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Rate of propagation for the Fisher-KPP equation with nonlocal diffusion and free boundaries

Abstract. In this paper, we obtain sharp estimates for the rate of propagation of the Fisher-KPP equation with nonlocal diffusion and free boundaries. The nonlocal diffusion operator is given by $\int_{\mathbb{R}} J(x - y)u(t, y)\, dy - u(t, x)$, and our estimates hold for some typical classes of kernel functions $J(x)$. For example, if for $|x| \gg 1$ the kernel function satisfies $J(x) \sim |x|^{-\gamma}$ with $\gamma > 1$, then it follows from [Y. Du et al., J. Math. Pures Appl. 154, 30–66 (2021)] that there is a finite spreading speed when $\gamma > 2$, namely the free boundary $x = h(t)$ satisfies $\lim_{t \to \infty} h(t)/t = c_0$ for some uniquely determined positive constant c_0 depending on J, and when $\gamma \in (1, 2]$, $\lim_{t \to \infty} h(t)/t = \infty$; the estimates in the current paper imply that, for $t \gg 1$,

$$c_0 t - h(t) \sim \begin{cases} 1 & \text{when } \gamma > 3, \\ \ln t & \text{when } \gamma = 3, \\ t^{3-\gamma} & \text{when } \gamma \in (2, 3), \end{cases} \qquad h(t) \sim \begin{cases} t \ln t & \text{when } \gamma = 2, \\ t^{1/(\gamma-1)} & \text{when } \gamma \in (1, 2). \end{cases}$$

Our approach is based on subtle integral estimates and constructions of upper and lower solutions, which rely crucially on guessing correctly the order of growth of the term to be estimated. The techniques developed here lay the groundwork for extensions to more general situations.

Keywords. Nonlocal diffusion, free boundary, rate of propagation

Contents

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1. Introduction

In this paper we determine the spreading rate for the Fisher-KPP equation with non-local diffusion and free boundaries considered in [11, 17]. The problem is a "nonlocal diffusion" version of the following free boundary problem with "local diffusion":

$$
\begin{cases}
u_t - d u_{xx} = f(u), & t > 0,\ g(t) < x < h(t), \\
u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\
g'(t) = -\mu u_x(t, g(t)),\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
g(0) = g_0,\ h(0) = h_0,\ u(0, x) = u_0(x), & g_0 \leq x \leq h_0,
\end{cases}
\tag{1.1}
$$

where f is a C^1 function satisfying $f(0) = 0$, $\mu > 0$ and $g_0 < h_0$ are constants, and u_0 is a C^2 function which is positive in (g_0, h_0) and vanishes at $x = g_0$ and $x = h_0$. For Fisher-KPP type of $f(u)$, (1.1) was first studied in [18], as a model for the spreading of a new or invasive species with population density $u(t, x)$, whose population range $(g(t), h(t))$ expands through its boundaries $x = g(t)$ and $x = h(t)$ according to the Stefan conditions $g'(t) = -\mu u_x(t, g(t))$, $h'(t) = -\mu u_x(t, h(t))$. A deduction of these conditions based on some ecological assumptions can be found in [9].

By [18], problem (1.1) admits a unique solution $(u(t, x), g(t), h(t))$ defined for all $t > 0$, and its long-time dynamical behaviour is characterised by a "spreading-vanishing dichotomy": Either $(g(t), h(t))$ is contained in a bounded subset of $\mathbb{R}$ for all $t > 0$ and $u(t, x) \to 0$ uniformly as $t \to \infty$ (called the *vanishing case*), or $(g(t), h(t))$ expands to $\mathbb{R}$ and $u(t, x)$ converges to the unique positive steady state of the ODE $v' = f(v)$ locally uniformly in $x \in \mathbb{R}$ as $t \to \infty$ (the *spreading case*). Moreover, when spreading occurs,

$$
\lim_{t \to \infty} \frac{-g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} = k_0 > 0,
$$

and k_0 is uniquely determined by a semi-wave problem associated to (1.1).

Problem (1.1) is closely related to the corresponding Cauchy problem

$$
\begin{cases}
U_t - d U_{xx} = f(U), & t > 0,\ x \in \mathbb{R}, \\
U(0, x) = U_0(x), & x \in \mathbb{R},
\end{cases}
\tag{1.2}
$$

where

$$
U_0(x) := \begin{cases}
u_0(x), & x \in [g_0, h_0], \\
0, & x \in \mathbb{R} \setminus [g_0, h_0].
\end{cases}
$$

Indeed, it follows from [16] that the unique solution (u, g, h) of (1.1) and the unique solution U of (1.2) are related in the following way: For any fixed $T > 0$, as $\mu \to \infty$, $(g(t), h(t)) \to \mathbb{R}$ and $u(t, x) \to U(t, x)$ locally uniformly in $(t, x) \in (0, T] \times \mathbb{R}$. Thus (1.2) may be viewed as the limiting problem of (1.1) (as $\mu \to \infty$).

Problem (1.2) with U_0 a nonnegative function having nonempty compact support has long been used to describe the spreading of a new or invasive species; see, for example, classical works of Fisher [25], Kolmogorov–Petrovskiĭ–Piskunov (KPP) [33] and Aronson–Weinberger [2].

In both (1.1) and (1.2), the dispersal of the species is described by the diffusion term du_{xx}, widely called a "local diffusion" operator, which is obtained from the assumption that individuals of the species move in space according to the rule of Brownian motion. The nonlocal diffusion version of (1.1) considered in [11] has the following form:[1]

$$
\begin{cases}
u_t = d \displaystyle\int_{\mathbb{R}} J(x - y)u(t, y)\,\mathrm{d}y - du(t, x) + f(u), & t > 0, \ x \in (g(t), h(t)), \\
u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\
h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)\,\mathrm{d}y\,\mathrm{d}x, & t > 0, \\
g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x - y)u(t, x)\,\mathrm{d}y\,\mathrm{d}x, & t > 0, \\
u(0, x) = u_0(x), \ h(0) = -g(0) = h_0, & x \in [-h_0, h_0],
\end{cases}
\tag{1.3}
$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, which is always assumed to be identically 0 for $x \in \mathbb{R} \setminus [g(t), h(t)]$;[2] d and μ are positive constants. The initial function $u_0(x)$ satisfies

$$
u_0 \in C([-h_0, h_0]), \quad u_0(-h_0) = u_0(h_0) = 0, \quad u_0(x) > 0 \text{ in } (-h_0, h_0), \tag{1.4}
$$

with $[-h_0, h_0]$ representing the initial population range of the species. The basic assumptions on the kernel function $J : \mathbb{R} \to \mathbb{R}$ are

(**J**) $J \in C(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, $J \geq 0$, $J(0) > 0$, $\int_{\mathbb{R}} J(x)dx = 1$, J is even.

The nonlocal free boundary problem (1.3) may be viewed as describing the spreading of a new or invasive species with population density $u(t, x)$, whose population range $[g(t), h(t)]$ expands according to the free boundary conditions

$$
\begin{cases}
h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)\,\mathrm{d}y\,\mathrm{d}x, \\
g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x - y)u(t, x)\,\mathrm{d}y\,\mathrm{d}x,
\end{cases}
$$

that is, the expanding rate of the range $[g(t), h(t)]$ is proportional to the outward flux of

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[1]The case $f(u) \equiv 0$ was considered in [13], where the long-time dynamics are completely different from the Fisher-KPP case in [11].

[2]Therefore $\int_{\mathbb{R}} J(x - y)u(t, y)\,\mathrm{d}y = \int_{g(t)}^{h(t)} J(x - y)u(t, y)\,\mathrm{d}y$.

the population across the boundary of the range (see [11] for further explanations and justification).

One advantage of the nonlocal problem (1.3) over the local problem (1.1) is that the nonlocal diffusion term

$$d \int_{\mathbb{R}} J(x - y)u(t, y)\, \mathrm{d}y - du(t, x)$$

in (1.3) is capable of including spatial dispersal strategies of the species beyond random diffusion modelled by the term du_{xx} in (1.1). Here $J(x - y)$ may be interpreted as the probability that an individual of the species moves from x to y in a time unit.

If f is a Fisher-KPP function, namely it satisfies

(f) $\begin{cases} f \in C^1, \ f > 0 = f(0) = f(1) \text{ in } (0, 1), \ f'(0) > 0 > f'(1), \\ f(u)/u \text{ is nonincreasing in } u > 0, \end{cases}$

then the long-time dynamical behaviour of (1.3), similar to that of (1.1), is determined by a "spreading-vanishing dichotomy" (see [11, Theorem 1.2]): As $t \to \infty$, either

(i) *Spreading:* $\lim_{t\to\infty}(g(t), h(t)) = \mathbb{R}$ and $\lim_{t\to\infty} u(t, x) = 1$ locally uniformly in $\mathbb{R}$, or

(ii) *Vanishing:* $\lim_{t\to\infty}(g(t), h(t)) = (g_\infty, h_\infty)$ is a finite interval and $\lim_{t\to\infty} u(t, x) = 0$ uniformly for $x \in [g(t), h(t)]$.

Criteria for spreading and vanishing are also obtained in [11, Theorem 1.3]. In particular, if the size $2h_0$ of the initial population range is larger than a certain critical number, then spreading always happens.

1.1. Threshold condition, spreading speed, and accelerated spreading

When spreading happens, the question of spreading speed was considered in [17]. In sharp contrast to the corresponding local diffusion problem (1.1), it was shown in [17] that (1.3) may spread superlinearly in time (a phenomenon known as accelerated spreading), depending on whether the following threshold condition is satisfied by the kernel function J:

(J1) $\int_0^\infty x J(x)\, \mathrm{d}x < \infty.$

More precisely, we have

Theorem A ([17]). *Suppose that* **(J)** *and* **(f)** *are satisfied, and spreading happens to the unique solution* (u, g, h) *of* (1.3). *Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = -\lim_{t\to\infty} \frac{g(t)}{t} = \begin{cases} c_0 \in (0, \infty) & \text{if } (\mathbf{J1}) \text{ holds}, \\ \infty & \text{if } (\mathbf{J1}) \text{ does not hold}. \end{cases}$$

As usual, when **(J1)** holds, we call c_0 the *spreading speed* of (1.3), which is determined by the semi-wave solutions to (1.3). These are pairs $(c, \phi) \in (0, \infty) \times C^1((-\infty, 0])$

determined by the following two equations:

$$\begin{cases} d \displaystyle\int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \end{cases}$$
(1.5)

and

$$c = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\phi(x)\,dy\,dx. \tag{1.6}$$

If (c, ϕ) solves (1.5), then we call ϕ a *semi-wave* with speed c, since the function $v(t, x) := \phi(x - ct)$ satisfies

$$\begin{cases} v_t = d \displaystyle\int_{-\infty}^{ct} J(x-y)v(t, y)\,dy - dv(t, x) + f(v(t, x)), & t > 0, \ x < ct, \\ v(t, -\infty) = 1, \quad v(t, ct) = 0, & t > 0. \end{cases}$$

However, only the semi-wave satisfying (1.6) meets the free boundary condition along the moving front $x = ct$, and hence is useful for determining the long-time dynamical behaviour of (1.3).

The spreading speed c_0 is given by the following result:

Theorem B ([17]). *Suppose that* (**J**) *and* (**f**) *are satisfied. Then* (1.5)–(1.6) *has a solution pair* $(c, \phi) = (c_0, \phi^{c_0}) \in (0, \infty) \times C^1((-\infty, 0])$ *with* $\phi^{c_0}(x)$ *nonincreasing in* x *if and only if* (**J1**) *holds. Moreover, when* (**J1**) *holds, there exists a unique such solution pair, and* $(\phi^{c_0})'(x) < 0$ *in* $(-\infty, 0]$.

It was also proved in [17, Theorem 5.3] that as $\mu \to \infty$, the limiting problem of (1.3) is the following nonlocal version of (1.2):

$$\begin{cases} u_t = d \displaystyle\int_{\mathbb{R}} J(x-y)u(t, y)\,dy - du(t, x) + f(u), & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$
(1.7)

Problem (1.7) and its many variations have been extensively studied in the literature; see, for example, [1, 3–6, 12, 14, 15, 23, 24, 26, 31, 32, 35, 38, 40, 43] and the references therein. In particular, if (**J**) and (**f**) are satisfied, and if the nonnegative initial function u_0 has non-empty compact support, then the basic long-time dynamical behaviour of (1.7) is given by

$$\lim_{t \to \infty} u(t, x) = 1 \quad \text{locally uniformly for } x \in \mathbb{R}.$$

Similar to (1.2), the nonlocal Cauchy problem (1.7) does not give a finite population range when $t > 0$. To understand the spreading behaviour of (1.7), one examines the level set

$$E_\lambda(t) := \{x \in \mathbb{R} : u(t, x) = \lambda\} \quad \text{with fixed } \lambda \in (0, 1),$$

by considering the large time behaviour of

$$x_\lambda^+(t) := \sup E_\lambda(t) \quad \text{and} \quad x_\lambda^-(t) = \inf E_\lambda(t).$$

As $t \to \infty$, $|x_\lambda^\pm(t)|$ may go to ∞ linearly in t or superlinearly in t, depending on whether the following threshold condition is satisfied by the kernel function, apart from (**J**):

(**J2**) There exists $\lambda > 0$ such that $\int_{\mathbb{R}} J(x)e^{\lambda x} \, dx < \infty$.

Yagisita [43] has proved the following result on traveling wave solutions to (1.7):

Theorem C ([43]). *Suppose that f satisfies (**f**) and J satisfies (**J**). If additionally J satisfies (**J2**), then there is a constant $c_* > 0$ such that (1.7) has a traveling wave solution with speed c if and only if $c \geq c_*$.*

Condition (**J2**) is often called a "thin tail" condition for J. When f satisfies (**f**), and J satisfies (**J**) and (**J2**), it is well known (see, for example, [23, 41]) that

$$\lim_{t \to \infty} \frac{|x_\lambda^\pm(t)|}{t} = c_*, \tag{1.8}$$

with c_* given by Theorem C. On the other hand, if (**f**) and (**J**) hold but (**J2**) is not satisfied, then it follows from [41, Theorem 6.4] that $|x_\lambda^\pm(t)|$ grows faster than any linear function of t as $t \to \infty$, namely, accelerated spreading happens:

$$\lim_{t \to \infty} \frac{|x_\lambda^\pm(t)|}{t} = \infty.$$

See also [1, 6, 7, 10, 22, 24, 26, 30, 39, 42] and references therein for further progress on accelerated spreading for (1.7) and related problems.

It is easily seen that (**J2**) implies (**J1**), but the reverse is not true; for example, $J(x) = C(1 + x^2)^{-\sigma}$ with $\sigma > 1$ satisfies (**J1**) (for some suitable $C > 0$) but not (**J2**).

The relationship between $c_0 = c_0(\mu)$ obtained in Theorem B and c_* in Theorem C is given in the following result (see [17, Theorems 5.1 and 5.2]):

Theorem D ([17]). *Suppose that (**J**), (**J1**) and (**f**) hold. Then $c_0(\mu)$ increases to c_* as $\mu \to \infty$, where we define $c_* = \infty$ when (**J2**) does not hold.*

For the local diffusion problem (1.1), sharp estimate for the spreading profile has been obtained in [19]: When spreading happens,

$$\lim_{t \to \infty} [h(t) - k_0 t] = C_1, \quad \lim_{t \to \infty} [g(t) + k_0 t] = C_2$$

for some $C_1, C_2 \in \mathbb{R}$ depending on u_0. Moreover, the solution $u(t, x)$ exhibits the corresponding semi-wave profile as $t \to \infty$. This is strikingly different from the situation of (1.2), where a well known logarithmic delay happens [8]:

$$m(t) := \sup \{x \in \mathbb{R} : U(t, x) = 1/2\}$$
$$= 2\sqrt{f'(0)d}\, t - \frac{3d}{2\sqrt{f'(0)d}} \ln t + C_0 + o(1) \quad \text{as } t \to \infty$$

for some $C_0 \in \mathbb{R}$ depending on the initial function U_0. We refer to [27–29, 34, 36, 37] for further advances in that direction.

In this paper, we aim to obtain sharp estimates for (1.3) in a similar spirit. It turns out that when **(J1)** holds and so a finite spreading speed c_0 exists for (1.3), the functions $h(t) - c_0 t$ and $g(t) + c_0 t$ need not be bounded as $t \to \infty$. For some rather general classes of J, we will find the exact rate of growth for $h(t) - c_0 t$ and $g(t) + c_0 t$ when **(J1)** holds, and determine the exact rate of growth of $h(t)$ and $g(t)$ when **(J1)** does not hold.

1.2. Description of the main results

We now describe our main results precisely. For $\alpha > 1$, we introduce the condition

($\mathbf{J}^\alpha$) $\int_0^\infty x^{\alpha-1} J(x) \, dx < \infty.$

Let us note that $(\mathbf{J}^2)$ is equivalent to **(J1)**, and if **(J2)** holds, then $(\mathbf{J}^\alpha)$ is satisfied for all $\alpha > 1$.

Theorem 1.1. *In Theorem A, suppose additionally $(\mathbf{J}^\alpha)$ holds for some $\alpha \geq 3$, and $f'(v)$ is locally Lipschitz in $[0, \infty)$. Then there exists $C > 0$ such that for $t \gg 1$,*

$$|h(t) - c_0 t| + |g(t) + c_0 t| \leq C,$$

$$\begin{cases} \phi^{c_0}(x - c_0 t + C) + o(1) \leq u(t, x) \leq \phi^{c_0}(x - c_0 t - C) + o(1) & \text{for } x \in [0, h(t)], \\ \phi^{c_0}(-x + c_0 t + C) + o(1) \leq u(t, x) \leq \phi^{c_0}(-x + c_0 t - C) + o(1) & \text{for } x \in [g(t), 0], \end{cases}$$

where (c_0, ϕ^{c_0}) is the unique semi-wave pair in Theorem B, and $o(1) \to 0$ uniformly as $t \to \infty$.

Further estimates on $g(t)$ and $h(t)$ can be obtained for more specific classes of kernel functions. We will write

$$\eta(t) \sim \xi(t) \quad \text{if } C_1 \xi(t) \leq \eta(t) \leq C_2 \xi(t) \tag{1.9}$$

for some positive constants $C_1 \leq C_2$ and all t in the concerned range.

Our next two theorems are about kernel functions satisfying, for some $\gamma > 0$,

($\hat{\mathbf{J}}^\gamma$) $J(x) \sim |x|^{-\gamma}$ for $|x| \gg 1$.

Note that for kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$, condition **(J)** is satisfied only if $\gamma > 1$, and **(J1)** is satisfied only if $\gamma > 2$. Thus accelerated spreading can happen exactly when $\gamma \in (1, 2]$. We have the following result on the exact growth rate of $h(t)$ and $g(t)$ in this case:

Theorem 1.2. *In Theorem A, if additionally the kernel function satisfies $(\hat{\mathbf{J}}^\gamma)$ for some $\gamma \in (1, 2]$, then for $t \gg 1$,*

$$-g(t), h(t) \sim \begin{cases} t \ln t & \text{if } \gamma = 2, \\ t^{1/(\gamma-1)} & \text{if } \gamma \in (1, 2). \end{cases}$$

We recall that the meaning of "$\sim$" is given in (1.9).

For kernel functions satisfying $(\hat{\mathbf{J}}^{\gamma})$, clearly $(\mathbf{J}^{\alpha})$ holds if $0 < \alpha < \gamma$. Therefore when $\gamma > 3$ the conclusions in Theorem 1.1 hold. The following theorem is concerned with the remaining case $\gamma \in (2, 3]$, which indicates that the result in Theorem 1.1 is sharp.

