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# Classification of links with Khovanov homology of minimal rank

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**Abstract.** If $L$ is an oriented link with $n$ components, then the rank of its Khovanov homology is at least $2^n$. We classify all links that achieve this lower bound and show that such links can be obtained by iterated connected sums and disjoint unions of Hopf links and unknots. This gives a positive answer to a question asked by Batson and Seed (2015).

**Keywords.** Khovanov homology, instanton Floer homology, knot theory, low-dimensional topology, gauge theory

## Contents



## 1. Introduction

Let $L$ be an oriented link in $S^3$ and $R$ be a ring. Khovanov homology [14] assigns a bi-graded $R$-module $\mathrm{Kh}(L; R)$ to the link $L$. When $R$ is an integral domain, the Euler

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characteristics of $\mathrm{Kh}(L;R)$ are given by the coefficients of the unreduced Jones polynomial of $L$. If $L$ has $n$ components, then the value of the unreduced Jones polynomial at $t=1$ equals $(-2)^n$. Therefore,

$$\operatorname{rank}_R \mathrm{Kh}(L;R) \geq 2^n. \tag{1.1}$$

The rank of $\mathrm{Kh}(L;R)$ is independent of the orientation of $L$. This paper classifies all links $L$ such that equality is achieved in (1.1).

If $L$ is the unlink, then $\operatorname{rank}_R \mathrm{Kh}(L;R) = 2^n$ and (1.1) attains equality. However, there are other examples where equality holds in (1.1). In graph theory, a finite simple graph is called a *forest* if it contains no cycles. Given a forest $G$, we define the link $L_G$ by placing an unknot at each vertex of $G$ and linking two unknots as a Hopf link whenever there is an edge connecting the corresponding vertices (see Figures 1–3 for examples). The link $L_G$ is called the *forest of unknots* defined by $G$. By definition, every forest of unknots can be obtained by iterated connected sums and disjoint unions of Hopf links and unknots. By [1, Corollary 6.6], if $L_G$ is a forest of unknots with $n$ components, then $\operatorname{rank}_R \mathrm{Kh}(L_G;R) = 2^n$.

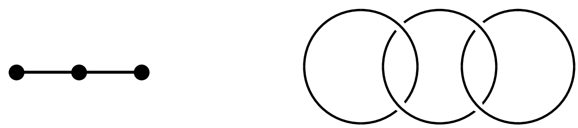

**Fig. 1.** Example of a forest of unknots.

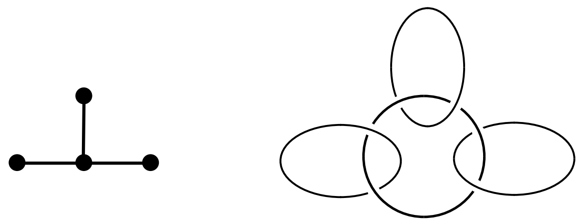

**Fig. 2.** Example of a forest of unknots.

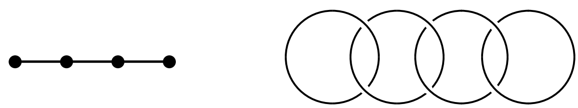

**Fig. 3.** Example of a forest of unknots.

The following question was asked by Batson and Seed:

**Question 1.1** ([5, Question 7.2]). *Are forests of unknots the only $n$-component links with Khovanov homology of rank $2^n$ in $\mathbb{Z}/2$-coefficients?*

The main result of the paper gives an affirmative answer to the above question.

**Theorem 1.2.** *If $L$ is an $n$-component link such that $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L;\mathbb{Z}/2) = 2^n$, then $L$ is a forest of unknots.*

**Remark 1.3.** The special case of Theorem 1.2 when $L$ is alternating was proved by Shumakovitch [30, Lemma 3.3.C].

The detection property of Khovanov homology has been a central question since the introduction of the theory. The first breakthrough in this field was the landmark paper by Kronheimer and Mrowka [20], which proved that Khovanov homology detects the unknot (see also [11,12]). Since then, several other detection results have been proved. It is known that Khovanov homology detects the unlink [5, 13], the trefoil [3] and the Hopf link [4].

Theorem 1.2 recovers all the previous detection results on links [4, 5, 13]. In fact, we have the following two corollaries of Theorem 1.2. They will be proved in Section 12.

**Corollary 1.4.** *Suppose $L_1$ and $L_2$ are two oriented links with $n$ components and $L_1$ is a forest of unknots. If there exist $e, f \in \mathbb{Z}$ such that*

$$\operatorname{Kh}^{i,j}(L_1;\mathbb{Z}/2) \cong \operatorname{Kh}^{i+e,j+f}(L_2;\mathbb{Z}/2)$$

*for all $i, j \in \mathbb{Z}$, then $L_2$ is isotopic to a forest of unknots whose graph has the same number of edges as the graph of $L_1$. In particular, if*

$$\operatorname{Kh}^{i,j}(L_1;\mathbb{Z}/2) \cong \operatorname{Kh}^{i,j}(L_2;\mathbb{Z}/2)$$

*for all $i, j \in \mathbb{Z}$, and if $L_1$ is the unlink, or a Hopf link, or the disjoint union of a Hopf link and the unlink, or a connected sum of two Hopf links, then $L_2$ is isotopic to $L_1$.*

Given a link $L$ with $n$ components, one can equip $\operatorname{Kh}(L;\mathbb{Z}/2)$ with a module structure over the ring

$$R_n := (\mathbb{Z}/2)[X_1,\ldots,X_n]/(X_1^2,\ldots,X_n^2).$$

For the definition of the module structure, the reader may refer to [13, Section 2] and [15, Section 3].

**Corollary 1.5.** *Suppose $L_1$ and $L_2$ are two links with $n$ components, and suppose $L_1$ is a forest of unknots with graph $G_1$. If $\operatorname{Kh}(L_1;\mathbb{Z}/2)$ is isomorphic to $\operatorname{Kh}(L_2;\mathbb{Z}/2)$ as $R_n$-modules, then $L_2$ is isotopic to a forest of unknots with graph $G_2$ such that*

(1) *there is a one-to-one correspondence between the connected components of $G_1$ and the connected components of $G_2$;*

(2) *the corresponding components of $G_1$ and $G_2$ have the same number of vertices.*

*If we further assume the number of vertices of every connected component of $G_1$ is less than or equal to 3, then $L_2$ is isotopic to $L_1$ as unoriented links.*

**Remark 1.6.** Suppose $L_1$ and $L_2$ are two forests of unknots which are associated to trees with the same number of vertices (see Figures 2 and 3 for examples). We may also orient $L_1, L_2$ so that the linking number of any two components is non-negative. Then $\operatorname{Kh}(L_1;\mathbb{Z}/2)$ and $\operatorname{Kh}(L_2;\mathbb{Z}/2)$ are isomorphic as bi-graded vector spaces according to

[1, Theorem 6.2]. Moreover, the module structures of $\mathrm{Kh}(L_1;\mathbb{Z}/2)$ and $\mathrm{Kh}(L_2;\mathbb{Z}/2)$ are also isomorphic according to Lemma 12.1.

The proof of Theorem 1.2 consists of five main steps. By Kronheimer–Mrowka's spectral sequence [20] and Batson–Seed's inequality, if $L$ satisfies the condition of Theorem 1.2, then all the components of $L$ are unknots. Using instanton Floer homology and various topological arguments, we establish the following results:

(1) The linking number of any two components of $L$ is 0 or 1. This is proved in Section 6 using a braid-detection result for annular instanton homology by the authors [34, Corollary 8.4] together with an Alexander polynomial argument for exchangeably braided links.

(2) Let $G$ be the graph such that each vertex of $G$ corresponds to a component of $L$ and there is an edge between two vertices of $G$ if and only if the linking number of the corresponding components of $L$ is non-zero. In Section 5, we show that if $G$ is a forest, then $L$ must be a forest of unknots. This uses the authors' previous result on the relationship between the generalized Thurston norm and singular instanton Floer homology [34, Theorem 8.2].

(3) If $G$ is not a forest, we may assume without loss of generality that $G$ is a cycle by passing to a sublink. By step (2), deleting a component of $L$ gives a connected sum of Hopf links which is fibered. In Section 7.2, we show that the deleted component can be made disjoint from a fiber. This step uses the properties of instanton homology with local coefficients developed in Section 3, and it is the most difficult gauge-theoretic step in the proof.

(4) It can be shown that the removed component in step (3) is the boundary of a disk that intersects the fiber in a single arc. We explicitly determine the arc using results in Sections 8 and 9. This step does not use gauge theory but depends on an in-depth analysis of the fundamental group of the complement of the connected sum of Hopf links.

(5) The link is now known explicitly enough to be ruled out by hand using a computation in Section 10.

The method in step (1) above can also be used to prove other new detection results for links with small Khovanov homology. The proof of Theorem 1.7 below is given in Section 6.

**Theorem 1.7.** *Let $L_1$ be the oriented link given by Figure 4, and let $L_2$ be the disjoint union of a trefoil and an unknot. Let $L = K_1 \cup K_2$ be a 2-component oriented link.*

(1) *If*
$$\mathrm{Kh}(L;\mathbb{Z}/2) \cong \mathrm{Kh}(L_1;\mathbb{Z}/2)$$
*as bi-graded vector spaces, then $L$ is isotopic to $L_1$.*

(2) *Let $q \in K_2$ and $p$ be a basepoint on the unknotted component of $L_2$. If*
$$\mathrm{Khr}(L,q;\mathbb{Z}) \cong \mathrm{Khr}(L_2,p;\mathbb{Z})$$

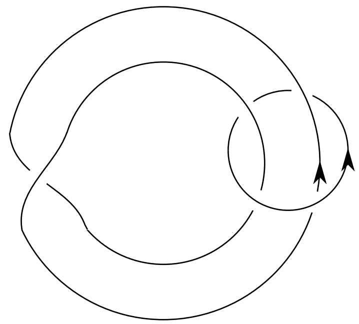

**Fig. 4.** The link L4a1 in the Thistlethwaite table with an orientation.

*as bi-graded abelian groups, then $L$ splits into the disjoint union of a trefoil $K_1$ and an unknot $K_2$.*

**Remark 1.8.** By the trefoil detection result of Baldwin and Sivek [3], the essential content of part (2) of Theorem 1.7 is the splitness of $L$. Shortly after the first version of this paper was posted to arXiv, Lipshitz and Sarkar [24] proved that the splitness of a general link can be detected by the module structure of Khovanov homology.

The proof of Theorem 1.2 does not immediately apply to other coefficient rings because it relies on [30, Corollary 3.2.C], which only holds for $\mathbb{Z}/2$-coefficients. In Section 12, we will use a purely algebraic argument to extend Theorem 1.2 to arbitrary coefficient rings and prove the following theorem.

**Theorem 1.9.** *Suppose $R$ is an integral domain. If $L$ is an $n$-component link such that* $\operatorname{rank}_R \operatorname{Kh}(L;R) = 2^n$, *then $L$ is a forest of unknots.*

## 2. Annular instanton Floer homology

The singular instanton Floer homology theory was introduced by Kronheimer and Mrowka [20,21]. Let $(Y, L, \omega)$ be a triple where $Y$ is a closed oriented 3-manifold, $L \subset Y$ is a link and $\omega \subset Y$ is an embedded 1-manifold such that $\partial\omega = \omega \cap L$. The triple $(Y, L, \omega)$ is called *admissible* if there is an embedded closed surface $\Sigma \subset Y$ satisfying either one of the following conditions:

- $\Sigma$ is disjoint from $L$ and the intersection number of $\omega$ and $\Sigma$ is odd,
- the intersection number of $L$ and $\Sigma$ is odd.

If $(Y, L, \omega)$ is admissible, the instanton Floer homology $\mathrm{I}(Y, L, \omega)$ is defined to be a Morse homology of the Chern–Simons functional on a certain space of orbifold SO(3)-connections over $Y$, where $Y$ is equipped with an orbifold structure with cone angle $\pi$ along $L$, and $\omega$ represents the second Stiefel–Whitney class of the SO(3)-bundle. In this article, we will always take $\mathbb{C}$-coefficients for instanton Floer homology.

The homology group $\mathrm{I}(Y, L, \omega)$ carries a relative $\mathbb{Z}/4$-homological grading. Given an embedded closed surface $F \subset Y$, there is an operator $\mu^{\mathrm{orb}}(F)$ defined on $\mathrm{I}(Y, L, \omega)$ with degree 2. For more details the reader may refer to, for example, [34, Section 2] or [31, Section 2.3.2].

The rest of this section gives a brief review of the annular instanton Floer homology introduced in [33]. Let $L$ be a link in the solid torus $S^1 \times D^2$. The annular instanton Floer homology $\mathrm{AHI}(L)$ is defined by the following procedure:

(1) Let $\mathcal{K}_2$ be the product link $S^1 \times \{p_1, p_2\}$ in $S^1 \times D^2$, and let $u$ be an arc in $S^1 \times D^2$ connecting $S^1 \times \{p_1\}$ and $S^1 \times \{p_2\}$.

(2) Form the new link $L \cup \mathcal{K}_2$ in

$$S^1 \times S^2 = S^1 \times D^2 \cup_{S^1 \times S^1} S^1 \times D^2,$$

where $L$ lies in the first copy of $S^1 \times D^2$, and $\mathcal{K}_2$ lies in the second copy.

(3) Define

$$\mathrm{AHI}(L) := \mathrm{I}(S^1 \times S^2, L \cup \mathcal{K}_2, u).$$

The vector space $\mathrm{AHI}(L)$ is equipped with an absolute $\mathbb{Z}$-grading (called the *f-grading*). By definition, the component of $\mathrm{AHI}(L)$ with f-degree $i$ is given by the generalized eigenspace of $\mu^{\mathrm{orb}}(S^2)$ for the eigenvalue $i$, and is denoted by $\mathrm{AHI}(L, i)$. Since $\mu^{\mathrm{orb}}(S^2)$ has degree 2 with respect to the $\mathbb{Z}/4$-homological grading of $\mathrm{AHI}(L)$, the subspace $\mathrm{AHI}(L, i)$ carries a $\mathbb{Z}/2$-homological grading, and we have

$$\mathrm{AHI}(L, i) \cong \mathrm{AHI}(L, -i). \tag{2.1}$$

There is a product formula for split links in $S^1 \times D^2$.

**Proposition 2.1** ([33, Proposition 4.3]). *Suppose $L_1$ and $L_2$ are two links in $S^1 \times D_1$ and $S^1 \times D_2$ respectively, where $D_1$ and $D_2$ are disjoint subdisks of $D^2$. Then*

$$\mathrm{AHI}(L_1 \cup L_2) \cong \mathrm{AHI}(L_1) \otimes \mathrm{AHI}(L_2).$$

*Moreover, the isomorphism above preserves the f-gradings.*

In the following, we will use $\mathcal{U}_n$ to denote the unlink with $n$ components in $S^1 \times D^2$, and use $\mathcal{K}_n$ to denote the closure of the trivial braid with $n$ strands in $S^1 \times D^2$. We will use $\mathcal{U}_k \cup \mathcal{K}_l$ to denote the union of $\mathcal{U}_k$ and $\mathcal{K}_l$ such that $\mathcal{U}_k$ is included in a solid 3-ball disjoint from $\mathcal{K}_l$.

**Example 2.2** ([33, Example 4.2]). The critical set of the unperturbed Chern–Simons functional for $\mathrm{AHI}(\mathcal{U}_1)$ is diffeomorphic to $S^2$, and after perturbation the critical set consists of two points whose homological degrees differ by 2. Therefore there is no differential, and we have

$$\mathrm{AHI}(\mathcal{U}_1) \cong \mathbb{C} \oplus \mathbb{C}.$$

The vector space $\mathrm{AHI}(\mathcal{U}_1)$ is supported in f-grading 0.

The critical set for $\mathrm{AHI}(\mathcal{K}_1)$ consists of two points whose homological degrees differ by 2, so there is no differential and

$$\mathrm{AHI}(\mathcal{K}_1) \cong \mathbb{C} \oplus \mathbb{C}.$$

The vector space $\mathrm{AHI}(\mathcal{K}_1)$ is supported in f-gradings $\pm 1$.

By Proposition 2.1, we have

$$\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l) \cong \mathrm{AHI}(\mathcal{U}_1)^{\otimes k} \otimes \mathrm{AHI}(\mathcal{K}_1)^{\otimes l},$$

and the above isomorphism preserves the f-gradings.

We also have $\mathrm{AHI}(\emptyset) \cong \mathbb{C}$, because the critical set consists of a single point.

**Definition 2.3.** A properly embedded, connected, oriented surface $S \subset S^1 \times D^2$ is called a *meridional surface* if $\partial S$ is a meridian of $S^1 \times D^2$.

The annular instanton Floer homology detects the generalized Thurston norm of meridional surfaces.

**Theorem 2.4** ([34, Theorem 8.2]). *Fix a link $L$ in $S^1 \times D^2$ and suppose $S$ is a meridional surface that intersects $L$ transversely. Let $g$ be the genus of $S$ and let $n := |S \cap L|$. If $S$ minimizes the value of $2g + n$ among meridional surfaces, then*

$$\begin{aligned} &\mathrm{AHI}(L, i) = 0 \quad \textit{for all } |i| > 2g + n, \\ &\mathrm{AHI}(L, \pm(2g + n)) \neq 0. \end{aligned}$$

We also need the following result.

**Proposition 2.5** ([34, Corollary 8.4]). *Let $L$ be a link in $S^1 \times D^2$. Then $L$ is isotopic to the closure of a braid with $n$ strands if and only if the top f-grading of $\mathrm{AHI}(L)$ is $n$ and $\mathrm{AHI}(L, n) \cong \mathbb{C}$.*

Although annular instanton Floer homology is defined for links in the solid torus, it can be used to study links in $S^3$. Let $L$ be a link in $S^3$ and let $p$ be a basepoint on $L$. In [20], Kronheimer and Mrowka defined the link invariant

$$\mathrm{I}^{\natural}(L, p) := \mathrm{I}(S^3, L \cup m, u),$$

where $m$ is a small meridian of $L$ around $p$ and $u$ is an arc joining $m$ and $p$. The following result is a consequence of the excision property of instanton Floer homology.

**Proposition 2.6** ([33, Section 4.3]). *Suppose $L$ has an unknotted component $U$ and let $p \in U$. Let $N(U)$ be a tubular neighborhood of $U$. Then $L_0 := L - U$ is a link in the solid torus $S^3 - N(U)$, and*

$$\mathrm{AHI}(L_0) \cong \mathrm{I}^{\natural}(L, p). \tag{2.2}$$

The above isomorphism does not preserve the f-grading of $\mathrm{AHI}(L_0)$ since there is no such grading on $\mathrm{I}^{\natural}(L, p)$. Notice that a meridional surface in the solid torus $S^3 - N(U)$ is a Seifert surface of $U$.

## 3. Local coefficients

This section reviews the singular instanton Floer homology theory with local coefficients, which was introduced in [21, Section 3.9] (see also [22, Section 3]). Let $\mathcal{B}(Y, L, \omega)$ be the space of gauge-equivalence classes of orbifold connections over $(Y, L, \omega)$. Let $\mathcal{R}$ be the ring

$$\mathcal{R} := \mathbb{C}[t, t^{-1}],$$

and suppose

$$\mu : \mathcal{B}(Y, L, \omega) \to \mathbb{R}/\mathbb{Z}$$

is a continuous function. For each $a \in \mathcal{B}(Y, L, \omega)$, define a rank-1 free $\mathcal{R}$-module by the formal multiplication

$$\Gamma^\mu_a := t^{\tilde{\mu}(a)} \cdot \mathcal{R},$$

where $\tilde{\mu}(a)$ is a lift of $\mu(a)$ in $\mathbb{R}$. Let $\mathrm{Crit}(CS)$ be the set of critical points of the (perturbed) Chern–Simons functional $CS$, and define a free $\mathcal{R}$-module $\mathbf{C}^\mu$ by

$$\mathbf{C}^\mu := \bigoplus_{\alpha \in \mathrm{Crit}(CS)} \Gamma^\mu_\alpha.$$

To make $\mathbf{C}^\mu$ a chain complex, we need to define a differential on it. For $\alpha, \beta \in \mathrm{Crit}(CS)$, let $M_d(\alpha, \beta)$ be the $d$-dimensional moduli space of trajectories of $CS$ from $\alpha$ to $\beta$. This space carries an $\mathbb{R}$-action and we denote the quotient space by

$$\breve{M}_d(\alpha, \beta) := M_d(\alpha, \beta)/\mathbb{R}.$$

A trajectory $z \in M_d(\alpha, \beta)$ determines a path $p_z$ in $\mathcal{B}(Y, L, \omega)$ from $\alpha$ to $\beta$. The map

$$\mu \circ p_z : [0, 1] \to \mathbb{R}/\mathbb{Z}$$

can be lifted to a map

$$\widetilde{\mu \circ p_z} : [0, 1] \to \mathbb{R}.$$

Although $\widetilde{\mu \circ p_z}$ is not unique, the difference

$$\nu(z) := \widetilde{\mu \circ p_z}(1) - \widetilde{\mu \circ p_z}(0)$$

is well-defined. We define an $\mathcal{R}$-module homomorphism by

$$d^\alpha_\beta : \Gamma^\mu_\alpha \to \Gamma^\mu_\beta, \quad t^s \mapsto \sum_{[z] \in \breve{M}_1(\alpha, \beta)} \mathrm{sign(z)} \cdot t^{s+\nu(z)}.$$

The differential $D$ on $\mathbf{C}^\mu$ is then given by

$$D := \bigoplus_{\alpha, \beta \in \mathrm{Crit}(CS)} d^\alpha_\beta,$$

and the instanton Floer homology with local coefficients is defined by

$$\mathrm{I}(Y, L, \omega; \Gamma^\mu) := H^*(\mathbf{C}^\mu, D). \tag{3.1}$$

If $F \subset Y$ is an embedded closed surface, the operator $\mu^{\mathrm{orb}}(F)$ can be defined in the setting with local coefficients. Roughly speaking, the surface $F$ defines a two-dimensional cohomology class on $\mathcal{B}(Y, L, \omega)$ (see [34, formula (10)]), and its Poincaré dual is given by a linear combination of divisors on $\mathcal{B}(Y, L, \omega)$ as

$$\sum a_i V_i, \quad a_i \in \mathbb{Q}.$$

There is a map from $M_d(\alpha, \beta)$ to $\mathcal{B}(Y, L, \omega)$ by restricting the trajectories at time 0. The divisors $V_i$ can be chosen to be generic in the sense that they are transverse to the restriction map $M_d(\alpha, \beta) \to \mathcal{B}(Y, L, \omega)$ for all $\alpha, \beta \in \mathrm{Crit}(CS)$ and $d \in \mathbb{N}$. We define an $\mathcal{R}$-module homomorphism by

$$f^\alpha_\beta : \Gamma^\mu_\alpha \to \Gamma^\mu_\beta, \quad t^s \mapsto \sum_i a_i \sum_{z \in M_2(\alpha,\beta) \cap V_i} \mathrm{sign}(z) \cdot t^{s+\nu(z)}.$$

A standard argument shows that the map

$$\bigoplus_{\alpha,\beta \in \mathrm{Crit}(CS)} f^\alpha_\beta : \mathbf{C}^\mu \to \mathbf{C}^\mu \tag{3.2}$$

is a chain map. The map (3.2) induces the operator $\mu^{\mathrm{orb}}(F)$ on $\mathrm{I}(Y, L, \omega; \Gamma^\mu)$. When it does not cause confusion, we will also use $\mu^{\mathrm{orb}}(F)$ to denote the chain map (3.2) by abuse of notation.

The tensor products

$$\big(\mathbf{C}^\mu \otimes_{\mathcal{R}} \mathcal{R}/(t-1), D \otimes_{\mathcal{R}} \mathcal{R}/(t-1)\big), \quad \mu^{\mathrm{orb}}(F) \otimes_{\mathcal{R}} \mathcal{R}/(t-1)$$

recover the ordinary Floer chain complex $(C, d)$ and the chain map defining the ordinary operator $\mu^{\mathrm{orb}}(F)$ on $\mathrm{I}(Y, L, \omega)$ with $\mathbb{C}$-coefficients.

