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Cohomology of the moduli of Higgs bundles on a curve via positive characteristic

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Abstract. For a curve of genus g and any two degrees coprime to the rank, we construct a family of ring isomorphisms parameterized by the complex Lie group $\mathrm{GSp}(2g)$, between the cohomology of the moduli spaces of stable Higgs bundles which preserve the perverse filtrations. As a consequence, we prove two structural results concerning the cohomology of Higgs moduli which are predicted by the P = W Conjecture in Non-Abelian Hodge Theory: (1) Galois conjugation for character varieties preserves the perverse filtrations for the corresponding Higgs moduli spaces. (2) The restriction of the Hodge–Tate decomposition for a character variety to each piece of the perverse filtration for the corresponding Higgs moduli space also gives a decomposition.

Our proof uses reduction to positive characteristic and relies on the non-abelian Hodge correspondence in characteristic p between Dolbeault and de Rham moduli spaces.

Keywords. Higgs bundles, non-abelian Hodge, the P = W conjecture

1. Introduction

1.1. Cohomology of the moduli of Higgs bundles

Let C be a non-singular irreducible projective curve over $\mathbb{C}$ of genus $g \geq 2$. Throughout the paper, we fix a positive integer $n > 0$. For any positive integer d coprime to n, we denote by $M_{\mathrm{Dol}}(C, d)$ the moduli of (slope-)stable Higgs bundles of rank n and degree d:

$$(\mathcal{E}, \theta) : \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega_C, \quad \mathrm{rank}(\mathcal{E}) = n,\ \deg(\mathcal{E}) = d.$$

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It is a non-singular quasi-projective variety which admits a Lagrangian fibration

$$h : M_{\mathrm{Dol}}(C,d) \to A := \bigoplus_{i=1}^{n} H^0(C, \Omega_C^{\otimes i}), \quad (\mathcal{E},\theta) \mapsto \mathrm{char}(\theta) \in A,$$

known as the *Hitchin fibration* [19, 20]. Here $\mathrm{char}(\theta)$ stands for the coefficients of the characteristic polynomial of the Higgs field θ, i.e.,

$$\mathrm{char}(\theta) = (a_1, \ldots, a_n), \quad a_i := \mathrm{trace}(\textstyle\bigwedge^i \theta) \in H^0(C, \Omega_C^{\otimes i}).$$

The singular cohomology of $M_{\mathrm{Dol}}(C,d)$ carries an increasing filtration

$$P_0 H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}) \subset P_1 H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}) \subset \cdots \subset H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}),$$

called the *perverse filtration*, which is governed by the topology of the Hitchin fibration [8].

The cohomology, along with the perverse filtration, has been the subject of intense study in recent years, motivated by the so-called P = W Conjecture in Non-Abelian Hodge Theory, which we recall in Section 1.3.

1.2. Main results

Before stating our main results, we first recall a natural set of generators for the cohomology algebra of $M_{\mathrm{Dol}}(C,d)$.

Let $(\mathcal{U}, \theta)$ be a universal family over $C \times M_{\mathrm{Dol}}(C,d)$. A natural way to construct cohomology classes on $M_{\mathrm{Dol}}(C,d)$ is to integrate the k-th component $\mathrm{ch}_k(\mathcal{U})$ over a class γ on C, so that the resulting class is determined by $k \in \mathbb{N}$ and $\gamma \in H^*(C,\mathbb{C})$. While the choice of a universal family is not unique, we may normalize the universal family to get tautological classes

$$c(\gamma,k) \in H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}), \quad \gamma \in H^*(C,\mathbb{C}),\ k \in \mathbb{N}, \tag{1}$$

so that they do not depend on the choice of a universal family; see [10, Section 0.3] and Section 2.1 below. Markman [22] proved that the tautological classes (1) generate $H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C})$ as a $\mathbb{C}$-algebra.