Theorem 1.3. *In Theorem A, suppose additionally the kernel function satisfies $(\hat{\mathbf{J}}^{\gamma})$ for some $\gamma \in (2, 3]$, $f'(v)$ is locally Lipschitz in $[0, \infty)$, and*

$$[f(v)/v]' < 0 \quad for \ v > 0. \tag{1.10}$$

Then for $t \gg 1$,

$$c_0 t + g(t), c_0 t - h(t) \sim \begin{cases} \ln t & if \ \gamma = 3, \\ t^{3-\gamma} & if \ \gamma \in (2, 3). \end{cases}$$

Note that (**f**) implies $[f(v)/v]' \le 0$ for $v > 0$, and (1.10) is satisfied, for example, by $f(v) = av - bv^p$ with $a, b > 0$ and $p > 1$.

The proofs of Theorems 1.1 and 1.3 rely on some of the following estimates on the semi-wave solutions of (1.5), which are of independent interest.

Theorem 1.4. *Suppose that f satisfies (**f**) and the kernel function satisfies (**J**), and $\phi(x)$ is a monotone solution of (1.5) for some $c > 0$. Then the following conclusions hold:*

(i) *If $(\mathbf{J}^{\alpha})$ holds for some $\alpha > 1$, then*

$$\int_{-\infty}^{-1} [1 - \phi(x)]|x|^{\alpha-2} \, \mathrm{d}x < \infty,$$

which implies, by the monotonicity of $\phi(x)$,

$$0 < 1 - \phi(x) \le C|x|^{1-\alpha} \quad for \ some \ C > 0 \ and \ all \ x < 0.$$

(ii) *If $(\mathbf{J}^{\alpha})$ does not hold for some $\alpha > 1$, then*

$$\int_{-\infty}^{-1} [1 - \phi(x)]|x|^{\alpha-2} \, \mathrm{d}x = \infty.$$

The conditions on f and ϕ in Theorem 1.4 can be considerably relaxed; see Section 2 for details.

Remark 1.5. This paper seems the first to establish estimates of the type in Theorems 1.1 and 1.3 for nonlocal diffusion problems, with or without free boundary.

Remark 1.6. The proofs of Theorems 1.1–1.3 are based on subtle constructions of upper and lower solutions. These constructions rely on first guessing correctly the order of growth of the term to be estimated, which is perhaps the most difficult part of this research. The techniques developed here lay the groundwork for extensions to more general situations.

1.3. Organisation of the paper

The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.4, where subtle analysis is used to find out the relationship between the behaviour of the semi-wave solution and that of the kernel function. Theorem 1.1 is proved in Section 3, through careful constructions of upper and lower solutions, based on the estimate obtained in Section 2. Section 4 is devoted to the proof of Theorem 1.3, where we completely determine the growth rate of $c_0 t - h(t)$ when $J(x) \sim |x|^{-\gamma}$ with γ in the range $(2, 3]$; note that the case $\gamma > 3$ is already covered by the more general Theorem 1.1. In Section 5, we prove Theorem 1.2 by giving the exact growth rate of $h(t)$ when $J(x) \sim |x|^{-\gamma}$ with $\gamma \in (1, 2]$.

2. Proof of Theorem 1.4

The purpose of this section is to prove the following two theorems, which imply Theorem 1.4. For possible applications elsewhere, we prove the results under much less restrictions on ϕ and f. We assume that f is C^1 and $f(1) = 0 > f'(1)$, and ϕ satisfies, for some $c > 0$,

$$\begin{cases} d \int_{-\infty}^{0} J(x-y)\phi(y)\,dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, \quad \phi(x) \in [0, 1], \\ \hspace{7cm} -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi'(x) \le 0 \text{ for } x \ll -1. \end{cases} \tag{2.1}$$

Theorem 2.1. *Suppose that the kernel function satisfies* **(J)** *and* **(J$^\alpha$)** *for some $\alpha > 1$, and f is C^1 with $f(1) = 0 > f'(1)$. If $\phi(x)$ satisfies* (2.1) *for some $c > 0$, then*

$$\int_{-\infty}^{-1} [1 - \phi(x)]|x|^{\alpha-2}\,dx < \infty,$$

and therefore, by the monotonicity of $\phi(x)$ near $-\infty$,

$$0 < 1 - \phi(x) \le C|x|^{1-\alpha} \quad \text{for some } C > 0 \text{ and all } x \le -1.$$

The next result shows that Theorem 2.1 is sharp.

Theorem 2.2. *Suppose that f is C^1 with $f(1) = 0 > f'(1)$ and the kernel function satisfies* **(J)**. *If* **(J$^\alpha$)** *is not satisfied for some $\alpha > 1$, and $\phi(x)$ satisfies* (2.1) *for some $c > 0$, then*

$$\int_{-\infty}^{-1} [1 - \phi(x)]|x|^{\alpha-2}\,dx = \infty. \tag{2.2}$$

The following three lemmas play a crucial role in the proof of Theorem 2.1.

Lemma 2.3. *Suppose $J(x)$ satisfies* **(J)** *and* **(J$^\alpha$)** *for some $\alpha \ge 2$, and $\psi \in L^1((-\infty, 0])$ is nonnegative and continuous in $(-\infty, 0]$. If ψ is nondecreasing near $-\infty$, and satisfies*

$$\int_{-\infty}^{0} |x|^{\beta-1}\psi(x)\,dx < \infty \quad \text{for some } \beta \ge 1, \tag{2.3}$$

then for any $\sigma \in (0, \min\{\beta, \alpha - 1\}]$, there exists $C > 0$ such that

$$I = I_M := \int_{-M}^{0} |x|^\sigma \left[\int_{-\infty}^{0} J(x - y)\psi(y) \, dy - \psi(x) \right] dx \in [-C, C] \quad \text{for all } M > 0.$$

Proof. For fixed $M > 0$ we have

$$\int_{-M}^{0} \int_{-\infty}^{0} |x|^\sigma J(x - y)\psi(y) \, dy \, dx = \int_{0}^{M} \int_{-\infty}^{x} x^\sigma J(y)\psi(y - x) \, dy \, dx$$

$$= \int_{0}^{M} \int_{-\infty}^{0} x^\sigma J(y)\psi(y - x) \, dy \, dx + \int_{0}^{M} \int_{0}^{x} x^\sigma J(y)\psi(y - x) \, dy \, dx$$

$$= \int_{-\infty}^{0} \int_{0}^{M} x^\sigma J(y)\psi(y - x) \, dx \, dy + \int_{0}^{M} \int_{y}^{M} x^\sigma J(y)\psi(y - x) \, dx \, dy$$

$$= \int_{-\infty}^{0} \int_{-y}^{M-y} (x + y)^\sigma J(y)\psi(-x) \, dx \, dy + \int_{0}^{M} \int_{0}^{M-y} (x + y)^\sigma J(y)\psi(-x) \, dx \, dy,$$

and

$$\int_{-M}^{0} |x|^\sigma \psi(x) \, dx = \int_{\mathbb{R}} \int_{0}^{M} x^\sigma J(y)\psi(-x) \, dx \, dy.$$

Therefore we can write $I = \sum_{j=1}^{3} I_j$ with

$$I_1 := \int_{-\infty}^{0} \int_{-y}^{M-y} [(x + y)^\sigma - x^\sigma] J(y)\psi(-x) \, dx \, dy$$

$$+ \int_{0}^{M} \int_{0}^{M-y} [(x + y)^\sigma - x^\sigma] J(y)\psi(-x) \, dx \, dy,$$

$$I_2 := \int_{-\infty}^{0} \int_{M}^{M-y} x^\sigma J(y)\psi(-x) \, dx \, dy - \int_{-\infty}^{0} \int_{0}^{-y} x^\sigma J(y)\psi(-x) \, dx \, dy,$$

$$I_3 := -\int_{0}^{M} \int_{M-y}^{M} x^\sigma J(y)\psi(-x) \, dx \, dy - \int_{M}^{\infty} \int_{0}^{M} x^\sigma J(y)\psi(-x) \, dx \, dy.$$

To estimate I_1 we will make use of some elementary inequalities. If $s, t > 0$ and $\sigma \in (0, 1]$, then it is easily checked that

$$(s + t)^\sigma - s^\sigma \leq t^\sigma. \tag{2.4}$$

If $\sigma = n + \theta$ with $n \geq 1$ an integer, and $\theta \in (0, 1]$, then by the mean value theorem,

$$(s + t)^\sigma - s^\sigma = \sigma(s + \zeta t)^{\sigma-1} t \leq \sigma t(s + t)^{\sigma-1} = \sigma t s^{\sigma-1} + \sigma t[(s + t)^{\sigma-1} - s^{\sigma-1}]$$

$$\leq \sum_{k=1}^{n} \left[\prod_{j=0}^{k-1} (\sigma - j)\right] t^k s^{\sigma-k} + \left[\prod_{j=0}^{n-1} (\sigma - j)\right] t^n [(s^\theta + t^\theta) - s^\theta]$$

$$\leq \sum_{k=1}^{n} \left[\prod_{j=0}^{k-1} (\sigma - j)\right] t^k s^{\sigma-k} + \left[\prod_{j=0}^{n-1} (\sigma - j)\right] t^{n+\theta}$$

$$= \sum_{k=1}^{n} c_k t^k s^{\sigma-k} + c_{n+1} t^\sigma,$$

where $\zeta \in [0, 1]$ and $c_k = c_k(\sigma) > 0$ for $k \in \{1, \ldots, n + 1\}$.

Applying this inequality to $(x + y)^\sigma - x^\sigma$ with $x + y > 0$ and $x > 0$, we obtain, for the case $\sigma > 1$,

$$|(x + y)^\sigma - x^\sigma| \leq \sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma$$

with $\sigma - n = \theta \in (0, 1]$ and $n \geq 1$ an integer, $c_k = c_k(\sigma) > 0$ for $k \in \{1, \ldots, n+1\}$.

Therefore, in the case $\sigma > 1$,

$$
\begin{aligned}
|I_1| &\leq \int_{-\infty}^{0} \int_{-y}^{M-y} \left[\sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma\right] J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{0}^{M} \int_{0}^{M-y} \left[\sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma\right] J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq 2 \sum_{k=1}^{n} c_k \int_{0}^{\infty} x^{\sigma-k}\psi(-x)\,\mathrm{d}x \int_{0}^{\infty} y^k J(y)\,\mathrm{d}y \\
&\quad + 2c_{n+1} \int_{0}^{\infty} \psi(-x)\,\mathrm{d}x \int_{0}^{\infty} y^\sigma J(y)\,\mathrm{d}y \\
&=: C_1.
\end{aligned}
$$

Since $1 \leq k \leq n < \sigma \leq \min\{\beta, \alpha - 1\}$, by the assumptions on J and ψ we see that C_1 is a finite number.

If $\sigma \in (0, 1]$, then

$$
\begin{aligned}
|I_1| &\leq \int_{-\infty}^{0} \int_{-y}^{M-y} |y|^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y + \int_{0}^{M} \int_{0}^{M-y} |y|^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq 2 \int_{0}^{\infty} \psi(-x)\,\mathrm{d}x \int_{0}^{\infty} y^\sigma J(y)\,\mathrm{d}y =: \tilde{C}_1 < \infty.
\end{aligned}
$$

Since $\psi(x) \geq 0$ is continuous in $x \leq 0$ and nondecreasing near $-\infty$, from (2.3) we easily deduce

$$\psi(-x) \leq M_1/x^\sigma \quad \text{for some } M_1 > 0 \text{ and all } x > 0.$$

Due to $(\mathbf{J}^\alpha)$ $(\alpha \geq 2)$, we have

$$\int_{0}^{\infty} y J(y)\,\mathrm{d}y < \infty.$$

Therefore

$$
\begin{aligned}
|I_2| &\leq \int_{-\infty}^{0} \int_{M}^{M-y} M_1 J(y)\,\mathrm{d}x\,\mathrm{d}y + \int_{-\infty}^{0} \int_{0}^{-y} M_1 J(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2M_1 \int_{0}^{\infty} y J(y)\,\mathrm{d}y =: C_2 < \infty,
\end{aligned}
$$

and

$$|I_3| \leq \int_0^M M_1 y J(y)\,\mathrm{d}y + \int_M^\infty M_1 M J(y)\,\mathrm{d}y$$
$$\leq M_1 \int_0^\infty y J(y)\,\mathrm{d}y =: C_3 < \infty.$$

We thus have

$$|I| \leq C_1 + \tilde{C}_1 + C_2 + C_3 =: C < \infty \quad \text{for all } M > 0.$$

The proof is complete.   ∎

Lemma 2.4. *Suppose that $J(x)$ satisfies* (**J**) *and* (**J**$^\alpha$) *for some $\alpha \in (1,2)$. Let ψ be non-negative, continuous in $(-\infty, 0]$, and nondecreasing near $-\infty$. Then there exists $C > 0$ such that*

$$S = S_M := \int_{-M}^0 |x|^{\alpha-1}\left[\int_{-\infty}^0 J(x - y)\psi(y)\,\mathrm{d}y - \psi(x)\right]\mathrm{d}x \leq C \quad \text{for all } M > 0.$$

Proof. As in the proof of Lemma 2.3, we deduce for fixed $M > 0$ and $\sigma > -1$,

$$\int_{-M}^0 \int_{-\infty}^0 |x|^\sigma J(x - y)\psi(y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_{-\infty}^0 \int_{-y}^{M-y} (x + y)^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y + \int_0^M \int_0^{M-y} (x + y)^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y.$$

and

$$\int_{-M}^0 |x|^\sigma \psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \int_0^M |x|^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y.$$

Hence $S = \sum_{i=1}^3 \tilde{I}_i$ with

$$\tilde{I}_1 := \int_{-\infty}^0 \int_{-y}^{M-y} [(x + y)^\sigma - x^\sigma]J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_0^M \int_0^{M-y} [(x + y)^\sigma - x^\sigma]J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$\tilde{I}_2 := \int_{-\infty}^0 \int_M^{M-y} x^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y - \int_{-\infty}^0 \int_0^{-y} x^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y,$$
$$\tilde{I}_3 := -\int_0^M \int_{M-y}^M x^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y - \int_M^\infty \int_0^M x^\sigma J(y)\psi(-x)\,\mathrm{d}x\,\mathrm{d}y.$$

Take $\sigma = \alpha - 2$. It is clear that $\tilde{I}_3 \leq 0$. For $\tilde{I}_1$, since $\sigma < 0$,

$$(x + y)^\sigma - x^\sigma < 0 \quad \text{when } x > 0 \text{ and } y > 0,$$

and hence, by $(\mathbf{J}^\alpha)$ and $\sigma + 1 = \alpha - 1 \in (0, 1)$,

$$
\begin{aligned}
\tilde{I}_1 &\leq \int_{-\infty}^0 \int_{-y}^{M-y} [(x + y)^\sigma - x^\sigma] J(y) \psi(-x) \, dx \, dy \\
&\leq \|\psi\|_\infty \int_{-\infty}^0 \int_{-y}^{M-y} [(x + y)^\sigma - x^\sigma] J(y) \, dx \, dy \\
&= \frac{\|\psi\|_\infty}{\sigma + 1} \int_{-\infty}^0 [M^{\sigma+1} - (M - y)^{\sigma+1} + (-y)^{\sigma+1}] J(y) \, dy \\
&\leq \frac{\|\psi\|_\infty}{\sigma + 1} \int_{-\infty}^0 (-y)^{\sigma+1} J(y) \, dy = \frac{\|\psi\|_\infty}{\sigma + 1} \int_0^\infty y^{\sigma+1} J(y) \, dy =: C_1 < \infty.
\end{aligned}
$$

Moreover, by $(\mathbf{J}^\alpha)$, $\sigma + 1 = \alpha - 1 \in (0, 1)$ and (2.4),

$$
\begin{aligned}
\tilde{I}_2 &\leq \int_{-\infty}^0 \int_M^{M-y} x^\sigma J(y) \psi(-x) \, dx \, dy \leq \|\psi\|_\infty \int_{-\infty}^0 \int_M^{M-y} x^\sigma J(y) \, dx \, dy \\
&= \frac{\|\psi\|_\infty}{\sigma + 1} \int_{-\infty}^0 [(M - y)^{\sigma+1} - M^{\sigma+1}] J(y) \, dy \\
&\leq \frac{\|\psi\|_\infty}{\sigma + 1} \int_0^\infty y^{\sigma+1} J(y) \, dy =: C_2 < \infty.
\end{aligned}
$$

Therefore, $S \leq C_1 + C_2 := C < \infty$ for all $M > 0$. The proof is complete. $\blacksquare$

Let $\phi(x)$ be a solution of (2.1) with some $c > 0$, and define

$$
\psi(x) := 1 - \phi(x), \quad \tilde{f}(u) := -f(1 - u).
$$

Then ψ satisfies

$$
\begin{cases}
0 = d \int_{-\infty}^0 J(x - y) \psi(y) \, dy - d\psi(x) + d \int_0^\infty J(x - y) \, dy \\
\quad + c\psi'(x) + \tilde{f}(\psi(x)) \quad \text{for } x < 0, \\
\psi(-\infty) = 0, \quad \psi'(x) \geq 0 \quad \text{for } x \ll -1.
\end{cases}
\tag{2.5}
$$

Since $\tilde{f}'(0) = f'(1) < 0$, there exist $\varepsilon > 0$ sufficiently small and some $b > 0$ such that

$$
\tilde{f}(u) \leq -bu \quad \text{for } u \in [0, \varepsilon].
$$

As $\psi(-\infty) = 0$ and $\psi(x) \geq 0$ for $x < 0$, we thus have $0 \leq \psi(x) < \varepsilon$ for $x \ll -1$, and so

$$
\tilde{f}(\psi(x)) \leq -b\psi(x) \quad \text{for } x \ll -1.
\tag{2.6}
$$

Lemma 2.5. *Suppose $(\mathbf{J})$ is satisfied, and f is C^1 with $f(1) = 0 > f'(1)$. If $(\mathbf{J}^\alpha)$ holds for some $\alpha \geq 2$, then the above defined ψ satisfies*

$$
\int_{-\infty}^0 \psi(x) \, dx < \infty.
$$

Proof. A simple calculation gives

$$\int_{-\infty}^{0} J(z-w)\psi(w)\,dw - \psi(z) + \int_{0}^{\infty} J(z-w)\,dw = -\int_{-\infty}^{0} J(z-w)\phi(w)dw + \phi(z).$$

Integrating the equation satisfied by ψ over the interval (x, y) with $x < y \ll -1$, and making use of (2.6), we obtain

$$c(\psi(y) - \psi(x)) + d\int_{x}^{y}\left[\int_{-\infty}^{0} J(z-w)\psi(w)\,dw - \psi(z) + \int_{0}^{\infty} J(z-w)\,dw\right]dz$$
$$= -\int_{x}^{y} \tilde{f}(\psi(z))\,dz \geq b\int_{x}^{y} \psi(z)\,dz.$$

We extend ϕ to a C^1 function $\tilde{\phi}$ over $\mathbb{R}$ satisfying $\tilde{\phi}(x) = 0$ for $x > 1$ and $|\tilde{\phi}(x)| \leq 2\|\phi\|_\infty$ for $x \in [0, 1]$. Then, due to ($\mathbf{J}^\alpha$), we have

$$\left|\int_{x}^{y}\left(\int_{-\infty}^{0} J(z-w)\phi(w)\,dw - \phi(z)\right)dz\right|$$
$$= \left|\int_{x}^{y}\left(\int_{\mathbb{R}} J(z-w)\tilde{\phi}(w)\,dw - \tilde{\phi}(z)\right)dz - \int_{x}^{y}\int_{0}^{1} J(z-w)\tilde{\phi}(w)\,dw\,dz\right|$$
$$\leq \left|\int_{x}^{y}\int_{\mathbb{R}} J(w)(\tilde{\phi}(z+w) - \tilde{\phi}(z))\,dw\,dz\right| + 2\|\phi\|_\infty$$
$$= \left|\int_{x}^{y}\int_{\mathbb{R}} J(w)\int_{0}^{1} w\tilde{\phi}'(z+sw)\,ds\,dw\,dz\right| + 2\|\phi\|_\infty$$
$$= \left|\int_{\mathbb{R}} wJ(w)\int_{0}^{1}[\tilde{\phi}(y+sw) - \tilde{\phi}(x+sw)]\,ds\,dw\right| + 2\|\phi\|_\infty$$
$$\leq 2\|\tilde{\phi}\|_\infty\int_{\mathbb{R}} |y|J(y)\,dy + 2\|\phi\|_\infty =: M < \infty.$$

Thus, for $x < y \ll -1$,

$$b\int_{x}^{y} \psi(z)\,dz \leq c(\psi(y) - \psi(x)) + dM \leq c\|\psi\|_\infty + dM,$$

which implies $\int_{-\infty}^{0} \psi(z)\,dz < \infty$. ∎

Proof of Theorem 2.1. *Case 1*: $\alpha \geq 2$. We want to show

$$\int_{-\infty}^{0} \psi(x)|x|^{\alpha-2}\,dx < \infty.$$

By Lemma 2.5 we have

$$\int_{-\infty}^{0} \psi(x)\,dx < \infty.$$

So there is nothing to prove if $\alpha = 2$, and we only need to consider the case $\alpha > 2$.