Suppose there is a component $K \subset L$ such that $K \cap \omega = \emptyset$. Fix an orientation and a framing of $K$. We can define a continuous map

$$\mu_K : \mathcal{B}(Y, L, \omega) \to U(1) = \mathbb{R}/\mathbb{Z}$$

by taking the limit holonomy of the orbifold connections along the longitude of $K$. The map $\mu_K$ then gives a local system. The local systems defined by different framings of $K$ are isomorphic via multiplications by powers of $t$, therefore the choice of the framing is not important. More generally, suppose there is a sublink $L' = K_1 \cup \cdots \cup K_l$ of $L$ such that $\omega \cap L' = \emptyset$. We can choose a framing for each $K_j$ and define the map $\mu_{K_j}$ as above, and hence we obtain a local system $\Gamma$ associated with $L'$ defined by

$$\mu_{L'} := \mu_{K_1} \cdots \mu_{K_l}.$$

If $L'$ is the empty link, then $\mu_{L'} = 1$, thus the local system $\Gamma$ is the trivial system with $\mathcal{R}$-coefficients. In this case, we have

$$\mathrm{I}(Y, L, \omega; \Gamma) = \mathrm{I}(Y, L, \omega) \otimes_{\mathbb{C}} \mathcal{R}. \tag{3.3}$$

Suppose $(Y_0, L_0, \omega_0)$ and $(Y_1, L_1, \omega_1)$ are two admissible triples with local systems $\Gamma_0$ and $\Gamma_1$ associated with oriented sublinks $L_0' \subset L_0$ and $L_1' \subset L_1$ respectively. Let

$$(W, S, \eta) = (W, S' \sqcup S'', \eta) : (Y_0, L_0, \omega_0) \to (Y_1, L_1, \omega_1)$$

be a cobordism such that $\partial S' = L_0' \cup L_1'$ and $\eta \cap S' = \emptyset$. Then $(W, S, \eta)$ induces a map

$$\mathrm{I}(W, S, \eta) : \mathrm{I}(Y_0, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y_1, L_1, \omega_1; \Gamma_1).$$

This makes the instanton Floer homology with local coefficients a functor. By the definition of cobordism of triples, $S$ and $\eta$ are required to be embedded surfaces in $W$. We can also consider the situation where $S$ is an immersed surface with transverse double points, as discussed in [21, Section 5] and [18]. In this situation, one can blow up $W$ at the self-intersection points of $S$ to resolve the double points and obtain an ordinary cobordism $(\widetilde{W}, \widetilde{S}, \eta)$, and then define

$$\mathrm{I}(W, S, \eta) := \mathrm{I}(\widetilde{W}, \widetilde{S}, \eta).$$

Now suppose $S = S' \sqcup S''$ and $\hat{S} = \hat{S}' \sqcup \hat{S}''$ are two immersed surfaces with transverse double points in $W$ such that $\eta \cap S' = \hat{\eta} \cap \hat{S}' = \emptyset$, $\eta = \hat{\eta}$, $\partial S = \partial \hat{S} = L_0 \cup L_1$, and $\partial S' = \partial \hat{S}' = L_0' \cup L_1'$. We consider the following five situations:

(i) $\hat{S}$ is obtained from $S$ by an ambient isotopy;

(ii) $\hat{S}'' = S''$, and $\hat{S}'$ is obtained from $S'$ by a twist move introducing a positive double point (see [10, Section 1.3] for the definition of twist move);

(iii) $\hat{S}'' = S''$, and $\hat{S}'$ is obtained from $S'$ by a twist move introducing a negative double point;

(iv) $\hat{S}'' = S''$, and $\hat{S}'$ is obtained from $S'$ by a finger move introducing two double points of opposite signs (see Figure 5 for a schematic picture and see [10, Section 1.5] for the precise definition of finger move);

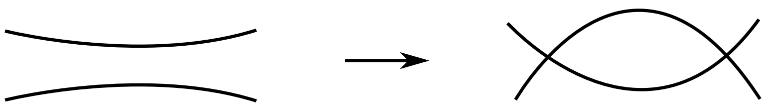

**Fig. 5.** A schematic picture of the finger move.

(v) $\hat{S}$ is obtained from $S$ by a finger move introducing two double points of opposite signs in $\hat{S}' \cap \hat{S}''$.

All the moves and isotopies above are assumed to be supported outside a neighborhood of $\eta = \hat{\eta}$.

**Proposition 3.1.** *Let* $(Y_0, L_0, \omega_0)$, $(Y_1, L_1, \omega_1)$, $W, S, \hat{S}, \eta, \Gamma_0, \Gamma_1$ *be as above, and let*

$$\mathrm{I}(W, S, \eta) : \mathrm{I}(Y_0, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y_1, L_1, \omega_1; \Gamma_1),$$
$$\mathrm{I}(W, S', \eta) : \mathrm{I}(Y_0, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y_1, L_1, \omega_1; \Gamma_1)$$

*be the induced cobordism maps. For the five cases listed above, the following equations hold respectively:*

(i) $\mathrm{I}(W, \hat{S}, \eta) = \mathrm{I}(W, S, \eta)$*;*

(ii) $\mathrm{I}(W, \hat{S}, \eta) = (1 - t^2)\,\mathrm{I}(W, S, \eta)$*;*

(iii) $\mathrm{I}(W, \hat{S}, \eta) = \mathrm{I}(W, S, \eta)$*;*

(iv) $\mathrm{I}(W, \hat{S}, \eta) = (1 - t^2)\,\mathrm{I}(W, S, \eta)$*;*

(v) $\mathrm{I}(W, \hat{S}, \eta) = \theta(t)\,\mathrm{I}(W, S, \eta)$ *for a universal non-zero polynomial* $\theta(t) \in \mathcal{R}$*.*

*Proof.* Part (i) is trivial. (ii)–(iv) are from [22, Proposition 3.1]. The proof of (v) is similar to that of (iv). We first review Kronheimer and Mrowka's proof of (iv) briefly. For simplicity, consider a special case that $(W, S, \eta)$ is closed, thus it can be viewed as a cobordism from the empty set to the empty set. We also assume $S'' = \hat{S}'' = \emptyset$. In this case,

$$\mathrm{I}(W, S, \eta) \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{R}) = \mathcal{R}$$

is the singular Donaldson invariant (from 0-dimensional moduli spaces) introduced by Kronheimer and Mrowka [19]. More precisely, we have

$$\mathrm{I}(W, S, \eta) = \sum_{k,l} q_{k,l}(W, S, \eta) t^{-l}, \tag{3.4}$$

where $q_{k,l}$ denotes the singular Donaldson invariant defined by counting the number of points in the 0-dimensional moduli spaces over all orbifold bundles with instanton number $k$ and monopole number $l$. The restriction of an orbifold SO(3)-bundle to $S$ has a reduction to $K \oplus \underline{\mathbb{R}}$ where $K$ is an SO(2)-bundle. By definition, the monopole number $l$ is given by

$$l = -\tfrac{1}{2} e(K)[S].$$

If $(W, \hat{S})$ is obtained from $(W, S)$ by a finger move, [18, Proposition 3.1] uses a gluing argument to prove that

$$q_{k,l}(W, \hat{S}, \eta) = q_{k,l}(W, S, \eta) - q_{k-1,l+2}(W, S, \eta). \tag{3.5}$$

When $(W, S, \eta)$ is closed and $S'' = \hat{S}'' = \emptyset$, part (iv) follows immediately from (3.4) and (3.5). Since the gluing argument only depends on the local structure of the finger move, it is straightforward to extend the argument to the general (relative) case.

To prove (v), we first assume $(W, S, \eta)$ is closed and $S'' \neq \emptyset$. We give a refined definition of the monopole number $l$ by taking

$$l_0 := -\tfrac{1}{2} e(K)[S'], \quad l_1 := -\tfrac{1}{2} e(K)[S''].$$

It is clear from the definition that $l = l_0 + l_1$. With this definition, we have a refined singular Donaldson invariant $q_{k,l_0,l_1}$. Similar to (3.4), we define the polynomial

$$Q(W, S, \eta)(t_0, t_1) := \sum_{k,l_0,l_1} q_{k,l_0,l_1}(W, S, \eta) t_0^{-l_0} t_1^{-l_1}.$$

If $\hat{S}$ is obtained by a finger move that introduces intersection points between $S'$ and $S''$, then the proof of [18, equation (23)] shows that there exist universal constants $a_{i,j} \in \mathbb{Z}$ such that

$$q_{k,l_0,l_1}(W,\hat{S},\eta) = \sum_{2|i+j} a_{i,j} q_{k-\frac{i+j}{2},l_0+i,l_1+j}(W,S,\eta).$$

By the Uhlenbeck compactness theorem, only finitely many $a_{i,j}$'s are non-zero. This implies that

$$Q(W,\hat{S},\eta)(t_0,t_1) = P(t_0,t_1)Q(W,S,\eta)(t_0,t_1) \tag{3.6}$$

for a universal polynomial $P(t_0,t_1) \in \mathbb{C}[t_0,t_0^{-1},t_1,t_1^{-1}]$. Notice that

$$q_{k,l} = \sum_{l_0+l_1=l} q_{k,l_0,l_1},$$

therefore (3.5) implies

$$P(t,t) = 1-t^2. \tag{3.7}$$

We also have $P(t_0,t_1) = P(t_1,t_0)$ because there is no difference between the roles of $S'$ and $S''$ in the finger move.

We claim that

$$P(t,1) = P(1,t) \neq 0. \tag{3.8}$$

In fact, suppose the contrary; then

$$(t_0-1) \mid P(t_0,t_1), \quad (t_1-1) \mid P(t_0,t_1),$$

thus we have

$$(t_0-1)(t_1-1) \mid P(t_0,t_1), \quad \text{therefore} \quad (t-1)(t-1) \mid P(t,t),$$

which contradicts (3.7), hence the claim is proved.

Now, let $\theta(t) := P(t,1)$. We have

$$\mathrm{I}(W,S,\eta)(t) = Q(W,S,\eta)(t,1),$$

therefore in the closed case, part (v) of the proposition follows from (3.6) and (3.8). By the gluing argument, the same result holds for the non-closed case. ■

Suppose $(Y,L_0,\omega)$ is an admissible triple, and let $L_0'$ be a sublink of $L_0$ such that $L_0' \cap \omega = \emptyset$. Fix an orientation of $L_0'$. By the previous discussion, $L_0'$ defines a local system $\Gamma_0$ with $\mathcal{R}$-coefficients. Suppose $L_1$ is obtained from $L_0$ by a local crossing change in $Y-\omega$, where the crossing is either within $L_0'$ or between $L_0'$ and $L_0 - L_0'$, and let $\Gamma_1$ be the local system of $(Y,L_1,\omega)$ associated with the image of $L_0'$ after the crossing change.

The crossing change induces an immersed cobordism $S: L_0 \to L_1$, where $S$ is an immersed surface in $[0,1]\times Y$ with one double point. Reversing $S$, we obtain an immersed cobordism $\overline{S}: L_1 \to L_0$ with one double point. The composition $S \cup \overline{S} \subset$

$[0,2] \times Y$ can be obtained from the product cobordism $[0,2] \times L_0$ by a finger move decribed by case (iv) or case (v) of Proposition 3.1. Therefore by Proposition 3.1, the map

$$\mathrm{I}([0,2] \times Y, S \cup \bar{S}, [0,2] \times \omega) : \mathrm{I}(Y, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y, L_0, \omega_0; \Gamma_0) \tag{3.9}$$

is equal to $(1-t^2)\,\mathrm{id}$ or $\theta(t)\,\mathrm{id}$. Similarly, the map

$$\mathrm{I}([0,2] \times Y, \bar{S} \cup S, [0,2] \times \omega) : \mathrm{I}(Y, L_1, \omega; \Gamma_1) \to \mathrm{I}(Y, L_1, \omega; \Gamma_1) \tag{3.10}$$

is equal to $(1-t^2)\,\mathrm{id}$ or $\theta(t)\,\mathrm{id}$. As a consequence, we have the following result.

**Proposition 3.2.** *Suppose $(Y, L_0, \omega)$ is an admissible triple and $L_0' \subset L_0$ is a sublink with $L_0 \cap \omega = \emptyset$. Fix an orientation on $L_0'$ and let $\Gamma_0$ be the local system of $(Y, L_0, \omega)$ defined by $L_0'$. Suppose $L_1'$ is an oriented link that is homotopic to $L_0'$ in $Y - \omega$ and is disjoint from $L_0 - L_0'$. Let $L_1 := L_1' \cup (L_0 - L_0')$. Let $\Gamma_1$ be the local system of $(Y, L_1, \omega)$ defined by $L_1'$. Then*

$$T^{-1}\,\mathrm{I}(Y, L_0, \omega; \Gamma_0) \cong T^{-1}\,\mathrm{I}(Y, L_1, \omega; \Gamma_1),$$

*where $T$ is the multiplicative system generated by $(1-t^2)\theta(t)$.*

*Proof.* Since $L_0'$ is homotopic to $L_1'$ in $Y - \omega$, the link $L_1$ can be obtained from $L_0$ by a finite sequence of crossing changes in $Y - \omega$ such that the sublink $L_0 - L_0'$ remains fixed. Without loss of generality, we may assume that $L_1$ is obtained from $L_0$ by one such crossing change. Let $S \subset Y \times [0,1]$ be the immersed cobordism from $L_0$ to $L_1$ given by the crossing change, and let $\bar{S}$ be the reverse of $S$. By (3.9), (3.10), and the functoriality of the instanton Floer homology with local coefficients, we have

$$\mathrm{I}(Y \times [0,1], \bar{S}, \omega \times [0,1]) \circ \mathrm{I}(Y \times [0,1], S, \omega \times [0,1]) = (1-t^2)\,\mathrm{id} \text{ or } \theta(t)\,\mathrm{id}$$

on $\mathrm{I}(Y, L_0, \omega; \Gamma_0)$, and

$$\mathrm{I}(Y \times [0,1], S, \omega \times [0,1]) \circ \mathrm{I}(Y \times [0,1], \bar{S}, \omega \times [0,1]) = (1-t^2)\,\mathrm{id} \text{ or } \theta(t)\,\mathrm{id}$$

on $\mathrm{I}(Y, L_1, \omega; \Gamma_1)$. Hence $T^{-1}\,\mathrm{I}(Y, L_0, \omega; \Gamma_0)$ and $T^{-1}\,\mathrm{I}(Y, L_1, \omega; \Gamma_1)$ are isomorphic. ■

**Corollary 3.3.** *Let $Y, \omega, L_0, L_1, \Gamma_0, \Gamma_1$ be as in Proposition 3.2. Then*

$$\mathrm{rank}_{\mathcal{R}}\,\mathrm{I}(Y, L_0, \omega; \Gamma_0) = \mathrm{rank}_{\mathcal{R}}\,\mathrm{I}(Y, L_1, \omega; \Gamma_1). \quad ■$$

Given an oriented link $L$ in $S^1 \times D^2$, we define the annular instanton Floer homology with local coefficients by

$$\mathrm{AHI}(L; \Gamma) := \mathrm{I}(S^1 \times S^2, L \cup \mathcal{K}_2, u; \Gamma),$$

where $\Gamma$ is the local system associated with $L$. The operator $\mu^{\mathrm{orb}}(S^2)$ on $\mathrm{AHI}(L;\Gamma)$ is now an $\mathcal{R}$-module homomorphism instead of a $\mathbb{C}$-linear map, therefore $\mathrm{AHI}(L;\Gamma)$ no longer carries the f-grading. The torus excision theorem [20, Theorem 5.6] still holds for instanton Floer homology with local coefficients, as long as the excision surface is disjoint from the sublink defining the local system. Therefore, Proposition 2.1 still holds for the annular instanton Floer homology with local coefficients, except that there is no f-grading anymore.

**Example 3.4.** By Example 2.2, the critical points of the (perturbed) Chern–Simons functional for $\mathrm{AHI}(\mathcal{U}_1)$ (or $\mathrm{AHI}(\mathcal{K}_1)$) consist of two points whose homological degrees differ by 2. Therefore there are no differentials in the Floer chain complex, and we have

$$\mathrm{AHI}(\mathcal{U}_1;\Gamma) \cong \mathcal{R} \oplus \mathcal{R}, \quad \mathrm{AHI}(\mathcal{K}_1;\Gamma) \cong \mathcal{R} \oplus \mathcal{R}.$$

By Proposition 2.1,

$$\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l;\Gamma) \cong \mathcal{R}^{2^{k+l}}.$$

Proposition 2.6 follows from the torus excision theorem, therefore it also works in the case with local coefficients. Let $L \subset S^1 \times D^2$ be a link with $n$ components and view $S^1 \times D^2$ as the complement of a neighborhood of the unknot $U$ in $S^3$, let $p \in U$ and let $\Gamma_L$ be the local system associated with $L$. Then

$$\mathrm{AHI}(L;\Gamma) \cong \mathrm{I}^\natural(L \cup U, p; \Gamma_L).$$

Suppose the annular link $L$ has $n$ components. Then the embedded image of $L$ in $S^3$ is homotopic to the embedded image of $\mathcal{U}_n$ in $S^3$. By Corollary 3.3, we have

$$\operatorname{rank}_{\mathcal{R}} \mathrm{I}^\natural(L \cup U, p; \Gamma_L) = \operatorname{rank}_{\mathcal{R}} \mathrm{I}^\natural(\mathcal{U}_n \cup U, p; \Gamma_{\mathcal{U}_n}).$$

By Proposition 2.6 again,

$$\mathrm{I}^\natural(\mathcal{U}_n \cup U, p; \Gamma) \cong \mathrm{AHI}(\mathcal{U}_n;\Gamma) \cong \mathcal{R}^{2^n}.$$

In conclusion, we obtain

$$\operatorname{rank}_{\mathcal{R}} \mathrm{AHI}(L;\Gamma) = 2^n. \tag{3.11}$$

By the universal coefficient theorem,

$$\operatorname{rank}_{\mathbb{C}} \mathrm{I}^\natural(L \cup U, p) = \operatorname{rank}_{\mathbb{C}} \mathrm{AHI}(L) \geq \operatorname{rank}_{\mathcal{R}} \mathrm{AHI}(L;\Gamma) = 2^n. \tag{3.12}$$

## 4. Limits of chain complexes

This section discusses a simple observation from linear algebra and its consequences in instanton Floer homology.

Suppose $\{C_n\}_{n\in\mathbb{Z}}$ is a sequence of finite-dimensional complex vector spaces. For each $n \in \mathbb{Z}$ and $k \geq 0$, and let $\partial_n^{(k)}$ be a $\mathbb{C}$-linear map from $C_n$ to $C_{n-1}$, let $f_n^{(k)}$ be an endomorphism of $C_n$. Suppose that for each pair $(n,k)$ we have $\partial_n^{(k)} \circ \partial_{n+1}^{(k)} = 0$ and $\partial_n^{(k)} \circ f_n^{(k)} = f_{n-1}^{(k)} \circ \partial_n^{(k)}$. Moreover, suppose that for each $n$ the limits

$$\lim_{k\to\infty} f_n^{(k)} \quad \text{and} \quad \lim_{k\to\infty} \partial_n^{(k)}$$

exist. Let

$$\partial_n := \lim_{k\to\infty} \partial_n^{(k)}, \quad f_n := \lim_{k\to\infty} f_n^{(k)},$$
$$H_n^{(k)} := \ker \partial_n^{(k)}/\operatorname{Im} \partial_{n+1}^{(k)}, \quad H_n := \ker \partial_n/\operatorname{Im} \partial_{n+1}.$$

The maps $f_n^{(k)}$ and $f_n$ induce maps on $H_n^{(k)}$ and $H_n$ respectively. For $\Lambda \subset \mathbb{C}$, define $E_{n,\Lambda}^{(k)} \subset H_n^{(k)}$ to be the direct sum of the generalized eigenspaces of $f_n^{(k)}$ with eigenvalues in $\Lambda$. Similarly, define $E_{n,\Lambda} \subset H_n$ to be the direct sum of the generalized eigenspaces of $f_n$ with eigenvalues in $\Lambda$.

**Lemma 4.1.** *Let $C_n$, $\partial_n^{(k)}$, $\partial_n$, $f_n^{(k)}$, $f_n$, $H_n^{(k)}$, $H_n$, $E_{n,\Lambda}^{(k)}$, and $E_{n,\Lambda}$ be as above.*

(1) *If $\Lambda \subset \mathbb{C}$ is a closed subset, then*

$$\dim E_{n,\Lambda} \geq \limsup_{k\to\infty} \dim E_{n,\Lambda}^{(k)}.$$

(2) *For $\varepsilon > 0$, let $N(\Lambda, \varepsilon)$ be the closed $\varepsilon$-neighborhood of $\Lambda$. If $\Lambda$ is closed, then*

$$\dim E_{n,\Lambda} \geq \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \dim E_{n,N(\Lambda,\varepsilon)}^{(k)}.$$

(3) *If we further assume that* $\dim H_n^{(k)} = \dim H_n$ *for all $k$, then*

$$\dim E_{n,\Lambda} = \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \dim E_{n,N(\Lambda,\varepsilon)}^{(k)}.$$

*Proof.* (1) Let $Z_n := \ker \partial_n^{(k)}$, $B_n^{(k)} := \operatorname{Im} \partial_{n+1}^{(k)}$, $Z_n := \ker \partial_n$, $B_n := \operatorname{Im} \partial_{n+1}$. After taking a subsequence, we may assume that the dimensions of $Z_n^{(k)}$ and $B_n^{(k)}$ are independent of $k$, and that they are convergent in the corresponding Grassmannians as $k \to \infty$. The spectrum of $f_n$ (with multiplicities) on

$$\Big(\lim_{k\to\infty} Z_n^{(k)}\Big)/\Big(\lim_{k\to\infty} B_n^{(k)}\Big) \tag{4.1}$$

is the limit of the spectra of $f_n^{(k)}$ on $Z_n^{(k)}/B_n^{(k)}$ as $k \to \infty$. Let

$$E'_{n,\Lambda} \subset \Big(\lim_{k\to\infty} Z_n^{(k)}\Big)/\Big(\lim_{k\to\infty} B_n^{(k)}\Big)$$

be the direct sum of the generalized eigenspaces of $f_n$ for the eigenvalues in $\Lambda$. Since $\Lambda$ is closed, the previous argument implies

$$\dim E'_{n,\Lambda} \geq \limsup_{k\to\infty} \dim E_{n,\Lambda}^{(k)}.$$

On the other hand, we have

$$Z_n \supset \lim_{k\to\infty} Z_n^{(k)}, \quad B_n \subset \lim_{k\to\infty} B_n^{(k)},$$

therefore (4.1) is a subquotient of $H_n = Z_n/B_n$, and hence

$$\dim E_{n,\Lambda} \geq \dim E'_{n,\Lambda} \geq \limsup_{k\to\infty} \dim E^{(k)}_{n,\Lambda},$$

which completes the proof.

