\mathscr{D}_{\mathscr{K}}} \Lambda\cdot[\mathscr{U}] & \xrightarrow{\mathrm{sum}} & \Lambda & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & \mathscr{L} & \xrightarrow{\mathrm{div}_\Psi} & \bigoplus_{\mathscr{V}\in\mathscr{D}_{\mathscr{L}}} \Lambda\cdot[\mathscr{V}] & \xrightarrow{\mathrm{sum}} & \Lambda & \longrightarrow & 0 \end{array}$$

The ϕ in the middle is shorthand for the morphism defined by $[\mathscr{U}] \mapsto [\phi\mathscr{U}]$.

Note that $\mathrm{div}_\Phi\, \overline{x} = [\mathscr{U}_{x,0}] - [\mathscr{U}_{x,\infty}]$, hence $\mathrm{div}_\Psi(\phi\overline{x}) = [\phi\mathscr{U}_{x,0}] - [\phi\mathscr{U}_{x,\infty}]$. By the discussion above, there exists a general element y of $L|l$ such that $\mathscr{K}_y = \mathscr{L}$ and

$$\phi\mathscr{U}_{x,0} = \mathscr{U}_{y,0}, \quad \phi\mathscr{U}_{x,\infty} = \mathscr{U}_{y,\infty}.$$

Replacing y with an element of the form $c \cdot y$ for some $c \in l^\times$ we may assume furthermore that $\phi\mathscr{U}_{x,1} = \mathscr{U}_{y,1}$. Define the bijection $\eta : k \cong l$ as the unique one satisfying $\phi\mathscr{U}_{x,a} = \mathscr{U}_{y,\eta a}$ for $a \in k$. Then for all $a \in k$, we have

$$\begin{aligned} \mathrm{div_{can}}(\varepsilon^{-1} \cdot \phi\overline{(x-a)}) &= \mathrm{div}_\Psi(\phi\overline{(x-a)}) = [\phi\mathscr{U}_{x,a}] - [\phi\mathscr{U}_{x,\infty}] \\ &= [\mathscr{U}_{y,\eta a}] - [\mathscr{U}_{y,\infty}] = \mathrm{div_{can}}\,\overline{(y-\eta a)}. \end{aligned}$$

The injectivity of $\mathrm{div_{can}}$ implies that $\phi\overline{(x-a)} = \varepsilon \cdot \overline{(y-\eta a)}$, as required. ∎

5.6. *Multiplicative synchronization*

At this point, our proof of Theorem 5.1 will use an adaptation of arguments due to Pop [24, Section 6]. Following Proposition 5.7, we will say that $\phi \in \mathrm{Isom}^{\rm dep}_{\rm rat}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(L|l))$ is *synchronized* provided that there exists some general element x of $K|k$, some general element y of $L|l$, and some bijection $\eta : k \cong l$ with $\eta 0 = 0$ and $\eta 1 = 1$, such that

$$\phi\overline{(x-a)} = \overline{(y-\eta a)}$$

for all $a \in k$. To specify x, y and η, we may say that ϕ is *synchronized by x and y via η*. By Proposition 5.7, for any $\phi \in \mathrm{Isom}^{\rm dep}_{\rm rat}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(L|l))$, there exists some $\varepsilon \in \Lambda^\times$ such that $\varepsilon \cdot \phi$ is synchronized.

The canonical map $K^\times/k^\times \to \mathscr{K}_\Lambda(K|k)$ is injective, and we will identify $K^\times/k^\times$ with its image in $\mathscr{K}_\Lambda(K|k)$, and similarly for $L^\times/l^\times$. We show that a synchronized ϕ is compatible with these lattices.

Proposition 5.8. *Assume that $\phi \in \mathrm{Isom}^{\rm dep}_{\rm rat}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(L|l))$ is synchronized. Then $\phi(K^\times/k^\times) = L^\times/l^\times$.*

Proof. It suffices to prove that $\phi(K^\times/k^\times) \subset L^\times/l^\times$ since ϕ^{-1} is also synchronized whenever ϕ is. Put $\mathscr{M} := \phi^{-1}(L^\times/l^\times) \cap (K^\times/k^\times)$ and let $M^\times$ denote the preimage of $\mathscr{M}$ in $K^\times$. Our goal is to show that $M := M^\times \cup \{0\} = K$.

Suppose ϕ is synchronized by x and y via η. Note that $k(x)^\times \subset M^\times$ since $k(x)^\times$ is multiplicatively generated by elements of the form $x - a$, $a \in k$.

More generally, assume that $u \in M^\times$ is general in $K|k$. By Proposition 5.7, there exists a bijection $\gamma : k \cong l$, a general element w of $L|l$, and an $\varepsilon \in \Lambda^\times$, such that $\gamma 0 = 0$, $\gamma 1 = 1$ and

$$\phi\overline{(u-a)} = \varepsilon \cdot \overline{(w - \gamma a)}$$

for all $a \in k$. Note in particular that $\phi\bar{u} = \varepsilon \cdot \bar{w}$, while $\phi\bar{u} \in L^\times/l^\times$.

We claim that $\varepsilon \in \mathbb{Z}$. Write $\varepsilon = m/n$ for integers m, n and $n > 0$. Since $l^\times$ is divisible, the above observations show that there exists $g \in L^\times$ such that $w^m = g^n$, hence w and g are algebraically dependent. As w is general in $L|l$, it follows that $g \in k(w)$ and comparing w-adic valuations we find $\varepsilon = m/n \in \mathbb{Z}$.

With $\varepsilon \in \mathbb{Z}$ as above, one has

$$\phi\overline{(u-a)} = \varepsilon \cdot \overline{(w - \gamma a)} = \overline{(w - \gamma a)^\varepsilon}$$

for all $a \in k$, hence $u - a \in M^\times$. As $k(u)^\times$ is multiplicatively generated by $u - a$ for $a \in k$, we deduce that $k(u)^\times$ is contained in $M^\times$.

Finally, since $\Lambda \subset \mathbb{Q}$, we see that for all $t \in K^\times$ there exists some integer $n > 0$ such that $n \cdot \phi\bar{t} \in L^\times/l^\times$. In other words, $t^n \in M^\times$, so that $K^\times/M^\times$ is torsion. Bringing together the above observations:

(1) The quotient $K^\times/M^\times$ is torsion.

(2) If $u \in M^\times$ is general in $K|k$, then $k(u)^\times$ is contained in $M^\times$.

(3) The element x is contained in $M^\times$ and x is general in $K|k$.

We claim that M is additively closed in K. As M is multiplicatively closed and $M^\times = M \smallsetminus \{0\}$ is a subgroup of $K^\times$, it suffices to prove that for all $u \in M$, one has $1 + u \in M$. As $k(x) \subset M$, we may also assume that $u \notin k(x)$, hence u and x are algebraically independent over k. For $a, b, c \in k$, put

$$A_{b,c} := \frac{b \cdot x + c}{u}, \quad B_{a,b,c} := a \cdot u + b \cdot x + c = u \cdot (A_{b,c} + a).$$

Let $c \in k^\times$ be an arbitrary element. Since x/u and c/u are algebraically independent, it follows from Fact 5.4 that

$$A_{b,c} = b \cdot \frac{x}{u} + \frac{c}{u}$$

is general in $K|k$ for all but finitely many $b \in k$. Since $b \cdot x + c \in M$ and $u \in M$, we see that $A_{b,c} \in M$ as well since M is multiplicatively closed. Since $A_{b,c}$ is general in $K|k$ for all but finitely many $b \in k$, the properties above show that $A_{b,c} + a \in M$ for such b and arbitrary $a \in k$. To summarize, for a given $a \in k$ and $c \in k^\times$, one has

$$B_{a,b,c} = u \cdot (A_{b,c} + a) \in M$$

for all but finitely many $b \in k$.

Next suppose that $a, c \in k$ and that $a \neq c$. Thus $(a \cdot u + c)/(u+1)$ and $x/(u+1)$ are algebraically independent. Using Fact 5.4 again, we see that for all but finitely many $b \in k$, the element

$$\frac{a \cdot u + c}{u+1} + b \cdot \frac{x}{u+1} = \frac{a \cdot u + b \cdot x + c}{u+1} = \frac{B_{a,b,c}}{u+1}$$

is general in $K|k$. In this case, the element

$$\frac{u+1}{B_{a,b,c}} + 1 = \frac{(a+1) \cdot u + b \cdot x + (c+1)}{B_{a,b,c}} = \frac{B_{a+1,b,c+1}}{B_{a,b,c}}$$

is also general in $K|k$.

