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Complex powers of the wave operator and the spectral action on Lorentzian scattering spaces

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Abstract. We consider perturbations of Minkowski space as well as more general spacetimes on which the wave operator $\square_g$ is known to be essentially self-adjoint. We define complex powers $(\square_g - i\varepsilon)^{-\alpha}$ by functional calculus, and show that the trace density exists as a meromorphic function of α. We relate its poles to geometric quantities, in particular to the scalar curvature. The results allow us to formulate a spectral action principle which serves as a simple Lorentzian model for the bosonic part of the Chamseddine–Connes action. Our proof combines microlocal resolvent estimates, including radial propagation estimates, with uniform estimates for the Hadamard parametrix. The arguments work in Lorentzian signature directly and do not rely on transition from the Euclidean setting.

Keywords. Spectral zeta functions, microlocal analysis, Lorentzian geometry, hyperbolic partial differential equations, wave equation, Hadamard parametrix

1. Introduction

1.1. Introduction and main result

The relationships between the geometry of compact Riemannian manifolds and the spectral theory of elliptic operators have been a rich ground for discovery for decades, owing to powerful methods based on heat kernel and resolvent expansions, complex powers, residue traces, zeta functions and related notions [9, 56, 86, 90, 97, 101, 103, 110]. They have also profoundly influenced the world of relativistic physics, relying on the presumption that a generalization to Lorentzian manifolds is possible [22, 26, 27, 62, 126]. This generalization was however found to be problematic on many levels. In particular, while it is possible to make sense of, e.g., formal heat kernel coefficients $\{a_i\}$ for the wave operator $\square_g$ on a Lorentzian manifold (M, g) by writing transport equations analogous

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to the Riemannian case, and interpret them in terms of a Lorentzian Hadamard parametrix [21,91], their relation to global objects defined by spectral theory is vastly unclear.

On the other hand, it was recently found that if the spacetime (M,g) has special symmetries, or if instead it is well-behaved at large distances, then it is possible to interpret $\Box_g$ as a self-adjoint operator in the sense of the canonical $L^2(M)$ space defined using the volume form of g. In fact, the essential self-adjointness of $\Box_g$ on *static* spacetimes was proved by Dereziński–Siemssen [34], and on *non-trapping Lorentzian scattering spaces* by Vasy [131]; this was then generalized by Nakamura–Taira to other differential operators of real principal type on long-range perturbations of Minkowski space [93]; cf. [25, 35, 74, 119, 120] for related recent work on self-adjointness of non-elliptic operators. As a consequence of self-adjointness, it is possible to define, e.g., $(\Box_g - i\varepsilon)^{-\alpha}$ for $\varepsilon > 0$ and $\alpha \in \mathbb{C}$ abstractly by functional calculus. However, the relation to the local geometry is then an open question.

In the present paper we demonstrate that globally defined complex powers of $\Box_g$ are in fact related to spacetime geometry in a way that parallels to a large extent the results known in the Riemannian case. We consider the setting of non-trapping Lorentzian scattering spaces introduced by Vasy [131], which assumes that the metric and the null geodesic flow behave asymptotically in a certain way; see Section 2.3. The main feature is that this class contains perturbations of Minkowski space and other asymptotically Minkowski spacetimes (including the class considered in [7,49]), and no particular symmetry of (M,g) nor real analyticity is assumed. We also make the assumption of *global hyperbolicity* of (M,g), which arises naturally in, e.g., the solvability of the Cauchy problem. Our main result is the following theorem.

Theorem 1.1 (cf. Theorem 8.3 and Proposition 8.5). *Assume (M,g) is a globally hyperbolic, non-trapping Lorentzian scattering space (or (M,g) is an ultrastatic spacetime) and assume its dimension n is even. Then for all $\varepsilon > 0$, the Schwartz kernel of $(\Box_g - i\varepsilon)^{-\alpha}$ has for $\operatorname{Re}\alpha > n/2$ a well-defined on-diagonal restriction $(\Box_g - i\varepsilon)^{-\alpha}(x,x)$, which extends as a meromorphic function of $\alpha \in \mathbb{C}$ with poles at $\{n/2, n/2-1, n/2-2, \dots, 1\}$. Furthermore,*

$$\lim_{\varepsilon\to 0^+} \operatorname{res}_{\alpha=n/2-1} (\Box_g - i\varepsilon)^{-\alpha}(x,x) = \frac{R_g(x)}{i6(4\pi)^{n/2}\Gamma(n/2-1)}, \tag{1.1}$$

where $R_g(x)$ is the scalar curvature at $x \in M$.

Theorem 1.1 can be seen as the Lorentzian version of a result attributed to Kastler [76] and Kalau–Walze [75] in the Riemannian case (and announced previously by Connes as a consequence of classical theorems in elliptic theory due to Minakshisundaram–Pleijel [90], Seeley [108], Wodzicki [134] and other authors; see [27, Theorem 1.148] for the heat kernel based argument and [56, Section 1.7] for an approach in the spirit of Atiyah–Bott–Patodi [4]). This type of relationships has been used to justify definitions of curvature in non-commutative geometry: see Connes–Marcolli [27, Definition 1.147] and Connes–Moscovici [29].

The ε regularizer in (1.1) deals with the fact that in contrast to the compact Riemannian setting, $\Box_g$ is not bounded from below. It is also responsible for the relationship to *Feynman inverses*; see Section 1.2.

The importance of (1.1) in physics stems from the fact that the r.h.s. is proportional to the *Einstein–Hilbert Lagrangian*, and the variational principle $\delta_g R_g(x) = 0$ is equivalent to the Einstein equations for g. The l.h.s., on the other hand, refers to the spectral theory of the self-adjoint operator $\Box_g$.

Remark 1.2. Our main cases of interest are perturbations of Minkowski space (in arbitrary spatial and time directions) as well as more general Lorentzian scattering spaces, but the results are also valid for *ultrastatic* spacetimes (M, g) in the sense that $M = \mathbb{R} \times Y$ and $g = dt^2 - h$ for some (t-independent) complete Riemannian manifold (Y, h). In that case, essential self-adjointness follows from [34] and the proof of Theorem 1.1 simplifies considerably; see Remark 1.4.

The residues at the other poles can also be computed, and we show the following result.

Theorem 1.3 (cf. Theorem 8.4). *For any Schwartz function f with Fourier transform supported in $]0, +\infty]$ and any $N \in \mathbb{N}_{\geqslant 0}$, we have for $\varepsilon > 0$ the large $\lambda > 0$ expansion*

$$f((\Box_g + i\varepsilon)/\lambda^2)(x, x) = \sum_{j=0}^{N} \lambda^{n-2j} C_j(f)\, a_j(x) + \mathcal{O}(\varepsilon, \lambda^{n-2N-1}),$$

where each $C_j(f)$ depends only on $j \in \mathbb{N}_{\geqslant 0}$, the space-time dimension n and f, and $a_j(x)$ are directly related to the Hadamard coefficients, in particular $a_0(x) = (4\pi)^{-n/2}$, $C_0(f) = i^{-1} e^{in\pi/4} \int_0^\infty \hat{f}(t) t^{n/2-1} dt$ and

$$a_1(x) = -(4\pi)^{-n/2} \tfrac{1}{6} R_g(x), \quad C_1(f) = i^{-1} e^{i(n-2)\pi/4} \int_0^\infty \hat{f}(t) t^{n/2-2}\, dt.$$

We refer to Theorem 8.4 for a precise calculation of the first three terms of the asymptotic expansion of $f((\Box_g + m^2 + i\varepsilon)/\lambda^2)(x, x)$ in terms of the regularizer ε and also the mass m. This formulation parallels as closely as possible the *spectral action principle* established in the Riemannian case by Chamseddine–Connes [22, 26], which has become a milestone in high energy physics developments driven by the noncommutative geometry program; see e.g. [23, 27, 43, 126, 127]. A feature of Theorem 1.3 is that in contrast to results in the Riemannian setting, we do not allow for functions f supported away from zero or in a half-line: intuitively, the reason is that the bottom of the spectrum plays a rôle which cannot be harmlessly disregarded in the Lorentzian case.

1.2. Structure of proof

The primary difficulty is unquestionably the non-ellipticity of $\Box_g$, which makes known methods from the Riemannian setting inapplicable to our situation. We stress that except at the very final stage (where we work *locally* with quadratic forms on $\mathbb{R}^n$ to compute numerical factors in the residues), our proof does not involve any kind of transition from Euclidean to Lorentzian signature. Instead, we use techniques from partial differential equations and microlocal analysis that emphasize the structure of the null geodesic flow on the cotangent space and its asymptotic behavior; see Section 1.3 for bibliographical remarks.

Theorems 1.1 and 1.3 rely on precise regularity estimates for the resolvent $(\Box_g - z)^{-1}$. The key feature is that for $\operatorname{Im} z > 0$, $(\Box_g - z)^{-1}$ is a *Feynman inverse*, meaning that the singularities of the Schwartz kernel (characterized by its wavefront set) are the same as for Duistermaat–Hörmander's Feynman parametrix [39] and Feynman propagators in quantum field theory and related contexts [49, 102]. In consequence, close to the diagonal in $M \times M$, the Schwartz kernel of $(\Box_g - z)^{-1}$ can be approximated by the Feynman version of the *Hadamard parametrix*, which is sufficiently explicit for the extraction of local geometrical quantities. The complex powers $(\Box_g - i\varepsilon)^{-\alpha}$ are then expressed in terms of the resolvent as integrals over an infinite contour in the complex upper half-plane. To be useful, however, this requires the estimates for the resolvent, parametrix and errors to be *uniform in* z, with sufficient decay along the integration contour. This complicates the analysis of the Hadamard parametrix, since apart from difficulties due to light-cone singularities, there is competition between regularity and decay in $|\operatorname{Im} z|$. It is also worth stressing that it is not possible to eliminate any error term by solving a Cauchy problem for $\Box_g - z$ because the associated retarded and advanced fundamental solutions badly behave as $|\operatorname{Im} z| \to +\infty$.

With these issues in mind, the proofs (in the Lorentzian scattering space case) are organized as follows:

(1) Setting $P = \Box_g$ or $P = \Box_g + m^2$, we use radial estimates in weighted scattering Sobolev spaces (due in the present context to Vasy [131] and generalizing results by Melrose [87]) to derive the mapping properties of the resolvent $(P - z)^{-1}$, uniformly in z. By integrating on a contour γ_ε in the upper half-plane (see Figure 1 in Section 2.6) we deduce the local mapping properties of $(P - i\varepsilon)^{-\alpha}$ in Sobolev spaces.

(2) In Section 3.5, for $\operatorname{Im} z \geqslant 0$ we construct a z-dependent parametrix of $P - z$ which is the sum of two independent parts, each with singularities propagating in only one of the two components of the characteristic set Σ. We show that the parametrix has *Feynman wavefront set* uniformly along the contour γ_ε. This step uses a time-dependent factorization of $P - z$ in Shubin's parameter-dependent pseudo-differential calculus [110] and the hyperbolicity of P.

(3) In Section 3.6 we relate $(P - z)^{-1}$ to the parametrix from step (2). The argument first emphasizes common behavior at the radial sets, and then uses radial estimates and propagation of singularities to obtain a global result. The main conclusion is that $(P - z)^{-1}$ has Feynman wavefront set uniformly in z along γ_ε.

(4) In Sections 4–5 we construct a z-dependent version $H_N(z, \cdot)$ of the Hadamard parametrix of $P - z$, and show in Section 5 that it also has Feynman wavefront set uniformly in z. We prove regularity estimates for $H_N(z, \cdot)$ and the remainders, with

control of the decay for large $|z|$ and the behavior near the real axis. Important prerequisites are Hölder–Zygmund and microlocal estimates shown in Section 4 for an elementary family of distributions $F_\alpha(z,\cdot)$ on $\mathbb{R}^n$, which serves as the building block of the parametrix in normal coordinates.

(5) For $\operatorname{Im} z > 0$, we relate the resolvent $(P-z)^{-1}$ to the uniform Hadamard parametrix $H_N(z,\cdot)$ using the estimates from step (4) and the Feynman form of the wavefront set proved in step (3) by a composition argument. The local analysis of the Schwartz kernel of $(P-i\varepsilon)^{-\alpha}$ and other functions of P is reduced in this way to contour integrals involving $H_N(z,\cdot)$.

(6) The meromorphic continuation of $\alpha \mapsto (P-i\varepsilon)^{-\alpha}$ and its poles are computed on the level of contour integrals of $H_N(z,\cdot)$. To compute the residues we use a homological argument which can be interpreted as a local Wick rotation of quadratic forms. Theorem 1.3 is deduced from the full version of Theorem 1.1 by a Mellin transform argument.

Various auxiliary proofs are collected in the appendices.

We stress that although the occurrence of local geometric quantities in the Hadamard parametrix is a well-known phenomenon, the relationship to globally defined functions of P proved in steps (1)–(6) is new.

Remark 1.4. The case of (M,g) ultrastatic is simpler because one can then give a quasi-explicit formula for $(P-z)^{-1}$ in terms of the Laplace–Beltrami operator Δ_h on the Cauchy surface. The formula implies that $(P-z)^{-1}$ is already of the form of the parametrix in step (2), so step (3) is no longer needed, and resolvent estimates can be derived directly; see Appendix C.1. From that point on, steps (4)–(6) apply verbatim.

1.3. Bibliographical remarks

The construction of complex powers of elliptic operators is due to Seeley [108] in the case of classical pseudo-differential operators on compact manifolds, and was extended to various other elliptic settings, among others in works by Rempel–Schulze [104], Guillemin [59], Grubb [58], Schrohe [106, 107], Loya [81, 82], Coriasco–Schrohe–Seiler [30] and Ammann–Lauter–Nistor–Vasy [2]; cf. recent work by Hintz [64]. Various results in the spirit of the Kastler–Kalau–Walze identity were obtained, e.g., by Ponge [98–100] (in particular, [100] discusses lower-dimensional geometric invariants) and Battisti–Coriasco [8]. The residues of the spectral zeta function have a natural interpretation in terms of the Guillemin–Wodzicki residue [59, 134] (cf. Connes–Moscovici [28], Lesch [78], Lesch–Pflaum [79], Paycha [96, 97], Maeda–Manchon–Paycha [83]); see [32] for a generalization to the Lorentzian case considered here.

Complex powers of non-elliptic first order pseudo-differential operators were obtained as paired Lagrangian distributions by Antoniano–Uhlmann [3]; see also Greenleaf–Uhlmann [57, Section 3]. Using the calculus of paired Lagrangian distributions, complex powers of the wave operator corresponding to a retarded or advanced problem were constructed by Joshi [73] in the case of time-independent coefficients. Enciso–González–Vergara [44] later showed that a particular fractional power coincides with the Dirichlet-to-Neumann map on static anti-de Sitter spacetimes. We remark that our complex powers are different from Joshi's, as the former are associated to a self-adjoint operator and are related to a Feynman problem rather than to a retarded or advanced one. However, it can be conjectured that they are paired Lagrangian distributions as well.

The approach to Lorentzian complex powers in the present paper builds on the non-elliptic Fredholm theory introduced by Vasy [128], originally in the context of the retarded and advanced problem on (Kerr–)de Sitter spaces, and further developed in a series of works tailored to the study of wave and Einstein equations (see e.g. [7, 63, 65, 66]), culminating in the resolution of the Kerr–de Sitter stability conjecture by Hintz–Vasy [67] and the proof of linear stability of Kerr black holes by Häfner–Hintz–Vasy [61]. The global approach to the Feynman problem for the wave equation on a class of Lorentzian scattering spaces was pioneered by Gell-Redman–Haber–Vasy [49] (including a non-linear version); cf. Baskin–Vasy–Wunsch [7] for previous work on the retarded and advanced problem in that setting. The construction also applies to the Klein–Gordon operator on de Sitter spaces, and in both settings, its positivity and microlocal properties were studied by Vasy [129] and Vasy–Wrochna [133]. The Feynman invertibility of the Klein–Gordon operator $\Box_g + m^2$ with $m > 0$ on asymptotically Minkowski spacetimes was proved by Gérard–Wrochna [52, 53] (cf. [54] for a brief account) using an approximate diagonalization of the evolution, related to the parametrix in Section 3.5 (though the focus here is on the behavior in z). In the already mentioned work of Vasy [131] on essential self-adjointness of $\Box_g$, the resolvent is constructed in terms of a Feynman problem which coincides with the Gérard–Wrochna definition by a result of Taira [121]. Vasy [131] also shows a limiting absorption principle for $\Box_g + m^2$, followed by an improvement by Taira [121]; cf. the earlier work of Dereziński–Siemssen [34] for the limiting absorption principle in the static case (possibly with electromagnetic potentials).

Related developments connecting the global theory of hyperbolic operators with space-time geometry have included a Lorentzian Atiyah–Patodi–Singer index theorem due to Bär–Strohmaier [5]; see also Braverman [18] for a spatially non-compact generalization, and very recently a local version was shown by Bär–Strohmaier [6]. Furthermore, Strohmaier–Zelditch proved a Gutzwiller–Duistermaat–Guillemin trace formula and a Weyl law for time-like Killing vector fields on stationary space-times [116–118], which in particular provides a spectral-theoretic way of recovering the scalar curvature and thus a spectral action in the stationary case. It is worth emphasizing that Feynman inverses appear naturally in all these developments (and in [6] the relationship to the Hadamard parametrix is used; see below).

We remark that non-elliptic Fredholm problems and radial estimates have arisen in many contexts outside relativistic settings; see e.g. [40, 42, 50]. In particular, we emphasize similarities to the work of Dyatlov–Zworski [40] on Anosov flows, which proves the meromorphic continuation of the Ruelle zeta function using microlocal resolvent estimates; we expect that semi-classical methods could provide useful alternatives to the arguments in Section 3.6.

The Hadamard parametrix for Laplace–Beltrami operators on pseudo-Riemannian manifolds is a classical tool in analysis; see e.g. [69, 112, 137] for textbook accounts focused on the Riemannian or Lorentzian time-independent case. This parametrix plays a fundamental rôle in quantum field theory on curved spacetimes, where it is used to subtract singularities from N-point functions to get well-defined non-linear quantities [20,37, 46,68,77,91,102]. In particular, Radzikowski [102] proved a relationship between the bisolution Hadamard parametrix and the Feynman parametrix of Duistermaat–Hörmander. In the present article we work directly with the (z-dependent) Feynman version of the Hadamard parametrix. Its analogue for fixed z was constructed by Zelditch [136] in the ultrastatic case, and by Lewandowski [80] and Bär–Strohmaier [6] in the general case using a family of distributions with distinguished wavefront set (the former construction also gives a unified treatment of even and odd dimensions). The Hadamard parametrix is also useful in spectral theory, and was applied e.g. by Sogge [111], Dos Santos Ferreira–Kenig–Salo [38] and Bourgain–Shao–Sogge–Yao [17] in the context of L^p resolvent estimates on compact Riemannian manifolds (including estimates uniform in the spectral parameter z), and by Zelditch [136] in the problem of analytic continuation of eigenfunctions.

Finally, we mention, only very non-exhaustively, works in noncommutative geometry aimed at establishing a Lorentzian theory [14, 15, 31, 36, 45, 92, 95, 115, 124, 125]. In contrast to the problem considered here, their focus is mostly on the formalism of spectral triples or on distance formulæ. In the last few years this has included progress on spectral actions by D'Andrea–Kurkov–Lizzi [31], Devastato–Farnsworth–Lizzi–Martinetti [36] and Martinetti–Singh [84], which involves however transition from Euclidean signature and relies on special symmetries or analyticity, very differently from the present paper's result.

1.4. Remarks on assumptions; outlook

The essential self-adjointness in [131] and the results of the present paper extend in a straightforward way to the Hermitian bundle setting provided that the principal symbol of P is a scalar wave operator and formal self-adjointness of P holds true for a *positive* scalar product. We further comment on this in Remark 3.21.

The assumptions on spacetime geometry in Theorems 1.1 and 1.3 are not expected to be sharp. In fact, only steps (1) and (3) of the proofs use the hypothesis that (M, g) is a Lorentzian scattering space, and one could try to adapt the arguments depending on estimates available in a given class of spacetimes. For instance, in view of the estimates in [128], a natural candidate could be the class of asymptotically de Sitter spacetimes. The essential self-adjointness of $\Box_g$ is also conjectured to be true for asymptotically static spacetimes (see Dereziński–Siemssen [33, Sections 5.8 and 8.6]), and it is therefore natural to ask whether (1) and (3) remain valid in that general setting.

A mathematically delicate point is the $\varepsilon \to 0^+$ limit of the Schwartz kernel of $(P - i\varepsilon)^{-\alpha}$ and of other functions of $P - i\varepsilon$. Namely, observe that in Theorems 1.1 and 1.3 we *first* compute a residue or an expansion and *then* take the $\varepsilon \to 0^+$ limit, but one could ask whether the order of these operations can be reversed. We give an affirmative answer in the setting of Theorem 1.3 if $\Box_g$ is replaced by $P = \Box_g + m^2$ with $m > 0$ (this then produces extra terms in the expansion which vanish as $m \to 0^+$; see Theorem 8.4) – this, however, requires stronger assumptions including *non-trapping at energy* m^2; see Sections 2.3 and 2.7. For the sake of illustration, we also prove in Appendix C.2 a limiting absorption principle for $(P - i\varepsilon)^{-\alpha}$ in the case of space-compact ultrastatic spacetimes, with similar conclusions on the possibility of taking the $\varepsilon \to 0^+$ limit before computing residues.

The study of the $\varepsilon \to 0^+$ limit with $m = 0$ rather than $m > 0$ requires a different approach, based for instance on recent work by Vasy [132]; cf. Bouclet–Burq [16].

Finally, we do not consider here functions of Dirac operators or generalizations needed to derive a spectral action principle for the whole Standard Model in Lorentzian signature; we expect this however to be a fruitful topic of research in the near future.

2. Complex powers on Lorentzian scattering spaces

2.1. Klein–Gordon operator

Let (M, g) be a Lorentzian manifold. We use the convention $(+, -, \dots, -)$ for the signature of g. We denote by $L^2(M)$ the canonical L^2 space associated to the volume density $d\mathrm{vol}_g$ of g, i.e. the $L^2(M)$ norm is

$$\|u\| = \left(\int_M |u(x)|^2 \, d\mathrm{vol}_g \right)^{1/2}.$$

Let $P = \Box_g + m^2$ be the *wave* or *Klein–Gordon operator*, i.e.

$$\Box_g = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}\, g^{\mu\nu} \partial_\nu)$$

is the Laplace–Beltrami operator in Lorentzian signature, $|g| = |\det g|$ and $m^2 \geqslant 0$.

2.2. Lorentzian scattering spaces

We will need to make assumptions on the asymptotic structure of (M, g) at spacetime infinity. To that end it is convenient to assume that M is the interior of a *compact manifold with boundary* $\overline{M}$.

We use the notation $C^\infty(\overline{M})$ for the space of smooth function on $\overline{M}$, meant in the usual sense of smooth extensibility across the boundary (denoted in what follows by $\partial\overline{M}$).

Let ρ be a boundary-defining function of $\partial\overline{M}$, i.e. a function $\rho \in C^\infty(\overline{M})$ such that $\rho > 0$ on M, $\partial\overline{M} = \{\rho = 0\}$, and $d\rho \neq 0$ on $\partial\overline{M}$. Recall that by the collar neighborhood theorem, there exists $W \supseteq \partial\overline{M}$, $\epsilon > 0$ and a diffeomorphism $\phi : [0, \epsilon[\times \partial\overline{M} \to W$ such that $\rho \circ \phi$ agrees with the projection to the first component of $[0, \epsilon[\times \partial\overline{M}$. We will drop ϕ in the notation when working close to the boundary, i.e. in the collar neighborhood W. In this sense we can find local coordinates of the form $(\rho, y_1, \dots, y_{n-1})$, where $(y_1, \dots, y_{n-1})$ are local coordinates on $\partial\overline{M}$.

We use the framework of Melrose's sc-geometry [87], mostly following the presentation in [130, 131]. Let ${}^{\mathrm{sc}}T\overline{M}$ be the *scattering tangent bundle* (or for short, the sc-*tangent bundle*) of $\overline{M}$. Recall that ${}^{\mathrm{sc}}T\overline{M}$ can be defined as the unique vector bundle over $\overline{M}$ such that all of its smooth sections $V \in C^\infty(\overline{M}; {}^{\mathrm{sc}}T\overline{M})$ are locally of the form

$$V = V_0(\rho, y)\rho^2\partial_\rho + \sum_{i=1}^{n-1} V_i(\rho, y)\rho\partial_{y_j}, \quad V_0, V_i \in C^\infty(\overline{U}),\ i = 1, \dots, n-1, \quad (2.2)$$

on a chart neighborhood $\overline{U}$ with local coordinates $(\rho, y_1, \dots, y_{n-1})$. Away from the boundary, ${}^{\mathrm{sc}}T\overline{M}$ is defined in the same way as the tangent bundle TM, and there is indeed a canonical isomorphism ${}^{\mathrm{sc}}T_M\overline{M} \to TM$.

The sc-*cotangent bundle* ${}^{\mathrm{sc}}T^*\overline{M}$ is by definition the dual bundle of ${}^{\mathrm{sc}}T\overline{M}$. Thus, in local coordinates $(\rho, y_1, \dots, y_{n-1})$, the smooth sections of ${}^{\mathrm{sc}}T^*\overline{M}$ are $C^\infty(\overline{M})$-generated by $(\rho^{-2}d\rho, \rho^{-1}dy_1, \dots, \rho^{-1}dy_{n-1})$. Again, over the interior there is a canonical isomorphism

$${}^{\mathrm{sc}}T^*_M\overline{M} \to T^*M. \quad (2.3)$$

Next, an sc-*metric* is by definition a non-degenerate smooth section of the fiberwise symmetrized tensor product bundle ${}^{\mathrm{sc}}T^*\overline{M} \otimes_{\mathrm{s}} {}^{\mathrm{sc}}T^*\overline{M}$.

Definition 2.1. $(\overline{M}, \overline{g})$ is a *Lorentzian scattering space* (or for short, a *Lorentzian* sc-*space*) if $\overline{g} \in C^\infty(\overline{M}; {}^{\mathrm{sc}}T^*\overline{M} \otimes_{\mathrm{s}} {}^{\mathrm{sc}}T^*\overline{M})$ is of Lorentzian signature.

Example 2.2. The standard example is $M = \mathbb{R}^n$, with $\overline{M} = \overline{\mathbb{R}^n}$ the *radial compactification of* $\mathbb{R}^n$. Recall that $\overline{\mathbb{R}^n}$ is defined as the quotient of $\mathbb{R}^n \sqcup ([0,1[_\rho \times \mathbb{S}^{n-1}_y)$ by the relation which identifies any non-zero $x \in \mathbb{R}^n$ with the point (ρ, y), where $\rho = r^{-1}$ and (r, y) are the polar coordinates of x. The smooth structure near $\{\rho = 0\}$ is the obvious one in (ρ, y) coordinates. Observe that the vector field $\partial_r = -\rho^2\partial_\rho$ is of the form (2.2). More generally, switching now to standard coordinates $(x_0, \dots, x_{n-1})$ on $\mathbb{R}^n$, the frame $(\partial_{x_0}, \dots, \partial_{x_{n-1}})$ smoothly extends to ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}^n}$, and any $V \in C^\infty(\overline{\mathbb{R}^n}; {}^{\mathrm{sc}}T\,\overline{\mathbb{R}^n})$ is in the $C^\infty(\overline{\mathbb{R}^n})$-span of $(\partial_{x_0}, \dots, \partial_{x_{n-1}})$, i.e. the coefficients smoothly extend across $\{\rho = 0\}$ in (ρ, y) coordinates on top of being smooth in $\mathbb{R}^n$. Similarly, any $\overline{g} \in C^\infty(\overline{\mathbb{R}^n}; {}^{\mathrm{sc}}T^*\overline{\mathbb{R}^n} \otimes_{\mathrm{s}} {}^{\mathrm{sc}}T^*\overline{\mathbb{R}^n})$ is in the $C^\infty(\overline{\mathbb{R}^n})$-span of $dx^\mu \otimes_{\mathrm{s}} dx^\nu$ for $\mu, \nu = 0, \dots, n-1$. In particular, the Minkowski metric $\eta = dx_0^2 - (dx_1^2 + \dots + dx_{n-1}^2)$ on $\mathbb{R}^n$ extends to an sc-metric on $\overline{\mathbb{R}^n}$ and in this sense Minkowski space is a Lorentzian sc-space.

We will assume that g (the Lorentzian metric on the boundaryless manifold M) extends to an sc-metric on $\overline{M}$, and so $(\overline{M}, \overline{g})$ is a Lorentzian sc-space. The volume density of (M, g), $d\mathrm{vol}_g$, extends then to an sc-density on $\overline{M}$, meaning that in local coordinates (ρ, y) it is of the form $\mu(\rho, y)|\rho^{-2}d\rho\, \rho^{-n+1}dy|$ with $\mu \in C^\infty(\overline{M})$.

2.3. Bicharacteristics and Hamiltonian flow

When discussing microlocalization it is useful to compactify the fibers of ${}^{\mathrm{sc}}T^*\overline{M}$. The base manifold $\overline{M}$ having a boundary already, the fiberwise radial compactification of

${}^{\rm sc}T^*\overline{M}$ yields a manifold with corners (see [87, Section 6.4] for details), which we will denote by $\overline{{}^{\rm sc}T^*M}$.

As a manifold with corners, $\overline{{}^{\rm sc}T^*M}$ has two boundary hypersurfaces: the first one is *base infinity* or *spacetime infinity*, which we denote by $\partial\,{}^{\rm sc}T^*\overline{M}$ (instead of using the more pedantic, but heavier notation $\overline{{}^{\rm sc}T^*_{\partial\overline{M}}\overline{M}}$), and the other one is *fiber infinity*, which we denote by $\partial\overline{{}^{\rm sc}T^*}M$. We stress that despite what the notation could suggest, these two boundary hypersurfaces *do* intersect at the corner $\partial\,{}^{\rm sc}T^*\overline{M}\cap\partial\overline{{}^{\rm sc}T^*}M\neq\emptyset$, and we have of course

$$\partial\overline{{}^{\rm sc}T^*M}=\partial\,{}^{\rm sc}T^*\overline{M}\cup\partial\overline{{}^{\rm sc}T^*}M.$$

Let $\langle\xi\rangle^{-1}$ be the formal notation for a boundary-defining function of fiber infinity. For $z\in\mathbb{C}$, the *principal symbol* of $\Box_g-z$ in the sense of the sc-calculus is the function p_z on $\partial\overline{{}^{\rm sc}T^*M}$ given by

$$p_z(x,\xi)=\begin{cases}-|\xi|^{-2}(\xi\cdot g^{-1}\xi) & \text{on } \partial\overline{{}^{\rm sc}T^*}M,\\ \langle\xi\rangle^{-2}(-\xi\cdot g^{-1}\xi-z) & \text{on } \partial\,{}^{\rm sc}T^*\overline{M}.\end{cases}\tag{2.4}$$

This is well-defined at $\partial\overline{{}^{\rm sc}T^*}M$ thanks to the $|\xi|^{-2}$ factor that compensates for the degree 2 homogeneity of $\xi\cdot g^{-1}\xi$ in ξ. This is also well-defined at $\partial\,{}^{\rm sc}T^*\overline{M}$ as a consequence of the assumption that g extends to an sc-metric. Furthermore, the definition is consistent at the corner.

The *characteristic set* of $\Box_g-z$, denoted by Σ_z, is defined as the closure of $p_z^{-1}(\{0\})$ in $\partial\overline{{}^{\rm sc}T^*M}$. Note that $\Sigma_z\subset\partial\overline{{}^{\rm sc}T^*}M$ unless z is real. Furthermore, $\Sigma_z\cap\partial\overline{{}^{\rm sc}T^*}M=\Sigma_0\cap\partial\overline{{}^{\rm sc}T^*}M$ is always non-empty but does not depend on z. This is why various hypotheses can be simply written in terms of Σ_σ with $\sigma\in\mathbb{R}$.

The *Hamiltonian vector field* of p_0 on ${}^{\rm sc}T^*\overline{M}$, denoted by H_{p_0}, is the extension of the usual Hamiltonian vector field defined in the interior, i.e. the standard definition on T^*M induces a vector field on ${}^{\rm sc}T^*_M\overline{M}$ via the isomorphism (2.3), and this then extends to a vector field H_{p_0} on ${}^{\rm sc}T^*\overline{M}$. Similarly, ${}^{\rm sc}T^*\overline{M}$ has a symplectic and contact structure inherited from T^*M by extension; see e.g. [85, Section 2]. In local coordinates on ${}^{\rm sc}T^*\overline{M}$ of the form (ρ,y,ϱ,η), where (ϱ,η) are the dual coordinates of (ρ,y), H_{p_0} is given by

$$H_{p_0}=\rho\Big((\partial_\varrho p)(\rho\partial_\rho+\eta\cdot\partial_\eta)-(\rho\partial_\rho+\eta\cdot\partial_\eta)p\partial_\varrho+\sum_{i=1}^{n-1}\big((\partial_{\eta_i}p)\partial_{y_i}-(\partial_{y_i}p)\partial_{\eta_i}\big)\Big).$$

The *rescaled Hamiltonian vector field* $\overline{H}_{p_0}:=\langle\xi\rangle^{-1}\rho^{-1}H_{p_0}$ extends to a smooth vector field on $\overline{{}^{\rm sc}T^*M}$ which is tangent to $\partial\overline{{}^{\rm sc}T^*M}$. We call its flow on $\partial\overline{{}^{\rm sc}T^*M}$ the *Hamiltonian flow*, and for $\sigma\in\mathbb{R}$, the *bicharacteristics* are the integral curves of the rescaled Hamiltonian vector field within Σ_σ.

Definition 2.3. For $\sigma\in\mathbb{R}$ we say that (M,g) is *non-trapping at energy* σ if the following conditions are satisfied:

(1) There are two submanifolds $L_-\subset\partial\,{}^{\rm sc}T^*\overline{M}$ and $L_+\subset\partial\,{}^{\rm sc}T^*\overline{M}$, each transversal to $\partial\,{}^{\rm sc}T^*\overline{M}\cap\partial\overline{{}^{\rm sc}T^*}M$, which are sources, resp. sinks for the Hamiltonian flow in Σ_σ. More precisely, this means that within Σ_σ,

(a) $dp_0 \neq 0$ on $L_\pm$ and $\overline{H}_{p_0}$ is tangent to $L_\pm$,

(b) there exists a quadratic defining function $\rho_\pm$ of $L_\pm$ and a smooth function $\beta_\pm > 0$ satisfying

$$\mp \overline{H}_{p_0}\rho_\pm = \beta_\pm \rho_\pm + s_\pm + r_\pm$$

for some smooth $s_\pm, r_\pm$ such that $s_\pm > 0$ on $L_\pm$ and $r_\pm$ vanishes cubically at $L_\pm$,

(c) there exists $\beta_{0,\pm} \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ such that $\beta_{0,\pm} > 0$ on $L_\pm$ and $\mp \overline{H}_{p_0}\rho = \beta_{0,\pm}\rho$.

(2) Within Σ_σ, each bicharacteristic either goes from L_+ to L_-, or goes from L_- to L_+, or stays within L_+ or L_-.

We say that (M, g) is *non-trapping* if (1) and (2) hold true with $\Sigma_0 \cap \partial\overline{{}^{\mathrm{sc}}T^*}M$ instead of Σ_σ.

We will simply refer to L_- as *sources* and to L_+ as *sinks*. In (1) of Definition 2.3, by saying that $\rho_\pm \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ is a *quadratic defining function of* $L_\pm$ we mean that $\rho_\pm = \sum_i \rho_{\pm,i}^2$ for finitely many $\rho_{\pm,i}$ such that $L_\pm = \bigcap_i \{\rho_{\pm,i} = 0\}$ within $\Sigma_\sigma \cap \partial^{\mathrm{sc}}T^*\overline{M}$, and the differentials $d\rho_{\pm,i}$ are linearly independent on $L_\pm \cap \Sigma_\sigma$. A more detailed discussion of conditions (b)–(c) can be found in [130, Section 5.4.7].

Example 2.4. A (non-exhaustive) class of examples is provided by the non-trapping Lorentzian scattering metrics introduced in [7] and further studied in the context of the Feynman problem in [49]. Namely, one assumes the existence of a function $v \in C^\infty(M)$ such that for all $V \in C^\infty(\overline{M}; {}^{\mathrm{sc}}T\overline{M})$ the signs of $g(V, V)$ and v are the same at $\partial M = \{\rho = 0\}$. Furthermore, near $\{v = \rho = 0\}$, the sc-metric g is assumed to be of the form

$$g = v\frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes_{\mathrm{s}} \frac{\omega}{\rho}\right) - \frac{\tilde{g}}{\rho^2},$$

where ω is a smooth 1-form such that $\omega = dv + \mathcal{O}(v) + \mathcal{O}(\rho)$, and the restriction of $\tilde{g} \in C^\infty(\overline{M}; T^*\overline{M} \otimes_{\mathrm{s}} T^*\overline{M})$ to the joint annihilator of $d\rho, dv$ is positive. As discussed in [7, Section 3.6], this implies the existence of sources/sinks at $\{\rho = v = 0, \varrho = \gamma = 0, \mp\gamma > 0\}$ in coordinates $(\rho, v, w, \varrho, \beta, \gamma) \in {}^{\mathrm{sc}}T^*\overline{M}$. One then needs to ensure that the non-trapping property (1) of Definition 2.3 holds true in $\Sigma_0 \cap \partial\overline{{}^{\mathrm{sc}}T^*}M$; see [7, Section 3.2]. Minkowski space is a special case (see [7, Section 3.1]), and we also note that in practice it is possible to consider perturbations that do not have the structure of sinks and sources, but for which the propagation estimates used in the sequel remain valid nevertheless. We also refer to [131, Section 2] for remarks on the assumption of non-trapping at $\sigma \neq 0$.

In [131], Vasy proves the following theorem; cf. the work of Nakamura–Taira [93] for the case of real principal type operators of arbitrary orders on $\mathbb{R}^n$ under a similar non-trapping condition.

Theorem 2.5 ([131, Theorem 1]). *Assume (M, g) is non-trapping. Then P acting on $C_{\mathrm{c}}^\infty(M)$ is essentially self-adjoint in $L^2(M)$.*

As a consequence, if we denote in the same way the closure of P acting on $C_{\mathrm{c}}^\infty(M)$, functions of P can be defined using the functional calculus for self-adjoint operators.

We are particularly interested in Schwartz kernels of functions of P, and therefore we need to know more precise mapping properties of the resolvent, also basing on the results from [131].