Suppose $\alpha > 2$ and

$$\int_{-\infty}^{0} |x|^{\gamma-1} \psi(x) \, dx < \infty \quad \text{for some } \gamma \geq 1. \tag{2.7}$$

Then by Lemma 2.3, for any β satisfying $0 < \beta \leq \min\{\gamma, \alpha - 1\}$,

$$\int_{-M}^{0} \left[\int_{-\infty}^{0} J(x-y)\psi(y) \, dy - \psi(x) \right] |x|^{\beta} \, dx \leq C \tag{2.8}$$

for some $C > 0$ and all $M > 0$. Moreover, if we fix $M_0 > 1$ such that (2.6) holds for $x \leq -M_0$, then for $M > M_0$ and β as above, we have

$$b \int_{-M}^{-M_0} \psi(x)|x|^{\beta} \, dx \leq - \int_{-M}^{-M_0} \tilde{f}(\psi(x))|x|^{\beta} \, dx$$

$$= c \int_{-M}^{-M_0} \psi'(x)|x|^{\beta} \, dx + d \int_{-M}^{-M_0} \left[\int_{-\infty}^{0} J(x-y)\psi(y) \, dy - \psi(x) \right] |x|^{\beta} \, dx$$

$$+ d \int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J(x-y) \, dy \, dx.$$

By (2.8),

$$d \int_{-M}^{-M_0} \left[\int_{-\infty}^{0} J(x-y)\psi(y) \, dy - \psi(x) \right] |x|^{\beta} \, dx$$

$$\leq dC - d \int_{-M_0}^{0} \left[\int_{-\infty}^{0} J(x-y)\psi(y) \, dy - \psi(x) \right] |x|^{\beta} \, dx$$

$$=: C_1 < \infty \quad \text{for all } M > M_0.$$

Moreover, if we assume additionally that $\beta \leq \alpha - 2$, then we have

$$\int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J(x-y) \, dy \, dx$$

$$\leq \int_{0}^{M} \int_{0}^{\infty} x^{\beta} J(x+y) \, dy \, dx = \int_{0}^{M} \int_{x}^{\infty} x^{\beta} J(y) \, dy \, dx$$

$$\leq \int_{0}^{\infty} \int_{x}^{\infty} x^{\beta} J(y) \, dy \, dx = \frac{1}{\beta+1} \int_{0}^{\infty} y^{\beta+1} J(y) \, dy =: C_2 < \infty.$$

Therefore, for $\beta \in (0, \min\{\gamma, \alpha - 2\}]$ and $M > M_0$,

$$b \int_{-M}^{-M_0} \psi(x)|x|^{\beta} \, dx \leq c \int_{-M}^{-M_0} \psi'(x)|x|^{\beta} \, dx + C_1 + dC_2$$

$$\leq c \int_{1}^{M} x^{\beta} \psi'(-x) \, dx + C_3 \leq c \int_{1}^{M} x^{\gamma} \psi'(-x) \, dx + C_3$$

$$\leq c\psi(-1) + c \int_{1}^{M} \gamma x^{\gamma-1} \psi(-x) \, dx + C_3 =: C_4 < \infty \text{ by (2.7)}.$$

It follows that

$$\int_{-\infty}^{0} \tilde{\psi}(x)|x|^{\beta} \, dx < \infty. \tag{2.9}$$

Thus we have proved that (2.7) implies (2.9) for any $\beta \in (0, \min\{\gamma, \alpha - 2\}]$.

If we write $\alpha - 2 = n + \theta$ with $n \geq 0$ an integer and $\theta \in (0, 1]$, then by the above conclusion and an induction argument we see that (2.9) holds with $\beta = n$. Thus (2.7) holds for $\gamma = n + 1$. So applying the above conclusion once more we see that (2.9) holds for every $\beta \in (0, \min\{n + 1, \alpha - 2\}] = (0, \alpha - 2]$, as desired.

Case 2: $\alpha \in (1, 2)$. Let $\beta = \alpha - 2$. As in Case 1, for $M > M_0$,

$$b \int_{-M}^{-M_0} \psi(x)|x|^{\beta} \, dx$$

$$\leq c \int_{-M}^{-M_0} \psi'(x)|x|^{\beta} \, dx + d \int_{-M}^{-M_0} \left[\int_{-\infty}^{0} J(x-y)\psi(y) \, dy - \psi(x) \right] |x|^{\beta} \, dx$$

$$+ d \int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J(x-y) \, dy \, dx$$

$$\leq c \int_{-M}^{-M_0} \psi'(x)|x|^{\beta} \, dx + \tilde{C}_1 + d \int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J(x-y) \, dy \, dx,$$

where $\tilde{C}_1 > 0$ is obtained by making use of Lemma 2.4. By $(\mathbf{J}^{\alpha})$ and $\beta + 1 = \alpha - 1$,

$$\int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J(x-y) \, dy \, dx \leq \int_{0}^{\infty} \int_{x}^{\infty} x^{\beta} J(y) \, dy \, dx$$

$$= \frac{1}{\alpha - 1} \int_{0}^{\infty} y^{\alpha - 1} J(y) \, dy =: \tilde{C}_2 < \infty.$$

Due to $\beta < 0$, we have

$$\int_{-M}^{-M_0} \psi'(x)|x|^{\beta} \, dx = \int_{M_0}^{M} \psi'(-x)x^{\beta} \, dx$$

$$= \psi(-M_0)M_0^{\beta} - \psi(-M)M^{\beta} + \beta \int_{M_0}^{M} \psi(-x)x^{\beta - 1} \, dx$$

$$\leq \psi(-M_0)M_0^{\beta} =: \tilde{C}_3 < \infty.$$

Hence

$$b \int_{-M}^{-M_0} \psi(x)|x|^{\beta} \, dx \leq \tilde{C}_1 + \tilde{C}_2 d + c\tilde{C}_3 < \infty$$

for all $M > M_0$, which implies

$$\int_{-\infty}^{-1} \psi(x)|x|^{\alpha - 2} \, dx < \infty.$$

The proof is complete. ∎

Proof of Theorem 2.2. We have

$$|g(\psi(x))| \le L\psi(x) \quad \text{for some } L > 0 \text{ and all } x < 0.$$

Now for $M > M_0 \gg 1$ and $\beta = \alpha - 2$,

$$L \int_{-M}^{-M_0} \psi(x)|x|^\beta \, dx \ge -\int_{-M}^{-M_0} g(\psi(x))|x|^\beta \, dx$$

$$= c \int_{-M}^{-M_0} \psi'(x)|x|^\beta \, dx + d \int_{-M}^{-M_0} \left[\int_{-\infty}^0 J(x-y)\psi(y) \, dy - \psi(x) \right] |x|^\beta \, dx$$

$$+ d \int_{-M}^{-M_0} \int_0^\infty |x|^\beta J(x-y) \, dy \, dx$$

$$\ge -d \int_{-M}^{-M_0} \psi(x)|x|^\beta \, dx + d \int_{-M}^{-M_0} \int_0^\infty |x|^\beta J(x-y) \, dy \, dx$$

Therefore, with $\widetilde{L} := L + d$, we have

$$\widetilde{L} \int_{-M}^{-M_0} \psi(x)|x|^\beta \, dx$$

$$\ge d \int_{-M}^{-M_0} \int_0^\infty |x|^\beta J(x-y) \, dy \, dx = d \int_{M_0}^M \int_x^\infty x^\beta J(y) \, dy \, dx$$

$$= d \left[\int_{M_0}^M \int_{M_0}^\infty - \int_{M_0}^M \int_{M_0}^x \right] x^\beta J(y) \, dy \, dx$$

$$= \frac{d}{\beta+1} \left[\int_{M_0}^\infty (M^{\beta+1} - M_0^{\beta+1}) J(y) \, dy + \int_{M_0}^M (y^{\beta+1} - M^{\beta+1}) J(y) \, dy \right]$$

$$\ge \frac{d}{\beta+1} \left[\int_{M_0}^M y^{\beta+1} J(y) \, dy - M_0^{\beta+1} \int_{M_0}^\infty J(y) \, dy \right] \to \infty \text{ as } M \to \infty,$$

since $\beta + 1 = \alpha - 1$. Therefore (2.2) holds, as we wanted. $\qquad\square$

3. Proof of Theorem 1.1

Let us first observe that it suffices to estimate $h(t) - c_0 t$, since the estimate for $g(t) + c_0 t$ follows by a simple change of the initial function: $(\tilde{u}(t,x), \tilde{g}(t), \tilde{h}(t)) := (u(t,-x), -h(t), -g(t))$ is the unique solution of (1.3) with initial function $\tilde{u}_0(x) := u_0(-x)$.

Theorem 1.1 will follow easily from the following two lemmas and their proofs, where more general and stronger conclusions are proved.

Lemma 3.1. *In Theorem A, if additionally* $(\mathbf{J}^\alpha)$ *holds for some* $\alpha \ge 2$, *and* f' *is locally Lipschitz in* $[0, \infty)$, *then there exists* $C > 0$ *such that for* $t \ge 0$,

$$h(t) - c_0 t \ge -C \left[1 + \int_0^t (1+x)^{1-\alpha} \, dx + \int_0^{\frac{c_0}{2}t} x^2 J(x) \, dx + t \int_{\frac{c_0}{2}t}^\infty x J(x) \, dx \right],$$

where $c_0 > 0$ *is given in Theorem B.*

Proof. Let (c_0, ϕ^{c_0}) be the unique semi-wave pair in Theorem B. To simplify notations we will write $\phi^{c_0}(x) = \phi(x)$. By $(\mathbf{J}^\alpha)$ ($\alpha \geq 2$) and Theorem 2.1 there is $C > 0$ such that

$$\int_0^\infty J(y) y^{\alpha-1} \, dy \leq C, \quad 0 < 1 - \phi(x) \leq \frac{C}{x^{\alpha-1}} \quad \text{for } x < 0. \tag{3.1}$$

Define

$$\begin{cases} \underline{h}(t) := c_0 t + \delta(t), & t \geq 0, \\ \underline{u}(t, x) := (1 - \varepsilon(t))[\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1], & t \geq 0, \ x \in [-\underline{h}(t), \underline{h}(t)], \end{cases}$$

where $\varepsilon(t) := (t + \theta)^{1-\alpha}$ and

$$\delta(t) := K_1 - K_2 \int_0^t \varepsilon(\tau) \, d\tau - 2\mu \int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)} \int_0^\infty J(x - y) \, dy \, dx \, d\tau,$$

with θ, K_1 and K_2 large positive constants to be determined.

For any $M > 0$,

$$\int_{-\infty}^{-M} \int_0^\infty J(x - y) \, dy \, dx = \int_M^\infty \int_x^\infty J(y) \, dy \, dx$$

$$= \int_M^\infty \int_M^y J(y) \, dx \, dy = \int_M^\infty (y - M) J(y) \, dy$$

$$\leq \int_M^\infty y J(y) \, dy.$$

Hence, due to $\displaystyle\int_0^\infty y J(y) \, dy < \infty$ (because $\alpha \geq 2$), we have

$$2\mu \int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)} \int_0^\infty J(x - y) \, dy \, dx \, d\tau \leq 2\mu \int_0^t \int_{-\infty}^{-\frac{c_0}{2}\theta} \int_0^\infty J(x - y) \, dy \, dx \, d\tau$$

$$\leq \left[2\mu \int_{\frac{c_0}{2}\theta}^\infty y J(y) \, dy \right] t \leq \frac{c_0}{4} t$$

provided that $\theta > 0$ is large enough, say $\theta \geq \theta_0$.

For any given small $\varepsilon_0 > 0$, due to $\phi(-\infty) = 1$ there is $K_0 = K_0(\varepsilon_0) > 0$ such that

$$1 - \varepsilon_0 \leq \phi(x) \quad \text{for } x \leq -K_0,$$

which implies that

$$\phi(x - \underline{h}(t)), \phi(-x - \underline{h}(t)) \in [1 - \varepsilon_0, 1] \quad \text{for } x \in [-\underline{h}(t) + K_0, \underline{h}(t) - K_0], \tag{3.2}$$

where we have assumed $\underline{h}(0) = K_1 > K_0$.

Clearly

$$K_2 \int_0^t (\tau + \theta)^{1-\alpha} \, d\tau \leq K_2 \theta^{1-\alpha} t \leq \frac{c_0}{4} t$$

provided $\theta \geq (4K_2/c_0)^{1/(\alpha-1)}$. Therefore

$$\underline{h}(t) \geq \frac{c_0}{2}t + K_1 \geq \frac{c_0}{2}(t + \theta) > K_0 \quad \text{for all } t \geq 0 \tag{3.3}$$

provided that

$$K_1 \geq \frac{c_0}{2}\theta \quad \text{and} \quad \theta \geq \max\{(4K_2/c_0)^{1/(\alpha-1)}, \theta_0, 2K_0/c_0\}. \tag{3.4}$$

Define

$$\varepsilon_1 := \inf_{x \in [-K_0, 0]} |\phi'(x)| > 0.$$

Then

$$\begin{cases} \phi'(x - \underline{h}(t)) \leq -\varepsilon_1 & \text{for } x \in [\underline{h}(t) - K_0, \underline{h}(t)], \\ \phi'(-x - \underline{h}(t)) \leq -\varepsilon_1 & \text{for } x \in [-\underline{h}(t), -\underline{h}(t) + K_0]. \end{cases} \tag{3.5}$$

Claim 1. *For suitably chosen θ, K_1, K_2, we have*

$$\underline{h}'(t) \leq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x)\,dy, \quad t > 0, \tag{3.6}$$

and

$$-\underline{h}'(t) \geq -\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x - y)\underline{u}(t, x)\,dy, \quad t > 0.$$

Due to $\underline{u}(t, x) = \underline{u}(t, -x)$ and $J(x) = J(-x)$, we just need to verify (3.6). We calculate

$$\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x)\,dy\,dx$$

$$= (1 - \varepsilon(t))\mu \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J(x - y)\phi(x)\,dy\,dx$$

$$+ (1 - \varepsilon(t))\mu \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J(x - y)[\phi(-x - 2\underline{h}(t)) - 1]\,dy\,dx$$

$$= (1 - \varepsilon(t))c_0 - (1 - \varepsilon(t))\mu \int_{-\infty}^{-2\underline{h}(t)} \int_{0}^{\infty} J(x - y)\phi(x)\,dy\,dx$$

$$- (1 - \varepsilon(t))\mu \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J(x - y)[1 - \phi(-x - 2\underline{h}(t))]\,dy\,dx.$$

From (3.3), for $t \geq 0$,

$$(1 - \varepsilon(t))\mu \int_{-\infty}^{-2\underline{h}(t)} \int_{0}^{\infty} J(x - y)\phi(x)\,dy\,dx$$

$$+ (1 - \varepsilon(t))\mu \int_{-2\underline{h}(t)}^{-\underline{h}(t)} \int_{0}^{\infty} J(x - y)[1 - \phi(-x - 2\underline{h}(t))]\,dy\,dx$$

$$\leq 2\mu \int_{-\infty}^{-\underline{h}(t)} \int_{0}^{\infty} J(x - y)\,dy\,dx \leq 2\mu \int_{-\infty}^{-\frac{c_0}{2}(t+\theta)} \int_{0}^{\infty} J(x - y)\,dy\,dx.$$

And by (3.1) we have, for $t > 0$,

$$(1 - \varepsilon(t))\mu \int_{-\underline{h}(t)}^{0} \int_{0}^{\infty} J(x - y)[1 - \phi(-x - 2\underline{h}(t))] \, dy \, dx$$

$$\leq \mu[1 - \phi(-\underline{h}(t))] \int_{-\underline{h}(t)}^{0} \int_{0}^{\infty} J(x - y) \, dy \, dx$$

$$\leq \mu \frac{C}{h(t)^{\alpha-1}} \int_{-\infty}^{0} \int_{0}^{\infty} J(x - y) \, dy \, dx$$

$$= \mu \frac{C}{h(t)^{\alpha-1}} \int_{0}^{\infty} y J(y) \, dy$$

$$\leq \mu \frac{C^2}{(c_0/2)^{\alpha-1}(t + \theta)^{\alpha-1}} \leq \frac{K_2 - c_0}{(t + \theta)^{\alpha-1}}$$

if

$$K_2 \geq c_0 + \frac{C^2}{(c_0/2)^{\alpha-1}} \mu. \tag{3.7}$$

Hence, when θ, K_1 and K_2 are chosen such that (3.4) and (3.7) hold, then

$$\mu \int_{-\underline{h}(t)}^{h(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x) \, dy \, dx$$

$$\geq (1 - \varepsilon(t))c_0 - 2\mu \int_{-\infty}^{-\frac{c_0}{2}(t+\theta)} \int_{0}^{\infty} J(x - y)\phi(x) \, dy \, dx - \frac{K_2 - c_0}{(t + \theta)^{\alpha-1}}$$

$$= c_0 - K_2\varepsilon(t) - 2\mu \int_{-\infty}^{-\frac{c_0}{2}(t+\theta)} \int_{0}^{\infty} J(x - y)\phi(x) \, dy \, dx$$

$$= h'(t) \quad \text{for all } t > 0,$$

which finishes the proof of (3.6).