(2) Let $\varepsilon_0$ be sufficiently small such that $E_{n,N(\Lambda,\varepsilon_0)} = E_{n,\Lambda}$. By (1), we have

$$\begin{aligned}\dim E_{n,\Lambda} &= \dim E_{n,N(\Lambda,\varepsilon_0)} \\ &\geq \limsup_{k\to\infty} \dim E^{(k)}_{n,N(\Lambda,\varepsilon_0)} \geq \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}.\end{aligned}$$

(3) Let $\varepsilon_0$ be sufficiently small such that $E_{n,N(\Lambda,\varepsilon_0)} = E_{n,\Lambda}$. Suppose $\varepsilon < \varepsilon_0$. Then $E_{n,N(\Lambda,\varepsilon)} = E_{n,\Lambda}$. Let $\Lambda_1 := \overline{\mathbb{C} - N(\Lambda,\varepsilon)}$. By the condition on $\varepsilon$,

$$E_{n,\partial N(\Lambda,\varepsilon)} = E_{n,\partial\Lambda_1} = \{0\}.$$

Hence by (1), for $k$ sufficiently large we have

$$\dim E^{(k)}_{n,\partial N(\Lambda,\varepsilon)} = \dim E^{(k)}_{n,\partial\Lambda_1} = 0.$$

Applying (1) again on $N(\Lambda,\varepsilon)$ and $\Lambda_1$, we deduce that if $k$ is sufficiently large, then $\dim E_{n,N(\Lambda,\varepsilon)} \geq \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}$ and $\dim E_{n,\Lambda_1} \geq \dim E^{(k)}_{n,\Lambda_1}$. Therefore

$$\begin{aligned}\dim H_n &= \dim E_{n,N(\Lambda,\varepsilon)} + \dim E_{n,\Lambda_1} \\ &\geq \dim E^{(k)}_{n,N(\Lambda,\varepsilon)} + \dim E^{(k)}_{n,\Lambda_1} \\ &= \dim H^{(k)}_n = \dim H_n.\end{aligned}$$

As a consequence, for $k$ sufficiently large, $\dim E_{n,N(\Lambda,\varepsilon)} = \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}$, and hence

$$\dim E_{n,\Lambda} = \dim E_{n,N(\Lambda,\varepsilon)} = \limsup_{k\to\infty} \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}.$$

Since the above equation holds for all $\varepsilon < \varepsilon_0$, part (3) of the lemma is proved. ■

Recall that given an admissible triple $(Y, L, \omega)$ and a continuous function $\mu : \mathcal{B}(Y, L, \omega) \to \mathbb{R}/\mathbb{Z}$, there is a local system $\Gamma^\mu$ on $\mathcal{B}(Y, L, \omega)$ defined by $\mu$. The Floer chain complex $\mathbf{C}^\mu$ is a finitely generated free $\mathcal{R}$-module, where $\mathcal{R} = \mathbb{C}[t, t^{-1}]$. The differential $D$ is an $\mathcal{R}$-endomorphism of $\mathbf{C}^\mu$. For $h \in \mathbb{C} - \{0\}$, define

$$(C_h, d_h) := (\mathbf{C}^\mu \otimes_{\mathcal{R}} \mathcal{R}/(t-h), D \otimes_{\mathcal{R}} \mathrm{id}_{\mathcal{R}/(t-h)}).$$

Notice that $\mathcal{R}/(t-h) \cong \mathbb{C}$ via the map $t \mapsto h$, so $(C_h, d_h)$ is a finite-dimensional chain complex over $\mathbb{C}$. Let $C := \mathbb{C}^{\mathrm{rank}_{\mathcal{R}} \mathbf{C}^\mu}$. We identify $C_h$ with $C$ using the isomorphism $\mathcal{R}/(t-h) \cong \mathbb{C}$. The differentials $d_h$ become a continuous family of linear maps on $C$. Given an embedded surface $F \subset Y$, define

$$\mu^{\mathrm{orb}}(F)_h := \mu^{\mathrm{orb}}(F) \otimes_{\mathcal{R}} \mathcal{R}/(t-h).$$

Then $\mu^{\text{orb}}(F)_h$ is continuous with respect to $h$ and is a chain map on $(C, d_h)$. Therefore, for each $h \in \mathbb{C} - \{0\}$, the map $\mu^{\text{orb}}(F)_h$ induces a map on the Floer homology

$$\mathrm{I}(Y, L, \omega; \Gamma^\mu \otimes_{\mathcal{R}} \mathcal{R}/(t-h)) = H^*(C, d_h).$$

To simplify notations, we will use $\Gamma^\mu(h)$ to denote $\Gamma^\mu \otimes_{\mathcal{R}} \mathcal{R}/(t-h)$ for the rest of this article. If $h = 1$, then $\mathrm{I}(Y, L, \omega; \Gamma^\mu(1))$ is the ordinary instanton Floer homology without local coefficients, and $\mu^{\text{orb}}(F)_1$ coincides with the ordinary $\mu$ map.

**Proposition 4.2.** *Let $Y, \omega, L_0, L_1, L_0', L_1', \Gamma_0, \Gamma_1$ be as in Proposition 3.2. Let $\theta(t)$ be the polynomial given by Proposition 3.1* (v). *Suppose $h \in \mathbb{C} - \{0\}$ satisfies*

$$(1-h^2)\theta(h) \neq 0. \tag{4.2}$$

*Then*

$$\mathrm{I}(Y, L_0, \omega; \Gamma_0(h)) \cong \mathrm{I}(Y, L_1, \omega; \Gamma_1(h)). \tag{4.3}$$

*Moreover, if $F \subset Y$ is a closed embedded surface in $Y$, then the isomorphism* (4.3) *intertwines with $\mu^{\text{orb}}(F)_h$.*

*Proof.* Let $T \subset \mathcal{R}$ be the multiplicative system generated by $(1-t^2)\theta(t)$ as in Proposition 3.2. By (4.2), the elements of $T$ have non-zero images in $\mathcal{R}/(t-h) \cong \mathbb{C}$, hence $\mathcal{R}/(t-h)$ is isomorphic to $(T^{-1}\mathcal{R})/(t-h)$. Therefore, for $i = 0, 1$, we have

$$\mathrm{I}(Y, L_i, \omega; \Gamma_i(h)) \cong \mathrm{I}\big(Y, L_i, \omega; \Gamma_i \otimes_{\mathcal{R}} T^{-1}\mathcal{R} \otimes_{T^{-1}\mathcal{R}} T^{-1}\mathcal{R}/(t-h)\big). \tag{4.4}$$

On the other hand, since localization is an exact functor, we have

$$\mathrm{I}(Y, L_i, \omega; \Gamma_i \otimes_{\mathcal{R}} T^{-1}\mathcal{R}) \cong T^{-1}\,\mathrm{I}(Y, L_i, \omega; \Gamma_i).$$

Therefore by Proposition 3.2,

$$\mathrm{I}(Y, L_0, \omega; \Gamma_0 \otimes_{\mathcal{R}} T^{-1}\mathcal{R}) \cong \mathrm{I}(Y, L_1, \omega; \Gamma_1 \otimes_{\mathcal{R}} T^{-1}\mathcal{R}). \tag{4.5}$$

Since $\mathcal{R}$ is a principal ideal domain, the localization $T^{-1}\mathcal{R}$ is also a principal ideal domain, hence (4.3) follows from the universal coefficient theorem and the isomorphisms (4.4) and (4.5).

It remains to prove that (4.3) intertwines with $\mu^{\text{orb}}(F)_h$. Since the isomorphism (4.5) is induced by a cobordism in which the two copies of the surface $F$ on the two ends are homologous, it intertwines the $\mu^{\text{orb}}(F)_h$ on the in-coming end with the $\mu^{\text{orb}}(F)_h$ on the out-going end, hence the statement is proved. ∎

Lemma 4.1 and Proposition 4.2 have the following application.

**Proposition 4.3.** *Suppose $L \subset S^1 \times D^2$ is an oriented link such that every component of $L$ has winding number* 0 *or* $\pm 1$. *Assume there are $k$ components with winding number* 0 *and $l$ components with winding number* $\pm 1$. *Then*

$$\dim_{\mathbb{C}} \mathrm{AHI}(L, i) \geq \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i) \quad \textit{for all } i \in \mathbb{Z}.$$

*Proof.* For $\lambda \in \mathbb{C}$, let $N(\lambda, \varepsilon)$ be the closed $\varepsilon$-neighborhood of $\lambda$ in $\mathbb{C}$. Given a vector space $V$ over $\mathbb{C}$, a linear map $f : V \to V$, and a subset $\Lambda \subset \mathbb{C}$, we use $E(V, f, \Lambda)$ to denote the direct sum of the generalized eigenspaces of $f$ with eigenvalues in $\Lambda$.

Recall that $\mathrm{AHI}(L; \Gamma)$ is defined to be the instanton Floer homology

$$\mathrm{I}(S^1 \times S^2, L \cup \mathcal{K}_2, u; \Gamma),$$

where $\Gamma$ is the local coefficient system associated with $L$. For $h \in \mathbb{C} - \{0\}$, recall that $\Gamma(h)$ is the local system over $\mathbb{C}$ given by $\Gamma \otimes_{\mathcal{R}} \mathcal{R}/(t-h)$. For every $i \in \mathbb{Z}$, Lemma 4.1 (2) and Proposition 4.2 give

$$\begin{aligned} \dim \mathrm{AHI}(L, i) &\geq \lim_{\varepsilon \to 0} \limsup_{h \to 1} E\big(\mathrm{AHI}(L; \Gamma(h)), \mu^{\mathrm{orb}}(S^2)_h, N(i, \varepsilon)\big) \\ &= \lim_{\varepsilon \to 0} \limsup_{h \to 1} E\big(\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma(h)), \mu^{\mathrm{orb}}(S^2)_h, N(i, \varepsilon)\big). \end{aligned}$$

According to Example 3.4, $\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma)$ is a free $\mathcal{R}$-module of rank $2^{k+l}$. By the universal coefficient theorem, $\dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma(h)) = 2^{k+l}$ for all $h \in \mathbb{C} - \{0\}$. Therefore Lemma 4.1 (3) gives

$$\lim_{\varepsilon \to 0} \limsup_{h \to 1} E\big(\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma(h)), \mu^{\mathrm{orb}}(F)_h, N(i, \varepsilon)\big) = \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i),$$

and the proposition is proved. ∎

**Corollary 4.4.** *Suppose $L \subset S^1 \times D^2$ is an oriented link such that every component of $L$ has winding number $0$ or $\pm 1$. Assume there are $k$ components with winding number $0$ and $l$ components with winding number $\pm 1$. Moreover, assume*

$$\dim_{\mathbb{C}} \mathrm{AHI}(L) = 2^{k+l}. \tag{4.6}$$

*Then there exists a meridional disk $S$ in $S^1 \times D^2$ such that $S$ intersects every component of $L$ with winding number $\pm 1$ transversely in one point, and $S$ is disjoint from every component of $L$ with winding number $0$.*

*Proof.* By Example 3.4, $\dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l) = 2^{k+l}$, therefore (4.6) and Proposition 4.3 imply

$$\dim_{\mathbb{C}} \mathrm{AHI}(L, i) = \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i)$$

for all $i \in \mathbb{Z}$. The top f-grading of $\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i)$ is $l$. By Theorem 2.4, there exists a meridional surface $S$ with genus $g$ such that $S$ intersects $L$ transversely in $n$ points, and $2g + n = l$. On the other hand, every component with a non-zero winding number must intersect $S$, therefore we have $g = 0$ and $n = l$, and the surface $S$ is the desired meridional disk. ∎

## 5. Linking numbers and forests of unknots

This section proves a weaker version of Theorem 1.2:

**Theorem 5.1.** *Suppose $L = K_1 \cup \cdots \cup K_n$ is an oriented link with $n$ components in $S^3$ such that*

(1) $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n$*;*

(2) *there exists a forest of unknots $L_G = K_1' \cup \cdots \cup K_n'$ such that*

$$\operatorname{lk}(K_i, K_j) = \operatorname{lk}(K_i', K_j') \quad \text{for all } i \neq j.$$

*Then $L$ is isotopic to $L_G$.*

Before starting the proof, we need to make some preparations. Notice that Batson and Seed's work [5] implies the following useful result.

**Proposition 5.2** ([5]). *If $L$ is a link in $S^3$ with $n$ components and*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n,$$

*then* $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_0; \mathbb{Z}/2) = 2^{|L_0|}$ *for every sublink $L_0$ of $L$, where $|L_0|$ is the number of components of $L_0$.*

*Proof.* Suppose $L = K_1 \cup \cdots \cup K_n$. Let $I$ be a subset of $\{1, \ldots, n\}$ with $|I|$ components. By [5, Theorem 1.1] (cf. the proof of [5, Proposition 7.1]), we have

$$\begin{aligned} 2^n &= \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K; \mathbb{Z}/2) \\ &\geq \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \notin I} K_i; \mathbb{Z}/2\Big) \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \in I} K_i; \mathbb{Z}/2\Big) \\ &\geq 2^{n-|I|} \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \in I} K_i; \mathbb{Z}/2\Big) \geq 2^n. \end{aligned}$$

Hence the inequalities above are equalities, and we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \in I} K_i; \mathbb{Z}/2\Big) = 2^{|I|}. \qquad \blacksquare$$

The above result together with Kronheimer–Mrowka's unknot detection theorem in [20] implies the following proposition.

**Proposition 5.3** ([5, Proposition 7.1]). *If $L$ is a link in $S^3$ with $n$ components and*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n,$$

*then each component of $L$ is an unknot.* $\blacksquare$

**Proposition 5.4.** *Suppose $L$ is a link in $S^3$ with $n$ components and*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n.$$

*Then for every point $p \in L$, we have* $\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) = 2^{n-1}$.

*Proof.* Given a point $p \in L$, we use $\mathrm{Khr}(L,p)$ to denote the reduced Khovanov homology with basepoint $p$. By [30, Corollary 3.2.C],

$$\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Khr}(L,p;\mathbb{Z}/2) = \tfrac{1}{2}\,\mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Kh}(L;\mathbb{Z}/2) = 2^{n-1}.$$

By the universal coefficient theorem,

$$\mathrm{rank}_{\mathbb{Q}}\,\mathrm{Khr}(L,p;\mathbb{Q}) \le \mathrm{rank}_{\mathbb{Z}/2}\,\mathrm{Khr}(L,p;\mathbb{Z}/2) = 2^{n-1}.$$

Let $\overline{L}$ be the mirror image of $L$. By [14, Corollary 11],

$$\mathrm{rank}_{\mathbb{Q}}\,\mathrm{Khr}(\overline{L},p;\mathbb{Q}) = \mathrm{rank}_{\mathbb{Q}}\,\mathrm{Khr}(L,p;\mathbb{Q}) \le 2^{n-1}.$$

Using Kronheimer–Mrowka's spectral sequence [20, Theorem 8.2] whose $E_2$-page is $\mathrm{Khr}(\overline{L},p;\mathbb{Z})$ and which converges to $\mathrm{I}^\natural(L,p;\mathbb{Z})$, we obtain

$$\dim_{\mathbb{C}} \mathrm{I}^\natural(L,p) = \mathrm{rank}_{\mathbb{Z}}\,\mathrm{I}^\natural(L,p;\mathbb{Z}) \le \mathrm{rank}_{\mathbb{Z}}\,\mathrm{Khr}(\overline{L},p;\mathbb{Z}) \le 2^{n-1}.$$

On the other hand, Proposition 5.3 and (3.12) imply that $\dim_{\mathbb{C}} \mathrm{I}^\natural(L,p) \ge 2^{n-1}$. Therefore we obtain $\dim_{\mathbb{C}} \mathrm{I}^\natural(L,p) = 2^{n-1}$. ∎

*Proof of Theorem 5.1.* We prove the theorem by induction on $n$. When $n = 1$, it is the unknot detection theorem of Kronheimer and Mrowka [20].

Assume the theorem holds when the number of components is smaller than $n$. Since $G$ is a forest, we can find a vertex of $G$ with degree less than or equal to 1. We discuss two cases.

*Case 1:* There is a vertex of $G$ with degree 1. Without loss of generality, assume this vertex corresponds to the component $K_n'$ of $L_G$. By the assumption of Theorem 5.1, there exists $i \in \{1,\dots,n-1\}$ such that $\mathrm{lk}(K_i,K_n) = \pm 1$ and $\mathrm{lk}(K_j,K_n) = 0$ when $1 \le j \le n-1, j \ne i$.

Pick a basepoint $p \in K_n$ and use $L'$ to denote $K_1 \cup \cdots \cup K_{n-1}$. According to Proposition 5.3, $K_n$ is an unknot. Let $N(K_n)$ be a tubular neighborhood of $K_n$. Then $L'$ can be viewed as a link in the solid torus $S^3 - N(K_n)$. By Propositions 2.6 and 5.4 we have

$$\mathrm{AHI}(L') \cong \mathrm{I}^\natural(L,p) \cong \mathbb{C}^{2^{n-1}}$$

According to Corollary 4.4, we can find a meridional disk $S$ in the solid torus $S^3 - N(K_n)$ which intersects $K_i$ in a single point and is disjoint from the other components. The meridional disk $S$ is a Seifert disk of $K_n$. By the induction hypothesis, $L'$ is a forest of unknots. We can shrink $K_n$ via $S$ to a small meridian of $K_i$. Therefore $L$ is also a forest of unknots. Since the linking numbers uniquely determine a forest of unknots, we conclude that $L$ is isotopic to $L_G$.

*Case 2:* There is a vertex of $G$ with degree 0. Without loss of generality, assume this vertex corresponds to the component $K_n'$ of $L_G$. By the assumption of Theorem 5.1, we have $\mathrm{lk}(K_j,K_n) = 0$ for all $1 \le j \le n-1$. Let $L' := K_1 \cup \cdots \cup K_{n-1}$ and let $N(K_n)$ be a tubular neighborhood of $K_n$. We can view $L'$ as a link in the solid torus $S^3 - N(K_n)$, and the same argument as above gives $\mathrm{AHI}(L') \cong \mathbb{C}^{2^{n-1}}$. By Proposition 4.4, we can find a meridional disk $S$ in the solid torus $S^3 - N(K_n)$ which is disjoint from $L'$. Therefore $L$ is the disjoint union of $L'$ and the unknot, and the result follows from the induction hypothesis on $L'$. ∎

## 6. The case of 2-component links

### *6.1. The linking numbers of 2-component links*

This subsection proves that condition (2) of Theorem 5.1 is implied by condition (1) when $n = 2$. The main result of this subsection is the following lemma.

**Lemma 6.1.** *Suppose $L = K_1 \cup K_2$ is a link with two components such that*

$$\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 4.$$

*Then $|\mathrm{lk}(K_1, K_2)| \le 1$.*

Combining this lemma with Theorem 5.1, we have the following corollary.

**Corollary 6.2.** *Suppose $L$ is a link with two components and* $\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 4$. *Then $L$ is either the* 2*-component unlink or the Hopf link.*

We start the proof of Lemma 6.1 with the following lemma.

**Lemma 6.3.** *Suppose $L = K_1 \cup K_2$ satisfies the assumption of Lemma* 6.1*, and suppose $L$ is not the unlink. Then $K_1$ is an unknot, and $K_2$ is a braid closure with axis $K_1$. Similarly, $K_2$ is an unknot, and $K_1$ is a braid closure with axis $K_2$.*

*Proof.* Proposition 5.3 implies that $K_1$ and $K_2$ are both unknots. Let $N(K_1)$ be a tubular neighborhood of $K_1$. Then $K_2$ is a knot in the solid torus $S^3 - N(K_1)$. Proposition 5.4 yields

$$\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2$$

for every $p \in L$. By Proposition 2.6,

$$\dim_{\mathbb{C}} \mathrm{AHI}(K_2) = \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2. \tag{6.1}$$

If $\mathrm{AHI}(K_2)$ is supported in f-degree 0, then by Theorem 2.4, there exists a meridional disk which is disjoint from $K_2$. This means $K_2$ is included in a 3-ball in the solid torus $S^3 - N(K_1)$, hence $K_1$ and $K_2$ are split, and therefore the link $L$ is the unlink, contradicting the assumption. Therefore $\mathrm{AHI}(K_2)$ is supported in f-degrees $\pm l$ for $l > 0$. By (6.1), we have $\mathrm{AHI}(K_2, \pm l) \cong \mathbb{C}$ and $\mathrm{AHI}(K_2)$ vanishes in all the other f-degrees. According to Proposition 2.5, $K_2$ is the closure of an $l$-braid in $S^3 - N(K_1)$.

The same argument for $S^3 - N(K_2)$ proves the second half of the lemma. ∎

**Remark 6.4.** A link described by the conclusion of Lemma 6.3 is called an *exchangeably braided link*. This concept was first introduced and studied by Morton [25].

Let $l > 1$ be an integer. Recall that the braid group $B_l$ is given by

$$B_l = \langle \sigma_1, \dots, \sigma_{l-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (j - i \geq 2)\rangle.$$

The reduced Burau representation (see [6]) is a group homomorphism

$$\rho : B_l \to \mathrm{GL}(l-1, \mathbb{Z}[t, t^{-1}])$$

defined by

$$\rho(\sigma_i) := \begin{pmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{l-i-2} \end{pmatrix}, \quad 2 \leq i \leq l-2,$$

$$\rho(\sigma_1) := \begin{pmatrix} -t & 1 & \\ 0 & 1 & \\ & & I_{l-3} \end{pmatrix}, \quad \rho(\sigma_{n-1}) := \begin{pmatrix} I_{l-3} & & \\ & 1 & 0 \\ & t & -t \end{pmatrix}$$

for $l > 2$, while for $l = 2$ it is defined by $\rho(\sigma_1) := (-t)$. Notice that for every $\beta \in B_l$, there exists an integer $a$ such that

$$\det(\rho(\beta)) = \pm t^a. \tag{6.2}$$

We also need the follow result by Morton.

**Theorem 6.5** ([25, Theorem 3]). *Suppose $L = U \cup \widehat{\beta}$ is a 2-component link where $U$ is the unknot and $\widehat{\beta}$ is the closure of a braid $\beta \in B_l$ with axis $U$. Then the multi-variable Alexander polynomial $\Delta_L(x,t)$ of $L$ is given by*

$$\Delta_L(x,t) \doteq \det(xI - \rho(\beta)(t)),$$

*where $x$ and $t$ are variables corresponding to $U$ and $\hat{\beta}$ respectively.*

**Remark 6.6.** The sign "$\doteq$" in Theorem 6.5 means the two sides are equal up to multiplication by $\pm x^a t^b$. This is necessary because the multi-variable Alexander polynomial is only defined up to multiplication by $\pm x^a t^b$.

**Lemma 6.7.** *Suppose $L = K_1 \cup K_2$ is an exchangeably braided link with linking number $l \geq 2$. Let $\Delta_L(x,y)$ be the multi-variable Alexander polynomial of $L$. Then the expansion of the Laurent polynomial $(x-1)(y-1)\Delta_L(x,y)$ has (strictly) more than four terms.*

*Proof.* Without loss of generality, assume $x$ and $y$ are the variables corresponding to $K_1$ and $K_2$ respectively. Let $\beta \in B_l$ be the braid whose closure is isotopic to $K_2$ as a link in the solid torus $S^3 - K_1$. By (6.2) and Theorem 6.5, we have

$$\begin{aligned}\Delta_L(x,y) &\doteq (-1)^{l-1}\det(\rho(\beta)(y)) + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} \\ &= \pm y^a + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} \end{aligned} \tag{6.3}$$

for some $a \in \mathbb{Z}$ and $f_i \in \mathbb{Z}[y, y^{-1}]$.

Switching the roles of $K_1$ and $K_2$, we have

$$\Delta_L(x,y) \doteq \pm x^b + g_1(x)y + \cdots + g_{l-2}(x)y^{l-2} + y^{l-1} \tag{6.4}$$

for some $b \in \mathbb{Z}$ and $g_i(x) \in \mathbb{Z}[x, x^{-1}]$.

By (6.3), we have

$$\begin{aligned}&(y-1)\Delta_L(x,y) \\ &\quad \doteq \pm(y-1)y^a + (y-1)f_1(y)x + \cdots + (y-1)f_{l-2}(y)x^{l-2} + (y-1)x^{l-1},\end{aligned}$$

hence we have the following expansion in increasing powers of $x$:

$$(x-1)(y-1)\Delta_L(x,y) \doteq \pm(y-1)y^a + h_1(y)x + \cdots + h_{l-1}(y)x^{l-1} + (y-1)x^l.$$

The right-hand side has at least four terms after expansion, which come from the lowest and highest powers of $x$. Suppose it has only four terms in total; then all the terms in between must vanish, thus we have

$$(x-1)(y-1)\Delta_L(x,y) \doteq \pm(y-1)y^a + (y-1)x^l. \tag{6.5}$$

Plugging in $x = 1$, we have

$$0 = \pm(y-1)y^a + (y-1),$$

therefore $a = 0$, and (6.5) gives

$$\Delta_L(x,y) \doteq \frac{-(y-1) + (y-1)x^l}{(x-1)(y-1)} = 1 + x + \cdots + x^{l-1},$$

which contradicts (6.4) when $l \geq 2$. ■

*Proof of Lemma* 6.1. Suppose $l \geq 2$. We use $\widehat{HFK}$ and $\widehat{HFL}$ to denote the Heegaard knot Floer homology [26, 28] and link Floer homology [27] respectively. The link Floer homology was originally defined only for $\mathbb{Z}/2$-coefficients, and was generalized to $\mathbb{Z}$-coefficients in [29]. It is known that

$$\operatorname{rank}_{\mathbb{Q}} \widehat{HFK}(L;\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}} \widehat{HFL}(L;\mathbb{Q}),$$

but $\widehat{HFL}$ carries more refined gradings.