2.4. *Sobolev spaces*

If $s \in \mathbb{Z}_{\geqslant 0}$, then the sc-*Sobolev space of order s* is by definition

$$H_{\rm sc}^{s,0}(M) = \{u \in L^2(M) \mid \forall k \leqslant s \text{ and } V_1, \dots, V_k \in C^\infty(\overline{M}; {}^{\rm sc}T\overline{M}),\ V_1 \dots V_k u \in L^2(M)\}.$$

The definition of $H_{\rm sc}^{s,0}(M)$ and of its norm $\|\cdot\|_{s,0}$ for arbitrary $s \in \mathbb{R}$ is most efficiently formulated with the help of sc-pseudo-differential operators; see Appendix A.2. The *weighted Sobolev spaces* are defined for non-zero $\ell \in \mathbb{R}$ by

$$H_{\rm sc}^{s,\ell}(M) = \rho^\ell H_{\rm sc}^{s,0}(M),$$

with norm $\|u\|_{s,\ell} = \|\rho^{-\ell} u\|_{s,0}$, where ρ is as before a boundary-defining function of $\partial\overline{M}$. Thus, higher s means more regularity, and higher ℓ means more decay at spacetime infinity, i.e. at $\partial\overline{M}$. In the special case of Minkowski space modelled on $\overline{\mathbb{R}^n}$, the space $H_{\rm sc}^{s,0}(M)$ coincides with the usual Sobolev space $H^s(\mathbb{R}^n)$, and if we choose as boundary-defining function $\rho = (1+|x|^2)^{-1/2} =: \langle x\rangle^{-1}$ then $H_{\rm sc}^{s,\ell}(M)$ coincides with the weighted Sobolev space $\langle x\rangle^{-\ell} H^s(\mathbb{R}^n)$.

The definition of $H_{\rm sc}^{s,\ell}(M)$ can be usefully generalized to weight orders ℓ that vary in phase space, i.e. to $\ell \in C^\infty(\overline{{}^{\rm sc}T^*M})$ rather than just $\ell \in \mathbb{R}$; see Appendix A.2. We will also use the Fréchet spaces

$$H_{\rm sc}^{\infty,\ell}(M) := \bigcap_{s\geqslant 0} H_{\rm sc}^{s,\ell}(M), \quad H_{\rm sc}^{s,\infty}(M) := \bigcap_{\ell \geqslant 0} H_{\rm sc}^{s,\ell}(M).$$

We stress that unless $s = \ell = 0$, the definition of $H_{\rm sc}^{s,\ell}(M)$ refers to the manifold with boundary $\overline{M}$. As a rule, we do not necessarily emphasize the dependence on $\overline{M}$ or $\overline{g}$ in the notation if there is an "sc" subscript, which indicates the dependence on the scattering structure already. Apart from the spaces with an "sc" subscript, we use standard notation. For instance, $C_{\rm c}^\infty(M)$, $C^\infty(M)$, $H_{\rm c}^s(M)$ and $H_{\rm loc}^s(M)$ are the standard spaces on the *boundaryless* manifold M (in contrast to the space of smooth functions $C^\infty(\overline{M})$ on the manifold with boundary $\overline{M}$), and similarly for the space $\mathcal{D}'(M)$ of distributions and $\mathcal{E}'(M)$ of compactly supported distributions on M.

Note that for all $s \in \mathbb{R}$ and $\ell \in C^\infty(\overline{{}^{\rm sc}T^*M})$ we have the continuous inclusions

$$H_{\rm c}^s(M) \subset H_{\rm sc}^{s,\ell}(M) \subset H_{\rm loc}^s(M) \subset \mathcal{D}'(M).$$

2.5. *Estimates for imaginary spectral parameter*

We now consider the operator $P - z$ for $z \in \mathbb{C}$, focusing first on the case $\operatorname{Im} z > 0$.

For $\operatorname{Im} z \neq 0$, $P - z \in \Psi^{2,0}_{\mathrm{sc}}(M)$ is microlocally elliptic in the sense of the sc-calculus except at fiber infinity $\partial \overline{{}^{\mathrm{sc}}T^*M}$; propagation estimates take place inside $\Sigma_0 \cap \partial \overline{{}^{\mathrm{sc}}T^*M}$.

For $\ell \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ we set $\ell_\pm = \ell|_{L_\pm}$. We will say that ℓ is *monotone in* Σ_z if it is monotone along the Hamiltonian flow restricted to Σ_z. In various estimates, $S, L \in \mathbb{R}$ will always be sufficiently negative numbers, which can be taken arbitrarily negative.

For an arbitrary $c > 0$ let $Z = \{\operatorname{Im} z \geqslant c|\operatorname{Re} z|\}$, and let $\delta > 0$.

Proposition 2.6 ([131, Proposition 2]). *Let $s \in \mathbb{R}$, and let $\ell \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ be monotone in Σ_0 and such that $\ell_- > -1/2$ and $\ell_+ < -1/2$. Then for all $s' \in \mathbb{R}$, all $\ell' \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ with $\ell'_- \in\,]-1/2, \ell_-]$ and all $u \in H^{s',\ell'}_{\mathrm{sc}}(M)$,*

$$\|u\|_{s,\ell} + (\operatorname{Im} z)^{1/2}\|u\|_{s-1/2,\ell+1/2} \leqslant C(\|(P-z)u\|_{s-1,\ell+1} + \|u\|_{S,L}), \tag{2.5}$$

uniformly for $z \in Z \cap \{|z| \geqslant \delta\}$.

Proof. The proof is based on a slight modification of the estimates in [130, Section 5.4] and can be found in [131]. We only sketch it very briefly for the reader's convenience.

The basic ingredients are the *higher decay radial estimate* at sources and the *lower decay radial estimate* into the sinks, recalled in more detail in Appendix A.3; see also Appendix A.2 for prerequisites on scattering pseudo-differential calculus. The first estimate (see Proposition A.3) reads

$$\|Au\|_{s,\ell} + (\operatorname{Im} z)^{1/2}\|Au\|_{s-1/2,\ell+1/2} \leqslant C(\|B(P-z)u\|_{s-1,\ell+1} + \|u\|_{S,L}),$$

for all $u \in H^{s',\ell'}_{\mathrm{sc}}(M)$, $\ell > \ell' > -1/2$, $s, s' \in \mathbb{R}$, $L_- \subset \operatorname{Ell}_{\mathrm{sc}}(A)$, $\mathrm{WF}'_{\mathrm{sc}}(A)$ contained in a small neighborhood of L_- in $\operatorname{Ell}_{\mathrm{sc}}(B)$, and within $\operatorname{Ell}_{\mathrm{sc}}(B)$, bicharacteristics from $\mathrm{WF}'_{\mathrm{sc}}(A)$ tend to L_- in the forward direction along the flow. The second estimate (see Proposition A.2) is

$$\begin{aligned}\|Au\|_{s,\ell} &+ (\operatorname{Im} z)^{1/2}\|Au\|_{s-1/2,\ell+1/2}\\ &\leqslant C(\|B_1u\|_{s,\ell} + \|B(P-z)u\|_{s-1,\ell+1} + \|Bu\|_{s',\ell'} + \|u\|_{S,L})\end{aligned}$$

for all $u \in H^{s',\ell'}_{\mathrm{sc}}(M)$, $\ell < -1/2$, $\ell', s, s' \in \mathbb{R}$, $L_+ \subset \operatorname{Ell}_{\mathrm{sc}}(A)$, $\mathrm{WF}'_{\mathrm{sc}}(A)$ contained in a small neighborhood of L_+ in $\operatorname{Ell}_{\mathrm{sc}}(B)$, and within $\operatorname{Ell}_{\mathrm{sc}}(B)$, bicharacteristics from $\mathrm{WF}'_{\mathrm{sc}}(A) \setminus L_+$ tend to L_+ in the forward direction along the flow, and intersect $\operatorname{Ell}_{\mathrm{sc}}(B_1)$ in the backward direction.

Thanks to the non-trapping assumption, by taking ℓ as in the assumption of the proposition, the two estimates applied in a neighborhood of $\Sigma_0 \cap \partial\overline{{}^{\mathrm{sc}}T^*M}$ can be combined with propagation of singularities estimates (Proposition A.1) and with the elliptic estimate to yield (2.5) (see e.g. [50, Section 3.2] for a pedagogical explanation of how to combine this type of estimates). ∎

By iterating (2.5) we can conclude that for all $\operatorname{Im} z > 0$ and all $N \in \mathbb{N}_{\geqslant 0}$,

$$(P-z)^{-N} : L^2_{\mathrm{c}}(M) \to H^N_{\mathrm{loc}}(M). \tag{2.6}$$

By replacing P by $-P$ (the rôle of L_+ and L_- is then exchanged) we also obtain (2.6) for $\operatorname{Im} z < 0$. Note that in contrast to the elliptic case, we cannot expect that the image is in $H^{2N}_{\rm loc}(M)$.

To show regularity properties of non-integer powers we will need the following more precise statement.

Proposition 2.7. *Let $\varepsilon > 0$, $N \in \mathbb{N}_{>0}$, $s \in \mathbb{R}$, and let $\ell \in C^\infty(\overline{{}^{\rm sc}T^*M})$ be monotone in Σ_0 and such that $\ell_- > -1/2$ and $\ell_+ < -N$. Then*

$$\|(P - i\varepsilon)^{-N}(P - z)^{-1} f\|_{s,\ell} \leqslant C(\operatorname{Im} z)^{-1/2}(\|f\|_{s-N-1/2,\ell+N+1/2} + \|f\|), \quad (2.7)$$

uniformly for all $z \in Z \cap \{|z| \geqslant \delta\}$ and for all $f \in L^2(M) \cap H^{s-N-1/2,\ell+N+1/2}_{\rm sc}(M)$.

Proof. Since $\ell_- > -1/2$ and $\ell_+ < -N + 1/2$, we can apply Proposition 2.6 with $u = (P - i\varepsilon)^{-N}(P - z)^{-1} f \in L^2(M)$ and then iterate the estimate, N times in total. By dropping the part proportional to $\operatorname{Im} z$ from the l.h.s. each time, we obtain

$$\|(P - i\varepsilon)^{-N}(P - z)^{-1} f\|_{s,\ell} \leqslant C(\|(P - z)^{-1} f\|_{s-N,\ell+N} + \|u\|_{S,L}).$$

Since $\ell_- + N - 1/2 > -1/2$ and $\ell_+ + N - 1/2 < -1/2$, we can apply Proposition 2.6 to $v = (P - z)^{-1} f \in L^2(M)$. By keeping only the part proportional to $(\operatorname{Im} z)^{1/2}$ on the l.h.s. we obtain

$$(\operatorname{Im} z)^{1/2}\|(P - z)^{-1} f\|_{s-N,\ell+N} \leqslant C(\|f\|_{s-N-1/2,\ell+N+1/2} + \|v\|_{S,L}).$$

The terms $\|u\|_{S,L}$ and $\|v\|_{S,L}$ above can be estimated by $\|(P - z)^{-1} f\|$ and thus by $(\operatorname{Im} z)^{-1}$ using the self-adjointness of P. Combining the estimates yields (2.7). ∎

Remark 2.8. Proposition 2.7 is also valid for $N = 0$ if $\ell_- > 0$, as can easily be seen by dropping the first part of the proof.

2.6. From resolvent to complex powers

As P is a self-adjoint operator, the complex powers $(P - i\varepsilon)^{-\alpha}$ are well-defined by functional calculus for $\varepsilon > 0$ and $\alpha \in \mathbb{C}$, and also for $\varepsilon = 0$ if $\operatorname{Re} \alpha < 0$. We deduce below various regularity properties of $(P - i\varepsilon)^{-\alpha}$ from resolvent estimates.

We will express $(P - i\varepsilon)^{-\alpha}$ as an integral of $(P - z)^{-1}$ over a contour γ_ε defined as follows. Let $\tilde{\gamma}_\varepsilon$ be a contour going from $\operatorname{Re} z \ll 0$ to $\operatorname{Re} z \gg 0$ in the upper half-plane, of the form

$$\tilde{\gamma}_\varepsilon = e^{i(\pi-\theta)}]-\infty, \varepsilon/2] \cup \{\varepsilon/2 e^{i\omega} \mid \pi - \theta < \omega < \theta\} \cup e^{i\theta}[\varepsilon/2, +\infty[$$

for some fixed $\theta \in]0, \pi/2]$. We then define $\gamma_\varepsilon := \tilde{\gamma}_\varepsilon + i\varepsilon$ (see Figure 1). We also define its degenerate version γ_0, which also goes from $\operatorname{Re} z \ll 0$ to $\operatorname{Re} z \gg 0$ in the upper half-plane and is of the form

$$\gamma_0 = e^{i(\pi-\theta)}]-\infty, 0] \cup e^{i\theta}[0, +\infty[.$$

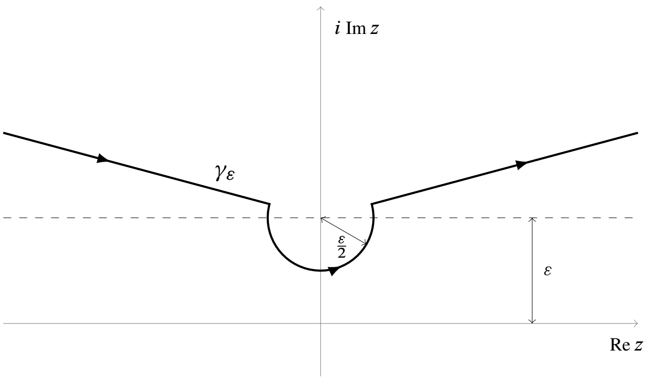

Fig. 1. The contour γ_ε used to express $(P - i\varepsilon)^{-\alpha}$ as an integral of the resolvent $(P - z)^{-1}$ for P self-adjoint. If $\varepsilon = 0$ the contour degenerates to two half-lines intersecting the real line at 0.

Proposition 2.9. *Assume (M, g) is non-trapping.*

(1) *For all $\varepsilon > 0$ and $\alpha \in \mathbb{C}$, $H^{\infty,0}_{\mathrm{sc}}(M) \subset \mathrm{Dom}\,(P - i\varepsilon)^{-\alpha}$.*

(2) *For all $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha < 0$, $H^{\infty,0}_{\mathrm{sc}}(M) \subset \mathrm{Dom}\,(P - i0)^{-\alpha}$.*

(3) *For all $u \in H^{\infty,0}_{\mathrm{sc}}(M)$ and all $\varepsilon > 0$, the functions $\mathbb{C} \ni \alpha \mapsto (P - i\varepsilon)^{-\alpha} u \in L^2(M)$ and $\{\mathrm{Re}\,\alpha < 0\} \ni \alpha \mapsto (P - i0)^{-\alpha} u \in L^2(M)$ are holomorphic.*

(4) *For all $\varepsilon \geqslant 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha < 0$, $s \geqslant 0$ and $\epsilon > 0$,*

$$(P - i\varepsilon)^{-\alpha} : H^{s,\infty}_{\mathrm{sc}}(M) \to H^{s+2\lfloor \mathrm{Re}\,\alpha \rfloor - 3/2, -\epsilon}_{\mathrm{sc}}(M)$$

continuously.

(5) *For all $\varepsilon > 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 1/2$, $s \geqslant 0$ and $\epsilon > 0$,*

$$(P - i\varepsilon)^{-\alpha} : H^{s,\infty}_{\mathrm{sc}}(M) \to H^{s+\lfloor \mathrm{Re}\,\alpha - 1/2 \rfloor + 1/2, -\lfloor \mathrm{Re}\,\alpha - 1/2 \rfloor + 1/2 - \epsilon}_{\mathrm{sc}}(M)$$

continuously.

Proof. (1) We write $(P - i\varepsilon)^{-\alpha} = (P - i\varepsilon)^{-\alpha - N}(P - i\varepsilon)^N$ for some $N \in \mathbb{N}_0$ with $N \geqslant -\mathrm{Re}\,\alpha$. Then $(P - i\varepsilon)^{-\alpha - N}$ is bounded on $L^2(M)$ and $(P - i\varepsilon)^N : H^{\infty,0}_{\mathrm{sc}}(M) \to H^{\infty,0}_{\mathrm{sc}}(M)$ continuously, hence $(P - i\varepsilon)^{-\alpha} : H^{\infty,0}_{\mathrm{sc}}(M) \to L^2(M)$ continuously and the claim follows.

(2) The assertion follows directly from (1) and the fact that (see Appendix B) for all $\mathrm{Re}\,\alpha < 0$ and all $\varepsilon > 0$, $\mathrm{Dom}\,(P - i0)^{-\alpha} = \mathrm{Dom}\,(P - i\varepsilon)^{-\alpha}$.

(3) This follows easily from (1) and functional calculus.

(4) Let $N = -\lfloor \mathrm{Re}\,\alpha \rfloor + 1$. By (B.6),

$$(P - i\varepsilon)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_0 + i\varepsilon} \frac{(z - i\varepsilon)^{-\alpha}}{(z - i)^N} (P - i)^N (P - z)^{-1}\, dz. \qquad (2.8)$$

By $P - i \in \Psi^{2,0}_{\rm sc}(M)$ and by Remark 2.8, for $L \in \mathbb{R}$ sufficiently large,

$$(P-i)^N (P-z)^{-1} : H^{s,L}_{\rm sc}(M) \to H^{s+1/2-2N,-\epsilon}_{\rm sc}(M) \text{ has } \mathcal{O}(|\mathrm{Im}\, z|^{-1/2}) \text{ norm.} \tag{2.9}$$

By dominated convergence, the integral (2.8) is bounded on the same spaces.

(5) We write $(P - i\varepsilon)^{-\alpha} = (P - i\varepsilon)^{-N}(P - i\varepsilon)^{-\mu}$, where $N = \lfloor \mathrm{Re}\, \alpha - 1/2 \rfloor$ and $\mathrm{Re}\, \mu > 1/2$, and then we express $(P - i\varepsilon)^{-\mu}$ in terms of the resolvent of iP as a contour integral using (B.4). This gives

$$(P - i\varepsilon)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\mu} (P - i\varepsilon)^{-N} (P - z)^{-1}\, dz. \tag{2.10}$$

By Proposition 2.7 (with $s + N + 1/2$ instead of s and $\ell_+ = -N - \epsilon$), for $L \in \mathbb{R}$ sufficiently large,

$$(P - i\varepsilon)^{-N}(P - z)^{-1} : H^{s,L}_{\rm sc}(M) \to H^{s+N+1/2,-N-\epsilon}_{\rm sc}(M) \text{ has } \mathcal{O}(|\mathrm{Im}\, z|^{-1/2}) \text{ norm.}$$

By dominated convergence, the integral (2.10) is bounded on the same spaces. ■

2.7. *Estimates uniform down to the real axis*

In [131], estimates uniform down to the real axis are obtained under the extra hypothesis that $P = \Box_g + m^2$ with $m^2 \neq 0$ and (M, g) is *non-trapping at energy* $\sigma = m^2$ (see Definition 2.3). This is not necessary for our main results. However, we briefly discuss the improved estimates here as they lead to stronger results (in terms of the dependence on ε for functions of $P - i\varepsilon$) in later sections.

The non-trapping at energy $\sigma = m^2$ ensures that the Fredholm estimates for $P - i\varepsilon$ are uniform down to $\varepsilon = 0$. Let us state this as a proposition (proved in analogy to Proposition 2.6) for further reference.

Proposition 2.10 ([131, Proposition 2]). *Let (M, g) be a non-trapping Lorentzian scattering space and assume it is non-trapping at energy $\sigma = m^2 \neq 0$. Let $s \in \mathbb{R}$, and let $\ell \in C^\infty(\overline{{}^{\rm sc}T^*M})$ be monotone in Σ_σ and such that $\ell_- > -1/2$ and $\ell_+ < -1/2$. Then there exists $\delta > 0$ such that for all $s' \in \mathbb{R}$, all $\ell' \in C^\infty(\overline{{}^{\rm sc}T^*M})$ with $\ell'_- \in]-1/2, \ell_-]$ and all $u \in H^{s',\ell'}_{\rm sc}(M)$,*

$$\|u\|_{s,\ell} \leqslant C(\|(P - z)u\|_{s-1,\ell+1} + \|u\|_{S,L}),$$

uniformly in $z \in \{\mathrm{Im}\, z > 0\} \cup \{|z| \leqslant \delta\}$.

An injectivity property is needed to get an invertibility statement in weighted Sobolev spaces down to $\mathrm{Im}\, z \to 0^+$.

Definition 2.11. We say that *injectivity* holds at $\sigma = m^2 \neq 0$ if for some $s \in \mathbb{R}$ and some $\ell \in C^\infty(\overline{{}^{\rm sc}T^*M})$ monotone with $\ell_- > -1/2$ and $\ell_+ < -1/2$,

$$(u \in H^{s,\ell}_{\rm sc}(M),\ Pu = 0) \implies (u = 0).$$

Sufficient conditions for injectivity with $-1 < \ell_+ < -1/2$ are discussed in [131]. As a consequence one concludes a *limiting absorption principle*, i.e. the existence of the limiting operator $(P - i0)^{-1}$ on weighted Sobolev spaces. We state here the following variant.

Proposition 2.12. *Let (M, g) be a non-trapping Lorentzian scattering space and assume non-trapping at energy $\sigma = m^2$ and injectivity. Let $s \in \mathbb{R}$ and $\ell \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ be as in Hypothesis 2.11. There exists $\delta > 0$ such that*

$$\|(P - z)^{-1} f\|_{s,\ell} \leqslant C \|f\|_{s-1,\ell+1}, \tag{2.11}$$

uniformly for $\operatorname{Im} z \geqslant 0$, $|z| < \delta$.

Proof. By [131, Theorem 5], if $\operatorname{Im} z \geqslant 0$ and $|z| < \delta$ with δ sufficiently small then $(P - z)^{-1}$ tends to $(P - i0)^{-1}$ in the weak operator topology as $z \to 0$. By compactness of the embeddings $H_{\mathrm{sc}}^{s_1,\ell_1}(M) \hookrightarrow H_{\mathrm{sc}}^{s_2,\ell_2}(M)$ for any $s_1 > s_2$ and $\ell_1 > \ell_2$, this gives boundedness in norm, i.e. (2.11). ∎

3. Wavefront set of the resolvent

3.1. Summary

Our next goal is to estimate the wavefront set of the resolvent $(P - z)^{-1}$ and give sufficient conditions for a given parametrix of $P - z$ to be equal $(P - z)^{-1}$ modulo smooth terms (in the sense of having smooth Schwartz kernel in $M \times M$). This needs to be true *uniformly in z* in an appropriate sense because we will then be interested in integrating in z when considering complex powers.

We remark that techniques to deduce the wavefront set of resolvents from propagation estimates were developed by Dyatlov–Zworski [40] in the semi-classical case, originally in the context of Anosov flows. Here, we use an argument more similar to the work of Vasy–Wrochna [133] and we also construct a parametrix related to that of Gérard–Wrochna [52]. The disadvantage as compared to the semi-classical approach is that it is less evident how to deal with possible singularities of Schwartz kernels which are microlocally at $o \times T^*M$ or $T^*M \times o$, where o is the zero section. This issue is however circumvented by considering an operator version of the wavefront set, similarly to [47, 135].

3.2. Uniform operator wavefront set

Let $Z \subset \mathbb{C}$ and let h be a strictly positive function on Z. Suppose $G_z : \mathcal{E}'(M) \to \mathcal{D}'(M)$ for $z \in Z$.

Definition 3.1. For $s \in \mathbb{R}$, we write

$$G_z = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$$

if for all $l \in \mathbb{R}$, $h(z)^{-1} G_z$ is a uniformly bounded family of continuous operators $H^l_{\rm c}(M) \to H^{l+s}_{\rm loc}(M)$. We write $G_z = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$ if $G_z = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$ for some $s \in \mathbb{R}$.

Note that $G_z = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$ implies $G_z^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$, and also $G_z = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$ implies $G_z^* = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$.

We will be mostly interested in wavefront set estimates in the interior M of $\overline{M}$. Over the interior, $\partial^{\rm sc}\overline{T^*M}$ is isomorphic to $\partial \overline{T^*M}$, the boundary of the fiber compactification of T^*M. We denote by $\Psi^s(M)$ the class of *properly supported* pseudo-differential operators of order $s \in \mathbb{R}$ on M (in the sense of the usual calculus on the boundaryless manifold M). One says that $A \in \Psi^s(M)$ is *elliptic at* $q \in \partial\overline{T^*M}$ if its principal symbol is non-zero at q.

Definition 3.2. If $G_z = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$ then its *uniform operator wavefront set of order* $s \in \mathbb{R}$ is the set $\mathrm{WF}'^{(s)}_{h(z)}(G_z) \subset \partial\overline{T^*M} \times \partial\overline{T^*M}$ defined as follows: $(q_1, q_2) \notin \mathrm{WF}'^{(s)}_{h(z)}(G_z)$ iff there exist $B_i \in \Psi^0(M)$ elliptic at q_i $(i = 1, 2)$ and such that

$$B_1 G_z B_2^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z)). \tag{3.12}$$

We show several elementary properties of the uniform operator wavefront set, the proof of which is to a large extent analogous to [135, Section 5.1].

Let us recall that for $A \in \Psi^s(M)$, there is a closely related notion of operator wavefront set $\mathrm{WF}'(A)$ which characterizes the directions $q \in \partial\overline{T^*M}$ in which microlocally, A does not behave as a regularizing operator.

Lemma 3.3. *For any* $q_1, q_2 \in \partial\overline{T^*M} \times \partial\overline{T^*M}$, $(q_1, q_2) \notin \mathrm{WF}'^{(s)}_{h(z)}(G_z)$ *if and only if for* $i = 1, 2$ *there exist neighborhoods* Γ_i *of* q_i *such that* (3.12) *holds true for all* $B_i \in \Psi^0(M)$ *elliptic at* q_i *and satisfying* $\mathrm{WF}'(B_i) \subset \Gamma_i$.

Proof. Suppose $(q_1, q_2) \notin \mathrm{WF}'^{(s)}_{h(z)}(G_z)$, so that there exists $A_i \in \Psi^0(M)$, $i = 1, 2$, elliptic at q_i, such that $A_1 G_z A_2^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$. There exists a compact neighborhood Γ_i of q_i on which A_i is elliptic. Therefore, there exists $A_i^{(-1)} \in \Psi^0(M)$ such that

$$\mathrm{WF}'(A_i^{(-1)} A_i - \mathbf{1}) \cap \Gamma_i = \emptyset.$$

Let $B_i \in \Psi^0(M)$ be elliptic at q_i and such that $\mathrm{WF}'(B_i) \subset \Gamma_i$. These conditions imply that

$$B_1(A_1^{(-1)} A_1 - \mathbf{1}) \in \Psi^{-\infty}(M), \quad (A_2^*(A_2^{(-1)})^* - \mathbf{1}) B_2^* \in \Psi^{-\infty}(M). \tag{3.13}$$

We can write

$$\begin{aligned} B_1 G_z B_2^* = {} & B_1 A_1^{(-1)} A_1 G_z A_2^* (A_2^{(-1)})^* B_2^* + B_1(\mathbf{1} - A_1^{(-1)} A_1) G_z A_2^* (A_2^{(-1)})^* B_2^* \\ & + B_1 A_1^{(-1)} A_1 G_z (\mathbf{1} - A_2^*(A_2^{(-1)})^*) B_2^* \\ & + B_1(\mathbf{1} - A_1^{(-1)} A_1) G_z (\mathbf{1} - A_2^*(A_2^{(-1)})^*) B_2^*. \end{aligned}$$

By $A_1 G_z A_2^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$ and (3.13), all the summands are $\mathcal{O}_{H^* \to H^{*+s}}(h(z))$, hence

$$B_1 G_z B_2^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z)).$$

The opposite direction is trivial. ∎

Lemma 3.4. *If* $G_z, G_z' = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$, *then*

$$\mathrm{WF}'^{(s)}_{h(z)}(G_z + \tilde{G}_z) \subset \mathrm{WF}'^{(s)}_{h(z)}(G_z) \cup \mathrm{WF}'^{(s)}_{h(z)}(\tilde{G}_z).$$

Proof. If $(q_1, q_2) \notin \mathrm{WF}'^{(s)}_{h(z)}(G_z)$ and $(q_1, q_2) \notin \mathrm{WF}'^{(s)}_{h(z)}(\tilde{G}_z)$ then by Lemma 3.3 we can choose B_1, B_2 elliptic at resp. q_1, q_2 such that

$$B_1 G_z B_2^* \text{ and } B_1 \tilde{G}_z B_2^* \text{ are both } \mathcal{O}_{H^* \to H^{*+s}}(h(z)).$$

Hence $B_1(G_z + \tilde{G}_z)B_2^*$ is $\mathcal{O}_{H^* \to H^{*+s}}(h(z))$ and thus $(q_1, q_2) \notin \mathrm{WF}'^{(s)}_{h(z)}(G_z + \tilde{G}_z)$. ∎

Proposition 3.5. *Suppose* $\mathrm{WF}'^{(s)}_{h(z)}(G_z) = \emptyset$. *Then* $G_z = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$.

Proof. It suffices to show that for any $x_1, x_2 \in M$ there exist $\chi_1, \chi_2 \in C_c^\infty(M)$ with $\chi_i \equiv 1$ near x_i such that $\chi_1 G_z \chi_2 = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$.

By definition of $\mathrm{WF}'^{(s)}_{h(z)}(G_z)$, for any $q, q' \in \partial \overline{T}^* M$ there exist $B_{1,q}, B_{2,q'} \in \Psi^0(M)$ elliptic at resp. q, q' such that $B_{1,q} G_z B_{2,q'}^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$. Let $\Gamma_{1,q}$ be the set on which $B_{1,q}$ is elliptic.

Then $\{\Gamma_{1,q} \mid q \in \partial \overline{T^*_{x_1}} M\}$ is an open cover of $\partial \overline{T^*_{x_1}} M$. By compactness we can find a finite subcover $\{\Gamma_{1,q_j}\}_{j=1}^N$. Then $B_1 = \sum_j B_{1,q_j}^* B_{1,q_j} \in \Psi^0(M)$ is elliptic on $\partial \overline{T^*_{x_1}} M$. In a similar way we construct $B_2 = \sum_l B_{2,q_l'}^* B_{2,q_l'} \in \Psi^0(M)$ elliptic on $\partial \overline{T^*_{x_2}} M$. This gives

$$B_1 G_z B_2^* = \sum_{j,l} B_{1,q_j}^* B_{1,q_j} G_z B_{2,q_l'}^* B_{2,q_l'} = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$$

using the fact that the sum is finite.

We can find a microlocal parametrix of B_1 and B_2, i.e. $B_i^{(-1)} \in \Psi^0(M)$ such that $R_1 = \mathbf{1} - B_1^{(-1)} B_1$ and $R_2 = \mathbf{1} - B_2 B_2^{(-1)}$ satisfy $\mathrm{WF}'(R_i) \cap \partial \overline{T^*_{x_i}} M = \emptyset$. This implies that there is a neighborhood O_i of x_i in M such that $\mathrm{WF}'(R_i) \cap \partial \overline{T^*_{O_i}} M = \emptyset$. Let $\chi_i \in C_c^\infty(M)$ be such that $\operatorname{supp} \chi_i \subset O_i$ and $\chi_i \equiv 1$ near x_i. We have

$$\begin{aligned} \chi_1 G_z \chi_2 = {} & \chi_1 B_1^{(-1)}(B_1 G_z B_2^*) B_2^{(-1)*} \chi_2 + \chi_1 R_1 G_z B_2^* B_2^{(-1)*} \chi_2 \\ & + \chi_1 B_1^{(-1)} B_1 G_z R_2^* \chi_2 + \chi_1 R_1 G_z R_2^* \chi_2, \end{aligned}$$

where all the summands are $\mathcal{O}_{H^* \to H^{*+s}}(h(z))$, hence $\chi_1 G_z \chi_2 \in \mathcal{O}_{H^* \to H^{*+s}}(h(z))$. ∎

Lemma 3.6. *If* $G_z = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$ *then* $(q_1, q_2) \in \mathrm{WF}'^{(s)}_{h(z)}(G_z)$ *if and only* $(q_2, q_1) \in \mathrm{WF}'^{(s)}_{h(z)}(G_z^*)$.

Proof. If $B_1 G_z B_2^*$ is $\mathcal{O}_{H^* \to H^{*+s}}(h(z))$ then so is its formal adjoint $B_2^* G_z^* B_1$, where B_2^* is elliptic at q_2 and B_1^* is elliptic at q_1. ∎

Lemma 3.7. *Let $G_{1,z} = \mathcal{O}_{C^\infty \to C^\infty}(h_1(z))$ and $G_{2,z} = \mathcal{O}_{C^\infty \to C^\infty}(h_2(z))$ and suppose that the operators $G_{2,z}$ are properly supported for all $z \in Z$. Then the composition $G_{1,z}G_{2,z} = \mathcal{O}_{C^\infty \to C^\infty}(h_1 h_2(z))$ is well-defined and satisfies*

$$\mathrm{WF}'^{(s)}_{h_1 h_2(z)}(G_{1,z}G_{2,z}) \subset \mathrm{WF}'^{(s)}_{h_1(z)}(G_{1,z}) \circ \mathrm{WF}'^{(s)}_{h_2(z)}(G_{2,z}), \tag{3.14}$$

where the composition of $\Gamma_1, \Gamma_2 \subset \partial\overline{T^*}M \times \partial\overline{T^*}M$ is defined by

$$\Gamma_1 \circ \Gamma_2 = \{(q_1, q_2) \in \partial\overline{T^*}M \times \partial\overline{T^*}M \mid \exists q \in \partial\overline{T^*}M : (q_1, q) \in \Gamma_1,\ (q, q_2) \in \Gamma_2\}.$$

Proof. For all $A_1, A_2 \in \Psi^0(M)$,

$$A_1 G_{1,z} G_{2,z} A_2^* = \sum_k (A_1 G_{1,z} B_k^*)(B_k G_{2,z} A_2^*), \tag{3.15}$$

where $B_k \in \Psi^0(M)$ is an arbitrary family such that $\sum_k B_k^* B_k = \mathbf{1}$ as a locally finite sum. By taking $\mathrm{WF}'(B_k)$ sufficiently small and using (3.15) we obtain (3.14). ∎

Let us now explain the relation with more standard notions which will be used in later sections.

Definition 3.8. Let X be a smooth (boundaryless) manifold and let $\Lambda \subset T^*X \setminus o$ be conic. Let $\{u_z\}_{z\in Z}$ be a family of distributions on X. We write $u_z = \mathcal{O}_{\mathcal{D}'_\Lambda}(h(z))$ iff for all $A \in \Psi^0(X)$ satisfying $\mathrm{WF}'(A) \cap \Lambda = \emptyset$ we have $Au_z = \mathcal{O}_{C^\infty}(h(z))$.

Definition 3.9. Let $\kappa : T^*M \setminus o \to \partial\overline{T^*}M$ be the quotient map for the $\mathbb{R}_{>0}$ action by fiberwise dilations. For each conic set $\Lambda \subset T^*(M \times M) \setminus o$ we define

$$\Lambda' = \{(\kappa(x_1; \xi_1), \kappa(x_1; -\xi_2)) \mid (x_1, x_2; \xi_1, \xi_2) \in \Lambda,\ \xi_1 \neq 0,\ \xi_2 \neq 0\},$$

which is a subset of $\partial\overline{T^*}M \times \partial\overline{T^*}M$.

Lemma 3.10. *Suppose $\Lambda \subset T^*(M \times M) \setminus o$ is conic and $G_z = \mathcal{O}_{C^\infty \to C^\infty}(h(z))$. If the associated family of Schwartz kernels satisfies $G_z(\cdot) = \mathcal{O}_{\mathcal{D}'_\Lambda}(h(z))$ then $\mathrm{WF}'^{(s)}_{h(z)}(G_z) \subset \Lambda'$ for all $s \in \mathbb{R}$.*

Proof. For ease of notation we identify $T^*M \setminus o$ with $\partial\overline{T^*}M$ using the quotient map κ. Let $q = (x_1, x_2; \xi_1, \xi_2) \in T^*(M \times M) \setminus \Lambda$ with $\xi_1 \neq 0$ and $\xi_2 \neq 0$, and let Γ_i, $i = 1, 2$, be a small conic neighborhood of $(x_i; \xi_i)$, to be fixed later on. Let $B_i \in \Psi^0(M)$ be elliptic at $(x_i; \xi_i)$, with $\mathrm{WF}'(B_i) \subset \Gamma_i$. Let $A \in \Psi^0(M \times M)$ be elliptic on $\Gamma_1 \times \Gamma_2$ and with symbol vanishing in a conical neighborhood of $o \times T^*M$ and $T^*M \times o$. This implies that $A(B_1 \otimes B_2) \in \Psi^0(M)$ and that $A(B_1 \otimes B_2)$ is elliptic at q. Since $G_z(\cdot) = \mathcal{O}_{\mathcal{D}'_\Lambda}(h(z))$ and $q \notin \Lambda$, we can take Γ_1, Γ_2 such that $\mathrm{WF}'(A(B_1 \otimes B_2))$ is in a small enough neighborhood of q so that $A(B_1 \otimes B_2)G_z(x, y) = \mathcal{O}_{C^\infty}(h(z))$. By ellipticity of A on $\Gamma_1 \times \Gamma_2$, this implies $(B_1 \otimes B_2)G_z(x, y) = \mathcal{O}_{C^\infty}(h(z))$, where B_1, B_2 acts on the first, resp. second variable of the Schwartz kernel of G_z. Hence $B_1 G_z \bar{B}_2^* = \mathcal{O}_{H^* \to H^{*+s}}(h(z))$ for all $s \in \mathbb{R}$, where $\bar{B}_2^* \in \Psi^0(M)$ is defined via complex conjugation of the Schwartz kernel of B_2^* and so is elliptic at $(x_2; -\xi_2)$. This implies $((x_1; \xi_1), (x_2; -\xi_2)) \notin \mathrm{WF}'^{(s)}_{h(z)}(G_z)$ for all $s \in \mathbb{R}$ as claimed. ∎

3.3. Uniform parametrices

After these general considerations we return to the setting of the operator $P - z$, though for the moment the only relevant assumption is that P is of real principal type.

Recall that $\partial \overline{{}^{\rm sc}T^*} M$ is identified with $\partial \overline{T^*} M$ over the interior M of $\overline{M}$, and in this sense the characteristic set of $\Box_g - z$ over M is

$$\Sigma := \Sigma_z \cap \partial \overline{T^*} M$$

and does not depend on z. Let us denote by $t \mapsto \Phi_t$ the bicharacteristic flow in Σ. For $q_1, q_2 \in \partial \overline{T^*} M$, we write

$$q_1 \sim q_2 \quad (\text{resp. } q_1 \prec q_2, \text{ or } q_1 \succ q_2)$$

if $q_1, q_2 \in \Sigma$ and $\Phi_t(q_1) = q_2$ for some $t \in \mathbb{R}$ (resp. $t > 0$, or $t < 0$). If $q_1 \sim q_2$ we denote by $\gamma_{q_1 \sim q_2}$ the closed bicharacteristic segment in Σ from q_1 to q_2.

We consider families $\{G_z\}_{z \in Z}$ of operators parametrized by some $Z \subset \mathbb{C}$. If the reference weight $h(z)$ is identically 1 we simply write ${\rm WF}'^{(s)}(G_z)$ for the uniform Sobolev wavefront set instead of ${\rm WF}_1^{(s)}(G_z)$. Other particularly useful weights are $h(z) = |{\rm Im}\, z|^{-1/2}$, $h(z) = \langle {\rm Im}\, z \rangle^{-1/2}$ and $h(z) = \langle z \rangle^{-1/2}$.