Let $\Lambda := H^1(C,\mathbb{Z})$ be the lattice with the intersection pairing, and let $\Lambda_{\mathbb{K}}$ be $\Lambda \otimes \mathbb{K}$ for $\mathbb{K}$ a field. We consider the similitude group

$$\mathrm{GSp}(\Lambda_{\mathbb{C}}) = \{A \in \mathrm{GL}_{2g}(\Lambda_{\mathbb{C}}) \mid \exists \lambda_A \in \mathbb{C}^*,\ \langle Aw_1, Aw_2\rangle = \lambda_A \langle w_1, w_2\rangle,\ \forall w_1, w_2 \in \Lambda_{\mathbb{C}}\}, \tag{2}$$

which is a subgroup of $\mathrm{GL}_{2g}(\Lambda_{\mathbb{C}})$. Here, $\langle -,- \rangle$ is the intersection pairing on the vector space $\Lambda_{\mathbb{C}}$, and λ_A is a nonzero constant uniquely determined by A. The symplectic group $\mathrm{Sp}(\Lambda_{\mathbb{C}}) \subset \mathrm{GSp}(\Lambda_{\mathbb{C}})$ is characterized by $\lambda_A = 1$. The natural action of $\mathrm{GSp}(\Lambda_{\mathbb{C}})$ on $\Lambda_{\mathbb{C}}$ admits an extension to the total cohomology of the curve

$$H^*(C,\mathbb{C}) = H^0(C,\mathbb{C}) \oplus \Lambda_{\mathbb{C}} \oplus H^2(C,\mathbb{C})$$

which acts as id on $H^2(C,\mathbb{C})$ and as multiplication by λ_A on $H^0(C,\mathbb{C})$. We note that this choice of extension of the $\mathrm{GSp}(\Lambda_{\mathbb{C}})$-action to $H^*(C,\mathbb{C})$ is not the natural extension compatible with cup product; nevertheless, this choice is essential for our purpose in view of Theorem 4.1; see Remark 4.2. For $A \in \mathrm{GSp}(\Lambda_{\mathbb{C}})$ and $\gamma \in H^*(C,\mathbb{C})$, we use $A\gamma \in H^*(C,\mathbb{C})$ to denote the class given by the action of A on γ.

The main result of this paper is the following:

Theorem 1.1. (a) *For each integer d coprime to n, there exists an action of* $\mathrm{GSp}(\Lambda_{\mathbb{C}})$ *on $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$ by $\mathbb{C}$-algebra automorphisms. For each $A \in \mathrm{GSp}(\Lambda_{\mathbb{C}})$, the corresponding automorphism G_A acts on tautological generators compatibly with the action on $H^*(C,\mathbb{C})$:*

$$G_A(c(\gamma,k)) = c(A\gamma,k).$$

Furthermore, it preserves the perverse filtrations:

$$G_A\big(P_k H^i(M_{\mathrm{Dol}}(C,d),\mathbb{C})\big) = P_k H^i(M_{\mathrm{Dol}}(C,d'),\mathbb{C}), \quad \forall k,i.$$

(b) *Given two integers d and d' coprime to n, there exists a $\mathbb{C}$-algebra isomorphism*

$$\phi_{d,d'} : H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C,d'),\mathbb{C})$$

which acts on tautological generators via the relation

$$\phi_{d,d'}(c(\gamma,k)) = c(\gamma,k).$$

Furthermore, this isomorphism intertwines the actions of $\mathrm{GSp}(\Lambda_{\mathbb{C}})$ *constructed in part* (a) *for d and d', and also preserves the perverse filtrations.*

We note that the action described above is uniquely characterized by its action on the tautological generators. However, it is not obvious that such an action exists, i.e., that it is compatible with the relations between these tautological generators or preserves the perverse filtration.

In part (a), if we restrict to the arithmetic subgroup $\mathrm{Sp}(\Lambda) \subset \mathrm{GSp}(\Lambda_{\mathbb{C}})$, the corresponding action can be constructed geometrically as we describe next. If we take the universal curve $\mathcal{C} \to \mathcal{M}_g$, then the monodromy of the corresponding family of Higgs moduli spaces defines an action of the mapping class group $\pi_1(\mathcal{M}_g)$ on $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$. This action is compatible with perverse filtrations by the main result of [9]. Finally, since the kernel of the symplectic representation of the mapping class group has no effect on the tautological generators, this action descends to an action of $\mathrm{Sp}(\Lambda)$.

We do not know a direct construction of the assignment $A \mapsto G_A$ for $A \in \mathrm{GSp}(\Lambda_{\mathbb{C}})$. Instead, our approach is to use techniques from positive characteristic to construct some of these G_A, and to then couple this construction with density arguments.

1.3. Implications for the P = W Conjecture

The motivation for our results in this paper comes from the P = W Conjecture in Non-Abelian Hodge Theory, proposed in 2010 by de Cataldo, Hausel, and Migliorini [8]. This conjecture predicts that the topology of the Hitchin fibration (particularly the perverse filtration) interacts surprisingly, via *Non-Abelian Hodge Theory*, with the Hodge theory of the corresponding character variety (particularly the weight filtration).