Choose $a, c \in k$ satisfying $a \neq c$, $c \neq 0$ and $c + 1 \neq 0$. By the discussion above, it follows that for all but finitely many b, the following two conditions hold:

(1) $B_{a,b,c} \in M$ and $B_{a+1,b,c+1} \in M$.

(2) $B_{a+1,b,c+1}/B_{a,b,c}$ is general in $K|k$.

Let $b \in k$ be such an element. Since M is multiplicatively closed, we see that

$$\frac{u+1}{B_{a,b,c}} + 1 = \frac{B_{a+1,b,c+1}}{B_{a,b,c}} \in M,$$

and since this element is general in $K|k$, it follows that $(u+1)/B_{a,b,c} \in M$ as well. Since $B_{a,b,c} \in M$, this shows that indeed $u + 1 \in M$, as contended.

The argument above shows that M is a subfield of K which contains k while $K^\times/M^\times$ is torsion. Since $K|k$ is a function field and k has characteristic 0, it follows that $K = M$, and this concludes the proof of the proposition. ∎

5.7. Collineation

At this point of the argument we have seen that for any $\phi \in \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$, there exists some $\varepsilon \in \Lambda^\times$ such that $\varepsilon \cdot \phi$ is synchronized, and in this case $\varepsilon \cdot \phi$ restricts to an isomorphism $K^\times/k^\times \cong L^\times/l^\times$ which is compatible with algebraic dependence. Note also that $K^\times/k^\times$ is the projectivization of K as a k-module, and similarly for $L|l$.

The proof of Theorem 5.1 can now be easily obtained from [11, Theorem 4] once we know that the ε above is unique (see Lemma 5.11 below). For the sake of completeness, we will explain how the work above can be used to deduce that the isomorphism $K^\times/k^\times \cong L^\times/l^\times$ is compatible with *projective lines*, after which we conclude the proof of Theorem 5.1 by applying the *fundamental theorem of projective geometry* [4, 18], similarly to the approach taken in [10, 11, 24].

Suppose now that $\phi \in \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$ is synchronized, so ϕ restricts to an isomorphism $K^\times/k^\times \cong L^\times/l^\times$ by Proposition 5.8.

Proposition 5.9. *In the above context, the isomorphism*

$$\phi : K^\times/k^\times \cong L^\times/l^\times$$

is a collineation, i.e. ϕ sends projective lines in $K^\times/k^\times$ to projective lines in $L^\times/l^\times$.

Proof. For $x \in K^\times/k^\times$, $x \neq 1$ and $\tilde{x} \in K^\times$ a representative of x, write

$$\mathcal{L}_x := \frac{(k + \tilde{x} \cdot k) \cap K^\times}{k^\times}$$

for the unique projective line in $K^\times/k^\times$ containing 1 and x. Since ϕ is compatible with multiplication, it suffices to show that $\phi\mathcal{L}_x = \mathcal{L}_{\phi x}$ for all such x.

Assume first that $x \in K \smallsetminus k$ and $y \in L \smallsetminus l$ are such that $\phi\mathcal{L}_{\bar{x}} = \mathcal{L}_{\bar{y}}$. Let $t \in K$ be algebraically independent from x over k, and choose $u \in L \smallsetminus l$ such that $\phi\bar{t} = \bar{u}$. Choose a divisorial valuation v of $K|k$ such that v is trivial on $\mathrm{acl}_K(x)$ and on $\mathrm{acl}_K(t)$, and such that x and t have the same image in $(Kv)^\times/k^\times$; such a v exists since x and t are algebraically independent over k. Put $w = v^\phi$, as in Fact 5.2. The same fact implies that y and u have the same image in $(Lw)^\times/l^\times$, while w is trivial on $\mathrm{acl}_L(y)$ and on $\mathrm{acl}_L(u)$ since ϕ is compatible with dependence (see Lemma 5.3).

Note that both maps

$$\mathrm{acl}_K(x)/k^\times \to (Kv)^\times/k^\times \leftarrow \mathrm{acl}_K(t)/k^\times$$

are injective. Letting $\bar{x} = \bar{t}$ denote the images of x and t in $(Kv)^\times/k^\times$, the images of $\mathcal{L}_{\bar{x}}$ and $\mathcal{L}_{\bar{t}}$ both agree with $\mathcal{L}_{\bar{x}}$ under these two maps. Since $\mathscr{U}_v^1 \cap (K^\times/k^\times) = \mathrm{U}_v^1 \cdot k^\times/k^\times$ and $\mathrm{acl}_K(t)^\times/k^\times = \mathscr{K}_t \cap (K^\times/k^\times)$, we deduce that

$$\mathcal{L}_{\bar{t}} = \mathscr{K}_t \cap (K^\times/k^\times) \cap (\mathcal{L}_{\bar{x}} \cdot (\mathscr{U}_v^1 \cap (K^\times/k^\times))).$$

Since ϕ identifies $\mathscr{K}_t$ with $\mathscr{K}_u$, $K^\times/k^\times$ with $L^\times/l^\times$, $\mathcal{L}_{\bar{x}}$ with $\mathcal{L}_{\bar{y}}$, and $\mathscr{U}_v^1$ with $\mathscr{U}_w^1$, it follows that $\phi\mathcal{L}_{\bar{t}} = \mathcal{L}_{\bar{u}}$. Finally, since ϕ is synchronized, there exist some x and y as above such that $\phi\mathcal{L}_{\bar{x}} = \mathcal{L}_{\bar{y}}$. Thus, the argument above shows that for every $t \in K$ algebraically independent from x, we have $\phi\mathcal{L}_{\bar{t}} = \mathcal{L}_{\phi\bar{t}}$.

If $t \in K \smallsetminus k$ is not algebraically independent from x, simply choose some $s \in K \smallsetminus k$ which is algebraically independent from x, so the argument above shows $\phi\mathcal{L}_{\bar{s}} = \mathcal{L}_{\phi\bar{s}}$. Since t is independent from s, the argument above then shows that $\phi\mathcal{L}_{\bar{t}} = \mathcal{L}_{\phi\bar{t}}$. ∎

5.8. *Concluding the proof*

We now conclude the proof of Theorem 5.1.

Proposition 5.10. *Assume that $\phi \in \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(L|l))$ is synchronized. Then there exists a unique isomorphism of fields $\Gamma_\phi : K \cong L$ such that $\phi\bar{t} = \overline{\Gamma_\phi t}$ for all $t \in K^\times$.*

Proof. Since ϕ is synchronized, it induces an isomorphism

$$\phi : K^\times/k^\times \cong L^\times/l^\times$$

by Proposition 5.8 which is a collineation by Proposition 5.9. By the *fundamental theorem of projective geometry* (see [4, 18]), there exists a unique isomorphism of fields $\gamma : k \cong l$ and a γ-semilinear isomorphism $\Gamma : K \cong L$ of additive groups such that $\overline{\Gamma(x)} = \phi\bar{x}$ for all $x \in K^\times$. Moreover, Γ is unique with these properties up to homotheties. Note that we must have $\Gamma(1) \in l^\times$ since $\phi(1) = 1$. Replace Γ with $(1/\Gamma(1)) \cdot \Gamma$ to assume furthermore that $\Gamma(1) = 1$, and note that this Γ is then the unique γ-semilinear isomorphism $K \cong L$ satisfying $\Gamma(1) = 1$ and $\overline{\Gamma(x)} = \phi\bar{x}$ for all $x \in K^\times$.

We follow an argument which is similar to [9, Theorem 7.3] to show that this Γ is a field isomorphism. First, since $\Gamma(1) = 1$, it follows that Γ restricts to $\gamma : k \cong l$ on k. In particular, for $x \in K$ and $a \in k$, one has

$$\Gamma(a \cdot x) = \gamma(a) \cdot \Gamma(x) = \Gamma(a) \cdot \Gamma(x).$$

Also, Γ is already an additive homomorphism, so we only need to show its compatibility with multiplication.