Claim 2. *With θ, K_1, K_2 chosen such that (3.4) and (3.7) hold, and K_2 suitably further enlarged (see (3.8) below), $\theta_0 \gg 1$ and $0 < \varepsilon_0 \ll 1$, we have, for all $t > 0$ and $x \in (-\underline{h}(t), \underline{h}(t))$,*

$$\underline{u}_t(t, x) \leq d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y) \, dy - d\underline{u}(t, x) + f(\underline{u}(t, x)).$$

A simple calculation gives

$$\underline{u}_t = -\varepsilon'(t)[\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1]$$
$$- (1 - \varepsilon(t))h'(t)[\phi'(x - \underline{h}(t)) + \phi'(-x - \underline{h}(t))]$$
$$= (\alpha - 1)(t + \theta)^{-\alpha}[\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1]$$
$$- (1 - \varepsilon(t))[c_0 + \delta'(t)][\phi'(x - \underline{h}(t)) + \phi'(-x - \underline{h}(t))],$$

and using the equation satisfied by ϕ we deduce

$$
\begin{aligned}
&- (1 - \varepsilon(t))c_0[\phi'(x - \underline{h}(t)) + \phi'(-x - \underline{h}(t))] \\
&\quad = (1 - \varepsilon)\left[d \int_{-\infty}^{\underline{h}(t)} J(x - y)\phi(y - \underline{h}(t))\,dy - d\phi(x - \underline{h}(t)) \right. \\
&\qquad\qquad \left. + d \int_{-\underline{h}(t)}^{\infty} J(-x - y)\phi(-y - \underline{h}(t))\,dy - d\phi(-x - \underline{h}(t)) \right] \\
&\qquad + (1 - \varepsilon(t))[f(\phi(x - \underline{h}(t))) + f(\phi(-x - \underline{h}(t)))] \\
&\quad = d\left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\,dy - \underline{u}(t, x) \right] \\
&\qquad + (1 - \varepsilon(t))\left[d \int_{-\infty}^{-\underline{h}(t)} J(x - y)\phi(y - \underline{h}(t)) - 1]\,dy \right. \\
&\qquad\qquad \left. + d \int_{\underline{h}(t)}^{\infty} J(-x - y)\phi(-y - \underline{h}(t))\,dy - 1]\,dy \right] \\
&\qquad + (1 - \varepsilon(t))[f(\phi(x - \underline{h}(t))) + f(\phi(-x - \underline{h}(t)))] \\
&\quad \leq d\left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\,dy - \underline{u}(t, x) \right] \\
&\qquad + (1 - \varepsilon(t))[f(\phi(x - \underline{h}(t))) + f(\phi(-x - \underline{h}(t)))].
\end{aligned}
$$

Hence

$$
\underline{u}_t \leq d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\,dy - \underline{u}(t, x) + f(\underline{u}(t, x)) + A_1(t, x) + A_2(t, x),
$$

where

$$
\begin{aligned}
A_1(t, x) &:= (\alpha - 1)(t + \theta)^{-\alpha}[\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1], \\
A_2(t, x) &:= -(1 - \varepsilon(t))\delta'(t)[\phi'(x - \underline{h}(t)) + \phi'(-x - \underline{h}(t))] \\
&\qquad + (1 - \varepsilon(t))[f(\phi(x - \underline{h}(t))) + f(\phi(-x - \underline{h}(t)))] - f(\underline{u}(t, x)).
\end{aligned}
$$

To finish the proof of Claim 2, it remains to check that

$$
A_1(t, x) + A_2(t, x) \leq 0 \quad \text{for } t > 0, \ x \in (-\underline{h}(t), \underline{h}(t)).
$$

We next prove this inequality for x in the following three intervals, separately:

$$
\begin{aligned}
I_1(t) &:= [\underline{h}(t) - K_0, \underline{h}(t)], \\
I_2(t) &:= [-\underline{h}(t), -\underline{h}(t) + K_0], \\
I_3(t) &:= [-\underline{h}(t) + K_0, \underline{h}(t) - K_0].
\end{aligned}
$$

For $x \in I_1(t)$, by (3.1),

$$
0 \geq \phi(-x - \underline{h}(t)) - 1 \geq \phi(K_0 - 2\underline{h}(t)) - 1 \geq \phi(-\underline{h}(t)) - 1 \geq \frac{-C}{h(t)^{\alpha - 1}}.
$$

Then by (**f**),

$$f(\phi(-x - \underline{h}(t))) = f(\phi(-x - \underline{h}(t))) - f(1) \le L\frac{C}{h(t)^{\alpha-1}}$$

and

$$f(\underline{u}(t,x)) \ge (1 - \varepsilon(t)) f\big(\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1\big)$$
$$\ge (1 - \varepsilon(t))\left[f(\phi(x - \underline{h}(t))) - L\frac{C}{h(t)^{\alpha-1}}\right].$$

Thus from the definition of $\delta(t)$, (3.3) and (3.5), we deduce

$$A_2(t,x) \le (1 - \varepsilon(t))\big[\delta'(t)[\phi'(x - \underline{h}(t)) + \phi'(-x - \underline{h}(t))] + f(\phi(x - \underline{h}(t)))$$
$$+ f(\phi(-x - \underline{h}(t))) - f\big(\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1\big)\big]$$
$$\le (1 - \varepsilon(t))\left[-\delta'(t)\varepsilon_1 + 2L\frac{C}{h(t)^{\alpha-1}}\right]$$
$$\le (1 - \varepsilon(t))\left[-K_2(t + \theta)^{1-\alpha}\varepsilon_1 + \frac{2LC}{h(t)^{\alpha-1}}\right]$$
$$\le (1 - \varepsilon(t))(t + \theta)^{1-\alpha}[-K_2\varepsilon_1 + 2LC(2/c_0)^{\alpha-1}].$$

Moreover,

$$A_1(t,x) \le (\alpha - 1)(t + \theta)^{-\alpha} \le 2(1 - \varepsilon(t))(\alpha - 1)(t + \theta)^{-\alpha},$$

where by enlarging θ_0 we have assumed that $\varepsilon(t) \le \theta_0^{1-\alpha} < 1/2$. Hence

$$A_1(t,x) + A_2(t,x) \le (1 - \varepsilon)(t + \theta)^{1-\alpha}[-K_2\varepsilon_1 + 2LC(2/c_0)^{\alpha-1} + 2(\alpha - 1)\theta_0^{-1}]$$
$$\le 0$$

if additionally

$$K_2 \ge \varepsilon_1^{-1}[2LC(2/c_0)^{\alpha-1} + 2(\alpha - 1)\theta_0^{-1}]. \tag{3.8}$$

This proves the desired inequality for $x \in I_1(t)$.

Since $A_1(t,x) + A_2(t,x)$ is even in x, the desired inequality is also valid for $x \in I_2(t) = -I_1(t)$. It remains to prove the inequality for $x \in I_3(t)$.

The case $x \in I_3(t)$ requires some preparations. Define, for $0 < \varepsilon \ll 1$,

$$g(u,v) := (1 - \varepsilon)[f(u) + f(v)] - f((1 - \varepsilon)(u + v - 1)), \quad u, v \in \mathbb{R}.$$

For $u, v \in [0, 1]$, we may apply the mean value theorem to the function

$$\xi(t) := g\big(1 + t(u - 1), 1 + t(v - 1)\big)$$

to obtain

$$\xi(1) = \xi(0) + \xi'(\zeta) \quad \text{for some } \zeta \in [0, 1].$$

Denote

$$\tilde{u} := 1 + \zeta(u - 1), \quad \tilde{v} := 1 + \zeta(v - 1).$$

Then the above identity is equivalent to

$$g(u, v) = g(1, 1) + \partial_u g(\tilde{u}, \tilde{v})(u - 1) + \partial_v g(\tilde{u}, \tilde{v})(v - 1)$$
$$= -f(1 - \varepsilon) + (1 - \varepsilon) f'(\tilde{u})(u - 1) + (1 - \varepsilon) f'(\tilde{v})(v - 1)$$
$$- (1 - \varepsilon) f'\big((1 - \varepsilon)(\tilde{u} + \tilde{v} - 1)\big)(u - 1)$$
$$- (1 - \varepsilon) f'\big((1 - \varepsilon)(\tilde{u} + \tilde{v} - 1)\big)(v - 1).$$

Note that $\tilde{u} \in [u, 1]$ and $\tilde{v} \in [v, 1]$. Since f' is locally Lipschitz, there is C_1 such that

$$|f'(u) - f'(v)| \leq C_1 |u - v| \quad \text{for } u, v \in [0, 1].$$

It follows that

$$(1 - \varepsilon) f'(\tilde{u})(u - 1) - (1 - \varepsilon) f'\big((1 - \varepsilon)(\tilde{u} + \tilde{v} - 1)\big)(u - 1)$$
$$= (1 - \varepsilon)\big[f'(\tilde{u}) - f'\big((1 - \varepsilon)(\tilde{u} + \tilde{v} - 1)\big)\big](u - 1)$$
$$\leq (1 - \varepsilon) b_1 (1 - u),$$

where

$$b_1 := C_1 |\tilde{u} - (1 - \varepsilon)(\tilde{u} + \tilde{v} - 1)| = C_1 |\varepsilon \tilde{u} - (1 - \varepsilon)(\tilde{v} - 1)|$$
$$\leq C_1 (\varepsilon + 1 - v).$$

Similarly,

$$(1 - \varepsilon) f'(\tilde{v})(v - 1) - (1 - \varepsilon) f'\big((1 - \varepsilon)(\tilde{u} + \tilde{v} - 1)\big)(v - 1) \leq (1 - \varepsilon) b_2 (1 - v),$$

where

$$b_2 := C_1 |\varepsilon \tilde{v} - (1 - \varepsilon)(\tilde{u} - 1)| \leq C_1 (\varepsilon + 1 - u).$$

Thus

$$g(u, v) \leq -f(1 - \varepsilon) + (1 - \varepsilon) b_1 (1 - u) + (1 - \varepsilon) b_2 (1 - v)$$
$$\leq -f(1 - \varepsilon) + C_1 (\varepsilon + 1 - v)(1 - u) + C_1 (\varepsilon + 1 - u)(1 - v)$$
$$= \varepsilon [f'(1) + o(1) + C_1 (1 - u + 1 - v)] + 2C_1 (1 - u)(1 - v),$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

For our discussions below, it is convenient to introduce the notations

$$p(t, x) := 1 - \phi(x - \underline{h}(t)), \quad q(t, x) := 1 - \phi(-x - \underline{h}(t)).$$

Then by (3.2) we have

$$p(t, x), q(t, x) \in [0, \varepsilon_0] \quad \text{for } x \in I_3(t), t > 0. \tag{3.9}$$

Moreover, since $\min \{x - \underline{h}(t), -x - \underline{h}(t)\} \leq -\underline{h}(t)$ always holds, by (3.1) and (3.3), if we denote $C_2 := C(c_0/2)^{1-\alpha}$, then

$$p(t, x) q(t, x) \leq \frac{C \varepsilon_0}{\underline{h}(t)^{\alpha-1}} \leq C_2 \varepsilon_0 \varepsilon(t) \quad \text{for } x \in I_3(t), \ t > 0. \tag{3.10}$$

Now due to $\delta'(t) < 0$ and $\phi' < 0$, we have, by (3.9) and (3.10),

$$
\begin{aligned}
A_2(t, x) &\le g(1 - p, 1 - q) \\
&\le \varepsilon(t)[f'(1) + o(1) + C_1(p + q)] + 2C_1 pq \\
&\le \varepsilon(t)[f'(1) + o(1) + C_3\varepsilon_0] \quad \text{for } x \in I_3(t), \ t > 0,
\end{aligned}
$$

with $C_3 := 2(C_1 + C_1 C_2)$. Since

$$
A_1(t, x) \le (\alpha - 1)(t + \theta)^{-\alpha} \le (\alpha - 1)\theta_0^{-1}\varepsilon(t)
$$

and $f'(1) < 0$, we thus obtain

$$
A_1 + A_2 \le \varepsilon(t)\big(f'(1) + [o(1) + C_3\varepsilon_0 + (\alpha - 1)\theta_0^{-1}]\big) < 0 \quad \text{for } x \in I_3(t), \ t > 0
$$

provided that θ_0 is sufficiently large and ε_0 is sufficiently small. The proof of Claim 2 is now complete.

Claim 3. *There exists $t_0 > 0$ such that*

$$
\begin{cases}
g(t + t_0) \le -\underline{h}(t), \ h(t + t_0) \ge \underline{h}(t) & \text{for } t \ge 0, \\
u(t + t_0, x) \ge \underline{u}(t, x) & \text{for } t \ge 0, \ x \in [-\underline{h}(t), \underline{h}(t)].
\end{cases} \tag{3.11}
$$

It is clear that

$$
\underline{u}(t, \pm\underline{h}(t)) = (1 - \varepsilon(t))[\phi(-2\underline{h}(t)) - 1] \le 0 \quad \text{for } t \ge 0.
$$

Since spreading happens for (u, g, h), there exists a large constant $t_0 > 0$ such that

$$
\begin{aligned}
g(t_0) &< -K_1 = -\underline{h}(0) \quad \text{and} \quad \underline{h}(0) = K_1 < h(t_0), \\
u(t_0, x) &\ge (1 - \theta^{1-\alpha}) \ge \underline{u}(0, x) \quad \text{for } x \in [-\underline{h}(0), \underline{h}(0)],
\end{aligned}
$$

which, together with the inequalities proved in Claims 1 and 2, allows us to apply the comparison principle in [11] to conclude that (3.11) is valid.

Claim 4. *There exists $C > 0$ such that*

$$
\delta(t) \ge -C\left[1 + \int_0^t (1 + x)^{1-\alpha}\,dx + \int_0^{\frac{c_0}{2}t} x^2 J(x)\,dx + t\int_{\frac{c_0}{2}t}^\infty x J(x)\,dx\right].
$$

Clearly

$$
\int_0^t \varepsilon(\tau)\,d\tau = \int_0^t (x + \theta)^{1-\alpha}\,dx < \int_0^t (x + 1)^{1-\alpha}\,dx.
$$

By changing the order of integrations we have

$$
\begin{aligned}
\int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)} \int_0^\infty J(x - y)\,dy\,dx\,d\tau
&\le \int_0^t \int_{-\infty}^{-\frac{c_0}{2}\tau} \int_0^\infty J(x - y)\,dy\,dx\,d\tau \\
&= \int_0^t \int_{\frac{c_0}{2}\tau}^\infty \left[y - \frac{c_0}{2}\tau\right] J(y)\,dy\,d\tau \le \int_0^t \int_{\frac{c_0}{2}\tau}^\infty y J(y)\,dy\,d\tau \\
&= \frac{c_0}{2}\int_0^{\frac{c_0}{2}t} y^2 J(y)\,dy + t\int_{\frac{c_0}{2}t}^\infty y J(y)\,dy.
\end{aligned}
$$

The desired inequality now follows directly from the definition of $\delta(t)$. ∎

Next we prove an upper bound for $h(t) - c_0 t$. Let us note that we do not need the condition $(\mathbf{J}^\alpha)$ in the following result.

Lemma 3.2. *Under the assumptions of Theorem A, if $(\mathbf{J1})$ holds, and additionally f' is locally Lipschitz in $[0, \infty)$, then there exists $C > 0$ such that*

$$h(t) - c_0 t \leq C \quad \text{for all } t > 0. \tag{3.12}$$

Proof. As in the proof of Lemma 3.1, (c_0, ϕ^{c_0}) denotes the unique semi-wave pair in Theorem B, and to simplify notations we write $\phi^{c_0}(x) = \phi(x)$.

For fixed $\beta > 1$, and some large constants $\theta > 0$ and $K_1 > 0$ to be determined, define

$$\begin{cases} \bar{h}(t) := c_0 + \delta(t), & t \geq 0, \\ \overline{u}(t, x) := (1 + \varepsilon(t))\phi(x - \bar{h}(t)), & t \geq 0, \ x \leq \bar{h}(t), \end{cases}$$

where $\varepsilon(t) := (t + \theta)^{-\beta}$ and

$$\delta(t) := K_1 + \frac{c_0}{1 - \beta}[(t + \theta)^{1-\beta} - \theta^{1-\beta}].$$

By comparing $u(t, x)$ with a suitable ODE solution, we see that there is a large constant $t_0 > 0$ such that

$$u(t + t_0, x) \leq 1 + \tfrac{1}{2}\varepsilon(0) \quad \text{for } t \geq 0, \ x \in [g(t + t_0), h(t + t_0)].$$

Due to $\phi(-\infty) = 1$, we may choose sufficiently large $K_1 > 0$ such that $\underline{h}(0) = K_1 > 2h(t_0)$, $-\underline{h}(0) = -K_1 < 2g(t_0)$, and also

$$\overline{u}(0, x) = (1 + \varepsilon(0))\phi(-K_1/2) \geq 1 + \tfrac{1}{2}\varepsilon(0) \geq u(t_0, x). \tag{3.13}$$

for $x \in [g(t_0), h(t_0)]$.

Claim 1. *We have*

$$\bar{h}'(t) \geq \mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\overline{u}(t, x) \, dy \quad \text{for } t > 0.$$

A direct calculation shows

$$\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\overline{u}(t, x) \, dy \leq \mu \int_{-\infty}^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\overline{u}(t, x) \, dy$$

$$= (1 + \varepsilon(t))\mu \int_{-\infty}^0 \int_0^\infty J(x - y)\phi(x) \, dy = (1 + \varepsilon(t))c_0 = \bar{h}'(t),$$

as desired.

Claim 2. *If $\theta > 0$ is sufficiently large, then for $t > 0$ and $x \in (g(t + t_0), \underline{h}(t))$,*

$$\overline{u}_t(t, x) \geq d \int_{g(t+t_0)}^{\bar{h}(t)} J(x - y)\overline{u}(t, y) \, dy - d\overline{u}(t, x) + f(\overline{u}(t, x)). \tag{3.14}$$

We calculate

$$\bar{u}_t(t, x) = -(1 + \varepsilon(t))[c_0 + \delta'(t)]\phi'(x - \bar{h}(t)) + \varepsilon'(t)\phi(x - \underline{h}(t))$$
$$= -(1 + \varepsilon(t))c_0\phi'(x - \bar{h}(t)) - (1 + \varepsilon(t))\delta'(t)\phi'(x - \bar{h}(t))$$
$$- \beta(t + \theta)^{-\beta-1}\phi(x - \underline{h}(t))$$
$$\geq d \int_{\underline{g}(t_0+t)}^{\bar{h}(t)} J(x - y)\bar{u}(t, y)\, dy - d\bar{u}(t, x) + f(\bar{u}(t, x)) + A(t, x)$$

with

$$A(t, x) := (1 + \varepsilon(t)) f(\phi(x - \bar{h}(t))) - f\big((1 + \varepsilon(t))\phi(x - \bar{h}(t))\big)$$
$$- (1 + \varepsilon(t))\delta'(t)\phi'(x - \bar{h}(t)) - \beta(t + \theta)^{-\beta-1}\phi(x - \underline{h}(t)).$$

To prove the claim, we need to show

$$A(t, x) \geq 0 \quad \text{for } x \in [\underline{g}(t_0 + t), \bar{h}(t)] \text{ and } t > 0.$$

Let ε_0, ε_1 and K_0 be given as in the proof of Lemma 3.1. For $x \in [\bar{h}(t) - K_0, \bar{h}(t)]$ and $t > 0$, by (3.5), we have

$$A(t, x) \geq -(1 + \varepsilon)\delta'(t)\phi'(x - \bar{h}(t)) - \beta(t + \theta)^{-\beta-1}\phi(x - \underline{h}(t))$$
$$= -(1 + \varepsilon)c_0(t + \theta)^{-\beta}\phi'(x - \bar{h}(t)) - \beta(t + \theta)^{-\beta-1}\phi(x - \underline{h}(t))$$
$$\geq c_0(t + \theta)^{-\beta}\varepsilon_1 - \beta(t + \theta)^{-\beta-1} \geq (t + \theta)^{-\beta-1}[c_0\theta\varepsilon_1 - \beta] \geq 0$$

provided θ is large enough.

We next estimate $A(t, x)$ for $x \in [\underline{g}(t + t_0), \underline{h}(t) - K_0]$. Define, for $0 < \varepsilon \ll 1$ and $u, v \geq 0$,

$$g(u) := (1 + \varepsilon) f(u) - f((1 + \varepsilon)u).$$

Then for $u, v \in [0, 1]$,

$$g(u) = g(1) + g'(\tilde{u})(u - 1)$$
$$= -f(1 + \varepsilon) + (1 + \varepsilon) f'(\tilde{u})(u - 1) - (1 + \varepsilon) f'((1 + \varepsilon)\tilde{u})(u - 1)$$
$$= -f(1 + \varepsilon) + (1 + \varepsilon)[f'(\tilde{u}) - f'((1 + \varepsilon)\tilde{u})](u - 1)$$

for some $\tilde{u} \in [u, 1]$. Since f' is locally Lipschitz, there exists $C_1 > 0$ such that

$$|f'(u) - f'(v)| \leq C_1|u - v| \quad \text{for } u, v \in [0, 2].$$

Therefore

$$g(u) \geq -f(1 + \varepsilon) - (1 + \varepsilon)\varepsilon C_1(1 - u) \geq -\varepsilon f'(1) + o(\varepsilon) - 2C_1\varepsilon(1 - u).$$

By (3.2) we have

$$-\varepsilon_0 \leq \phi(x - \bar{h}(t)) - 1 < 0 \quad \text{for } x \in [\underline{g}(t_0 + t), \underline{h}(t) - K_0], \ t > 0. \tag{3.15}$$

Using (3.2), $\delta' > 0$, $\phi' \leq 0$ and $\varepsilon(t) = (t + \theta)^{-\beta} \leq \theta^{-\beta}$, we obtain

$$
\begin{aligned}
A(t, x) &\geq (1 + \varepsilon(t)) f(\phi(x - \bar{h}(t))) - f((1 + \varepsilon)\phi(x - \bar{h}(t))) \\
&\quad - \beta(t + \theta)^{-\beta-1}\phi(x - \underline{h}(t)) \\
&= g(\phi(x - \bar{h}(t))) - \beta(t + \theta)^{-\beta-1}\phi(x - \underline{h}(t)) \\
&\geq \varepsilon(t)[-f'(1) + o(1) - 2\varepsilon_0 C_1 - \beta\theta^{-\beta-1}] \\
&> 0 \quad \text{for } x \in [g(t_0 + t), \underline{h}(t) - K_0], \ t > 0
\end{aligned}
$$

provided θ is large enough and $\varepsilon_0 > 0$ is small enough, since $f'(1) < 0$. We have now proved (3.14).