We state below a variant of Hörmander's propagation of singularities theorem for the uniform operator wavefront set. For convenience, we formulate it as a corollary of propagation estimates in weighted Sobolev spaces recalled in Appendix A.3 in our setting. Note that the statement is valid in greater generality (aspects at infinity of M being irrelevant), as one can give a direct proof by adapting Hörmander's positive commutator estimates [70, Section 6.5] along the lines of Vasy's work [131] to account for the case ${\rm Im}\, z \neq 0$.

Proposition 3.11. *Let $Z \subset \{{\rm Im}\, z \geqslant 0\}$. Assume $G_z = \mathcal{O}_{C^\infty \to C^\infty}(1)$, and suppose that $(P - z)G_z = \mathcal{O}_{C^\infty \to C^\infty}(1)$ and*

$$(q_1, q_2) \in {\rm WF}'^{(s)}(G_z) \setminus {\rm WF}'^{(s-1)}((P - z)G_z). \tag{3.16}$$

Then $q_1 \in \Sigma$, and $(q_1', q_2) \in {\rm WF}'^{(s)}(G_z)$ for all q_1' such that $q_1' \prec q_1$ provided that $(q, q_2) \notin {\rm WF}'^{(s-1)}((P - z)G_z)$ for all $q \in \gamma_{q_1 \sim q_1'}$. Similarly, if we assume $G_z(P - z) = \mathcal{O}_{C^\infty \to C^\infty}(1)$ and

$$(q_1, q_2) \in {\rm WF}'^{(s)}(G_z) \setminus {\rm WF}'^{(s-1)}(G_z(P - z)), \tag{3.17}$$

then $q_2 \in \Sigma$, and $(q_1, q_2') \in {\rm WF}'^{(s)}(G_z)$ for all q_2' such that $q_2' \succ q_2$ provided that $(q_1, q) \notin {\rm WF}'^{(s-1)}(G_z(P - z))$ for all $q \in \gamma_{q_2 \sim q_2'}$.

Proof. For the first statement, suppose $(q_1', q_2) \notin {\rm WF}'^{(s)}(G_z)$. Then by definition there exist $B_1', B_2 \in \Psi^0(M)$ elliptic at q_1', q_2 respectively such that for any bounded subset $H_{\rm c}^l(M)$, the set $B_1' G_z B_2^* \mathcal{U}$ is uniformly bounded in $H_{\rm loc}^{l+s}(M)$. Without loss of generality we can assume B_1' is supported away from $\partial \overline{M}$, so that $B_1' \in \Psi_{\rm sc}^0(M)$ and ${\rm WF}_{\rm sc}'(B_1') \cap \partial^{\rm sc} T^* \overline{M} = \emptyset$ (see Appendix A.2). We apply Proposition A.1 to each $u \in G_z B_2^* \mathcal{U}$. More precisely, we take ℓ arbitrary, $A_2 = B_1'$, and $B, A_1 \in \Psi^0(M)$ supported away from $\partial \overline{M}$

such that A_1 is elliptic at q_1, $\mathrm{WF}'(A_1)$ is a small neighborhood of q_1, and B is elliptic on $\gamma_{q_1 \sim q_1'}$ (note that in view of A_1, A_2, B being supported away from $\partial \overline{M}$, this simply corresponds to propagation of singularities at fiber infinity $\partial \overline{T^*M}$). In consequence (using the sign condition $\operatorname{Im} z \geqslant 0$ to get rid of the $(\operatorname{Im} z)^{1/2} \|A_1 u\|_{s-1/2,\ell+1/2}$ term in the estimate), we conclude that $A_1 G_z B_2^* \mathcal{U}$ is bounded in $H^{l+s}_{\mathrm{loc}}(M)$. Hence, $(q_1, q_2) \notin \mathrm{WF}'^{(s)}(G_z)$. This proves the first statement.

The second statement follows by applying the analogue of the first statement for $\operatorname{Im} z \leqslant 0$ to the adjoint families G_z^* and $(P-z)^*$ and then using Lemma 3.6. ∎

Note that if $Z \subset \{\operatorname{Im} z = 0\}$, then by considering $-P$ instead of P one obtains propagation in the other direction, and in consequence, in that case $q_1' \prec q_1$ and $q_2' \succ q_2$ can be replaced by $q_1' \sim q_1$ and $q_2' \sim q_2$ in the statement of Proposition 3.11.

Remark 3.12. In (3.16) and (3.17) the set $\mathrm{WF}'^{(s)}(G_z)$ can be replaced by

$$\mathrm{WF}'^{(s)}(G_z) \cup \mathrm{WF}'^{(s-1/2)}_{|\operatorname{Im} z|^{-1/2}}(G_z),$$

and therefore by $\mathrm{WF}'^{(s-1/2)}_{\langle \operatorname{Im} z \rangle^{-1/2}}(G_z)$. This is a consequence of the fact that for $\operatorname{Im} z \neq 0$, the propagation estimates become stronger: indeed, instead of getting rid of the term $(\operatorname{Im} z)^{1/2} \|A_1 u\|_{s-1/2,\ell+1/2}$ in the proof of Proposition 3.11, we can use it to get an estimate for $\|A_1 u\|_{s-1/2}$.

We use the notation $q_i = (x_i; \xi_i)$ for points in $\partial \overline{T^*_{x_i}} M$. The *enlarged diagonal* in $\partial \overline{T^*} M \times \partial \overline{T^*} M$ is the set

$$T^*_\Delta(M \times M) := \{(q_1, q_2) \in \partial \overline{T^*} M \times \partial \overline{T^*} M \mid x_1 = x_2\}.$$

By abuse of notation, in later sections the image of $T^*_\Delta(M \times M)$ under the identification of $\partial \overline{T^*} M$ with $T^*M \setminus o$ will also be denoted by $T^*_\Delta(M \times M)$ (note that by definition it does not contain the zero section of $T^*(M \times M)$).

Definition 3.13. We say that G_z is a *uniform parametrix of order* $s \in \mathbb{R}$ in $M_0 \subset M$ (more precisely, a right parametrix) for the family $\{(P-z)\}_{z \in Z}$ if $G_z = \mathcal{O}_{C^\infty \to C^\infty}(\langle z \rangle^{-1/2})$ and

$$(P-z) G_z = \mathbf{1} + R_z \quad \text{on } M_0, \tag{3.18}$$

for some $R_z = \mathcal{O}_{H^* \to H^{*+s}}(1)$. We say that G_z is a *uniform local parametrix of order* $s \in \mathbb{R}$ if (3.18) holds true for some $R_z = \mathcal{O}_{C^\infty \to C^\infty}(1)$ satisfying merely

$$\mathrm{WF}'^{(s-1/2)}(R_z) \cap U = \emptyset,$$

where U is some neighborhood of $T^*_\Delta(M \times M)$.

Proposition 3.14. *Suppose that G_z is a uniform local parametrix of order s in M which satisfies*

$$\mathrm{WF}'^{(s)}(G_z) \subset \{(q_1, q_2) \in \Sigma \times \Sigma \mid q_1 \succ q_2\} \cup T^*_\Delta(M \times M). \tag{3.19}$$

If $\widetilde{G}_z$ is a local uniform parametrix of order s which also satisfies (3.19), then

$$\mathrm{WF}'^{(s)}_{\langle \operatorname{Im} z\rangle^{-1/2}}(\widetilde{G}_z - G_z) \cap U = \emptyset \tag{3.20}$$

for some neighborhood U of $T^*_\Delta(M\times M)$.

Proof. There exists a neighborhood U of $T^*_\Delta(M\times M)$ such that

$$\mathrm{WF}'^{(s-1/2)}((P-z)(\widetilde{G}_z - G_z)) \cap U = \emptyset.$$

Suppose $(q_1, q_2) \in \mathrm{WF}'^{(s)}_{\langle \operatorname{Im} z\rangle^{-1/2}}(\widetilde{G}_z - G_z) \cap U$. Then by Proposition 3.11 and Remark 3.12, $q_1 \in \Sigma$ and $(q_1', q_2) \in \mathrm{WF}'^{(s)}(\widetilde{G}_z - G_z)$ for some $q_1' \sim q_1$ with $x_1' \neq x_2$ and such that $q_1' \prec q_2$ if $q_1 \sim q_2$. On the other hand, (3.19) and the analogous assumption for $\widetilde{G}_z$ imply that $q_1' \succ q_2$ or $x_1' = x_2$, which gives a contradiction. This proves (3.20). ∎

3.4. Global hyperbolicity

From now on we make the additional assumption that (M, g) is a *global hyperbolic spacetime*.

Recall that (M, g) is a spacetime if it is equipped with a time orientation. It is a *globally hyperbolic spacetime* (or for short, globally hyperbolic space) if in addition it admits a *Cauchy surface*, i.e., a closed subset of M which is intersected exactly once by each maximally extended time-like curve. By a result of Geroch [55] and Bernal–Sánchez [10,11], there exists an $(n-1)$-dimensional smooth manifold Y and an isometric diffeomorphism $\varphi : M \to \mathbb{R}\times Y$ such that

$$\varphi^* g = c^2(t, y)dt^2 - h_t(y)dy^2, \tag{3.21}$$

where $c \in C^\infty(M)$, $c > 0$, $\mathbb{R} \ni t \mapsto h_t(y)dy^2$ is a smooth family of Riemannian metrics, and for all $t_0 \in \mathbb{R}$, $\{t_0\}\times Y$ is a smooth space-like Cauchy surface in $\mathbb{R}\times Y$.

3.5. Uniform parametrix construction

We will prove the existence of a *Feynman parametrix* in the sense of Duistermaat–Hörmander, which has a special form and is uniform in z.

Thanks to (3.21) we can work on the n-dimensional manifold $\mathbb{R}\times Y \cong M$ with coordinates denoted by $x = (t, y)$. We will need a t-dependent variant of the parameter-dependent pseudo-differential calculus developed by Shubin [110].

Let $Z \subset \mathbb{C}$. Let $U \subset \mathbb{R}^{n-1}$ be an open set and $s \in \mathbb{R}$. Recall that the symbol space $S^s(T^*U)$ consists of functions $a(y, \eta) \in C^\infty(T^*U)$ such that

$$(1+|\eta|)^{-s+|\beta|}\partial_y^\alpha \partial_\eta^\beta a(y,\eta) \text{ is bounded on } U\times\mathbb{R}^{n-1}$$

for all $\alpha, \beta \in \mathbb{N}^{n-1}_{\geqslant 0}$. We denote by $C^\infty(\mathbb{R}; S^s_Z(T^*U))$ the space of functions $a(t, z, y, \eta)$ such that $a(t, z_0, y, \eta) \in C^\infty(\mathbb{R}\times T^*U)$ for each fixed $z_0 \in Z$, and

$$((1+|\eta|+|z|^{1/2})^{-s+|\beta|}\partial_t^\gamma \partial_y^\alpha \partial_\eta^\beta a(t,z,y,\eta) \text{ is bounded on } I\times Z\times U\times\mathbb{R}^{n-1} \tag{3.22}$$

for all $\gamma \in \mathbb{N}_{\geqslant 0}$, all $\alpha, \beta \in \mathbb{N}_{\geqslant 0}^{n-1}$ and all intervals $I \Subset \mathbb{R}$. Note that taking the square root of $|z|$ is natural from the point of view of the spectral theory of *second order* elliptic differential operators. The space $C^\infty(\mathbb{R}; S^s(T^*U))$ is defined by replacing (3.22) by

$$(1+|\eta|)^{-s+|\beta|}\partial_t^\gamma \partial_y^\alpha \partial_\eta^\beta a(t,z,y,\eta) \text{ is bounded on } I \times Z \times U \times \mathbb{R}^{n-1}$$

for all $\gamma \in \mathbb{N}_{\geqslant 0}$, all $\alpha, \beta \in \mathbb{N}_{\geqslant 0}^{n-1}$ and all intervals $I \Subset \mathbb{R}$. Thus, elements of the space $C^\infty(\mathbb{R}; S^s(T^*U))$ depend on $z \in Z$, but only in a very mild way, which is why we do not indicate it in the notation explicitly.

Recall that all pseudo-differential operators in $\Psi^s(Y)$ can be obtained first by reduction to the case of an open set $U \subset \mathbb{R}^{n-1}$ (using a partition of unity subordinate to a locally finite cover by charts), then by quantization of elements of $S^s(T^*U)$, and finally by adding the ideal of smoothing operators. By applying an analogous procedure to t- and z-dependent elements of $S^s(T^*U)$ we obtain classes of t- and z-dependent pseudo-differential operators on Y. We denote by $C^\infty(\mathbb{R}; \Psi^s_Z(Y))$ the class obtained from elements of $C^\infty(\mathbb{R}; S^s_Z(T^*U))$ (plus the ideal of t,z-dependent smoothing operators, with all t-derivatives bounded and rapidly decaying in $|z|$), and by $C^\infty(\mathbb{R}; \Psi^s(Y))$ the class obtained from elements of $C^\infty(\mathbb{R}; S^s(T^*U))$ (plus the ideal of t,z-dependent smoothing operators, with all t-derivatives bounded and rapidly decaying in $|z|$, denoted by $C^\infty(\mathbb{R}; \Psi^{-\infty}_Z(Y))$ by abuse of notation). We say that A is *properly supported* if there exists a closed set $K \subset Y \times Y$ with proper projections on each factor of $Y \times Y$ and such that the Schwartz kernel of A is supported in K for all $t \in \mathbb{R}$ and $z \in Z$.

By the exact t-dependent analogue of the proofs in [110, Section 9] we can prove properties of the $C^\infty(\mathbb{R}; \Psi^s_Z(Y))$ and $C^\infty(\mathbb{R}; \Psi^s(Y))$ classes under composition and taking adjoints. In particular, for all $s_1, s_2 \in \mathbb{R}$, for properly supported operators we have

$$[A \in C^\infty(\mathbb{R}; \Psi^{s_1}_Z(Y)),\ B \in C^\infty(\mathbb{R}; \Psi^{s_2}_Z(Y))] \implies AB \in C^\infty(\mathbb{R}; \Psi^{s_1+s_2}_Z(Y)).$$

Of particular use for us are operators in $C^\infty(\mathbb{R}; \Psi^s_Z(Y))$ with symbols that are one-step poly-homogeneous in $(\eta, z^{1/2})$ (for $|\eta| + |z|^{1/2} \geqslant 1$), see [110, Section 9.1]. We say that such an operator A is *elliptic with parameter* if it is properly supported and

$$a_s(t,z,y,\eta) \neq 0 \quad \text{if } |\eta| + |z|^{1/2} \neq 0,$$

where a_s is the leading order term in the poly-homogeneous expansion. Standard poly-homogeneous expansion arguments can be used to show that if $A \in C^\infty(\mathbb{R}; \Psi^s_Z(Y))$ is elliptic with parameter, then it has a parametrix in $C^\infty(\mathbb{R}; \Psi^{-s}_Z(Y))$ which is also elliptic with parameter, and the error is in $C^\infty(\mathbb{R}; \Psi^{-\infty}_Z(Y))$.

Example 3.15 (cf. [110, Example 9.1]). If $L(t)$ is a second order differential operator on Y with coefficients in $C^\infty(\mathbb{R}; C^\infty(Y))$, then the leading order term in the poly-homogeneous expansion of the symbol of $L(t) - z$ is simply $\sigma_{\rm pr}(L(t)) - z$, where $\sigma_{\rm pr}(L(t))$ is the principal symbol in the usual $\Psi^s(Y)$ sense. Therefore, $L(t) - z$ is elliptic with parameter if $\sigma_{\rm pr}(L(t))$ does not intersect Z at $\{|\eta| = 1\}$.

It is also occasionally useful to work with pseudo-differential operators of order not consistent with the order of decay in z. Namely, we write $R(t,z) \in C^\infty(\mathbb{R}; \Psi_Z^{s_1,s_2}(Y))$ if

$$R(t,z) = \sum_{i=1}^{k} R_{1,i}(t,z) R_{2,i}(t,z)$$

for some $k \in \mathbb{N}_{>0}$ and $R_{1,i} \in C^\infty(\mathbb{R}; \Psi^{s_1}(Y))$, $R_{2,i} \in C^\infty(\mathbb{R}; \Psi_Z^{s_2}(Y))$, $i = 1, \dots, k$. Using the trivial inclusion $C^\infty(\mathbb{R}; \Psi_Z^s(Y)) \subset C^\infty(\mathbb{R}; \Psi^s(Y))$ for $s < 0$ whenever needed, we can show that for all $s_1, r_1, r_2 \in \mathbb{R}$ and all $s_2 < 0$,

$$\begin{aligned}
&[A \in C^\infty(\mathbb{R}; \Psi_Z^{s_1,s_2}(Y)),\ B \in C^\infty(\mathbb{R}; \Psi_Z^{r_1,r_2}(Y))] \\
&\qquad\qquad\qquad \implies AB \in C^\infty(\mathbb{R}; \Psi_Z^{s_1+s_2+r_1,r_2}(Y)), \\
&[A \in C^\infty(\mathbb{R}; \Psi_Z^{s_1,s_2}(Y)),\ B \in C^\infty(\mathbb{R}; \Psi_Z^{s_2}(Y))] \\
&\qquad\qquad\qquad \implies [A,B] \in C^\infty(\mathbb{R}; \Psi_Z^{s_1+s_2-1,s_2}(Y)).
\end{aligned} \tag{3.23}$$

Notation. We denote by Σ^+ and Σ^- the two connected components of Σ, distinguished by the property that within $\Sigma^\pm$, bicharacteristics flow in the past/future direction.

Proposition 3.16. *Assume global hyperbolicity. Let $Z \subset \mathbb{C}$ be an angle in the upper half-plane $\{\operatorname{Im} z \geqslant 0\}$ with vertex at the origin. Let $M_0 \subset M$ be an open subset such that for each $t \in \mathbb{R}$, $\varphi^* M_0 \cap (\{t\} \times Y)$ is included in a compact set. Then for all $s \geqslant 0$, the family $\{(P - z)\}_{z \in Z}$ has a uniform parametrix G_z of order $s \in \mathbb{R}$ in M_0 of the form $G_z = G_z^+ + G_z^-$, where $G_z^\pm = \mathcal{O}_{C^\infty \to C^\infty}(\langle z \rangle^{-1/2})$ has the property that for each $f \in \mathcal{E}'(M)$ there exists a Cauchy surface $t_0 \in \mathbb{R}$ such that*

$$\operatorname{supp} G_z^\pm f \subset (\varphi^{-1})^* \{\pm t \geqslant \pm t_0\}, \tag{3.24}$$

and furthermore

$$\mathrm{WF}'^{(s)}_{\langle z \rangle^{-1/2}}(G_z^\pm) \subset \{(q_1,q_2) \in \Sigma^\mp \times \Sigma^\mp \mid q_1 \succ q_2\} \cup T^*_\Delta(M \times M). \tag{3.25}$$

Proof. A straightforward computation shows that the differential operator

$$Q(t,z) := -c^2(t)(\varphi^*(P - z))$$

is of the form

$$Q(t,z) = D_t^2 + Q_0(t) D_t - Q_2(t,z),$$

where $Q_0(t) = \partial_t (c^{-1}(t) |h(t)|^{1/2}) \in C^\infty(\mathbb{R}; \Psi^0(Y))$ is a multiplication operator and

$$Q_2(t,z) = c(t) |h(t)|^{-1/2} \sum_{i,j=1}^{n-1} D_i c(t) h(t)^{ij} |h(t)|^{1/2} D_j - z c^2(t) \in C^\infty(\mathbb{R}; \Psi_Z^2(Y))$$

is elliptic with parameter. Our proof is divided into several steps.

Step 1. We claim that for each $i \in \mathbb{N}_{\geqslant 0}$ there exist $A_i(t,z), B_i(t,z) \in C^\infty(\mathbb{R}; \Psi^1_Z(Y))$, each elliptic with parameter, and $R_i(t,z) \in C^\infty(\mathbb{R}; \Psi^{1-i,0}_Z(Y))$, such that

$$Q(t,z) = (D_t - A_i(t,z))(D_t + B_i(t,z)) + R_i(t,z). \tag{3.26}$$

We show this inductively by adapting the arguments in [69, Section 23.2] and [51, Section 6] to our setting.

Namely, suppose that (3.26) holds true for some $i \in \mathbb{N}_{\geqslant 0}$. We then set

$$C_i := -R_i(A_i + B_i)^{(-1)}, \quad L_i := R_i(\mathbf{1} - (A_i + B_i)^{(-1)}(A_i + B_i)), \tag{3.27}$$

$$R_{i+1} := [C_i, D_t] + [A_i, C_i] + C_i^2 + L_i, \tag{3.28}$$

where $(A_i + B_i)^{(-1)} \in C^\infty(\mathbb{R}; \Psi^{-1}_Z(Y))$ is an elliptic parametrix of $A_i + B_i$, and the dependence on t, z is disregarded in the notation. Using (3.23), we obtain

$$C_i, L_i, R_{i+1} \in C^\infty(\mathbb{R}; \Psi^{1-(i+1),0}_Z(Y)).$$

These operators are defined in (3.27)–(3.28) in such a way that they satisfy the identities

$$C_i B_i + R_i = -C_i A_i + L_i, \quad C_i(D_t - A_i) + L_i = (D_t - A_i - C_i)R_{i+1},$$

which entail

$$(D_t - A_i)(D_t + B_i) + R_i = (D_t - A_i - C_i)(D_t + B_i + C_i) + R_{i+1}.$$

Thus, by setting

$$A_{i+1} := A_i + C_i \in C^\infty(\mathbb{R}; \Psi^1_Z(Y)) \quad \text{and} \quad B_{i+1} := B_i + C_i \in C^\infty(\mathbb{R}; \Psi^1_Z(Y))$$

we conclude that (3.26) holds true for $i+1$ in place of i.

Now, to show (3.26) it remains to check the base case $i = 0$. To that end we set

$$A_0(t,z) := (Q_2(t,z))^{(1/2)} - \tfrac{1}{2}Q_0(t), \quad B_0(t,z) := (Q_2(t,z))^{(1/2)} + \tfrac{1}{2}Q_0(t),$$

where $Q_2^{(1/2)}$ is an approximate square root obtained from the poly-homogeneous expansion of Q_2 in the parameter-dependent sense. By construction, $Q_2^{(1/2)}$, A_0, B_0 are in $C^\infty(\mathbb{R}; \Psi^1_Z(Y))$. Furthermore,

$$Q(z,t) = (D_t - A_0(t,z))(D_t + B_0(t,z)) + R_0(t,z),$$

where

$$R_0 = \tfrac{1}{2}[Q_0, D_t] + \tfrac{1}{4}Q_0^2 + [Q_2^{(1/2)}, D_t] \bmod C^\infty(\mathbb{R}; \Psi^{-\infty}_Z(Y)). \tag{3.29}$$

We want to show that $R_0 \in C^\infty(\mathbb{R}; \Psi^{1,0}_Z(Y))$. The first two terms on the l.h.s. of (3.29) clearly belong to that space as they are z-independent. The third term equals

$$\begin{aligned} i[\partial_t, Q_2^{(1/2)}] &= -\tfrac{1}{2}\partial_t(Q_2)Q_2^{(-1/2)} \bmod C^\infty(\mathbb{R}; \Psi^{-\infty}_Z(Y)) \\ &= -\tfrac{1}{2}\partial_t(Q_2)U^{-1}UQ_2^{(-1/2)} \bmod C^\infty(\mathbb{R}; \Psi^{-\infty}_Z(Y)), \end{aligned} \tag{3.30}$$

where $Q_2^{(-1/2)} \in C^\infty(\mathbb{R}; \Psi_Z^{-1}(Y))$ is an elliptic parametrix of $Q_2^{(1/2)}$ and where $U \in C^\infty(\mathbb{R}; \Psi^1(Y))$ is chosen elliptic and invertible. Then $U^{-1} \in C^\infty(\mathbb{R}; \Psi^{-1}(Y))$ and we have

$$(\partial_t Q_2)U^{-1} \in C^\infty(\mathbb{R}; \Psi^1(Y)), \quad UQ_2^{(-1/2)} \in C^\infty(\mathbb{R}; \Psi_Z^0(Y)),$$

where the second estimate is crude, but sufficient for our purpose. We conclude that the operator in (3.30) is in $C^\infty(\mathbb{R}; \Psi_Z^{1,0}(Y))$ as desired.

Step 2. We have proved in Step 1 that there exist $A, B \in C^\infty(\mathbb{R}; \Psi_Z^1(Y))$ elliptic with parameter, and $R \in C^\infty(\mathbb{R}; \Psi_Z^{-s-1,0}(Y))$, such that

$$Q(t,z) = (D_t - A(t,z))(D_t + B(t,z)) + R(t,z). \tag{3.31}$$

We can repeat the construction in Step 1 with the rôles of A_0 and B_0 reversed to obtain $\tilde{A}, \tilde{B} \in C^\infty(\mathbb{R}; \Psi_Z^1(Y))$ elliptic with parameter, and $\tilde{R} \in C^\infty(\mathbb{R}; \Psi_Z^{-s-1,0}(Y))$, such that

$$Q(t,z) = (D_t + \tilde{B}(t,z))(D_t - \tilde{A}(t,z)) + \tilde{R}(t,z). \tag{3.32}$$

To prove our main assertion, without loss of generality we can assume that $Y = \mathbb{R}^d$, and that $\tilde{A}(t,z), B(t,z) \in C^\infty(\mathbb{R}; \Psi_Z^1(\mathbb{R}^d))$ take values in the uniform pseudo-differential class on $\mathbb{R}^d$. In fact, at each time $t \in \mathbb{R}$ we can modify the definition of $Q(t,z)$ outside a compact set and then apply the construction from Step 1 to the modified operator. This brings about an extra error term on the r.h.s. of (3.31) and (3.32), but this error term vanishes outside M_0 and for this reason it will be of no relevance in the rest of the proof (as we only want a parametrix in M_0). We disregard it for simplicity of notation.

In the simplified situation with $Y = \mathbb{R}^d$, we want to show that $D_t - \tilde{A}(t,z)$ has an advanced inverse acting on $H_c^{-N}(\mathbb{R} \times Y)$, denoted in the sequel by $U_{\tilde{A}}^-(z)$, and $D_t + B(t,z)$ has a retarded inverse $U_{-B}^+(z)$. This means that $U_{\tilde{A}}^-(z)$ and $U_{-B}^+(z)$ are left inverses on $H_c^{-N}(\mathbb{R} \times Y)$ of the respective operators, and for all $f \in H_c^{-N}(\mathbb{R} \times Y)$ there exists $t_0 \in \mathbb{R}$ such that

$$\operatorname{supp} U_{\tilde{A}}^-(z) f \subset \{t \leqslant t_0\}, \quad \operatorname{supp} U_{-B}^+(z) f \subset \{t \geqslant t_0\}. \tag{3.33}$$

Modulo a reparametrization of the time interval, the existence of $U_{-B}^+(z)$ follows from well-posedness of the inhomogeneous Cauchy problem for $t \in [0,T]$ (with $T > 0$ arbitrary) shown in [69, Theorem 23.1.4], provided the assumptions are satisfied. To that end we need to check the boundedness from below:

$$-\operatorname{Im} B(t,z) \geqslant -C\mathbf{1} \quad \text{on } H^1(\mathbb{R}^d) \tag{3.34}$$

for all $t \in [0,T]$, where the constant $C > 0$ depends only on T (we use here the notation $\operatorname{Im} B = \frac{1}{2}(B + B^*)$ and $\operatorname{Re} B = \frac{1}{2}(B - B^*)$). Indeed, by a direct computation we find frac

$$\operatorname{Im} B = C_{1/4}^*(\operatorname{Im} Q)C_{1/4} + C_0, \tag{3.35}$$

where $C_{1/4}(t,z) \in C^\infty(\mathbb{R}; \Psi_Z^{1/4}(\mathbb{R}^d))$ is an approximate square root of $\operatorname{Re} Q^{(1/2)}(t,z)$

and $C_0(t,z) \in C^\infty(\mathbb{R}; \Psi^0_Z(\mathbb{R}^d))$, with values in the uniform pseudo-differential class on $\mathbb{R}^d$. In view of

$$-\operatorname{Im} Q(t,z) = -(\operatorname{Im} z) c^2(t) \geqslant 0,$$

(3.35) implies (3.34) with $C = \sup_{z\in Z} \sup_{t\in[0,T]} \|C_0(t,z)\| < +\infty$. The existence of $U^-_{\tilde{A}}(z)$ is shown analogously, with obvious sign changes.[1] Furthermore, thanks to the fact that the constant C does not depend on z, the proof of [69, Theorem 23.1.4] implies that

$$U^-_{\tilde{A}}(z) = \mathcal{O}_{H^*\to H^*}(1), \quad U^+_{-B}(z) = \mathcal{O}_{H^*\to H^*}(1), \tag{3.36}$$

where the notation refers to the mapping properties $H^l_{\rm c}(\mathbb{R}\times Y) \to H^l_{\rm loc}(\mathbb{R}\times Y)$ for all $l \in \mathbb{R}$, uniformly in z.

Next, using (3.31)–(3.32) we compute

$$\begin{aligned} Q(U^-_{\tilde{A}} - U^+_{-B}) &= ((D_t + \tilde{B})(D_t - \tilde{A}) + \tilde{R})U^-_{\tilde{A}} - ((D_t - A)(D_t + B) + R)U^+_{-B} \\ &= \tilde{B} + A + \tilde{R}U^-_{\tilde{A}} + RU^+_{-B}. \end{aligned}$$

If now $(\tilde{B} + A)^{(-1)} \in C^\infty(\mathbb{R}; \Psi^{-1}_Z(Y))$ is an elliptic parametrix of $\tilde{B} + A$, we conclude that

$$U^-_{\tilde{A}}(\tilde{B} + A)^{(-1)} \text{ and } U^+_{-B}(\tilde{B} + A)^{(-1)} \text{ are } \mathcal{O}_{H^*\to H^*}(\langle z\rangle^{-1}), \tag{3.37}$$

and

$$Q(U^-_{\tilde{A}} - U^+_{-B})(\tilde{B} + A)^{(-1)} = \mathbf{1} + E,$$

where $E = \mathcal{O}_{H^*\to H^{*+s}}(1)$.

Step 3. The wavefront sets of (3.36) can be estimated by a variant of Egorov's theorem. More precisely, let us first show

$$\mathrm{WF}'^{(s)}(U^+_{-B}(z))) \subset \{(q_1, q_2) \in \Sigma^- \times \Sigma^- \mid q_1 \sim q_2\} \cup T^*_\Delta(M\times M). \tag{3.38}$$

Let $q_1, q_2 \in \partial\overline{T^*}M$ with base points $(t_1, y_1) \neq (t_2, y_2)$. If $q_1 \notin \Sigma^-$ or $q_2 \notin \Sigma^-$ then $(q_1, q_2) \notin \mathrm{WF}'^{(s)}(U^+_{-B}(z))$ by microlocal ellipticity. Consider now $q_1 = (t_1, y_1; \tau_1, \eta_1) \in \Sigma^-$. By the arguments in the proof of [69, Theorem 23.1.4] there exists $S(t) \in C^\infty(\mathbb{R}; \Psi^0(Y))$ such that

$$[D_t + B(t,z), S(t)] \in C^\infty(\mathbb{R}; \Psi^{-\infty}(Y)), \tag{3.39}$$

$S(t_1) \in \Psi^0(Y)$ is elliptic at $(y_1; \eta_1)$ and $\mathrm{WF}'(S(t))$ is a neighborhood of $\Phi^{t-t_1}((y_1; \eta_1))$. Consider the tensor product operator $S \otimes \mathbf{1}$ acting on $M = \mathbb{R} \times Y$.[2] Furthermore, if $q_1 \nsim q_2$ then that neighborhood can be chosen in such a way that $(S \otimes \mathbf{1})S_2^* \in \Psi^{-\infty}(M)$ is smoothing for some $S_2 \in \Psi^0(M)$ elliptic at q_2 since $\eta_2 \neq 0$ with the symbol of S_2

[1] Note that for the advanced problem we need to solve the inhomogeneous Cauchy problem backwards, so the analogue of (3.34) has reversed sign, hence the necessity of considering $\tilde{A}$ in place of $-B$.

[2] Note that the tensor product of two pseudo-differential operators is not necessarily in the usual calculus.

vanishing in some conical neighborhood of $\eta = 0$ by [69, Theorem 18.1.35, p. 94]. In view of $q_1 \in \Sigma^-$ we have $\eta_1 \neq 0$ and therefore we can find $T \in \Psi^0(M)$ such that $S_1 := T \circ (S \otimes \mathbf{1}) \in \Psi^0(M)$ and S_1 is elliptic at q_1 again by [69, Theorem 18.1.35, p. 94], the symbol of T is also chosen to vanish in some conical neighborhood of $\eta = 0$.[3] Using (3.36), (3.39) and the fact that $(S \otimes \mathbf{1})S_2^*$ is smoothing, we obtain

$$\begin{aligned} S_1 U_{-B}^+(z) S_2^* = T(S \otimes \mathbf{1}) U_{-B}^+(z) S_2^* &= T U_{-B}^+(z)(S \otimes \mathbf{1}) S_2^* + T[(S \otimes \mathbf{1}), U_{-B}^+(z)] S_2^* \\ &= \mathcal{O}_{H^* \to H^{*+s}}(1). \end{aligned}$$

Since S_i is elliptic at q_i this shows that $(q_1, q_2) \notin \mathrm{WF}'^{(s)}(U_{-B}^+(z))$, and in this way we get (3.38).

Using the support properties (3.33) we can improve on (3.38) and eliminate points (q_1, q_2) in the wavefront set such that, writing $q_i = (x_i; \xi_i)$, x_1 is in the past of x_2. We can also observe that for $(q_1, q_2) \in \Sigma^+ \times \Sigma^+$ (resp. $\Sigma^- \times \Sigma^-$) with $q_1 \sim q_2$, x_1 is in the past of x_2 (resp. in the future) if and only if $q_1 \succ q_2$. Therefore,

$$\mathrm{WF}'^{(s)}(U_{-B}^+(z)) \subset \{(q_1, q_2) \in \Sigma^- \times \Sigma^- \mid q_1 \succ q_2\} \cup T_\Delta^*(M \times M).$$

In an analogous way we prove

$$\mathrm{WF}'^{(s)}(U_{\tilde{A}}^-(z)) \subset \{(q_1, q_2) \in \Sigma^+ \times \Sigma^+ \mid q_1 \succ q_2\} \cup T_\Delta^*(M \times M).$$

Since $(\tilde{B} + A)^{(-1)} \in C^\infty(\mathbb{R}; \Psi_Z^{-1}(Y))$ it follows that

$$\begin{aligned} \mathrm{WF}'^{(s)}_{\langle z \rangle^{-1/2}}(U_{-B}^+(\tilde{B}+A)^{(-1)}) \subset \{(q_1, q_2) \in \Sigma^- \times \Sigma^- \mid q_1 \succ q_2\} \cup T_\Delta^*(M \times M), \\ \mathrm{WF}'^{(s)}_{\langle z \rangle^{-1/2}}(U_{\tilde{A}}^-(\tilde{B}+A)^{(-1)}) \subset \{(q_1, q_2) \in \Sigma^+ \times \Sigma^+ \mid q_1 \succ q_2\} \cup T_\Delta^*(M \times M). \end{aligned} \tag{3.40}$$

We have therefore constructed a parametrix with properties (3.33), (3.40) and (3.37) analogous to the ones in the statement of the proposition, but for the auxiliary operator $Q(t, z)$ instead of $P - z$.

Step 4. It now remains to reformulate the parametrix construction in terms of $P - z$. Recall that P and Q are related by $Q = -c^2(\varphi^*(P - z))$, so by setting

$$\begin{aligned} G_z^- &:= -(\varphi^{-1})^* \big(U_{\tilde{A}}^-(z)(\tilde{B} + A)^{(-1)}(t, z) \big) c(t)^{-2}, \\ G_z^+ &:= -(\varphi^{-1})^* \big(-U_{-B}^+(z)(\tilde{B} + A)^{(-1)}(t, z) \big) c(t)^{-2}, \end{aligned}$$

and $G_z := G_z^+ + G_z^-$ we obtain a parametrix for $P - z$ with the desired properties. ∎

3.6. *Wavefront set of the resolvent*

We now proceed to estimate the uniform wavefront set of $(P - z)^{-1}$. Recall that γ_ε is the contour in the complex upper half-plane defined in Section 2.6 (see Figure 1).

[3] We have followed the method of Hörmander [69, p. 390] to convert space to spacetime wavefront bounds.

Lemma 3.17. *Assume that* (M, g) *is non-trapping and* $\varepsilon > 0$. *Then* $\{(P-z)^{-1}\}_{z\in\gamma_\varepsilon}$ *satisfies*

$$(P-z)^{-1} = \mathcal{O}_{C^\infty\to C^\infty}(\langle z\rangle^{-1/2}). \tag{3.41}$$

Assuming in addition non-trapping at energy $\sigma = m^2 \neq 0$ *and injectivity,* (3.41) *also holds true for* $\{(P-z)^{-1}\}_{z\in\gamma_0}$.

Proof. By (2.9) with $N = 0$, $(P-z)^{-1}$ and $(P-\bar{z})^{-1}$ are $\mathcal{O}(|\mathrm{Im}\, z|^{-1/2})$ for $z \in \mathbb{C}\setminus\mathbb{R}$ as bounded maps $H^l_{\rm c}(M) \to H^{l+1/2}_{\rm loc}(M)$ for all $l \geqslant 0$. The analogous claim for l negative follows by duality. Finally, the γ_0 case is shown similarly using (2.11) to get control for z down to the real axis. ∎

We now state the key lemma.

Lemma 3.18. *Assume that* (M, g) *is globally hyperbolic, non-trapping, and let* $\varepsilon > 0$. *For each bicharacteristic* γ *there exists* $U_\gamma \subset M$ *such that if* G_z, $z \in Z$, *is a uniform parametrix in* U_γ *as in Proposition* 3.16, *then for all* $s \in \mathbb{R}$,

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P-z)^{-1} - G_z) \cap (\gamma \times \partial\overline{T^*}M) \subset T^*_\Delta(M\times M).$$

Proof. Let U_γ be a small enough neighborhood of the base projection of γ so that for each $t \in \mathbb{R}$, $\varphi^* U_\gamma \cap (\{t\}\times Y)$ is included in a compact set. Then Proposition 3.16 yields a uniform parametrix $G_z = G_z^+ + G_z^-$ in U_γ, where $G_z^\pm$ solves a pseudo-differential retarded/advanced problem. Let $L_-^\pm = L_- \cap \Sigma^\pm$ be the future/past component of the sources L_-. By Fredholm estimates, i.e. Proposition 2.6, for any $s \in \mathbb{R}$ and any bounded subset $\mathcal{U} \subset H^{s-1}_{\rm c}(M)$, the set $(P-z)^{-1}\mathcal{U}$ is uniformly bounded in $H^{s,\ell}_{\rm sc}(M)$ for arbitrary $s \in \mathbb{R}$ and for some ℓ with $\ell_- > -1/2$, thus in particular with $\ell > -1/2$ in a neighborhood of $L_-^\mp$. By support properties of $G_z^\pm$, i.e. by (3.24), $G_z^\pm\mathcal{U}$ vanishes in the far past/future. Therefore, $G_z^\pm\mathcal{U}$ is uniformly bounded in $H^{s,\ell}_{\rm sc}(M)$ (after possibly modifying the definition of ℓ outside a neighborhood of $L_-^\pm$).