More precisely, we consider the character variety $M_{\mathrm{B}}(C,d)$ of rank n and degree d:

$$M_{\mathrm{B}}(C,d) := \Big\{(a_1,\dots,a_g,b_1,\dots,b_g) \,\Big|\, a_k,b_k \in \mathrm{GL}_n \text{ for } k=1,\dots,g, \\ \prod_{j=1}^{g}[a_j,b_j] = \zeta_n^d\cdot \mathrm{Id}_n\Big\} /\!\!/ \mathrm{GL}_n, \quad \zeta_n := e^{\frac{2\pi\sqrt{-1}}{n}},$$

obtained as an affine GIT quotient with respect to the action by conjugation. Non-Abelian Hodge Theory [25] (see [17] for the twisted case) then induces a diffeomorphism between the moduli spaces $M_{\mathrm{Dol}}(C,d)$ and $M_{\mathrm{B}}(C,d)$, which identifies their cohomology rings:

$$H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) = H^*(M_{\mathrm{B}}(C,d),\mathbb{C}). \tag{3}$$

The P = W Conjecture refines (3) incorporating the perverse filtration for $M_{\mathrm{Dol}}(C,d)$ and the mixed Hodge structure for $M_{\mathrm{B}}(C,d)$; it predicts that

$$P_kH^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) = W_{2k}H^*(M_{\mathrm{B}}(C,d),\mathbb{C}) = W_{2k+1}H^*(M_{\mathrm{B}}(C,d),\mathbb{C}), \quad \forall k \tag{4}$$

with $W_\bullet$ the weight filtration.

The P = W Conjecture makes the following predictions for the perverse filtration, which we will deduce as consequences of our main theorem.

1.3.1. Galois. The first prediction comes from Galois conjugation on the character variety.

Fix n, and consider two integers d,d' coprime to n. The Betti moduli spaces $M_{\mathrm{B}}(C,d)$ and $M_{\mathrm{B}}(C,d')$ are Galois conjugate via an automorphism of $\mathbb{Q}[\zeta]$ sending ζ^d to $\zeta^{d'}$ [16, Section 4]. Galois conjugation induces an isomorphism preserving the weight filtrations:

$$\widetilde{\phi}_{d,d'} : H^*(M_{\mathrm{B}}(C,d),\mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{B}}(C,d'),\mathbb{C}), \quad W_k \mapsto W_k. \tag{5}$$

By passing through the non-abelian Hodge isomorphism (3), this induces a ring isomorphism between the cohomology groups of the Dolbeault moduli spaces:

$$\widetilde{\phi}_{d,d'} : H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C,d'),\mathbb{C}), \tag{6}$$

which, assuming the P = W Conjecture, should preserve the respective perverse filtrations.

The first consequence of our main theorem is that this is indeed the case:

Theorem 1.2. *The Galois conjugation $\widetilde{\phi}_{d,d'}$ of (6) preserves perverse filtrations:*

$$\widetilde{\phi}_{d,d'}\big(P_kH^i(M_{\mathrm{Dol}}(C,d),\mathbb{C})\big) = P_kH^i(M_{\mathrm{Dol}}(C,d'),\mathbb{C}), \quad \forall k,i. \tag{7}$$

We will show this in Sections 2.2 and 6.2 by matching $\widetilde{\phi}_{d,d'}$ of (6) and $\phi_{d,d'}$ of Theorem 1.1 (b). The following corollary of Theorem 1.2 is immediate.

Corollary 1.3. *For fixed rank n, if the P = W Conjecture holds for some degree d coprime to n, then it holds for every degree d' coprime to n.*

1.3.2. Weight decomposition. The second prediction of Theorem 1.1 relates the perverse filtration to the weight decomposition of the character variety.

The Hodge structure for $M_{\mathrm{B}}(C,d)$ was shown to be of Hodge–Tate type [24], and we have a canonical Hodge–Tate decomposition

$$H^*(M_{\mathrm{B}}(C,d),\mathbb{C}) = \bigoplus_{i,k} \mathrm{Hdg}^i_k, \quad \mathrm{Hdg}^i_k := W_{2k} \cap F^k\big(H^i(M_{\mathrm{B}}(C,d),\mathbb{C})\big). \tag{8}$$

In particular, we have a canonical decomposition $W_{2k}H^i = \bigoplus_{k' \leq k} \mathrm{Hdg}^i_{k'}$ for the weight filtration. The P = W Conjecture implies immediately the same equality with P_k replacing W_{2k}.