Due to the inequalities proved in Claims 1 and 2, (3.13) and

$$
\overline{u}(t, g(t + t_0)) > 0, \quad \overline{u}(t, \bar{h}(t)) = (1 + \varepsilon)\phi(\bar{h}(t) - \bar{h}(t)) = 0 \quad \text{for } t \geq 0,
$$

we are now able to apply the comparison principle to conclude that

$$
\begin{aligned}
h(t + t_0) &\leq \bar{h}(t), \quad && t \geq 0, \\
u(t + t_0, x) &\leq \overline{u}(t, x), \quad && t \geq 0, \ x \in [g(t + t_0), \underline{h}(t)].
\end{aligned}
$$

The desired inequality (3.12) follows directly from $\delta(t) \leq K_1 + \frac{c_0}{\beta-1}\theta^{1-\beta}$ and $h(t + t_0) \leq \bar{h}(t)$. The proof is complete. ∎

Proof of Theorem 1.1. Since $\alpha \geq 3$, from the definitions of $\bar{h}(t)$ and $\underline{h}(t)$ in the proofs of Lemmas 3.1 and 3.2 it is easily seen that

$$
C_0 := \sup_{t > 0}[|\bar{h}(t) - c_0 t| + |\underline{h}(t) - c_0 t|] < \infty.
$$

Hence for large fixed $\theta > 0$ and all large t, say $t \geq t_0$,

$$
[g(t), h(t)] \supset [-\underline{h}(t - t_0), \underline{h}(t - t_0)] \supset [-c_0 t + C, c_0 t - C] \quad \text{with } C := C_0 + c_0 t_0,
$$

and

$$
u(t, x) \geq \underline{u}(t, x) \geq (1 - \varepsilon(t))[\phi^{c_0}(x - c_0 t + C) + \phi^{c_0}(-x - c_0 t + C) - 1]
$$

for $x \in [-c_0 t + C, c_0 t - C]$, where $\varepsilon(t) = (t + \theta)^{1-\alpha}$. This inequality for $u(t, x)$ also holds for $x \in [g(t), h(t)]$ if we assume that $\phi^{c_0}(x) = 0$ for $x > 0$, since when x lies outside of $[-c_0 t + C, c_0 t - C]$ the right side is negative.

From the proof of Lemma 3.2 we see that the following analogous inequalities hold:

$$
g(t) \geq -\bar{h}(t - t_0), \quad u(t, x) \leq (1 + \varepsilon(t))\phi^{c_0}(-x - \bar{h}(t - t_0))
$$

for $t > t_0$ and $x \in [g(t), h(t)]$. We thus have

$$
[g(t), h(t)] \subset [-\bar{h}(t - t_0), \bar{h}(t - t_0)] \subset [-c_0 t - C, c_0 t + C],
$$

and

$$u(t, x) \leq \overline{u}(t, x) \leq (1 - \varepsilon(t)) \min \{\phi^{c_0}(x - c_0 t - C), \phi^{c_0}(-x - c_0 t - C)\}$$

for $t > t_0$ and $x \in [g(t), h(t)]$.

Finally we note that as $t \to \infty$,

$$\begin{cases} \phi^{c_0}(-x - c_0 t \pm C) \to 1 & \text{uniformly in } [0, \infty), \\ \phi^{c_0}(x - c_0 t \pm C) \to 1 & \text{uniformly in } (-\infty, 0], \end{cases}$$

and the conclusions for $u(t, x)$ in Theorem 1.1 thus follow directly. ∎

4. Proof of Theorem 1.3

In this section we determine the growth rate of $c_0 t - h(t)$ and $c_0 t + g(t)$ when the kernel function satisfies, for some $\gamma \in (2, 3]$,

$$J(x) \sim |x|^{-\gamma} \quad \text{for } |x| \gg 1. \tag{4.1}$$

That is, $(\hat{\mathbf{J}}^{\gamma})$ holds with $\gamma \in (2, 3]$. As before, we will only estimate $c_0 t - h(t)$, since the estimate for $c_0 t + g(t)$ follows by making the variable change $x \to -x$ in the initial function.

The upper bound for $c_0 t - h(t)$ follows directly from Lemma 3.1, so we only need to obtain a suitable lower bound. It turns out that the case $f'(0) \geq d$ is more difficult to treat than the case $f'(0) < d$. Therefore we will consider the case $f'(0) < d$ first, and then handle the more difficult case $f'(0) \geq d$ by appropriate modifications of the proof for the first case.

4.1. The case $f'(0) < d$

Lemma 4.1. *Suppose that the assumptions in Theorem 1.3 are satisfied and $f'(0) < d$. Then there exists $\sigma = \sigma(\gamma) > 0$ such that for all large $t > 0$,*

$$c_0 t - h(t) \geq \begin{cases} \sigma t^{3-\gamma} & \text{if } \gamma \in (2, 3), \\ \sigma \ln t & \text{if } \gamma = 3. \end{cases} \tag{4.2}$$

Proof. Let $\beta := \gamma - 2 \in (0, 1]$, and (c_0, ϕ) be the semi-wave pair in Theorem B. Define

$$\varepsilon(t) := K_1(t + \theta)^{-\beta}, \quad \delta(t) := K_2 - K_3 \int_0^t \varepsilon(\tau) \, d\tau$$

and

$$\begin{cases} \overline{h}(t) := c_0 t + \delta(t), & t \geq 0, \\ \overline{u}(t, x) := (1 + \varepsilon(t))\phi(x - \overline{h}(t)) + \rho(t, x), & t \geq 0, \ x \leq \overline{h}(t), \end{cases}$$

where

$$\rho(t, x) := K_4 \xi(x - \bar{h}(t)) \varepsilon(t),$$

with $\xi \in C^2(\mathbb{R})$ satisfying

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\varepsilon}, \quad \xi(x) = 0 \text{ for } |x| > 2\tilde{\varepsilon}, \tag{4.3}$$

and the positive constants θ, K_1, K_2, K_3, K_4, $\tilde{\varepsilon}$ are to be determined.

We are going to show that it is possible to choose these constants and some $t_0 > 0$ such that

$$\bar{u}_t \geq d \int_{g(t+t_0)}^{\bar{h}(t)} J(x - y) \bar{u}(t, y) \, dy - \bar{u} + f(\bar{u}) \quad \text{for } t > 0, \ x \in (g(t + t_0), \bar{h}(t)), \tag{4.4}$$

$$\bar{h}'(t) \geq \mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y) \bar{u}(t, x) \, dy \qquad \text{for } t > 0, \tag{4.5}$$

$$\bar{u}(t, g(t + t_0)) \geq 0, \quad \bar{u}(t, \bar{h}(t)) \geq 0 \qquad \text{for } t \geq 0, \tag{4.6}$$

$$\bar{u}(0, x) \geq u(t_0, x), \quad \bar{h}(0) \geq h(t_0) \qquad \text{for } x \in [g(t_0), h(t_0)]. \tag{4.7}$$

If these inequalities are proved, then by the comparison principle, we obtain

$$\bar{h}(t) \geq h(t + t_0), \quad \bar{u}(t, x) \geq u(t + t_0, x) \quad \text{for } t > 0, \ x \in [g(t + t_0), h(t + t_0)],$$

and the desired inequality for $c_0 t - h(t)$ follows easily from the definition of $\bar{h}(t)$.

Therefore, to complete the proof, it suffices to prove the above inequalities. We divide the arguments below into several steps.

Firstly, by Theorem A, there is $C_1 > 1$ such that

$$-g(t), h(t) \leq (c_0 + 1)t + C_1 \quad \text{for } t \geq 0. \tag{4.8}$$

Let us also note that (4.6) holds trivially.

Step 1. Choose $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (4.7) holds.

For later analysis, we need to find $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (4.7) holds and at the same time they grow slower than linearly in θ.

Since $f'(1) < 0$, there exists small $\varepsilon_* > 0$ such that for any $k \in (0, \varepsilon_*]$,

$$f(1 + k) \leq \frac{f'(1)}{2} k < 0 < -\frac{f'(1)}{2} k \leq f(1 - k).$$

It follows that, for $\tilde{\sigma} := f'(1)/2$,

$$\overline{w}(t) = 1 + \varepsilon_* e^{\tilde{\sigma} t}, \quad \underline{w}(t) = 1 - \varepsilon_* e^{\tilde{\sigma} t}$$

are a pair of upper and lower solutions of the ODE $w' = f(w)$ with initial data $w(0) \in [1 - \varepsilon_*, 1 + \varepsilon_*]$. By (**f**), the unique solution of the ODE

$$W' = F(W), \quad W(0) = \|u_0\|_\infty$$

satisfies $\lim_{t \to \infty} W(t) = 1$. Hence there exists $t_* > 0$ such that

$$W(t_*) \in [1 - \varepsilon_*, 1 + \varepsilon_*].$$

Using the above defined upper solution $\overline{w}(t)$ we obtain

$$W(t + t_*) \leq 1 + \varepsilon_* e^{\tilde{\sigma} t} \quad \text{for } t \geq 0.$$

By the comparison principle we deduce

$$u(t + t_*, x) \leq W(t + t_*) \leq 1 + \varepsilon_* e^{\tilde{\sigma} t} \quad \text{for } t \geq 0, \; x \in [g(t + t_*), h(t + t_*)].$$

Hence

$$u(t_0, x) \leq 1 + \varepsilon(0)/2 \quad \text{for } x \in [g(t_0), h(t_0)]$$

provided that

$$t_0 = t_0(\theta) := \frac{\beta}{|\tilde{\sigma}|} \ln \theta + \frac{\ln(2\tilde{\varepsilon}_*/K_1)}{|\tilde{\sigma}|} + t_*.$$

By (4.1), for any fixed $\omega_* \in (\beta, \gamma - 1)$, we have

$$\int_{\mathbb{R}} J(x)|x|^{\omega_*} \, dx < \infty.$$

Then by Theorem 1.4, there is C_2 such that

$$1 - \phi(x) \leq C_2/|x|^{\omega_*} \quad \text{for } x \leq -1.$$

Hence, for $K > 1$ we have

$$(1 + \varepsilon(0))\phi(-K) - (1 + \varepsilon(0)/2) \geq (1 + \varepsilon(0))[1 - C_2 K^{-\omega_*}] - (1 + \varepsilon(0)/2)$$
$$= K_1 \theta^{-\beta}/2 - C_2 K^{-\omega_*}(1 + K_1 \theta^{-\beta}) \geq 0$$

provided that

$$K^{\omega_*} \geq 2C_2 + \frac{2C_2}{K_1} \theta^{\beta}.$$

Therefore, for all $K_1 \in (0, 1]$, $\theta \geq 1$ and $K \geq (4C_2/K_1)^{1/\omega_*} \theta^{\beta/\omega_*}$, we have

$$(1 + \varepsilon(0))\phi(-K) - (1 + \varepsilon(0)/2) \geq 0.$$

Now define

$$K_2(\theta) := 2 \max \{(4C_2/K_1)^{1/\omega_*} \theta^{\beta/\omega_*}, (c_0 + 1)t_0(\theta) + C_1\}. \tag{4.9}$$

Then for $K_2 = K_2(\theta)$ we have

$$\overline{h}(0) = K_2 > K_2/2 \geq (c_0 + 1)t_0 + C_1 \geq h(t_0),$$

and for $x \in [g(t_0), h(t_0)]$,

$$\overline{u}(0, x) = (1 + \varepsilon(0))\phi(x - K_2) \geq (1 + \varepsilon(0))\phi(-K_2/2) \geq 1 + \varepsilon(0)/2.$$

Thus (4.7) holds if t_0 and K_2 are chosen as above, for any $\theta \geq 1$, $K_1 \in (0, 1]$.

Step 2. We verify that (4.5) holds if θ, K_1, K_3 and K_4 are chosen suitably.

Denote

$$C_3 := \mu \int_{-\infty}^0 \int_0^\infty J(x - y)\,\mathrm{d}y\,\mathrm{d}x = \mu \int_0^\infty J(y)y\,\mathrm{d}y. \tag{4.10}$$

A direct calculation shows, writing $\varepsilon(t) = \varepsilon$,

$$\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\bar{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \mu \int_{-\infty}^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\bar{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x - \mu \int_{-\infty}^{g(t+t_0)} \int_{\bar{h}(t)}^\infty J(x - y)\bar{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \mu \int_{-\infty}^0 \int_0^\infty J(x - y)[(1 + \varepsilon)\phi(x) + \rho(t, x + \bar{h}(t))]\,\mathrm{d}y\,\mathrm{d}x$$

$$\quad - \mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_0^\infty J(x - y)[(1 + \varepsilon)\phi(x) + \rho(t, x + \bar{h}(t))]\,\mathrm{d}y\,\mathrm{d}x$$

$$\leq (1 + \varepsilon)c_0 + C_3 K_4 \varepsilon - \mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_0^\infty J(x - y)(1 + \varepsilon)\phi(x)\,\mathrm{d}y\,\mathrm{d}x$$

$$\leq (1 + \varepsilon)c_0 + C_3 K_4 \varepsilon - \mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_0^\infty J(x - y)\phi(x)\,\mathrm{d}y\,\mathrm{d}x.$$

By elementary calculus, for any $k > 1$,

$$\int_{-\infty}^{-k} \int_0^\infty \frac{1}{|x - y|^{2+\beta}}\,\mathrm{d}y\,\mathrm{d}x = \int_{-\infty}^{-k} \int_{-x}^\infty \frac{1}{y^{2+\beta}}\,\mathrm{d}y\,\mathrm{d}x = \beta^{-1}(1 + \beta)^{-1}k^{-\beta}. \tag{4.11}$$

Due to (4.1) and (4.8), there exists $C_4 > 0$ such that

$$\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_0^\infty J(x - y)\phi(x)\,\mathrm{d}y\,\mathrm{d}x$$

$$\geq C_4 \phi(g(t + t_0) - \bar{h}(t)) \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_0^\infty \frac{1}{|x - y|^{2+\beta}}\,\mathrm{d}y\,\mathrm{d}x$$

$$\geq \phi_* C_4 \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_0^\infty \frac{1}{|x - y|^{2+\beta}}\,\mathrm{d}y\,\mathrm{d}x = \frac{\phi_* C_4}{\beta(1 + \beta)}(|g(t + t_0)| + \bar{h}(t))^{-\beta}$$

$$\geq \frac{\phi_* C_4}{\beta(1 + \beta)}[(c_0 + 1)(t + t_0) + C_1 + c_0 t + K_2]^{-\beta}$$

$$= \frac{\phi_* C_4}{\beta(1 + \beta)(2c_0 + 1)^\beta}\left[t + \frac{(c_0 + 1)t_0 + C_1 + K_2}{(2c_0 + 1)}\right]^{-\beta}, \tag{4.12}$$

where $\phi_* := \phi(-1) \leq \phi(-K_2) \leq \phi(g(t + t_0) - \bar{h}(t))$. Therefore, for all $\theta > 0$ so large that

$$\theta > \frac{(c_0 + 1)t_0 + C_1 + K_2}{(2c_0 + 1)}, \tag{4.13}$$

which is possible since $t_0(\theta)$ and $K_2(\theta)$ grow slower than linearly in θ, we have

$$\mu \int_{\underline{h}(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x-y)\bar{u}(t,x)\,\mathrm{d}y\,\mathrm{d}x$$

$$\leq (1+\varepsilon(t))c_0 + C_3 K_4 \varepsilon(t) - \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta}$$

$$= c_0 + \varepsilon(t)\left[c_0 + C_3 K_4 - \frac{\phi_* C_4}{K_1 \beta(1+\beta)(2c_0+1)^\beta}\right]$$

$$\leq c_0 - K_3 \varepsilon(t) = h'(t)$$

provided that K_1, K_3 and K_4 are small enough that

$$K_1(c_0 + C_3 K_4 + K_3) \leq \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}. \tag{4.14}$$

Therefore (4.5) holds if we first fix K_1, K_3, K_4 small so that (4.14) holds, and then choose θ large such that (4.13) is satisfied.

Step 3. We show that (4.4) holds when K_3 and K_4 are chosen suitably small and θ is large.

We have

$$\bar{u}_t(t,x) = -(1+\varepsilon(t))[c_0 + \delta'(t)]\phi'(x-\bar{h}(t)) + \varepsilon'(t)\phi(x-\underline{h}(t)) + \rho_t(t,x),$$

and, writing $\varepsilon(t) = \varepsilon$ to simplify the notation,

$$-(1+\varepsilon)c_0\phi'(x-\bar{h}(t))$$

$$= (1+\varepsilon)\left[d\int_{-\infty}^{\bar{h}(t)} J(x-y)\phi(y-\bar{h}(t))\,\mathrm{d}y - d\phi(x-\bar{h}(t)) + f(\phi(x-\bar{h}(t)))\right]$$

$$= d\int_{-\infty}^{\bar{h}(t)} J(x-y)[\bar{u}(t,y)-\rho(t,y)]\,\mathrm{d}y - d[\bar{u}(t,x)-\rho(t,x)] + (1+\varepsilon)f(\phi(x-\bar{h}(t)))$$

$$\geq d\int_{\underline{g}(t+t_0)}^{\bar{h}(t)} J(x-y)\bar{u}(t,y)\,\mathrm{d}y - d\bar{u}(t,x) + f(\bar{u}(t,x))$$

$$+ d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\,\mathrm{d}y\right] + (1+\varepsilon)f(\phi(x-\bar{h}(t))) - f(\bar{u}(t,x)).$$

Hence

$$\bar{u}_t(t,x) \geq d\int_{\underline{g}(t+t_0)}^{\bar{h}(t)} J(x-y)\bar{u}(t,y)\,\mathrm{d}y - d\bar{u}(t,x) + f(\bar{u}(t,x)) + A(t,x)$$

with

$$A(t,x) := d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\,\mathrm{d}y\right] + (1+\varepsilon)f(\phi(x-\bar{h}(t))) - f(\bar{u}(t,x))$$

$$- (1+\varepsilon)\delta'(t)\phi'(x-\bar{h}(t)) + \varepsilon'(t)\phi(x-\underline{h}(t)) + \rho_t(t,x).$$

Therefore to complete this step, it suffices to show that we can choose K_3, K_4 and θ such that $A(t, x) \geq 0$. We will do that for $x \in [\bar{h}(t) - \tilde{\varepsilon}, \bar{h}(t)]$ and for $x \in [g(t_0 + t), \bar{h}(t) - \tilde{\varepsilon}]$ separately.