By [7, Corollary 1.7], we have

$$\operatorname{rank}_{\mathbb{Q}} \widehat{HFK}(L;\mathbb{Q}) \leq 2\operatorname{rank}\mathrm{Khr}(L;\mathbb{Q}) \leq 2\operatorname{rank}\mathrm{Khr}(L;\mathbb{Z}/2) = 4. \tag{6.6}$$

On the other hand, let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of $L$. It was proved in [27] that the graded Euler characteristic of $\widehat{HFL}(L;\mathbb{Q})$ satisfies

$$\chi(\widehat{HFL}(L;\mathbb{Q})) \doteq (x-1)(y-1)\Delta_L(x, y).$$

By Lemma 6.7, we have

$$\operatorname{rank}_{\mathbb{Q}} \widehat{HFK}(L;\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}} \widehat{HFL}(L;\mathbb{Q}) > 4,$$

which contradicts (6.6). ∎

**Remark 6.8.** The proof of Lemma 6.1 relies on Dowlin's inequality [7, Corollary 1.7] and the fact that the graded Euler characteristic of link Floer homology recovers the multi-variable Alexander polynomial [27]. The instanton analogue of Dowlin's inequality is proved in [32]. When the first version of this paper was written, it was not clear how to recover the multi-variable Alexander polynomial from instanton knot homology. Recently, such a result has been established by Zhenkun Li and Fan Ye [23, Theorem 1.4]. Therefore, Lemma 6.1 can also be proved using instanton Floer homology by a similar argument using results from [23, 32] instead.

We introduce the following condition on a link $L \subset S^3$:

**Condition 6.9.** (1) *$L$ has $n \geq 3$ connected components.*

(2) *The rank of* $\mathrm{Kh}(L;\mathbb{Z}/2)$ *is* $2^n$.

(3) *The components of $L$ can be arranged as a sequence $K_1, \ldots, K_n$ such that the linking number of $K_i$ and $K_j$ $(i \neq j)$ is $\pm 1$ when $|i-j| = 1$ or $n-1$, and is zero otherwise.*

Theorem 5.1 and Lemma 6.1 have the following consequence.

**Lemma 6.10.** *If $L_0$ is an $m$-component link with* $\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L_0;\mathbb{Z}/2) = 2^m$, *then either $L_0$ is a forest of unknots, or $L_0$ contains a sublink $L$ satisfying Condition 6.9.*

*Proof.* Let $K_1, \ldots, K_m$ be the components of $L_0$. By Proposition 5.2 and Lemma 6.1, for each pair $i \neq j$, the linking number of $K_i$ and $K_j$ is equal to 0 or $\pm 1$. Let $G$ be a simple graph with $m$ vertices $p_1, \ldots, p_m$ such that $p_i$ and $p_j$ are connected by an edge if and only if $|\mathrm{lk}(K_i, K_j)| = \pm 1$. If $G$ is a forest, then Theorem 5.1 implies that $L_0$ is a forest of unknots. If $G$ contains a cycle, then the vertices of the shortest cycle of $G$ correspond to a sublink of $L_0$ satisfying Condition 6.9. ∎

The next subsection gives a proof of Theorem 1.7. The rest of this article is devoted to proving that there is no link satisfying Condition 6.9, therefore Theorem 1.2 will follow from Lemma 6.10.

### 6.2. *Some 2-component links with small Khovanov homology*

This subsection gives a proof of Theorem 1.7, and shows that the bi-graded Khovanov homology detects some simple 2-component links other than the unlink and the Hopf link. The results of this subsection will not be used in the proof of Theorem 1.2.

Recall that the internal grading of the Khovanov homology of a link $L$ is introduced in [5, Section 2] as $h - q$, where $h$ is the homological grading and $q$ is the quantum grading. The following is a special case of a more general result due to Batson and Seed.

**Theorem 6.11** ([5, Corollary 4.4]). *Suppose $L = K_1 \cup K_2$ is a 2-component oriented link. Then*

$$\operatorname{rank}^l_{\mathbb{F}} \operatorname{Kh}(L;\mathbb{F}) \geq \operatorname{rank}^{l+2\operatorname{lk}(K_1,K_2)}_{\mathbb{F}}(\operatorname{Kh}(K_1;\mathbb{F}) \otimes \operatorname{Kh}(K_2;\mathbb{F})),$$

*where $\mathbb{F}$ is an arbitrary field and* $\operatorname{rank}^k$ *denotes the rank of the summand with internal grading $k$.*

Let $L_1$ be the oriented link given in Figure 4. Then its Khovanov homology is given by

$$\operatorname{Kh}(L_1;\mathbb{Z}) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(2)} \oplus (\mathbb{Z}/2)_{(3)} \oplus \mathbb{Z}^2_{(4)} \oplus \mathbb{Z}_{(6)}, \tag{6.7}$$

where the subscripts represent the internal gradings.

**Theorem 6.12.** *Let $L = K_1 \cup K_2$ be a 2-component oriented link. Suppose*

$$\operatorname{Kh}(L;\mathbb{Z}/2) \cong \operatorname{Kh}(L_1;\mathbb{Z}/2)$$

*as bi-graded vector spaces. Then $L$ is isotopic to $L_1$.*

*Proof.* To simplify notation, we use $\mathbb{F}$ to denote the field $\mathbb{Z}/2$. By (6.8) and the universal coefficient theorem, we have

$$\operatorname{Kh}(L;\mathbb{F}) = \mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)} \oplus \mathbb{F}^2_{(2)} \oplus \mathbb{F}_{(3)} \oplus \mathbb{F}^2_{(4)} \oplus \mathbb{F}_{(6)}, \tag{6.8}$$

where the subscripts represent the internal grading. By Theorem 6.11, we have

$$8 = \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(L;\mathbb{F}) \geq \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_1;\mathbb{F}) \cdot \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_2;\mathbb{F}),$$

hence $\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_i;\mathbb{F}) \leq 4$. On the other hand, $\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_i;\mathbb{F}) = 2\operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_i;\mathbb{F})$, and $\operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_i;\mathbb{F})$ is always odd for knots. Therefore

$$\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_1;\mathbb{F}) = \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_2;\mathbb{F}) = 2,$$
$$\operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_1;\mathbb{F}) = \operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_2;\mathbb{F}) = 1.$$

By Kronheimer–Mrowka's unknot detection theorem, both $K_1$ and $K_2$ are unknots. Hence

$$\operatorname{Kh}(K_1;\mathbb{F}) \otimes \operatorname{Kh}(K_2;\mathbb{F}) = \mathbb{F}_{(-2)} \oplus \mathbb{F}^2_{(0)} \oplus \mathbb{F}_{(2)}.$$

By Theorem 6.11 and (6.8), we have $\operatorname{lk}(K_1, K_2) = 1$ or $2$, hence $K_1$ is homotopic to the closure of a 1-braid or a 2-braid in the solid torus $S^3 - N(K_2)$. Let $l := \operatorname{lk}(K_1, K_2)$, and let $\hat{\beta}_l$ be the closure of an arbitrary $l$-braid subject to the condition that $\hat{\beta}_l$ is connected. By [33, Section 4.4],

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1, l) \equiv \dim_{\mathbb{C}} \operatorname{AHI}(\hat{\beta}_l, l) = 1 \bmod 2, \tag{6.9}$$

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1, j) \equiv \dim_{\mathbb{C}} \operatorname{AHI}(\hat{\beta}_l, j) = 0 \bmod 2 \quad \text{if } j > l. \tag{6.10}$$

By Proposition 2.6 and Kronheimer–Mrowka's spectral sequence [20, Theorem 8.2], we have

$$4 = \operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(L_1, p; \mathbb{F}) \geq \dim_{\mathbb{C}} \operatorname{AHI}(K_1), \tag{6.11}$$

where $p \in K_2$.

If $l = 1$, we have $\operatorname{AHI}(K_1, \pm 1) = \mathbb{C}$ by (6.9) and (6.11). By (6.10) and (6.11), we have $\operatorname{AHI}(K_1, j) = 0$ for all $j > l$. Hence the top f-grading of $\operatorname{AHI}(K_1)$ is 1, and $K_1$ is the closure of a 1-braid by Theorem 2.5. This implies that $L$ is a Hopf link, whose Khovanov homology is different from $\operatorname{Kh}(L_1)$, which yields a contradiction. Hence we must have $l = 2$. By similar arguments, we obtain $\operatorname{AHI}(K_1, \pm 2) = \mathbb{C}$ and the top f-grading of $\operatorname{AHI}(K_1)$ is 2. Therefore $K_1$ is the closure of a 2-braid. Since $K_1$ is an unknot, this 2-braid must be given by a generator of the 2-braid group. The proof is then completed by directly checking all the possible choices of the generator and orientations. ∎

Now we prove the second part of Theorem 1.7. Let $T$ be the left-handed trefoil. We have

$$\operatorname{Kh}(T; \mathbb{Z}) \cong \mathbb{Z}_{(-3,-9)} \oplus \mathbb{Z}_{(-2,-5)} \oplus \mathbb{Z}_{(0,-3)} \oplus \mathbb{Z}_{(0,-1)} \oplus (\mathbb{Z}/2)_{(-2,-7)},$$

where the subscripts represent the $(h, q)$-bigrading. Let $L_2$ be the disjoint union of $T$ and an unknot $U$. Then

$$\operatorname{Khr}(L_2, p; \mathbb{Z}) \cong \operatorname{Kh}(T; \mathbb{Z}),$$

where the basepoint $p$ is in $U$.

**Theorem 6.13.** *Let $L = K_1 \cup K_2$ be a 2-component link with a basepoint $q \in K_2$. Suppose*

$$\operatorname{Khr}(L, q; \mathbb{Z}) \cong \operatorname{Khr}(L_2, p; \mathbb{Z})$$

*as bi-graded abelian groups. Then the link $L$ splits into a left-handed trefoil $K_1$ and an unknot $K_2$.*

*Proof.* By Kronheimer–Mrowka's spectral sequence [20, Theorem 8.2], we have

$$4 = \operatorname{rank}_{\mathbb{Z}} \operatorname{Khr}(L, q; \mathbb{Z}) \geq \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, q). \tag{6.12}$$

Let $\Gamma$ be the local system associated with $K_1$. We have

$$\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, q) \geq \operatorname{rank}_{\mathcal{R}} \mathrm{I}^{\natural}(L, q; \Gamma).$$

By Corollary 3.3,

$$\operatorname{rank}_{\mathcal{R}} \mathrm{I}^{\natural}(L, q; \Gamma) = \operatorname{rank}_{\mathcal{R}} \mathrm{I}^{\natural}(K_2 \cup U, q; \Gamma).$$

By excision,

$$\operatorname{rank}_{\mathcal{R}} \mathrm{I}^{\natural}(K_2 \cup U, q; \Gamma) = 2 \dim_{\mathbb{C}} \mathrm{I}^{\natural}(K_2, q).$$

Hence $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(K_2, q) \leq 2$. We know that $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(K_2, q)$ is always odd since crossing changes do not change the parity of $\mathrm{I}^{\natural}$ and $\mathrm{I}^{\natural}(\text{unknot}) \cong \mathbb{C}$. Hence $K_2$ is the unknot by

[20, Proposition 1.4]. Let $\mathbb{F}=\mathbb{Z}/2$. The universal coefficient theorem implies

$$\begin{aligned}\mathrm{Khr}(L,q;\mathbb{F}) &\cong \mathrm{Khr}(L_2,p;\mathbb{F})\\ &\cong \mathbb{F}_{(-3,-9)}\oplus\mathbb{F}_{(-2,-5)}\oplus\mathbb{F}_{(0,-3)}\oplus\mathbb{F}_{(0,-1)}\oplus\mathbb{F}_{(-2,-7)}\oplus\mathbb{F}_{(-3,-7)}\\ &\cong \mathbb{F}_{(6)}\oplus\mathbb{F}_{(5)}\oplus\mathbb{F}_{(4)}\oplus\mathbb{F}^2_{(3)}\oplus\mathbb{F}_{(1)},\end{aligned}$$

where the subscripts in the second to last row represent the $(h,q)$-bigrading, and the subscripts in the last row represent the internal grading. According to [30, Corollary 3.2.C],

$$\mathrm{Kh}^{i,j}(L;\mathbb{F})\cong \mathrm{Khr}^{i,j-1}(L,q;\mathbb{F})\oplus\mathrm{Khr}^{i,j+1}(L,q;\mathbb{F}).$$

Hence

$$\mathrm{Kh}(L;\mathbb{F})\cong\mathbb{F}_{(0)}\oplus\mathbb{F}^2_{(2)}\oplus\mathbb{F}_{(3)}\oplus\mathbb{F}^3_{(4)}\oplus\mathbb{F}^2_{(5)}\oplus\mathbb{F}_{(6)}\oplus\mathbb{F}_{(7)}. \tag{6.13}$$

By Theorem 6.11,

$$12=\operatorname{rank}_{\mathbb{F}}\mathrm{Kh}(L;\mathbb{F})\geq\operatorname{rank}_{\mathbb{F}}\mathrm{Kh}(K_1;\mathbb{F})\operatorname{rank}_{\mathbb{F}}\mathrm{Kh}(K_2;\mathbb{F})=2\operatorname{rank}_{\mathbb{F}}\mathrm{Kh}(K_1;\mathbb{F}).$$

Hence $\operatorname{rank}_{\mathbb{F}}\mathrm{Kh}(K_1;\mathbb{F})\leq 6$ and $\operatorname{rank}_{\mathbb{F}}\mathrm{Khr}(K_1;\mathbb{F})\leq 3$. Kronheimer–Mrowka's unknot detection theorem [20] and Baldwin–Sivek's trefoil detection theorem [3] imply $K_1$ is either an unknot or a trefoil.

Suppose $K_1$ is an unknot. Then $\mathrm{lk}(K_1,K_2)=1$ or 2 by Theorem 6.11 and (6.13). Now $K_1$ is a knot in the solid torus $S^3-N(K_2)$ with winding number 1 or 2, and by Proposition 2.6,

$$4\geq\dim_{\mathbb{C}}\mathrm{AHI}(K_1).$$

The argument in the proof of Theorem 6.12 shows that $K_1$ is either the closure of the 1-braid or the closure of a generator of the 2-braid group. In either case, $\mathrm{Khr}(L,q;\mathbb{Z})$ is not isomorphic to $\mathrm{Khr}(L_2,p;\mathbb{Z})$, which is a contradiction.

By the discussion above, $K_1$ must be a trefoil, hence Theorem 6.11 and (6.13) imply $\mathrm{lk}(K_1,K_2)=0$. Hence $K_1$ is homotopic to the unknot in $S^3-N(K_2)$. By [33, Section 4.4], $\dim_{\mathbb{C}}\mathrm{AHI}(K_1,j)$ is even for all $|j|>0$. We also have $\dim_{\mathbb{C}}\mathrm{AHI}(K_1,0)\geq\dim_{\mathbb{C}}\mathrm{AHI}(\mathcal{U}_1,0)=2$ by Proposition 4.3. By (6.12), $\dim_{\mathbb{C}}\mathrm{AHI}(K_1)=\dim_{\mathbb{C}}\mathrm{I}^\natural(L,q)\leq 4$, hence the argument above implies that the top f-grading of $\mathrm{AHI}(K_1)$ is 0. By Theorem 2.4, $K_2$ has a Seifert disk which is disjoint from $K_1$, hence $L$ is a split link. ∎

## 7. Topological properties from instanton Floer homology

From now on, let $L=K_1\cup\cdots\cup K_n$ be a hypothetical link that satisfies Condition 6.9. The goal is to deduce a contradiction from Condition 6.9. This section derives several topological properties of $L$ using instanton Floer homology.

In Section 7.1, we show that every $K_i$ bounds a disk $D_i$ such that when $i\neq j$, the disk $D_i$ intersects $K_j$ transversely in $|\mathrm{lk}(K_i,K_j)|$ points. If $n\geq 4$, we show that this property implies that (after isotopy) the link $L$ has the form given by Figure 8. In Section 7.2, we show that $K_n$ can be isotoped in the complement of $K_1\cup\cdots\cup K_{n-1}$ in such a way that it becomes disjoint from the surface shown in Figure 11 or the surface shown in Figure 12.

Let $L' = K_1 \cup \cdots \cup K_{n-1}$. In Sections 8 and 9, we will use the two topological properties summarized above to study the possible isotopy classes of $K_n$ in $S^3 - L'$. The results proved in this section will imply that the *homotopy* class of $K_n$ in $S^3 - L'$ has a specific form and that $K_n$ bounds a disk that intersects a fibered Seifert surface of $K_1 \cup \cdots \cup K_{n-1}$ in an arc. Sections 8 and 9 will find all the possible isotopy classes of the intersection arc by solving the word problem in $\pi_1(S^3 - L')$. This will imply that $L$ must be isotopic to the link $L_{n,1-n}$ or $L_{n,2-n}$ defined in Section 10. We will then show that the ranks of the Khovanov homology of $L_{n,1-n}$ and $L_{n,2-n}$ are both greater than $2^n$, which yields a contradiction.

## *7.1. Seifert surfaces of $K_i$*

**Proposition 7.1.** *For each $K_i$, there exists an embedded disk $D_i$ such that*

(1) *$\partial D_i = K_i$;*

(2) *for each $j \neq i$, if $|i - j| = 1$ or $n - 1$, then $D_i$ intersects $K_j$ transversely in one point; otherwise, the disk $D_i$ is disjoint from $K_j$.*

*Proof.* Pick a basepoint $p \in K_i$. By Proposition 5.4, we have $\dim \mathrm{I}^\natural(L, p) = 2^{n-1}$. By Proposition 5.3, $K_i$ is an unknot. Let $N(K_i)$ be a tubular neighborhood of $K_i$, and view $L - K_i$ as a link in the solid torus $S^3 - N(K_i)$. By Proposition 2.6,

$$\dim \mathrm{AHI}(L - K_i) = \mathrm{I}^\natural(L, p) = 2^{n-1}.$$

By Corollary 4.4, there exists a meridional disk $\hat{D}_i$ which is disjoint from $K_j$ if $|i-j| \neq 1$ or $n-1$ and intersects $K_j$ transversely at one point if $|i - j| = 1$ or $n - 1$. The meridional disk $\hat{D}_i$ extends to the desired Seifert disk of $K_i$. ■

**Definition 7.2.** Let $D_1, \ldots, D_n$ be a sequence of immersed disks in $\mathbb{R}^3$ such that $\partial D_i = K_i$ for all $i$. We say that the sequence $D_1, \ldots, D_n$ is *generic* if every self-intersection point of $\bigsqcup D_i$ is locally diffeomorphic to one of the following models in $\mathbb{R}^3$ at $(0, 0, 0)$:

(1) the intersection of $\{(x, y, z) \mid z = 0,\ y \geq 0\}$ and the $yz$-plane;

(2) the intersection of the $xy$-plane and the $yz$-plane;

(3) the intersection of the $xy$-, $yz$-, and $xz$-planes.

If $D = (D_1, \ldots, D_n)$ is generic, let $\Sigma_1(D)$, $\Sigma_2(D)$, $\Sigma_3(D)$ be the sets of self-intersection points described by (1), (2), (3) above respectively.

**Definition 7.3.** If $D = (D_1, \ldots, D_n)$ is generic, define the *complexity* of $D$ to be the number of components of $\Sigma_2(D)$.

Notice that if $D$ is generic, then the complexity of $D$ is greater than or equal to $\frac{1}{2}\,\#\,\Sigma_1(D)$, which is at least $n$.

**Definition 7.4.** We say that the sequence $D = (D_1, \dots, D_n)$ is *admissible* if

(1) $D$ is generic;

(2) $\#\Sigma_1(D) = 2n$;

(3) every point in $\Sigma_3(D)$ is contained in at least two different disks in $D$.

**Remark 7.5.** In the definitions above, the disks $\{D_i\}$ are only required to be immersed. Condition (2) in the definition above is equivalent to the following statement: for each $j \neq i$, if $|i - j| = 1$ or $n - 1$, then $D_i$ intersects $K_j$ transversely at one point; otherwise, the immersed disk $D_i$ is disjoint from $K_j$. Moreover, the interior of $D_i$ is disjoint from $K_i$.

**Proposition 7.6.** *If $n \geq 4$, then there exists a sequence $D = (D_1, \dots, D_n)$ of disks such that $\partial D_i = K_i$ for all $i$, and $D$ is admissible with complexity $n$.*

*Proof.* By Proposition 7.1, there exists a sequence of disks $\hat{D} = (\hat{D}_1, \dots, \hat{D}_n)$ such that for all $i$, $\hat{D}_i$ is embedded, $\partial \hat{D}_i = K_i$, and $\#\Sigma_1(\hat{D}) = 2n$. Perturb $\hat{D}_1, \dots, \hat{D}_n$ in such a way that they are generic. Since all the disks are embedded, every point in $\Sigma_3(\hat{D})$ is contained in three different disks. Therefore $\hat{D}$ is admissible, so admissible configurations exist. Let $D = (D_1, \dots, D_n)$ be an admissible configuration with minimal complexity.

We first show that all the $D_i$'s are embedded. Suppose there exists $i$ such that $D_i$ is not embedded. Then by admissibility, $D_i$ does not have triple self-intersections, and the self-intersection locus of $D_i$ is a disjoint union of circles. Let $\gamma \subset D_i$ be a circle in the self-intersection of $D_i$.

Let $B^2$ be the unit disk in $\mathbb{R}^2$, and let $f_i : B^2 \to \mathbb{R}^3$ be an immersion that parametrizes $D_i$. Then $f_i^{-1}(\gamma)$ is a double cover of $\gamma$. There are three possibilities:

*Case 1:* $f_i^{-1}(\gamma)$ is a disjoint union of two circles, and they bound disjoint disks $B_1$ and $B_2$. In this case, take a diffeomorphism $\iota$ from $B_1$ to $B_2$ such that

$$(f_i \circ \iota)|_{\partial B_1} = f_i|_{\partial B_1}.$$

Define

$$f_i'(p) := \begin{cases} f_i(p) & \text{if } p \notin B_1 \cup B_2, \\ f_i(\iota(p)) & \text{if } p \in B_1, \\ f_i(\iota^{-1}(p)) & \text{if } p \in B_2. \end{cases}$$

By smoothing $f_i'$, we obtain an immersed disk with the same boundary as $D_i$ but has fewer self-intersection components. Figure 6 shows a local picture of $f_i'$ after the

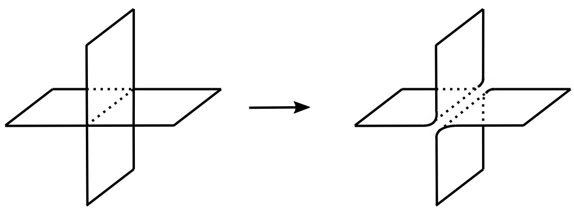

**Fig. 6.** The local construction of $f_i'$ after smoothing.

smoothing. Replacing $D_i$ by the image of the smoothed $f_i'$ decreases the complexity of $D$ and preserves the admissibility condition.

*Case 2:* $f_i^{-1}(\gamma)$ is a disjoint union of two circles, and they bound disks $B_1$ and $B_2$ with $B_1 \supset B_2$. In this case, take a diffeomorphism $\iota$ from $B_1$ to $B_2$ such that

$$(f_i \circ \iota)|_{\partial B_1} = f_i|_{\partial B_1}.$$

Define

$$f_i'(p) := \begin{cases} f_i(p) & \text{if } p \notin B_1, \\ f_i(\iota(p)) & \text{if } p \in B_1. \end{cases}$$

Replacing $f_i$ by the smoothed version of $f_i'$ gives an admissible configuration with smaller complexity.

*Case 3:* $f_i^{-1}(\gamma)$ is one circle, and it bounds a disk $B_1$. We show that this case is impossible. Let $\tau : f_i^{-1}(\gamma) \to f_i^{-1}(\gamma)$ be the deck transformation of the double cover. Fix an orientation of $f_i^{-1}(\gamma)$. For each $p \in f_i^{-1}(\gamma)$, let $v(p)$ be the unit tangent vector of $f_i^{-1}(\gamma)$ with positive orientation, and let $w(p)$ be the unit normal vector of $f_i^{-1}(\gamma)$ pointing outward of $B_1$. Then the images of $v(p), w(p), w(\tau(p))$ under the tangent map of $f_i$ form a basis of $\mathbb{R}^3$. However, isotoping the point $p$ to $\tau(p)$ on $f_i^{-1}(\gamma)$ reverses the orientation of the basis, which yields a contradiction.

Since $D$ is assumed to have minimal complexity among admissible configurations, we conclude that $D_i$ has to be embedded.