Assume therefore that $x, y \in K$ are given. We must show that $\Gamma(x \cdot y) = \Gamma(x) \cdot \Gamma(y)$. Since Γ restricts to the field isomorphism γ on k, we may assume furthermore that $x \cdot y$ and y are k-linearly independent. Since Γ induces $\phi : K^\times/k^\times \cong L^\times/l^\times$ which is multiplicative, there exists some $c \in l^\times$ such that

$$\Gamma(x \cdot y) = c \cdot \Gamma(x) \cdot \Gamma(y).$$

Since $x \cdot y$ and y are k-linearly independent, we see that $c^{-1} \cdot \Gamma(x \cdot y) = \Gamma(x) \cdot \Gamma(y)$ and $\Gamma(y)$ are again l-linearly independent.

Consider $\Gamma(x \cdot y + y)$. On the one hand, we have

$$\Gamma(x \cdot y + y) = \Gamma(x \cdot y) + \Gamma(y) = c \cdot \Gamma(x) \cdot \Gamma(y) + \Gamma(y),$$

and on the other there exists some $d \in l^\times$ such that

$$\begin{aligned}\Gamma(x \cdot y + y) = \Gamma((x+1) \cdot y) = d \cdot \Gamma(x+1) \cdot \Gamma(y) &= d \cdot (\Gamma(x) + 1) \cdot \Gamma(y) \\ &= d \cdot \Gamma(x) \cdot \Gamma(y) + d \cdot \Gamma(y).\end{aligned}$$

In particular, we see that $c = d = 1$, and hence $\Gamma(x \cdot y) = \Gamma(x) \cdot \Gamma(y)$, as required. ∎

We now conclude the proof of Theorem 5.1. Let $\phi \in \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$ be given. By Proposition 5.7, there exists some $\varepsilon \in \Lambda^\times$ such that $\psi := \varepsilon \cdot \phi$ is synchronized, while Proposition 5.10 shows there is a unique isomorphism $\Gamma_\psi : K \cong L$ of fields satisfying $\psi\bar{t} = \overline{\Gamma_\psi t}$. If furthermore ϕ arises from a given isomorphism $\Gamma : K \cong L$, then ϕ is synchronized and it is easy to see that $\Gamma = \Gamma_\phi$.

Lemma 5.11. *Let $\phi : \mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathcal{K}_\Lambda(K|k), \mathcal{K}_\Lambda(L|l))$ be given, and suppose that $\varepsilon_1, \varepsilon_2 \in \Lambda^\times$ are such that both $\varepsilon_1 \cdot \phi$ and $\varepsilon_2 \cdot \phi$ are synchronized. Then $\varepsilon_1 = \varepsilon_2$.*

Proof. Put $\phi_i := \varepsilon_i \cdot \phi$ and $\Gamma_i := \Gamma_{\phi_i}$ as in Proposition 5.10. Consider $\Gamma = \Gamma_1^{-1} \circ \Gamma_2$ and put $\delta := \varepsilon_1^{-1} \cdot \varepsilon_2$. Write $\delta = m/n$ with m, n integers such that $n > 0$. Note that Γ is an automorphism of K satisfying $\overline{\Gamma(x)} = \delta \cdot \bar{x}$ for all $x \in K^\times$.

Let $x \in K^\times$ be general in $K|k$. Since $\Gamma k = k$, it follows that $y := \Gamma(x)$ is also general in $K|k$, and one has $\bar{y} = \delta \cdot \bar{x}$, hence $y^n = c \cdot x^m$ for some $c \in k^\times$. Comparing x-adic valuations of x and y with this equality, we find that $\delta \in \mathbb{Z}$ and $y = c \cdot x^\delta$. In particular,

x and y are algebraically dependent while both x and y are general in $K|k$. This can only happen if $y \in \{c \cdot x, c \cdot x^{-1}\}$. The map $\varepsilon \mapsto \varepsilon \cdot \overline{x}$, $\varepsilon \in \Lambda^\times$, is injective since x is general in $K|k$, hence $\delta \in \{-1, 1\}$. We will conclude by showing $\delta = 1$.

Assume otherwise, hence $\delta = -1$, so $y = c \cdot x^{-1}$ and there exists $d \in k^\times$ such that $\Gamma(1 + x) = d \cdot (1 + x)^{-1}$. But Γ is a field automorphism, hence

$$1 + c \cdot x^{-1} = 1 + y = \Gamma(1 + x) = d \cdot (1 + x)^{-1}.$$

This is clearly impossible as x is transcendental over k. ∎

We have thus constructed a left inverse of the map

$$\mathrm{Isom}(K, L) \to \underline{\mathrm{Isom}}^{\mathrm{dep}}_{\mathrm{rat}}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(L|l))$$

appearing in Theorem 5.1, which is easily seen to be functorial with respect to isomorphisms. To conclude, we must prove that this left inverse is injective, and for this it suffices to take $K = L$ and prove that the group homomorphism we produced in this case,

$$\underline{\mathrm{Isom}}^{\mathrm{dep}}_{\mathrm{rat}}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(K|k)) \to \mathrm{Aut}(K),$$

is injective. So suppose that ϕ is an automorphism of $\mathscr{K}_\Lambda(K|k)$ which is compatible with dependence and with rational submodules, and which represents an element in the kernel of this map. Replacing ϕ with $\varepsilon \cdot \phi$ for some uniquely determined $\varepsilon \in \Lambda^\times$ (see Lemma 5.11), we may assume ϕ is synchronized and by assumption $\Gamma_\phi = \mathbf{1}$. Note $\phi\overline{t} = \overline{\Gamma_\phi t} = \overline{t}$ for all $t \in K^\times$, hence ϕ acts as the identity on $K^\times/k^\times$. But $\mathscr{K}_\Lambda(K|k)$ is generated as a Λ-module by this subgroup, so it follows that ϕ is the identity on $\mathscr{K}_\Lambda(K|k)$ as well. This concludes the proof of Theorem 5.1.

6. Proof of the main theorem

We now turn to the proof of the main theorem, which we state precisely as follows.

Theorem 6.1. *Let Λ be a subring of $\mathbb{Q}$, and let k, l be algebraically closed fields. Let $\sigma : k \hookrightarrow \mathbb{C}$ be a complex embedding. Let K be a function field of transcendence degree ≥ 2 over k and L a function field over l. There exists an isomorphism $K \cong L$ of fields which restricts to an isomorphism $k \cong l$ if and only if there exists a complex embedding $\tau : l \hookrightarrow \mathbb{C}$ and an isomorphism of mixed Hodge structures*

$$\phi : \mathrm{H}^1(K|k, \Lambda(1)) \cong \mathrm{H}^1(L|l, \Lambda(1))$$

such that the map ϕ induces a bijection $\mathscr{R}(K|k, \Lambda) \cong \mathscr{R}(L|l, \Lambda)$. Here $\mathrm{H}^*(K|k, \Lambda(*))$ is computed via σ while $\mathrm{H}^*(L|l, \Lambda(*))$ is computed via τ.

The rest of the section is devoted to proving this theorem. First, recall that any isomorphism of fields $K \cong L$ restricts to an isomorphism $k \cong l$. Given such an isomorphism of fields, say $\Gamma : K \cong L$ with restriction $\gamma : k \cong l$, one can define $\tau : l \hookrightarrow \mathbb{C}$ as the composition

$$l \xrightarrow{\gamma^{-1}} k \xrightarrow{\sigma} \mathbb{C},$$

and the existence of ϕ as in the statement of the theorem is then trivial.

Let us now fix an isomorphism ϕ as in the statement of the theorem. Our goal is to produce an isomorphism of fields $K \cong L$, which will automatically restrict to $k \cong l$ as observed above. The strategy is as follows:

(1) First, show that ϕ restricts to an isomorphism $\phi : \mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$ via Kummer theory. This will follow from the compatibility with the mixed Hodge structures.

(2) Second, show that this isomorphism is compatible with dependence and with rational submodules.

(3) Conclude by applying Theorem 5.1 to obtain an isomorphism $K \cong L$.

From now on, we put ourselves in the context of Theorem 6.1, and fix ϕ as in the statement.