Using our main Theorem 1.1, we show the following weaker compatibility of the perverse filtration on the Dolbeault side, with the Hodge–Tate decomposition (8) on the Betti side.

Theorem 1.4. *We have the following compatibility between the decomposition (8) and the perverse filtration:*

$$P_k H^i(M_{\mathrm{Dol}}(C,d),\mathbb{C}) = \bigoplus_{k'} (P_k H^i(M_{\mathrm{Dol}}(C,d),\mathbb{C})) \cap \mathrm{Hdg}^i_{k'}.$$

Here we use the non-abelian Hodge isomorphism (3) to transfer the decomposition (8) to $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$. In general, the restriction of a decomposition of a vector space $Q = \bigoplus_i Q_i$ to a vector subspace $Q' \subset Q$ may fail to be a decomposition, i.e., in general $Q' \neq \bigoplus_i (Q' \cap Q_i)$. We prove that instead this is the case for the subspaces of the perverse filtration in Section 6.2, by relating the $\mathbb{G}_{\mathrm{m}}$ which induces the Hodge–Tate decomposition with the central $\mathbb{G}_{\mathrm{m}}$ in the GSp-action we construct.

Finally, the perverse filtration admits a natural splitting, known as the first Deligne splitting. In Section 6.3, we explain that the operators of our main theorem are compatible with this splitting; see Corollary 6.1.

2. Tautological classes

2.1. Tautological classes

Let C be a complex curve as in the introduction; we assume that $(n,d) = 1$. We first review the construction of the tautological classes

$$c(\gamma,k) \in H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}), \quad \gamma \in H^*(C,\mathbb{C}),\ k \in \mathbb{Z}_{\geq 0},$$

as integrals of normalized classes, following [10]. In this paper, as is standard, when we push forward cohomology classes, it is always for proper l.c.i. morphisms, and we use the proper pushforward in cohomology stemming from Chow bi-variant theory coupled with the cycle class map, as in [14].

Let

$$p_C : C \times M_{\mathrm{Dol}}(C,d) \to C, \quad p_M : C \times M_{\mathrm{Dol}}(C,d) \to M_{\mathrm{Dol}}(C,d)$$

be the projections. We say that a triple $(\mathcal{U}, \theta, \alpha)$ is a *twisted universal family* over $C \times M_{\mathrm{Dol}}(C,d)$ if $(\mathcal{U}, \theta)$ is a universal family and

$$\alpha = p_C^* \alpha_C + p_M^* \alpha_M \in H^2(C \times M_{\mathrm{Dol}}(C,d), \mathbb{C}) \tag{9}$$

with $\alpha_C \in H^2(C, \mathbb{C})$ and $\alpha_M \in H^2(M_{\mathrm{Dol}}(C,d), \mathbb{C})$. For a twisted universal family $(\mathcal{U}, \theta, \alpha)$, we define the *twisted Chern character* $\mathrm{ch}^\alpha(\mathcal{U})$ as

$$\mathrm{ch}^\alpha(\mathcal{U}) = \mathrm{ch}(\mathcal{U}) \cup \exp(\alpha) \in H^*(C \times M_{\mathrm{Dol}}(C,d), \mathbb{C}),$$

and we denote by

$$\mathrm{ch}_k^\alpha(\mathcal{U}) \in H^{2k}(C \times M_{\mathrm{Dol}}(C,d), \mathbb{C})$$

its degree $2k$ part. The class $\mathrm{ch}^\alpha(\mathcal{U})$ is called *normalized* if

$$\begin{aligned} &\mathrm{ch}_1^\alpha(\mathcal{U})|_{x \times M_{\mathrm{Dol}}(C,d)} = 0 \in H^2(M_{\mathrm{Dol}}(C,d), \mathbb{C}), \\ &\mathrm{ch}_1^\alpha(\mathcal{U})|_{C \times y} = 0 \in H^2(C, \mathbb{C}), \end{aligned} \tag{10}$$

with $x \in C$ and $y \in M_{\mathrm{Dol}}(C,d)$ points. For two universal families $(\mathcal{U}_1, \theta_1)$ and $(\mathcal{U}_2, \theta_2)$, there is a line bundle $\mathcal{L}$ pulled back from $M_{\mathrm{Dol}}(C,d)$ so that $\mathcal{U}_1 = \mathcal{U}_2 \otimes \mathcal{L}$. By condition (10), normalized classes do not depend on the choice of a universal family. For any $\gamma \in H^i(C, \mathbb{Q})$, the *tautological class* $c(\gamma, k)$ is defined by integrating the degree k normalized class

$$c(\gamma, k) := \int_\gamma \mathrm{ch}_k^\alpha(\mathcal{U}) = p_{M*}(p_C^* \gamma \cup \mathrm{ch}_k^\alpha(\mathcal{U})) \in H^{i+2k-2}(M_{\mathrm{Dol}}(C,d), \mathbb{C}). \tag{11}$$