The intersection of $D_i$ and $D_j$ $(i \neq j)$ is a disjoint union of compact 1-manifolds, possibly with boundary. Assume there exist $i \neq j$ such that the intersection of $D_i$ and $D_j$ contains a circle $\gamma$. The circle $\gamma$ bounds a disk $B_1$ in $D_i$, and bounds a disk $B_2$ in $D_j$. Let

$$D_i' := (D_i - B_i) \cup B_j, \quad D_j' := (D_j - B_j) \cup B_i.$$

Replacing $D_i$ and $D_j$ by $D_i'$ and $D_j'$ and smoothing the corners, we obtain a generic configuration with smaller complexity. Since neither $D_i'$ nor $D_j'$ has triple self-intersection points, the new configuration is still admissible, contradicting the definition of $D$. We conclude that the intersection of $D_i$ and $D_j$ $(i \neq j)$ does not contain any circle.

By the admissibility assumption, the intersection $D_i \cap D_j$ $(i \neq j)$ consists of circles and at most one arc, and there is an arc if and only if $|i - j| = 1$ or $n - 1$. As a consequence, $D_i$ and $D_j$ are disjoint if $|i - j| \neq 1, n - 1$, and $D_i \cap D_j$ is an arc if $|i - j| = 1$ or $n - 1$. Since $n \geq 4$, this implies that $D_i \cap D_{i+1}$ is disjoint from $D_j \cap D_{j+1}$ whenever $i \neq j$ (where the subscripts are taken modulo $n$), so the complexity of $D$ is $n$. ∎

Let $L' := K_1 \cup \cdots \cup K_{n-1}$. Proposition 7.6 has the following corollary.

**Corollary 7.7.** *The link $L'$ is a connected sum of $n - 2$ Hopf links as given by Figure 7. If $n \geq 4$, then the link $L$ has a diagram described by Figure 8.*

*Proof.* The first part of the statement follows from Theorem 5.1 and Proposition 5.2. For the second part, by Proposition 7.6, there exists a sequence of disks $D_1, \ldots, D_{n-1}$ such that (1) $D_i$ is embedded and $\partial D_i = K_i$ for $i = 1, \ldots, n-1$, (2) if $|i - j| = 1$, the disks $D_i$

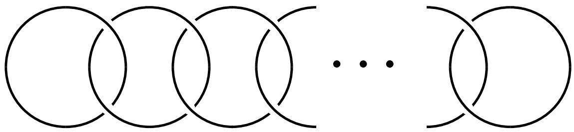

**Fig. 7.** The link $L'$.

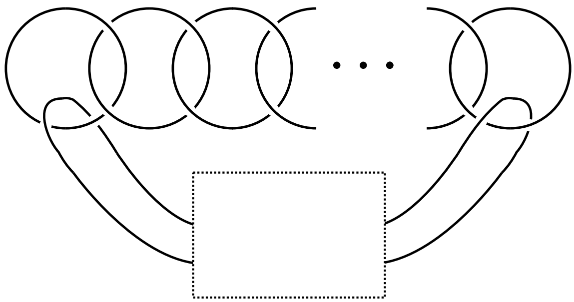

**Fig. 8.** The link $L$.

and $D_j$ intersect in an arc, (3) if $i \neq j$ and $|i-j| \neq 1$, the disks $D_i$ and $D_j$ are disjoint. It follows that if we arrange Figure 7 in such a way that each component is contained in a (flat) plane, then after an isotopy, the link $L'$ is given by Figure 7, and $D_i$ is the disk bounded by $K_i$ on the corresponding plane. Moreover, by Proposition 7.6 again, we may assume that $K_n$ bounds an embedded disk $D_n$ which intersects $D_1$ and $D_{n-1}$ respectively in an arc and is disjoint from $D_2 \cup \cdots \cup D_{n-2}$, so $L$ is isotopic to a link described by Figure 8. ∎

*7.2. Seifert surfaces of $L'$*

We recall the following property of fibered links.

**Lemma 7.8.** *Suppose $L_1$ and $L_2$ are two oriented fibered links with oriented Seifert surfaces $S_1$ and $S_2$ respectively. Let $f_1 : S_1 \to S_1$ and $f_2 : S_2 \to S_2$ be the monodromies. Take $p_1 \in L_1$, $p_2 \in L_2$, and form the connected sum $L_1 \# L_2$ and the boundary connected sum $S_1 \#_b S_2$ with respect to $(p_1, p_2)$. Then $L_1 \# L_2$ is a fibered link with Seifert surface $S_1 \#_b S_2$ and monodromy $f_1 \#_b f_2$.*

*Proof.* Given a compact surface $S$ with boundary, and given a diffeomorphism $f : S \to S$ that restricts to the identity on a neighborhood of $\partial S$, define

$$\mathcal{M}_f := S \times [0,1]/\sim,$$

where $\sim$ is defined by $(x,0) \sim (f(x),1)$ for $x \in S$, and $(x,t_1) \sim (x,t_2)$ for $x \in \partial S$, $t_1, t_2 \in [0,1]$. By the assumptions of the lemma,

$$\mathcal{M}_{f_1} \cong S^3, \quad \mathcal{M}_{f_2} \cong S^3,$$

and the images of $\partial S_1$ and $\partial S_2$ are isotopic to $L_1$ and $L_2$ respectively. Therefore,

$$\mathcal{M}_{f_1 \#_b f_2} \cong \mathcal{M}_{f_1} \# \mathcal{M}_{f_2} \cong S^3,$$

and the image of $\partial(S_1 \#_b S_2)$ is isotopic to $L_1 \# L_2$. ∎

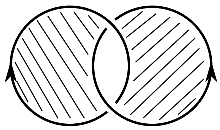

**Fig. 9.** Seifert surface of the Hopf link with linking number $-1$.

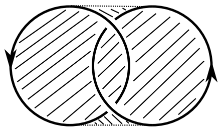

**Fig. 10.** Seifert surface of the Hopf link with linking number $1$.

Notice that the Hopf link is fibered. Depending on the orientations of the components, the corresponding Seifert surface is given by Figure 9 or Figure 10. Both Seifert surfaces are diffeomorphic to the annulus, and the monodromies are Dehn twists along the core circles.

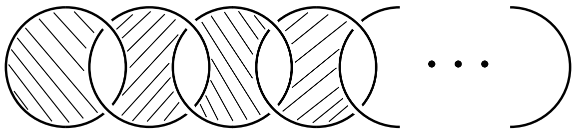

**Fig. 11.** The Seifert surface $S_1$.

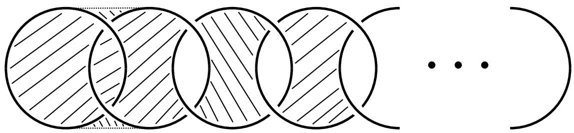

**Fig. 12.** The Seifert surface $S_2$.

Let $S_1$ and $S_2$ be the Seifert surfaces of $L'$ given by Figure 11 and Figure 12 respectively. By Lemma 7.8, the link $L'$ is fibered with respect to both $S_1$ and $S_2$. For each $j = 1, 2$, endow the components $K_1, \dots, K_{n-1}$ with the boundary orientation of $S_j$ and choose an arbitrary orientation for $K_n$. Then the algebraic intersection number of $K_n$ and $S_j$ is equal to the sum of the linking numbers $\sum_{i=1}^{n-1} \operatorname{lk}(K_n, K_i)$. Therefore, Condition 6.9 (3) implies that there exists exactly one $j \in \{1, 2\}$ such that the algebraic intersection number of $S_j$ and $K_n$ is zero. The main result of this subsection is the following proposition.

**Proposition 7.9.** *Suppose $j \in \{1,2\}$ and the algebraic intersection number of $K_n$ with $S_j$ is zero. Then there exists a knot $K_n'$ such that $K_n'$ is disjoint from $S_j$, and $K_n$ is isotopic to $K_n'$ in $\mathbb{R}^3 - L'$.*

Before proving Proposition 7.9, we need to prove some results on instanton Floer homology. Let $U$ be an unknot included in a 3-ball which is disjoint from $L'$, let $m_i$ be a small meridian circle around $K_i$ $(1 \leq i \leq n-1)$ and $u_i$ be a small arc joining $K_i$ and $m_i$.

**Lemma 7.10.** *We have*

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) = 2^{2n-4}, \tag{7.1}$$

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) = 2^{2n-3}, \tag{7.2}$$

*and*

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \cong \mathcal{R}^{2^{2n-3}}, \tag{7.3}$$

*where $\Gamma_U$ is the local system associated with $U$.*

*Proof.* Picking a crossing between $m_1$ and $K_1$ and applying Kronheimer–Mrowka's unoriented skein exact triangle [20, Section 6], we obtain a 3-periodic exact sequence

$$\cdots \to \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \to \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=2}^{n-1} m_i, \sum_{i=2}^{n-1} u_i\Big)$$
$$\to \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=2}^{n-1} m_i, \sum_{i=2}^{n-1} u_i\Big) \to \cdots .$$

See [32, Section 3] for more details. The above exact triangle implies

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \leq 2 \dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=2}^{n-1} m_i, \sum_{i=2}^{n-1} u_i\Big).$$

Repeating this argument for the other meridians, we obtain

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big)$$
$$\leq 2^{n-2} \dim_{\mathbb{C}} \mathrm{I}(S^3, L' \cup m_{n-1}, u_{n-1}) = 2^{n-2} \dim_{\mathbb{C}} \mathrm{I}^\natural(L', p).$$

By Propositions 5.2 and 5.4, $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L', p) = 2^{n-2}$, therefore

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \le 2^{2(n-2)}. \tag{7.4}$$

A similar earrings-removal argument yields

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \le 2^{n-2} \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2^{2n-3}. \tag{7.5}$$

We recall some properties of the instanton knot Floer homology KHI for oriented links, which was introduced in [17, Definition 2.4]. Given an oriented link $M \subset S^3$, the homology group $\mathrm{KHI}(M)$ carries an Alexander $\mathbb{Z}$-grading and a homological $\mathbb{Z}/2$-grading. The rank of $\mathrm{KHI}(M)$ does not depend on the orientation of $M$. We use $\mathrm{KHI}(M, i)$ to denote the summand of $\mathrm{KHI}(M)$ with Alexander degree $i$, and use $\chi(\mathrm{KHI}(M, i))$ to denote its Euler characteristic with respect to the homological grading. Recall that we always take coefficients in $\mathbb{C}$ for instanton Floer homology in this article. According to [17, Theorem 3.6 and (14)], we have

$$\sum_i \chi(\mathrm{KHI}(M, i)) t^i = \pm (t^{1/2} - t^{-1/2})^{|M|-1} \Delta_M(t),$$

where $\Delta_M(t)$ denotes the single-variable Alexander polynomial of $M$. Notice that the Alexander polynomial for $L'$ satisfies $|\Delta_{L'}(-1)| = 2^{n-2}$ for every orientation of $L'$. Therefore, taking $M = L'$, we have

$$\dim_{\mathbb{C}} \mathrm{KHI}(L') \ge 2^{n-2} |\Delta_{L'}(-1)| = 2^{2n-4}.$$

By [32, Proposition 5.1],

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) = \dim_{\mathbb{C}} \mathrm{KHI}(L') \ge 2^{2n-4}. \tag{7.6}$$

Inequalities (7.4) and (7.6) imply (7.1).

Consider the two admissible triples

$$\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i\Big), \quad (S^1 \times S^2, S^1 \times \{p_1, p_2\}, v),$$

where $v$ is an arc joining the two components of $S^1 \times \{p_1, p_2\}$. Let $N(K_1)$ be a small tubular neighborhood of $K_1$ in the first triple, and deform $U$ into $N(K_1)$ by an isotopy. Let $N(S^1 \times \{p_1\})$ be a small tubular neighborhood of $S^1 \times \{p_1\}$ in the second triple. Cutting out $N(K_1)$ and $N(S^1 \times \{p_1\})$, exchanging them, and gluing back, we obtain two new triples

$$\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big), \quad (S^1 \times S^2, S^1 \times \{p_1, p_2\} \cup U', v),$$

where $U'$ is an unknot included in a 3-ball disjoint from $S^1 \times \{p_1, p_2\}$. By the torus excision theorem and the definition of AHI, we have

$$\begin{aligned} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \otimes_{\mathcal{R}} \mathrm{AHI}(\emptyset; \Gamma) \\ \cong \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i; \Gamma_\emptyset\Big) \otimes_{\mathcal{R}} \mathrm{AHI}(U'; \Gamma), \end{aligned} \tag{7.7}$$

where $\Gamma_\emptyset$ is the trivial local system with coefficients $\mathcal{R}$. By (3.3) and Example 3.4, $\mathrm{AHI}(\emptyset; \Gamma) \cong \mathcal{R}$ and $\mathrm{AHI}(U'; \Gamma) \cong \mathcal{R}^2$. By (3.3) and (7.1),

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i; \Gamma_\emptyset\Big) \cong \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \otimes_{\mathbb{C}} \mathcal{R} \cong \mathcal{R}^{2^{2n-4}}.$$

Therefore by (7.7),

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \cong \mathcal{R}^{2^{2n-4}} \otimes_{\mathcal{R}} \mathcal{R}^2 \cong \mathcal{R}^{2^{2n-3}}.$$

This completes the proof of (7.3).

Let $\Gamma_{K_n}$ be the local system on $\mathcal{R}(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i)$ associated with $K_n$. By Corollary 3.3 and the universal coefficient theorem, we have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) &\geq \operatorname{rank}_{\mathcal{R}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i; \Gamma_{K_n}\Big) \\ &= \operatorname{rank}_{\mathcal{R}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) = 2^{2n-3}. \end{aligned}$$

The above inequality together with (7.5) implies (7.2). ■

Choose $j \in \{1, 2\}$ such that the algebraic intersection number of $S_j$ and $K_n$ is zero. Choose an orientation of $S_j$, and endow $L'$ with the boundary orientation. For each $i = 1, \dots, n-1$, let $N(K_i)$ be a sufficiently small tubular neighborhood of $K_i$ that is disjoint from $m_i$. Cut $N(K_i)$ from $S^3$ and glue it back using a diffeomorphism that identifies the meridian of $N(K_i)$ to $S_j \cap \partial N(K_i)$. Since $S^3 - L'$ is fibered over $S^1$ with fiber $S_j$, the manifold obtained from the cutting-and-pasting (which are, of course, Dehn surgeries) is fibered over $S^1$ with fiber $S^2$. Since the orientation-preserving mapping class group of $S^2$ is trivial, the resulting manifold is diffeomorphic to $S^1 \times S^2$ with the product fibration. Let

$$\hat{K}_1, \dots, \hat{K}_n, \hat{m}_1, \dots, \hat{m}_{n-1}, \hat{U} \subset S^1 \times S^2$$

be the images of $K_1, \dots, K_n, m_1, \dots, m_{n-1}, U$ respectively by the cutting-and-pasting. Let $\hat{L}' := \hat{K}_1 \cup \dots \cup \hat{K}_{n-1}$ be the image of $L'$, let $m := m_1 \cup \dots \cup m_{n-1}$ be the union of the earrings, and let $\hat{m} := \hat{m}_1 \cup \cdots \cup \hat{m}_{n-1}$ be the image of $m$. We further require that the diffeomorphism used to glue back $N(K_i)$ fixes $u_i \cap \partial N(K_i)$ for all $i = 1, \ldots, n-1$, so the image of $u_i$ is an arc connecting $\hat{K}_i$ and $\hat{m}_i$, and we denote the image of $u_i$ by $\hat{u}_i$.

Given a $\mathbb{C}$-vector space $V$, a linear map $f : V \to V$, and $\lambda \in \mathbb{C}$, we will use $E(V, f, \lambda)$ to denote the generalized eigenspace of $f$ with eigenvalue $\lambda$.

**Lemma 7.11.** *For all $\lambda \in \mathbb{C}$, we have*

$$\begin{aligned}\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big) \\ = \dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big),\end{aligned}$$

*where the operator $\mu^{\mathrm{orb}}(S^2)$ is the $\mu$-map defined by $\{p\} \times S^2 \subset S^1 \times S^2$ for an arbitrary $p \in S^1$.*

*Proof.* By the torus excision theorem and Lemma 7.10, we have

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big) \cong \mathrm{I}\Big(S^3, L \cup m, \sum_{i=1}^{n-1} u_i\Big) \cong \mathbb{C}^{2^{2n-3}}, \tag{7.8}$$

and

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}\Big) \cong \mathrm{I}\Big(S^3, L' \cup m \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \cong \mathcal{R}^{2^{2n-3}}, \tag{7.9}$$

where $\Gamma_{\hat{U}}$ is the local system associated with $\hat{U}$, and $\Gamma_U$ is the local system associated with $U$. Since the algebraic intersection number of $K_n$ and $S_j$ is zero, we conclude that $\hat{K}_n$ is homotopic to $\hat{U}$ in $S^1 \times S^2 - \bigcup_{i=1}^{n-1} \hat{u}_i$. Let $\Gamma_{\hat{K}_n}$ be the local system associated with $\hat{K}_n$. By Proposition 4.2, we have

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{K}_n}(h)\Big) \cong \mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}(h)\Big) \tag{7.10}$$

for every $h \in \mathbb{C} - \{0\}$ satisfying $(1 - h^2)\theta(h) \neq 0$, and this isomorphism commutes with $\mu^{\mathrm{orb}}(S^2)$. As a consequence, for every $\lambda \in \mathbb{C}$ and $h \in \mathbb{C} - \{0\}$ satisfying $(1 - h^2)\theta(h) \neq 0$, we have

$$\begin{aligned}\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i; \hat{\Gamma}_{\hat{K}_n}(h)\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big) \\ = \dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}(h)\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big).\end{aligned} \tag{7.11}$$

When $h(1-h^2)\theta(h) \neq 0$, the universal coefficient theorem and (7.9), (7.10) imply

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{K}_n}(h)\Big) \cong \mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}(h)\Big) \cong \mathbb{C}^{2^{2n-3}}.$$

On the other hand, notice that when $h = 1$, the local systems $\Gamma_{\hat{K}_n}(h)$ and $\Gamma_{\hat{U}}(h)$ become the trivial system with coefficients $\mathbb{C}$, hence by (7.8) and (7.9),

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{K}_n}(1)\Big) \cong \mathbb{C}^{2^{2n-3}} \cong \mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}(1)\Big).$$

Therefore the desired result follows from (7.11) by taking the limit $h \to 1$ and invoking Proposition 4.1 (3). ∎

Notice that $\hat{L}' \cup \hat{m}$ is a braid in $S^1 \times S^2$. In fact, the projection of $\hat{L}' \cup \hat{m}$ to $S^1$ is a diffeomorphism on each component. Therefore, after an isotopy, we may write $S^1 \times S^2$ as $A_0 \cup_{S^1 \times S^1} A_1$, where $A_0, A_1$ are diffeomorphic to $S^1 \times D^2$, such that

(1) $\hat{K}_1$, $\hat{m}_1$ are included in $A_0$ and are given by $S^1 \times \{p_1\}$ and $S^2 \times \{p_2\}$ with $p_1, p_2$ in $D^2$,

(2) $\hat{u}_1$ is an arc connecting $\hat{K}_1$ and $\hat{m}_1$, and $\hat{u}_1$ is included in $A_0$;

(3) $\hat{K}_2, \ldots, \hat{K}_n, \hat{m}_2, \ldots, \hat{m}_{n-1}, \hat{U}$ are included in $A_1$.

Let

$$\mathcal{L}_0 := \bigcup_{i=2}^{n-1} \hat{K}_i \cup \bigcup_{i=2}^{n-1} \hat{m}_i \cup \hat{K}_n, \tag{7.12}$$

$$\mathcal{L}_1 := \bigcup_{i=2}^{n-1} \hat{K}_i \cup \bigcup_{i=2}^{n-1} \hat{m}_i \cup \hat{U}. \tag{7.13}$$

Then $\mathcal{L}_0$ and $\mathcal{L}_1$ are two annular links in $A_1$. By the definition of annular instanton Floer homology, we have

$$\mathrm{AHI}(\mathcal{L}_0) \cong \mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \tag{7.14}$$

$$\mathrm{AHI}(\mathcal{L}_1) \cong \mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \hat{u}_1). \tag{7.15}$$

**Lemma 7.12.** *Assume there exists a connected oriented Seifert surface $S \subset S^3$ of $L'$ such that $S$ is compatible with the orientation of $L'$, and $S$ has genus $g$ and is disjoint from $K_n$. Then*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g + 2n - 4\Big)$$
$$= \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + 2n - 4\big),$$

*and*

$$E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),j\Big)=0$$

*for all integers* $j>2g+2n-4$.

*Proof.* After an isotopy, we may assume that $S$ intersects each $m_i$ transversely in one point. The image of $S-\bigcup_{i=1}^{n-1}N(K_i)$ in $S^1\times S^2$ is a connected surface with $n-1$ boundary components, where the boundary components are given by the meridians of $\hat{K}_1,\ldots,\hat{K}_{n-1}$. Therefore we can glue disks to the boundary of the image of $S-\bigcup_{i=1}^{n-1}N(K_i)$ and obtain a connected closed surface in $S^1\times S^2$ with genus $g$ that is disjoint from $\hat{K}_n$ and intersects each of $\hat{K}_1,\ldots,\hat{K}_{n-1},\hat{m}_1,\ldots,\hat{m}_{n-1}$ transversely in one point. Denote this surface by $\hat{S}$. After a further isotopy, we may assume that the arcs $\hat{m}_1,\ldots,\hat{m}_{n-1}$ lie on $\hat{S}$.

Recall that $\hat{K}_1$ and $\hat{m}_1$ are contained in $A_0\cong S^1\times D^2$ and are given by $S^1\times\{p_1\}$ and $S^1\times\{p_2\}$ for $p_1,p_2\in D^2$. Take a point $p_0\in D^2-\{p_1,p_2\}$, and let $\hat{K}_0\subset A_0$ be the knot $S^1\times\{p_0\}$. After a further isotopy, we may assume that $\hat{S}$ intersects $\hat{K}_0$ transversely in one point. Let $c$ be a simple closed curve on $D^2$ such that $p_0,p_1$ are inside $c$ and $p_2$ is outside. Let $T_1\subset A_0$ be the torus given by $T_1:=S^1\times c$.

Notice that $\hat{S}$ is homologous to the slice of $S^2$ in $S^1\times S^2$, therefore $\mu^{\mathrm{orb}}(\hat{S})=\mu^{\mathrm{orb}}(S^2)$. The surface $\hat{S}$ intersects $\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0$ transversely in $2n-1$ points. Applying [34, Theorem 6.1] to the surface $\hat{S}$, we deduce that the set of eigenvalues of $\mu^{\mathrm{orb}}(S^2)$ on

$$\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0,\sum_{i=1}^{n-1}\hat{u}_i\Big)$$

is included in

$$\{-(2g+2n-3),-(2g+2n-5),\ldots,2g+2n-5,2g+2n-3\}.$$

Consider the triple $(S^1\times S^2,S^1\times\{q_1,q_2\},v)$, where $q_1,q_2\in S^2$ and $v$ is an arc connecting $S^1\times\{q_1\}$ and $S^1\times\{q_2\}$. Let $T_2$ be a torus given by the boundary of a tubular neighborhood of $S^1\times\{q_1\}$. Recall that $T_1\subset A_0$ is the torus $S^1\times c$ as defined above. Applying the torus excision on the triple

$$\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0,\sum_{i=1}^{n-1}\hat{u}_i\Big)\sqcup(S^1\times S^2,S^1\times\{q_1,q_2\},v)$$

along $T_1\cup T_2$ yields

$$\begin{aligned}
&\dim_{\mathbb{C}}E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),\lambda\Big)\\
&\qquad=\dim_{\mathbb{C}}E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),\lambda-1\Big)\\
&\qquad\quad+\dim_{\mathbb{C}}E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),\lambda+1\Big)
\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Therefore

$$\begin{aligned}&\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g + 2n - 4\Big)\\ &= \dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g + 2n - 3\Big) \quad (7.16)\end{aligned}$$

and

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0 \quad (7.17)$$

for all integers $j > 2g + 2n - 4$.