6.1. *Compatibility with Kummer theory*

Note that the Kummer map

$$\kappa_K^\Lambda : \mathscr{K}_\Lambda(K|k, \Lambda) \to \mathrm{H}^1(K|k, \Lambda(1))$$

is injective since Λ is flat over $\mathbb{Z}$.

Lemma 6.2. *Let $\gamma : \Lambda \to \mathrm{H}^1(K|k, \Lambda(1))$ be a morphism of Λ-modules. Then γ arises from a morphism of mixed Hodge structures, where Λ is the underlying module of $\Lambda(0)$, if and only if $\gamma(1)$ is contained in the image of κ_K^Λ.*

Proof. First suppose $t \in K^\times$ is given, and consider the map

$$\gamma_t : \Lambda \to \mathrm{H}^1(K|k, \Lambda(1))$$

given by the composition

$$\Lambda = \mathrm{H}^1(\mathbb{G}_\mathrm{m}, \Lambda(1)) \xrightarrow{t^*} \mathrm{H}^1(U, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1)),$$

where U is any model of $K|k$ with $t \in \mathscr{O}^\times(U)$, and t^* is the map on cohomology induced by the associated morphism $t : U \to \mathbb{G}_\mathrm{m}$. Since both morphisms in this composition are compatible with the mixed Hodge structures, the same holds for γ_t. If γ is as in the statement of the lemma, and $\gamma(1)$ is in the image of κ_K^Λ, then it is a linear combination of morphisms of the form γ_t for $t \in K^\times$, so again γ_t is compatible with the mixed Hodge structures.

Conversely, suppose that γ is compatible with the mixed Hodge structures, and let X be a smooth projective model of $K|k$ and U a sufficiently small nonempty open k-subvariety of X such that γ factors through $\mathrm{H}^1(U, \Lambda(1)) \hookrightarrow \mathrm{H}^1(K|k, \Lambda(1))$. Consider the Picard 1-motive $\mathbf{M}^{1,1}(U)$ associated to the inclusion $U \hookrightarrow X$, as well as the 1-motive

$\mathbb{Z} := [\mathbb{Z} \to 0]$. Since Λ is flat over $\mathbb{Z}$, by Theorem 4.2 and Lemma 4.1, the Hodge realization functor induces a canonical isomorphism

$$\mathrm{Hom}_k(\mathbb{Z}, \mathbf{M}^{1,1}(U)) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{MHS}}(\Lambda(0), \mathrm{H}^1(U, \Lambda(1))).$$

Unfolding the definitions, we see that

$$\mathrm{Hom}_k(\mathbb{Z}, \mathbf{M}^{1,1}(U)) \otimes_{\mathbb{Z}} \Lambda = (\mathcal{O}^\times(U)/k^\times) \otimes_{\mathbb{Z}} \Lambda,$$

and so γ gives rise to some element $\alpha \in (\mathcal{O}^\times(U)/k^\times) \otimes_{\mathbb{Z}} \Lambda \subset \mathscr{K}_\Lambda(K|k)$ via this bijection. Tracing through the definitions, it is easy to see that $\gamma(1) = \kappa_K^\Lambda(\alpha)$. ∎

As a consequence of this lemma, we deduce that ϕ restricts to an isomorphism

$$\phi : \mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$$

which fits in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1(K|k, \Lambda(1)) & \xrightarrow{\phi} & \mathrm{H}^1(L|k, \Lambda(1)) \\ {\scriptstyle \kappa_K^\Lambda}\uparrow & & \uparrow{\scriptstyle \kappa_L^\Lambda} \\ \mathscr{K}_\Lambda(K|k) & \xrightarrow{\phi} & \mathscr{K}_\Lambda(L|l) \end{array} \tag{6.1}$$

6.2. Concluding the proof

Our final two tasks are to show that the isomorphism $\phi : \mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$ discussed in (6.1) is compatible with dependence and with rational submodules. We prove these in two separate lemmas.

Lemma 6.3. *In the above context, $\phi : \mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$ is compatible with dependence.*

Proof. Let $x, y \in \mathscr{K}_\Lambda(K|k)$ be given. By Lemma 3.2, we see that x, y are dependent if and only if $\kappa_K^\Lambda x \cup \kappa_K^\Lambda y = 0$ in $\mathrm{H}^2(K|k, \Lambda(2))$. The lemma follows from the compatibility of ϕ with $\mathscr{R}$ and the commutativity of (6.1). ∎

We will need the following lemma to deduce the compatibility with rational submodules.

Lemma 6.4. *Let E be a function field over k satisfying* $\mathrm{trdeg}(E|k) = 1$. *Then $E|k$ is rational if and only if the Kummer map*

$$\kappa_E^\Lambda : \mathscr{K}_\Lambda(E|k) \to \mathrm{H}^1(E|k, \Lambda(1))$$

is an isomorphism.

Proof. Let C be the smooth projective model of $E|k$. For each closed subset Z of C, one has an exact sequence of the form

$$0 \to \mathrm{H}^1(C, \Lambda(1)) \to \mathrm{H}^1(C \smallsetminus Z, \Lambda(1)) \to \mathrm{Div}_Z^0(C) \otimes_{\mathbb{Z}} \Lambda \to 0,$$

which is obtained (after tensoring with Λ) from (4.3) in the case where one considers the Picard 1-motive $[\mathrm{Div}^0_Z(C) \to \mathbf{Pic}^0_C]$. Passing to the colimit over all $U := C \smallsetminus Z$, we obtain an exact sequence

$$0 \to \mathrm{H}^1(C, \Lambda(1)) \to \mathrm{H}^1(E|k, \Lambda(1)) \to \mathrm{Div}^0(C) \otimes_{\mathbb{Z}} \Lambda \to 0.$$

Note that the map $\mathrm{H}^1(E|k, \Lambda(1)) \to \mathrm{Div}^0(C) \otimes_{\mathbb{Z}} \Lambda$ in this exact sequence is given by the sum of the residue maps ∂_x associated to $x \in C(k)$.

This exact sequence, and the Kummer map κ_E^Λ in question, both fit in the following commutative diagram with exact rows/columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{K}_\Lambda(E|k) & = & \mathscr{K}_\Lambda(E|k) & & \\
 & & & & \downarrow{\scriptstyle \kappa_E^\Lambda} & & \downarrow{\scriptstyle \mathrm{div}\otimes\Lambda} & & \\
0 & \longrightarrow & \mathrm{H}^1(C, \Lambda(1)) & \longrightarrow & \mathrm{H}^1(E|k, \Lambda(1)) & \longrightarrow & \mathrm{Div}^0(C) \otimes_{\mathbb{Z}} \Lambda & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^1(C, \Lambda(1)) & \longrightarrow & \mathrm{coker}(\kappa_E^\Lambda) & \longrightarrow & \mathrm{Pic}^0(C) \otimes_{\mathbb{Z}} \Lambda & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now, $E|k$ is rational if and only if C has genus 0, which is equivalent to $\mathrm{Pic}^0(C) \otimes_{\mathbb{Z}} \Lambda = 0$ and $\mathrm{H}^1(C, \Lambda(1)) = 0$. The assertion of the lemma follows from the exactness of the bottom row in the above diagram. ∎

Lemma 6.5. *In the above context, $\phi : \mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$ is compatible with rational submodules.*

Proof. Let x be a general element of $K|k$ so that $\mathscr{K}_x := \mathscr{K}_\Lambda(k(x)|k)$ is a rational submodule of $\mathscr{K}_\Lambda(K|k)$. Put $\mathscr{L} := \phi\mathscr{K}$, a submodule of $\mathscr{K}_\Lambda(L|l)$. Since ϕ is compatible with dependence by Lemma 6.3, $\mathscr{L}$ must have the form $\mathscr{K}_\Lambda(\mathrm{acl}_L(y)|l)$ for some $y \in L \smallsetminus l$ by Lemma 5.3. We must show that $\mathrm{acl}_L(y)$ is rational over l.