We can therefore apply the higher decay radial estimate (Proposition A.3) to the family $(P-z)^{-1} - G_z^\pm$, which is a uniform bi-solution of $P - z$ microlocally near $\gamma\cap\Sigma^\mp$. This allows us to conclude that $B^\pm((P-z)^{-1} - G_z^\pm)\mathcal{U}$ is $\mathcal{O}_{C^\infty}(\langle z\rangle^{-1/2})$ for some $B^\pm \in \Psi^{0,0}_{\rm sc}(M)$ elliptic on $L_-^\mp$. Thus, in $\gamma \times \partial\overline{T^*}M$, $L_-^\mp \times \partial\overline{T^*}M$ is disjoint from $\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P-z)^{-1} - G_z^\pm)$. By the non-trapping assumption and propagation of singularities, the whole flow-out of $L_-^\mp \times \partial\overline{T^*}M$ (in the first variable) within $\Sigma^\mp$ is disjoint from $\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P-z)^{-1} - G_z^\pm)$ in $\gamma \times \partial\overline{T^*}M$. This means that

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P-z)^{-1} - G_z^\pm) \subset \Sigma^\pm \times \partial\overline{T^*}M$$

in $\gamma \times \partial\overline{T^*}M$. We now combine this with (3.25) to conclude that

$$\begin{aligned}\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P-z)^{-1} - G_z) &\subset \mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P-z)^{-1} - G_z^\pm) + \mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}(G_z^\mp)\\ &\subset (\Sigma^\pm \times \partial\overline{T^*}M) \cup T^*_\Delta(M\times M)\end{aligned}$$

in $\gamma \times \partial\overline{T^*}M$. Since $\Sigma^+ \cap \Sigma^- = \emptyset$, this implies the assertion of the lemma. ∎

Theorem 3.19. *Assume that (M, g) is non-trapping, globally hyperbolic, and let $\varepsilon > 0$. Then for any $s \in \mathbb{R}$, the family $\{(P - z)^{-1}\}_{z \in \gamma_\varepsilon}$ satisfies*

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}((P - z)^{-1}) \subset \{(q_1, q_2) \in \Sigma \times \Sigma \mid q_1 \succ q_2\} \cup T^*_\Delta(M \times M). \tag{3.42}$$

Moreover, suppose that H_z is a local uniform parametrix of order s for $P - z$ in the sense of Definition 3.13, and H_z also satisfies (3.42). Then for all $x \in M$ there exists $\chi \in C^\infty_{\mathrm{c}}(M)$ with $\chi(x) = 1$ such that

$$\chi(P - z)^{-1}\chi = \chi H_z \chi + \mathcal{O}_{H^* \to H^{*+s}}(\langle z\rangle^{-1/2}). \tag{3.43}$$

Proof. The estimate (3.42) follows now directly from Lemma 3.18 applied to all bicharacteristics γ and from the fact that

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-1/2}}(G_z) \subset \{(q_1, q_2) \in \Sigma \times \Sigma \mid q_1 \succ q_2\} \cup T^*_\Delta(M \times M)$$

by Proposition 3.16. The second assertion follows directly from (3.42) and Proposition 3.14. ∎

We will show that a local uniform parametrix of arbitrarily high order can be obtained by a z-dependent variant of the Hadamard parametrix construction.

The result (3.43) is satisfactory for many purposes, we remark however that it does not give stronger decay of the error term on the r.h.s. even if $(P - z)H_z - \mathbf{1}$ has better decrease in z. For this reason in Section 6 we will use a more precise composition argument.

Remark 3.20. Assuming in addition non-trapping and injectivity at energy $\sigma = m^2 \neq 0$, Lemma 3.18 and Theorem 3.19 also hold true for γ_0 instead of γ_ε.

Remark 3.21. All the results in Sections 2–3 generalize in a straightforward way to the case when P is a principally scalar wave operator on a finite-dimensional Hermitian bundle E, provided that P is formally self-adjoint for the canonical scalar product induced by the Hermitian form on fibers and by the volume form. We stress that this requires having a scalar product which is in particular *positive*. In more general situations such as the wave equation on tensors, the propagation estimates need to be modified (see e.g. [63]).

4. The elementary family $F_\alpha(z, x)$

4.1. Definition of the family $F_\alpha(z, x)$

In this section we define a family $F_\alpha(z, \cdot)$ of distributions on $\mathbb{R}^n$ which is the first ingredient in the Hadamard parametrix construction. We analyze its regularity properties and its dependence on the complex parameter z. More precisely, we control the wavefront set uniformly in z along the contour γ_ε defined in Section 2.6. We also study the Hölder regularity asymptotically in the parameter z on the upper half-plane $\{\operatorname{Im} z > 0\}$ down to $z \in \mathbb{R} \setminus \{0\}$.

Let $\alpha \in \mathbb{C}$. When writing complex powers we always use the usual branch of the log defined on $\mathbb{C} \setminus]-\infty, 0]$. For $\operatorname{Im} z > 0$, we define the distribution in the $x \in \mathbb{R}^n$ variable

$$F_\alpha(z, x) = \frac{\Gamma(\alpha+1)}{(2\pi)^n} \int e^{i\langle x,\xi\rangle} (|\xi|_\eta^2 - z)^{-\alpha-1} \, d^n\xi \tag{4.44}$$

in the sense of an inverse Fourier transform, where $\eta = dx_0^2 - (dx_1^2 + \cdots + dx_{n-1}^2)$ is the flat Minkowski metric, and $|\xi|_\eta^2 = -\xi \cdot \eta^{-1}\xi = -\xi_0^2 + \sum_{i=1}^{n-1} \xi_i^2$ is defined for convenience with a *minus* sign. The distribution (4.44) is Lorentz invariant.

Next, we extend the definition (4.44) to $z \in \mathbb{R} \setminus \{0\}$. To that end we define the family of distributions $(|\xi|_\eta^2 - z - i0)^{-\alpha-1}$ corresponding to taking the limit of $(|\xi|_\eta^2 - z)^{-\alpha-1}$ as $\operatorname{Im} z \to 0^+$. More precisely, denoting $Q(\xi) = |\xi|_\eta^2$, for $z \in \mathbb{R}$ we define as in [48, III, Section 2.4],

$$(Q(\xi) - z - i0)^{-\alpha} = \lim_{\varepsilon \to 0^+} (Q(\xi) - z - i\varepsilon)^{-\alpha},$$

considered first as a distribution on $\mathbb{R}^n \setminus \{0\}$.

Proposition 4.1. *The family $\{(Q(\xi) - i0)^{-\alpha}\}_{\alpha\in\mathbb{C}}$ of distributions is well-defined on $\mathbb{R}^n \setminus \{0\}$ by pull-back. It extends homogeneously to $\mathbb{R}^n$ as a meromorphic family in $\alpha \in \mathbb{C}$ with simple poles contained in $\mathbb{N} + n/2$. The residues at the poles are distributions supported at $0 \in \mathbb{R}^n$.*

On the other hand, if $z \in \mathbb{R} \setminus \{0\}$, then $\{(Q(\xi) - z - i0)^{-\alpha}\}_{\alpha\in\mathbb{C}}$ is a holomorphic family of distributions on $\mathbb{R}^n$.

Proof. The meromorphic family of distributions $(t - i0)^{-\alpha}$ in $\mathcal{S}'(\mathbb{R})$ has singular support only at $t = 0$. Observe that along the cone $Q = 0$, we have $dQ(\xi) \neq 0$ when $\xi \neq 0$. Therefore, the pull-back $Q^*(t - i0)^{-\alpha}$ is well-defined on $\mathbb{R}^n \setminus \{0\}$ with wavefront set contained in $\{(\xi; \hat{\xi}) \mid Q(\xi) = 0, \ \hat{\xi} = \tau dQ(\xi), \ \tau < 0\}$ by the pull-back theorem [71, Theorem 8.2.4, p. 263]; see also [71, (8.2.6), p. 265]. The distribution $Q^*(t - i0)^{-\alpha}$ is homogeneous of degree -2α, hence by [71, Theorem 3.2.3, p. 75], it has a unique extension as a holomorphic family of distributions in $\alpha \in \mathbb{C} \setminus \{0, 1, 2, \dots\}$ defined on $\mathbb{R}^n$. It has poles which are contained in $\{0, 1, 2, \dots\}$ by [71, Theorem 3.2.4]. Furthermore, [48, III, Section 2.4] tells us that the poles are simple, they are actually contained in $\mathbb{N} + n/2$ [48, p. 275] and the residues are derivatives of δ_0 [48, (1), (1)', p. 276].

In the case of $(Q(\xi) - z - i0)^{-\alpha}_{\alpha\in\mathbb{C}}$, we start from the holomorphic family of distributions $(t - z - i0)^{-\alpha}$ which has singular support at $t = z \neq 0$. The difference is that the pull-back by the map

$$\mathbb{R}^n \ni \xi \mapsto Q(\xi) - z \in \mathbb{R}$$

can be applied everywhere since for all ξ such that $Q(\xi) - z = 0$ we have $dQ(\xi) \neq 0$. ∎

Corollary 4.2. *By inverse Fourier transform,*

$$F_\alpha(z, x) = \frac{\Gamma(\alpha+1)}{(2\pi)^n} \int e^{i\langle x,\xi\rangle} (|\xi|_\eta^2 - i0 - z)^{-\alpha-1} \, d^n\xi$$

is a well-defined family of distributions on $\mathbb{R}^n$, holomorphic in $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$ for $z \in \{\operatorname{Im} z \geqslant 0\} \setminus \{0\}$.

Thus, to regulate the infrared poles of the family $F_\alpha(0,x)$ one can introduce a mass $m>0$ and consider $F_\alpha(-m^2,x)$.

4.2. *Hölder estimate on $F_\alpha(z,\cdot)$*

For large $\operatorname{Re}\alpha$, $F_\alpha(z,\cdot)$ has Fourier transform $(Q-z)^{-\alpha-1}$, which has good decay at infinity except along the light-cone, so the pressing question is: can we control $F_\alpha(z,\cdot)$ in Sobolev or Hölder spaces of high regularity? The answer is yes, but the price to pay is that we need to lose in terms of the decay in z. *We trade decay in z for regularity in the $x\in\mathbb{R}^n$ variable.*

4.2.1. Estimates on $(Q-z)^{-\alpha}$ as distributions. We first discuss the case of $(Q(\xi)-z)^{-\alpha}$ for $\alpha\in\mathbb{N}$. We start from the family $\log(t-i\varepsilon)$ for $\varepsilon>0$, which is a well-defined distribution on $\mathbb{R}$. Uniformly in ε, we have $|\langle\log(t-i\varepsilon),\varphi\rangle|\leqslant C_K\|\varphi\|_{L^2(\mathbb{R})}$ for all test functions φ supported in a fixed compact set K. It follows that for all test functions φ supported in a fixed compact set K and for all $\alpha\in\mathbb{N}$,

$$|\langle(t-i\varepsilon)^{-\alpha},\varphi\rangle| = C_\alpha|\langle\partial_t^\alpha\log(t-i\varepsilon),\varphi\rangle| = C_\alpha|\langle\log(t-i\varepsilon),\partial_t^\alpha\varphi\rangle| \leqslant CC_\alpha\|\varphi\|_{H^\alpha(\mathbb{R})},$$

where the estimate still holds uniformly in $\varepsilon>0$. For large $\operatorname{Im}z>0$ and φ supported in a fixed compact set K,

$$\begin{aligned}|\langle(t-z)^{-\alpha},\varphi\rangle| &= |\langle(t-\operatorname{Re}z-i\operatorname{Im}z)^{-\alpha},\varphi\rangle|\\ &= |\langle(t-i\operatorname{Im}z)^{-\alpha},\varphi(\cdot-\operatorname{Re}z)\rangle| \leqslant |\operatorname{Im}z|^{-\alpha}C_{K,\alpha}\|\varphi\|_{L^2(\mathbb{R})}.\end{aligned}$$

The case of small $\operatorname{Im}z$ is handled by the previous estimates. So in general, for $\varphi\in C_c^\infty(\mathbb{R}^n)$ supported in a fixed compact set K, we have the estimate

$$|\langle(t-z)^{-\alpha},\varphi\rangle| \leqslant (1+|\operatorname{Im}z|)^{-\alpha}C_K\|\varphi\|_{H^\alpha(\mathbb{R})}, \tag{4.45}$$

where C_K does not depend on z on the upper half-plane. As before, let Q be the quadratic form of signature $(n-1,1)$ for the Minkowski metric and let $\alpha\in\mathbb{N}$. The pull-back $Q^*(t-z)^{-\alpha}=(Q-z)^{-\alpha}$ is well-defined as a distribution of order α in $\mathcal{D}'(\mathbb{R}^n\setminus\{0\})$, uniformly in $\operatorname{Im}z>0$ since $dQ(\xi)\neq0$ for all $\xi\neq0$. It follows that for any function χ supported in a compact set K which does not intersect 0, we have $|\langle(Q(\xi)-z)^{-\alpha},\chi\rangle|\leqslant(1+|\operatorname{Im}z|)^{-\alpha}C_K\|\chi\|_{H^\alpha(\mathbb{R}^n)}$ where the pull-back is well-defined. For non-integer α, it suffices to start from $(t-z)^{-\alpha}$ which is well-defined in $L^1_{\mathrm{loc}}(\mathbb{R})$ for $\operatorname{Re}\alpha<1$, hence defines a holomorphic family of distributions of order 0 in $\mathcal{D}'(\mathbb{R})$ in the half-plane $\operatorname{Re}\alpha<1$. This description is *uniform* in $z\in\{\operatorname{Im}z>0\}$. Then, to extend to all $\alpha\in\mathbb{C}\setminus\mathbb{Z}$, for $k<\operatorname{Re}\alpha<k+1$, we use successive integration by parts:

$$(t-z)^{-\alpha} = \frac{1}{(-\alpha+k)\dots(-\alpha+1)}\partial_t^k(t-z)^{-\alpha+k}$$

for $k=\lfloor\operatorname{Re}\alpha\rfloor$, which shows that the l.h.s. is a well-defined holomorphic family of distributions of order k, *uniformly* in $z\in\{\operatorname{Im}z>0\}$. Again by pull-back, this shows that for any compactly supported function χ supported in a compact set K which does not meet 0, we have

$$|\langle (Q(\xi)-z)^{-\alpha}, \chi\rangle| \leqslant (1+|\operatorname{Im} z|)^{-\operatorname{Re}\alpha} C_K \|\chi\|_{H^{\lfloor \operatorname{Re}\alpha \rfloor}(\mathbb{R}^n)},$$

where the pull-back is well-defined.

4.2.2. The Hölder–Zygmund estimate on $F_\alpha(z,\cdot)$. In this subsection, we deal with Euclidean harmonic analysis of the holomorphic family $F_\alpha(z,\cdot) \in \mathcal{D}'(\mathbb{R}^n)$.

Recall that the Littlewood–Paley decomposition starts from a partition of unity $1 = \chi_0 + \sum_{j=0}^{\infty} \chi(2^{-j}\cdot)$. A function u belongs to the *Zygmund class* $\mathcal{C}^r(\mathbb{R}^n)$ [88, p. 294], [72, Section 8.6, p. 201], [122, Section 8, p. 40] iff

$$\|\chi_0(2^{-j}\sqrt{-\Delta})u\|_{L^\infty} + \sup_j 2^{jr}\|\chi(2^{-j}\sqrt{-\Delta})u\|_{L^\infty} < +\infty, \tag{4.46}$$

and this also defines a Banach norm $\|\cdot\|_{\mathcal{C}^r}$ on $\mathcal{C}^r(\mathbb{R}^n)$ (if $r \geqslant 0$ is not an integer then $\mathcal{C}^r(\mathbb{R}^n)$ coincides with the usual Hölder class). The local version of $\mathcal{C}^r(\mathbb{R}^n)$ is denoted by $\mathcal{C}^r_{\mathrm{loc}}(\mathbb{R}^n)$. The equivalence of (4.46) with a Fourier transform characterization is recalled in Appendix D.3.

We will use the dyadic decomposition to analyze the family of distributions $F_\alpha(z,\cdot)$. For $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we estimate the norm of $\psi\chi(2^{-j}\sqrt{-\Delta})F_\alpha$ for $\operatorname{Im} z \geqslant 0$, namely

$$\begin{aligned}
&\left\|\psi(x)\int_{\mathbb{R}^n}(Q(\xi)-z)^{-\alpha-1}\chi(2^{-j}|\xi|)e^{i\langle x,\xi\rangle}d^n\xi\right\|_{L^\infty_x} \\
&\quad = 2^{jn}\left\|\psi(x)\int_{\mathbb{R}^n}2^{-2j(\alpha+1)}(Q(\xi)-2^{-2j}z)^{-\alpha-1}\chi(\|\xi\|)e^{i\langle 2^j x,\xi\rangle}\,d^n\xi\right\|_{L^\infty_x}.
\end{aligned} \tag{4.47}$$

Note the important $2^{-2j}z$ term which explains why at high frequencies, even if z has large imaginary part, the dyadic scaling will push $2^{-2j}z$ arbitrarily close to the real axis so that for large j, $(Q(\xi)-2^{-2j}z)^{-\alpha}$ behaves more and more like the distribution $(Q(\xi)-i0)^{-\alpha}$. For $k = \lfloor \operatorname{Re}\alpha \rfloor + 1$, by (4.47) we find

$$\begin{aligned}
&\|\chi(2^{-j}\sqrt{-\Delta})F_\alpha(z,\cdot)\psi\|_{L^\infty} \\
&\quad = 2^{j(n-2\operatorname{Re}\alpha-2)}\left\|\psi(x)\int_{\mathbb{R}^n}(Q(\xi)+2^{-2j}z)^{-\alpha-1}\chi(\|\xi\|)e^{i\langle 2^j x,\xi\rangle}\,d^n\xi\right\|_{L^\infty_x} \\
&\quad \leqslant 2^{j(n-2\operatorname{Re}\alpha-2)}(1+2^{-2j}|\operatorname{Im} z|)^{-\operatorname{Re}\alpha-1} \\
&\qquad \times C\sup_{x\in\operatorname{supp}\psi}\|\chi(\|\xi\|)e^{i\langle 2^j x,\xi\rangle}\|_{H^k_\xi(\mathbb{R}^n)} \\
&\quad \leqslant C_1 2^{j(n-2\operatorname{Re}\alpha-2)}(1+2^{-2j}|\operatorname{Im} z|)^{-\operatorname{Re}\alpha-1}(1+2^j R)^k \\
&\quad \leqslant C_2 2^{j(n-2\operatorname{Re}\alpha+k-2)}(1+2^{-2j}|\operatorname{Im} z|)^{-\operatorname{Re}\alpha-1}.
\end{aligned}$$

In the last two inequalities, we have made crucial use of the fact that χ is supported in a compact ball $\{|\xi| \leqslant R\}$ and also that the support of ψ is compact so that we have the simple bound $\sup_{x\in\operatorname{supp}\psi}\|\chi(\|\xi\|)e^{i\langle 2^j x,\xi\rangle}\|_{H^k_\xi(\mathbb{R}^n)} \lesssim (1+2^j R)^k$. Let us now interpolate the above inequality to show the interplay between decay in $\operatorname{Im} z$ and also decay in the dyadic scaling, which expresses Hölder regularity. Choosing some $a \in [0,1]$, we get

$$\begin{aligned}\|\chi(2^{-j}\sqrt{-\Delta})F_\alpha(z,\cdot)\psi\|_{L^\infty} &\leqslant C_2 2^{j(n-2\operatorname{Re}\alpha+k-2)}(1+2^{-2j}|\operatorname{Im} z|)^{-\operatorname{Re}\alpha-1}\\ &\leqslant C_2 2^{j(n-2\operatorname{Re}\alpha+k-2)}2^{2ja(\operatorname{Re}\alpha+1)}(2^{2j}+|\operatorname{Im} z|)^{-a(\operatorname{Re}\alpha+1)}\\ &\quad\times(1+2^{-2j}|\operatorname{Im} z|)^{-(1-a)(\operatorname{Re}\alpha+1)}\\ &\leqslant C_2 2^{j(n-2\operatorname{Re}\alpha+k-2+2a(\operatorname{Re}\alpha+1))}(1+|\operatorname{Im} z|)^{-a(\operatorname{Re}\alpha+1)}.\end{aligned}$$

To estimate the low energy part $\psi\chi_0(\sqrt{-\Delta})F_\alpha$, we first need that $\operatorname{Im} z \geqslant 0$, $|z| \geqslant \varepsilon > 0$ to avoid the infrared pole in the massless case when α is an integer. However, for $\alpha \in \mathbb{C} \setminus (\mathbb{N} + n/2)$, we can still let $\varepsilon \to 0$, which means the real part of z is allowed to vanish. The element $\lim_{\operatorname{Im} z\to 0^+,\,|z|\geqslant\varepsilon}(Q(\cdot)-z)^{-\alpha-1}$ extends as a distribution weakly homogeneous of degree $-2\alpha-2$, hence it extends by [89, Section 2] as a distribution of order $p = \lfloor 2\operatorname{Re}\alpha + 2 - n\rfloor + 1$. This implies that

$$\begin{aligned}&\left\|\psi(x)\int_{\mathbb{R}^n}(Q(\xi)-z)^{-\alpha-1}\chi_0(\xi)e^{i\langle x,\xi\rangle}\,d^n\xi\right\|_{L^\infty_x}\\ &\qquad\leqslant C\|\psi\|_{L^\infty}(1+|\operatorname{Im} z|)^{-\operatorname{Re}\alpha-1}\sup_{x\in\operatorname{supp}\psi}\|\chi_0 e^{i\langle x,\xi\rangle}\|_{C^p_\xi(\mathbb{R}^n)}.\end{aligned}$$

Now we can deduce the Hölder regularity estimates of our family $F_\alpha(z,\cdot)$ for $\operatorname{Im} z \geqslant 0$, $|z| \geqslant \varepsilon > 0$:

$$\begin{aligned}\|F_\alpha(z,\cdot)\psi\|_{C^s(\mathbb{R}^n)} &= \sup_{j\in\mathbb{N}} 2^{js}\|\chi(2^{-j}\sqrt{-\Delta})F_\alpha(z,\cdot)\psi\|_{L^\infty} + \|\chi_0(\sqrt{-\Delta})F_\alpha(z,\cdot)\psi\|_{L^\infty}\\ &\leqslant C(1+|\operatorname{Im} z|)^{-a(\operatorname{Re}\alpha+1)}\end{aligned}$$

if $n - 2\operatorname{Re}\alpha + k - 2 + 2a(\operatorname{Re}\alpha+1) + s \leqslant 0$, hence if $s \leqslant (2-2a)(\operatorname{Re}\alpha+1) - k - n$. So if we want high Hölder regularity, we have to choose large $\operatorname{Re}\alpha$.

The proof also shows that for $\operatorname{Re}\alpha > L$, $L \in \mathbb{R}$, the series

$$\sum_{j=1}^{\infty}\chi(2^{-j}\sqrt{-\Delta})F_\alpha(z,\cdot)\psi + \chi_0(\sqrt{-\Delta})F_\alpha(z,\cdot)\psi$$

converges absolutely in $\mathcal{C}^{(2-2a-1)(L+1)-n}(\mathbb{R}^n)$ where each term is *holomorphic* in α. Therefore $F_\alpha(z,\cdot)\psi$ is holomorphic in α with values in the Banach space $\mathcal{C}^{(2-2a-1)(L+1)-n}(\mathbb{R}^n)$.

We summarize the estimates as follows for $\alpha \in \mathbb{C} \setminus \{-1,-2,\dots\}$.

Proposition 4.3. *Let* $k = \lfloor\operatorname{Re}\alpha\rfloor + 1$ *and* $F_\alpha(z,\cdot) \in \mathcal{D}'(\mathbb{R}^n)$ *as defined in* (4.44). *For all* $\varepsilon > 0$, *if* $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon\}$ *then* $F_\alpha(z,\cdot) \in \mathcal{C}^{(2-2a)(\operatorname{Re}\alpha+1)-k-n}_{\mathrm{loc}}(\mathbb{R}^n)$ *with decay in* z *of order* $\mathcal{O}((1+|\operatorname{Im} z|)^{-a(\operatorname{Re}\alpha+1)})$ *for* $a \in [0,1]$.

For $\operatorname{Re}\alpha > L$, $L \in \mathbb{R}_{>-1}$, *the family* $F_\alpha(z,\cdot)$ *is holomorphic in* $\alpha \in \{\operatorname{Re}\alpha > L\}$ *with values in the Fréchet space* $\mathcal{C}^{(2-2a-1)(L+1)-n}_{\mathrm{loc}}(\mathbb{R}^n)$, *with decay in* z *of order* $\mathcal{O}(|\operatorname{Im} z|^{-a(L+1)})$ *for* $a \in [0,1]$.

As expected, we always have to trade regularity for decay in $\operatorname{Im} z$.

4.3. *Microlocal estimates*

To prove microlocal bounds, we will need to represent the distribution $F_\alpha(z,\cdot)$ as the sum of two oscillatory integrals which account for the high (UV) versus low (IR) frequency parts. Both require a careful treatment of the $\operatorname{Im} z \to 0^+$ limit.

4.3.1. *Oscillatory integral representation formula.*

We first prove the following important technical lemma.

Lemma 4.4. *Let $\psi \in C_c^\infty(\mathbb{R};[0,1])$ be such that $\psi = 1$ near 0. For $-1 < \operatorname{Re}\alpha < 0$ and $\operatorname{Im} z > 0$, we have*

$$F_\alpha(z,\cdot) = I_{\rm IR} + I_{\rm UV},$$

where

$$I_{\rm IR} = \frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty e^{u\frac{Q(x)}{4i}} e^{iz/u} \psi(u) u^{n/2-\alpha-2}\, du,$$

$$I_{\rm UV} = \frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty e^{u\frac{Q(x)}{4i}} e^{iz/u} (1-\psi)(u) u^{n/2-\alpha-2}\, du.$$

Furthermore, in the sense of distributions in $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, the α, z-dependent oscillatory integral $I_{\rm UV}$ extends uniquely to a holomorphic family in $\alpha \in \mathbb{C}$, uniformly in $z \in \{\operatorname{Im} z \geqslant 0\}$. The term $I_{\rm IR}$ extends uniquely as a distribution in $\mathcal{D}'(\mathbb{R}^n)$, depending holomorphically in α in the half-plane $\operatorname{Re}\alpha < n/2 - 1$, uniformly in $z \in \{\operatorname{Im} z \geqslant 0\}$.

The difficulty is in proving that the two oscillatory integrals on the r.h.s. have well-defined distributional limits for all $\alpha \in \mathbb{C}$.

Proof of Lemma 4.4. We start from an elementary representation formula for $\operatorname{Im} z > 0$:

$$(Q(\xi) - z)^{-\alpha-1} = \frac{e^{-i(\alpha+1)\pi/2}}{\Gamma(\alpha+1)} \int_0^\infty e^{-iu(Q(\xi)-z)} u^\alpha\, du,$$

where Q is the quadratic form of signature $(n-1, 1)$. When $\operatorname{Im} z > 0$ and $\operatorname{Re}\alpha > -1$, the r.h.s. converges absolutely and is holomorphic in α. Let φ be a Schwartz function. We study the following integral:

$$\int_0^\infty u^\alpha \left(\int_{\mathbb{R}^n} e^{-iu(Q(\xi)-z)} \widehat{\varphi}(\xi)\, d^n\xi \right) du$$
$$= \int_0^\infty \int_{\mathbb{R}^n} \varphi(x) \frac{(2\pi)^n e^{\frac{Q(x)}{4ui}}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} e^{iuz} u^{\alpha-n/2}\, d^n x\, du,$$

where we have used the Plancherel formula and the Fourier transform of complex Gaussians to obtain the last equality. The integral on the r.h.s. is well-defined for $\operatorname{Re}\alpha - n/2 > -1$. So after a change of variables in u, we get another oscillatory integral representation for $\operatorname{Re}\alpha > n/2 - 1$, $\operatorname{Im} z > 0$:

$$\int_0^\infty u^\alpha \left(\int_{\mathbb{R}^n} e^{-iu(Q(\xi)-z)} \widehat{\varphi}(\xi)\ d^n\xi \right) du = C(\alpha) \int_0^\infty \langle e^{u\frac{Q(\cdot)}{4i}}, \varphi \rangle e^{iz/u} u^{n/2-\alpha-2}\, du,$$

where we have absorbed all normalizations in the holomorphic constant $C(\alpha)$ for simplicity, since they play no rôle in the oscillatory bounds. Then we just use the test function ψ to divide the integration in $u \in \mathbb{R}_{\geqslant 0}$ into two parts to separate the IR and UV problems:

$$\underbrace{\frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty \big\langle e^{u\frac{Q(\cdot)}{4i}}, \varphi\big\rangle e^{iz/u}(1-\psi(u))u^{n/2-\alpha-2}\,du}_{\text{UV part}} \\ + \underbrace{\frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty \big\langle e^{u\frac{Q(\cdot)}{4i}}, \varphi\big\rangle e^{iz/u}\psi(u)u^{n/2-\alpha-2}\,du}_{\text{IR part}}.$$

The IR part is well-defined for all values of α and all z s.t. $\operatorname{Im} z > 0$ since $e^{iz/u} = \mathcal{O}(u^\infty)$ as $u \to 0^+$. Another observation is that we can take the $\operatorname{Im} z \to 0^+$ limit when $\operatorname{Re}\alpha < n/2 - 1$, since $u^{n/2-\alpha-2}$ is Riemann integrable near $u = 0$ so there is no problem to let $\operatorname{Im} z \to 0^+$ and there are no constraints on $\operatorname{Re} z$.

Now we need to justify that the integral representation of the UV part is well-defined as a distribution on the half-plane $\operatorname{Im} z > 0$ which is holomorphic in $\alpha \in \mathbb{C}$. For $\operatorname{Re}\alpha > -1$ and $\operatorname{Im} z > 0$ and for any test function $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$,

$$\int_0^\infty \big\langle e^{u\frac{Q(\cdot)}{4i}}, \varphi\big\rangle e^{iz/u}(1-\psi)(u)u^{n/2-\alpha-2}\,du \\ = \int_0^\infty \big\langle e^{\frac{uQ(\cdot)}{4i}+iz/u}, ({}^{\mathrm{t}}L)^N\varphi\big\rangle (1-\psi)(u)u^{n/2-\alpha-2}\,du,$$

where $L = \frac{(4i)\langle\nabla Q, \nabla\rangle}{u\|\nabla Q\|^2}$ is a well-defined differential operator since $dQ \neq 0$ on $\mathbb{R}^n \setminus \{0\}$, N is an arbitrary integer and the integral is holomorphic in α on the half–plane $\operatorname{Re}\alpha > N + n/2 - 1$ since $({}^{\mathrm{t}}L)^N\varphi = \mathcal{O}(u^{-N})$ uniformly in $z \in \{\operatorname{Im} z \geqslant 0\}$. ■

Next, we make an observation on the large $\operatorname{Im} z$ behavior of $F_\alpha(z,\cdot)$ outside $\{0\} \subset \mathbb{R}^n$ which follows from the oscillatory integral representation.

Lemma 4.5. *For all* $\alpha \in \mathbb{C}$, *all* $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ *and all* $\operatorname{Im} z > 0$, *we have* $\langle F_\alpha(z,\cdot), \varphi\rangle = \mathcal{O}(|\operatorname{Im} z|^{-\infty})$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. For the UV part, outside $x = 0$, set $L = 4i\frac{\langle\nabla Q,\nabla\rangle}{u\|\nabla Q\|^2}$. Then

$$\Big|\int_0^\infty \big\langle e^{uQ(\cdot)/4i}, \varphi\big\rangle e^{iz/u}(1-\psi(u))u^{n/2-\alpha-2}\,du\Big| \\ = \Big|\int_0^\infty \big\langle e^{u\frac{Q(\cdot)}{4i}}, ({}^{\mathrm{t}}L)^N\varphi\big\rangle e^{iz/u}(1-\psi(u))u^{n/2-\alpha-2}\,du\Big| \\ \leqslant C\int_0^\infty e^{-\operatorname{Im} z/u}(1-\psi(u))\underbrace{\|({}^{\mathrm{t}}L)^N\varphi\|_{L^\infty(\mathbb{R}^n)}u^{n/2-\operatorname{Re}\alpha-2}}_{=\mathcal{O}(u^{n/2-\operatorname{Re}\alpha-2-N})}\,du \\ \leqslant C|\operatorname{Im} z|^{n/2-\operatorname{Re}\alpha-2-N}\int_0^\infty e^{-u^{-1}}(1-\psi(u\operatorname{Im} z))u^{n/2-\operatorname{Re}\alpha-2-N}\,du,$$

where the integral on the r.h.s. is uniformly bounded as $\operatorname{Im} z \to +\infty$. Therefore we get $I_{\mathrm{UV}} = \mathcal{O}_{\mathcal{D}'(\mathbb{R}^n \setminus \{0\})}(|\operatorname{Im} z|^{-\infty})$.[4]

For the infrared part, one immediately deduces from the integral representation that

$$\left| \int_0^\infty \langle e^{u\frac{Q(\cdot)}{4i}}, \varphi \rangle e^{iz/u} \psi(u) u^{n/2-\alpha-2}\, du \right| \leqslant C \|\varphi\|_{L^\infty} \int_0^\infty \psi(u) e^{-\operatorname{Im} z/u} u^{n/2-\operatorname{Re}\alpha-2}\, du,$$

hence $I_{\mathrm{IR}} = \mathcal{O}_{\mathcal{D}'(\mathbb{R}^n)}(|\operatorname{Im} z|^{-\infty})$.[5] ∎

4.3.2. Bounds on the seminorms $\|\cdot\|_{N,V,\chi}$. Recall that for a closed conic set $\Gamma \subset T^*\mathbb{R}^n \setminus o$, the topology of $\mathcal{D}'_\Gamma$ is given by the continuous seminorms $\|\cdot\|_{N,V,\chi}$,

$$\|t\|_{N,V,\chi} = \sup_{\xi \in V} (1 + \|\xi\|)^N |\widehat{t\chi}(\xi)|,$$

where $\operatorname{supp}(\chi) \times V \cap \Gamma = \emptyset$, plus the weak or strong topology of distributions. Note that throughout the paper, we typically control the size of distributions in a stronger topology than the weak or strong topology on $\mathcal{D}'$ because we operate with Hölder norms.

We need to estimate the seminorms $\|\cdot\|_{N,V,\chi}$ for $F_\alpha(z,\cdot)$ uniformly in $z \in \{\operatorname{Im} z > 0\}$. We will also need to control these seminorms down to $\operatorname{Im} z \to 0^+$ with $\operatorname{Re} z \neq 0$. Now we can bound the wavefront set of $F_\alpha(z,\cdot)$ using the oscillatory integral representation of Lemma 4.4 which involves oscillatory integrals with complex phase. For $\operatorname{Im} z > 0$, they have exponential decay and the oscillatory integrals are well-defined for all $\alpha \in \mathbb{C}$. But when $\operatorname{Im} z \to 0^+$, they converge to some oscillatory integrals with real phase so we can control the integration in u for $\operatorname{Im} z \to 0^+$ only when $\operatorname{Re}\alpha < n/2$.

Step 1 (ultraviolet part). We first deal with the UV part

$$\int_0^\infty e^{u\frac{Q(x)}{4i}} e^{iz/u} (1-\psi)(u) u^{n/2-\alpha-2}\, du.$$

Assume that $\operatorname{supp}\chi \times V$ does not meet $\{(x;\xi) \mid Q(x) = 0,\ \xi = \tau dQ,\ \tau < 0\} \cup T_0^*\mathbb{R}^n$, which in particular implies that $0 \notin \operatorname{supp}\chi$. Our proof is inspired by [113, Theorem 0.5.1, p. 38, Theorem 0.4.6, p. 34]. We choose some smooth ψ and a smooth bump function β supported in $[1/2, 2]$ such that $\psi(u) + \sum_{j=0}^\infty \beta(2^{-j}u) = 1$; this is a dyadic partition of unity. Set $\beta_j(\cdot) = \beta(2^{-j}\cdot)$. Then for $\xi \in V$, we need to consider the series

$$\sum_{j=0}^\infty \int_0^\infty \left(\int_{\mathbb{R}^n} \chi(x) e^{-i(\langle \xi, x\rangle + uQ(x)/4)} d^n x \right) e^{iu^{-1}z} \beta_j(u) u^{n/2-\alpha-2}\, du.$$

[4] To get large decay in $\operatorname{Im} z$ the distribution is viewed as an element of high order, and we need to differentiate the test function φ many times.

[5] Here it is a distribution of order 0 for all orders of decay in $\operatorname{Im} z$.