Claim 1. *If $\tilde{\varepsilon} > 0$ in (4.3) is sufficiently small and θ is sufficiently large, then*

$$
\left[\rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y)\rho(t, y)\, dy \right] + (1 + \varepsilon) f(\phi(x - \bar{h}(t))) - f(\overline{u}(t, x))
$$

$$
\geq \frac{d - f'(0)}{2} \rho(t, x) > 0 \quad \text{for } x \in [\bar{h}(t) - \tilde{\varepsilon}, \bar{h}(t)]. \quad (4.15)
$$

We have, for $x \in [\bar{h}(t) - \tilde{\varepsilon}, \bar{h}(t)]$,

$$
d \left[\rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y)\rho(t, y)\, dy \right]
$$

$$
= K_4 \varepsilon(t) \left[d - d \int_{-\infty}^{0} J(x - \bar{h}(t) - y)\xi(y)\, dy \right]
$$

$$
\geq K_4 \varepsilon(t) \left[d - d \int_{-2\tilde{\varepsilon}}^{0} J(x - \bar{h}(t) - y)\, dy \right] = K_4 \varepsilon(t) \left[d - d \int_{\bar{h}(t)-x-2\tilde{\varepsilon}}^{\bar{h}(t)-x} J(y)\, dy \right]
$$

$$
\geq K_4 \varepsilon(t) \left[d - d \int_{-2\tilde{\varepsilon}}^{\tilde{\varepsilon}} J(y)\, dy \right] \geq K_4 \varepsilon(t) \left[d - \frac{d - f'(0)}{4} \right]
$$

$$
= \left[d - \frac{d - f'(0)}{4} \right] \rho(t, x)
$$

provided $\tilde{\varepsilon} \in (0, \varepsilon_1]$ for some small $\varepsilon_1 > 0$.

On the other hand, for $x \in [\bar{h}(t) - \tilde{\varepsilon}, \bar{h}(t)]$, by **(f)** we obtain

$$
(1 + \varepsilon) f(\phi(x - \bar{h}(t))) - f(\overline{u}(t, x)) \geq f((1 + \varepsilon)\phi(x - \bar{h}(t))) - f(\overline{u}(t, x))
$$

$$
= f(\overline{u}(t, x) - \rho(t, x)) - f(\overline{u}(t, x)),
$$

and due to $0 < K_4 \ll 1$,

$$
0 \leq \overline{u}(t, x) \leq (1 + \varepsilon)\phi(\tilde{\varepsilon}) + K_4 \varepsilon \leq 2\phi(\tilde{\varepsilon}) + \theta^{-\beta}.
$$

So $\overline{u}(t, x)$ and $\rho(t, x)$ are small for small $\tilde{\varepsilon}$ and large θ. It follows that

$$
f(\overline{u}(t, x) - \rho(t, x)) - f(\overline{u}(t, x)) = -\rho(t, x)[f'(\overline{u}(t, x)) + o(1)]
$$

$$
= -\rho(t, x)[f'(0) + o(1)] \geq -\left[f'(0) + \frac{d - f'(0)}{4} \right] \rho(t, x)
$$

for $x \in [\bar{h}(t) - \tilde{\varepsilon}, \bar{h}(t)]$ provided that $\tilde{\varepsilon}$ is small and θ is large. Hence, (4.15) holds.

Denote

$$
M := \sup_{x \leq 0} |\phi'(x)|.
$$

For $x \in [\bar{h}(t) - \tilde{\varepsilon}, \bar{h}(t)]$, by (4.15) we have

$$A(t, x) \geq \frac{d - f'(0)}{2} \rho(t, x) - (1 + \varepsilon(t)) \delta'(t) \phi'(x - \bar{h}(t)) + \varepsilon'(t) \phi(x - \underline{h}(t)) + \rho_t(t, x)$$

$$\geq \varepsilon(t) \left[\frac{d - f'(0)}{2} K_4 - 2K_3 M - \beta(t + \theta)^{-1} - K_4 \beta(t + \theta)^{-1} \right]$$

$$\geq \varepsilon(t) \left[\frac{d - f'(0)}{2} K_4 - 2K_3 M - \theta^{-1} \beta (1 + K_4) \right] \geq 0$$

provided that we first fix K_3 and K_4 so that (4.14) holds and at the same time

$$\frac{d - f'(0)}{2} K_4 - 2K_3 M > 0, \tag{4.16}$$

and then choose θ sufficiently large.

Next, for fixed small $\tilde{\varepsilon} > 0$, we estimate $A(t, x)$ for $x \in [g(t + t_0), \bar{h}(t) - \tilde{\varepsilon}]$.

Claim 2. *For any given $1 \gg \eta > 0$, there is $c_1 = c_1(\eta)$ such that*

$$(1 + \varepsilon) f(v) - f((1 + \varepsilon) v) \geq c_1 \varepsilon \quad \text{for } v \in [\eta, 1] \text{ and } 0 < \varepsilon \ll 1. \tag{4.17}$$

Indeed, by (1.10) there exists $c_1 > 0$ depending on η such that

$$f(v) - v f'(v) \geq 2c_1 \quad \text{for } v \in [\eta, 1].$$

Since

$$\lim_{\varepsilon \to 0} \frac{(1 + \varepsilon) f(v) - f((1 + \varepsilon) v)}{\varepsilon} = f(v) - v f'(v) \geq 2c_1$$

uniformly for $v \in [\eta, 1]$, there exists $\varepsilon_0 > 0$ so small that

$$\frac{(1 + \varepsilon) f(v) - f((1 + \varepsilon) v)}{\varepsilon} \geq c_1$$

for $v \in [\eta, 1]$ and $\varepsilon \in (0, \varepsilon_0]$. This proves Claim 2.

By Claim 2 and $f \in C^1$, there is a positive constant C_f such that, for $v = \phi(x - \bar{h}(t)) \in [\phi(-\tilde{\varepsilon}), 1]$,

$$(1 + \varepsilon) f(v) - f((1 + \varepsilon) v + \rho)$$
$$= (1 + \varepsilon) f(v) - f((1 + \varepsilon) v) + f((1 + \varepsilon) v) - f((1 + \varepsilon) v + \rho)$$
$$\geq c_1 \varepsilon - C_f K_4 \varepsilon$$

when $\varepsilon = \varepsilon(t)$ is small.

We also have

$$d \left[\rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y) \rho(t, y) \, dy \right] \geq -d \int_{-\infty}^{\bar{h}(t)} J(x - y) \rho(t, y) \, dy \geq -d K_4 \varepsilon(t),$$

and

$$\rho_t(t, x) = -\xi' \bar{h}' K_4 \varepsilon(t) + \xi K_4 \varepsilon'(t) \geq -\xi_* K_4 \varepsilon(t) - K_4 \beta(t + \theta)^{-1} \varepsilon(t)$$
$$\geq -(\xi_* + \beta\theta^{-1}) K_4 \varepsilon(t)$$

with $\xi_* := c_0 \max_{x \in \mathbb{R}} |\xi'(x)|$.

Using these we obtain, for $x \in [g(t_0 + t), \bar{h}(t) - \tilde{\varepsilon}]$,

$$A(t, x) \geq -dK_4 \varepsilon(t) + (1 + \varepsilon) f(\phi(x - \bar{h}(t))) - f(\bar{u}(t, x)) + 2M\delta'(t) + \varepsilon'(t) + \rho_t(t, x)$$
$$\geq \varepsilon(t)[c_1 - K_4(C_f + d) - 2MK_3 - \beta(t + \theta)^{-1} - (\xi_* + \beta\theta^{-1})K_4]$$
$$\geq \varepsilon(t)[c_1 - K_4(C_f + d) - 2MK_3 - \xi_* K_4 - \beta\theta^{-1}(1 + K_4)] \geq 0$$

provided that we first choose K_3 and K_4 so small that

$$c_1 - K_4(C_f + d) - 2MK_3 - \xi_* K_4 > 0$$

while keeping both (4.14) and (4.16), and then choose $\theta > 0$ sufficiently large.

Therefore, (4.4) holds when K_3, K_4 and θ are chosen as above. The proof of the lemma is now complete. ∎

4.2. The case $f'(0) \geq d$

Lemma 4.2. *In Lemma 4.1, if $f'(0) \geq d$, then (4.2) still holds.*

Proof. This is a modification of the proof of Lemma 4.1, where in the definition of $\bar{u}$, we add a new term $\lambda(t)$ and change $\rho(t, x)$ to $-\rho(t, x)$; see the details below.

We will use similar notations. Let $\beta = \gamma - 2 \in (0, 1]$, and for fixed $\tilde{\varepsilon} > 0$, let $\xi \in C^2(\mathbb{R})$ satisfy

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\varepsilon}, \quad \xi(x) = 0 \text{ for } |x| > 2\tilde{\varepsilon}.$$

Define

$$\begin{cases} \bar{h}(t) := c_0 t + \delta(t), & t \geq 0, \\ \bar{u}(t, x) := (1 + \varepsilon(t))\phi(x - \bar{h}(t) - \lambda(t)) - \rho(t, x), & t \geq 0, \ x \leq \bar{h}(t), \end{cases}$$

where

$$\varepsilon(t) := K_1(t + \theta)^{-\beta}, \quad \delta(t) := K_2 - K_3 \int_0^t \varepsilon(\tau) \, d\tau,$$

$$\rho(t, x) := K_4 \xi(x - \bar{h}(t))\varepsilon(t), \quad \lambda(t) := K_5 \varepsilon(t),$$

and the positive constants θ, $\tilde{\varepsilon}$ and K_1, K_2, K_3, K_4, K_5 are to be determined.

Let

$$C_{\tilde{\varepsilon}} := \min_{x \in [-2\tilde{\varepsilon}, 0]} |\phi'(x)|.$$

Then for $x \in [\bar{h}(t) - 2\tilde{\varepsilon}, \bar{h}(t)]$,

$$\bar{u}(t, x) \geq \phi(-\lambda(t)) - \rho(t, x) \geq C_{\tilde{\varepsilon}}\lambda(t) - K_4\varepsilon(t) \geq \varepsilon(t)(C_{\tilde{\varepsilon}}K_5 - K_4) > 0$$

if

$$K_4 = C_{\tilde{\varepsilon}} K_5 / 2, \tag{4.18}$$

which combined with $\xi(x) = 0$ for $|x| \geq 2\tilde{\varepsilon}$ implies

$$\bar{u}(t, x) \geq 0 \quad \text{for } t \geq 0, \ x \leq \bar{h}(t). \tag{4.19}$$

Let $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ be given by Step 1 in the proof of Lemma 4.1. Then $[g(t_0), h(t_0)] \subset (-\infty, K_2/2)$, and due to $\rho(0, x) = 0$ for $x \leq h(t_0) < K_2/2 < K_2 = \bar{h}(0)$, we have

$$\bar{u}(0, x) = (1 + \varepsilon(0))\phi(x - K_2 - \lambda(0)) \geq (1 + \varepsilon(0))\phi(-K_2/2)$$
$$\geq 1 + \varepsilon(0)/2 \geq u(t_0, x) \quad \text{for } x \in [g(t_0), h(t_0)]. \tag{4.20}$$

Step 1. We verify that by choosing K_1, K_3 and K_5 suitably small,

$$\bar{h}'(t) \geq \mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y)\bar{u}(t, x)\, \mathrm{d}y\, \mathrm{d}x \quad \text{for all } t > 0. \tag{4.21}$$

By direct calculations we have

$$\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y)\bar{u}(t, x)\, \mathrm{d}y\, \mathrm{d}x$$

$$\leq \mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y)(1 + \varepsilon)\phi(x - \bar{h}(t) - \lambda(t))\, \mathrm{d}y\, \mathrm{d}x$$

$$= (1 + \varepsilon)\mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x - y)\phi(x - \lambda(t))\, \mathrm{d}y\, \mathrm{d}x$$

$$\quad - (1 + \varepsilon)\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_{0}^{\infty} J(x - y)\phi(x - \lambda(t))\, \mathrm{d}y\, \mathrm{d}x$$

$$\leq (1 + \varepsilon)c_0 + (1 + \varepsilon)\mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x - y)[\phi(x - \lambda(t)) - \phi(x)]\, \mathrm{d}y\, \mathrm{d}x$$

$$\quad - (1 + \varepsilon)\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_{0}^{\infty} J(x - y)\phi(x)\, \mathrm{d}y\, \mathrm{d}x.$$

Let $M_1 := \sup_{x \leq 0} |\phi'(x)|$ and C_3 be given by (4.10). Then

$$(1 + \varepsilon)\mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x - y)[\phi(x - \lambda(t)) - \phi(x)]\, \mathrm{d}y\, \mathrm{d}x \leq 2C_3 M_1 \lambda(t).$$

By (4.12),

$$\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_{0}^{\infty} J(x - y)\phi(x)\, \mathrm{d}y\, \mathrm{d}x$$

$$\geq \frac{\phi_* C_4}{\beta(1 + \beta)(2c_0 + 1)^\beta} \left[t + \frac{(c_0 + 1)t_0 + C_1 + K_2}{(2c_0 + 1)} \right]^{-\beta}.$$

Therefore, as in the proof of Lemma 4.1, for θ so large that

$$\theta > \frac{(c_0 + 1)t_0 + C_1 + K_2}{(2c_0 + 1)}. \tag{4.22}$$

we have

$$\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y)\bar{u}(t, x) \, dy \, dx$$

$$\leq (1 + \varepsilon)c_0 + 2C_3 M_1 \lambda(t) - \frac{\phi_* C_4}{\beta(1 + \beta)(2c_0 + 1)^\beta}(t + \theta)^{-\beta}$$

$$= c_0 + \varepsilon(t)\left[c_0 + 2C_3 M_1 K_5 - \frac{\phi_* C_4}{K_1 \beta(1 + \beta)(2c_0 + 1)^\beta}\right]$$

$$\leq c_0 - K_3 \varepsilon(t) = \bar{h}'(t)$$

provided that K_1, K_3 and K_5 are suitably small so that

$$K_1(c_0 + 2C_3 M_1 K_5 + K_3) \leq \frac{\phi_* C_4}{\beta(1 + \beta)(2c_0 + 1)^\beta}. \tag{4.23}$$

Step 2. We show that by choosing K_3, K_5 suitably small and θ sufficiently large, for $t > 0$ and $x \in [g(t + t_0), \bar{h}(t)]$,

$$\bar{u}_t(t, x) \geq d \int_{g(t+t_0)}^{\bar{h}(t)} J(x - y)\bar{u}(t, y) \, dy - \bar{u}(t, x) + f(\bar{u}(t, x)). \tag{4.24}$$

Using the definition of $\bar{u}$, we have

$$\bar{u}_t(t, x) = -(1 + \varepsilon)(\bar{h}' + \lambda')\phi'(x - \bar{h} - \lambda) + \varepsilon'\phi(x - \bar{h} - \lambda) - \rho_t$$

$$= -(1 + \varepsilon)[c_0 + \delta' + \lambda']\phi'(x - \bar{h} - \lambda) + \varepsilon'\phi(x - \bar{h} - \lambda) - \rho_t$$

and

$$-(1 + \varepsilon)c_0\phi'(x - \bar{h} - \lambda)$$

$$= (1 + \varepsilon)\left[d \int_{-\infty}^{\bar{h}+\lambda} J(x - y)\phi(y - \bar{h} - \lambda) \, dy - d\phi(x - \bar{h} - \lambda) + f(\phi(x - \bar{h} - \lambda))\right]$$

$$\geq (1 + \varepsilon)\left[d \int_{-\infty}^{\bar{h}} J(x - y)\phi(y - \bar{h} - \lambda) \, dy - d\phi(x - \bar{h} - \lambda) + f(\phi(x - \bar{h} - \lambda))\right]$$

$$= d \int_{-\infty}^{\bar{h}} J(x - y)[\bar{u}(t, y) + \rho] \, dy - d[\bar{u}(t, x) + \rho] + (1 + \varepsilon)f(\phi(x - \bar{h} - \lambda))$$

$$= \int_{-\infty}^{\bar{h}(t)} J(x - y)\bar{u}(t, y) \, dy - d\bar{u}(t, x)$$

$$- d\left[\rho(t, x) - \int_{-\infty}^{\bar{h}(t)} J(x - y)\rho(t, y) \, dy\right] + (1 + \varepsilon)f(\phi(x - \bar{h} - \lambda))$$

$$\geq d \int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\,\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x))$$

$$- d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\,\mathrm{d}y\right] + (1+\varepsilon)f(\phi(x-\bar{h}-\lambda)) - f(\overline{u}(t,x)).$$

Hence

$$\overline{u}_t(t,x) \geq d \int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\,\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x)) + B(t,x)$$

with

$$B(t,x) := -d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}} J(x-y)\rho(t,y)\,\mathrm{d}y\right]$$
$$+ (1+\varepsilon)f(\phi(x-\bar{h}-\lambda)) - f(\overline{u}(t,x))$$
$$- (1+\varepsilon)(\delta' + \lambda')\phi'(x-\bar{h}-\lambda) + \varepsilon'\phi(x-\underline{h}-\lambda) - \rho_t.$$

To show (4.24), it remains to choose suitable K_3, K_5 and θ such that $B(t,x) \geq 0$ for $t > 0$ and $x \in [g(t+t_0), \bar{h}(t)]$.

Claim. *There exist small $\tilde{\varepsilon}_0 \in (0, \tilde{\varepsilon}/2)$ and some $\tilde{J}_0 > 0$ depending on $\tilde{\varepsilon}$ but independent of $\tilde{\varepsilon}_0$ such that*

$$-d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}} J(x-y)\rho(t,y)\,\mathrm{d}y\right] + (1+\varepsilon)f(\phi(x-\bar{h}-\lambda)) - f(\overline{u}(t,x))$$
$$\geq \tilde{J}_0\,\rho(t,x) \quad \text{for } x \in [\bar{h}(t) - \tilde{\varepsilon}_0, \bar{h}(t)]. \quad (4.25)$$

Indeed, for $x \in [\bar{h}(t) - \tilde{\varepsilon}_0, \bar{h}(t)]$,

$$d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\,\mathrm{d}y\right] = K_2\varepsilon(t)\left[d - d\int_{-\infty}^{\bar{h}(t)} J(x-y)\xi(y-\bar{h}(t))\,\mathrm{d}y\right]$$
$$\leq K_2\varepsilon(t)\left[d - d\int_{\bar{h}(t)-\tilde{\varepsilon}}^{\bar{h}(t)} J(x-y)\,\mathrm{d}y\right] = K_2\varepsilon(t)\left[d - d\int_{\bar{h}(t)-\tilde{\varepsilon}-x}^{\bar{h}(t)-x} J(x-y)\,\mathrm{d}y\right]$$
$$\leq d\rho\left[1 - \int_{-\tilde{\varepsilon}+\tilde{\varepsilon}_0}^{0} J(y)\,\mathrm{d}y\right] \leq d\rho\left[1 - \int_{-\tilde{\varepsilon}/2}^{0} J(y)\,\mathrm{d}y\right].$$

On the other hand, for $x \in [\bar{h}(t) - \tilde{\varepsilon}_0, \bar{h}(t)]$, we have

$$(1+\varepsilon)f(\phi(x-\bar{h}-\lambda) - f(\overline{u}) \geq f((1+\varepsilon)\phi(x-\bar{h}-\lambda)) - f(\overline{u})$$
$$= f(\overline{u}+\rho) - f(\overline{u}) = \rho(f'(\overline{u}) + o(1)) = (f'(0) + o(1))\rho$$

since both $\overline{u}(t,x)$ and $\rho(t,x)$ are close to 0 for $x \in [\bar{h}(t) - \tilde{\varepsilon}_0, \bar{h}(t)]$ with $\tilde{\varepsilon}_0$ small.

Hence, for such x and $\tilde{\varepsilon}_0$, since $f'(0) \geq d$,

$$-d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\,dy\right] + (1+\varepsilon)f(\phi(x-\bar{h}(t))) - f(\overline{u}(t,x))$$

$$\geq d\rho\left[-1 + \int_{-\tilde{\varepsilon}/2}^{0} J(y)\,dy\right] + f'(0)\rho + o(1)\rho$$

$$\geq \tilde{J}_0\,\rho(t,x) \quad \text{with} \quad \tilde{J}_0 := \frac{d}{2}\int_{-\tilde{\varepsilon}/2}^{0} J(y)\,dy.$$

This proves (4.25).