By Fact 3.1 and Proposition 3.4, we see that the image of the injective map

$$\mathrm{H}^1(\mathrm{acl}_L(y)|l, \Lambda(1)) \to \mathrm{H}^1(L|l, \Lambda(1))$$

can be characterized as the collection of elements α of $\mathrm{H}^1(L|l, \Lambda(1))$ such that for all $\beta \in \mathscr{K}_\Lambda(\mathrm{acl}_L(y)|l)$, one has $\kappa_L^\Lambda \beta \cup \alpha = 0$. The image of

$$\mathrm{H}^1(k(x)|k, \Lambda(1)) \to \mathrm{H}^1(K|k, \Lambda(1))$$

can be similarly characterized as the elements which pair trivially with $\mathscr{K}_x = \mathscr{K}_\Lambda(k(x)|k)$. The compatibility of ϕ with $\mathscr{R}$ ensures that ϕ restricts to a commutative diagram whose horizontal morphisms are injective and whose vertical morphisms are isomorphisms which are all induced by ϕ:

$$\begin{array}{ccccc} \mathscr{K}_x = \mathscr{K}_\Lambda(k(x)|k) & \hookrightarrow & \mathrm{H}^1(k(x)|k,\Lambda(1)) & \hookrightarrow & \mathrm{H}^1(K|k,\Lambda(1)) \\ \cong\downarrow & & \downarrow\cong & & \downarrow\cong \\ \mathscr{L} = \mathscr{K}_\Lambda(\mathrm{acl}_L(y)|k) & \hookrightarrow & \mathrm{H}^1(\mathrm{acl}_L(y)|l,\Lambda(1)) & \hookrightarrow & \mathrm{H}^1(L|l,\Lambda(1)) \end{array}$$

The fact that $\mathscr{L}$ is a rational submodule now follows easily from Lemma 6.4. ∎

Putting everything together, ϕ induces an isomorphism $\mathscr{K}_\Lambda(K|k) \cong \mathscr{K}_\Lambda(L|l)$ via Kummer theory (see (6.1)) which is compatible with dependence by Lemma 6.3 and with rational submodules by Lemma 6.5. Thus $\mathrm{Isom}^{\mathrm{dep}}_{\mathrm{rat}}(\mathscr{K}_\Lambda(K|k), \mathscr{K}_\Lambda(L|l))$ is nonempty, hence $\mathrm{Isom}(K, L)$ is nonempty by Theorem 5.1. This concludes the proof of Theorem 6.1 as any isomorphism $K \cong L$ restricts to an isomorphism $k \cong l$.

Appendix A. The local theory

The *local theory* in "almost-abelian" anabelian geometry has been extensively developed by Bogomolov [7], Bogomolov–Tschinkel [8], Pop [22], and the author [28, 30]. Despite this, the precise statement which is needed in the above paper has not appeared in the literature, since previous results have mostly focused on the "classical" anabelian point of view of decomposition and inertia groups in Galois groups (of function fields, in this case). In this appendix we give an essentially self-contained account of the *local theory*, which is required in the main body of the present paper. The arguments we give here are merely a distillation of the ideas developed in the references mentioned above.

We use the notation introduced in the body of the paper. The main result in the local theory reads as follows.

Theorem A.1. *Let $K|k$ and $L|l$ be two function fields over algebraically closed fields, and let Λ be a subring of $\mathbb{Q}$. Assume that* $\mathrm{trdeg}(K|k) \geq 2$. *Let*

$$\phi : \mathscr{K}_\Lambda(K|k) \xrightarrow{\cong} \mathscr{K}_\Lambda(L|l)$$

be an isomorphism of Λ-modules which is compatible with dependence, and let v be a divisorial valuation of $K|k$. Then there exists a unique divisorial valuation w of $L|l$ such that $\phi(\mathscr{U}_v) = \mathscr{U}_w$ and $\phi(\mathscr{U}_v^1) = \mathscr{U}_w^1$.

We work in the context of this theorem for the rest of the appendix. For a field extension $M|F$, we put

$$\mathscr{G}(M|F) := \mathrm{Hom}(M^\times/F^\times, \mathbb{Q}),$$

considered as a $\mathbb{Q}$-module endowed with the pointwise convergence topology, and we will tacitly only consider submodules which are closed with respect to this topology. We consider elements of $\mathscr{G}(M|F)$ as morphisms $M^\times \to \mathbb{Q}$ which are trivial on $F^\times$.

For a valuation v of M, we define

$$\mathcal{I}_v := \mathrm{Hom}(M^\times/(\mathrm{U}_v \cdot F^\times), \mathbb{Q}), \quad \mathcal{D}_v := \mathrm{Hom}(M^\times/(\mathrm{U}_v^1 \cdot F^\times), \mathbb{Q}),$$

both considered as subspaces of $\mathcal{G}(M|F)$ with $\mathcal{I}_v \subset \mathcal{D}_v$.

Note that the isomorphism ϕ from Theorem A.1 induces an isomorphism

$$\psi := (\phi^{-1})^* : \mathcal{G}(K|k) \cong \mathcal{G}(L|l),$$

and, for a divisorial valuation v of $K|k$, $\mathcal{U}_v$ resp. $\mathcal{U}_v^1$ is orthogonal to $\mathcal{I}_v$ resp. $\mathcal{D}_v$ with respect to the canonical Λ-bilinear pairing $\mathcal{G}(K|k) \times \mathcal{K}_\Lambda(K|k) \to \mathbb{Q}$. It therefore suffices to provide a characterization of $\mathcal{I}_v \subset \mathcal{D}_v$ for divisorial valuations v of $K|k$ in terms of the following data:

(1) The Λ-module $\mathcal{K}_\Lambda(K|k)$ and the $\mathbb{Q}$-module $\mathcal{G}(K|k)$.

(2) The canonical pairing $\mathcal{G}(K|k) \times \mathcal{K}_\Lambda(K|k) \to \mathbb{Q}$.

(3) The binary relation on $\mathcal{K}_\Lambda(K|k)$ given by dependence.

A.1. Abhyankar's inequality

Let v be any valuation of K. *Abhyankar's inequality* [1] asserts that

$$\dim_\mathbb{Q}((vK/vk) \otimes_\mathbb{Z} \mathbb{Q}) + \mathrm{trdeg}(Kv|kv) \le \mathrm{trdeg}(K|k).$$

If equality holds, then we say that v is *defectless*, and in this case vK/vk is a finitely-generated group and $Kv|kv$ is a finitely-generated field extension.

Note that $K^\times/(\mathrm{U}_v \cdot k^\times) = vK/vk$ and thus

$$\mathcal{I}_v \cong \mathrm{Hom}_\mathbb{Q}((vK/vk) \otimes_\mathbb{Z} \mathbb{Q}, \mathbb{Q})$$

is finite-dimensional and $\dim_\mathbb{Q} \mathcal{I}_v = \dim_\mathbb{Q}((vK/vk) \otimes_\mathbb{Z} \mathbb{Q}) \le \mathrm{trdeg}(K|k)$. Also, since the quotient $\mathcal{D}_v/\mathcal{I}_v$ is naturally isomorphic to $\mathcal{G}(Kv|kv)$ and kv is algebraically closed, we see that $\mathrm{trdeg}(Kv|kv) = 0$ if and only if $\mathcal{I}_v = \mathcal{D}_v$. We will use these observations several times in the discussion below.

A.2. acl-*pairs*

Let $f, g \in \mathcal{G}(K|k)$ be given. We say that (f, g) is an acl-*pair* provided that one of the following equivalent (since Λ is a subring of $\mathbb{Q}$) conditions holds true:

(1) For all dependent $x, y \in \mathcal{K}_\Lambda(K|k)$, one has $f(x) \cdot g(y) = f(y) \cdot g(x)$.

(2) For all k-algebraically dependent $x, y \in K^\times$, one has $f(x) \cdot g(y) = f(y) \cdot g(x)$.

A subspace $\mathcal{H} \subset \mathcal{G}(K|k)$ is an acl-*subspace* if any pair of elements of $\mathcal{H}$ is an acl-pair.

Lemma A.2. *Let v be a valuation of K, and let $f \in \mathcal{I}_v$ and $g \in \mathcal{D}_v$ be given. Then (f, g) is an* acl-*pair.*

Proof. Suppose $x, y \in K^\times$ are algebraically dependent, and let $t \in K \smallsetminus k$ be such that $x, y \in \mathrm{acl}_K(t) =: M$. We must show that $f(x) \cdot g(y) = f(y) \cdot g(x)$.