We fix j and rewrite one term of the series after a change of variables:

$$\int_0^\infty \left(\int_{\mathbb{R}^n} \chi(x)e^{-i(\langle\xi,x\rangle+uQ(x)/4)}d^nx\right)e^{iu^{-1}z}\beta_j(u)u^{n/2-\alpha-2}\,du$$
$$= 2^{j(n/2-\alpha-1)}\int_0^\infty \left(\int_{\mathbb{R}^n} \chi(x)e^{-i(\langle\xi,x\rangle+2^juQ(x)/4)}d^nx\right)e^{i2^{-j}u^{-1}z}\beta_1(u)u^{n/2-\alpha-2}\,du.$$

The phase function $\phi(x,\xi,j,u) = \langle\xi,x\rangle + 2^juQ(x)/4$ is non-degenerate since $d_x\phi = \xi + 2^judQ(x)/4 \in C^\infty(\mathbb{R}^n \times V \times]0,+\infty]; \mathbb{R}^n)$ never vanishes because $\xi \in V$ does not meet $\mathbb{R}_{<0}dQ(x)$ for all x in the support of the test function χ. Define the differential operator $\mathcal{L} = \frac{1+\langle\nabla_x\phi,\nabla_x\rangle}{1+\langle\nabla_x\phi,\nabla_x\phi\rangle}$. Observe that the term $\langle\nabla_x\phi,\nabla_x\phi\rangle$ in the denominator is bounded from below by

$$\left\|\xi + 2^ju\frac{dQ(x)}{4}\right\|^2 \geqslant C(\|\xi\| + 2^j)^2$$

for some $C > 0$ uniformly in $\xi \in V$ and $u \in [1,4]$ since u lives in the support of β_1 and $dQ(x) \neq 0$ because $0 \notin \operatorname{supp}\chi$. By $3N$ integrations by parts with respect to $\mathcal{L}$ as in [113, Lemma 0.4.7, p. 35], since $j \geqslant 1$, we get the bound

$$\sup_{u\in]0,+\infty]}\left|\int_{\mathbb{R}^n} \chi(x)e^{-i(\langle\xi,x\rangle+2^juQ(x)/4)}\beta_1(u)\,d^nx\right| \leqslant C(\|\xi\| + 2^j)^{-3N}$$
$$\leqslant C(1+\|\xi\|)^{-N}2^{-j2N}.$$

Therefore for $j \geqslant 1$,

$$\left|2^{j(n/2-\alpha-1)}\int_0^\infty \left(\int_{\mathbb{R}^n} \chi(x)e^{-i(\langle\xi,x\rangle+2^juQ(x)/4)}\,d^nx\right)e^{-2^{-j}u^{-1}z}\beta_1(u)u^{n/2-\alpha-2}\,du\right|$$
$$\leqslant C(1+\|\xi\|)^{-N}\left(\int_1^4 e^{-2^{-j}u^{-1}\operatorname{Im} z}u^{n/2-\operatorname{Re}\alpha-2}\,du\right)2^{j(n/2-\operatorname{Re}\alpha-1-2N)}.$$

Using the elementary estimates

$$\int_1^4 e^{-2^{-j}u^{-1}\operatorname{Im} z}u^{n/2-\operatorname{Re}\alpha-2}\,du = \int_{1/4}^1 e^{-2^{-j}u\operatorname{Im} z}u^{\operatorname{Re}\alpha-n/2}\,du$$
$$\leqslant C_1\int_{1/4}^1 e^{-2^{-j}u\operatorname{Im} z}\,du \leqslant \frac{3}{4}C_1e^{-\frac{\operatorname{Im} z}{4\cdot 2^j}} \leqslant C_{2,N}\left(1+\frac{|\operatorname{Im} z|}{2^j}\right)^{-N}$$
$$\leqslant C_{2,N}2^{jN}(1+|\operatorname{Im} z|)^{-N}$$

combined with the above stationary phase estimate, we deduce that

$$\left|\int_0^\infty \left(\int_{\mathbb{R}^n} \chi(x)e^{-i(\langle\xi,x\rangle+uQ(x)/4)}d^nx\right)e^{iu^{-1}z}\beta_j(u)u^{n/2-\alpha-2}\,du\right|$$
$$\leqslant C_{3,N}(1+\|\xi\|)^{-N}(1+|\operatorname{Im} z|)^{-N}2^{j(n/2-\operatorname{Re}\alpha-1-N)}.$$

Now, for all $N > n/2 - \operatorname{Re}\alpha - 1$, the series in j converges absolutely and yields an estimate of the form

$$\Big|\sum_{j=1}^{\infty}\int_0^{\infty}\Big(\int_{\mathbb{R}^n}\chi(x)e^{-i(\langle\xi,x\rangle+uQ(x)/4)}d^nx\Big)e^{iu^{-1}z}\beta_j(u)u^{n/2-\alpha-2}\,du\Big| \\ \leqslant C_{3,N}(1+\|\xi\|)^{-N}(1+|\operatorname{Im}z|)^{-N}.$$

Step 2 (infrared part). To conclude the estimate, we still need to deal with the infrared part

$$\langle I_{\mathrm{IR}},\varphi e^{i\langle\xi,\cdot\rangle}\rangle = \frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}}\int_0^{\infty}\big\langle e^{\frac{Q(\cdot)}{4iu}},\varphi e^{i\langle\xi,\cdot\rangle}\big\rangle e^{izu}\psi(u^{-1})u^{\alpha-n/2}\,du$$

for $\xi\in V$ and $\varphi\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$, where we have made the change of variable $u\mapsto u^{-1}$. We first assume that $\operatorname{Im}z\geqslant\varepsilon>0$. The function $e^{\frac{Q(x)}{4iu}}\varphi(x)$ is smooth in x uniformly in $u\in\operatorname{supp}\psi(u^{-1})$. Therefore $|\langle e^{\frac{Q(\cdot)}{4iu}},\varphi e^{i\langle\xi,\cdot\rangle}\rangle|\leqslant C_N(1+\|\xi\|)^{-N}$ for all $N\in\mathbb{N}$. If $\operatorname{Im}z>0$, then $\mathbb{R}\ni u\mapsto e^{izu}\psi(u^{-1})u^{\alpha-n/2}$ is Riemann integrable on $\mathbb{R}$, hence we immediately find that for all N,

$$|\langle I_{\mathrm{IR}},\varphi e^{i\langle\xi,\cdot\rangle}\rangle|\leqslant|C(\alpha)|\int_0^{\infty}\big|\big\langle e^{\frac{Q(\cdot)}{4iu}},\varphi e^{i\langle\xi,\cdot\rangle}\big\rangle\big|e^{-\operatorname{Im}zu}\psi(u^{-1})u^{\operatorname{Re}\alpha-n/2}\,du \\ = \mathcal{O}(e^{-\operatorname{Im}z\delta/2}\|\xi\|^{-N}) = \mathcal{O}(|\operatorname{Im}z|^{-N}\|\xi\|^{-N}),$$

where $\delta>0$ is such that $[0,\delta]\cap\operatorname{supp}(\psi(u^{-1}))=\emptyset$. Now when $\operatorname{Re}\alpha<n/2-1$, then the above bound holds true uniformly on $\{\operatorname{Im}z\geqslant 0\}$ since $\psi(u^{-1})u^{\operatorname{Re}\alpha-n/2}$ is Riemann integrable and

$$\int_0^{\infty}\big|\big\langle e^{\frac{Q(\cdot)}{4iu}},\varphi e^{i\langle\xi,\cdot\rangle}\big\rangle\big|e^{-\operatorname{Im}zu}\psi(u^{-1})u^{\operatorname{Re}\alpha-n/2}\,du \\ \leqslant C_N(1+\|\xi\|)^{-N}e^{-\delta\operatorname{Im}z/2}\underbrace{\int_0^{\infty}\psi(u^{-1})u^{\operatorname{Re}\alpha-n/2}\,du}_{<+\infty} \\ \leqslant C_{2,N}(1+\|\xi\|)^{-N}(1+|\operatorname{Im}z|)^{-N}.$$

Step 3 (conclusion). Let

$$\Lambda_0=\{(x;\xi)\mid\xi=\tau dQ(x),\ Q(x)=0,\ \tau<0\}\cup(T_0^*\mathbb{R}^n\setminus o)\subset T^*\mathbb{R}^n.$$

For all $\chi\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n\setminus\{0\})$ and all cones V such that $\operatorname{supp}\chi\times V$ does not meet $\{(x;\xi)\mid Q(x)=0,\ \xi=\tau dQ,\ \tau<0\}\cup(T_0^*\mathbb{R}^n\setminus o)$, for $\operatorname{Im}z\geqslant\varepsilon>0$, we deduce an estimate of the form

$$|\mathcal{F}(F_\alpha(z,\cdot)\chi)(\xi)|\leqslant C(1+\|\xi\|)^{-N}|\operatorname{Im}z|^{-N}$$

uniformly in $\xi\in V$, where $\mathcal{F}$ denotes the Fourier transform. In other words, in terms of the continuous seminorms $\|\cdot\|_{N,V,\chi}$ of the $\mathcal{D}'_{\Lambda_0}$ topology, the above estimate reads $\|F_\alpha(z,\cdot)\|_{N,V,\chi}\leqslant C|\operatorname{Im}z|^{-N}$ for $\operatorname{Im}z\geqslant\varepsilon>0$. When $\operatorname{Re}\alpha<n/2-1$, we have a stronger

estimate which holds true for $\operatorname{Im} z > 0$:

$$\| F_\alpha(z,\cdot)\|_{N,V,\chi} \leqslant C(1+|\operatorname{Im} z|)^{-N}. \tag{4.48}$$

Combining this with the Hölder estimates of Section 4.2.2 and Lemma 4.5, we get the following result.

Proposition 4.6. *Let* $\Lambda_0 = \{(x;\xi) \mid \xi = \tau dQ(x),\ Q(x)=0,\ \tau<0\} \cup (T_0^*\mathbb{R}^n \setminus o)$*. Then:*

(1) *The family* $(1+|\operatorname{Im} z|)^{\operatorname{Re}\alpha+1} F_\alpha(z,\cdot)$, $\operatorname{Im} z \geqslant 0$, $|z| \geqslant \varepsilon > 0$, *is bounded in* $\mathcal{D}'_{\Lambda_0}(\mathbb{R}^n)$.

(2) *For all* $\varepsilon > 0$ *the family* $|\operatorname{Im} z|^N F_\alpha(z,\cdot)$, $\operatorname{Im} z \geqslant \varepsilon$, *is bounded in* $\mathcal{D}'_{\Lambda_0}(\mathbb{R}^n \setminus \{0\})$ *for all* $N \in \mathbb{N}$.

(3) *If* $\operatorname{Re}\alpha < n/2 - 1$ *then the family* $(1+|\operatorname{Im} z|)^{\operatorname{Re}\alpha+1} F_\alpha(z,\cdot)$, $\operatorname{Im} z \geqslant 0$, *is bounded in* $\mathcal{D}'_{\Lambda_0}(\mathbb{R}^n)$.

(4) *If* $\operatorname{Re}\alpha < n/2 - 1$ *then the family* $(1+|\operatorname{Im} z|)^N F_\alpha(z,\cdot)$, $\operatorname{Im} z \geqslant 0$, *is bounded in* $\mathcal{D}'_{\Lambda_0}(\mathbb{R}^n \setminus \{0\})$ *for all* $N \in \mathbb{N}$.

Using the notation introduced in Definition 3.8, statement (1) is equivalent to $F_\alpha(z,\cdot) = \mathcal{O}_{\mathcal{D}'_{\Lambda_0}}(|\operatorname{Im} z|^{-\operatorname{Re}\alpha-1})$ in $\operatorname{Im} z > 0$, and we can rephrase (2)–(4) similarly.

4.4. The holomorphic family of distributions $F_\alpha(z,\cdot)$ *for* $\operatorname{Re}\alpha \geqslant 0$

We need to verify algebraic relations satisfied by the holomorphic family of Lorentz invariant distributions $F_\alpha(z,\cdot) \in \mathcal{D}'(\mathbb{R}^n)$ which will appear in the asymptotic expansion of Feynman powers.

Let γ_ε be the contour in the upper half-plane introduced in Section 2.6. We will need the following lemma when inserting the parametrix for $(P-z)^{-1}$ in contour integrals along γ_ε. It is precisely the family of distributions $F_\alpha(z,\cdot)$ that will contribute to the singularities near the diagonal of the Schwartz kernel of the complex powers.

Lemma 4.7. *Let* $F_\alpha(z,\cdot) \in \mathcal{D}'(\mathbb{R}^n)$, $\alpha \in \mathbb{C}$, $\operatorname{Im} z > 0$, *be the family of distributions defined by* (4.44)*. For all* $k \in \mathbb{N}$, $m \in \mathbb{R}$, $\varepsilon > 0$, *they satisfy the contour integral identity*

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon} (z \pm i\varepsilon)^{-\alpha} F_k(z-m^2,\cdot)\, dz = \frac{(-1)^k \Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)} F_{\alpha+k-1}(-m^2 \mp i\varepsilon,\cdot),$$

where both sides converge in $\mathcal{D}'(\mathbb{R}^n)$ *for* $\operatorname{Re}\alpha > 0$*. For* $\Box_\eta = \eta^{ij}\partial_{x^i}\partial_{x^j}$*, we also have*

$$(\Box_\eta - z)F_\alpha(z,\cdot) = \alpha F_{\alpha-1}(z,\cdot).$$

Proof. We claim that by density of compactly supported functions in $L^2(\mathbb{R}^n)$ and the Cauchy residue formula,[6]

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon} (z+i\varepsilon)^{-\alpha-1}(Q(\xi)-z)^{-1}\, dz = (Q(\xi)+i\varepsilon)^{-\alpha-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

where the l.h.s. is norm convergent in $\mathcal{B}(L^2(\mathbb{R}^n))$ when $\operatorname{Re}\alpha > 0$.

[6]Beware that our contour γ_ε is oriented counterclockwise but we integrate against $(Q(\xi)-z)^{-1}$ instead of $(z-Q(\xi))^{-1}$.

Indeed, when we multiply $\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha-1}(Q(\xi)-z)^{-1}\,dz$ by some compactly supported $\varphi(\xi)$, the values of $Q(\xi)$ when multiplied in $(Q(\xi)-z)^{-1}\varphi(\xi)$ lie in a compact $K\subset\mathbb{C}$. We can enclose K with a large piece of $\gamma_\varepsilon\cap B(0,R)$, which we close up with a circle arc of the form $\{Re^{i\theta}\mid\theta\in[-\omega,\omega]\}$. This arc has size $\sim R$ but the integrand on it decays like $R^{-\operatorname{Re}\alpha-1}$ so this large portion tends to 0 as $R\to+\infty$. The part $\gamma_\varepsilon\cap B(0,R)^{\mathrm{c}}$ on the complement of the ball of radius R also decays when $R\to+\infty$. Therefore, Cauchy's formula tells us that the identity

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha-1}(Q(\xi)-z)^{-1}\,dz=(Q(\xi)+i\varepsilon)^{-\alpha-1}$$

holds true as operators acting on compactly supported smooth functions of ξ. But since these are dense in L^2, this identity extends to operators in $\mathcal{B}(L^2(\mathbb{R}^n))$. By inverse Fourier transform, this yields

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha-1}F_0(z,\cdot)\,dz=\Gamma(\alpha+1)^{-1}F_\alpha(-i\varepsilon,\cdot).$$

We have to extend the above discussion to the case

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha}F_k(z,\cdot)\,dz$$

still in the region $\operatorname{Re}\alpha>1$. Recall that in Fourier space,

$$\mathcal{F}F_k(z,\cdot)(\xi)=k!(Q(\xi)-z)^{-k-1}.$$

For every holomorphic f, Cauchy's formula says that $\frac{i}{2\pi}\int_\gamma f(z)(z-z_0)^{-k-1}\,dz=\frac{f^{(k)}(z_0)}{k!}$ if γ is a clockwise oriented contour around z_0. Therefore arguing as above yields

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_\varepsilon}&(z+i\varepsilon)^{-\alpha}k!(Q(\xi)-z)^{-k-1}\,dz\\&=(-1)^k\frac{i}{2\pi}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha}k!(z-Q(\xi))^{-k-1}\,dz\\&=(-1)^k(-\alpha)\dots(-\alpha-k+1)(Q(\xi)+i\varepsilon)^{-\alpha-k},\end{aligned}$$

where both sides converge when $\operatorname{Re}\alpha>0$ as multiplication operators in $\mathcal{B}(L^2(\mathbb{R}^n))$. By inverse Fourier transform, using the definition of $F_{\alpha+k-1}(-i\varepsilon,\cdot)$ yields

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha}F_k(z,\cdot)\,dz=\frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)}F_{\alpha+k-1}(-i\varepsilon,\cdot),$$

where the integral makes sense as a bounded operator acting on $L^2(\mathbb{R}^n)$. ■

4.4.1. Analytic continuation of the microlocal estimates and Bernstein–Sato polynomial. Our next goal is to prove an analytic continuation of the microlocal estimates on $F_\alpha(z,\cdot)$

for all $\alpha \in \mathbb{C}$ and $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$. The idea is to prove the existence of a functional equation satisfied by $F_\alpha(z,\cdot)$ involving Bernstein–Sato polynomials [12, 13, 105].

If $z \neq 0$, observe that

$$\partial^2_{\xi_i}(Q(\xi)-z)^{-\alpha} = -2\alpha\eta_{ii}(Q(\xi)-z)^{-\alpha-1} + 4\alpha(\alpha+1)\eta_{ii}^2\,\xi_i^2(Q(\xi)-z)^{-\alpha-2},$$

which implies, after division by η_{ii} and summation over i,

$$\sum_{i=1}^{n}\frac{\partial^2_{\xi_i}}{\eta_{ii}}(Q(\xi)-z)^{-\alpha} = -2\alpha(Q(\xi)-z)^{-\alpha-1} + 4\alpha(\alpha+1)Q(Q(\xi)-z)^{-\alpha-2},$$

hence

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2_{\xi_i}}{\eta_{ii}}(Q(\xi)-z)^{-\alpha} = -2\alpha(Q(\xi)-z)^{-\alpha-1} &+ 4\alpha(\alpha+1)(Q(\xi)-z)^{-\alpha-1}\\ &+ 4\alpha(\alpha+1)z(Q(\xi)-z)^{-\alpha-2},\end{aligned}$$

and consequently

$$\big(Q(\xi)Q(\partial_\xi) + 2\alpha - 4\alpha(\alpha+1)\big)(Q(\xi)-z)^{-\alpha} = 4\alpha(\alpha+1)z(Q(\xi)-z)^{-\alpha-1}.$$

By inverse Fourier transform this yields the functional equation satisfied by the family $F_\alpha(z,\cdot)$,

$$A(\alpha,x,D_x)F_\alpha(z,\cdot) = F_{\alpha+1}(z,\cdot), \tag{4.49}$$

where A is the differential operator with polynomial coefficients

$$A(\alpha,x,D_x) = \frac{\alpha\big(Q(\partial_x^2)Q(x) + 2(\alpha+1) - 4(\alpha+1)(\alpha+2)\big)}{4(\alpha+1)(\alpha+2)z},$$

which is holomorphic on the half-plane $\operatorname{Re}\alpha > -1$. Using the functional equation (4.49), denoting by $\mathcal{F}$ the Fourier transform, we deduce the identity

$$\begin{aligned}\mathcal{F}(F_{\alpha+1}(z,\cdot)\chi)(\xi) &= \int_{\mathbb{R}^n}\chi(x)e^{i\langle\xi,x\rangle}A(\alpha,x,D_x)F_\alpha(z,x)\,d^nx\\ &= \int_{\mathbb{R}^n}F_\alpha(z,x)\,{}^{\mathrm{t}}\!A(\alpha,x,D_x)(\chi(x)e^{i\langle\xi,x\rangle})\,d^nx\\ &= \sum\int_{\mathbb{R}^n}F_\alpha(z,x)(A_{(1)}(s,x,D_x)\chi(x))\big(A_{(2)}(s,x,D_x)e^{i\langle\xi,x\rangle}\big)\,d^nx\\ &= \sum\int_{\mathbb{R}^n}F_\alpha(z,x)(A_{(1)}(s,x,D_x)\chi(x))\big(A_{(2)}(s,x,\xi)e^{i\langle\xi,x\rangle}\big)\,d^nx,\end{aligned}$$

where we split the differential operator ${}^{\mathrm{t}}A$ into two pieces in all possible ways using the Leibniz rule.[7] The above sum is finite and the degree of $A_{(2)}$ in both x and ξ is always

[7] This can also be stated in terms of the coproduct $\Delta\,{}^{\mathrm{t}}A = \sum A_{(1)} \otimes A_{(2)}$ in the coalgebra of differential operators with polynomial coefficients, following Sweedler's notation.

less than 2. Therefore for all $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$, for $\operatorname{Re}\alpha < n/2 - 1$ and all $N_2 \in \mathbb{N}$, we can bound the seminorms of $F_{\alpha+1}(z,\cdot)$ in terms of those of $F_\alpha(z,\cdot)$:

$$\begin{aligned}\|F_{\alpha+1}(z,\cdot)\|_{N,V,\chi} &= \sup_{\xi\in V}(1+\|\xi\|)^N |\mathcal{F}(F_{\alpha+1}(z,\cdot)\chi)(\xi)| \\ &= \sup_{\xi\in V}(1+\|\xi\|)^N \left| \sum \int_{\mathbb{R}^n} F_\alpha(z,x)(A_{(1)}(s,x,D_x)\chi(x))\big(A_{(2)}(s,x,\xi)e^{i\langle\xi,x\rangle}\big)\, d^n x \right| \\ &\leqslant C\sum \|F_\alpha(z,\cdot)\|_{N+2,V,\chi_{1,2}} \leqslant C(1+|\operatorname{Im} z|)^{-N_2},\end{aligned}$$

where all above sums are *finite* and the smooth test functions $\chi_{1,2}$ depend on the operators $A_{(1)}, A_{(2)}$. Integrating these bounds, we propagate the microlocal estimates from the half-plane $\operatorname{Re}\alpha < n/2-1$ to $\alpha \in \mathbb{C}$. We deduce that for any continuous seminorm $\|\cdot\|_{N,V,\chi}$ of $\mathcal{D}'_{\Lambda_0}$, for all $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$ and all $\alpha \in \mathbb{C}$, $N_2 \in \mathbb{N}$,

$$\|F_{\alpha+1}(z,\cdot)\|_{N,V,\chi} \leqslant C(1+|\operatorname{Im} z|)^{-N_2}.$$

We summarize this in the next theorem, together with other results from this section.

Theorem 4.8 (Analytic properties of F_α). *Let $k = \lfloor \operatorname{Re}\alpha \rfloor + 1$ and $F_\alpha(z,\cdot) \in \mathcal{D}'(\mathbb{R}^n)$ as in (4.44). Then for $a \in [0,1]$ and $\varepsilon > 0$, $F_\alpha(z,\cdot) \in \mathcal{C}^{(2-2a)(\operatorname{Re}\alpha+1)-k-n}_{\rm loc}(\mathbb{R}^n)$ with decay in $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon\}$ of order $\mathcal{O}((1+|\operatorname{Im} z|)^{-a(\operatorname{Re}\alpha+1)})$.*

Let $\Lambda_0 = \{(x;\xi) \,|\, \xi = \tau dQ(x),\ Q(x) = 0,\ \tau < 0\} \cup (T^*_0\mathbb{R}^n \setminus o) \subset T^*\mathbb{R}^n$. For all $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$ and all $\alpha \in \mathbb{C}$, the family $(1+|\operatorname{Im} z|)^{1+\operatorname{Re}\alpha} F_\alpha(z,\cdot)$ is bounded in $\mathcal{D}'_{\Lambda_0}(\mathbb{R}^n)$. Moreover, for every $N \in \mathbb{N}$ and $\varepsilon > 0$, $\{|\operatorname{Im} z|^N F_\alpha(z,\cdot)\}_{z\in\{\operatorname{Im} z \geqslant \varepsilon\}}$ is bounded in $\mathcal{D}'_{\Lambda_0}(\mathbb{R}^n \setminus \{0\})$.

5. Formal Hadamard parametrix for the resolvent

5.1. Pull-back by exponential maps

Next, we introduce the main ingredient in the construction of the formal Hadamard parametrix, namely, the pull-back of the distributions $F_\alpha(z,\cdot)$ near the diagonal $\Delta \subset M \times M$ using the exponential map.

5.1.1. Moving frame. We use the notation $(x;v)$ for elements of TM, where $x \in M$ and $v \in T_xM$. Let $\mathcal{N}$ be a neighborhood of the zero section o in TM for which the map $\mathcal{N} \ni (x;v) \mapsto (x, \exp_x(v)) \in M^2$ is a local diffeomorphism onto its image ($\exp_x : T_xM \to M$ is the exponential geodesic map).

The construction of the exponential in the pseudo-Riemannian setting is explained in detail in [72, Appendix A]. The subset $\mathcal{U} = \exp\mathcal{N} \subset M^2$ is a neighborhood of Δ and the inverse map $\mathcal{U} \ni (x_1,x_2) \mapsto (x_1; \exp^{-1}_{x_1}(x_2)) \in \mathcal{N}$ is a well-defined diffeomorphism. Let Ω be an open subset of M and let $(e_0,\dots,e_n)$ be *a time oriented orthonormal moving frame* on Ω (i.e. for all $x \in \Omega$, $g_x(e_\mu(x), e_\nu(x)) = \eta_{\mu\nu}$, and e_0 is future directed), and $(s^\mu)_\mu$ the corresponding orthonormal moving coframe.

5.1.2. Pull-back. We denote by ϵ_μ the canonical basis of $\mathbb{R}^n$. The data of the orthonormal moving coframe $(s^\mu)_\mu$ allows us to define for $(x_1, x_2) \in \mathcal{U}$ the submersion

$$G : (x_1, x_2) \mapsto G^\mu(x_1, x_2)\epsilon_\mu = \underbrace{s^\mu_{x_1}}_{\in T^*_{x_1}M} \underbrace{(\exp^{-1}_{x_1}(x_2))}_{\in T_{x_1}M}\epsilon_\mu \in \mathbb{R}^n. \tag{5.50}$$

For any distribution f in $\mathcal{D}'(\mathbb{R}^n)$, the composition $\mathcal{U} \ni (x_1, x_2) \mapsto G^* f(x_1, x_2)$ defines the pull-back of f on $\mathcal{U} \subset M^2$. If f is $O(1, n-1)^\uparrow_+$-invariant, then the pull-back defined above *does not depend on the choice of orthonormal moving frame* $(e_\mu)_\mu$ and is thus *intrinsic* (since all orthonormal moving frames are related by gauge transformations in $C^\infty(M; O(1, n-1)^\uparrow_+)$).

This allows us to canonically pull back $O(1, n-1)^\uparrow_+$-invariant distributions to distributions defined on a neighborhood $\mathcal{U}$ of Δ.

Definition 5.1. We apply this construction to the family $F_\alpha(z, \cdot) \in \mathcal{D}'(\mathbb{R}^n)$ constructed in Proposition 4.1, and we obtain the distribution $\mathbf{F}_\alpha(z, \cdot) = G^* F_\alpha(z, \cdot) \in \mathcal{D}'(\mathcal{U})$.

Lemma 5.2. *Let (M, g) be a globally hyperbolic Lorentzian manifold, $\mathcal{U}$ the neighborhood of the diagonal $\Delta \subset M \times M$ defined in Section 5.1.1, $G : \mathcal{U} \to \mathbb{R}^n$ the map defined in (5.50), and $F_\alpha(z, \cdot) \in \mathcal{D}'(\mathbb{R}^n)$ the family of distributions defined in (4.44). Then the wavefront set of the distribution $\mathbf{F}_\alpha(z, \cdot) = G^* F_\alpha(z, \cdot)$ is contained in the* Feynman wavefront[8] *$\Lambda \subset (T^*M \setminus o) \times (T^*M \setminus o)$, defined by*

$$\Lambda' = \{(q_1, q_2) \in \Sigma \times \Sigma \mid q_1 \succ q_2\} \cup T^*_\Delta(M \times M) \tag{5.51}$$

using the notation introduced in Definition 3.9.

The proof of Lemma 5.2 will be given in Appendix D.2.

We conclude this section by our main result on the regularity of the family $\mathbf{F}_\alpha(z) = G^* F_\alpha(z) \in \mathcal{D}'_\Lambda(\mathcal{U})$ which follows from continuity in the normal topology [19, Proposition 5.1, p. 211] of the pull-back $G^* : \mathcal{D}'_{\Lambda_0}(\mathbb{R}^n) \to \mathcal{D}'_\Lambda(\mathcal{U})$ and Proposition 4.6.

Proposition 5.3 (Boundedness of $\mathbf{F}_\alpha$). *Let (M, g) be a globally hyperbolic Lorentzian manifold, $\mathcal{U}$ the neighborhood of the diagonal $\Delta \subset M \times M$ defined in Section 5.1.1, $G : \mathcal{U} \to \mathbb{R}^n$ the map defined by (5.50), and $F_\alpha(z, \cdot) \in \mathcal{D}'(\mathbb{R}^n)$ the family of distributions (4.44). For every $\alpha \in \mathbb{C}$, the family of distributions $\mathbf{F}_\alpha(z) = \mathbf{F}_\alpha(z, \cdot) = G^* F_\alpha(z, \cdot)$, $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$, has the property that $(1 + |\operatorname{Im} z|)^{\operatorname{Re}\alpha+1}\mathbf{F}_\alpha(z)$ is bounded in $\mathcal{D}'_\Lambda(\mathcal{U})$.*

5.1.3. Preliminary identities. Recall that our differential operator of interest is of the form

$$P - z = (\partial_{x^j} g^{jk} \partial_{x^k} + m^2 - z) + b^j \partial_{x^j}. \tag{5.52}$$

[8] Note that in the literature on quantum field theory on curved spacetime, the opposite convention is often used for Feynman propagators and for the Feynman wavefront.

In the formal calculus[9] used in the Hadamard parametrix construction, the part in parentheses on the r.h.s. of (5.52) has weight 2, in particular the parameter z is included in the weight 2 part.

We first state the key identities satisfied by the family $F_\alpha(z,\cdot)$ on $\mathbb{R}^n$.

Lemma 5.4. *For all* $z \in \{\operatorname{Im} z \geqslant 0, z \neq 0\}$, *the family of distributions* $F_\alpha(z,\cdot)$ *on* $\mathbb{R}^n$ *satisfies the identities*

$$\begin{aligned}(\eta^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} - z)F_\alpha(z,\cdot) &= \alpha F_{\alpha-1}(z,\cdot) \quad \textit{if}\ \alpha \neq 0,\\ (\eta^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} - z)F_0(z,|\cdot|_g) &= \delta_0,\end{aligned} \tag{5.53}$$

$$2\partial_{x^\mu} F_\alpha(z,\cdot) = \eta_{\mu\nu}x^\nu F_{\alpha-1}(z,|\cdot|_g). \tag{5.54}$$

Proof. The first two identities follow from Lemma 4.7. The third identity follows from the representation formula from Lemma 4.4, namely

$$F_\alpha(z,x) = \frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty e^{\frac{uQ(x)}{4i}+izu^{-1}} u^{n/2-\alpha-2}\, du,$$

and we differentiate under the integral and use the chain rule to obtain the desired result. In more detail, this reads[10]

$$\begin{aligned}&\partial_{x^j} \frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty e^{\frac{uQ(x)}{4i}+iu^{-1}z} u^{n/2-\alpha-2}\, du\\ &= -2\eta_{ji}x^i \frac{e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty \frac{u}{4i} e^{\frac{uQ(x)}{4i}+iu^{-1}z} u^{n/2-\alpha-2}\, du\\ &= \frac{\eta_{ji}x^i}{2} \frac{e^{i\pi/2}e^{-i(\alpha+1)\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}} \int_0^\infty e^{\frac{uQ(x)}{4i}+iu^{-1}z} u^{n/2-\alpha-1}\, du = \frac{\eta_{ji}x^i}{2} F_{\alpha-1}(z,x).\end{aligned}$$

The fact that we can differentiate under the integral is justified for $\operatorname{Im} z > 0$ and $\operatorname{Re}\alpha > n/2 - 2$ (this guarantees all integrals converge absolutely and we can differentiate under the integral) and the general result follows from analytic continuation of the identity $2\partial_{x^j} F_\alpha(z,\cdot) = \eta_{ji}x^i F_{\alpha-1}(z,\cdot)$ in α and the weak convergence of both sides in the distribution sense when $\operatorname{Im} z \to 0^+$. ∎

5.1.4. Identities in normal coordinates. We consider the family of distributions $\mathbf{F}_\alpha(z,\cdot) \in \mathcal{D}'(\mathcal{U})$ introduced in Definition 5.1, which play the role of the building blocks of the parametrix.

The parametrix is constructed in normal charts. This means that we fix a point $x_0 \in M$, and then we express the distribution $x \mapsto \mathbf{F}_\alpha(z, x_0, x)$ in normal coordinates centered

[9] We remark here that in applications it could be advantageous to make the connection with a systematic calculus tailored to computations near the diagonal; see [33].

[10] The minus sign comes from $Q(x) = -\eta_{ij}x^i x^j$.

at x_0. The fact that we can freeze x_0 and view $x \mapsto \mathbf{F}_\alpha(z, x_0, x)$ as a *distribution of the second variable* x comes from the wavefront set of $\mathbf{F}_\alpha(z, \cdot, \cdot) \in \mathcal{D}'(\mathcal{U})$, which is contained in $\Lambda \subset T^*(M \times M)$. Near the element (x_0, x_0) on the diagonal, the set Λ is close to the conormal $N^*\Delta$ and therefore Λ is locally transverse to the conormal $N^*(\{x_0\} \times M) = T^*_{x_0}M \times o$ of the submanifold $\{x_0\} \times M$ near the diagonal (x_0, x_0). Hence, the pull-back theorem of Hörmander allows us to restrict the distribution $\mathbf{F}_\alpha(z, \cdot, \cdot)$ to $\{x_0\} \times M$, which means in practice that we freeze the variable x_0 and consider $\mathbf{F}_\alpha(z, x_0, \cdot)$ as a distribution of the second variable. In the sequel, we work in normal coordinates centered at x_0.

Definition 5.5. Instead of $T_{x_0}M \supset U \ni v \mapsto \mathbf{F}_\alpha(z, x_0, \exp_{x_0}(v))$, we use the simplified notation $\mathbf{F}_\alpha(z, |\cdot|_g) \in \mathcal{D}'(U)$, where $|y|_g^2$ is the pseudodistance squared from y to $0 \in T_{x_0}M$ which represents the point x_0 in the normal chart around x_0 and g is the metric pulled back onto $T_{x_0}M$ by the exponential map.

The fundamental equation satisfied by the normal coordinates reads [72, equation A 2.3, p. 271]

$$g_{jk}(x)x^k = g_{jk}(0)x^k = x^j, \tag{5.55}$$

and this very general result is valid in pseudo-Riemannian geometry. This implies that $|y|_g^2 = g_{jk}(0)y^j y^k = \eta_{jk}y^j y^k$. The second key observation is the statement of the next lemma.

Lemma 5.6. *Let $\mathbf{F}_\alpha(z, |\cdot|_g) \in \mathcal{D}'(U)$ be the family of distributions from Definition 5.5. In the normal coordinate system $(x^j)_{j=0}^n$ on U defined in (5.55), we have the identities*

$$\begin{aligned} 2g^{jk}(x)\partial_{x^k}\mathbf{F}_\alpha(z, |x|_g) &= x^j\mathbf{F}_{\alpha-1}(z, |x|_g), \\ (\partial_{x^j} g^{jk}\partial_{x^k} - z)\mathbf{F}_0(z, |x|_g) &= |g(x)|^{-1/2}\delta_0(x), \\ (\partial_{x^j} g^{jk}\partial_{x^k} - z)\mathbf{F}_\alpha(z, |x|_g) &= \alpha\mathbf{F}_{\alpha-1}(z, |x|_g). \end{aligned}$$

Proof. The proof is completely analogous to the Riemannian case (see [69, (17.4.2), (17.4.3), pp. 31–32]). The important property used to derive these identities is that in the normal coordinate system, for any function $f(|\cdot|_g^2)$ of the square geodesic length, we have

$$g^{jk}(x)\partial_{x^k} f(|x|_g^2) = g^{jk}(0)\partial_{x^k} f(|x|_g^2).$$

The first equation follows from the fact that

$$\begin{aligned} 2g^{jk}(x)\partial_{x^k}\mathbf{F}_\alpha(z, |x|_g) = 2g^{jk}(0)\partial_{x^k}\mathbf{F}_\alpha(z, |x|_g) &= g^{jk}(0)\eta_{ki}x^i\mathbf{F}_{\alpha-1}(z, |x|_g) \\ &= x^j\mathbf{F}_{\alpha-1}(z, |x|_g), \end{aligned}$$

where we have used (5.54) and $g^{jk}(0) = \eta^{jk}$. The second equation follows from the first equation, (5.53) and the properties of the normal coordinate chart:

$$(\partial_{x^j} g^{jk}\partial_{x^k} - z)\mathbf{F}_0(z, |x|_g) = |g(x)|^{-1/2}\delta_0(x). \qquad \blacksquare$$

5.2. Deriving the transport equations

Recall that $\Box_g$ is the Lorentzian Laplace–Beltrami operator and $P = \Box_g + m^2$ is the Klein–Gordon operator. The parametrix construction involves transport equations because even though the operator $\partial_{x^j} g^{jk}(x)\partial_{x^k} + m^2 - z$ has a fundamental solution which is $\mathbf{F}_0(-m^2 + z, |x|_g^2)$, the operator $P - z$ is not necessarily of the form $\partial_{x^j} g^{jk}(x)\partial_{x^k} + m^2 - z$ but is rather given by the more general expression

$$P - z = \partial_{x^j} g^{jk}(x)\partial_{x^k} + b^j(x)\partial_{x^j} + m^2 - z$$

with a *non-trivial subprincipal part* $b^j(x)\partial_{x^j}$. This subprincipal part will be responsible for the appearance of the scalar curvature as we will later see.

For $\operatorname{Im} z > 0$, let $f(z,\cdot)$ be the unique Schwartz distribution such that

$$F_0(z,x) = \frac{1}{(2\pi)^n}\int e^{i\langle x,\xi\rangle}(|\xi|_\eta^2 - i0 - z)^{-1}\, d^n\xi = f(z,|x|_\eta^2).$$

We have the following Fourier integral representation for f:[11]

$$f(z,q) = \frac{e^{-i\pi/2}}{(4\pi i)^{n/2}(-1)^{(n-1)/2}}\int_0^\infty e^{\frac{q}{4ui} + iuz} u^{-n/2}\, du.$$

The existence of $f(z,\cdot) \in \mathcal{D}'(\mathbb{R})$ and of $f(z,|\cdot|_g^2) \in \mathcal{D}'(U)$ for $\operatorname{Im} z > 0$ follows from the oscillatory integral proofs of Lemma 4.4. We have the following key lemma, which again parallels the Riemannian case [69, (17.4.5), p. 32].

Lemma 5.7. *Let $\mathbf{F}_\alpha(z - m^2, |\cdot|_g) \in \mathcal{D}'(U)$ be the family of distributions from Definition 5.5, in the normal coordinate system $(x^j)_{j=0}^n$ on $U \subset T_{x_0}M$ defined in (5.55), and let $\operatorname{Re}\alpha > 0$. For any $u \in C^\infty(U)$,*

$$(P - z)(u\mathbf{F}_\alpha) = \alpha u\mathbf{F}_{\alpha-1} + (Pu)\mathbf{F}_\alpha + (hu + 2\rho u)\frac{\mathbf{F}_{\alpha-1}}{2}, \tag{5.56}$$

where

$$h(x) = b^j(x)\eta_{jk}x^k \quad \text{and} \quad \rho = x^k\partial_{x^k}.$$

For $\alpha = 0$ and all $u_0 \in C^\infty(U)$,

$$\begin{aligned}(P - z)u_0\mathbf{F}_0 = {} & u_0|g(x)|^{-1/2}\delta_0(x) + (Pu_0)\mathbf{F}_0 + 2hu_0 f'(z,|\cdot|_g) \\ & + 4x^j\frac{\partial u_0}{\partial x_j} f'(z,|\cdot|_g),\end{aligned} \tag{5.57}$$

where f' is the distributional derivative of f.

[11] It is related to Bessel–Macdonald K functions, sometimes called modified Bessel functions of the second kind.