There is also an alternative way in [17] to obtain canonically defined classes in $H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C})$ which we briefly review; this will only be used in Section 2.2 to characterize the action of Galois conjugation on the tautological classes of the character varieties. We let $\mathcal{T}$ be the projective bundle $\mathbb{P}(\mathcal{U})$ associated with *any* universal family $(\mathcal{U}, \theta)$. If we assume that $\xi_1, \ldots, \xi_n$ are the Chern roots of $\mathcal{U}$, then the Chern roots for $\mathcal{T}$ are

$$\xi_1 - \overline{\xi},\ \xi_2 - \overline{\xi},\ \ldots,\ \xi_n - \overline{\xi},$$

with $\overline{\xi}$ the average of the ξ_i. We may consider Chern classes $c_k(\mathcal{T})$ and Chern characters $\mathrm{ch}_k(\mathcal{T})$ via the Chern roots. In particular, $c_1(\mathcal{T}) = 0$. For any twisted universal family $(\mathcal{U}, \theta, \alpha)$, we have

$$\mathrm{ch}^\alpha(\mathcal{U}) = \mathrm{ch}(\mathcal{T}) \cup \exp\left(\frac{c_1(\mathcal{U})}{n} + \alpha \right), \tag{12}$$

and for a normalized class $\mathrm{ch}^\alpha(\mathcal{U})$ we have

$$\mathrm{ch}_1^\alpha(\mathcal{U}) = c_1(\mathcal{U}) + n\alpha \in H^1(C, \mathbb{C}) \otimes H^1(M_{\mathrm{Dol}}(C,d), \mathbb{C}).$$

Therefore the degree 1 tautological classes

$$c(\gamma, 1) \in H^1(M_{\mathrm{Dol}}(C,d), \mathbb{C}), \quad \gamma \in H^1(C, \mathbb{C}), \tag{13}$$

recover all the classes in $H^1(M_{\mathrm{Dol}}(C,d), \mathbb{C})$, and by (12) any tautological class (11) can be expressed in terms of (13) and

$$\int_\gamma c_k(\mathcal{T}) \in H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}), \quad \gamma \in H^*(C, \mathbb{C}),\ k \geq 2. \tag{14}$$

Remark 2.1. Recall that we have the product formula (see [8, the equation following Remark 2.4.4])

$$H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}) = H^*(\mathrm{Jac}(C), \mathbb{C}) \otimes H^*(\hat{M}_{\mathrm{Dol}}(C,d), \mathbb{C})$$

with $\hat{M}_{\mathrm{Dol}}(C,d)$ the PGL_n-Higgs moduli space and $\mathrm{Jac}(C)$ the Jacobian of the curve C. The classes (13) generate the first factor; in turn these are generated by the tautological classes associated with the normalized Poincaré line bundle over $C \times \mathrm{Jac}(C)$. The classes (14) generate the second factor.

2.2. *Character varieties*

In [17], Hausel and Thaddeus described the tautological classes directly on the character variety $M_{\mathrm{B}}(C,d)$ side. By their description, these classes are preserved under Galois conjugation (5).

Proposition 2.2. *For d and d' coprime to n, a morphism of $\mathbb{C}$-algebras*

$$H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C,d'), \mathbb{C})$$

is induced by the Galois conjugation (5) *if and only if it preserves the tautological classes* (11),

$$H^*(M_{\mathrm{Dol}}(C,d), \mathbb{C}) \ni c(\gamma, k) \mapsto c(\gamma, k) \in H^*(M_{\mathrm{Dol}}(C,d'), \mathbb{C}).$$

This was observed by Hausel [16, Remark 4.8]; we give the proof for the reader's convenience.

Proof of Proposition 2.2. We first review the construction of Hausel–Thaddeus [17]. We consider the map

$$\mu : \mathrm{GL}_n^{2g} \to \mathrm{GL}_n, \quad (a_1, \dots, a_g, b_1, \dots, b_g) \mapsto \prod_{j=1}^{g} [a_j, b_j],$$

so that $M_{\mathrm{B}}(C,d)$ is the geometric quotient of $\mu^{-1}(\zeta_n^d \,\mathrm{Id}_n)$ by the conjugation action. Now we follow [17, Section 1] to describe the principal PGL_n-bundle $\mathcal{T}_d'$ corresponding to $\mathcal{T}$ in Section 2.1.