Let w denote the restriction of v to M. If $x, y \in \mathrm{U}_v \cdot k^\times$, then the claim is trivial since $f(x) = f(y) = 0$. Otherwise wM/wk has rational rank at least 1, and by Abhyankar's inequality it must be exactly 1, while $\mathrm{trdeg}(Mw|kw) = 0$, hence $\mathrm{U}_w \cdot k^\times = \mathrm{U}_w^1 \cdot k^\times$. This implies that x, y must have $\mathbb{Q}$-linearly dependent images in

$$(M^\times/\mathrm{U}_w \cdot k^\times) \otimes_{\mathbb{Z}} \mathbb{Q} = (M^\times/\mathrm{U}_w^1 \cdot k^\times) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since both f, g act trivially on $\mathrm{U}_w^1 \cdot k^\times$, it follows again that $f(x) \cdot g(y) = f(y) \cdot g(x)$. ∎

Theorem A.3. *Let $\mathcal{H} \subset \mathcal{G}(K|k)$ be a subspace. Then $\mathcal{H}$ is an* acl*-subspace of $\mathcal{G}(K|k)$ if and only if there exists a valuation v of $K|k$ such that $\mathcal{H} \subset \mathcal{D}_v$ and $\mathcal{I}_v \cap \mathcal{H}$ has codimension at most* 1 *in $\mathcal{H}$.*

Proof. If v exists as in the statement of the theorem, then the assertion about $\mathcal{H}$ follows from Lemma A.2. Conversely, suppose that $\mathcal{H}$ is an acl-subspace. Let $T \subset K^\times$ be the orthogonal of $\mathcal{H}$ with respect to the pairing $\mathcal{G}(K|k) \times K^\times \to \mathbb{Q}$. By [3, Theorem 2.16], it suffices to prove that for all $x, y \in K^\times \smallsetminus T$ such that $1 + x \notin T \cup x \cdot T$ and $1 + y \notin T \cup y \cdot T$, and all $f, g \in \mathcal{H}$, one has

$$f(x) \cdot g(y) = f(y) \cdot g(x).$$

Indeed, letting H denote the subgroup of $K^\times$ generated by T and all $x \in K^\times \smallsetminus T$ such that $1 + x \notin T \cup x \cdot T$, it follows from [3, Theorem 2.16] that there exists a subgroup $\widetilde{H}$ of $K^\times$ containing H with $[\widetilde{H} : H] \le 2$, and a valuation v of K such that $\mathrm{U}_v^1 \subset T$ and $\mathrm{U}_v \subset \widetilde{H}$. The condition above would then imply that $\mathcal{I} := \mathrm{Hom}(K^\times/\widetilde{H}, \mathbb{Q}) = \mathrm{Hom}(K^\times/H, \mathbb{Q})$ has codimension at most 1 in $\mathrm{Hom}(K^\times/T, \mathbb{Q}) = \mathcal{H}$, while $\mathcal{H} \subset \mathcal{D}_v$ and $\mathcal{I} \subset \mathcal{I}_v$.

Assume toward a contradiction that this does not hold, and let $x, y \in K^\times$ and $f, g \in \mathcal{H}$ witness this. Put $\Phi := (f, g) : K^\times \to \mathbb{Q}^2$. By our assumption on $\mathcal{H}$, we see that whenever $u, v \in K^\times$ with $u \pm v \in K^\times$, the triple $(\Phi(u \pm v), \Phi(u), \Phi(v))$ is collinear (in the affine sense). Under our assumption on x, y and f, g, the pair $\Phi(x), \Phi(y)$ is linearly independent and $\Phi(1 + x) \notin \{\Phi(1), \Phi(x)\}$, $\Phi(1 + y) \notin \{\Phi(1), \Phi(y)\}$.

Embed $\mathbb{Q}^2 = \mathbb{A}^2(\mathbb{Q})$ into $\mathbb{P}^2(\mathbb{Q})$ via $(a, b) \mapsto (1 : a : b)$, and compose with this inclusion and the unique projective-linear automorphism Σ of $\mathbb{P}^2(\mathbb{Q})$ satisfying

$$\Sigma(1:0:0) = (1:0:0),\ \ \Sigma(\Phi(x)) = (1:1:0),\ \ \Sigma(\Phi(y)) = (1:0:1),$$

$$\Sigma(\Phi(1+x)) = (0:1:0),\ \ \Sigma(\Phi(1+y)) = (0:0:1).$$

Denote the resulting map by $\Psi : K^\times \to \mathbb{P}^2(\mathbb{Q})$. The following conditions hold:

(1) For all $u, v \in K^\times$ with $u \pm v \in K^\times$, the triple $(\Psi(u), \Psi(v), \Psi(u \pm v))$ is collinear.

(2) $\Psi(1) = (1 : 0 : 0)$, $\Psi(x) = (1 : 1 : 0)$ and $\Psi(y) = (1 : 0 : 1)$.

(3) $\Psi(1 + x) = (0 : 1 : 0)$ and $\Psi(1 + y) = (0 : 0 : 1)$.

We will show that the image of Ψ contains a complete projective line, which is impossible by the construction of Ψ, as the image of Ψ misses the image under Σ of the line at infinity. For $u, v \in K^\times$ with $\Psi(u) \neq \Psi(v)$, write $\mathfrak{L}(u, v)$ for the unique projective line containing $\Psi(u)$ and $\Psi(v)$.

The proof that follows is elementary but technical, and comes down to computing intersections of pairs of lines of the form $\mathfrak{L}(u, v)$ based on a sum/difference decomposition of the corresponding elements in $K^\times$. For example, $\Psi(1 + x + y) = (1 : 1 : 1)$ since $1 + x + y = (1 + x) + y = (1 + y) + x$, hence $\Psi(1 + x + y)$ lies in the intersection $\mathfrak{L}(1 + x, y) \cap \mathfrak{L}(1 + y, x)$, which contains the unique point $(1 : 1 : 1)$. In the steps below, we give the sum/difference decomposition, leaving the straightforward computation of the intersection of the corresponding lines to the reader.

Step 1. *One has* $\Psi(1 + x + y) = (1 : 1 : 1)$.

Proof. This follows from the equation $1 + x + y = (1 + x) + y = (1 + y) + x$. ■

Step 2. *One has* $\Psi(2 + x + y) = (0 : 1 : 1)$.

Proof. This follows from

$$2 + x + y = 1 + (1 + x + y) = (1 + x) + (1 + y)$$

and Step 1. ■

Step 3. *For all integers* $n \geq 1$,

$$\Psi((2 - n) + x + y) = (n : 1 : 1), \quad \Psi((1 - n) + x) = (n : 1 : 0).$$

Proof. We proceed by induction on n with the base case $n = 1$ having been done above. For the inductive case, first calculate $\Psi((2 - (n + 1)) + x + y)$ using

$$(2 - (n + 1)) + x + y = ((1 - n) + x) + y = ((2 - n) + x + y) - 1$$

combined with the inductive hypothesis, showing that

$$\Psi((2 - (n + 1)) + x + y) = (n + 1 : 1 : 1).$$

Conclude by calculating $\Psi((1 - (n + 1)) + x)$ using

$$(1 - (n + 1)) + x = ((2 - (n + 1)) + x + y) - (1 + y) = ((1 - n) + x) - 1,$$

along with the calculation above and the inductive hypothesis to obtain

$$\Psi((1 - (n + 1)) + x) = (n + 1 : 1 : 0).$$

This concludes the proof of this step. ■

Step 4. *For all integers* $n, m \geq 1$,

$$\Psi((1 + m - n) + m \cdot x + y) = (n : m : 1), \quad \Psi((m - n) + m \cdot x) = (n : m : 0).$$

Proof. Proceed by induction on m with the case $m = 1$ taken care of by Step 3. For the inductive case, first use

$$\begin{aligned}(1 + (m + 1) - n) + (m + 1) \cdot x + y &= ((1 + m - n) + m \cdot x + y) + (1 + x)\\ &= ((m - n) + m \cdot x) + (2 + x + y)\end{aligned}$$

along with the inductive hypothesis and Step 1 to deduce

$$\Psi((1 + (m + 1) - n) + (m + 1) \cdot x + y) = (n : m + 1 : 1).$$

Then use the equation

$$\begin{aligned}((m + 1) - n) + (m + 1) \cdot x &= ((m - n) + m \cdot x) + (1 + x)\\ &= ((1 + (m + 1) - n) + (m + 1) \cdot x + y) - (1 + y)\end{aligned}$$

with the inductive hypothesis and the above calculation to deduce

$$\Psi(((m + 1) - n) + (m + 1) \cdot x) = (n : m + 1 : 0).$$

This concludes the proof of this step. ■

Step 5. *One has* $\Psi(2 + x) = (1 : -1 : 0)$.