Proof. By definition and using all identities from Lemma 5.6,

$$
\begin{aligned}
(P-z)(u\mathbf{F}_\alpha) &= (Pu)\mathbf{F}_\alpha + u\big((\partial_{x^k} g^{jk}(x)\partial_{x^k} - z)\mathbf{F}_\alpha\big) + 2(\partial_{x^j} u) g^{jk}(x)(\partial_{x^k}\mathbf{F}_\alpha) \\
&\quad + u b^j(x)(\partial_{x^j}\mathbf{F}_\alpha) \\
&= (Pu)\mathbf{F}_\alpha + u\alpha[-6pt]\mathbf{F}_{\alpha-1} + (x^j\partial_{x^j} u)\mathbf{F}_{\alpha-1} + \frac{u b^j(x)\eta_{jk}x^k}{2}\mathbf{F}_{\alpha-1}
\end{aligned}
$$

since

$$2g^{jk}(x)\partial_{x^k}\mathbf{F}_\alpha = x^j\mathbf{F}_{\alpha-1},$$

which implies

$$2\partial_{x^i}\mathbf{F}_\alpha = 2g_{ik}(x)g^{kj}(x)\partial_{x^j}\mathbf{F}_\alpha = g_{ik}(x)x^k\mathbf{F}_{\alpha-1} = g_{ik}(0)x^k\mathbf{F}_{\alpha-1} = \eta_{ik}x^k\mathbf{F}_{\alpha-1}.$$

The second equation is obtained in the same way. ■

The existence of $f'(z, |\cdot|_g)$ follows from the same arguments as in the proof of Lemma 4.4.

5.2.1. Parametrix from transport equations. In this subsection we construct the formal parametrix in the normal coordinate chart $U \subset T_{x_0}M$ centered around $x_0 \in M$. We start from equation (5.57). We need to solve away the term in front of f' which reads $4x^j\frac{\partial u_0}{\partial x_j} + 2hu_0$, so we must look for $u_0 \in C^\infty(U)$ solving the first *transport equation*

$$2\rho u_0 + hu_0 = 0$$

with initial condition $u_0(0) = 1$. So we see immediately that there is a potential problem since there is still a term $(Pu_0)\mathbf{F}_0$ which is singular. To kill this singular term, we look for $u_1 \in C^\infty(U)$ satisfying

$$\rho u_1 + u_1 + \frac{h}{2}u_1 = -Pu_0,$$

since for such a pair of smooth functions $(u_0, u_1) \in C^\infty(U)^2$, we would immediately find that

$$
\begin{aligned}
(P-z)(u_0\mathbf{F}_0 + u_1\mathbf{F}_1) &= u_0|g|^{-1/2}\delta_0(x) + (Pu_0)\mathbf{F}_0 + u_1\mathbf{F}_0 + (Pu_1)\mathbf{F}_1 \\
&\quad + (hu_1 + 2\rho u_1)\frac{\mathbf{F}_0}{2} \\
&= |g|^{-1/2}\delta_0(x) + (Pu_1)\mathbf{F}_1.
\end{aligned}
$$

Applying the above algorithm recursively, we see that at order N we have to look for a parametrix $H_N(z)$ of the form

$$H_N(z) = \sum_{k=0}^{N} u_k\mathbf{F}_k(z-m^2, |\cdot|_g) \in \mathcal{D}'(U),$$

where the sequence $(u_k)_{k=0}^\infty$ of functions in $C^\infty(U)$ solves the well-known *hierarchy of transport equations*

$$2ku_k + hu_k + 2\rho u_k + 2Pu_{k-1} = 0, \tag{5.58}$$

where for $k = 0$, we choose the convention that $u_{k-1} = 0$. This would kill all terms in front of $f', \mathbf{F}_0, \dots, \mathbf{F}_{N-1}$, therefore in the normal chart near x, $H_N(z, \cdot)$ satisfies the equation

$$(P - z)H_N(z, \cdot) = |g|^{-1/2}\delta_0 + (Pu_N)\mathbf{F}_N.$$

Note that the solutions $(u_k)_{k=0}^\infty$ of the transport equations do not depend on z or on the mass term m. This dependence is absorbed in the distributions $\mathbf{F}_k(z - m^2, |\cdot|_g)$.

The next lemma is fully analogous to [69, Lemma 17.4.1, p. 33]:

Lemma 5.8. *The hierarchy of transport equations always has solutions in* $C^\infty(U)$ *where* $U \subset T_{x_0}M$ *is any open neighborhood of* $0 \in T_{x_0}M$ *such that* $\exp_{x_0} : U \to M$ *is* injective.

The formulation of Hörmander is practical for proving estimates and fairly general, but to extract the scalar curvature we will later have to specialize to the case of the pseudo-Riemannian Laplace–Beltrami operator.

5.2.2. Going back to a neighborhood of the diagonal Δ. For the moment we have constructed a parametrix $T_{x_0}M \supset U \ni x \mapsto H_N(z, x)$ around some fixed $x_0 \in M$. Now we need to treat x_0 as a parameter and prove that everything depends nicely on $x_0 \in M$. First, observe that the solutions $(u_k)_k$ of the transport equations are smooth in $C^\infty(U)$. Recall that $(s^\mu)_\mu$ is the coframe from Section 5.1.1 and $\mathcal{U} \subset M \times M$ is a neighborhood of the diagonal $\Delta \subset M \times M$. Therefore, $\mathcal{U} \ni (x_1, x_2) \mapsto u_k(s(\exp_{x_1}^{-1}(x_2)))$ is smooth in both arguments by composition and smoothness of the inverse exponential map on $\mathcal{U}$. The distributions $\mathbf{F}_\alpha(z, \cdot) = G^*F_\alpha(z, \cdot) \in \mathcal{D}'(\mathcal{U})$ are also well-defined on the neighborhood $\mathcal{U}$ of the diagonal (with wavefront set in Λ). Therefore the parametrix

$$H_N(z, x_1, x_2) = \sum_{k=0}^{N} u_k(s(\exp_{x_1}^{-1}(x_2)))G^*F_\alpha(z - m^2, x_1, x_2)$$

describes in fact an element of $\mathcal{D}'_\Lambda(\mathcal{U})$. For the sake of brevity, by slight abuse of notation we simply write u_k for the solution of the transport equation (the inverse exponential map is dropped), and the parametrix $H_N(z, x)$ depending on the variable x in the normal chart around x_0 or its pull-back $H_N(z, s(\exp_{x_1}^{-1}(x_2)))$ on $\mathcal{U}$ are both denoted by H_N. So from now on, one should always be aware that all objects are defined in terms of the exponential map. With these conventions the *Hadamard parametrix* $H_N(z, \cdot)$ reads

$$H_N(z, \cdot) = \sum_{k=0}^{N} u_k\mathbf{F}_k(z - m^2, |\cdot|_g) \in \mathcal{D}'(\mathcal{U}).$$

In the sequel, we shall use the notation δ_Δ for the distribution defined locally by pull-back as $\delta_\Delta = G^*\delta_0$ which we can extend globally by a partition of unity. By construction,

δ_Δ is a conormal distribution supported by the diagonal $\Delta \subset M \times M$ and $\delta_\Delta(x,y)|g|^{-1/2}$ is the Schwartz kernel of the identity map.[12]

6. The Hadamard parametrix approximates the resolvent

6.1. Summary

From now on, we assume that (M, g) is globally hyperbolic with non-trapping Lorentz scattering metric g.

The goal of this section is to prove that the formal parametrix $H_N(z, \cdot)$ constructed in Section 5 truly approximates the resolvent $(P - z)^{-1}$ in the space $\mathcal{D}'_\Lambda(\mathcal{U})$ of distributions defined near the diagonal, whose wavefront set is the Feynman wavefront Λ.

By Lemma D.3 proved in the appendix which shows that the Hölder regularity of $F_\alpha(z, \cdot)$ is preserved under pull-back by G, and by Theorem 4.8 giving the microlocal properties of the family $F_\alpha(z, \cdot)$, combined with Proposition 5.3, we have the following bounds.

Lemma 6.1. *Let $\mathcal{U} \subset M \times M$ be the neighborhood of the diagonal $\Delta \subset M \times M$ as defined in Section 5.1.1, and let $a \in [0, 1]$. The family of distributions $\langle \operatorname{Im} z\rangle^{\operatorname{Re}\alpha+1}\mathbf{F}_\alpha(z, \cdot)$ is bounded in $\mathcal{D}'_\Lambda(\mathcal{U})$ and the family $\langle \operatorname{Im} z\rangle^{a(\operatorname{Re}\alpha+1)}\mathbf{F}_\alpha(z, \cdot)$ is bounded in $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$ uniformly in $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$ for all $s \leqslant (2 - 2a)(\operatorname{Re}\alpha + 1) - k - n$ where $k = \lfloor \operatorname{Re}\alpha \rfloor + 1$.*

Moreover, for every $N \in \mathbb{N}$ and $\varepsilon > 0$, $(|\operatorname{Im} z|^N \mathbf{F}_\alpha(z, \cdot))_{z \in \operatorname{Im} z > \varepsilon}$ is bounded in $\mathcal{D}'_\Lambda(\mathcal{U} \setminus \Delta)$ uniformly in $\alpha \in \mathbb{C}$. If $z \in \{\operatorname{Im} z \geqslant 0,\ \operatorname{Re} z \geqslant m^2 > 0\}$, then $(\langle \operatorname{Im} z\rangle^N \mathbf{F}_\alpha(z, \cdot))_{z \in \{\operatorname{Im} z \geqslant 0, |z| \geqslant \varepsilon > 0\}}$ is bounded in $\mathcal{D}'_\Lambda(\mathcal{U} \setminus \Delta)$ uniformly in $\alpha \in \mathbb{C}$.

6.2. Resolvent approximation

Let $\chi \in C^\infty(M \times M; [0, 1])$ be such that $\chi = 1$ near the diagonal $\Delta \subset M \times M$ and $\chi(x, y) = 0$ outside the neighborhood $\mathcal{U}$ defined in (5.50). Recall that the Feynman wavefront set is defined in (5.51).

We interpret the family

$$H_N(z - m^2, \cdot)\chi = \sum_{k=0}^{N} u_k \mathbf{F}_k(z - m^2, \cdot)\chi \in \mathcal{D}'_\Lambda(\mathcal{U})$$

as a family of Schwartz kernels, and we show that near the diagonal, the corresponding family of operators is a parametrix which approximates the resolvent $(P - z)^{-1}$.

Let $\tilde{\mathcal{U}}$ be a neighborhood of the diagonal Δ such that $\chi|_{\tilde{\mathcal{U}}} = 1$ and $\tilde{\mathcal{U}} \subset \mathcal{U}$. Let

$$\Lambda_\chi = \{(x, y; \xi, \eta) \mid (x, y; \xi, \eta) \in \Lambda,\ (x, y) \notin \tilde{\mathcal{U}}\},$$

[12] We need to multiply by $|g|^{-1/2}$ in order to take into account integration against the volume element.

so $\Lambda_\chi = \Lambda \cap T^*((M \times M) \setminus \tilde{\mathcal{U}})$ is a *truncation of the Feynman wavefront set* Λ where we removed the neighborhood $\tilde{\mathcal{U}}$ near the diagonal.

Lemma 6.2. *Assume* (M, g) *is globally hyperbolic and set* $P = \Box_g + m^2$. *For every* $s \in \mathbb{R}_{\geqslant 0}$, $p \in \mathbb{Z}_{\geqslant 0}$, *and* $m \geqslant 0$ *there exists* N *large enough such that for every* $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$,

$$(P - z)\Big(\sum_{k=0}^{N} u_k \mathbf{F}_k(z - m^2, \cdot)\chi\Big) = |g|^{-1/2}\delta_\Delta + (Pu_N)\mathbf{F}_N(z - m^2, \cdot)\chi + r_N(z), \qquad (6.59)$$

where

(1) $|g|^{-1/2}\delta_\Delta \in \mathcal{D}'(M \times M)$ *is the Schwartz kernel of the identity map,*

(2) $\langle z\rangle^p (Pu_N)\mathbf{F}_N(z - m^2, \cdot)\chi$ *is bounded in* $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$,

(3) $r_N(z, \cdot) \in \mathcal{D}'(M \times M)$ *vanishes on* $\tilde{\mathcal{U}} \subset \mathcal{U}$ *and outside* $\mathcal{U} \subset M \times M$. *In particular,* $r_N(z, \cdot)$ *is the Schwartz kernel of a family of properly supported operators. Furthermore,* $r_N(z, \cdot)$ *is bounded in* $\mathcal{D}'_{\Lambda_\chi}(M \times M)$ *uniformly in* $z \in \{\operatorname{Im} z \geqslant 0,\ |z| \geqslant \varepsilon > 0\}$, *and* $r_N(z, \cdot) = \mathcal{O}_{\mathcal{D}'_{\Lambda_\chi}}(\langle \operatorname{Im} z\rangle^{-\infty})$.

Note that in the above formulæ, the coefficients $(u_k)_{k=0}^\infty$ of the transport equations do not depend on the mass m or on the spectral parameter z.

Proof of Lemma 6.2. We start from the result from Section 5 which says that in the sense of distributions in $\mathcal{D}'(\mathcal{U})$,

$$(P - z)\Big(\sum_{k=0}^{N} u_k \mathbf{F}_k(z - m^2, \cdot)\Big) = |g|^{-1/2}\delta_\Delta + (Pu_N)\mathbf{F}_N(z - m^2, \cdot), \qquad (6.60)$$

where $\mathbf{F}_N(z, \cdot) \in \mathcal{C}^{(2-2a)(N+1)-N-n}_{\mathrm{loc}}(\mathcal{U})$ with decay in z of the form $\langle \operatorname{Im} z\rangle^{-a(N+1)}$ for $a \in [0, 1]$ by Lemma 6.1, and $\mathbf{F}_N(z, \cdot)$ is bounded in $\mathcal{D}'_\Lambda(\mathcal{U})$ by Proposition 5.3. Now, multiplying the parametrix defined near the diagonal by the cut-off function χ creates an *additional term*. Namely, it turns (6.60) into

$$(P - z)\Big(\sum_{k=0}^{N} u_k \mathbf{F}_k(z - m^2, \cdot)\chi\Big) = |g|^{-1/2}\delta_\Delta + (Pu_N)\mathbf{F}_N(z - m^2, \cdot)\chi + r_N(z, \cdot),$$

where

$$r_N(z, \cdot) = 2\Big\langle \nabla\chi, \nabla\Big(\sum_{k=0}^{N} u_k \mathbf{F}_k(z - m^2, \cdot)\chi\Big)\Big\rangle + (P\chi)\Big(\sum_{k=0}^{N} u_k \mathbf{F}_k(z - m^2, \cdot)\Big).$$

The term $r_N(z, \cdot)$ vanishes whenever either $\chi = 1$ or $\chi = 0$, since it involves products with derivatives of the diagonal cut-off function χ. This means that $r_N(z, x, y) = 0$ for $(x, y) \in \tilde{\mathcal{U}}$, which is near the diagonal Δ, and also $r_N(z, x, y) = 0$ for $(x, y) \notin \mathcal{U}$.

This implies that the Schwartz kernel $r_N(z,\cdot) \in \mathcal{D}'(M \times M)$ defines a *proper operator*. Therefore, $r_N(z,\cdot)$ is bounded in $\mathcal{D}'_{\Lambda_\chi}$ with a bound of the form $r_N(z,\cdot) = \mathcal{O}_{\mathcal{D}'_{\Lambda_\chi}(M\times M)}(|\mathrm{Im}\, z|^{-\infty})$ since it is defined in terms of the distributions $\mathbf{F}_k(z,\cdot) = \mathcal{O}_{\mathcal{D}'_{\Lambda}(\mathcal{U}\setminus\Delta)}(|\mathrm{Im}\, z|^{-\infty})$ restricted *outside the diagonal* and because of its support that we have just discussed. ■

We are now ready to conclude that near the diagonal, our parametrix is a good approximation of the resolvent $(P-z)^{-1}$.

We denote by $r_N(z)$ the operator with Schwartz kernel $r_N(z,\cdot)$, and similarly for other Schwartz kernels. Recall that γ_ε is the integration contour needed to define the complex powers (see Section 2.6 and Figure 1 therein).

Proposition 6.3. *Assume that (M,g) is a globally hyperbolic non-trapping Lorentzian scattering space and let $\varepsilon > 0$. Set $P = \Box_g + m^2$. For every $s \in \mathbb{R}_{\geqslant 0}$, $p \in \mathbb{Z}_{\geqslant 0}$, and $m \geqslant 0$, there exists N large enough such that uniformly in $z \in \gamma_\varepsilon$, we have the identity*

$$(P-z)^{-1} = \Big(\sum_{k=0}^{N} u_k \mathbf{F}_k(z-m^2,\cdot)\chi\Big) + E_{N,1}(z) + E_{N,2}(z) \tag{6.61}$$

in the sense of operators $\mathcal{O}_{C^\infty\to C^\infty}(\langle z\rangle^{-1/2})$, where $E_{N,1}(z) = (P-z)^{-1} r_N(z)$ satisfies

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-\infty}}(E_{N,1}(z)) \subset \Lambda'_\chi, \tag{6.62}$$

and $E_{N,2}(z) = (P-z)^{-1}(Pu_N)\mathbf{F}_N(z)\chi$ satisfies $E_{N,2}(z) = \mathcal{O}_{H^*\to H^{*+s}}(\langle z\rangle^{-p})$. Furthermore if $m \neq 0$ and assuming in addition injectivity and non-trapping at $\sigma = m^2$, then the estimates and (6.61) hold uniformly in $z \in \gamma_0$.

In particular, (6.62) implies that $E_{N,1}(z)$ is *smooth near the diagonal*.

As explained earlier, there are inevitable losses in decay in z in the high regularity estimates, and the $\mathcal{O}(\langle z\rangle^{-p})$ bound requires choosing N extremely large.

Proof of Proposition 6.3. Recall that $(P-z)^{-1} = \mathcal{O}_{C^\infty\to C^\infty}(\langle z\rangle^{-1/2})$ along γ_ε by Lemma 3.17. By Lemma 6.2 (2), for every $(p,a) \in \mathbb{N}\times\mathbb{R}$, for N large enough, in $\mathcal{U}$ we have $\langle z\rangle^p (Pu_N)\mathbf{F}_N(z)\chi \in H^{a+n-0}_{\mathrm{loc}}(M\times M)$ uniformly in z along γ_ε by the Sobolev embeddings $\mathcal{C}^a_{\mathrm{loc}}(M\times M) \hookrightarrow H^{a+n-0}_{\mathrm{loc}}(M\times M)$ recalled in Lemma D.2 in the appendix. For every $b \in \mathbb{R}$ and $s \in \mathbb{R}_{\geqslant 0}$, the external tensor product $H^b_{\mathrm{c}}(M) \times H^{-b-s}_{\mathrm{c}}(M) \ni (v_1,v_2) \mapsto v_1 \otimes v_2 \in H^{\inf(b,-b-s,-s)}_{\mathrm{c}}(M\times M)$ is linear continuous [73, Theorem 3.2, p. 140], therefore choosing n large enough so that $a+n+\inf(b,-b-s,-s) > 0$, we find that

$$\begin{aligned} H^b_{\mathrm{c}}(M) \times H^{-b-s}_{\mathrm{c}}(M) \ni (v_1,v_2) &\mapsto \langle v_2, \langle z\rangle^p (Pu_N)\mathbf{F}_N(z)\chi v_1\rangle_M \\ &= \langle \langle z\rangle^p (Pu_N)\mathbf{F}_N(z)\chi, v_1\otimes v_2\rangle_{M\times M} \end{aligned}$$

is bilinear continuous by Sobolev duality uniformly in z along γ_ε. Therefore we have (for N large enough)

$$(Pu_N)\mathbf{F}_N(z)\chi = \mathcal{O}_{H^*\to H^{*+s}}(\langle z\rangle^{-p}). \tag{6.63}$$

Since the operators in (6.63) are properly supported,

$$E_{N,2}(z) = (P - z)^{-1}(Pu_N)\mathbf{F}_N(z)\chi = \mathcal{O}_{H^* \to H^{*+s}}(\langle z \rangle^{-p})$$

follows by composition.

Next, by Lemma 6.2 (3), $r_N(z, \cdot) = \mathcal{O}_{\mathcal{D}'_{\Lambda_\chi}}(\langle z \rangle^{-\infty})$ along γ_ε. Consequently, in terms of the operator wavefront set we have $\mathrm{WF}'^{(s)}_{\langle z \rangle^{-\infty}}(r_N(z)) \subset \Lambda'_\chi$ by Lemma 3.10. We also know from Theorem 3.19 that

$$\mathrm{WF}'^{(s)}_{\langle z \rangle^{-1/2}}((P - z)^{-1}) \subset \Lambda'.$$

By the composition rule for operator wavefront sets (Lemma 3.7), we obtain

$$\mathrm{WF}'^{(s)}_{\langle z \rangle^{-\infty}}(E_{N,1}(z)) \subset \Lambda' \circ \Lambda'_\chi. \tag{6.64}$$

It is easy to show using the transitivity of $\succ$ that Λ and Λ_χ have the remarkable property

$$\Lambda' \circ \Lambda' \subset \Lambda' \quad \text{and} \quad \Lambda' \circ \Lambda'_\chi \subset \Lambda'_\chi. \tag{6.65}$$

From (6.64) and the second property in (6.65) we conclude (6.62) immediately. The case of γ_0 is fully analogous. ■

7. Parametrix for complex powers and functions of the wave operator

7.1. The case of complex powers $(P - i\varepsilon)^{-\alpha}$

Our next objective is to study analytic properties of complex powers in a neighborhood $\mathcal{U} \subset M \times M$ of the diagonal using the relationship between the resolvent and the Hadamard parametrix shown in Proposition 6.3.

Let $\varepsilon > 0$. We already know that the contour integral

$$(P - i\varepsilon)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\alpha}(P - z)^{-1}\, dz$$

makes sense as an operator in $\mathcal{B}(L^2(M))$ for $\operatorname{Re}\alpha > 0$. Using the decay along γ_ε of the various terms, stated in Lemma 6.1 and Proposition 6.3, we can insert the r.h.s. of (6.61) into the above contour integral. For $\operatorname{Re}\alpha > 0$ this yields

$$\begin{aligned}
(P - i\varepsilon)^{-\alpha} &= \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\alpha}(P - z)^{-1}\, dz \\
&= \sum_{k=0}^{N} \chi u_k \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\alpha}\mathbf{F}_k(z)\, dz \\
&\quad + \frac{i}{2\pi} \int_{\gamma_\varepsilon} (z - i\varepsilon)^{-\alpha}(E_{N,1}(z) + E_{N,2}(z))\, dz,
\end{aligned}$$

and this extends to $\alpha \in \mathbb{C}$ provided we check that the summands have an analytic continuation.

By Lemma 4.7 and continuity of the pull-back by G, we know that

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z-i\varepsilon)^{-\alpha}\mathbf{F}_k(z,\cdot)\,dz = \frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)}\mathbf{F}_{k+\alpha-1}(i\varepsilon,\cdot),$$

which is a well-defined holomorphic family of distributions in $\mathcal{D}'(\mathcal{U})$. Therefore the finite sum $\sum_{k=0}^{N}\chi u_k\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(z+i\varepsilon)^{-\alpha}\mathbf{F}_k(z,\cdot)\,dz$ in fact reads

$$\sum_{k=0}^{N}\chi u_k\frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)}\mathbf{F}_{k+\alpha-1}(-i\varepsilon,\cdot)$$

and is a well-defined holomorphic family of distributions in the parameter $\alpha \in \mathbb{C}$.

The error term

$$R_N(z,\alpha) := \frac{i}{2\pi}\int_{\gamma_\varepsilon}(z-i\varepsilon)^{-\alpha}(E_{N,1}(z)+E_{N,2}(z))\,dz$$

is smooth near the diagonal.

It follows that we have a decomposition of the Schwartz kernel of $(P-i\varepsilon)^{-\alpha}$ which has to be understood in the sense of germs of distributions defined near the diagonal $\Delta \subset M\times M$. The germ is the only information we need to take the diagonal restrictions:

Lemma 7.1. *Let (M,g) and P be as in Proposition 6.3. Then for every $s \in \mathbb{R}_{\geqslant 0}$ and $p \in \mathbb{N}$, there exists $N \geqslant 0$ such that in $\mathcal{D}'(\tilde{\mathcal{U}})$ we have the decomposition*

$$(P-i\varepsilon)^{-\alpha} = \sum_{k=0}^{N}u_k\frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)}\mathbf{F}_{k+\alpha-1}(-m^2+i\varepsilon)+R_N(i\varepsilon,\alpha), \tag{7.66}$$

$$R_N(i\varepsilon,\alpha) \in C^s(\mathcal{U}), \tag{7.67}$$

where the terms on the r.h.s. depend holomorphically on α in the half-plane $\operatorname{Re}\alpha > -p$ *and the r.h.s. is well-defined as an element of $\mathcal{D}'(\tilde{\mathcal{U}})$.*

Corollary 7.2. *For* $\operatorname{Re}\alpha \geqslant n+s$ *(where* $n = \dim M$*) and* $s>0$*, the operator* $(P-i\varepsilon)^{-\alpha}$ *has Hölder regularity* $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$ *in a neighborhood of the diagonal* $\Delta\subset M\times M$*, and under the non-trapping and injectivity assumption at* $\sigma = m^2 \neq 0$ *the limit* $\lim_{\varepsilon\to 0^+}(P-i\varepsilon)^{-\alpha}$ *exists in the sense of* $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$.

Remark 7.3 (Checking the combinatorial factors). In the $\alpha \to 1$ limit, we get

$$\frac{(-1)^k\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)\Gamma(\alpha+k)} \to 1$$

since the poles of $\Gamma(-\alpha+1)$ and $\Gamma(-\alpha-k+1)$ compensate each other. Therefore, Lemma 7.1 is consistent with the formula $(P-i\varepsilon)^{-1} = \sum_{k=0}^{N}u_k\mathbf{F}_k(i\varepsilon)+R_N(i\varepsilon,\alpha)$ as expected.

7.2. The case of $f(\frac{P+i\varepsilon}{\lambda^2})$

With the spectral action in mind we now discuss other functions of P. For that purpose it is actually slightly more convenient to work with $P+i\varepsilon$ instead of $P-i\varepsilon$, which in practice amounts to considering $-P$ instead of P. Note that $(P+i\varepsilon)^{-1}$ and the corresponding Hadamard parametrix have *anti-Feynman* rather than Feynman wavefront set because the sinks and sources are interchanged, but this has no practical significance in the discussion below.

Definition 7.4. We denote by $S_+^{-\infty}(\mathbb{R})$ the set of Schwartz functions f such that $\widehat{f}$ is in $C_c^\infty(]0,+\infty])$.

First, observe that by the Paley–Wiener theorem, each $f\in S_+^{-\infty}(\mathbb{R})$ has a unique holomorphic extension to the upper half–plane $\{\operatorname{Im} z\geqslant 0\}$ and that the analytic extension, still denoted by f, has exponential decay when $\operatorname{Im} z\to+\infty$. Also note that $f(\cdot+i\varepsilon)\in L^\infty(\mathbb{R})\cap C^\infty(\mathbb{R})$. Recall that the Mellin transform of $\widehat{f}\in C_c^\infty(]0,+\infty])$ is by definition the function

$$\mathcal{M}\widehat{f}(\alpha)=\int_0^\infty \tau^{\alpha-1}\widehat{f}(\tau)\,d\tau.$$

By assumption on $\widehat{f}$, the Mellin transform $\mathcal{M}\widehat{f}$ has fast decay, and the Mellin inversion yields $\widehat{f}(\tau)=\frac{1}{2\pi i}\int_{\operatorname{Re}\alpha=c}\tau^{-\alpha}\mathcal{M}\widehat{f}(\alpha)\,ds$, where the integral is absolutely convergent uniformly in $\tau\in K\subset]0,+\infty]$ for any compact K. By inverse Fourier transform of $\tau_+^{-\alpha}$, for every $\varepsilon>0$, we have the formula

$$f(t+i\varepsilon)=\frac{1}{2\pi i}\int_{\operatorname{Re}\alpha=c} e^{i\alpha\pi/2}(t+i\varepsilon)^{-\alpha}\Gamma(\alpha)\mathcal{M}\widehat{f}(\alpha)\,d\alpha.$$

The l.h.s. makes sense when $\varepsilon>0$ since f has an analytic continuation to the upper half-plane $\{\operatorname{Im} z\geqslant 0\}$. Note that for $\varepsilon>0$ the integral on the r.h.s. converges absolutely and that for $t\in K$ in some compact $K\subset\mathbb{R}\setminus\{0\}$ away from 0, the integral in

$$f(t)=\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{\operatorname{Re}\alpha=c} e^{i\alpha\pi/2}(t+i\varepsilon)^{-\alpha}\Gamma(\alpha)\mathcal{M}\widehat{f}(\alpha)\,d\alpha$$

also converges absolutely. This allows us to give a representation formula for $f(P+i\varepsilon)$ involving complex powers:

$$f(P+i\varepsilon)=\frac{1}{2\pi i}\int_{\operatorname{Re}\alpha=c} e^{i\alpha\pi/2}(P+i\varepsilon)^{-\alpha}\Gamma(\alpha)\mathcal{M}\widehat{f}(\alpha)\,ds. \tag{7.68}$$

As long as $\varepsilon>0$, the integral on the r.h.s. is norm convergent in $\mathcal{B}(L^2(M))$ and the identity follows by Borel functional calculus. It should be noted that one can choose $c>0$ arbitrarily large on the r.h.s.; this does not affect the convergence properties.

Recall $\mathcal{U}$ is a neighborhood of the diagonal $\Delta\subset M\times M$. Setting now $c>\dim M+s$ for $s>0$, we know by Corollary 7.2 that the Schwartz kernel of $(P+i\varepsilon)^{-\alpha}$ belongs to $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$ uniformly in $\alpha\in\{\operatorname{Re}\alpha=c\}$. Note that to take the limit $\varepsilon\to 0^+$, we need to assume $m \neq 0$ and non-trapping and injectivity at m^2, in which case $\lim_{\varepsilon\to 0^+}(P+i\varepsilon)^{-\alpha} = (P+i0)^{-\alpha}$ has Schwartz kernel in $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$ uniformly in $\alpha \in \{\operatorname{Re}\alpha = c\}$.

Therefore by the fast decay of $\mathcal{M}\widehat{f}(\alpha)$ on the vertical line $\{\operatorname{Re}\alpha = c\}$, we find that $f(P+i\varepsilon)$ has Schwartz kernel which belongs to $\mathcal{C}^s_{\mathrm{loc}}(\mathcal{U})$. If m^2 is non-trapping, the same result holds for $\lim_{\varepsilon\to 0^+} f(P+i\varepsilon) = f(P+i0)$, which implies we can take $\varepsilon \to 0^+$ on the r.h.s. of (7.68), which makes sense as Schwartz kernel near $\Delta \subset M \times M$ and one can take the restriction to the diagonal as $f(P+i0)(x,x)$. If $\varepsilon > 0$, we do not need the mass term and both sides of (7.68) make sense as operators acting on L^2 whose Schwartz kernel is $\mathcal{C}^s$ near the diagonal Δ.

So if m^2 non-trapping, in the limit $\varepsilon \to 0^+$, we take the formula on the r.h.s. of (7.68) as the definition of $f(P+i0)$ and both sides are no longer viewed as operators but as germs of Schwartz kernels defined near the diagonal $\Delta \subset M \times M$:

$$f(P+i0)(\cdot,\cdot) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{\operatorname{Re}\alpha = c} e^{i\alpha\pi/2}(P+i\varepsilon)^{-\alpha}(\cdot,\cdot)\Gamma(\alpha)\mathcal{M}\widehat{f}(\alpha)\, ds \in \mathcal{C}^s_{\mathrm{loc}}(\mathcal{U}). \quad (7.69)$$

Since s and hence the parameter c can be chosen arbitrarily large, we have proved:

Lemma 7.5. *Let (M,g) be a globally hyperbolic non-trapping Lorentzian scattering space and let $\varepsilon > 0$ and $P = \square_g + m^2$, $m \geqslant 0$. Then for all $f \in S^{-\infty}_{+}(\mathbb{R})$, $f(P+i\varepsilon) : L^2(M) \to L^2(M)$ exists and has* smooth *Schwartz kernel in some neighborhood $\mathcal{U}$ of the diagonal $\Delta \subset M \times M$.*

Moreover, if $m \neq 0$ and assuming injectivity and non-trapping at energy $\sigma = m^2$, for all $f \in S^{-\infty}_{+}(\mathbb{R})$, the limit

$$f(P+i0)(\cdot,\cdot) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{\operatorname{Re}\alpha = c} e^{i\alpha\pi/2}(P+i\varepsilon)^{-\alpha}(\cdot,\cdot)\Gamma(\alpha)\mathcal{M}\widehat{f}(\alpha)\, ds \quad (7.70)$$

exists in $\mathcal{D}'_{\Lambda}(M\times M)$ and is smooth *in a neighborhood $\mathcal{U}$ of Δ.*

8. Diagonal restriction, poles and residues

8.1. Summary

We make the central observation that $F_\alpha(z,0) = \mathbf{F}_\alpha(z,x,x)$ for every $x \in M$ by construction, therefore to study the restriction to the diagonal we only need to study the analytic continuation in α of $F_\alpha(z,0)$.

8.2. Meromorphic continuation of $F_\alpha(z,0)$

The value at $0 \in \mathbb{R}^n$ of the distribution $F_\alpha(z,\cdot)$ that we denote by $F_\alpha(z,0)$ is studied. By the Hölder regularity shown in Proposition 4.6, $F_\alpha(z,\cdot) \in \mathcal{C}^s(\mathbb{R}^n)$ for $s \leqslant \operatorname{Re}\alpha + 1 - n$, therefore the value $F_\alpha(z,0)$ at 0 is well-defined when $\operatorname{Re}\alpha > n-1$ and depends holomorphically on $\alpha \in \{\operatorname{Re}\alpha > n-1\}$, and also the limit when $\operatorname{Im} z \to 0^+$, $|z| \geqslant m^2 > 0$

is well-defined. We prove it admits an analytic continuation as a meromorphic function with simple poles at $\alpha \in \{n/2-1, n/2-2, \dots\}$.

8.2.1. A warm-up calculation. The pole of $F_\alpha(z,0)$ comes from its representation as an integral of symbols on cones, where the decay of the symbol approaches the critical dimension of the cone. We will also have to take into account the Γ factor. A typical example reads

$$\int_{\mathbb{R}^n} (\|\xi\|^2 - z)^{-\alpha}\, d^n\xi,$$

where $\|\xi\|$ is the Euclidean norm of ξ. We assume for the moment $z < 0$ so that z acts as mass squared to regulate infrared divergences since we only want to deal with UV problems.

To compute residues of such integrals, observe that the poles of

$$\int_{\mathbb{R}^n} (\|\xi\|^2 - z)^{-\alpha}\, d^n\xi = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\int_{\mathbb{R}^n} e^{-t(\|\xi\|^2 - z)}\, d^n\xi \right) t^{\alpha-1}\, dt$$

are the poles of

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_{\mathbb{R}^n} e^{-t(\|\xi\|^2 - z)}\, d^n\xi \right) t^{\alpha-1}\, dt \\ &\quad = \frac{(2\pi)^n}{\Gamma(\alpha)(4\pi)^{n/2}} \sum_{k=0}^\infty \frac{z^k}{k!} \int_0^1 t^{\alpha-n/2+k-1} dt = \frac{\pi^{n/2}}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{z^k}{k!(\alpha - n/2 + k)}. \end{aligned}$$

All poles are simple and located at $\alpha = n/2, \dots, 1$ and have z^k in the numerator; there are compensations for $\alpha \in -\mathbb{N}$ due to the Γ factor. So the residue at $\alpha = k \in \{1, \dots, n/2-1\}$ is

$$\operatorname{res}_{\alpha=k} \int_{\mathbb{R}^n} (\|\xi\|^2 - z)^{-\alpha}\, d^n\xi = \frac{z^{n/2-k}\pi^{n/2}}{(n/2-k)!\Gamma(k)}.$$

8.3. The Wick rotation by homological methods

We need to deal with similar integrals to those in the above subsection but with the Minkowski quadratic form instead of the Euclidean one. We present a geometric approach to the analytic continuation of the residue which is close to the Wick rotation in the physics literature but is fully rigorous.

Consider $\mathbb{C}^n$ viewed as a Kähler manifold with coordinates $(z_1, \dots, z_n)$, and some complex parameter $u \in \mathbb{C}$ that will take values in the upper half-plane $\{\operatorname{Im} u > 0\}$. Set $Q(z) = \sum_{i=1}^n z_i^2$ and consider the complex valued n-form

$$\omega_\alpha = \Big(\sum_{i=1}^n z_i^2 - u \Big)^{-\alpha} dz_1 \wedge \cdots \wedge dz_n \in \Omega^{n,0}(U;\mathbb{C}),$$

which is well-defined on the Zariski open set $U = \{z \in \mathbb{C}^n \mid Q(z) - u \notin]-\infty, 0]\}$ since we chose the usual branch of log which avoids the negative reals.

For all $\theta \in]-\pi/2, \pi/2]$ let $P_\theta = \{(e^{i\theta} z_1, \dots, z_n) \mid (z_1, \dots, z_n) \in \mathbb{R}^n\}$, considered as an oriented n–chain.

Proposition 8.1 (Stokes' theorem). *For all $\alpha \in \mathbb{C}$, the differential form ω_α is closed, and for all $\theta \in]-\pi/2, \pi/2]$ and all $\operatorname{Re}\alpha > n/2$,*

$$\int_{P_\theta} \omega_\alpha = \int_{P_0} \omega_\alpha. \tag{8.71}$$

Proof. It is obvious that $d\omega_\alpha = 0$. One has to be careful when applying Stokes' theorem since our "cycles" have their boundaries along sectors at infinity. Denote by $B(R) = \{\|z\|^2 \leqslant R^2\}$ the ball of radius R in $\mathbb{C}^n$.