Clearly

$$-\rho_t(t,x) = \beta K_4 K_1(t+\theta)^{-\beta-1} \geq 0.$$

Denoting $M_1 := \sup_{x\leq 0}|\phi'(x)|$, we obtain, for $x \in [\bar{h}(t)-\tilde{\varepsilon}_0, \bar{h}(t)]$ and small $\tilde{\varepsilon}_0$,

$$\begin{aligned}
B(t,x) &\geq \tilde{J}_0 K_2\varepsilon(t) + 2(\delta'(t) + \lambda'(t))M_1 + \varepsilon'(t) \\
&= \tilde{J}_0 K_2\varepsilon(t) + 2\varepsilon(t)(-K_3 - K_5\beta(t+\theta)^{-1})M_1 - \beta(t+\theta)^{-1}\varepsilon(t) \\
&\geq \varepsilon(t)[\tilde{J}_0 K_2 - 2(K_3 + K_5\beta\theta^{-1})M_1 - \beta\theta^{-1}] \\
&= \varepsilon(t)[\tilde{J}_0 K_2 - 2K_3 M_1 - \theta^{-1}(K_5\beta M_1 + \beta)] \geq 0
\end{aligned}$$

provided that K_3 is chosen so small that (4.23) holds, and

$$\tilde{J}_0 K_2 - 2K_3 M_1 > 0, \tag{4.26}$$

and θ is chosen sufficiently large.[3]

We next estimate $B(t,x)$ for $x \in [g(t+t_0), \bar{h}(t)-\tilde{\varepsilon}_0]$. From Claim 2 in the proof of Lemma 4.1, and the Lipschitz continuity of f, there exist positive constants $C_l = C_l(\tilde{\varepsilon}_0)$ and C_f such that, for $v = \phi(x - \bar{h}(t-\lambda(t))) \in [\phi(-\tilde{\varepsilon}_0), 1]$,

$$\begin{aligned}
(1+\varepsilon)f(v) &- f((1+\varepsilon)v - \rho) \\
&= (1+\varepsilon)f(v) - f((1+\varepsilon)v) + f((1+\varepsilon)v) - f((1+\varepsilon)v - \rho) \\
&\geq C_l\varepsilon - C_f\rho \geq C_l\varepsilon - C_f K_4\varepsilon
\end{aligned}$$

when $\varepsilon = \varepsilon(t)$ is small. Hence

$$\begin{aligned}
(1+\varepsilon(t))&f(\phi(x - \bar{h}(t) - \lambda(t))) - f(\bar{u}(t,x)) \\
&\geq C_l\varepsilon(t) - C_f K_4\varepsilon(t) \quad \text{for } x \in [g(t+t_0), \bar{h}(t)-\tilde{\varepsilon}_0], \, 0 < \tilde{\varepsilon}_0 \ll 1.
\end{aligned}$$

Clearly,

$$-d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\,dy\right] \geq -dK_4\varepsilon(t),$$

---

[3]In fact, by the choice of $K_2 = K_2(\theta)$ in (4.9), for fixed K_3, (4.26) always holds for large enough θ.

and

$$\rho_t(t, x) = -K_4 \xi' \bar{h}'(t)\varepsilon(t) + K_4 \xi \varepsilon'(t) \leq \xi_* K_4 \varepsilon(t)$$

with $\xi_* := c_0 \max_{x \in \mathbb{R}} |\xi'(x)|$.

We thus obtain, for $x \in [g(t + t_0), \bar{h}(t) - \tilde{\varepsilon}_0]$ and $0 < \tilde{\varepsilon}_0 \ll 1$,

$$\begin{aligned}
B(t, x) &\geq -K_4 \varepsilon(t)d + (1 + \varepsilon)f(\phi(x - \bar{h})) - f(\overline{u}) + 2M_1(\delta' + \lambda') + \varepsilon' - \rho_t \\
&\geq C_l \varepsilon(t) - K_4 \varepsilon(t)(d + C_f + \xi_*) + 2M_1[-K_3 \varepsilon(t) + K_5 \varepsilon'(t)] + \varepsilon'(t) \\
&\geq \varepsilon(t)[C_l - K_4(d + C_f + \xi_*) - 2M_1(K_3 + K_5 \beta(t + \theta)^{-1}) - \beta(t + \theta)^{-1}] \\
&\geq \varepsilon(t)[C_l - K_4(d + C_f + \xi_*) - 2M_1 K_3 - \theta^{-1}\beta(2M_1 K_5 + 1)] \geq 0
\end{aligned}$$

if we choose K_3 and K_5 so small that (4.23) and (4.26) hold and at the same time, due to (4.18),

$$C_l - K_4(d + C_f + \xi_*) - 2M_1 K_3 > 0,$$

and then choose θ sufficiently large. Hence, (4.24) is satisfied if K_3 and K_5 are chosen small as above, and θ is sufficiently large.

From (4.19), we have

$$\overline{u}(t, g(t + t_0)) \geq 0, \quad \overline{u}(t, \bar{h}(t)) \geq 0 \quad \text{for } t \geq 0.$$

Together with (4.20), (4.21) and (4.24), this enables us to use the comparison principle to conclude that

$$h(t + t_0) \leq \bar{h}(t), \quad u(t + t_0, x) \leq \overline{u}(t, x) \quad \text{for } t \geq 0, \ x \in [g(t + t_0), \underline{h}(t)],$$

which implies (4.2). The proof of the lemma is now complete. ∎

4.3. Proof of Theorem 1.3

By Lemma 3.1 and then by (4.1), there exists $C_0 > 0$ such that

$$\begin{aligned}
h(t) - c_0 t &\geq -C\left[1 + \int_0^t (1 + x)^{1-\gamma} \, dx + \int_0^{\frac{c_0}{2}t} x^2 J(x) \, dx + t \int_{\frac{c_0}{2}t}^\infty x J(x) \, dx\right] \\
&\geq -C\left[1 + \frac{1}{\gamma - 2} + \int_0^1 J(x) \, dx + C_0 \int_1^{\frac{c_0}{2}t} x^{2-\gamma} \, dx + C_0 t \int_{\frac{c_0}{2}t}^\infty x^{1-\gamma} \, dx\right].
\end{aligned}$$

Therefore when $\gamma \in (2, 3)$ we have

$$h(t) - c_0 t \geq -C[\tilde{C} + \ln(t + 1) + \tilde{C}_1 t^{3-\gamma}] \geq -\hat{C}_1 t^{3-\gamma} \quad \text{for all } t \gg 1$$

and some $\hat{C}_1, \tilde{C}, \tilde{C}_1 > 0$, and when $\gamma = 3$,

$$h(t) - c_0 t \geq -\hat{C}_2 \ln t \quad \text{for all } t \gg 1 \text{ and some } \hat{C}_2 > 0.$$

These combined with Lemmas 4.1 and 4.2 yield the conclusion of Theorem 1.3. ∎

5. Proof of Theorem 1.2

Throughout this section, we assume that J satisfies $(\mathbf{J})$ and $(\hat{\mathbf{J}}^\gamma)$ for some $\gamma \in (1, 2]$. So there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{|x|^\gamma + 1} \le J(x) \le \frac{C_2}{|x|^\gamma + 1} \quad \text{for } x \in \mathbb{R} \text{ and some } \gamma \in (1, 2]. \tag{5.1}$$

Clearly now $(\mathbf{J1})$ is not satisfied.

The purpose of this section is to prove Theorem 1.2, and as before we will only prove the estimate for $h(t)$, since that for $g(t)$ follows by the change of variable $x \to -x$. Theorem 1.2 will follow directly from the lemmas in Sections 5.1 and 5.2 below.

5.1. Upper bound

This is the easy part of the proof.

Lemma 5.1. *Assume that* $(\mathbf{J})$ *and* $(\mathbf{f})$ *hold. If spreading happens, and* (5.1) *is satisfied, then there exists* $C = C(\gamma) > 0$ *such that*

$$h(t) \le \begin{cases} Ct^{1/(\gamma-1)} & \text{if } \gamma \in (1, 2), \\ Ct \ln t & \text{if } \gamma = 2. \end{cases} \tag{5.2}$$

Proof. Define, for $t \ge 0$,

$$\bar{h}(t) := \begin{cases} (Kt + \theta)^{1/(\gamma-1)} & \text{if } \gamma \in (1, 2], \\ (Kt + \theta) \ln(Kt + \theta) & \text{if } \gamma = 2, \end{cases}$$

$$\bar{u}(t, x) = \bar{u} := \max\{\|u_0\|_\infty, 1\}, \quad x \in [-\bar{h}(t), \bar{h}(t)],$$

with positive constants θ and K to be determined.

We start by showing

$$\bar{h}'(t) \ge \mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\bar{u}(t, x) \, dy \, dx \quad \text{for } t > 0, \tag{5.3}$$

and

$$-\bar{h}'(t) \le -\mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{-\infty}^{-\bar{h}(t)} J(x - y)\bar{u}(t, x) \, dy \, dx \quad \text{for } t > 0.$$

Since $\bar{u}(t, x) = \bar{u}(t, -x)$ and $J(x) = J(-x)$, it suffices to prove (5.3).

By simple calculations and (5.1), for any $k > 1$,

$$\int_{-k}^0 \int_0^\infty J(x - y) \, dy \, dx = \int_0^k \int_x^\infty J(y) \, dy \, dx = \int_0^k J(y)y \, dy + \mu k \int_k^\infty J(y) \, dy$$

$$\le \int_0^k \frac{C_2 y}{y^\gamma + 1} \, dy + k \int_k^\infty \frac{C_2}{y^\gamma + 1} \, dy \le \int_0^1 C_2 \, dy + \int_1^k \frac{C_2 y}{y^\gamma} \, dy + k \int_k^\infty \frac{C_2}{y^\gamma} \, dy,$$

and so

$$\int_{-k}^{0} \int_{0}^{\infty} J(x - y) \, dy \, dx \leq \begin{cases} C_2 + \frac{C_2}{2-\gamma}(k^{2-\gamma} - 1) + \frac{C_2 k^{2-\gamma}}{\gamma - 1} & \text{if } \gamma \in (1, 2), \\ 2C_2 + C_2 \ln k & \text{if } \gamma = 2. \end{cases} \tag{5.4}$$

Clearly

$$\int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y) \bar{u}(t, x) \, dy \, dx = \bar{u} \int_{-2\bar{h}(t)}^{0} \int_{0}^{\infty} J(x - y) \, dy \, dx.$$

Hence for $1 < \gamma < 2$, by (5.4),

$$\mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y) \bar{u}(t, x) \, dy \, dx$$

$$\leq \mu \bar{u} \left[C_2 + 2^{2-\gamma} \left(\frac{C_2}{2 - \gamma} + \frac{C_2}{\gamma - 1} \right) (Kt + \theta)^{(2-\gamma)/(\gamma-1)} \right]$$

$$\leq \frac{K}{\gamma - 1} (Kt + \theta)^{(2-\gamma)/(\gamma-1)} = \bar{h}'(t)$$

provided that $K > 0$ is large enough. And for $\gamma = 2$,

$$\mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y) \bar{u}(t, x) \, dy \, dx \leq \mu \bar{u} \big(2C_2 + C_2 \ln[2(Kt + \theta) \ln(Kt + \theta)] \big)$$

$$\leq \mu \bar{u} \big(2C_2 + C_2 \ln 2(Kt + \theta) + C_2 \ln[\ln(Kt + \theta)] \big) \leq K \ln(Kt + \theta) + K = \bar{h}'(t)$$

if $K \gg 1$. This finishes the proof of (5.3).

Since $\bar{u} \geq 1$ is a constant, we have, for $t > 0$, $x \in [-\bar{h}(t), \bar{h}(t)]$,

$$\bar{u}_t(t, x) \equiv 0 \geq d \int_{-\bar{h}(t)}^{\bar{h}(t)} J(x - y) \bar{u}(t, y) \, dy - d \bar{u}(t, x) + f(\bar{u}(t, x)). \tag{5.5}$$

Moreover, $\bar{h}(0) \geq h_0$ for large θ, and obviously

$$\bar{u}(t, \pm \bar{h}(t)) \geq 0 \quad \text{for } t \geq 0,$$
$$\bar{u}(0, x) \geq u(0, x) \quad \text{for } x \in [-h_0, h_0].$$

Hence we can apply the comparison principle to conclude that

$$[g(t + t_0), h(t + t_0)] \subset [-\bar{h}(t), \bar{h}(t)], \quad t \geq 0,$$
$$u(t + t_0, x) \leq \bar{u}(t, x), \quad t \geq 0, \ x \in [g(t + t_0), h(t + t_0)].$$

Thus (5.2) holds. ∎

5.2. Lower bound

We will consider the cases $\gamma \in (1, 2)$ and $\gamma = 2$ separately.

5.2.1. The case $\gamma \in (1, 2)$. We start with a result from [20].

Lemma 5.2 ([20, (2.11)]). *If J satisfies (**J**), then for any $\varepsilon > 0$, there is $L_\varepsilon > 0$ such that for all $l > L_\varepsilon$, the function $\psi_l(x) := l - |x|$ satisfies*

$$\int_{-l}^{l} J(x - y)\psi_l(y) \, dy \geq (1 - \varepsilon)\psi_l(x) \quad in \ [-l, l]. \tag{5.6}$$

Lemma 5.3. *Assume that the conditions in Theorem 1.2 are satisfied and $\gamma \in (1, 2)$. Then there exists $C = C(\gamma) > 0$ such that*

$$h(t) \geq Ct^{1/(\gamma-1)} \quad for \ t \gg 1. \tag{5.7}$$

Proof. Define

$$\underline{h}(t) := (K_1 t + \theta)^{1/(\gamma-1)}, \quad t \geq 0,$$

$$\underline{u}(t, x) := K_2 \frac{\underline{h}(t) - |x|}{\underline{h}(t)}, \quad t \geq 0, \ x \in [-\underline{h}(t), \underline{h}(t)],$$

with positive constants θ and K_1, K_2 to be determined.

Step 1. We show that, for large K_1,

$$\underline{h}'(t) \leq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x) \, dy \, dx \quad for \ t > 0. \tag{5.8}$$

By simple calculations and (5.1), we obtain

$$\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x) \, dy \, dx \geq \mu K_2 \int_{0}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\frac{\underline{h}(t) - x}{\underline{h}(t)} \, dy \, dx$$

$$= \frac{\mu K_2}{\underline{h}(t)} \int_{-\underline{h}(t)}^{0} \int_{0}^{\infty} J(x - y)(-x) \, dy \, dx = \frac{\mu K_2}{\underline{h}(t)} \int_{0}^{\underline{h}(t)} \int_{x}^{\infty} J(y)x \, dy \, dx$$

$$= \frac{\mu K_2}{\underline{h}(t)} \left(\int_{0}^{\underline{h}(t)} \int_{0}^{y} + \int_{\underline{h}(t)}^{\infty} \int_{0}^{\underline{h}(t)} \right) J(y)x \, dx \, dy \geq \frac{\mu K_2}{2\underline{h}(t)} \int_{0}^{\underline{h}(t)} J(y)y^2 \, dy$$

$$\geq \frac{\mu K_2 C_1}{2\underline{h}(t)} \int_{0}^{\underline{h}(t)} \frac{y^2}{y^\gamma + 1} \, dy \geq \frac{\mu K_2 C_1}{4\underline{h}(t)} \int_{1}^{\underline{h}(t)} y^{2-\gamma} \, dy \geq \frac{\mu K_2 C_1}{8\underline{h}(t)} \frac{\underline{h}(t)^{3-\gamma}}{3 - \gamma}$$

$$= \hat{C}_0 (K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} \geq \frac{K_1}{\gamma - 1} (K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} = \underline{h}'(t)$$

provided that $0 < K_1 \leq \hat{C}_0(\gamma - 1)$ and $\theta \gg 1$. This finishes the proof of Step 1.

Step 2. We show that, by choosing K_1, K_2 and θ properly, for $t > 0$ and $x \in (-\underline{h}(t), \underline{h}(t))$,

$$\underline{u}_t(t, x) \geq d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y) \, dy - d\underline{u}(t, x) + f(\underline{u}(t, x)). \tag{5.9}$$

From the definition of $\underline{u}$, for $t > 0$ and $x \in (-\underline{h}(t), \underline{h}(t))$,

$$\underline{u}_t(t, x) = K_2 \frac{|x|\underline{h}'(t)}{\underline{h}^2(t)} \le K_2 \frac{\underline{h}'(t)}{\underline{h}(t)} = \frac{K_1 K_2}{\gamma - 1} \underline{h}(t)^{1-\gamma}.$$

Claim 1. *For $x \in [-\underline{h}(t), \underline{h}(t)]$, there exists a positive constant $\hat{C}_1$ depending only on γ such that*

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\,dy \ge \hat{C}_1 K_2 \underline{h}(t)^{1-\gamma}. \tag{5.10}$$

By (5.1), writing $\underline{h}(t) = \underline{h}$ for simplicity of notation, we have

$$\int_{-\underline{h}}^{\underline{h}} J(x - y)\underline{u}(t, y)\,dy = \int_{-\underline{h}-x}^{\underline{h}-x} J(y)\underline{u}(t, y + x)\,dy$$

$$\ge K_2 \int_{-\underline{h}-x}^{\underline{h}-x} \frac{C_1}{|y|^\gamma + 1} \frac{\underline{h} - |y + x|}{\underline{h}}\,dy.$$

Thus, for $x \in [\underline{h}/4, \underline{h}]$,

$$\int_{-\underline{h}}^{\underline{h}} J(x - y)\underline{u}(t, y)\,dy \ge K_2 \int_{-\underline{h}/4}^{0} \frac{C_1}{|y|^\gamma + 1} \frac{\underline{h} - |y + x|}{\underline{h}}\,dy$$

$$= K_2 \int_{-\underline{h}/4}^{0} \frac{C_1}{|y|^\gamma + 1} \frac{\underline{h} - (y + x)}{\underline{h}}\,dy \ge K_2 \int_{-\underline{h}/4}^{0} \frac{C_1}{|y|^\gamma + 1} \frac{-y}{\underline{h}}\,dy$$

$$= \frac{K_2}{\underline{h}} \int_0^{\underline{h}/4} \frac{C_1 y}{y^\gamma + 1}\,dy \ge \frac{C_1 K_2}{2\underline{h}} \int_1^{\underline{h}/4} y^{1-\gamma}\,dy$$

$$\ge \frac{C_1 K_2}{4(2 - \gamma)\underline{h}} (\underline{h}/4)^{2-\gamma} =: \hat{C}_1 K_2 \underline{h}^{1-\gamma}.$$

And for $x \in [0, \underline{h}/4]$,

$$\int_{-\underline{h}}^{\underline{h}} J(x - y)\underline{u}(t, y)\,dy \ge K_2 \int_0^{\underline{h}/4} \frac{C_1}{|y|^\gamma + 1} \frac{\underline{h} - |y + x|}{\underline{h}}\,dy$$

$$\ge K_2 \int_0^{\underline{h}/4} \frac{C_1}{y^\gamma + 1} \frac{y}{\underline{h}}\,dy \ge \hat{C}_1 K_2 \underline{h}^{1-\gamma}$$

by repeating the last few steps in the previous calculations.

This proves (5.10) for $x \in [0, \underline{h}]$. It also holds for $x \in [-\underline{h}, 0]$ since both $J(x)$ and $\underline{u}(t, x)$ are even in x.

Claim 2. *We can choose small K_2 and large θ such that, for $x \in [-\underline{h}(t), \underline{h}(t)]$ and $t \ge 0$,*

$$d \int_{-\underline{h}}^{\underline{h}} J(x - y)\underline{u}(t, y)\,dy - d\underline{u}(t, x) + f(\underline{u}(t, x)) \ge F_* \int_{-\underline{h}}^{\underline{h}} J(x - y)\underline{u}(t, y)\,dy$$

for some positive constant F_*.