Similarly, applying torus excision on the triple

$$(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0, \hat{u}_1) \sqcup (S^1 \times S^2, S^1 \times \{q_1, q_2\}, v)$$

along $T_1 \cup T_2$ yields

$$\begin{aligned}&\dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + 2n - 3\big)\\ &\qquad = \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + 2n - 4\big). \quad (7.18)\end{aligned}$$

Let $\{z_1, \dots, z_{2n-1}\} \subset \hat{S}$ be the intersection of $\hat{S}$ with $\hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0$. Apply the singular excision theorem [34, Theorem 6.4] on the triple

$$(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0, \hat{u}_1) \sqcup \Big(S^1 \times \hat{S}, S^1 \times \{z_1, \dots, z_{2n-1}\}, \sum_{i=2}^{n-1} \hat{u}_i\Big)$$

along $\hat{S}$ in the first component, and a slice of $\hat{S}$ in the second component, and invoke [34, Proposition 6.7], to obtain

$$\begin{aligned}&\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(\hat{S}_0), 2g + 2n - 3\Big)\\ &\qquad = \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n \cup \hat{K}_0, \hat{u}_1), \mu^{\mathrm{orb}}(\hat{S}_0), 2g + 2n - 3\big). \quad (7.19)\end{aligned}$$

Since $\mu^{\mathrm{orb}}(\hat{S}_0) = \mu^{\mathrm{orb}}(S^2)$, the first part of the lemma is proved by (7.16), (7.18), and (7.19). The second part of the lemma is proved by (7.17). ∎

**Lemma 7.13.** *Let $\mathcal{L}_0 \subset A_1$ be the annular link defined by (7.12). Suppose there exists a meridional surface $S$ (cf. Definition 2.3) with genus $g$ such that $S$ intersects $\mathcal{L}_0$ transversely at $m$ points. Then there exists a connected Seifert surface $\hat{S}$ of $L'$ in $S^3$ such that $\hat{S}$ is compatible with the orientation of $L'$ and is disjoint from $K_n$, and the genus of $\hat{S}$ is equal to*

$$g + m/2 - n + 2.$$

*Proof.* Suppose there is a component $K$ of $\mathcal{L}_0$ whose intersection with $S$ has different signs. Then we can attach a tube to $S$ along a segment of $K$ to decrease the value of $m$ by 2 and increase the value of $g$ by 1. Repeating this process until the number of intersection points of $S$ with each component of $\mathcal{L}_1$ equals the absolute value of their algebraic intersection number, we obtain a new meridional surface $S' \subset A_1$ such that

(1) the genus of $S'$ equals $g + (m - 2n + 4)/2$;

(2) $S'$ intersects each of $\hat{K}_2, \dots, \hat{K}_{n-1}, \hat{m}_2, \dots, \hat{m}_{n-1}$ transversely in one point;

(3) $S'$ is disjoint from $\hat{K}_n$.

Since $S'$ is a meridional surface, by attaching a disk in $A_0$, we can complete the surface $S'$ to a closed surface with the same genus that intersects each of $\hat{K}_1, \dots, \hat{K}_{n-1}$ transversely in one point and is disjoint from $\hat{K}_n$, therefore it gives rise to a Seifert surface of $L'$ in $S^3$ with the same genus that is disjoint from $K_n$, hence the lemma is proved. ■

**Corollary 7.14.** *Let $g_0$ be the smallest integer with the property that there exists a connected oriented Seifert surface $S \subset S^3$ of $L'$ that is compatible with the orientation of $L'$, has genus $g_0$ and is disjoint from $K_n$. Then*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g_0 + 2n - 4\Big) \neq 0,$$

*and*

$$E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0$$

*for all integers $j > 2g_0 + 2n - 4$.*

*Proof.* By Theorem 2.4 and (7.14), there are integers $g, m$ such that there exists a meridional surface in $A_1$ with genus $g$ and intersecting $\mathcal{L}_1$ transversely in $m$ points such that

$$\dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + m\big) > 0. \tag{7.20}$$

Let $g' := g + m/2 - n + 2$. By Lemma 7.13, we may choose $g, m$ such that there exists a connected oriented Seifert surface of $L'$ in $S^3$ that is compatible with the orientation of $L'$, has genus $g'$, and is disjoint from $K_n$. Since $2g + m = 2g' + 2n - 4$, by Lemma 7.12 and (7.20) we have

$$\begin{aligned} &\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g' + 2n - 4\Big) \\ &\qquad = \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + m\big) > 0. \end{aligned} \tag{7.21}$$

By the definition of $g_0$, we have $g_0 \le g'$. On the other hand, the second part of Lemma 7.12 implies that

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0 \tag{7.22}$$

for all integers $j > 2g_0 + 2n - 4$. Therefore by (7.21) and (7.22), we have $g_0 \ge g'$. In conclusion, we must have $g = g_0$, and the lemma follows from (7.21) and (7.22). ■

Replacing $\hat{K}_n$ with $\hat{U}$ in the previous arguments, we also have the following lemma.

**Lemma 7.15.** *Let $g_1$ be the smallest integer with the property that there exists a connected oriented Seifert surface $S \subset S^3$ of $L'$ that is compatible with the orientation of $L'$, has genus $g_1$ and is disjoint from $U$. Then*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g_1 + 2n - 4\Big) \neq 0,$$

*and*

$$E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0$$

*for all integers $j > 2g_1 + 2n - 4$.* ∎

*Proof of Proposition 7.9.* It is obvious that the minimal genus $g_1$ in Lemma 7.15 is zero, therefore by Lemma 7.11, Corollary 7.14, and Lemma 7.15, the genus $g_0$ in Corollary 7.14 is also zero. As a result, there exists a connected oriented Seifert surface $S \subset S^3$ for $L'$ with genus zero that is disjoint from $K_n$ and is compatible with the orientation of $L'$. Since the minimal-genus Seifert surface for an oriented fibered link is unique up to isotopy, we conclude that there exists an ambient isotopy of $S^3$ that fixes $L'$ and takes $S$ to $S_j$. This ambient isotopy gives the desired isotopy from $K_n$ to $K_n'$. ∎

## 8. The fundamental group of $\mathbb{R}^3 - L'$

This section takes a detour to study the properties of $\pi_1(\mathbb{R}^3 - L')$. The results in this section (or more precisely, Lemma 8.12 and Corollary 8.14) will be used in the proof of the non-existence of $L$.

By the Wirtinger presentation, $\pi_1(\mathbb{R}^3 - L')$ is generated by

$$g_1, \dots, g_{n-1}, g_2', \dots, g_{n-2}'$$

as shown in Figure 13, where the basepoint is taken to be a point above and far away from the diagram. Notice that $g_i'$ and $g_i$ are homotopic relative to the basepoint because one

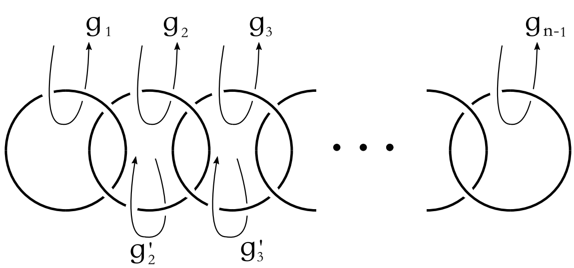


**Fig. 13.** Generators of $\pi_1(\mathbb{R}^3 - L')$.

can shrink $K_1 \cup \cdots \cup K_{i-1}$ into a small neighborhood of $K_i$. Therefore $\pi_1(\mathbb{R}^3 - L')$ is generated by $g_1, \dots, g_{n-1}$, and the Wirtinger presentation gives

$$\pi_1(\mathbb{R}^3 - L') = \langle g_1, \dots, g_{n-1} \mid [g_i, g_{i+1}] = 1 \text{ for } i = 1, \dots, n-2\rangle.$$

To simplify notation, for the rest of this section we will use $m$ to denote $n-1$, and use $G$ to denote the group $\pi_1(\mathbb{R}^3 - L')$. For $i = 1, \dots, m$, define the set

$$C_i := \begin{cases} \{g_1, g_2, g_1^{-1}, g_2^{-1}\} & \text{if } i = 1, \\ \{g_{i-1}, g_i, g_{i+1}, g_{i-1}^{-1}, g_i^{-1}, g_{i+1}^{-1}\} & \text{if } i = 2, \dots, m-1, \\ \{g_{m-1}, g_m, g_{m-1}^{-1}, g_m^{-1}\} & \text{if } i = m. \end{cases}$$

The first part of this section solves the word problem for $G$.

**Definition 8.1.** A *word* is a sequence $(x_1, \dots, x_N)$ such that

$$x_i \in \{g_1, \dots, g_m, g_1^{-1}, \dots, g_m^{-1}\} \quad \text{for all } i.$$

We call $x_1, \dots, x_N$ the *letters* of the word $(x_1, x_2, \dots, x_N)$.

**Definition 8.2.** The word $(x_1, x_2, \dots, x_N)$ is called *reduced* if for every pair $u < v$ with $(x_u, x_v) = (g_i, g_i^{-1})$ or $(g_i^{-1}, g_i)$, there exists $w$ such that $u < w < v$ and $x_w \notin C_i$.

**Definition 8.3.** Define an equivalence relation $\sim$ on the set of words using the following relations as generators:

$$\begin{aligned} (x_1, \dots, x_k, g_i, g_{i+1}, x_{k+3}, \dots, x_N) &\sim (x_1, \dots, x_k, g_{i+1}, g_i, x_{k+3}, \dots, x_N), \\ (x_1, \dots, x_k, g_i^{-1}, g_{i+1}, x_{k+3}, \dots, x_N) &\sim (x_1, \dots, x_k, g_{i+1}, g_i^{-1}, x_{k+3}, \dots, x_N), \\ (x_1, \dots, x_k, g_i, g_{i+1}^{-1}, x_{k+3}, \dots, x_N) &\sim (x_1, \dots, x_k, g_{i+1}^{-1}, g_i, x_{k+3}, \dots, x_N), \\ (x_1, \dots, x_k, g_i^{-1}, g_{i+1}^{-1}, x_{k+3}, \dots, x_N) &\sim (x_1, \dots, x_k, g_{i+1}^{-1}, g_i^{-1}, x_{k+3}, \dots, x_N). \end{aligned}$$

It is straightforward to verify that if two words are equivalent and one of them is reduced, then the other is also reduced. Therefore $\sim$ defines an equivalence relation on the set of reduced words.

Every word $(x_1, \dots, x_N)$ represents an element of $G$ by taking the product $x_1 \cdots x_N$. By the definition of $G$, equivalent words represent the same element.

**Proposition 8.4.** *Every element of $G$ is represented by a reduced word. Two reduced words represent the same element if and only if they are equivalent.*

*Proof.* Define another group $\widetilde{G}$ as follows. The elements of $\widetilde{G}$ are the equivalence classes of reduced words. If $(x_1, \dots, x_N)$ is a reduced word, we use $[x_1, \dots, x_N] \in \widetilde{G}$ to denote the equivalence class of $(x_1, \dots, x_N)$. Let $(x_1, \dots, x_N)$ be a reduced word, and let $y \in \{g_1, \dots, g_m, g_1^{-1}, \dots, g_m^{-1}\}$. If the word $(x_1, \dots, x_N, y)$ is reduced, define

$$[x_1, \dots, x_N] \cdot [y] := [x_1, \dots, x_N, y]. \tag{8.1}$$

If the word $[x_1, \dots, x_N, y]$ is not reduced, then there exists $u$ such that $x_u y = 1$ and every letter in $(x_{u+1}, \dots, x_N)$ commutes with both $x_u$ and $y$. In this case, define

$$[x_1, \dots, x_N] \cdot [y] := [x_1, \dots, x_{u-1}, x_{u+1}, \dots, x_N]. \tag{8.2}$$

For different choices of $x_u$, the right-hand side of (8.2) gives the same equivalence class. Moreover, if we take a different representative of $[x_1, \dots, x_N]$, the right-hand sides of (8.1) and (8.2) remain the same. It is also straightforward to verify that if $y_1$ and $y_2$ are commutative generators of $G$, then

$$[[x_1, \dots, x_N] \cdot y_1] \cdot y_2 = [[x_1, \dots, x_N] \cdot y_2] \cdot y_1. \tag{8.3}$$

Therefore, we obtain a well-defined product operator on $\widetilde{G}$ defined inductively by

$$[x_1, \dots, x_N] \cdot [y_1, \dots, y_M] := [[x_1, \dots, x_N] \cdot [y_1, \dots, y_{M-1}]] \cdot y_M.$$

We show that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{8.4}$$

for all $a, b, c \in \widetilde{G}$ by induction on the length of the reduced words representing $c$.

If $c$ is given by a word with length 1, write $b = [x_1, \dots, x_N]$ and $c = [y_1]$. We discuss two cases. If $(x_1, \dots, x_N, y_1)$ is a reduced word, then (8.4) follows from the definition. If $(x_1, \dots, x_N, y_1)$ is not reduced, let $y_1^{-1}$ be the reciprocal of $y_1$; then there exists $(x'_1, \dots, x'_{N-1})$ such that $b = [x'_1, \dots, x'_{N-1}, y_1^{-1}]$. Let $b' := [x'_1, \dots, x'_{N-1}]$. By definition and (8.3),

$$(a \cdot b) \cdot c = ((a \cdot b') \cdot y_1^{-1}) \cdot y_1 = a \cdot b' = a \cdot (b \cdot c).$$

Hence (8.4) is proved if $c$ is given by a word with one letter.

In general, if $c = [x_1, \dots, x_N]$ with $N \geq 2$, let $c' := [x_1, \dots, x_{N-1}]$. By definition and the induction hypothesis,

$$a \cdot (b \cdot c) = a \cdot ((b \cdot c') \cdot x_N) = (a \cdot (b \cdot c')) \cdot x_N = ((a \cdot b) \cdot c') \cdot x_N = (a \cdot b) \cdot c.$$

In conclusion, we have proved that $\widetilde{G}$ is associative.

For an element $[x_1, \dots, x_N] \in \widetilde{G}$, we have $[x_1, \dots, x_N] \cdot [x_N^{-1}, \dots, x_1^{-1}] = 1$, so every element in $\widetilde{G}$ has an inverse, therefore $\widetilde{G}$ is a group.

By the universal property, there is a unique homomorphism $\varphi$ from $G$ to $\widetilde{G}$ defined by $\varphi(g_i) := [g_i]$. We also have a map $\psi$ from $\widetilde{G}$ to $G$ defined by

$$\psi([x_1, \dots, x_N]) := x_1 \cdots x_N.$$

Since

$$\psi([x_1, \dots, x_N] \cdot [y_1, \dots, y_M]) = \psi([x_1, \dots, x_N]) \cdot \psi([y_1, \dots, y_M]),$$

the map $\psi$ is a group homomorphism. It is obvious from the definitions that $\varphi$ and $\psi$ are inverse to each other, therefore $\varphi$ and $\psi$ are isomorphisms, and the proposition is proved. ∎

**Definition 8.5.** If $(x_1, \ldots, x_N)$ is a reduced word and $w = x_1 \cdots x_N \in G$, we call $x_1 \cdots x_N$ a *reduced presentation* of $w$.

**Definition 8.6.** For $w \in G$, define length$(w)$ to be the length of a reduced presentation of $w$.

By Proposition 8.4, length$(\cdot)$ does not depend on the choice of the reduced presentation, so it is well-defined.

**Lemma 8.7.** *For $C \subset \{g_1, \ldots, g_m, g_1^{-1}, \ldots, g_m^{-1}\}$, let $G_C$ be the subgroup of $G$ generated by $C$. Suppose $(x_1, \ldots, x_N)$ is a reduced word. Then $x_1 \cdots x_N \in G_C$ if and only if $x_i \in C \cup C^{-1}$ for all $i$.*

*Proof.* Let $w = x_1 \cdots x_N$. Suppose $w \in G_C$. Then $w = y_1 \cdots y_M$ with $y_i \in C \cup C^{-1}$ for all $i$. If $(y_1, \ldots, y_M)$ is not reduced, there exist letters $y_u$ and $y_v$ such that $y_u y_v = 1$ and every letter between $y_u$ and $y_v$ commutes with both $y_u$ and $y_v$. Removing $y_u$ and $y_v$ from the word yields a shorter word representing the same element of $G$. Repeating this process, we obtain a reduced word representing $w$ which is given by a subsequence of $y_1, \ldots, y_M$. By Proposition 8.4, this word is equivalent to $(x_1, \ldots, x_N)$, and hence $x_i \in C \cup C^{-1}$ for all $i$.

The other direction of the lemma is obvious. ∎

**Lemma 8.8.** *For each $i$, the centralizer of $g_i$ in $G$ is generated by $C_i$.*

*Proof.* Suppose there exists an element $w$ in the centralizer of $g_i$ that is not generated by $C_i$; choose such a $w$ with $N :=$ length$(w)$ smallest possible. Let $w = x_1 \cdots x_N$ be a reduced presentation of $w$. Then there exists $u$ such that $x_u \notin C_i$. If $x_1 \in C_i$, then $x_2 \cdots x_N$ is an element in the centralizer of $g_i$, and by Lemma 8.7, the element $x_2 \cdots x_N$ is not generated by $C_i$, which contradicts the minimality of $N$. Therefore $x_1 \notin C_i$. Similarly, $x_N \notin C_i$. Moreover, the same property holds for every reduced word that is equivalent to $(x_1, \ldots, x_N)$. Therefore $(x_1, \ldots, x_N, g_i, x_N^{-1}, \ldots, x_1^{-1})$ is a reduced word. By Proposition 8.4, $x_1 \cdots x_N g_i x_N^{-1} \cdots x_1^{-1} \neq g_i$, therefore $w$ is not in the centralizer of $g_i$, contradicting the assumption. ∎

**Lemma 8.9.** *If $m \geq 4$, then the only element that commutes with both $g_1$ and $g_m$ is $1$.*

*Proof.* The lemma is an immediate consequence of Proposition 8.4, Lemma 8.7 and Lemma 8.8. ∎

**Lemma 8.10.** *Suppose $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ are reduced words such that $x_1$, $y_1$ do not commute. Then $x_1 \cdots x_N \neq y_1 \cdots y_N$.*

*Proof.* By Proposition 8.4, we only need to show that the two words $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ are not equivalent. Assume the contrary, and let $w_x$ be the word obtained by removing all the letters that are not equal to $x_1$ or $y_1$ from $(x_1, \ldots, x_N)$. Similarly, let $w_y$ be the word obtained by removing all the letters that are not equal to $x_1$ or $y_1$ from $(y_1, \ldots, y_N)$. Since $x_1$ and $y_1$ do not commute with each other, $w_x$ must be equal to $w_y$

if $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ are equivalent. On the other hand, $w_x$ starts with $x_1$, and $w_y$ starts with $y_1$, so $w_x \neq w_y$, a contradiction. ■

**Lemma 8.11.** *Suppose $m \geq 4$. Then the centralizer of $g_1 g_m$ is generated by $g_1 g_m$.*

*Proof.* Let $w = x_1 \cdots x_N$ be an element in the centralizer of $g_1 g_m$, and assume that $(x_1, \ldots, x_N)$ is a reduced word. We use induction on $N$ to show that $w$ is a power of $g_1 g_m$. If $N = 0$, then $w = 1$ and the property is trivial. From now, assume $N > 0$, and assume that the claim is proved when $\mathrm{length}(w) < N$.

By the assumptions on $w$, we have $g_1 g_m w g_m^{-1} g_1^{-1} = w$, so the word

$$(g_1, g_m, x_1, \ldots, x_N, g_m^{-1}, g_1^{-1})$$

is not reduced. Hence there are three possibilities:

(a1) $g_m^{-1}$ is a letter in $(x_1, \ldots, x_N)$, and every letter before the first appearance of $g_m^{-1}$ in $(x_1, \ldots, x_N)$ is in $C_m$;

(a2) $g_m$ is a letter in $(x_1, \ldots, x_N)$, and every letter after the last appearance of $g_m$ in $(x_1, \ldots, x_N)$ is in $C_m$;

(a3) every letter in $(x_1, \ldots, x_N)$ is contained in $C_m$.

Case (a3) implies $[w, g_m] = 1$, therefore $[w, g_1] = 1$, and by Lemma 8.9, $w = 1$. Since we are assuming $N > 0$, case (a3) is impossible.

Similarly, since $g_m^{-1} g_1^{-1} w g_1 g_m = w$, the word

$$(g_m^{-1}, g_1^{-1}, x_1, \ldots, x_N, g_1, g_m)$$

is not reduced. Applying the same argument as before, we conclude that there are two possibilities:

(b1) $g_1$ is a letter in $(x_1, \ldots, x_N)$, and every letter before the first appearance of $g_1$ in $(x_1, \ldots, x_N)$ is in $C_1$;

(b2) $g_1^{-1}$ is a letter in $(x_1, \ldots, x_N)$, and every letter after the last appearance of $g_1^{-1}$ in $(x_1, \ldots, x_N)$ is in $C_1$;

Since $m \geq 4$, we have $C_1 \cap C_m = \emptyset$, therefore (a1) and (b1) are mutually exclusive, and (a2) and (b2) are mutually exclusive. Hence either (a2) and (b1) hold, or (a1) and (b2) hold.

If (a2) and (b1) hold, then $(x_1, \ldots, x_N)$ is equivalent to a reduced word of the form $(g_1, x_2', \ldots, x_{N-1}', g_m)$. Let $w' := x_2' \cdots x_{N-1}'$. Then $[g_1 w' g_m, g_1 g_m] = [w, g_1 g_m] = 1$, and hence $[w', g_m g_1] = 1$. Let $\sigma : G \to G$ be the isomorphism of $G$ defined by $\sigma(g_k) := g_{m+1-k}$. Then $[\sigma(w'), g_1 g_m] = [\sigma(w'), \sigma(g_m g_1)] = 1$. By the induction hypothesis, $\sigma(w')$ is a power of $g_1 g_m$, therefore $w'$ is a power of $g_m g_1$, so $w = g_1 w' g_m$ is a power of $g_1 g_m$.

If (a1) and (b2) hold, then $(x_1, \ldots, x_N)$ is equivalent to a reduced word of the form $(g_m^{-1}, x_2', \ldots, x_{N-1}', g_1^{-1})$, and the result follows from a similar argument. ■

**Lemma 8.12.** *Suppose $m \geq 4$. The solutions $u, v \in G$ to the equation*

$$ug_1u^{-1} \cdot vg_mv^{-1} = g_1g_m \tag{8.5}$$

*are given by*

$$u = (g_1g_m)^k u', \tag{8.6}$$

$$v = (g_1g_m)^k v', \tag{8.7}$$

*where $k \in \mathbb{Z}$, $u'$ is in the centralizer of $g_1$, and $v'$ is in the centralizer of $g_m$.*

**Remark 8.13.** The expressions on the right-hand side of (8.6) and (8.7) are not required to be reduced. For example, we may have $k = 1$, $u' = 1$, $v' = g_m^{-1}$.

*Proof.* It is clear that every pair $(u, v)$ given by (8.6) and (8.7) is a solution to (8.5). To prove the reverse, we use induction on $\mathrm{length}(u) + \mathrm{length}(v)$. If $\mathrm{length}(u) + \mathrm{length}(v) = 0$, then $u = v = 1$, and the result is obvious.

Suppose $\mathrm{length}(u) + \mathrm{length}(v) = N > 0$, and assume the result is proved when $\mathrm{length}(u) + \mathrm{length}(v) < N$. We can write $u$ as $u = u_1u_2$ with the following properties:

- $\mathrm{length}(u) = \mathrm{length}(u_1) + \mathrm{length}(u_2)$;
- $u_2$ is in the centralizer of $g_1$;
- $u_1$ does not have a reduced presentation that ends with a letter in $C_1$.

Notice that $(u_1, v)$ is also a solution to (8.5), so the result is proved by the induction hypothesis if $u_2 \neq 1$.

Similarly, we can write $v$ as $v = v_1v_2$ with the following properties:

- $\mathrm{length}(v) = \mathrm{length}(v_1) + \mathrm{length}(v_2)$;
- $v_2$ is in the centralizer of $g_m$,
- $v_1$ does not have a reduced presentation that ends with a letter in $C_m$.

Since $(u, v_2)$ is also a solution to (8.5), the result is proved by the induction hypothesis if $v_2 \neq 1$.