Proof. Use the equation $2 + x = 1 + (1 + x) = (2 + x + y) - y$ along with Step 2. ■

We can now conclude, as follows. First, by Step 4, the image of Ψ contains the set

$$\{(1 : a : 0) \mid a \in \mathbb{Q}_{\geq 0}\} \cup \{(0 : 1 : 0)\}.$$

Repeat the argument above while replacing x with $x' := -2 - x$ and $1 + x' = -1 - x$, while noting that $\Psi(x') = \Psi(2 + x) = (1 : -1 : 0)$ by Step 5 and $\Psi(1 + x') = \Psi(1 + x) = (0 : 1 : 0)$, to see that $\{(1 : a : 0) \mid -a \in \mathbb{Q}_{\geq 0}\}$ is also contained in the image of Ψ. Hence the whole projective line $\mathfrak{L}(1, x)$ is contained in the image of Ψ, which is impossible as discussed above. This provides the required contradiction, hence proving the theorem. ■

We will need the following refinement of the above theorem.

Proposition A.4. *Put* $d := \operatorname{trdeg}(K|k)$*, and let* $\mathcal{H} \subset \mathcal{G}(K|k)$ *be an* acl-*subspace of dimension* d*. Then there exists a valuation* v *of* K *with no transcendence defect in* $K|k$ *such that* $\mathcal{I}_v \subset \mathcal{H} \subset \mathcal{D}_v$ *and* $\mathcal{I}_v$ *has codimension at most* 1 *in* $\mathcal{H}$.

Proof. By Theorem A.3, there exists a valuation v such that $\mathcal{H} \subset \mathcal{D}_v$ and $\mathcal{I}_v \cap \mathcal{H}$ has codimension at most 1 in $\mathcal{H}$. By Abhyankar's inequality, $\mathcal{I}_v$ has dimension $\leq d$. Furthermore, if $\dim_{\mathbb{Q}} \mathcal{I}_v = d$ then v is defectless and $\operatorname{trdeg}(Kv|kv) = 0$ so that $\mathcal{D}_v = \mathcal{I}_v$, hence $\mathcal{I}_v = \mathcal{H} = \mathcal{D}_v$. Otherwise $\dim_{\mathbb{Q}} \mathcal{I}_v \leq d - 1$, and we still have $\mathcal{I}_v \subset \mathcal{H}$ since $\mathcal{I}_v \cap \mathcal{H}$ has dimension at least $d - 1$. In any case, we have $\mathcal{I}_v \subset \mathcal{H} \subset \mathcal{D}_v$.

To conclude we must show that v is defectless. If $\mathcal{I}_v = \mathcal{H}$ then we are done, as observed above, so assume that $\mathcal{I}_v \neq \mathcal{H}$. The argument above shows that $\mathcal{I}_v$ has dimension $d - 1$, hence vK/vk has rational rank $d - 1$. Thus, the only way that v can have any transcendence defect is if $\mathrm{trdeg}(Kv|kv) = 0$, in which case $\mathcal{D}_v = \mathcal{I}_v$. The fact that $\mathcal{I}_v \subset \mathcal{H} \subset \mathcal{D}_v$, while $\mathcal{I}_v \neq \mathcal{H}$ shows that this cannot happen. ∎

Finally, we will need the following characterization of the transcendence degree of $K|k$, which follows fairly easily from Abhyankar's inequality and Theorem A.3.

Lemma A.5. *If $\mathcal{H} \subset \mathcal{G}(K|k)$ is an* acl-*subspace, then* $\dim_{\mathbb{Q}} \mathcal{H} \leq \mathrm{trdeg}(K|k)$. *In particular,* $\mathrm{trdeg}(K|k)$ *is the maximal integer d such that $\mathcal{G}(K|k)$ has an* acl-*subspace of dimension d.*

Proof. Let $\mathcal{H}$ be an acl-subspace of $\mathcal{G}(K|k)$. By Theorem 5.1, there exists a valuation v of $K|k$ such that $\mathcal{H} \subset \mathcal{D}_v$ and $\mathcal{H} \cap \mathcal{I}_v$ has codimension ≤ 1 in $\mathcal{H}$. By Abhyankar's inequality, $\dim_{\mathbb{Q}} \mathcal{I}_v \leq \mathrm{trdeg}(K|k)$ so we may assume without loss of generality that $\mathcal{H} \cap \mathcal{I}_v$ has codimension 1 in $\mathcal{H}$.

Since $\mathcal{H}/\mathcal{H} \cap \mathcal{I}_v$ embeds into $\mathcal{G}(Kv|kv)$ it then follows that $\mathrm{trdeg}(Kv|kv) \geq 1$ and thus

$$\dim_{\mathbb{Q}} \mathcal{H} = \dim_{\mathbb{Q}}(\mathcal{H} \cap \mathcal{I}_v) + 1 \leq \dim_{\mathbb{Q}} \mathcal{I}_v + \mathrm{trdeg}(Kv|kv) \leq \mathrm{trdeg}(K|k),$$

again using Abhyankar's inequality.

Finally, if $d = \mathrm{trdeg}(K|k)$ and v is any k-valuation of K whose value group is $\mathbb{Z}^d$ (e.g. a valuation arising from a complete flag of divisorial valuations), then $\mathcal{I}_v$ is an acl-subspace of $\mathcal{G}(K|k)$ of dimension d. Thus d does indeed agree with the maximal dimension of an acl-subspace of $\mathcal{G}(K|k)$. ∎

A.3. Quasi-divisorial valuations

A valuation v of K is called a *quasi-divisorial valuation of $K|k$* provided that it is minimal with respect to the following conditions:

(1) $vK/vk \cong \mathbb{Z}$.

(2) $\mathrm{trdeg}(Kv|kv) = \mathrm{trdeg}(K|k) - 1$.

Note that a quasi-divisorial valuation is divisorial if and only if it is trivial on k.

Proposition A.6. *Let $\mathcal{I} \subset \mathcal{G}(K|k)$ be a one-dimensional subspace, and put $d :=$* $\mathrm{trdeg}(K|k)$. *Assume that $d \geq 2$. Then there exists a quasi-divisorial valuation v of $K|k$ such that $\mathcal{I} = \mathcal{I}_v$ if and only if there exist two* acl-*subspaces $\mathcal{H}_1, \mathcal{H}_2$ of dimension d such that $\mathcal{I} = \mathcal{H}_1 \cap \mathcal{H}_2$.*

Proof. If $\mathcal{I} = \mathcal{I}_v$, then the existence of $\mathcal{H}_1$ and $\mathcal{H}_2$ is easy by choosing two independent kv-valuations w_1, w_2 of Kv whose value group is $\mathbb{Z}^{d-1}$ with the lexicographic ordering, letting $v_i = w_i \circ v$, and taking $\mathcal{H}_i = \mathcal{I}_{v_i}$.

For the converse, we know from Proposition A.4 that there exist defectless valuations v_1 and v_2 such that $\mathcal{I}_{v_i} \subset \mathcal{H}_i \subset \mathcal{D}_{v_i}$, with $\mathcal{I}_{v_i}$ having codimension at most 1 in $\mathcal{H}_i$. By our assumption on d, v_1 and v_2 are both nontrivial. Furthermore, the two valuations v_1, v_2 must be *dependent* since $\mathcal{I} \subset \mathcal{D}_{v_1} \cap \mathcal{D}_{v_2}$ with $\mathcal{I}$ being nontrivial, hence $\mathrm{U}^1_{v_1} \cdot \mathrm{U}^1_{v_2} \neq K^\times$.