It is clear that $0 \le \underline{u}(t, x) \le K_2$, and thus for small $K_2 > 0$,

$$f(\underline{u}(t, x)) = [f'(0) + o(1)]\underline{u}(t, x) \ge \tfrac{3}{4} f'(0)\underline{u}(t, x).$$

Moreover, by (5.6), there is $L_1 > 0$ such that for $\theta^{1/(\gamma-1)} \geq L_1$,

$$d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\,\mathrm{d}y + \frac{f'(0)}{4}\underline{u}(t,x) \geq d\underline{u}(t,x) \quad \text{for } x \in [-\underline{h}(t), \underline{h}(t)].$$

Therefore Claim 2 is valid with $F_* = f'(0)/2$.

Combining Claims 1 and 2, we obtain

$$d \int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\,\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x))$$

$$\geq F_* \hat{C}_1 K_2 \underline{h}(t)^{1-\gamma} \geq \frac{K_1 K_2}{\gamma - 1} \underline{h}(t)^{1-\gamma} \geq \underline{u}_t(t,x)$$

provided that $K_1 \leq F_* \hat{C}_1 (\gamma - 1)$. This proves (5.9).

Step 3. We prove (5.7) by the comparison principle.

It is clear that

$$\underline{u}(t, \pm\underline{h}(t)) = 0 \quad \text{for } t \geq 0.$$

Since spreading happens for (u, g, h), for fixed $\theta \gg 1$ and small K_1, K_2 as chosen above, there exists a large $t_0 > 0$ such that

$$[-\underline{h}(0), \underline{h}(0)] \subset [g(t_0)/2, h(t_0)/2],$$
$$u(t_0, x) \geq K_2 \geq \underline{u}(0, x) \quad \text{for } x \in [-\underline{h}(0), \underline{h}(0)].$$

Moreover, since $J(x)$ and $\underline{u}(t,x)$ are both even in x, (5.8) implies

$$-\underline{h}'(t) \geq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x-y)\underline{u}(t,x)\,\mathrm{d}y\,\mathrm{d}x \quad \text{for } t > 0.$$

These combined with the estimates in Steps 1 and 2 allow us to apply the comparison principle to conclude that

$$[-\underline{h}(t), \underline{h}(t)] \subset [g(t+t_0), h(t+t_0)], \quad t \geq 0,$$
$$\underline{u}(t, x) \geq u(t+t_0, x), \qquad\qquad t \geq 0, \ x \in [-\underline{h}(t), \underline{h}(t)].$$

Hence (5.7) holds. ∎

5.2.2. The case $\gamma = 2$. The following simple result will play an important role in our analysis later.

Lemma 5.4. *Let l_1 and l_2 with $0 < l_1 < l_2$ be two constants, and define*

$$\psi(x) = \psi(x; l_1, l_2) := \min\left\{ 1, \frac{l_2 - |x|}{l_1} \right\}, \quad x \in \mathbb{R}.$$

If J satisfies (**J**)*, then for any $\varepsilon > 0$, there is $L_\varepsilon > 0$ such that for all $l_1 > L_\varepsilon$ and $l_2 - l_1 > L_\varepsilon$,*

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\,\mathrm{d}y \geq (1-\varepsilon)\psi(x) \quad \text{in } [-l_2, l_2]. \tag{5.11}$$

Proof. Since $\int_{\mathbb{R}} J(x)\, dx = 1$, there exists $B > 0$ such that

$$\int_{-B}^{B} J(x)\, dx > 1 - \varepsilon/2. \tag{5.12}$$

In the following discussion we always assume that $l_1 \gg B$ and $l_2 - l_1 \gg B$. Clearly, for $x \in [-(l_2 - l_1) + B, (l_2 - l_1) - B]$, due to

$$\psi(x) = 1 \quad \text{in } [-(l_2 - l_1), l_2 - l_1],$$

we have

$$
\begin{aligned}
\int_{-l_2}^{l_2} J(x - y)\psi(y)\, dy &\geq \int_{-(l_2 - l_1)}^{l_2 - l_1} \tilde{J}(x - y)\psi(y)\, dy = \int_{-(l_2 - l_1)}^{l_2 - l_1} \tilde{J}(x - y)\, dy \\
&= \int_{-(l_2 - l_1) - x}^{l_2 - l_1 - x} J(y)\, dy \geq \int_{-B}^{B} \tilde{J}(y)\, dy \\
&\geq 1 - \varepsilon/2 > (1 - \varepsilon)\psi(x).
\end{aligned}
$$

It remains to prove (5.11) for $x \in [-l_2, -(l_2 - l_1) + B] \cup [(l_2 - l_1) - B, l_2]$. By the symmetric property of $\psi(x)$ and $J(x)$ with respect to x, we just need to verify (5.11) for $x \in [(l_2 - l_1) - B, l_2]$, which will be carried out according to the following three cases:

(i) $x \in [l_2 - l_1 - B, l_2 - l_1 + B]$,   (ii) $x \in [l_2 - l_1 + B, l_2 - B]$,   (iii) $x \in [l_2 - B, l_2]$.

(i) For $x \in [l_2 - l_1 - B, l_2 - l_1 + B]$, since $\psi(z)$ is nonincreasing for $z \geq 0$, we have

$$
\begin{aligned}
\int_{-l_2}^{l_2} J(x - y)\psi(y)\, dy &= \int_{-l_2 - x}^{l_2 - x} J(y)\psi(y + x)\, dy \\
&\geq \int_{-2l_2 + l_1 + B}^{B} J(y)\psi(y + x)\, dy \geq \int_{-B}^{B} J(y)\psi(y + x)\, dy \\
&\geq \int_{-B}^{B} J(y)\psi(y + l_2 - l_1 + B)\, dy.
\end{aligned}
$$

By the definition of ψ, for $y \in [-B, B]$ we have

$$\psi(y + l_2 - l_1 + B) = \frac{l_2 - (y + l_2 - l_1 + B)}{l_1} = 1 - \frac{y + B}{l_1}.$$

Hence,

$$
\begin{aligned}
\int_{-B}^{B} J(y)\psi(y + l_2 - l_1 + B)\, dy &= \int_{-B}^{B} J(y)\, dy - \int_{-B}^{B} J(y)\frac{y + B}{l_1}\, dy \\
&\geq 1 - \varepsilon/2 - \|J\|_{L^\infty(\mathbb{R})}\frac{2B^2}{l_1} \geq 1 - \varepsilon \geq (1 - \varepsilon)\psi(x)
\end{aligned}
$$

provided

$$l_1 \geq 4\|J\|_{L^\infty(\mathbb{R})}B^2/\varepsilon,$$

which then gives

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\,dy \geq (1-\varepsilon)\psi(x) \quad \text{for } x \in [l_2 - l_1 - B, l_2 - l_1 + B].$$

(ii) For $x \in [l_2 - l_1 + B, l_2 - B]$,

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\,dy = \int_{-l_2-x}^{l_2-x} J(y)\psi(y+x)\,dy$$

$$\geq \int_{-2l_2-B+l_1}^{B} J(y)\psi(y+x)\,dy \geq \int_{-B}^{B} J(y)\psi(y+x)\,dy.$$

From the definition of ψ, for $x \in [l_2 - l_1 + B, l_2 - B]$ and $y \in [-B, B]$,

$$\psi(y+x) = \frac{l_2 - (y+x)}{l_1} = \frac{l_2 - x}{l_1} - \frac{y}{l_1} = \psi(x) - \frac{y}{l_1}.$$

Thus, by (5.12),

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\,dy \geq \int_{-B}^{B} J(y)\psi(y+x)\,dy$$

$$= \psi(x) \int_{-B}^{B} J(y)\,dy - \int_{-B}^{B} J(y)\frac{y}{l_1}\,dy = \psi(x) \int_{-B}^{B} J(y)\,dy \geq (1-\varepsilon)\psi(x).$$

(iii) For $x \in [l_2 - B, l_2]$,

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\,dy = \int_{-l_2-x}^{l_2-x} J(y)\psi(y+x)\,dy$$

$$\geq \int_{-2l_2-B+l_1}^{l_2-x} J(y)\psi(y+x)\,dy \geq \int_{-B}^{l_2-x} J(y)\psi(y+x)\,dy$$

$$= \int_{-B}^{B} J(y)\psi(y+x)\,dy - \int_{l_2-x}^{B} J(y)\psi(y+x)\,dy$$

As in (ii), we see that

$$\int_{-B}^{B} J(y)\psi(y+x)\,dy = \psi(x) \int_{-B}^{B} J(y)\,dy \geq (1-\varepsilon)\psi(x).$$

By the definition of ψ,

$$\psi(y+x) \leq 0 \quad \text{for } x \in [l_2 - B, l_2],\ y \in [l_2 - x, B],$$

which indicates

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\,dy \geq \int_{-B}^{B} J(y)\psi(y+x)\,dy \geq (1-\varepsilon)\psi(x).$$

The proof is now complete. ∎

Lemma 5.5. *If the conditions in Theorem* 1.2 *are satisfied and* $\gamma = 2$, *then there exists* $C > 0$ *such that*

$$h(t) \geq Ct \ln t \quad for\ t \gg 1. \tag{5.13}$$

Proof. Fix $\beta \in (0, 1)$ and define

$$
\begin{cases}
\underline{h}(t) := K_1(t + \theta) \ln(t + \theta), & t \geq 0, \\
\underline{u}(t, x) := K_2 \min\left\{1, \dfrac{\underline{h}(t) - |x|}{(t + \theta)^\beta}\right\}, & t \geq 0,\ x \in [-\underline{h}(t), \underline{h}(t)],
\end{cases}
$$

with constants $\theta \gg 1$ and $1 \gg K_1 > 0, 1 \gg K_2 > 0$ to be determined. Obviously, for any $t > 0$, the function $\partial_t \underline{u}(t, x)$ exists for $x \in [-\underline{h}(t), \underline{h}(t)]$ except when $|x| = \underline{h}(t) - (t + \theta)^\beta$. However, the one-sided partial derivates $\partial_t \underline{u}(t \pm 0, x)$ always exist.

Step 1. We show that by choosing θ and K_1, K_2 suitably,

$$\underline{h}'(t) \leq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{for } t > 0, \tag{5.14}$$

$$-\underline{h}'(t) \geq -\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x - y)\underline{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x \quad \text{for } t > 0. \tag{5.15}$$

Since $\underline{u}(t, x) = \underline{u}(t, -x)$ and $J(x) = J(-x)$, we see that (5.15) follows from (5.14).

By elementary calculations and (5.1), we have

$$\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x \geq \mu \int_0^{\underline{h}(t) - (t+\theta)^\beta} \int_{\underline{h}(t)}^{\infty} J(x - y)\underline{u}(t, x)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \mu K_2 \int_{-\underline{h}(t)}^{-(t+\theta)^\beta} \int_0^{\infty} J(x - y)\,\mathrm{d}y\,\mathrm{d}x = \mu K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_x^{\infty} J(y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \mu K_2 \left(\int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{(t+\theta)^\beta}^y + \int_{\underline{h}(t)}^{\infty} \int_{(t+\theta)^\beta}^{\underline{h}(t)} \right) J(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq \mu K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{(t+\theta)^\beta}^y J(y)\,\mathrm{d}x\,\mathrm{d}y \geq \mu C_1 K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \frac{y - (t + \theta)^\beta}{y^2 + 1}\,\mathrm{d}y$$

$$\geq \mu C_1 K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \frac{y - (t + \theta)^\beta}{2y^2}\,\mathrm{d}y$$

$$= \mu C_1 K_2 \frac{1}{2}\left(\ln \underline{h}(t) - \beta \ln(t + \theta) + \frac{(t + \theta)^\beta}{\underline{h}(t)} - 1 \right)$$

$$\geq \mu C_1 K_2 \frac{1}{2}(\ln \underline{h}(t) - \beta \ln(t + \theta) - 1)$$

$$= \mu C_1 K_2 \frac{1}{2}\left(\ln K_1 + \ln(t + \theta) + \ln(\ln(t + \theta)) - \beta \ln(t + \theta) - 1 \right)$$

$$\geq \frac{\mu C_1 K_2 (1 - \beta)}{2}[\ln(t + \theta) + 1] \geq K_1 \ln(t + \theta) + K_1 = \underline{h}'(t)$$

provided

$$\ln(\ln \theta) \geq -\ln K_1 + 2 \quad \text{and} \quad 0 < K_1 \leq \mu C_1 K_2 (1 - \beta)/2, \tag{5.16}$$

which then finishes the proof of Step 1.

Step 2. We show that by choosing K_1, K_2 and θ suitably, for $t > 0$ and $x \in [-\underline{h}(t), \underline{h}(t)]$ with $|x| \neq \underline{h}(t) - (t + \theta)^\beta$,

$$\underline{u}_t(t, x) \leq d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\, dy - d\underline{u}(t, x) + f(\underline{u}(t, x)). \tag{5.17}$$

From the definition of $\underline{u}$, for $t > 0$,

$$\underline{u}_t(t, x) = \begin{cases} K_1 K_2 \frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta} + \frac{K_2 \beta |x|}{(t+\theta)^{1+\beta}} & \text{if } \underline{h}(t) - (t + \theta)^\beta < |x| \leq \underline{h}(t), \\ 0 & \text{if } |x| < \underline{h}(t) - (t + \theta)^\beta. \end{cases}$$

Claim 1. *For $x \in [-\underline{h}(t), -\underline{h}(t) + (t + \theta)^\beta] \cup [\underline{h}(t) - (t + \theta)^\beta, \underline{h}(t)]$ and large θ,*

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\, dy \geq \frac{C_1 K_2 \beta \ln(t + \theta)}{4(t + \theta)^\beta}, \tag{5.18}$$

where $C_1 > 0$ is given by (5.1).

A simple calculation yields, for $x \in [\underline{h}(t) - (t + \theta)^\beta, \underline{h}(t)]$,

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\, dy \geq K_2 \int_{\underline{h}(t)-(t+\theta)^\beta}^{\underline{h}(t)} J(x - y)\frac{\underline{h}(t) - y}{(t + \theta)^\beta}\, dy$$

$$= \frac{K_2}{(t + \theta)^\beta} \int_{\underline{h}(t)-(t+\theta)^\beta-x}^{\underline{h}(t)-x} J(y)[\underline{h}(t) - (y + x)]\, dy.$$

Hence, for $x \in [\underline{h}(t) - \frac{3}{4}(t + \theta)^\beta, \underline{h}(t)]$, by simple calculations and (5.1),

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y)\, dy \geq \frac{K_2}{(t + \theta)^\beta} \int_{-(t+\theta)^\beta/4}^{0} J(y)(-y)\, dy$$

$$= \frac{K_2}{(t + \theta)^\beta} \int_{0}^{(t+\theta)^\beta/4} J(y)y\, dy \geq \frac{C_1 K_2}{(t + \theta)^\beta} \int_{0}^{(t+\theta)^\beta/4} \frac{y}{y^2 + 1}\, dy$$

$$\geq \frac{C_1 K_2}{2(t + \theta)^\beta} \int_{1}^{(t+\theta)^\beta/4} y^{-1}\, dy = \frac{C_1 K_2}{2(t + \theta)^\beta} [\beta \ln(t + \theta) - \ln 4]$$

$$\geq \frac{C_1 K_2 \beta \ln(t + \theta)}{4(t + \theta)^\beta}$$

provided that

$$\frac{\beta}{2} \ln \theta \geq \ln 4. \tag{5.19}$$

And for $x \in [\underline{h}(t) - (t+\theta)^\beta, \underline{h}(t) - \frac{3}{4}(t+\theta)^\beta]$,

$$
\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\,\mathrm{d}y \geq \frac{K_2}{(t+\theta)^\beta} \int_0^{3(t+\theta)^\beta/4} J(y)[\underline{h}(t) - (y+x)]\,\mathrm{d}y
$$

$$
\geq \frac{K_2}{(t+\theta)^\beta} \int_0^{(t+\theta)^\beta/4} J(y)y\,\mathrm{d}y \geq \frac{C_1 K_2 \beta \ln(t+\theta)}{4(t+\theta)^\beta}.
$$

This proves (5.18) for $x \in [\underline{h}(t) - (t+\theta)^\beta, \underline{h}(t)]$.

For $x \in [-\underline{h}(t), -\underline{h}(t) + (t+\theta)^\beta]$, (5.10) also holds since both $J(x)$ and $\underline{u}(t,x)$ are even in x. Claim 1 is thus proved.

Claim 2. *We can choose small K_2 and large θ such that, for $x \in [-\underline{h}(t), \underline{h}(t)]$,*

$$
d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\,\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)) \geq F_* \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\,\mathrm{d}y
\tag{5.20}
$$

for some $F_* > 0$.

For small $K_2 > 0$, from $0 \leq \underline{u}(t,x) \leq K_2$ we obtain

$$
f(\underline{u}(t,x)) \geq \tfrac{3}{4} f'(0)\underline{u}(t,x).
$$

For large θ and $t \geq 0$, we have

$$
\underline{h}(t) - (t+\theta)^\beta \geq \theta^\beta (K_1 \theta^{1-\beta} \ln\theta - 1) \geq \theta^\beta.
\tag{5.21}
$$

Hence, by (5.11), there is large $L_1 > 0$ such that, for $\theta^\beta > L_1$,

$$
d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\,\mathrm{d}y + \frac{f'(0)}{4}\underline{u}(t,x) \geq d\underline{u}(t,x) \quad \text{for } x \in [-\underline{h}(t), \underline{h}(t)].
$$

Therefore (5.20) holds with $F_* = f'(0)/2$.

Applying (5.18) and (5.20), for $x \in [-\underline{h}(t), -\underline{h}(t) + (t+\theta)^\beta) \cup (\underline{h}(t) - (t+\theta)^\beta, \underline{h}(t)]$ we have

$$
d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\,\mathrm{d}y - \underline{u}(t,x) + f(\underline{u}(t,x))
$$

$$
\geq \frac{F_* C_1 K_2 \beta \ln(t+\theta)}{4(t+\theta)^\beta} \geq K_1 K_2 \frac{\ln(t+\theta)+1}{(t+\theta)^\beta}
$$

$$
= K_1 K_2 \frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta} + \frac{K_2 \beta \underline{h}(t)}{(t+\theta)^{1+\beta}}
$$

$$
\geq K_1 K_2 \frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta} + \frac{K_2 \beta |x|}{(t+\theta)^{1+\beta}}
$$

$$
= \underline{u}_t(t,x)
$$

if apart from the earlier requirements, we further assume

$$\ln \theta > 2 \quad \text{and} \quad K_1 \leq F_* C_1 \beta / 8. \tag{5.22}$$

For $|x| < \underline{h}(t) - (t + \theta)^\beta$, $\underline{u}(t, y) = K_2$ and

$$d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y) \underline{u}(t, y) \, dy - d \underline{u}(t, x) + f(\underline{u}(t, x))$$

$$\geq F_* \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x - y) \underline{u}(t, y) \, dy \geq 0 = \underline{u}_t(t, x).$$

Thus (5.17) holds. (Let us stress that it is possible to find K_1, K_2 and large θ such that (5.16), (5.19), (5.21) and (5.22) hold simultaneously.)

Step 3. We finally prove (5.13).

Clearly, $\underline{u}(t, \pm \underline{h}(t)) = 0$ for $t \geq 0$. Since spreading happens for (u, g, h) and $K_2 > 0$ is small, there is a large constant $t_0 > 0$ such that

$$[-\underline{h}(0), \underline{h}(0)] \subset [g(t_0)/2, h(t_0)/2],$$
$$\underline{u}(0, x) \leq K_2 \leq u(t_0, x) \quad \text{for } x \in [-\underline{h}(0), \underline{h}(0)].$$

By [21, Remark 2.4], we see that the comparison principle still applies to our situation here, even though $\partial \underline{u}_t(t, x)$ has a jumping discontinuity at $|x| = \underline{h}(t) - (t + \theta)^\beta$. Therefore we have

$$[-\underline{h}(t), \underline{h}(t)] \subset [g(t + t_0), h(t + t_0)], \quad t \geq 0,$$
$$\underline{u}(t, x) \leq u(t + t_0, x), \quad\quad\quad\quad t \geq 0, \ x \in [-\underline{h}(t), \underline{h}(t)].$$

So (5.13) holds. This completes the proof of the lemma. ∎

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