After intersecting our chains P_θ with $B(R)$, the usual Stokes' theorem yields

$$\int_{P_\theta \cap B(R)} \omega_\alpha - \int_{P_0 \cap B(R)} \omega_\alpha = \underbrace{\int_{D_\theta} d\omega_\alpha}_{=0} - \int_{R_\theta} \omega_\alpha,$$

where D_θ is the angular sector $D_\theta = \{(z_1, \dots, z_n) \mid 0 \leqslant \arg(z_1) \leqslant \theta,\ (z_2, \dots, z_n) \in \mathbb{R}^n\} \cap B(R)$ and $R_\theta = \{(z_1, \dots, z_n) \mid 0 \leqslant \arg(z_1) \leqslant \theta,\ (z_2, \dots, z_n) \in \mathbb{R}^n,\ \|z\| = R\}$. Let us bound the integral on the arc R_θ:

$$\begin{aligned} \int_{R_\theta} \omega_\alpha &= \int_{\{\sum_{i=2}^n z_i^2 \leqslant R^2\}} \left(\int_0^\theta \left(e^{2ia} (R^2 - \sum_{i=2}^n z_i^2 - u) + \sum_{i=2}^n z_i^2 \right)^{-\alpha} i e^{ia} da \right) dz_2 \dots dz_n \\ &\leqslant C R^{n-1} R^{-2\operatorname{Re}\alpha}, \end{aligned}$$

which tends to 0 as $R \to +\infty$. Since for all $\theta \in]-\pi/2, \pi/2]$ the integral $\int_{P_\theta} \omega_\alpha$ converges absolutely when $\operatorname{Re}\alpha > n/2$, we can take the limit $R \to +\infty$, which yields

$$\lim_{R \to +\infty} \int_{P_\theta \cap B(R)} \omega_\alpha = \lim_{R \to +\infty} \int_{P_0 \cap B(R)} \omega_\alpha. \qquad \blacksquare$$

It follows from the identity $\int_{P_\theta} \omega_\alpha = \int_{P_0} \omega_\alpha$ for $\operatorname{Re}\alpha > n/2$ as holomorphic functions, and from the fact that $\int_{P_0} \omega_\alpha$ is a meromorphic function with simple poles at $\alpha \in \{n/2, n/2 - 1, \dots, 1\}$, that both sides coincide in the sense of meromorphic functions for all $\alpha \in \mathbb{C} \setminus \{n/2, \dots, 1\}$ by analytic continuation in α. Define the linear invertible holomorphic map $\Phi_\theta : (z_1, \dots, z_n) \mapsto (e^{i\theta} z_1, \dots, z_n)$. Since Φ_θ is invertible and does not reverse orientation, by the pull-back theorem we get

$$\int_{P_\theta} \omega_\alpha = \int_{\Phi_\theta(P_0)} \omega_\alpha = \int_{P_0} \Phi_\theta^* \omega_\alpha.$$

Combined with the equality $\int_{P_\theta} \omega_\alpha = \int_{P_0} \omega_\alpha$, this means that

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\sum_{i=1}^n z_i^2 - u \Big)^{-\alpha} dz_1 \wedge \dots \wedge dz_n = \int_{P_0} \omega_\alpha = \int_{P_0} \Phi_\theta^* \omega_\alpha \\ = e^{i\theta} \int_{\mathbb{R}^n} \Big(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - u \Big)^{-\alpha} dz_1 \wedge \dots \wedge dz_n. \end{aligned}$$

When $\theta \to -(\pi/2)^+$, the term $(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - u)$ has non-positive imaginary part and the integrand $(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - u)^{-\alpha}$ converges to $(-z_1^2 + \sum_{i=2}^n z_i^2 - u - i0)^{-\alpha}$ in the sense of distributions by Lemma D.1 proved in the appendix. By weak homogeneity at infinity and using a Littlewood–Paley decomposition $1 = \sum_j \chi_0(|z|) + \psi(2^{-j}|z|)$ as in Section 4.2.2, one can show that

$$\lim_{\theta \to -(\pi/2)^+} \int_{\mathbb{R}^n} \Big(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - u\Big)^{-\alpha} dz_1 \wedge \cdots \wedge dz_n$$
$$= \int_{\mathbb{R}^n} \Big(-z_1^2 + \sum_{i=2}^n z_i^2 - u - i0\Big)^{-\alpha} d^n z, \qquad (8.72)$$

where the bound

$$\sup_{\theta \in]-\pi/2, 0]} \left| \int_{\mathbb{R}^n} \Big(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - u\Big)^{-\alpha} \psi(2^{-j}|z|)\, dz_1 \wedge \cdots \wedge dz_n \right|$$
$$= \sup_{\theta \in]-\pi/2, 0]} 2^{j(n - 2\operatorname{Re}\alpha)} \left| \left\langle \Big(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - \frac{u}{2^{2j}} - i0\Big)^{-\alpha}, \psi \right\rangle \right| \leqslant C 2^{j(n-2\operatorname{Re}\alpha)}$$

ensures that both sides of (8.72) can be written as convergent series and both are *holomorphic* in α when $\operatorname{Re}\alpha > n/2$.

By Proposition 8.1 and the warm-up calculation in Section 8.2.1, for all $\theta \in]-\pi/2, 0]$,

$$\int_{\mathbb{R}^n} \Big(e^{2i\theta} z_1^2 + \sum_{i=2}^n z_i^2 - u\Big)^{-\alpha} dz_1 \wedge \cdots \wedge dz_n = e^{-i\theta} \int_{P_0} \omega_\alpha$$

extends as a meromorphic function in α with simple poles at $\alpha = n/2, \ldots, 1$. Hence the limit on the l.h.s. of (8.72) which equals $i \int_{P_0} \omega_\alpha$ also does. Therefore the r.h.s. of (8.72) equals $i \int_{P_0} \omega_\alpha$, which is meromorphic with simple poles at $\alpha = n/2, \ldots, 1$; this finally yields

$$\operatorname{res}_{\alpha=k} \int_{\mathbb{R}^n} \Big(-z_1^2 + \sum_{i=2}^n z_i^2 - u - i0\Big)^{-\alpha} d^n z = i \operatorname{res}_{\alpha=k} \int_{\mathbb{R}^n} \Big(\sum_{i=1}^n z_i^2 - u\Big)^{-\alpha} d^n z$$
$$= \frac{i \pi^{n/2} u^{n/2-k}}{\Gamma(k)(n/2-k)!}.$$

8.3.1. Conclusion and structure of residues. Therefore, we can go back to the residue of the diagonal restriction of $\mathbf{F}_\alpha(z, x, x)$; for all $z \neq 0$, $\operatorname{Im} z > 0$, we find that

$$\operatorname{res}_{\alpha=n/2-k} \Gamma(\alpha+k)^{-1} \mathbf{F}_{\alpha+k-1}(\pm z, x, x)$$
$$= \frac{1}{(2\pi)^n} \operatorname{res}_{\alpha=n/2-k} \int_{\mathbb{R}^n} (Q(\xi) \mp (z + i0))^{-\alpha-k} d^n \xi$$
$$= \pm \frac{i}{2^n \pi^{n/2} (n/2-1)!}.$$

The poles of $\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}(\pm z,x,x)$ occur at $\alpha=\{1-k,\dots,n/2-k\}$. For $j\in\{1-k,\dots,n/2-k\}$,

$$\operatorname{res}_{\alpha=j}\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}(\pm z,x,x)=\pm\frac{i(\pm z)^{n/2-k-j}}{\Gamma(j+k)2^n\pi^{n/2}(n/2-k-j)!}.$$

Applying this result to the parametrix of the Feynman powers, we note that

$$\frac{\Gamma(-\alpha+1)}{\Gamma(-\alpha-k+1)}=(-\alpha)\dots(-\alpha-k+1),$$

which means that this term has no poles for every $\alpha\in\mathbb{C}$ and *does not contribute* to the residues of $(P\pm i\varepsilon)^{-\alpha}(x,x)$. We would like to study the residue in the variable s of $(P\pm i\varepsilon)^{-\alpha}$ at $p=n/2,n/2-1,\dots$ in two cases.

Case 1. When $p=n/2,\dots,1$, we find that for α near p,

$$(P\pm i\varepsilon)^{-\alpha}=\sum_{k=0}^{\infty}\underbrace{(-1)^k(-\alpha)\dots(-\alpha-k+1)}_{\neq 0}\underbrace{\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}(-m^2\mp i\varepsilon,x,x)}_{\text{simple poles at }\{1-k,\dots,n/2-k\}},$$

which implies that

$$\begin{aligned}&\operatorname{res}_{\alpha=p}(P\pm i\varepsilon)^{-\alpha}\\&=\sum_{k=0}^{n/2-p}(-1)^k(-p)\dots(-p-k+1)\operatorname{res}_{\alpha=p}\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}(-m^2\mp i\varepsilon,x,x).\end{aligned}$$

The only residue which is independent of ε,m reads

$$\begin{aligned}&\operatorname{res}_{\alpha=n/2-k}(-1)^k\frac{u_k(x,x)\mathbf{F}_{k+s-1}(-m^2\mp i\varepsilon,x,x)\Gamma(-\alpha+1)}{\Gamma(\alpha+k)\Gamma(-\alpha-k+1)}\\&=\mp u_k(x,x)\Big(k-\frac{n}{2}\Big)\dots\Big(1-\frac{n}{2}\Big)\frac{(-1)^k i}{2^n\pi^{n/2}\Gamma(n/2)}\\&=\mp u_k(x,x)\frac{(n/2-1)!}{(n/2-k-1)!}\,\frac{i}{2^n\pi^{n/2}\Gamma(n/2)}=\mp\frac{u_k(x,x)}{(n/2-k-1)!}\,\frac{i}{2^n\pi^{n/2}}.\end{aligned}$$

Case 2. When $p\leqslant 0$, we find for α near p that

$$(P\pm i\varepsilon)^{-\alpha}=\sum_{k=0}^{\infty}(-1)^k\underbrace{(-\alpha)\dots(-\alpha-k+1)}_{\text{simple zeroes for }k\geqslant 1-p}\underbrace{\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}(-m^2\mp i\varepsilon,x,x)}_{\text{simple poles at }\{1-k,\dots,n/2-k\}},$$

therefore

$$\begin{aligned}&\operatorname{res}_{\alpha=p}(P\pm i\varepsilon)^{-\alpha}\\&=\sum_{k=0}^{-p}(-1)^k(-p)\dots(-p-k+1)\operatorname{res}_{\alpha=p}\underbrace{\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}(-m^2\mp i\varepsilon,x,x)}_{\text{no poles at }\alpha=p}=0,\end{aligned}$$

where $\Gamma(\alpha+k)^{-1}\mathbf{F}_{\alpha+k-1}$ has no poles at $\alpha=p$ because $p+k\leqslant 0$ and $\Gamma(\alpha+k)^{-1}F_{\alpha+k-1}$ has no poles in the region $\operatorname{Re}\alpha+k\leqslant 0$.

Remark 8.2. We recover the well-known fact that on a compact Riemannian manifold, if $0 \notin \operatorname{sp}(-\Delta_g)$ then the meromorphic continuation of $\{\operatorname{Re}\alpha \gg 0\} \ni \alpha \mapsto \operatorname{Tr}((-\Delta_g)^{-\alpha})$ has no pole at $\alpha = 0, 1, 2, \dots$.

In summary, we have proved the following result.

Theorem 8.3. *Let (M, g) be a globally hyperbolic non-trapping Lorentzian scattering space of even dimension n and let $P = \Box_g$. Then the Schwartz kernel $K_s(\cdot,\cdot)$ of the operator $(P \pm i\varepsilon)^{-\alpha}$ exists as a family of distributions near the diagonal depending holomorphically to α on the half-plane* $\operatorname{Re}\alpha > -1$. *Its restriction to the diagonal $K_\alpha(x, x)$ exists and is holomorphic for* $\operatorname{Re}\alpha > n/2$, *and it extends as a meromorphic function of α with simple poles along the arithmetic progression $\{n/2, n/2-1, \dots, 1\}$. Furthermore,*

$$\lim_{\varepsilon\to 0^+} \operatorname{res}_{\alpha=n/2-k} (P \pm i\varepsilon)^{-\alpha}(x, x) = \mp \frac{i u_k(x, x)}{2^n \pi^{n/2}(n/2-k-1)!}.$$

8.4. Residues for the spectral action

To recover the spectral action principle, we must study the pole structure of the restriction $\Gamma(\alpha)(P + i\varepsilon)^{-\alpha}(x, x)$ where we must take into account the non-trivial effect of the Γ factor for $\alpha \leqslant 0$. For usual applications of the spectral action principle, we need the first three poles at $\alpha = n/2, n/2-1, n/2-2$ of $\Gamma(\alpha)(P \pm i\varepsilon)^{-\alpha}(x, x)$ that we will explicitly calculate in terms of the mass term m^2 and the regulator ε; after tedious bookkeping of all the formulæ from the previous subsection we find:

$$\begin{aligned}
\operatorname{res}_{\alpha=n/2} \Gamma(\alpha)(P \pm i\varepsilon)^{-\alpha}(x, x) &= \mp \frac{i}{2^n \pi^{n/2}},\\
\operatorname{res}_{\alpha=n/2-1} \Gamma(\alpha)(P \pm i\varepsilon)^{-\alpha}(x, x) &= \mp \frac{i(-m^2 \mp i\varepsilon)}{2^n \pi^{n/2}} \mp \frac{i u_1(x, x)}{2^n \pi^{n/2}},\\
\operatorname{res}_{\alpha=n/2-2} \Gamma(\alpha)(P \pm i\varepsilon)^{-\alpha}(x, x) &= \mp \frac{i(-m^2 \mp i\varepsilon)^2}{2^{n+1} \pi^{n/2}} \mp \frac{i(-m^2 \mp i\varepsilon) u_1(x, x)}{2^n \pi^{n/2}}\\
&\quad \mp \frac{i u_2(x, x)}{2^n \pi^{n/2}}.
\end{aligned}$$

In conclusion, this yields the following result.

Theorem 8.4. *Let (M, g) be a globally hyperbolic non-trapping Lorentzian scattering space of even dimension n and let $P = \Box_g + m^2$, $m \geqslant 0$, be the corresponding Klein–Gordon or wave operator. Then for every $f \in S_+^{-\infty}(\mathbb{R})$, the Schwartz kernel $f(\frac{P+i\varepsilon}{\lambda^2})(\cdot,\cdot)$ is smooth near the diagonal and admits an asymptotic expansion of the form*

$$\begin{aligned}
f\left(\frac{P + i\varepsilon}{\lambda^2}\right)(x, x) &= \frac{e^{in\pi/4} c_0}{i 2^n \pi^{n/2}} \lambda^n + \frac{e^{i((n-2)\pi)/4} c_1}{i 2^n \pi^{n/2}} \big((-m^2 - i\varepsilon) + u_1(x, x)\big)\lambda^{n-2}\\
&+ \frac{e^{i(n-4)\pi/4} c_2}{i 2^n \pi^{n/2}} \left(\frac{(-m^2 - i\varepsilon)^2}{2} + (-m^2 - i\varepsilon)u_1(x, x) + u_2(x, x)\right)\lambda^{n-4} + \mathcal{O}(\lambda^{n-5}),
\end{aligned}$$

where u_k are the Hadamard coefficients and $c_k = \int_0^\infty \hat{f}(t) t^{n/2-k-1}\, dt$.

We remark that in the case when $m \neq 0$, assuming injectivity and non-trapping at energy $\sigma = m^2$ we have an analogous result with $\varepsilon = 0$ and with $f(\frac{P+i0}{\lambda^2})(x,x)$ defined by (7.70).

Proof of Theorem 8.4. Since we are interested in the first three terms of the asymptotic expansion, we choose $n/2 - 3 < c_2 < n/2 - 2$. By Lemma 7.1, we can start from the diagonal expansion for $(P+i\varepsilon)^{-\alpha}$:

$$(P+i\varepsilon)^{-\alpha} = \sum_{k=0}^{N} (-1)^k u_k \frac{\Gamma(-\alpha+1)}{\Gamma(\alpha+k)\Gamma(-\alpha-k+1)} \mathbf{F}_{k+\alpha-1}(-m^2-i\varepsilon) + R_N(-i\varepsilon,\alpha). \tag{8.73}$$

By Lemma 7.1, for any $p \in \mathbb{N}$ with $-p < c_2$ and $s > 0$, we may always choose N large enough so that the remainder term $R_N(-i\varepsilon,\alpha)$ has $\mathcal{C}^s_{\text{loc}}$ regularity, hence it has a well-defined diagonal restriction which is holomorphic and bounded on $\operatorname{Re}\alpha \geqslant -p$. We have also proved that the term $\mathbf{F}_{k+\alpha-1}(-m^2-i\varepsilon,x,x)$ has a well-defined diagonal restriction which is holomorphic on the vertical line $\operatorname{Re}\alpha = c_2$ since this line does not meet the poles of $\mathbf{F}_{k+\alpha-1}(-m^2-i\varepsilon,x,x)$. This means that the term $e^{i\alpha\pi/2}(P+i\varepsilon)^{-\alpha}(x,x)\lambda^{2\alpha}\Gamma(\alpha)$ in

$$f\left(\frac{P+i\varepsilon}{\lambda^2}\right)(x,x) = \frac{1}{2\pi i}\int_{\operatorname{Re}\alpha = c} e^{i\alpha\pi/2}(P+i\varepsilon)^{-\alpha}(x,x)\lambda^{2\alpha}\Gamma(\alpha)\mathcal{M}\hat{f}(\alpha)\,ds$$

has simple poles at $n/2, n/2-1, n/2-2$.

Then the result follows by moving the contour from $\operatorname{Re}\alpha = c$ to $\operatorname{Re}\alpha = c_2$ and using the Cauchy residue formula (we are allowed to do so because of the fast decay of $\mathcal{M}\hat{f}(\alpha)$ when $|\operatorname{Im}\alpha| \to +\infty$) to get

$$\begin{aligned} f\left(\frac{P+i\varepsilon}{\lambda^2}\right)(x,x) &= \sum_{k=0}^{2} (\operatorname{res}_{\alpha=n/2-k}\Gamma(\alpha)(P+i\varepsilon)^{-\alpha}(x,x))e^{i\frac{\pi}{2}(n/2-k)}\lambda^{n-2k}\mathcal{M}\hat{f}(n/2-k) \\ &\quad + \underbrace{\frac{1}{2\pi i}\int_{\operatorname{Re}\alpha=c_2} e^{i\alpha\pi/2}(P+i\varepsilon)^{-\alpha}(x,x)\lambda^{2\alpha}\Gamma(\alpha)\mathcal{M}\hat{f}(\alpha)\,ds,}_{\mathcal{O}(\lambda^{2c_2})} \end{aligned}$$

where the underbraced $\mathcal{O}(\lambda^{2c_2})$ term is of lower order than the preceding ones. ∎

8.5. Extraction of the scalar curvature

Finally, we specialize the discussion of the formal parametrix construction in Section 5 to the Laplace–Beltrami operator $P = \Box_g$ to explain how one can extract the scalar curvature from the residue of $(P \pm i0)^{-\alpha}(x,x)$ at $\alpha = n/2 - 1$.

To that end we need to understand the geometric nature of the term $b^j\partial_{x^j}$ appearing in P, and also to interpret geometrically the first transport equation on u_0, u_1. Recall that the operator P is defined in any coordinate system as (see [72, p. 270])

$$Pu = |g|^{-1/2}\partial_{x^j}(|g|^{1/2}g^{jk}\partial_{x^k}u)$$

for all $u \in C^\infty(M)$, where we sum over repeated indices.[13] Therefore $P = \partial_{x^j}g^{jk}\partial_{x^k} + b^k(x)\partial_{x^k}$ where by [72, p. 270], $b^k(x) = |g(x)|^{-1/2}g^{jk}(x)(\partial_{x^j}|g(x)|^{1/2})$. This leads us to the identity

$$P = \partial_{x^k}g^{kj}(x)\partial_{x^j} + g^{jk}(x)(\partial_{x^j}\log|g(x)|^{1/2})\partial_{x^k}$$

which holds true in normal coordinates centered at an arbitrary point x_0. These formulæ are completely analogous to the well-known ones for the Laplace–Beltrami operator on Riemannian manifolds [112, pp. 41–42].

Recall that when we introduced the transport equations to study the parametrix, in Lemma 5.7 there was a function h defined in normal coordinates as $h(x) = b^j(x)\eta_{jk}x^k$. It can be written as

$$h(x) = b^j\eta_{jk}x^k = g^{lj}(x)(\partial_{x^l}\log|g|^{1/2})\eta_{jk}x^k = x^k\partial_{x^k}\log|g|^{1/2} = \rho\log|g|^{1/2},$$

where ρ is the Euler vector field induced by the pseudo-Riemannian metric. The first transport equation $2\rho u_0 + hu_0 = 0$ now reads [112, (2.4.18), p. 43]

$$2\rho u_0 = -\rho\log|g|^{1/2}u_0, \;\; u_0(0) = 1,$$

hence $u_0(x) = |g(0)|^{1/4}|g(x)|^{-1/4}$ The second transport equation is

$$\rho u_1 + u_1 + \frac{h}{2}u_1 = -Pu_0.$$

Since both ρu_1 and $h = \rho\log|g|^{1/2}$ vanish at the origin, this implies that

$$u_1(0) = -Pu_0(0) = -P(|g(0)|^{1/4}|g(x)|^{-1/4})|_{x=0}.$$

Now in normal coordinates $|g(0)|^{1/4} = 1$ and from the Taylor expansion of the metric in normal coordinates [1, (5.2), p. 82], [9, Proposition 1.28, p. 37],

$$g_{ij}(x) = \eta_{ij} + \tfrac{1}{3}R_{ikjl}x^kx^l + \mathcal{O}(|x|^3).$$

Therefore [1, p. 84] we get the Taylor expansion of $|g(x)|^{-1/4}$ in normal coordinates:

$$\begin{aligned}
|g(x)|^{-1/4} &= \big||\eta|\exp\big(\mathrm{Tr}\log\big(\delta_{ij} + \eta_i^{i_1}\tfrac{1}{3}R_{i_1kjl}(0)x^kx^l + \mathcal{O}(|x|^3)\big)\big)\big|^{-1/4} \\
&= \big(1 + \tfrac{1}{3}\mathrm{Tr}(\eta_i^{i_1}R_{i_1kjl}(0)x^kx^l)\big)^{-1/4} + \mathcal{O}(|x|^3) \\
&= 1 + \tfrac{1}{12}\mathbf{Ric}_{kl}(0)x^kx^l + \mathcal{O}(|x|^3),
\end{aligned}$$

[13] Our convention follows Hörmander [72, p. 270].

since $\mathrm{Tr}(\eta_i^{i_1} R_{i_1kjl}(0)x^k x^l) = \delta^{ij}\eta_i^{i_1} R_{i_1kjl}(0)x^k x^l = \eta^{ij} R_{ikjl}(0)x^k x^l = -\mathbf{Ric}_{kl}(0)x^k x^l$ where $\mathbf{Ric}_{kl}$ is the Ricci tensor. This implies that

$$-P|g(x)|^{-1/4} = -\tfrac{1}{6}\underbrace{g^{kl}\mathbf{Ric}_{kl}(0)}_{=R_g(0)} + \mathcal{O}(|x|),$$

where we recognize $R_g(0) = g^{kl}\mathbf{Ric}_{kl}$ to be the scalar curvature. Finally, $u_1(x,x) = -R_g(x)/6$.

We are done with extracting the scalar curvature from the coefficient $u_1(0)$ of the transport equations. We have thus deduced the following result.

Proposition 8.5. *As a particular case of Theorem 8.3, if in addition the dimension of M is $n \geqslant 4$ then*

$$\lim_{\varepsilon\to 0^+} \mathrm{res}_{\alpha=n/2-1}\,(P \pm i\varepsilon)^{-\alpha}(x,x) = \pm\frac{iR_g(x)}{6(4\pi)^{n/2}(n/2-2)!},$$

where $R_g(x)$ is the scalar curvature at x.

Put together with Theorem 8.3, this proves our main result stated in the introduction, i.e. Theorem 1.1.

Appendix A. Propagation estimates

A.1. Summary

The purpose of this appendix is to supplement the material in Sections 2.1–2.4 with a very brief summary on scattering calculus and propagation estimates.

Propagation estimates in the scattering setting are due to Melrose [87]. The generalization to variable weight orders presented here is due to Vasy [130, 131]; see [50, Sections 2–3] for a concise introduction, cf. [41, Appendix E.4]. The scattering calculus in the model case $\overline{\mathbb{R}^n}$ was earlier developed among others by Shubin [109] and Parenti [94].

A.2. Scattering calculus

We use the notation already introduced in Sections 2.1–2.4; recall in particular that ρ is a boundary-defining function and y are local coordinates on $\partial\overline{M}$, extended onto a collar neighborhood of $\partial\overline{M}$. Let (ρ, y, ϱ, η) be local coordinates on $\overline{{}^{\mathrm{sc}}T^*M}$ such that (ϱ, η) are the dual coordinates of (ρ, y). Recall that we introduced the formal notation $\langle\xi\rangle^{-1}$ for the boundary-defining function of fiber infinity.

The class of *scattering symbols* of order $s, \ell \in \mathbb{R}$, denoted by $S^{s,\ell}_{\mathrm{sc}}(T^*M)$, is defined away from $\partial\overline{M}$ in the same way as the usual symbol class $S^s(T^*M)$, whereas near the boundary, any $a \in S^{s,\ell}_{\mathrm{sc}}(T^*M)$ is a smooth section of T^*M that satisfies the estimate

$$\forall j,k \in \mathbb{N},\ \alpha,\beta \in \mathbb{N}^{n-1}, \quad |(\rho\partial_\rho)^j\partial_y^\alpha\partial_\varrho^k\partial_\eta^\beta a(\rho,y,\varrho,\eta)| \leqslant C_{jk\alpha\beta}\,\rho^{-\ell}\langle\xi\rangle^{s-k-|\beta|}. \tag{A.1}$$

The model example is as always $\overline{\mathbb{R}^n}$ with standard coordinates (x,ξ) on $T^*\mathbb{R}^n$. In this case, by using spherical coordinates $x=(r,y)$ and setting $\rho=r^{-1}$, $\xi=(\varrho,\rho^{-1}\eta)$, $\langle x\rangle=(1+|x|^2)^{1/2}$ and $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$, one finds that (A.1) is equivalent to

$$\forall\alpha,\beta\in\mathbb{N}^n,\quad |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leqslant C_{\alpha\beta}\langle x\rangle^{\ell-|\alpha|}\langle\xi\rangle^{s-|\beta|}.$$

The class $\Psi_{\rm sc}^{s,\ell}(M)$ of *scattering pseudo-differential operators* is obtained from sc-symbols $a\in S_{\rm sc}^{s,\ell}(T^*M)$ by reduction to quantization of symbols on $T^*\mathbb{R}^n$. This requires choosing a partition of unity $\{\psi_i\}_i$ subordinate to a finite chart covering of M as well as suitable diffeomorphisms close to $\partial\overline{M}$ (one can show that in the end different choices give the same operator modulo an element of $\Psi_{\rm sc}^{s-1,\ell-1}(M)$). One also includes in the definition of $\Psi_{\rm sc}^{s,\ell}(M)$ a class of regularizing operators in the sense that their Schwartz kernels $K_A(x,x')$ are smooth and decrease rapidly (with all derivatives) as the distance between $x\in M$ and $x'\in M$ tends to infinity. We refer the reader to, e.g., [130] and [123, Section 2] for an introduction; cf. [87] for the original, more geometric description of the Schwartz kernels of scattering pseudo-differential operators.

In the sc-calculus, the *principal symbol* of $A\in\Psi_{\rm sc}^{s,\ell}(M)$ is the equivalence class of the symbol of A in $S_{\rm sc}^{s,\ell}(T^*M)/S_{\rm sc}^{s-1,\ell-1}(T^*M)$. It is often useful to consider the more narrow class of *classical pseudo-differential operators* which is obtained from *classical symbols*, i.e. from symbols of the form $a=\langle\xi\rangle^s\rho^{-\ell}\tilde a$ with $\tilde a\in C^\infty(\overline{^{\rm sc}T^*M})$. In the simplest case of $A\in\Psi_{\rm sc}^{0,0}(M)$ classical, it is possible to identify the principal symbol with the restriction of $a\in C^\infty(\overline{^{\rm sc}T^*M})$ to $\partial\overline{^{\rm sc}T^*M}$. For classical $A\in\Psi_{\rm sc}^{s,\ell}(M)$ of arbitrary order there is also a natural identification of the principal symbol with a function on $\partial\overline{^{\rm sc}T^*M}$; see for instance (2.4) for the explicit formula for the principal symbol p_z of $\Box_g-z$.

The *microsupport* $\mathrm{WF}'_{\rm sc}(A)$ of $A\in\Psi_{\rm sc}^{s,\ell}(M)$ is the complement of the set of points $q\in\partial\overline{^{\rm sc}T^*M}$ such that the (full) symbol of A coincides in a neighborhood of A with a symbol in $S_{\rm sc}^{-N,-L}(T^*M)$ for all $N,L\in\mathbb{R}$. If A is classical, then its *elliptic set* is the complement $\mathrm{Ell}_{\rm sc}(A)=\partial\overline{^{\rm sc}T^*M}\setminus\Sigma_{\rm sc}(A)$ of the *characteristic set* $\Sigma_{\rm sc}(A)$, defined as the closure of the zero set of the principal symbol.

In the context of propagation estimates it is useful to allow for weight orders ℓ that vary on $\overline{^{\rm sc}T^*M}$. On top of the obvious modifications of the definitions of $S_{\rm sc}^{s,\ell}(T^*M)$ and $\Psi_{\rm sc}^{s,\ell}(M)$, the cost to pay is that one needs to slightly relax the decay stated in (A.1), and require instead that

$$|(\rho\partial_\rho)^j\partial_y^\alpha\partial_\varrho^k\partial_\eta^\beta a(\rho,y,\varrho,\eta)|\leqslant C_{jk\alpha\beta}\,\rho^{-\ell-\delta(j+k+|\alpha|+|\beta|)}\langle\xi\rangle^{s-k-|\beta|}. \tag{A.2}$$

for some $\delta>0$. This circumvents logarithmic losses one would otherwise have when differentiating ℓ. Apart from that, the change of definition (A.2) has however no big practical significance and will be disregarded in the notation entirely.

Now, if $s\geqslant 0$ and $\ell\in C^\infty(\overline{^{\rm sc}T^*M})$, we define the *weighted Sobolev space of variable weight order* to be

$$H_{\rm sc}^{s,\ell}(M)=\{u\in L^2(M)\mid Au\in L^2(M)\},$$

where $A \in \Psi_{\mathrm{sc}}^{s,\ell}(M)$ is a classical elliptic operator (i.e., $\mathrm{Ell}_{\mathrm{sc}}(A) = \partial\overline{{}^{\mathrm{sc}}T^*M}$) which can be chosen arbitrarily. One can fix in particular an invertible A, and the norm can then be defined as $\|u\|_{s,\ell} = \|Au\|$ (different choices of A give equivalent norms). This agrees with the definition for $s \in \mathbb{Z}_{\geqslant 0}$ given in the main part of the paper. For $s \leqslant 0$, $H_{\mathrm{sc}}^{s,\ell}(M)$ can be defined as the dual of $H_{\mathrm{sc}}^{-s,-\ell}(M)$. Then, for all $s, \ell \in \mathbb{R}$ and any elliptic $A \in \Psi_{\mathrm{sc}}^{s,\ell}(M)$,

$$H_{\mathrm{sc}}^{s,\ell}(M) = \Big\{ u \in \bigcup_{s',\ell'} H_{\mathrm{sc}}^{s',\ell'}(M) \;\Big|\; Au \in L^2(M) \Big\}.$$

A.3. Propagation estimates

We now consider the setting of the wave or Klein–Gordon operator $P - z$ on non-trapping Lorentzian scattering spaces introduced in Sections 2.1–2.3. We review various microlocal estimates for $P - z$, following [130, 131], with a particular emphasis on the dependence on the complex parameter z, which we assume to vary in some set $Z \subset \mathbb{C}$.

Recall that with the notation from Section 2.3, the characteristic set of $P - z$ is $\Sigma_{\mathrm{sc}}(P - z) = (\Sigma_0 \cap \partial\overline{{}^{\mathrm{sc}}T^*}M) \cup (\Sigma_z \cap \partial^{\mathrm{sc}}T^*\overline{M})$.

We first state the analogue of Hörmander's propagation of singularities theorem in our setting. The fixed z version is due to Melrose [87]; see [130, Theorem 5.4] and the remarks in [131] for the uniform version below including the $\operatorname{Im} z$ term. As in the main part of the text, we write $q \sim q'$ if q and q' are connected by a bicharacteristic in $\Sigma_{\mathrm{sc}}(P - z)$, and we denote the closed bicharacteristic segment from q to q' by $\gamma_{q \sim q'}$. The notation $q \succ q'$ means that $q \sim q'$ and q comes after q' along the flow.

Proposition A.1 (Propagation of singularities). *Let $s \in \mathbb{R}$ and let $\ell \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ be non-decreasing along the Hamiltonian flow. Let $A_1, A_2, B \in \Psi_{\mathrm{sc}}^{0,0}(M)$ be such that $\mathrm{WF}'_{\mathrm{sc}}(A_1) \subset \mathrm{Ell}_{\mathrm{sc}}(B)$ and for all $z \in Z$ the following control condition is satisfied:*

$$\forall q \in \mathrm{WF}'_{\mathrm{sc}}(A_1) \cap \Sigma_{\mathrm{sc}}(P - z),\ \exists q' \in \mathrm{Ell}_{\mathrm{sc}}(A_2) : q \succ q' \text{ and } \gamma_{q \sim q'} \subset \mathrm{Ell}_{\mathrm{sc}}(B). \quad \text{(A.3)}$$

Suppose $A_2 u \in H_{\mathrm{sc}}^{s,\ell}(M)$ and $\{B(P - z)u\}_{z \in Z}$ is bounded in $H_{\mathrm{sc}}^{s-1,\ell+1}(M)$. Then for all $u \in H_{\mathrm{sc}}^{S,L}(M)$,

$$\begin{aligned}\|A_1 u\|_{s,\ell} + (\operatorname{Im} z)^{1/2}\, \|A_1 u\|_{s-1/2,\ell+1/2} \\ \leqslant C(\|A_2 u\|_{s,\ell} + \|B(P - z)u\|_{s-1,\ell+1} + \|u\|_{S,L})\end{aligned}$$

uniformly in $z \in Z \cap \{\operatorname{Im} z \geqslant 0\}$.

The control condition (A.3) means in particular that the knowledge about u being in $H_{\mathrm{sc}}^{s,\ell}(M)$ microlocally is propagated *forward* (from $\mathrm{Ell}_{\mathrm{sc}}(A_2)$ to $\mathrm{Ell}_{\mathrm{sc}}(A_1)$), consistently with the sign of $\operatorname{Im} z$.

Beside propagation of singularities one can also show a uniform version of the simpler *elliptic estimate* [130, Corollary 5.5].

Next, recall that in our setting, L_+ are the sinks and L_- the sources. Below, $\ell \in C^\infty(\partial \overline{{}^{\rm sc}T^*M})$ and $\ell_\pm = \ell|_{L_\pm}$ as in the main part of the text.

We now state the radial estimates for $P-z$. The *low decay radial estimate* can be used to propagate decay properties of u into L_+ from a punctured neighborhood $U \setminus U_1$. The *higher decay radial estimate* serves to gain decay properties in a neighborhood of L_- provided it is already better than the *threshold value* $-1/2$.

We refer again to [87] and [130, Proposition 5.27] for the fixed z version, and to [131] for the modifications in the proof needed to accomodate the $\operatorname{Im} z$ term.

Proposition A.2 (Low decay radial estimate [131, (5)]). *Let $s \in \mathbb{R}$ and assume that ℓ is non-decreasing along the Hamiltonian flow and $\ell_+ < -1/2$. Let $A, B, B_1 \in \Psi^{0,0}_{\rm sc}(M)$ and let U_1, U be open neighborhoods of L_+ in $\partial \overline{{}^{\rm sc}T^*M}$ and assume $U_1 \subset \operatorname{Ell}_{\rm sc}(A)$, $\operatorname{WF}'_{\rm sc}(A) \subset \operatorname{Ell}_{\rm sc}(B) \subset U$ and $\operatorname{WF}'_{\rm sc}(B_1) \subset U \setminus U_1$. Assume that for all $z \in Z$ the following control condition is satisfied:*

$$\forall q \in \operatorname{WF}'_{\rm sc}(A) \cap \Sigma_{\rm sc}(P-z) \setminus L_+,\ \exists q' \in \operatorname{Ell}_{\rm sc}(B_1) : q \succ q' \text{ and } \gamma_{q \sim q'} \subset \operatorname{Ell}_{\rm sc}(B).$$

Suppose $B_1 u \in H^{s,\ell}_{\rm sc}(M)$ and $\{B(P-z)u\}_{z\in Z}$ is bounded in $H^{s-1,\ell+1}_{\rm sc}(M)$. Then for all $u \in H^{S,L}_{\rm sc}(M)$,

$$\|Au\|_{s,\ell} + (\operatorname{Im} z)^{1/2} \|Au\|_{s-1/2,\ell+1/2} \leqslant C(\|B_1 u\|_{s,\ell} + \|B(P-z)u\|_{s-1,\ell+1} + \|u\|_{S,L})$$

uniformly in $z \in Z \cap \{\operatorname{Im} z \geqslant 0\}$.

Proposition A.3 (Higher decay radial estimate [131, (4)]). *Let $s \in \mathbb{R}$ and assume that ℓ is non-decreasing along the Hamiltonian flow and $\ell_- > -1/2$. Let $A, B \in \Psi^{0,0}_{\rm sc}(M)$ and let U be a sufficiently small open neighborhood of L_- in $\partial \overline{{}^{\rm sc}T^*M}$. Assume $L_- \subset \operatorname{Ell}_{\rm sc}(A)$ and $\operatorname{WF}'_{\rm sc}(A) \subset \operatorname{Ell}_{\rm sc}(B) \subset U$. Suppose $\{B(P-z)u\}_{z\in Z}$ is bounded in $H^{s-1,\ell+1}_{\rm sc}(M)$. Then for all $s' \in \mathbb{R}$, $\ell' \in \left]-1/2, \ell\right]$ and $u \in H^{S,L}_{\rm sc}(M)$ such that $Bu \in H^{s',\ell'}_{\rm sc}(M)$,*

$$\|Au\|_{s,\ell} + (\operatorname{Im} z)^{1/2} \|Au\|_{s-1/2,\ell+1/2} \leqslant C(\|Bu\|_{s',\ell'} + \|B(P-z)u\|_{s-1,\ell+1} + \|u\|_{S,L})$$

uniformly in $z \in Z \cap \{\operatorname{Im} z \geqslant 0\}$.

Appendix B. Complex powers via functional calculus

Suppose P is a (possibly unbounded) self-adjoint operator acting in a Hilbert space $\mathcal{H}$.

If $\alpha \in \mathbb{C}$ and $\varepsilon > 0$, or if $\operatorname{Re}\alpha < 0$ and $\varepsilon \geqslant 0$, then the operator $(P - i\varepsilon)^{-\alpha}$ is well-defined by the Borel functional calculus for self-adjoint operators. In the particular case $\operatorname{Re}\alpha > 0$ and $\varepsilon > 0$, it satisfies

$$(P - i\varepsilon)^{-\alpha} = \frac{e^{-i\pi\alpha/2}}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-\varepsilon s} e^{iPs}\, ds.$$

in the sense of convergence of the integral in the strong operator topology.

From our point of view it is more useful to express $(P - i\varepsilon)^{-\alpha}$ in terms of the resolvent $(P - z)^{-1}$ as a contour integral, the precise form of which we briefly recall. It is actually more instructive to work with $A = iP$ instead of P. The reason is that A and $A + \varepsilon$ are *sectorial operators* (of angle $\pi/2$), and therefore their complex powers are special cases of a large and systematically studied functional calculus based on contour integrals; see [60, Section 2] and references therein. By [60, Proposition 7.1.3], the Borel functional calculus definition of $(A + \varepsilon)^{-\alpha}$ coincides with the sectorial calculus definition. This has the following immediate consequences.