Let $\widetilde{C} \to C$ be the universal cover with the natural $\pi_1(C)$-action on $\widetilde{C}$. Then there is an action of the group $\pi_1(C) \times \mathrm{GL}_n$ on the product

$$\mathrm{PGL}_n \times \mu^{-1}(\zeta_n^d \mathrm{Id}_n) \times \widetilde{C},$$

given by

$$(p, g) \cdot (h, \rho, x) = ([g]\rho(p)h, [g]\rho[g]^{-1}, p \cdot x).$$

Here $[g]$ is the projection of $g \in \mathrm{GL}_n$ to PGL_n, and we view $\rho \in \mu^{-1}(\zeta_n^d \mathrm{Id}_n)$ as a homomorphism $\pi_1(C) \to \mathrm{PGL}_n$. The resulting quotient, denoted by $\mathcal{T}'_d$, gives the desired PGL_n-principal bundle over the product $M_{\mathrm{B}}(C, d) \times C$. It corresponds to $\mathcal{T} = \mathbb{P}(\mathcal{U})$ on the Higgs side via the diffeomorphism given by Non-Abelian Hodge Theory; we refer to [17, Section 5] for more details.

The two complex varieties $M_B(C, d)$ and $M_B(C, d')$ can be obtained by base change of a scheme defined over $\mathbb{Q}[\zeta]$ via two complex embeddings $\mathbb{Q}[\zeta] \hookrightarrow \mathbb{C}$; the two embeddings differ by an automorphism of $\mathbb{Q}[\zeta]$ sending ζ^d to $\zeta^{d'}$. Hence we have identified the cohomology of the two moduli spaces as $\mathbb{C}$-algebras induced by this automorphism of $\mathbb{Q}[\zeta]$:

$$H^*(M_{\mathrm{B}}(C, d), \mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{B}}(C, d'), \mathbb{C}). \tag{15}$$

Furthermore, as an immediate consequence of the description of the bundle $\mathcal{T}'_d$, the pairs $(M_{\mathrm{B}}(C, d), \mathcal{T}'_d)$ and $(M_{\mathrm{B}}(C, d'), \mathcal{T}'_{d'})$ correspond via the automorphism $\zeta^d \mapsto \zeta^{d'}$ above. In particular, the isomorphism (15) preserves each class (14) if we pass through the non-abelian Hodge correspondence. Switching back to the tautological classes $c(\gamma, k)$, the "only if" direction is now clear. The "if" direction follows from Markman's theorem [22] that the classes $c(\gamma, k)$ generate the total cohomology as a $\mathbb{C}$-algebra. ■

2.3. *Change the degree by n*

Let $\mathcal{O}_C(1)$ be a degree 1 line bundle on the curve C. Taking tensor product with $\mathcal{O}_C(1)$ induces an isomorphism between the moduli spaces

$$M_{\mathrm{Dol}}(C, d) \xrightarrow{\simeq} M_{\mathrm{Dol}}(C, d+n), \quad (\mathcal{E}, \theta) \mapsto (\mathcal{E} \otimes \mathcal{O}_C(1), \theta). \tag{16}$$

Proposition 2.3. *The isomorphism of the cohomology*

$$H^*(M_{\mathrm{Dol}}(C, d), \mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C, d+n), \mathbb{C})$$

induced by (16) *preserves the tautological classes:*

$$H^*(M_{\mathrm{Dol}}(C, d), \mathbb{C}) \ni c(\gamma, k) \mapsto c(\gamma, k) \in H^*(M_{\mathrm{Dol}}(C, d+n), \mathbb{C}).$$

Proof. Under the isomorphism (16), a universal family for $M_{\mathrm{Dol}}(C, d+n)$ is obtained by taking the tensor product of a universal family for $M_{\mathrm{Dol}}(C, d)$ with the pullback of $\mathcal{O}_C(1)$ from C. Hence the isomorphism

$$H^*(C \times M_{\mathrm{Dol}}(C, d), \mathbb{C}) \xrightarrow{\simeq} H^*(C \times M_{\mathrm{Dol}}(C, d+n), \mathbb{C})$$

induced by (16) also preserves the normalized classes, and thus the tautological classes. ■

3. Similitude groups

3.1. Good elements

We fix two integers d, d' coprime to n. We call an element $A \in \mathrm{GSp}(\Lambda_{\mathbb{C}})$ *good* if there is an isomorphism of $\mathbb{C}$-algebras

$$G_A : H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C,d'),\mathbb{C})$$

such that:

(i) $G_A(c(\gamma,k)) = c(A\gamma,k)$ for any $\gamma \in H^*(C,\mathbb{C})$ and $k \in \mathbb{N}$;

(ii) $G_A(P_i H^j(M_{\mathrm{Dol}}(C,d),\mathbb{C})) = P_i H^j(M_{\mathrm{Dol}}(C,d'),\mathbb{C})$ for any $i, j \in \mathbb{N}$.