From now on, we assume $u_2 = v_2 = 1$. This implies $u = u_1$, $v = v_1$, so both $ug_1u^{-1}$ and $vg_m^{-1}v^{-1}$ are reduced presentations, and we have

$$\mathrm{length}(ug_1u^{-1}) = 2\,\mathrm{length}(u) + 1,$$

$$\mathrm{length}(vg_m^{-1}v^{-1}) = 2\,\mathrm{length}(v) + 1.$$

As a result,

$$\mathrm{length}(g_1^{-1}ug_1u^{-1}) = 2\,\mathrm{length}(u) + 2 \text{ or } 2\,\mathrm{length}(u),$$

$$\mathrm{length}(g_mvg_m^{-1}v^{-1}) = 2\,\mathrm{length}(v) + 2 \text{ or } 2\,\mathrm{length}(v).$$

By (8.5),

$$g_1^{-1}ug_1u^{-1} = g_mvg_m^{-1}v^{-1},$$

so there are four possibilities:

*Case 1:* $\operatorname{length}(u) = \operatorname{length}(v) + 1$, the expression $g_m v g_m^{-1} v^{-1}$ is reduced, and $g_1^{-1} u g_1 u^{-1}$ is not reduced. By the previous assumption on $u$, the element $u$ cannot be represented by a reduced word that ends with a letter in $C_1$, so for $g_1^{-1} u g_1 u^{-1}$ not to be reduced, $u$ must have a presentation of the form $u = g_1 \hat{u}$, where $\operatorname{length}(\hat{u}) = \operatorname{length}(u) - 1$. Thus

$$g_m v g_m^{-1} v^{-1} = \hat{u} g_1 \hat{u}^{-1} g_1^{-1}. \tag{8.8}$$

Since the left-hand side of (8.8) is reduced, and the right-hand side of (8.8) is given by a word with the same length, the right-hand side of (8.8) is also reduced. Therefore, by Proposition 8.4, the corresponding words given by the two sides of (8.8) are equivalent. By the assumption that $v = v_1$, every reduced presentation of $g_m v g_m^{-1} v^{-1}$ has the property that the product of the first $\operatorname{length}(v) + 1$ terms is $g_m v$. Similarly, since $u = u_1$, every reduced presentation of $\hat{u} g_1 \hat{u}^{-1} g_1^{-1}$ has the property that the product of the first $\operatorname{length}(\hat{u}) + 1$ terms is $\hat{u} g_1$. Therefore

$$g_m v = \hat{u} g_1, \quad g_m^{-1} v^{-1} = \hat{u}^{-1} g_1^{-1}.$$

Hence $g_1 \hat{u} = v g_m$, and

$$g_1 g_m v = g_1 \hat{u} g_1 = v g_m g_1,$$

and so $[g_1 g_m, v g_m] = 1$. By Lemma 8.11, we have $v = (g_1 g_m)^k g_m^{-1}$ for some integer $k$. By the previous equations, $\hat{u} = g_1^{-1} (g_m g_1)^k$, and $u = g_1 \hat{u} = (g_1 g_m)^k$, so the desired result is proved.

*Case 2:* $\operatorname{length}(v) = \operatorname{length}(u) + 1$, and $g_1^{-1} u g_1 u^{-1}$ is reduced, while $g_m v g_m^{-1} v^{-1}$ is not. This case follows from the same argument as Case 1.

*Case 3:* $\operatorname{length}(u) = \operatorname{length}(v)$, and both $g_1^{-1} u g_1 u^{-1}$ and $g_m v g_m^{-1} v^{-1}$ are reduced. This is impossible by Lemma 8.10.

*Case 4:* $\operatorname{length}(u) = \operatorname{length}(v)$, neither $g_1^{-1} u g_1 u^{-1}$ nor $g_m v g_m^{-1} v^{-1}$ is reduced. By the previous assumption that $u = u_1$, for $g_1^{-1} u g_1 u^{-1}$ not to be reduced, $u$ must have a presentation of the form $u = g_1 \hat{u}$, where $\operatorname{length}(\hat{u}) = \operatorname{length}(u) - 1$. Similarly by the assumption that $v = v_1$, there is a presentation of $v$ given by $v = g_m^{-1} \hat{v}$, where $\operatorname{length}(\hat{v}) = \operatorname{length}(v) - 1$. Equation (8.5) gives

$$\hat{u} g_1 \hat{u}^{-1} g_1^{-1} g_m^{-1} \hat{v} g_m \hat{v}^{-1} = 1,$$

therefore

$$\hat{v} g_m \hat{v}^{-1} \hat{u} g_1 \hat{u} = g_m g_1.$$

Let $\sigma : G \to G$ be the isomorphism of $G$ defined by $\sigma(g_k) := g_{m+1-k}$. Then

$$\sigma(\hat{v}) g_1 \sigma(\hat{v})^{-1} \sigma(\hat{u}) g_m \sigma(\hat{u})^{-1} = g_1 g_m.$$

By the induction hypothesis, $\sigma(\hat{v}) = (g_1 g_m)^k \hat{v}'$, $\sigma(\hat{u}) = (g_1 g_m)^k \hat{u}'$, where $k \in \mathbb{Z}$, $\hat{v}'$ is in the centralizer of $g_1$, and $\hat{u}'$ is in the centralizer of $g_m$. Therefore

$$u = g_1 \hat{u} = g_1 (g_m g_1)^k \sigma(\hat{u}') = (g_1 g_m)^k g_1 \sigma(\hat{u}'),$$
$$v = g_m^{-1} \hat{v} = g_m^{-1} (g_m g_1)^k \sigma(\hat{v}') = (g_1 g_m)^k g_m^{-1} \sigma(\hat{v}').$$

Since $g_1\sigma(\hat{u}')$ is in the centralizer of $g_1$, and $g_m^{-1}\sigma(\hat{v}')$ is in the centralizer of $g_m$, the desired result is proved.

In conclusion, every solution of (8.5) can be written as (8.6) and (8.7). ∎

**Corollary 8.14.** *Suppose $m \geq 4$. The solutions $u, v \in G$ to the equation*

$$ug_1u^{-1} \cdot vg_m^{-1} v^{-1} = g_1g_m^{-1} \tag{8.9}$$

*are given by*

$$u = (g_1g_m^{-1})^k u', \tag{8.10}$$

$$v = (g_1g_m^{-1})^k v', \tag{8.11}$$

*where $k \in \mathbb{Z}$, $u'$ is in the centralizer of $g_1$, and $v'$ is in the centralizer of $g_m$.*

*Proof.* Notice that there is an isomorphism $\sigma : G \to G$ defined by $\sigma(g_i) := g_i$ for $i < m$, and $\sigma(g_m) := g_m^{-1}$. Applying $\sigma$ to the formulas of Lemma 8.12 yields the result. ∎

## 9. Arcs on compact surfaces

This section collects several results about arcs on surfaces that will be used later. We first recall the following result of Feustel.

**Proposition 9.1** ([9]). *Let $S$ be a smooth compact surface with boundary, and let $\gamma_1$, $\gamma_2$ be two smoothly embedded arcs in $S$ such that $\gamma_i \cap \partial S = \partial \gamma_i$ and $\gamma_i$ is transverse to $\partial S$ for $i = 1, 2$. Suppose $\gamma_1$ and $\gamma_2$ are homotopic to each other in $S$ relative to $\partial\gamma_1 = \partial\gamma_2$. Then $\gamma_1$ and $\gamma_2$ are isotopic to each other in $S$ relative to $\partial\gamma_1 = \partial\gamma_2$.*

We also need the following result in Section 11.

**Lemma 9.2.** *Let $p \in S^1$, and let $D$ be a closed disk in $(S^1 - \{p\}) \times (0, 1)$. Let $S := S^1 \times [0, 1] - D$ and $\gamma_0 := \{p\} \times [0, 1]$. Let $f_1 : S \to S$ be the Dehn twist along a curve parallel to $S^1 \times \{0\}$, and let $f_2 : S \to S$ be the Dehn twist along a curve parallel to $S^1 \times \{1\}$. Suppose $\gamma$ is an arc on $S$ from $(p, 0)$ to $(p, 1)$. Then there exist integers $u$ and $v$ such that $\gamma$ is isotopic to $f_1^u f_2^v(\gamma_0)$ in $S$ relative to $\partial S$.*

*Proof.* Notice that $S$ can be embedded in $\mathbb{R}^2$. By the Jordan curve theorem, cutting $S$ open along $\gamma$ yields a closed annulus with corners, which is diffeomorphic to the manifold obtained by cutting $S$ open along $\gamma_0$. Hence there exists an orientation-preserving diffeomorphism $\varphi : S \to S$ such that $\varphi(\gamma_0) = \gamma$ and $\varphi|_{\partial S} = \mathrm{id}$. Since

- the mapping class group of $S$ is generated by Dehn twists (see, for example, [2] or [8, Corollary 4.16]);
- every simple closed curve on $S$ is parallel to the boundary;
- Dehn twists along curves parallel to $\partial D$ preserve the isotopy class of $\gamma_0$,

the desired result is proved. ∎

The rest of this section studies arcs on Seifert surfaces.

**Definition 9.3.** Let $L_0$ be a link in $\mathbb{R}^3$, let $S$ be a Seifert surface of $L_0$, and let $\gamma$ be an arc on $S$ such that $\gamma$ intersects $\partial S$ transversely in $S$ and $\gamma \cap \partial S = \partial\gamma$. Define $K(S,\gamma)$ to be the knot in $\mathbb{R} - L_0$ which bounds an embedded disk $D$ in $\mathbb{R}^3$ that intersects $S$ transversely in $\gamma$ and intersects $\partial S$ transversely in $\partial\gamma$.

**Remark 9.4.** Since $K(S,\gamma)$ can be isotoped to a neighborhood of $\gamma$ in $D - \gamma$, the knot $K(S,\gamma)$ satisfying Definition 9.3 is unique up to isotopy in $\mathbb{R}^3 - L_0$. An example of $K(S,\gamma)$ can be constructed as follows. Let $S'$ be an extension of $S$ to a slightly bigger embedded surface such that $S$ is in the interior of $S'$. Let $N(S') \subset \mathbb{R}^3$ be a small neighborhood of the zero section of the normal bundle of $S'$. Then $N(S')$ is a neighborhood of $S$. Let $\pi : N(S') \to S'$ be the bundle projection. Let $\gamma'$ be an extension of $\gamma$ in $S'$. Then $K(S,\gamma)$ can be taken to be the boundary of a neighborhood of $\gamma$ in $\pi^{-1}(\gamma')$.

By definition, $K(S,\gamma)$ is always an unknot in $\mathbb{R}^3$.

**Lemma 9.5.** *Let $S$, $\gamma$ be as in Definition 9.3. Suppose $K$ is a knot in $\mathbb{R}^3 - L_0$ such that $K$ bounds an embedded disk $D$ in $\mathbb{R}^3$. Moreover, assume $D$ intersects both $\partial S$ and $S$ transversely, and that $D \cap S$ is the disjoint union of $\gamma$ and a family of circles. Then $K$ is isotopic to $K(S,\gamma)$ in $\mathbb{R}^3 - L_0$.*

*Proof.* By the assumptions, $\gamma$ is an arc in the interior of $D$, and $D \cap L_0 = \partial(D \cap S) = \partial\gamma$. Let $K'$ be the boundary of a small neighborhood of $\gamma$ in $D$. Then $K$ is isotopic to $K'$ in $D - \gamma$. Since $(D - \gamma) \cap L_0 = \emptyset$, the isotopy remains in $\mathbb{R}^3 - L_0$. By the definition of $K(S,\gamma)$, the knot $K'$ is isotopic to $K(S,\gamma)$ in $\mathbb{R}^3 - L_0$, hence the lemma is proved. ∎

**Lemma 9.6.** *Let $S$, $\gamma$ be as in Definition 9.3. Suppose $L_0$ is a fibered link with respect to the Seifert surface $S$ and with monodromy $f : S \to S$. Then $K(S,\gamma)$ is isotopic to $K(S, f(\gamma))$ in $\mathbb{R}^3 - L_0$.*

*Proof.* By the definition of monodromy, there exists an isotopy $\tau : S \times [0,1] \to \mathbb{R}^3$ such that

- $\tau(x,t)$ is independent of $t$ for $x \in \partial S$;
- $\tau(x,0) = x$ for all $x \in S$;
- $\tau(x,1) = f(x)$ for all $x \in S$.

The map $\tau$ induces an isotopy from $K(S,\gamma)$ to $K(S, f(\gamma))$ in $\mathbb{R}^3 - L_0$ by the family of knots $K(\tau(S,t), \tau(\gamma,t))$. ∎

## 10. The link $L_{u,v}$

This section defines a family of links $L_{u,v}$ and computes their Jones polynomials at $t = -1$. The computation will be used in the proof of the non-existence of the hypothetical link $L$ satisfying Condition 6.9.

**Definition 10.1.** For a pair of integers $(u, v)$ with $u \geq 3$, we define a link $L_{u,v}$ as follows. If $v \geq 0$, define $L_{u,v}$ to be the link given by Figure 14 with $u$ components such that there are $v$ crossings in the dotted rectangle. If $v < 0$, define $L_{u,v}$ to be the link given by Figure 15 with $u$ components such that there are $|v|$ crossings in the dotted rectangle.

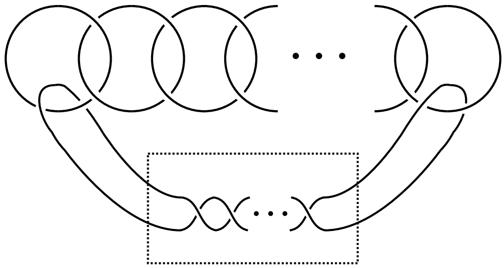

**Fig. 14.** $L_{u,v}$ when $v \geq 0$.

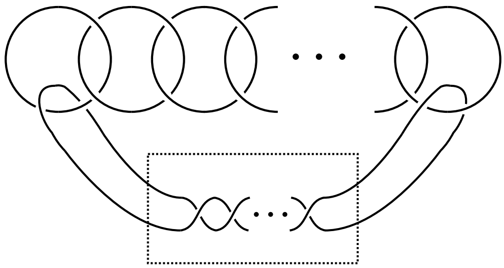

**Fig. 15.** $L_{u,v}$ when $v < 0$.

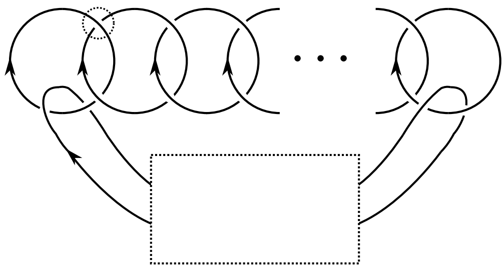

**Fig. 16.** An orientation of $L_{u,v}$.

The only difference between Figures 14 and 15 is that the crossings in the dotted rectangles are reversed. Notice that $L_{u,v}$ is alternating if $v \geq 0$.

Let $V(L_{u,v})$ be the reduced Jones polynomial of $L_{u,v}$ with the orientation given by Figure 16. The Jones polynomial is normalized so that if $U$ is the unknot then $V(U) = 1$. Let $V_{u,v}$ be the value of $V(L_{u,v})$ when plugging in $t^{1/2} = -i$. Although this will not be used in the proofs, we remark that $|V_{u,v}|$ is equal to the determinant of $L_{u,v}$.

Notice that the Hopf link with linking number $+1$ has Jones polynomial $-t^{1/2}-t^{5/2}$, and the Hopf link with linking number $-1$ has Jones polynomial $-t^{-1/2}-t^{-5/2}$. Moreover, the Jones polynomial of the connected sum of links is the product of the Jones polynomials of the summands. Therefore, if $v$ is even, by the skein relation at the dotted circle in Figure 16, we have

$$(t^{1/2}-t^{-1/2})V(L_{u-1,v}) = t^{-1}V(L_{u,v}) - t(-t^{1/2}-t^{5/2})^{u-1}.$$

If $v$ is odd, then the skein relation gives

$$(t^{1/2}-t^{-1/2})V(L_{u-1,v}) = t^{-1}V(L_{u,v}) - t(-t^{1/2}-t^{5/2})^{u-2}(-t^{-1/2}-t^{-5/2}).$$

Hence

$$V_{u,v} = \begin{cases} (2i)V_{u-1,v} + (2i)^{u-1} & \text{if } v \text{ is even,} \\ (2i)V_{u-1,v} - (2i)^{u-1} & \text{if } v \text{ is odd.} \end{cases}$$

On the other hand, if $v$ is even, the skein relation at a crossing in the dotted box in Figure 16 yields

$$(t^{1/2}-t^{-1/2})(-t^{1/2}-t^{5/2})^{u} = t^{-1}V(L_{u,v-2}) - tV(L_{u,v}).$$

If $v$ is odd, then the skein relation gives

$$(t^{1/2}-t^{-1/2})(-t^{1/2}-t^{5/2})^{u-1}(-t^{-1/2}-t^{-5/2}) = t^{-1}V(L_{u,v-2}) - tV(L_{u,v}).$$

Therefore

$$V_{u,v} = \begin{cases} V_{u,v-2} - (2i)^{u+1} & \text{if } v \text{ is even,} \\ V_{u,v-2} + (2i)^{u+1} & \text{if } v \text{ is odd.} \end{cases}$$

It can be directly computed that

$$\begin{aligned} V(L_{3,-1}) &= 2 + t^2 + t^4, \\ V(L_{3,0}) &= t^7 - t^6 + 3t^5 - t^4 + 3t^3 - 2t^2 + t, \end{aligned}$$

therefore

$$V_{3,-1} = 4, \quad V_{3,0} = -12.$$

Combining the computations above, we have

$$V_{u,v} = (-1)^v(2i)^{u-1}(u+2v). \tag{10.1}$$

As a consequence, we have the following result.

**Corollary 10.2.** *If* $|u+2v|>1$, *then* $\operatorname{rank}_{\mathbb{Z}/2}\operatorname{Kh}(L_{u,v};\mathbb{Z}/2) > 2^u$.

*Proof.* Since the coefficients of the Jones polynomial $V(L_{u,v})$ are the Euler characteristics of $\operatorname{Khr}(L_{u,v})$ at different $q$-gradings, we have

$$\operatorname{rank}_{\mathbb{Z}/2}\operatorname{Khr}(L_{u,v};\mathbb{Z}/2) \geq |V_{u,v}| = |2^{u-1}(u+2v)|.$$

If $|u+2v|>1$, then

$$\operatorname{rank}_{\mathbb{Z}/2}\operatorname{Kh}(L_{u,v};\mathbb{Z}/2) = 2\cdot\operatorname{rank}_{\mathbb{Z}/2}\operatorname{Khr}(L_{u,v};\mathbb{Z}/2) > 2^u. \qquad \blacksquare$$

## 11. The non-existence of $L$

This section combines the results from Sections 7–10 to prove that the hypothetical link $L$ satisfying Condition 6.9 does not exist. We will proceed by showing more properties of $L$ and eventually deduce a contradiction. By Lemma 6.10, this will finish the proof of Theorem 1.2.

Recall that the components of $L$ are $K_1, \dots, K_n$, and $L' = K_1 \cup \dots \cup K_{n-1}$. We have defined $S_1$ and $S_2$ to be the Seifert surfaces of $L'$ given by Figures 11 and 12 respectively. By the conditions on the linking numbers of $L$, there are two possibilities:

*Case 1:* The algebraic intersection number of $S_1$ and $K_n$ is zero.

*Case 2:* The algebraic intersection number of $S_2$ and $K_n$ is zero.

By Proposition 7.9, for $j \in \{1, 2\}$, if the algebraic intersection number of $S_j$ and $K_n$ is zero, then $K_n$ can be isotopically deformed in $\mathbb{R}^3 - L'$ into $\mathbb{R}^3 - S_j$. The first half of this section will focus on Case 1. The argument for Case 2 is similar and will be sketched afterwards.

Let $\gamma_0$ be the arc on $S_1$ as shown in Figure 17, where $\gamma_0$ starts from a point $p_1 \in K_1$ and travels from left to right, goes through the crossings of $L'$ in an alternating way, and ends at a point $p_2 \in K_{n-1}$.

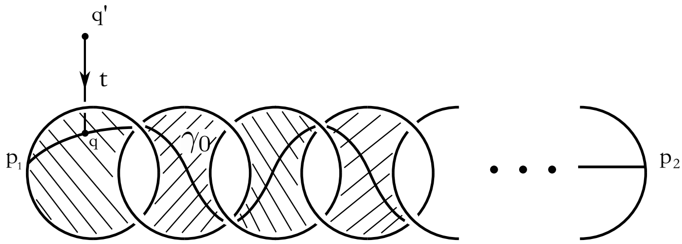


**Fig. 17.** The arc $\gamma_0$ on $S_1$.

**Lemma 11.1.** *Suppose Case 1 holds. Then there exists an arc $\gamma \subset S_1$ from $p_1$ to $p_2$ such that $K_n$ is isotopic to $K(S_1, \gamma)$ in $\mathbb{R}^3 - L'$.*

*Proof.* By Proposition 7.9, there exists a knot $K_n' \subset \mathbb{R}^3 - S_1$ such that $K_n$ is isotopic to $K_n'$ in $\mathbb{R}^3 - L'$. By Proposition 7.1, $K_n'$ bounds a disk $D_n$ such that $D_n$ intersects $K_1$ and $K_{n-1}$ in one point each, and is disjoint from $K_2 \cup \dots \cup K_{n-2}$. After a further isotopy, we may assume that $D_n \cap L' = \{p_1, p_2\}$, and that $D_n$ intersects $S_1$ transversely. Therefore $D_n \cap S_1$ consists of an arc $\gamma \subset S_1$ from $p_1$ to $p_2$ and a union of circles. By Lemma 9.5, $K_n'$ is isotopic to $K(S_1, \gamma)$ in $\mathbb{R}^3 - L'$. ∎

**Lemma 11.2.** *Suppose Case 1 holds. Fix an orientation on $S_1$, let $f_1$, $f_2$ be the Dehn twists on $S_1$ along an oriented curve parallel to $K_1$ and an oriented curve parallel to $K_{n-1}$ respectively, and let $f_3 : S_1 \to S_1$ be the monodromy of the fibered structure of $L'$. Let $\gamma$ be the arc given by Lemma 11.1. Then there exist integers $a, b, c$ such that $\gamma$ is isotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$ on $S_1$.*

*Proof.* If $n = 3$, then $S_1$ is an annulus, and every arc from $p_1$ to $p_2$ is isotopic to $f_1^a\gamma_0$ for some integer $a$. If $n = 4$, then $S_1$ is an annulus with a disk removed, and the result follows from Lemma 9.2 with $c = 0$. From now, we assume $n \geq 5$.

Fix a point $q$ in the interior of $\gamma_0$ as shown in Figure 17. Let $\gamma_1$ be the subarc of $\gamma_0$ from $p_1$ to $q$, and let $\gamma_2$ be the subarc of $\gamma_0$ from $q$ to $p_2$. Then there exists a closed curve $w$ in the interior of $S_1$, based at $q$, such that $\gamma$ is homotopic to $\gamma_1 \cdot w \cdot \gamma_2$ relative to $\{p_1, p_2\}$ on $S_1$. The loop $w$ is not necessarily simple.

Let $g_1, \dots, g_{n-1}$ be the generators of $\pi_1(\mathbb{R}^3 - L', q')$ defined in Section 8, where $q'$ is their basepoint. Fix an arc $t$ from $q'$ to $q$ as given by Figure 17, let $t^{-1}$ be the same arc with the reversed orientation, and let $[w] \in \pi_1(\mathbb{R}^3 - L', q')$ be the homotopy class of $t \cdot w \cdot t^{-1}$.

Every oriented knot in $\mathbb{R}^3 - L'$ defines a conjugacy class in $\pi_1(\mathbb{R}^3 - L', q')$. By Corollary 7.7, the conjugacy class defined by $K_n$ has the form $g_1^a g_{n-1}^b$, where $a, b \in \{-1, 1\}$ depend on the signs of the linking numbers and the orientation of $K_n$. On the other hand, under a suitable orientation, the conjugacy class defined by $K(S_1, \gamma)$ is given by $g_1[w]g_{n-1}^{b'}[w]^{-1}$, where $b' = (-1)^{n+1}$. Therefore, there exists $r \in \pi_1(\mathbb{R}^3 - L', q')$ such that

$$rg_1[w]g_{n-1}^{b'}[w]^{-1}r^{-1} = g_1^a g_{n-1}^b.$$

Comparing the images of both sides in $H_1(\mathbb{R}^3 - L'; \mathbb{Z})$ yields $a = 1$, $b = b'$, thus the equation can be rewritten as

$$rg_1r^{-1} \cdot (r[w])g_{n-1}^{b'}(r[w])^{-1} = g_1 g_{n-1}^{b'}.$$

Applying Lemma 8.12 and Corollary 8.14 for $u = r$, $v = r[w]$, and invoking Lemma 8.8, we have

$$[w] = g_1^\alpha g_2^\beta g_{n-2}^\delta g_{n-1}^\eta$$

for some $\alpha, \beta, \delta, \eta \in \mathbb{Z}$. Notice that the image of $H_1(\text{interior}(S_1); \mathbb{Z})$ in $H_1(\mathbb{R}^3 - L'; \mathbb{Z})$ is generated by $[g_1] + [g_2], [g_2] + [g_3], \dots, [g_{n-2}] + [g_{n-1}]$, therefore we have

$$\alpha - \beta + (-1)^{n-1}\delta + (-1)^n\eta = 0,$$

and hence

$$[w] = (g_1g_2)^\beta g_1^\theta (g_{n-1}^{(-1)^{n-1}})^\theta (g_{n-2}g_{n-1})^\delta, \tag{11.1}$$

where $\theta := \alpha - \beta = (-1)^{n-1}(\eta - \delta)$.