First, let $\varepsilon > 0$. If $\operatorname{Re}\alpha > 0$, then consistency with the sectorial calculus implies the formula

$$(A+\varepsilon)^{-\alpha} = \frac{1}{2\pi i}\int_{\eta_\delta} z^{-\alpha}(z-(A+\varepsilon))^{-1}\,dz, \tag{B.4}$$

where $0 < \delta < \varepsilon$ and η_δ is an arbitrary contour going from $\operatorname{Im} z \gg 0$ to $\operatorname{Im} z \ll 0$ of the form

$$\eta_\delta = e^{i\theta}]{+\infty}, \delta] \cup \{\delta e^{i\omega} \mid -\theta < \omega < \theta\} \cup e^{-i\theta}[\delta, +\infty[$$

for some $\theta \in]\pi/2, \pi]$. More generally, for any $\alpha \in \mathbb{C}$,

$$(A+\varepsilon)^{-\alpha} = (1+A)^N \frac{1}{2\pi i}\int_{\eta_\delta} \frac{z^{-\alpha}}{(1+z)^N}(z-(A+\varepsilon))^{-1}\,dz, \tag{B.5}$$

where $N \in \mathbb{N}_{\geqslant 0}$ is an arbitrary number such that $N > -\operatorname{Re}\alpha$. Furthermore, $\operatorname{Dom} A^N$ is a core for $(A+\varepsilon)^{-\alpha}$ [60, Proposition 3.1.1]. Observe that if $\operatorname{Re}\alpha < 1$ then it is not necessary to surround 0 in the integral, and so the contour η_δ in (B.5) can be replaced by

$$\eta_0 = e^{i\theta}]{+\infty}, 0] \cup e^{-i\theta}[0, +\infty[.$$

For $\operatorname{Re}\alpha < 0$, $(A+\varepsilon)^{-\alpha}$ is in general not bounded, it is however still a closed operator, with domain independent of $\varepsilon > 0$.

Let now $\varepsilon = 0$. If $\operatorname{Re}\alpha < 0$ then again

$$A^{-\alpha} = (1+A)^N \frac{1}{2\pi i}\int_{\eta_0} \frac{z^{-\alpha}}{(1+z)^N}(z-A)^{-1}\,dz, \tag{B.6}$$

and the domain is $\operatorname{Dom} A^{-\alpha} = \operatorname{Dom}(A+\varepsilon)^{-\alpha}$ for $\varepsilon > 0$ arbitrary [60, Proposition 3.1.9]. Furthermore,

$$A^{-\alpha}u = \lim_{\varepsilon\to 0+}(A+\varepsilon)^{-\alpha}u, \quad u \in \operatorname{Dom} A^{-\alpha}.$$

In the special situation $0 \notin \operatorname{sp}(A)$, $A^{-\alpha}$ is well-defined for all $\alpha \in \mathbb{C}$, and

$$A^{-\alpha} = (1+A)^N \frac{1}{2\pi i}\int_{\eta_\delta} \frac{z^{-\alpha}}{(1+z)^N}(z-A)^{-1}\,dz$$

for all sufficiently small $\delta \geqslant 0$.

Using back the relation $A = iP$ and changing the integration variable $z \to i(z+\varepsilon)$ one finds integrals with $(P-z)^{-1}$ over the contour γ_ε used in the main part of the text (see Section 2.6).

Appendix C. The ultrastatic case

C.1. Resolvent bounds and Feynman wavefront sets for ultrastatic spacetimes

Let (Y, h) be a complete Riemannian manifold of dimension $n-1$ and let

$$M = \mathbb{R} \times Y, \quad g = dt^2 - h \tag{C.7}$$

be the corresponding *ultrastatic* Lorentzian manifold of dimension n.

The wave operator is then $\Box_g = \partial_t^2 - \Delta_h$, where Δ_h is the Laplace–Beltrami operator on (Y, h) (with the convention $-\Delta_h \geqslant 0$). As explained in [34], the essential self-adjointness of $\Box_g$ in that case can be shown using Nelson's commutator theorem.

For $s \in \mathbb{R}$ and $p > 1/2$, we recall the definition of weighted Sobolev spaces $L_t^{2,p} H^s(Y)$ on M:

$$\|u\|_{L_t^{2,p} H^s(Y)} = \left(\int_{\mathbb{R}} \langle t \rangle^{2p} \|u(t,\cdot)\|^2_{H^s(Y)} \right)^{1/2},$$

where $H^s(Y) = \langle -\Delta_h \rangle^{-s} L^2(Y)$ is the usual Sobolev space on Y.

Theorem C.1. *Let (Y, h) be a complete Riemannian manifold, let (M, g) be as in (C.7) and let $P = \Box_g$. Let $s \in \mathbb{R}$ and $p > 1/2$. Then for $z \in \{\operatorname{Im} z \geqslant 0, |\operatorname{Re} z| \geqslant \varepsilon > 0\}$, $P - z$ admits a Feynman inverse $(P - z)^{-1} : L_t^{2,-p} H_y^s(Y) \to L_t^{2,p} H_y^{s+1}(Y)$ which satisfies a bound of the form*

$$\|(P - z)^{-1} u\|_{L_t^{2,-p} H_y^s} \leqslant C \|u\|_{L_t^{2,p} H_y^{s-1}}. \tag{C.8}$$

Furthermore, if $\operatorname{Re} \alpha > 0$ and $\operatorname{Im} z > 0$ then $(P - z)^{-\alpha} : H^s(M) \to H^s(M)$ is well-defined for all $s \in \mathbb{R}$.

In particular, the above bound (C.8) holds true when $\operatorname{Im} z \to 0^+$, $\operatorname{Re} z \neq 0$, which yields a limiting absorption principle for the Klein–Gordon resolvent $(\Box_g + m^2 - i0)^{-1}$ as also proved in [34].

Proof of Theorem C.1. The starting point is the well-known ansatz for $\operatorname{Im} z \geqslant 0$:

$$((P - z)^{-1} u)(t, \cdot) = -1/2 \int_{\mathbb{R}} \frac{e^{-i|t-s|\sqrt{-\Delta_h - z}}}{\sqrt{-\Delta_h - z}} u(s, \cdot)\, ds.$$

We first prove a rough bound for low values of $\operatorname{Im} z \geqslant 0$, $|\operatorname{Re} z| \geqslant \varepsilon$:

$$\begin{aligned}
&\|(P - z)^{-1} u\|^2_{L_t^{2,-p} H_y^s} \\
&\quad = \frac{1}{4} \int_{\mathbb{R}} \langle t \rangle^{-2p} \left\| \int_{\mathbb{R}} \frac{e^{-i|t-s|\sqrt{-\Delta_h - z}}}{\sqrt{-\Delta_h - z}} u(s, \cdot)\, ds \right\|^2_{H_y^s(Y)} dt \\
&\quad \leqslant \frac{1}{4} \int_{\mathbb{R}} \langle t \rangle^{-2p}\, dt\, \sup_t \left(\int_{\mathbb{R}} \langle s \rangle^{-p} \langle s \rangle^{p} \left\| \frac{e^{-i|s-t|\sqrt{-\Delta_h - z}}}{\sqrt{-\Delta_h - z}} u(s, \cdot) \right\|_{H_y^s(Y)} ds \right)^2.
\end{aligned}$$

Then by the Cauchy–Schwarz inequality in s,

$$\begin{aligned}
&\int_{\mathbb{R}} \langle s\rangle^{-p}\langle s\rangle^{p}\left\|\frac{e^{-i|s-t|\sqrt{-\Delta_h-z}}}{\sqrt{-\Delta_h-z}}u(s,\cdot)\right\|_{H^s_y(Y)} ds\\
&\quad\leqslant \left(\int_{\mathbb{R}}\langle s\rangle^{-2p}\,ds\right)^{1/2}\sup_t\left(\int_{\mathbb{R}}\langle s\rangle^{2p}\left\|\frac{e^{-i|s-t|\sqrt{-\Delta_h-z}}}{\sqrt{-\Delta_h-z}}u(s,\cdot)\right\|^2_{H^s_y(Y)} ds\right)^{1/2}\\
&\quad\leqslant C\left(\int_{\mathbb{R}}\langle s\rangle^{2p}\|u(s,\cdot)\|^2_{H^{s-1}_y(\mathbb{R}^{n-1})}\,ds\right)^{1/2} = C\|u\|_{L^{2,p}_t H^{s-1}_y}
\end{aligned}$$

using the fact that

$$\frac{e^{-i|t-s|\sqrt{-\Delta_h-z}}}{\sqrt{-\Delta_h-z}} : H^s_y(Y)\to H^{s+1}_y(Y)$$

is bounded for all $s\in\mathbb{R}$ uniformly in $\operatorname{Im} z\geqslant 0$, $\|\operatorname{Re} z\|\geqslant\varepsilon$. Finally, for small $\operatorname{Im} z$, we get

$$\|(P-z)^{-1}u\|^2_{L^{2,-p}_t H^s_y(\mathbb{R}^{n-1})}\leqslant C^2\|u\|^2_{L^{2,p}_t H^{s-1}_y},$$

which shows that $(P-z)^{-1}: L^{2,p}_t H^s_y(Y)\to L^{2,-p}_t H^{s+1}_y(Y)$ is invertible on the half-plane $\operatorname{Im} z\geqslant 0$, $|\operatorname{Re} z|\geqslant\varepsilon$.

Next, we refine the above bounds for large $|z|$ along the contour γ_ε defined in Section 2.6 to get decay in z. We denote by $E(\lambda)d\lambda$ the projection-valued measure associated to the functional calculus of $-\Delta_h$, which is well-known to be self-adjoint by completeness of Y [24, 114]. For $u\in C^\infty_{\mathrm{c}}(M)$, we define $\hat{u}=\int_{\mathbb{R}} e^{-i\tau t}E_\lambda(u(t,\cdot))\,dt$. Then we get

$$\begin{aligned}
\|(P-z)^{-1}u\|^2_{H^s(M)} &= \int_{\mathbb{R}\times\mathbb{R}_{\geqslant 0}}\frac{(1+|\tau|^2+\lambda)^s\|\hat{u}(\tau,\lambda)\|^2_{L^2(Y)}}{|-\tau^2+\lambda-z|^2}\,d\tau\,d\lambda\\
&\lesssim \frac{1}{|\operatorname{Im} z|^2}\|u\|^2_{H^s(M)}.
\end{aligned}$$

For $\varepsilon>0$, this implies by a contour integration argument as in the proof of Lemma 4.7 that the complex powers $(P-i\varepsilon)^{-\alpha}$, $\operatorname{Re}\alpha>0$, are well-defined and can be represented as

$$(P-i\varepsilon)^{-\alpha}u=\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}_{\geqslant 0}} e^{i\tau t}(-\tau^2+\lambda-i\varepsilon)^{-\alpha}\hat{u}(\tau,\lambda)\,d\tau\,d\lambda \tag{C.9}$$

for all $u\in H^s(M)$.

The bound on the wavefront set of $(P-z)^{-1}$ is an immediate consequence of the explicit formula for the Feynman inverse and follows the discussion in Section 3.5. ∎

C.2. *Limiting absorption principle for Feynman powers*

In this second part of Appendix C, for the sake of illustration we specialize to ultrastatic Lorentzian manifolds $\mathbb{R}\times Y$ which are *space-compact*, i.e. Y is compact Riemannian. We give an elementary proof of the limiting absorption principle for Feynman powers, by which we mean that the limit $\lim_{\varepsilon\to 0^+}(P - i\varepsilon)^{-\alpha} : C_c^\infty(M) \to \mathcal{D}'(M)$ exists for $P = \Box_g + m^2$ with $m > 0$. This amounts to showing that one can take the $\varepsilon \to 0^+$ limit in (C.9), and then to finding suitable weighted Sobolev spaces on which the complex Feynman powers are well-defined.

Definition C.2 (Weighted anisotropic Sobolev spaces). We define the following product-type weighted Sobolev norms on M depending on three indices, $H^{(s,\ell),p}(M)$, where s is the time regularity, ℓ is a weight on the time variable and p is the space regularity:

$$\|u\|_{H^{(s,\ell),p}(M)} = \left(\int_M |\langle D_t\rangle^s \langle t\rangle^\ell \langle -\Delta_h\rangle^{p/2} u|^2 d\,\mathrm{vol}_g\right)^{1/2}.$$

An important property of these spaces is that Fourier transform in the time variable exchanges the first two indices, i.e. $u \in H^{(s,\ell),p}(M)$ implies $\mathcal{F}_t(u) \in H^{(\ell,s),p}(M)$.

Lemma C.3. *Let $m > 0$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. Then $(-\tau^2+\lambda+m^2 - i0)^{-\alpha}_{\lambda\in\mathbb{R}_{\geqslant 0}}$ is a family of tempered distributions which satisfies*

$$\|(-\tau^2+\lambda+m^2 - i0)^{-\alpha}\|_{H^{\ell-\operatorname{Re}\alpha,-s_1}(\mathbb{R})} = \mathcal{O}(\lambda^{-\operatorname{Re}\alpha/2}) \tag{C.10}$$

for all $\ell \in [0, 1/2[$, $s_1 > 1/2$, where $H^{\ell-\operatorname{Re}\alpha,-s_1}(\mathbb{R})$ denotes the weighted Sobolev space $\langle t\rangle^{s_1} H^{\ell-\operatorname{Re}\alpha}(\mathbb{R})$.

Proof. We cut the domain into three regions using a smooth partition of unity $1 = \chi_1 + \chi_2 + \chi_3$, where $\operatorname{supp}\chi_3 \subset \{\tau \geqslant \delta > 0\}$, $\operatorname{supp}\chi_1 \subset \{\tau \leqslant -\delta < 0\}$ and $\operatorname{supp}\chi_2 \subset \{|\tau| \leqslant m/2\}$.

Observe that $\chi_2(-\tau^2+\lambda+m^2 - i0)^{-\alpha}$ is smooth, compactly supported and uniformly bounded in the space of Schwartz functions of $\lambda \in \mathbb{R}_{\geqslant 0}$, so this term satisfies (C.10).

Let us examine the term $\chi_1(-\tau^2+\lambda+m^2 - i0)^{-\alpha}$. On the support of χ_3, $\tau + \sqrt{\lambda + m^2} \geqslant \delta > 0$ uniformly in λ, therefore in the factorization

$$\chi_1(-\tau^2+\lambda+m^2 - i0)^{-\alpha} = \chi_1(\tau)(\tau + \sqrt{\lambda + m^2})^{-\alpha}(-\tau+\sqrt{\lambda + m^2} - i0)^{-\alpha},$$

the term $\chi_1(\tau)(\tau + \sqrt{\lambda + m^2})^{-\alpha}$ is $\mathcal{O}_{C^\infty(\mathbb{R})}(\lambda^{-\operatorname{Re}\alpha/2})$. By inverse Fourier transform in τ, we get

$$\mathcal{F}(-\tau+\sqrt{\lambda + m^2} - i0)^{-\alpha} = C e^{it\sqrt{\lambda+m^2}} 1_{\mathbb{R}_{\geqslant 0}}(t) t^{\alpha-1},$$

where C is some constant. Thus,

$$\begin{aligned}
|\langle \chi_1(-\tau+\sqrt{\lambda + m^2} - i0)^{-\alpha}, \psi\rangle| &= |C| \left| \int_{\mathbb{R}} e^{it\sqrt{\lambda+m^2}} 1_{\mathbb{R}_{\geqslant 0}}(t) t^{\alpha-1} \widehat{\psi}(t)\, dt \right| \\
&\leqslant |C| \int_0^\infty \left| t^{\alpha-1} \widehat{\psi}(t) \right| dt \\
&\leqslant |C| (\|\langle t\rangle^{\operatorname{Re}\alpha-\ell} \widehat{\psi}\|_{L^2(\mathbb{R})} + \|\widehat{\psi}\|_{H^{s_1}(\mathbb{R})}) \\
&\lesssim \|\widehat{\psi}\|_{H^{s_1,\operatorname{Re}\alpha-\ell}(\mathbb{R})} = \|\psi\|_{H^{\operatorname{Re}\alpha-\ell,s_1}(\mathbb{R})}
\end{aligned}$$

if $\ell \in [0, 1/2[$, $\ell \leqslant \operatorname{Re}\alpha$ and $s_1 > 1/2$. By duality, $\chi_1(-\tau+\sqrt{\lambda + m^2} - i0)^{-\alpha}$ is bounded in the dual weighted Sobolev space $H^{\ell-\operatorname{Re}\alpha,-s_1}(\mathbb{R})$.

The term with χ_3 is treated in the same way. ■

Our objective is to study the regularity of the distribution $\langle u_2, \lim_{\varepsilon\to 0^+}(P-i\varepsilon)^{-\alpha}u_1\rangle$ where P is the Klein–Gordon operator. By compactness of Y, one has a discrete spectral resolution of the Laplacian $-\Delta_h : C^\infty(M) \to L^2(M)$; there is an orthonormal basis $(e_\lambda)_{\lambda\in\mathrm{sp}(-\Delta_h)}$ of $L^2(M)$, where $-\Delta_h e_\lambda = \lambda e_\lambda$. By functional calculus, for any $f \in L^\infty(\mathbb{R}_{\geqslant 0})$, we have

$$f(-\Delta_h)u = \sum_{\lambda\in\mathrm{sp}(-\Delta_h)} f(\lambda)\langle u, e_\lambda\rangle e_\lambda,$$

where $f(-\Delta_h) : L^2(Y) \to L^2(Y)$ acts as a bounded operator.

For all test functions $u \in C_c^\infty(M)$, we define

$$\widehat{u}(\tau,\lambda) = \int_{\mathbb{R}} e^{-it\tau}\langle (u(t,\cdot), e_\lambda\rangle\, dt \in \mathcal{S}(\mathbb{R}).$$

By Fourier transform we are reduced to studying the pairing

$$\Big\langle u_2, \lim_{\varepsilon\to 0^+}(P-i\varepsilon)^{-\alpha}u_1\Big\rangle = \sum_{\lambda\in\mathrm{sp}(-\Delta_h)} \int_{\mathbb{R}} \frac{\overline{\widehat{u}_2}(\tau,\lambda)\widehat{u}_1(\tau,\lambda)}{(-\tau^2+\lambda+m^2-i0)^\alpha}\, d\tau$$

for all test functions u_1 and u_2. The functions $\widehat{u}_1, \widehat{u}_2$ are Schwartz in τ, more precisely

$$\sum_{\lambda\in\mathrm{sp}(-\Delta_h)} \int_{\mathbb{R}} (1+\lambda+\tau^2)^N \|\langle\partial_\tau\rangle^N \widehat{u}_i(\tau,\lambda)\|^2_{L^2(Y)}\, d\tau < \infty$$

for all N, hence the product $\widehat{u}_2(\tau,\lambda)\widehat{u}_1(\tau,\lambda)$ is Schwartz in τ with fast decay in λ, which implies the distributional pairing $\int_{\mathbb{R}} \frac{\overline{\widehat{u}_2}(\tau,\lambda)\widehat{u}_1(\tau,\lambda)}{(-\tau^2+\lambda+m^2-i0)^\alpha}\, d\tau$ is well-defined for all $\lambda \in \sigma(-\Delta_h)$.

Now using the crucial Lemma C.3, we get

$$\begin{aligned}\Big|\Big\langle u_2, \lim_{\varepsilon\to 0^+}(P-i\varepsilon)^{-\alpha}u_1\Big\rangle\Big| &\leqslant \sum_{\lambda\in\mathrm{sp}(-\Delta_h)} \bigg|\bigg(\int_{\mathbb{R}} \frac{\overline{\widehat{u}_2}(\tau,\lambda)\widehat{u}_1(\tau,\lambda)}{(-\tau^2+\lambda+m^2-i0)^\alpha}\, d\tau\bigg)\bigg| \\ &\leqslant C \sum_{\lambda\in\mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^{-\mathrm{Re}\,\alpha/2} \|\overline{\widehat{u}_2}(\cdot,\lambda)\widehat{u}_1(\cdot,\lambda)\|_{H^{\mathrm{Re}\,\alpha, s_1}(\mathbb{R})}\end{aligned}$$

for some $s_1 > 1/2$. Now if $s_2 > 1/2$ and $s_2 \geqslant \mathrm{Re}\,\alpha$, the Moser estimates yield

$$\|uv\|_{H^{s_2}(\mathbb{R})} \lesssim \|v\|_{H^{s_2}(\mathbb{R})}\|v\|_{H^{s_2}(\mathbb{R})},$$

or in the weighted version,

$$\|uv\|_{H^{s_2,s_1}(\mathbb{R})} \lesssim \|v\|_{H^{s_2,\ell_1}(\mathbb{R})}\|v\|_{H^{s_2,\ell_2}(\mathbb{R})}$$

where $\ell_1 + \ell_2 = s_1 > 1/2$. This implies that for each $\lambda \in \mathrm{sp}(-\Delta_h)$,

$$\|\widehat{u}_2(\cdot,\lambda)\widehat{u}_1(\cdot,\lambda)\|_{H^{\mathrm{Re}\,\alpha,s_1}(\mathbb{R})} \leqslant C\|\widehat{u}_1(\cdot,\lambda)\|_{H^{s_2,\ell_1}_\tau(\mathbb{R})}\|\widehat{u}_2(\cdot,\lambda)\|_{H^{s_2,\ell_2}_\tau(\mathbb{R})}.$$

Therefore, using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\Big|\Big\langle u_2, \lim_{\varepsilon\to 0^+} (P - i\varepsilon)^{-\alpha} u_1\Big\rangle\Big| \\
&\qquad \leqslant \sum_{\lambda\in \mathrm{sp}(-\Delta_h)} C\langle\lambda\rangle^{-\mathrm{Re}\,\alpha/2} \|\hat{u}_1(\cdot,\lambda)\|_{H_\tau^{s_2,\ell_1}(\mathbb{R})} \|\hat{u}_2(\cdot,\lambda)\|_{H_\tau^{s_2,\ell_2}(\mathbb{R})} \\
&\qquad \leqslant C\Big(\sum_{\lambda\in \mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^{-2p_1} \|\hat{u}_1(\cdot,\lambda)\|^2_{H_\tau^{s_2,\ell_1}(\mathbb{R})}\Big)^{1/2} \\
&\qquad\quad \times \Big(\sum_{\lambda\in \mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^{-2p_2} \|\hat{u}_2(\cdot,\lambda)\|^2_{H_\tau^{s_2,\ell_2}(\mathbb{R})}\Big)^{1/2},
\end{aligned}$$

where $p_1 + p_2 = \mathrm{Re}\,\alpha/2$.

To estimate the r.h.s. we need the following simple result.

Lemma C.4. *For all $u \in C_c^\infty(M)$,*

$$\sum_{\lambda\in \mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^p \|u_\lambda\|^2_{H_t^{s,\ell}(\mathbb{R})} = \|u\|^2_{H^{(s,\ell),p}(M)},$$

where $\|\cdot\|_{H^{(s,\ell),p}(M)}$ is the product-type weighted norm from Definition C.2.

Proof. By definition of $\|\cdot\|_{H^{(s,\ell),p}(M)}$ and Fubini's theorem,

$$\begin{aligned}
\sum_{\lambda\in \mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^p \|u_\lambda\|^2_{H_t^{s,\ell}(\mathbb{R})} &= \sum_{\lambda\in \mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^p \int_{\mathbb{R}} |\langle D_t\rangle^s \langle t\rangle^\ell \langle u(t), e_\lambda\rangle|^2\, dt \\
&= \int_{\mathbb{R}} \sum_{\lambda\in \mathrm{sp}(-\Delta_h)} \langle\lambda\rangle^p \big|\big\langle (\langle D_t\rangle^s \langle t\rangle^\ell u)(t), e_\lambda\big\rangle\big|^2\, dt \\
&= \int_{\mathbb{R}} \|(\mathbf{1} - \Delta_h)^{p/2} \langle D_t\rangle^s \langle t\rangle^\ell u\|^2_{L^2(Y)}\, dt = \|u\|^2_{H^{(s,\ell),p}(M)},
\end{aligned}$$

where we have used functional calculus and the fact that the spectral projection commutes with operators depending only on the t variable. ∎

Therefore, we find that

$$\Big|\Big\langle u_2, \lim_{\varepsilon\to 0^+} (P - i\varepsilon)^{-\alpha} u_1\Big\rangle\Big| \leqslant C\|u_1\|_{H^{(\ell_1,s_2),p_1}(M)} \|u_2\|_{H^{(\ell_2,s_2),p_2}(M)}$$

for all $s_2 > 1/2$, $s_2 \geqslant \mathrm{Re}\,\alpha$, $\ell_1 + \ell_2 > 1/2$ and $p_1 + p_2 = \mathrm{Re}\,\alpha/2$, which concludes the proof of the limiting absorption principle stated below.

Theorem C.5. *Let $M = \mathbb{R} \times Y$ be an ultrastatic Lorentzian manifold such that Y is compact, and let P be the Klein–Gordon operator with $m > 0$. The complex Feynman power acts as a continuous map between weighted Sobolev spaces, namely, the weak operator limit*

$$(P - i0)^{-\alpha} : H^{(\ell_1,s_2),p}(M) \to H^{(-\ell_2,-s_2),p-\mathrm{Re}\,\alpha/2}(M)$$

is well-defined and continuous for all $p \in \mathbb{R}$, $s_2 > 1/2$, $s_2 \geqslant \mathrm{Re}\,\alpha$ and $\ell_1 + \ell_2 > 1/2$.

Appendix D. Various auxiliary proofs

D.1. A Wick rotation lemma

We state below a lemma used several times in the main part of the text. As in Section 4, Q is the quadratic form $Q(\xi) = -\xi_0^2 + \sum_{i=1}^{n-1} \xi_i^2$ on $\mathbb{R}^n$.

Lemma D.1. *Let $\alpha \in \mathbb{C}$. When $\theta \to -\pi/2$, the distribution $(e^{i2\theta}\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{-\alpha}$ $\to (Q(\xi) - i0)^{-\alpha}$ in $\mathcal{D}'_\Gamma(\mathbb{R}^n \setminus \{0\})$, $\Gamma = \{(\xi; \tau dQ(\xi)) \mid Q(\xi) = 0, \tau < 0\}$.*

Proof. The proof follows closely the proof of [71, Theorem 3.1.15] for the convergence in $\mathcal{D}'$. For the control of the wavefront set, the proof follows closely [71, Theorem 8.4.8], which gives a wavefront bound in the sense of quasi-analytic classes. ■

D.2. Wavefront set of the pull-back $G^*F_\alpha(z,\cdot)$

We compute the wavefront set of the germ of distribution $G^*F_\alpha(z,\cdot)$ as stated in Lemma 5.2. Let us recall that $F_\alpha(z,\cdot)$ is the elementary family of distributions on $\mathbb{R}^n$ introduced in Section 4.1, and the pull-back by the submersion G defined in Section 5.1.2 gives a distribution defined on a neighborhood $\mathcal{U}$ of the diagonal Δ in $M \times M$.

Proof of Lemma 5.2. Step 1. An application of the pull-back theorem [71, Theorem 8.2.4] in our situation gives

$$\mathrm{WF}(G^*F_\alpha) \subset \{(x_1, x_2; k \circ d_{x_1}G, k \circ d_{x_2}G) \mid (G(x_1,x_2), k) \in \mathrm{WF}(F_\alpha)\} \tag{D.11}$$

We denote by $(x_1, x_2; \eta_1, \eta_2)$ an element of $T^*\mathcal{V} \subset T^*M^2$ and $(h^\mu; k_\mu)$ the coordinates in $T^*\mathbb{R}^n$. The pull-back with indices reads

$$(x_1, x_2; k \circ d_{x_1}G, k \circ d_{x_2}G) = (x_1, x_2; k_\mu d_{x_1}G^\mu, k_\mu d_{x_2}G^\mu).$$

Step 2. We first compute $\mathrm{WF}(G^*F_\alpha)$ outside the set $\Delta = \{x_1 = x_2\}$. The condition $(G(x_1,x_2), k) \in \mathrm{WF}(F_\alpha)$ in (D.11) reads

$$(G^\mu(x_1,x_2); k_\mu) = (G^\mu(x_1,x_2); \lambda\eta_{\mu\nu}G^\nu(x_1,x_2)),$$

where $\lambda < 0$. We obtain

$$(x_1, x_2; \lambda k \circ d_{x_1}G, \lambda k \circ d_{x_2}G) = (x_1, x_2; \lambda G^\mu \eta_{\mu\nu_2} d_{x_1}G^{\nu_2}, \lambda G^\mu \eta_{\mu\nu_2} d_{x_2}G^{\nu_2})$$

and also $G^\mu(x_1,x_2)\eta_{\mu\nu}G^\nu(x_1,x_2) = 0$. Now set $\Gamma(x_1,x_2) = G^\mu(x_1,x_2)\eta_{\mu\nu}G^\nu(x_1,x_2)$. The key observation is that $d_{x_1}\Gamma = 2G^\mu\eta_{\mu\nu}d_{x_1}G^\nu$ and $d_{x_2}\Gamma = 2G^\mu\eta_{\mu\nu}d_{x_2}G^\nu$, hence

$$\mathrm{WF}(G^*F_\alpha) \subset \{(x_1, x_2; \lambda d_{x_1}\Gamma, \lambda d_{x_2}\Gamma) \mid \Gamma(x_1,x_2) = 0,\ \lambda \in \mathbb{R}_{<0}\}.$$

We first interpret the term

$$\{(x_1, x_2; \lambda d_{x_1}\Gamma, \lambda d_{x_2}\Gamma) \mid \Gamma(x_1,x_2) = 0,\ \lambda < 0\}$$

appearing in the last formula as the subset of all elements in $T^*\mathcal{U}$ of the conormal bundle of the conoid $\{\Gamma = 0\}$ such that elements of positive energy are propagated in the future and elements of negative energy are propagated in the past: this is exactly the *Feynman condition*. In fact, if we use the metric to lift the indices, $d_{x_1}\Gamma(e_\mu(x_1))\eta^{\mu\nu}e_\nu(x_1)$ and $d_{x_2}\Gamma(e_\mu(x_2))\eta^{\mu\nu}e_\nu(x_2)$ are the Euler vector fields $\nabla_1\Gamma, \nabla_2\Gamma$ defined by Hadamard. The vectors $\nabla_1\Gamma, -\nabla_2\Gamma$ are parallel along the null geodesic connecting x_1 and x_2 (which is easily checked using normal coordinates centered at x_1, proving $(d_{x_1}\Gamma, -d_{x_2}\Gamma)$ are in fact *coparallel* along this null geodesic. Denoting $q_1 = (x_1; \lambda d_{x_1}\Gamma)$ and $q_2 = (x_2; \lambda d_{x_2}\Gamma)$, the relation $\exp_{x_1}(\nabla_1\Gamma) = x_2$ implies that $\nabla_1\Gamma$ points to the future (resp. past) if and only if $q_1 \succ q_2$ (resp. $q_2 \succ q_1$), which implies the Feynman condition.

Step 3 ("diagonal"). For any function G on M^2, we uniquely decompose the total differential into two parts as follows:

$$dG = d_{x_1}G + d_{x_2}G, \quad \text{where} \quad d_{x_1}G|_{\{0\}\times T_{x_2}M} = 0,\ d_{x_2}G|_{T_{x_1}M\times\{0\}} = 0.$$

Let i be the diagonal inclusion map $i := M \ni x \mapsto (x, x) \in \Delta \subset M$. Then for $x \in M$, $G \circ i(x) = 0$ implies $d_x G \circ i = 0$, which is equivalent to $d_{x_1}G \circ di + d_{x_2}G \circ di = 0$. Since

$$d_{x_2}G^\mu(x, x) = d_{x_2}s^\mu_{x_1}(\exp^{-1}_{x_1}(x_2))|_{x_1=x_2=x} = s^\mu_{x_1}(d_{x_2}\exp^{-1}_{x_1}(x_2))|_{x_1=x_2=x} = s^\mu(x),$$

because $d_{x_2}\exp^{-1}_{x_1}(x_2)|_{x_1=x_2=x} = \mathrm{id}_{T_xM\to T_xM} = e_\mu(x)s^\mu(x)$, we see that $d_{x_1}G^\mu(x,x) = -s^\mu(x)$ and

$$\{(x_1, x_2; k \circ d_{x_1}G, k \circ d_{x_2}G) \mid x_1 = x_2\} = \{(x, x; -k_\mu s^\mu(x), k_\mu s^\mu(x)) \mid x \in M\}.$$

This concludes the proof of Lemma 5.2. ∎

D.3. Hölder, scaling and Fourier decay

We now turn our attention to the proof of regularity estimates for $G^*F_\alpha(z,\cdot)$ which are uniform in z.

We first recall a position space definition of Hölder $\mathcal{C}^s(\mathbb{R}^n)$ functions which coincides with the Fourier definition for non-integer $s > 0$. The equivalence is proved in [122, Proposition 8.1], [72, Proposition 8.6.1]. Let us recall a version adapted to our discussion.

Lemma D.2. *Let $s \in \mathbb{R}$. Then $u \in \mathcal{C}^s_{\mathrm{loc}}(\mathbb{R}^n)$ iff for every test function $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$,*

$$|\widehat{u\chi}(\xi)| \leqslant C(1 + |\xi|)^{-s-n}.$$

As a consequence, we have a continuous injection $\mathcal{C}^s_{\mathrm{loc}}(\mathbb{R}^n) \hookrightarrow H^{s+n/2-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $\varepsilon > 0$.

Proof. If $u \in \mathcal{C}^s(\mathbb{R}^n)$ with $s > 0, k < s < k+1$ then for any x, there exists a polynomial P of degree k, which is nothing but the Taylor polynomial of u at x, such that for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ (one could also take φ in the Schwartz class)

$$\left| \int_{\mathbb{R}^n} (u - P)(\lambda(y - x) + x)\varphi(y)\, d^n y \right| \leqslant C\lambda^s \|\varphi\|_{L^\infty}.$$

Now let $u \in \mathcal{C}^s_{\mathrm{loc}}(\mathbb{R}^n)$, hence we may multiply u with some cut-off $\chi \in C_c^\infty(\mathbb{R}^n)$ so that $u\chi \in \mathcal{C}^s$. In particular, choosing the function $\varphi(x) = e^{i\langle \xi, x\rangle}$ on the r.h.s. yields

$$\sup_{0<\lambda\leqslant 1} \lambda^{-s} \sup_{1\leqslant|\xi|\leqslant 2} |\langle (u\chi - P)(\lambda\cdot), e^{i\langle\xi,\cdot\rangle}\rangle| \leqslant \|u\chi\|_{\mathcal{C}^s} \sup_{1\leqslant|\xi|\leqslant 2} \|e^{i\langle\xi,\cdot\rangle}\|_{L^\infty} = \|u\chi\|_{\mathcal{C}^s}.$$

Therefore, using the fact that $\langle (u\chi - P)(\lambda\cdot), e^{i\langle\xi,\cdot\rangle}\rangle = \langle u\chi(\lambda\cdot), e^{i\langle\xi,\cdot\rangle}\rangle$ since the Fourier transform restricted to $|\xi| \geqslant 1$ does not see the polynomial, and $\langle u\chi(\lambda\cdot), e^{i\langle\xi,\cdot\rangle}\rangle = \lambda^{-n}\widehat{u\chi}(\xi/\lambda)$, we get

$$\sup_{0<\lambda\leqslant 1} \lambda^{-s-n} \sup_{1\leqslant|\xi|\leqslant 2} |\widehat{u\chi}(\xi/\lambda)| \leqslant \|u\chi\|_{\mathcal{C}^s}.$$

Hence for $|\xi| \geqslant 1$, we get

$$|\widehat{u\chi}(\xi)| = |\widehat{u\chi}(\xi|\xi|/|\xi|)| \leqslant \|u\chi\|_{\mathcal{C}^s}|\xi|^{-s-n},$$

and finally this means that

$$|\widehat{u\chi}(\xi)| \leqslant C(1 + |\xi|)^{-s-n}.$$

Conversely, if we have the Fourier decay $|\widehat{u\chi}(\xi)| \leqslant C(1 + |\xi|)^{-r}$ for $r \in \mathbb{R}_{\geqslant 0}$, then the Littlewood–Paley blocks are bounded by

$$\begin{aligned}
\|\psi(2^{-j}\sqrt{-\Delta})(u\chi)\|_{L^\infty} &= \|\mathcal{F}^{-1}(\psi(2^{-j}|\xi|)\widehat{u\chi}(\xi))\|_{L^\infty} \leqslant \int_{\mathbb{R}^n} |\psi(2^{-j}|\xi|)\widehat{u\chi}(\xi)|\, d^n\xi \\
&\leqslant 2^{jn} \int_{\mathbb{R}^n} |\psi(|\xi|)\widehat{u\chi}(2^{-j}\xi)|\, d^n\xi \leqslant C 2^{jn} \int_{\mathbb{R}^n} \psi(|\xi|)(1 + 2^j|\xi|)^{-r}\, d^n\xi \\
&\leqslant C 2^{j(n-r)} \int_{\mathbb{R}^n} \psi(|\xi|)(2^{-j} + |\xi|)^{-r}\, d^n\xi \lesssim 2^{j(n-r)}.
\end{aligned}$$

This means that $u \in \mathcal{C}^{r-n}_{\mathrm{loc}}(\mathbb{R}^n)$. ∎

Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geqslant 0$. We consider the Hölder regularity under pull-back of $G^*F_\alpha \in \mathcal{D}'(\mathcal{U})$ where $\mathcal{U} \subset M \times M$ is the neighborhood of the diagonal and $G : \mathcal{U} \ni (x, y) \mapsto G(x, y) \in \mathbb{R}^n$ is the C^∞ submersive map defined by (5.2).

Lemma D.3. *Let $k = \lfloor \operatorname{Re}\alpha \rfloor + 1$ and $F_\alpha(z, \cdot) \in \mathcal{D}'(\mathbb{R}^n)$ as defined in (4.44). Let G be the C^∞ submersive map defined in (5.2). Then the pull-back $\mathbf{F}_\alpha(z, \cdot) = G^*F_\alpha(z, \cdot)$ is in $\mathcal{C}^{(2-2a)(\operatorname{Re}\alpha+1)-k-n}_{\mathrm{loc}}(\mathcal{U})$ with decay in z of order $\mathcal{O}(|\operatorname{Im} z|^{-a(\operatorname{Re}\alpha+1)})$ for $a \in [0, 1]$.*

Proof. Let U be a small geodesically convex open subset in M. We choose a test function $\chi \in C_c^\infty(U)$ in such a way that, in the support of $\chi \otimes \chi$, we have a local diffeomorphism

$E : U \times \mathbb{R}^n \ni (x,h) \mapsto (x, \exp_x(h)) \in U \times U$. Then by definition of G and of the exponential map, we have the identity

$$E^*(\chi \otimes \chi G^* F_\alpha(z,\cdot))(x;h) = F_\alpha(z,|h|_g)\chi(x)\chi(\exp_x(h)) \in \mathcal{D}'(U \times \mathbb{R}^n).$$

Now observe that $F_\alpha(z,|h|_g) = \mathcal{O}_{C^s}((1+|\operatorname{Im} z|)^{-a(\operatorname{Re}\alpha+1)})$ for $a \in [0,1]$ and $s \leqslant (2-2a)(\operatorname{Re}\alpha+1)-k-n$, and that $\chi(x)\chi(\exp_x(h)) \in C_c^\infty(U \times \mathbb{R}^n)$, hence the result follows. ∎

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