We denote by $\mathscr{G}$ the set of all good elements. It is clear that Theorem 1.1 is equivalent to

$$\mathscr{G} = \mathrm{GSp}(\Lambda_{\mathbb{C}}). \tag{17}$$

We prove some basic properties for $\mathscr{G}$ in Section 3.2, and conclude the section with a criterion for (17).

3.2. Basic properties

Lemma 3.1. *The set $\mathscr{G} \subset \mathrm{GSp}(\Lambda_{\mathbb{C}})$ is closed under the left or right action of* $\mathrm{Sp}(\Lambda)$.

Proof. This is given by the monodromy symmetry. More precisely, for fixed d, a monodromy operator obtained by varying the curve C is of the form

$$H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}), \quad c(\gamma,k) \mapsto c(M\gamma,k),$$

with $M \in \mathrm{Sp}(V)$ (cf. the end of Section 1.2), and it follows from [9, Theorem 1.1.1] that a monodromy operator preserves the perverse filtration. The lemma follows from composing G_A with a monodromy operator on the left or right. ∎

Lemma 3.2. *The set $\mathscr{G} \subset \mathrm{GSp}(\Lambda_{\mathbb{C}})$ is Zariski closed.*

Proof. For convenience, we pick a $\mathbb{C}$-basis $e_1, \dots, e_{2g}$ of $H^1(C,\mathbb{C})$. So we have the $\mathbb{C}$-algebra generators

$$c(1,k),\ c(e_i,k),\ c(\mathrm{pt},k), \quad i,k \in \mathbb{N}, \tag{18}$$

of $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$ and $H^*(M_{\mathrm{Dol}}(C,d'),\mathbb{C})$. In order to prove that $\mathscr{G}$ is Zariski closed in $\mathrm{GSp}(\Lambda_{\mathbb{C}})$, it suffices to show that (i) and (ii) are Zariski closed conditions for $A \in \mathrm{GSp}(\Lambda_{\mathbb{C}})$.

We first treat (i). Any $c(A\gamma,k)$ can be expressed in terms of the classes (18) with coefficients given by certain entries of the matrix A. Hence the condition that G_A is a $\mathbb{C}$-algebra isomorphism sending $c(\gamma,k)$ to $c(A\gamma,k)$ is equivalent to any relation between $\{c(\gamma,k)\}$ on $M_{\mathrm{Dol}}(C,d)$ being sent to a relation between $\{c(A\gamma,k)\}$ on $M_{\mathrm{Dol}}(C,d')$, which is clearly a Zariski closed condition on the entries of A.

For (ii), we consider a filtered basis of each cohomology group with respect to the perverse filtration; that is, we require that each piece P_j of the perverse filtration is spanned by a subset of the basis. Condition (ii) can be expressed completely in terms of the filtered basis, namely, the image of every vector of the basis which lies in P_j is sent to a vector which is only a linear combination of the subbasis giving a basis of P_j. In other words, condition (ii) is equivalent to the vanishing of certain coefficients for the linear transformation G_A under the filtered basis. By Markman [22] every vector of the filtered basis is expressed in terms of a linear combination of products of the classes (18). Hence the coefficients for the linear transformation G_A under the filtered basis are polynomials in the entries of the matrix A, whose vanishing is also a Zariski closed condition. ■

Combining the two lemmas above, we get the following criterion for $\mathscr{G} = \mathrm{GSp}(\Lambda_{\mathbb{C}})$.

Proposition 3.3. *If the set*

$$\{\lambda_A \mid A \in \mathscr{G} \subset \mathrm{GSp}(\Lambda_{\mathbb{C}})\} \subset \mathbb{G}_{\mathrm{m}}$$

is infinite, then $\mathscr{G} = \mathrm{GSp}(\Lambda_{\mathbb{C}})$.