We construct a set of generators of $\pi_1(\text{interior}(S_1), q)$ as follows. Let $u_1, \dots, u_{n-2}$ be the oriented simple closed curves on $S_1$ as given by Figure 18. Each $u_i$ intersects $\gamma_0$ at

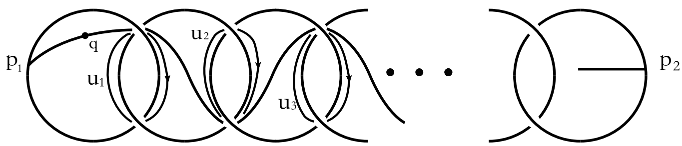


**Fig. 18.** The generators of $\pi_1(\text{interior}(S_1), q)$.

one point near one of the crossings of $L'$. Let $q_i$ be the intersection point of $u_i$ and $\gamma_0$, let $v_i$ be the subarc of $\gamma_0$ from $q$ to $q_i$, and let $v_i^{-1}$ be the same arc with reverse orientation. Let $u_i'$ be the loop based at $q$ defined by $v_i \cdot u_i \cdot v_i^{-1}$. Then $\pi_1(\text{interior}(S_1), q)$ is a free group generated by $[u_1'], \dots, [u_{n-2}']$. Equation (11.1) implies that $w$ is based homotopic to

$$u_1'^{\beta} \cdot (u_1'^{\theta} u_2'^{-\theta} u_3'^{\theta} \cdots u_{n-2}'^{(-1)^{n-1}\theta}) \cdot u_{n-2}'^{\delta} \tag{11.2}$$

in $\pi_1(\mathbb{R}^3 - L', q)$.

Since $\mathbb{R}^3 - L'$ is a fiber bundle over $S^1$ with a fiber being the interior of $S_1$, the map from $\pi_1(\text{interior}(S_1), q)$ to $\pi_1(\mathbb{R}^3 - L', q)$ is injective, hence $w$ is based homotopic to (11.2) in $S_1$. By Lemma 7.8, the monodromy $f_3$ is given by the composition of the Dehn twists along $u_1, \dots, u_{n-2}$. Therefore, under a suitable choice of the orientation for the monodromy $f_3$, the image of $\gamma_0$ under $f_3^c$ is homotopic to $\gamma_1 \cdot (u_1'^{c} u_2'^{-c} u_3'^{c} \cdots u_{n-2}'^{(-1)^{n-1}c}) \cdot \gamma_2$ relative to $\{p_1, p_2\}$, where the alternating signs in front of $c$ come from the fact that the normal vector field of $S_1$ switches directions at each crossing of the diagram. As a consequence, $\gamma$ is homotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$ on $S_1$ with $a = \pm\beta$, $b = \pm\delta$, $c = \theta$, where the signs depend on the orientations of the Dehn twists in the definitions of $f_1, f_2$. By Proposition 9.1, $\gamma_0$ is isotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$ on $S_1$. ■

**Corollary 11.3.** *Under the condition of Case* 1*, the knot $K_n$ is isotopic to $K(S_1, \gamma_0)$ in $\mathbb{R}^3 - L'$.*

*Proof.* Let $f_1, f_2, f_3$ be as in Lemma 11.2. By Lemmas 11.1 and 11.2, there exist integers $a, b, c$ such that $K_n$ is isotopic to $K(S_1, \gamma)$ in $\mathbb{R}^3 - L'$, where $\gamma$ is an arc on $S_1$ that is isotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$. Therefore $\gamma$ is isotopic to $f_3^c(\gamma_0)$ on $S_1$ if we allow its boundary points to move on $\partial S_1$. Hence $K(S_1, \gamma)$ is isotopic to $K(S_1, f_3^c(\gamma_0))$ in $\mathbb{R}^3 - L'$. By Lemma 9.6, $K(S_1, f_3^c(\gamma_0))$ is isotopic to $K(S_1, \gamma_0)$ in $\mathbb{R}^3 - L'$, hence the result is proved. ■

Recall that for a pair of integers $u, v$ with $u \geq 3$, the link $L_{u,v}$ is defined by Definition 10.1.

**Lemma 11.4.** *The link $L' \cup K(S_1, \gamma_0)$ is isotopic to $L_{n,1-n}$.*

*Proof.* Notice that $(S_1, \gamma_0)$ is isotopic to Figure 19. Removing the bands in the dotted boxes in Figure 19 from $S_1$ yields a disk, so the surface $S_1$ is given by a disk with $n - 1$

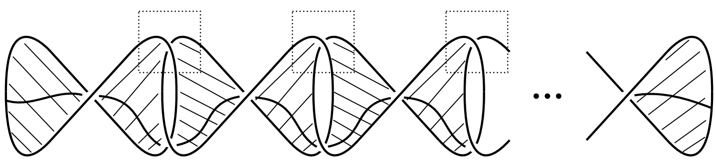

**Fig. 19.** Another diagram for $S_1$ and $\gamma_0$.

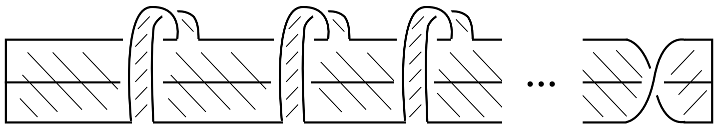

**Fig. 20**

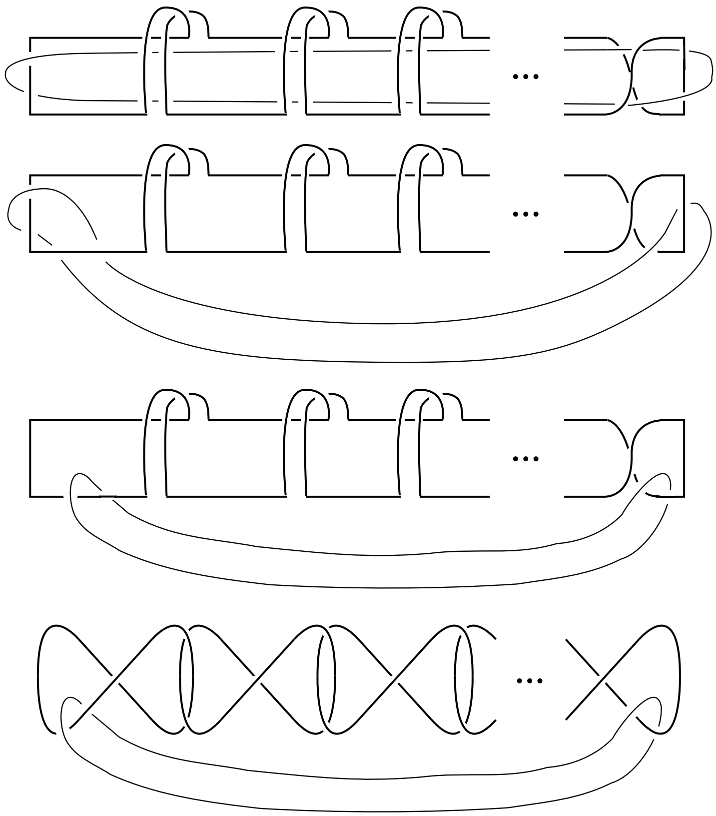

**Fig. 21.** Isotopy from $L' \cup K(S_1, \gamma_0)$ to $L_{n,1-n}$.

bands attached, and one can isotope Figure 19 to Figure 20. Figure 21 then shows an isotopy from $L' \cup K(S_1, \gamma_0)$ to $L_{n,1-n}$. ∎

By Corollary 10.2, if $n \geq 4$, then $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{n,1-n}; \mathbb{Z}/2) > 2^n$. It can be directly verified that $L_{3,-2}$ is isotopic (up to mirror image) to the link L6n1 in the Thistlethwaite link table, and the rank of $\operatorname{Kh}(L_{3,-2}; \mathbb{Z}/2)$ equals 12. Therefore the links $L_{n,1-n}$ all fail to satisfy Condition 6.9 (2). This proves the non-existence of $L$ for Case 1.

To prove the statement for Case 2, let $\gamma_0$ be the arc on $S_2$ given by Figure 22. Then the same argument as for Lemma 11.2 and Corollary 11.3 shows that $K_n$ is iso-

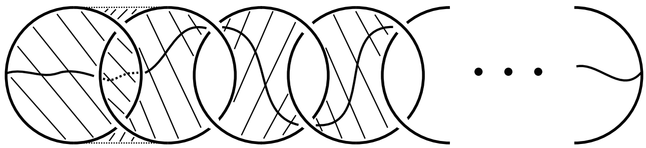

**Fig. 22.** The arc $\gamma_0$ on $S_2$.

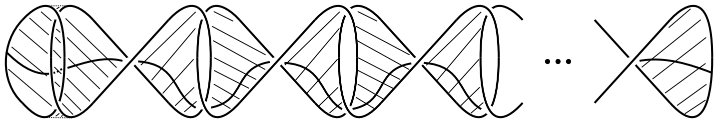

**Fig. 23.** Another diagram for $S_2$ and $\gamma_0$.

topic to $K(S_2, \gamma_0)$ in $\mathbb{R}^3 - L'$. A similar argument to the one for Lemma 11.4 shows that $L' \cup K(S_2, \gamma_0)$ is isotopic to $L_{n,2-n}$. By Corollary 10.2, when $n \geq 6$, we have $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{n,2-n}; \mathbb{Z}/2) > 2^n$. The link $L_{3,-1}$ is isotopic up to mirror image to the link L6n1 in the Thistlethwaite link table, and the rank of $\operatorname{Kh}(L_{3,-1}; \mathbb{Z}/2)$ equals 12. The link $L_{4,-2}$ is isotopic up to mirror image to L8n8, and the rank of $\operatorname{Kh}(L_{4,-2}; \mathbb{Z}/2)$ equals 24. The link $L_{5,-3}$ is isotopic up to mirror image to L10n113, and the rank of $\operatorname{Kh}(L_{5,-3}; \mathbb{Z}/2)$ equals 60. Therefore, $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{n,2-n}; \mathbb{Z}/2) > 2^n$ for all $n$, and this proves the desired result for Case 2.

In conclusion, we have proved that the link $L$ satisfying Condition 6.9 does not exist, therefore Theorem 1.2 follows from Lemma 6.10.

## 12. Algebraic results

In this section, we use algebraic arguments to prove Corollary 1.4, Corollary 1.5 and Theorem 1.9.

*Proof of Corollary* 1.4. Suppose $T$ is a tree with $k$ vertices and let $L_T$ be the forest of unknots given by $T$. If $L_T$ is oriented such that the linking numbers between the components of $L_T$ are all non-negative, then by [1, Corollary 6.6] we have

$$\begin{aligned} P(L_T) &:= \sum_{i,j} t^i q^j \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}^{i,j}(L_T; \mathbb{Z}/2) \\ &= t^{k-1} q^{3(k-1)} (q + q^{-1})(tq^2 + t^{-1}q^{-2})^{k-1}. \end{aligned}$$

Changing the orientation of $L_T$ will change $P(L_T)$ by multiplication by $\pm t^r q^s$, which is a unit in the ring $\mathbb{Z}[t, t^{-1}, q, q^{-1}]$. Theorem 1.2 implies $L_2$ is a forest of unknots. Let $G = T_1 \sqcup \cdots \sqcup T_l$ be the graph of $L_2$, where $T_i$ is a tree with $k_i$ vertices and $n = \sum k_i$. Then the Künneth formula shows that

$$P(L_G) = t^a q^b (q + q^{-1})^l (tq^2 + t^{-1}q^{-2})^{n-l},$$

where $a$ and $b$ are integers depending on the orientaion of $L_2$. Since $q + q^{-1}$ and $tq^2 + t^{-1}q^{-2}$ are irreducible polynomials in the unique factorization domain $\mathbb{Z}[t, t^{-1}, q, q^{-1}]$, the value of $n - l$ is determined by $P(L_G)$. Therefore the first part of the corollary is proved. For the second part, notice that in these four cases the graph for $L_1$ is uniquely determined by the number of edges. ■

**Lemma 12.1.** *Suppose $L$ is an oriented link and $L' = L \cup m$ where $m$ is a meridian near a point $p \in L$. Then*

$$\mathrm{Kh}(L'; \mathbb{Z}/2) \cong \mathrm{Kh}(L; \mathbb{Z}/2) \oplus \mathrm{Kh}(L; \mathbb{Z}/2) \tag{12.1}$$

*as un-graded $\mathbb{Z}/2$-vector spaces. Given any point $r \in L$, this isomorphism intertwines the basepoint operators $X_r$ on the two sides. If $q \in m$, then the isomorphism intertwines $X_q$ on the left-hand side with $X_p$ on the right-hand side.*

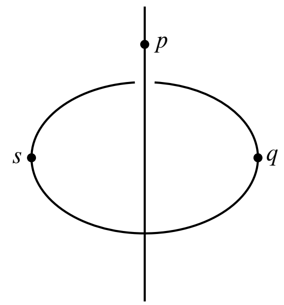


**Fig. 24.** The diagram $D' = D \cup m$

*Proof.* In the proof we set $R = \mathbb{Z}/2$. Let $D$ be a diagram of $L$, and $C$ be the associated Khovanov chain complex. The meridian $m$ is added to $D$ with two new crossings introduced as in Figure 24. We may also assume the point $r$ is away from the region drawn in Figure 24. According to Figure 25, the Khovanov chain complex $C'$ for $D \cup m$ is

$$\begin{tikzcd} C \otimes_R R[X_s]/(X_s^2) \arrow[r, "\mu"] \arrow[d, "\mu"'] & C \arrow[d, "-\Delta"] \\ C \arrow[r, "\Delta"] & C \otimes_R R[X_q]/(X_q^2) \end{tikzcd}$$

where $\mu(\alpha \otimes 1) = \alpha,\ \mu(\alpha \otimes X_s) = X_p\alpha$, and $\Delta(\alpha) = \alpha \otimes X_q + X_p\alpha \otimes 1$. It is clear that the subcomplex

$$\begin{tikzcd} C \otimes_R R\{1\} \arrow[r, "\mu"] \arrow[d, "\mu"'] & C \arrow[d, "-\Delta"] \\ C \arrow[r, "\Delta"] & \operatorname{Im}\Delta \end{tikzcd}$$

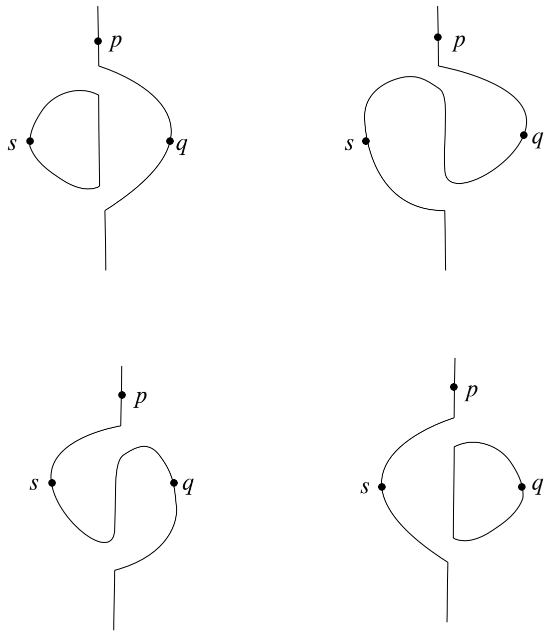


**Fig. 25.** Four resolutions of $D' = D \cup m$.

is acyclic. Therefore the quotient complex $C''$ given by

$$\begin{tikzcd}
C \otimes_R R\{X_s\} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \frac{C \otimes_R R[X_q]/(X_q^2)}{\operatorname{Im} \Delta}
\end{tikzcd}$$

is quasi-isomorphic to $C'$. This quasi-isomorphism respects the actions of $X_p, X_q, X_r$ since the subcomplex is indeed an $R[X]/(X^2)$-submodule for $X = X_p, X_q, X_r$. Now

$$H(C') \cong H(C'') \cong H(C \otimes_R R\{X_s\}) \oplus H\left(\frac{C \otimes_R R[X_q]/(X_q^2)}{\operatorname{Im} \Delta}\right) \cong H(C) \oplus H(C).$$

The actions of $X_p$ and $X_q$ on $C \otimes_R R\{X_s\}$ are the same since $p$ and $q$ lie on the same component of the resolved diagram in Figure 25. On the quotient

$$\frac{C \otimes_R R[X_q]/(X_q^2)}{\operatorname{Im} \Delta}$$

the actions of $X_p$ and $X_q$ are also the same. Therefore the actions of $X_p, X_q$ on $H(C'')$ also coincide. This completes the proof. ■

**Remark 12.2.** The change of grading in (12.1) is computed in [1, Theorem 6.2].

*Proof of Corollary* 1.5. Let $U_n$ be the $n$-component unlink and $H$ be the Hopf link. It is straightforward to calculate that

$$\mathrm{Kh}(U_n;\mathbb{Z}/2) \cong R_n \quad \text{and} \quad \mathrm{Kh}(H;\mathbb{Z}/2) \cong R_2/(X_1 - X_2) \oplus R_2/(X_1 - X_2).$$

For $k \in \mathbb{Z}^+$, let $H_{k-1}$ be a forest of unknots with $k$ components whose graph is a tree. Then for $k \geq 2$, the link $H_{k-1}$ is given by a connected sum of $k-1$ Hopf links. Lemma 12.1 implies that

$$\mathrm{Kh}(H_{k-1};\mathbb{Z}/2) \cong [R_k/(X_1 = \cdots = X_k)]^{\oplus 2^{k-1}}. \tag{12.2}$$

The Khovanov module of the disjoint union of two links is the tensor product of the Khovanov modules of the two links over $\mathbb{Z}/2$. Theorem 1.2 implies $L_2$ is a forest of unknots with a graph $G_2$. It is clear from the above discussion that the module structure of $\mathrm{Kh}(L_2;\mathbb{Z}/2)$ determines the number of vertices in each component of $G_2$, and hence the corollary is proved. ■

Now we prove Theorem 1.9. The proof of Theorem 1.2 does not immediately apply to the case of arbitrary coefficient rings because we have used the equation

$$\mathrm{rank}_{\mathbb{Z}/2} \mathrm{Khr}(L, p;\mathbb{Z}/2) = \tfrac{1}{2} \mathrm{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L;\mathbb{Z}/2)$$

in the proof of Proposition 5.4, and the above equation only holds for $\mathbb{Z}/2$-coefficients. For $\mathbb{Q}$-coefficients, the same proof would only give the following result.

**Theorem 12.3.** *If $L$ is an $n$-component link such that*

$$\begin{aligned} &\mathrm{rank}_{\mathbb{Q}} \mathrm{Kh}(L;\mathbb{Q}) = 2^n, \\ &\mathrm{rank}_{\mathbb{Q}} \mathrm{Khr}(L, p;\mathbb{Q}) = 2^{n-1} \quad \textit{for all basepoints } p, \end{aligned}$$

*then $L$ is a forest of unknots.*

*Sketch of proof.* By Batson–Seed's inequality, we have $\mathrm{rank}_{\mathbb{Q}} \mathrm{Kh}(K;\mathbb{Q}) = 2$ for every component $K$ of $L$. Therefore, Kronheimer–Mrowka's unknot detection theorem implies that every component of $L$ is an unknot. The assumption $\mathrm{rank}_{\mathbb{Q}} \mathrm{Khr}(L, p;\mathbb{Q}) = 2^{n-1}$ implies that $\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) \leq 2^{n-1}$ by Kronheimer–Mrowka's spectral sequence. By (3.12), we also have $\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) \geq 2^{n-1}$, therefore

$$\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) = 2^{n-1} \quad \text{for every basepoint } p \in L.$$

The proof then proceeds as for Theorem 1.2 to reduce to the three links $L_{3,-2}$, $L_{4,-2}$ and $L_{5,-3}$. Since

$$\begin{aligned} \mathrm{rank}_{\mathbb{Q}}(L_{3,-2};\mathbb{Q}) &= 10 > 2^3, \\ \mathrm{rank}_{\mathbb{Q}}(L_{4,-2};\mathbb{Q}) &= 20 > 2^4, \\ \mathrm{rank}_{\mathbb{Q}}(L_{5,-3};\mathbb{Q}) &= 46 > 2^5, \end{aligned}$$

all the three links are eliminated. ■

Using the equivariant Khovanov homology introduced in [16], we have the following lemma.

**Lemma 12.4.** *If $L$ is an $n$-component link such that* $\operatorname{rank}_{\mathbb{Q}} \operatorname{Kh}(L;\mathbb{Q}) = 2^n$*, then*

$$\operatorname{rank}_{\mathbb{Q}} \operatorname{Khr}(L,p;\mathbb{Q}) = 2^{n-1} \quad \textit{for all basepoints } p.$$

*Proof.* Given a diagram $D$ for $L$, a chain complex $\mathcal{F}_3(D)$ of free $\mathbb{Z}[t]$-modules is introduced in [16]. The tensor product $C_t(D) := \mathcal{F}_3(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a chain complex of free $\mathbb{Q}[t]$-modules. Its homology $\mathcal{H}(L)$ is a $\mathbb{Q}[t]$-module called the *equivariant Khovanov homology*. If a basepoint $p \in L$ is chosen, then $C_t(D)$ (hence $\mathcal{H}(L)$) becomes a $\mathbb{Q}[X]$-module, where the action of $X$ depends on $p$ and satisfies $X^2 = t$. The tensor product $C_t(D) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]/(t)$ is the chain complex defining $\operatorname{Kh}(L;\mathbb{Q})$. The tensor product $C_t(D) \otimes_{\mathbb{Q}[X]} \mathbb{Q}[X]/(X)$ is the chain complex defining $\operatorname{Khr}(L,p;\mathbb{Q})$. By [16, Proposition 7] (and the discussion before it), we have

$$\mathcal{H}(L) \cong \mathbb{Q}[t]^{\oplus 2^n} \oplus T,$$

where $T$ is a direct sum of torsion modules of the form $\mathbb{Q}[t]/(t^l)$. Since $\operatorname{Kh}(L;\mathbb{Q}) \cong \mathbb{Q}^{2^n}$, the universal coefficient theorem implies that $T = 0$. Therefore

$$\mathcal{H}(L) \cong \mathbb{Q}[X]^{\oplus 2^{n-1}}$$

as a $\mathbb{Q}[X]$-module since $X^2 = t$. Now applying the universal coefficient theorem again we obtain

$$\operatorname{Khr}(L,p;\mathbb{Q}) \cong \mathbb{Q}^{\oplus 2^{n-1}} \quad \text{for all basepoints } p. \qquad \blacksquare$$

Theorem 12.3 and Lemma 12.4 imply Theorem 1.9.

**Theorem 1.9.** *Suppose $R$ is an integral domain. If $L$ is an $n$-component link such that* $\operatorname{rank}_R \operatorname{Kh}(L;R) = 2^n$*, then $L$ is a forest of unknots.*

*Proof.* By the universal coefficient theorem,

$$2^n = \operatorname{rank}_R \operatorname{Kh}(L;R) \geq \operatorname{rank}_{\mathbb{Z}} \operatorname{Kh}(L;\mathbb{Z}) = \operatorname{rank}_{\mathbb{Q}} \operatorname{Kh}(L;\mathbb{Q}).$$

Therefore by (1.1), we have $\operatorname{rank}_{\mathbb{Q}} \operatorname{Kh}(L;\mathbb{Q}) = 2^n$. Lemma 12.4 and Theorem 12.3 then imply that $L$ is a forest of unknots. $\blacksquare$

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