Let v_0 denote the maximal common coarsening of v_1, v_2, so $\mathrm{U}_{v_0} = \mathrm{U}^1_{v_1} \cdot \mathrm{U}^1_{v_2}$, and thus $\mathcal{I} \subset \mathcal{I}_{v_0}$. Let v be the coarsening of v_0 associated to the maximal convex subgroup of $v_0(\mathcal{I}^\perp)$, where $\mathcal{I}^\perp$ is the orthogonal to $\mathcal{I}$ with respect to the pairing $\mathcal{G}(K|k) \times K^\times \to \mathbb{Q}$. Note that $\mathcal{I} \subset \mathcal{I}_v$ as well, while $\mathcal{D}_{v_i} \subset \mathcal{D}_v$ for $i = 1, 2$. We claim that v does the job.

As v is a coarsening of v_i, $i = 1, 2$, it is also defectless, hence vK/vk is finitely-generated as an abelian group and $Kv|kv$ is finitely-generated as a field extension. Note that $\mathcal{I} \subset \mathcal{I}_v \subset \mathcal{I}_{v_1} \cap \mathcal{I}_{v_2} \subset \mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{I}$, hence $\mathcal{I} = \mathcal{I}_v$. As vK/vk is a finitely-generated free abelian group, it follows that $vK/vk \cong \mathbb{Z}$ since $\dim_{\mathbb{Q}} \mathcal{I} = 1$. Since v is defectless, this then implies that $\mathrm{trdeg}(Kv|kv) = \mathrm{trdeg}(K|k) - 1$. The minimality of v is ensured from the construction since $v(\mathcal{I}_v^\perp)$ has no nontrivial convex subgroup. ∎

Lemma A.7. *Let v be a quasi-divisorial valuation of $K|k$. Then $\mathcal{D}_v$ is the subspace of $\mathcal{G}(K|k)$ consisting of elements f which form an* acl-*pair with every element of $\mathcal{I}_v$.*

Proof. By Lemma A.2, any element of $\mathcal{D}_v$ forms an acl-pair with any element of $\mathcal{I}_v$. Conversely, suppose that f forms an acl-pair with every element of $\mathcal{I}_v$. Let $x \in \mathfrak{m}_v$ be given. Note $\mathcal{I}_v$ is one-dimensional, and let g be a generator of $\mathcal{I}_v$. If $g(x) \neq 0$, then $f(1 + x) = 0$ since

$$f(1 + x) \cdot g(x) = f(x) \cdot g(1 + x) = 0$$

as $1 + x \in \mathrm{U}_v$. Otherwise, the minimality of v ensures that there exists some $y \in K^\times$ with $0 < v(y) < v(x)$ and $g(y) \neq 0$. Arguing as above, we have $f(1 + y) = 0$, and since $v(y + x \cdot (1 + y)) = v(y)$, so that $g(y + x \cdot (1 + y)) = g(y) \neq 0$ as well, we have

$$f(1 + x) + f(1 + y) = f(1 + (y + x \cdot (1 + y))) = 0,$$

hence $f(1 + x) = 0$. In other words, f acts trivially on U^1_v, so $f \in \mathcal{D}_v$, as required. ∎

A.4. Characterizing divisorial valuations

Let us summarize what we have done.

(1) We gave a characterization of $\mathrm{trdeg}(K|k)$ in terms of acl-pairs in Lemma A.5.

(2) We gave a characterization of $\mathcal{I}_v$ for quasi-divisorial valuations v of $K|k$ in terms of acl-pairs and $\mathrm{trdeg}(K|k)$ in Proposition A.6.

(3) For every such v, we gave a characterization of $\mathcal{D}_v$ in terms of acl-pairs and $\mathcal{I}_v$ in Lemma A.7.

In other words, we provided a characterization of $\mathcal{I}_v \subset \mathcal{D}_v$ for all quasi-divisorial valuations v of $K|k$ in terms of acl-pairs in $\mathcal{G}(K|k)$. To conclude the proof of Theorem A.1, we will provide a characterization of the divisorial valuations among the quasi-divisorial valuations using acl-pairs in $\mathcal{G}(K|k)$ and the dependence relation in $\mathcal{K}_\Lambda(K|k)$.

Lemma A.8. *Let v be a quasi-divisorial valuation of $K|k$ and assume* $\mathrm{trdeg}(K|k) \geq 2$. *Then v is divisorial if and only if there exists some $t \in K \smallsetminus k$ such that the composition*

$$\mathcal{D}_v \hookrightarrow \mathcal{G}(K|k) \to \mathcal{G}(\mathrm{acl}_K(t)|k)$$

is surjective, where the morphism $\mathcal{G}(K|k) \to \mathcal{G}(\mathrm{acl}_K(t)|k)$ *is the canonical one arising from the inclusion* $\mathrm{acl}_K(t) \hookrightarrow K$.

Proof. First, if v is divisorial, we can choose some $t \in \mathrm{U}_v \smallsetminus k^\times \cdot \mathrm{U}_v^1$ so that v is trivial on $\mathrm{acl}_K(t)$, and, letting s denote the image of t in Kv, we obtain an embedding $\iota : \mathrm{acl}_K(t) \hookrightarrow \mathrm{acl}_{Kv}(s)$. The map in question fits in a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_v & \twoheadrightarrow & \mathcal{G}(Kv|kv) & \twoheadrightarrow & \mathcal{G}(\mathrm{acl}_{Kv}(s)|k) \\ \downarrow & & & & \downarrow{\scriptstyle \iota^*} \\ \mathcal{G}(K|k) & & \twoheadrightarrow & & \mathcal{G}(\mathrm{acl}_K(t)|k) \end{array}$$

where the arrows decorated with two heads are surjective. Hence the map in question is indeed surjective.

Conversely, assume that v is nontrivial on k, and let $t \in K \smallsetminus k$ be given. Replacing t with $a \cdot t^{\pm 1}$ for some $a \in k^\times$ if needed, we may assume that $v(t) > 0$. With this in mind, we have $1 + a \cdot t \in \mathrm{U}_v^1$ for all $a \in k^\times$ such that $v(a) > 0$. There are infinitely many such a where $1 + a \cdot t$ have $\mathbb{Q}$-linearly independent images in $\mathcal{K}_{\mathbb{Q}}(\mathrm{acl}_K(t)|k)$. Dually, the map $\mathcal{D}_v \to \mathcal{G}(\mathrm{acl}_K(t)|k)$ mentioned in the statement must have a cokernel of infinite rank. ∎

A.5. *Concluding the proof*

The proof of Theorem A.1 is a simple matter of putting everything together. Let ϕ be as in the statement of the theorem. This ϕ induces an isomorphism $\psi := (\phi^{-1})^* : \mathcal{G}(K|k) \cong \mathcal{G}(L|l)$, which is compatible with acl-pairs since ϕ is compatible with dependence. By Lemma A.5, it follows that $\mathrm{trdeg}(K|k) = \mathrm{trdeg}(L|l)$, and thus both are ≥ 2 by assumption. By Theorem A.6 and Lemma A.7 it follows that for every divisorial valuation v of $K|k$ there exists some *quasi-divisorial* valuation w of $L|l$ such that $\psi \mathcal{D}_v = \mathcal{D}_w$ and $\psi \mathcal{I}_v = \mathcal{I}_w$. Dualizing with respect to the pairing

$$\mathcal{G}(-) \times \mathcal{K}_\Lambda(-) \to \Lambda,$$

we see that $\phi \mathcal{U}_v = \mathcal{U}_w$ and $\phi \mathcal{U}_v^1 = \mathcal{U}_w^1$.

Arguing as in Lemma 5.3 from the main body of the paper, for every $t \in K \smallsetminus k$, there exists some $s \in L \smallsetminus l$ such that $\phi \mathcal{K}_t = \mathcal{K}_s$. Note that the inclusion $\mathcal{K}_t \hookrightarrow \mathcal{K}_\Lambda(K|k)$ dualizes to $\mathcal{G}(K|k) \to \mathcal{G}(\mathrm{acl}_K(t)|k)$, and similarly $\mathcal{K}_s \hookrightarrow \mathcal{K}_\Lambda(L|l)$ dualizes to $\mathcal{G}(L|l) \to \mathcal{G}(\mathrm{acl}_L(s)|l)$. Hence, by Lemma A.8, w is divisorial. To conclude, simply note that the valuation w is uniquely determined by $\mathcal{U}_w$, since the value group of w is $\mathbb{Z}$.

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