Proof. The Borel density theorem entails that $\mathrm{Sp}(\Lambda) \subset \mathrm{Sp}(\Lambda_{\mathbb{C}})$ is a Zariski dense subset. We then deduce from Lemmas 3.1 and 3.2 that $\mathscr{G}$ is preserved by the right action of $\mathrm{Sp}(\Lambda_{\mathbb{C}})$ on $\mathrm{GSp}(\Lambda_{\mathbb{C}})$. Hence to prove $\mathscr{G} = \mathrm{GSp}(\Lambda_{\mathbb{C}})$, it suffices to prove that the image of $\mathscr{G}$ via the projection

$$\mathrm{GSp}(\Lambda_{\mathbb{C}}) \to \mathrm{GSp}(\Lambda_{\mathbb{C}})/\mathrm{Sp}(\Lambda_{\mathbb{C}})$$

is dense, where the quotient is with respect to the right action. This is exactly the assumption of the proposition, since the similitude character λ_A defines an isomorphism

$$\mathrm{GSp}(\Lambda_{\mathbb{C}})/\mathrm{Sp}(\Lambda_{\mathbb{C}}) = \mathbb{G}_{\mathrm{m}}.$$ ■

Next, we use techniques from positive characteristic to construct sufficiently many elements in $\mathscr{G}$.

4. Reduction to positive characteristic

In this section, we will always use $\overline{\mathbb{Q}}_\ell$-adic cohomology with the prime ℓ coprime to p. In order to compare $\overline{\mathbb{Q}}_\ell$-adic cohomology with singular cohomology with $\mathbb{C}$-coefficients, we fix an isomorphism $\overline{\mathbb{Q}}_l \xrightarrow{\simeq} \mathbb{C}$.

4.1. Non-abelian Hodge in positive characteristic

Our main tool to construct elements in $\mathscr{G}$ is to use the non-abelian Hodge correspondence in positive characteristic [4, 11, 15].

Let C_p be a curve over an algebraically closed field $\mathbf{k}$ of positive characteristic $p > 0$. Throughout the rest of the paper, we assume that p is large enough so that it is coprime to the rank n. We denote by $C_p^{(1)}$ the Frobenius twist of the curve C_p obtained from the base change of

$$\mathrm{Frob} : \mathbf{k} \to \mathbf{k}, \quad x \mapsto x^p,$$

and we denote by

$$\mathrm{Fr}_p : C_p \to C_p^{(1)}$$

the relative Frobenius $\mathbf{k}$-morphism, which is finite of degree p.

As in the case of $\mathbb{C}$, the moduli space $M_{\mathrm{Dol}}(C_p^{(1)}, d)$ of (slope-)stable Higgs bundles of rank n and degree d over the curve $C_p^{(1)}$ carries a Hitchin fibration $h_p : M_{\mathrm{Dol}}(C_p^{(1)}, d) \to A(C_p^{(1)})$. Compared with the characteristic zero case, a new feature in characteristic p is the existence of a Hitchin type fibration, which is called the *Hitchin–de Rham* morphism, from the moduli space $M_{\mathrm{dR}}(C_p, dp)$ of (slope-)stable flat connections

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_{C_p}, \quad \mathrm{rank}(\mathcal{E}) = n, \, \deg(\mathcal{E}) = dp, \tag{19}$$

to the Hitchin base $A(C_p^{(1)})$ associated with the Frobenius twist $C_p^{(1)}$. More concretely, the *p-curvature* of a flat bundle gives rise to a morphism $\mathcal{E} \to \mathcal{E} \otimes \mathrm{Fr}_p^* \Omega_{C_p^{(1)}}$, whose characteristic polynomial induces the Hitchin–de Rham morphism

$$h_p^{(1)} : M_{\mathrm{dR}}(C_p, dp) \to A(C_p^{(1)});$$

we refer to [21] and [15, Section 3] for more details about the p-curvature and the Hitchin–de Rham fibration. Consequently, the cohomology of both moduli spaces $M_{\mathrm{Dol}}(C_p^{(1)}, d)$ and $M_{\mathrm{dR}}(C_p, dp)$ admits perverse filtrations induced by h_p and $h_p^{(1)}$, respectively.

Groechenig's version of the non-abelian Hodge theorem in characteristic p [15, Theorem 1.1] asserts that the two morphisms (Hitchin for $C_p^{(1)}$, and Hitchin–de Rham for C_p)

$$h_p : M_{\mathrm{Dol}}(C_p^{(1)}, d) \to A(C_p^{(1)}), \quad h_p^{(1)} : M_{\mathrm{dR}}(C_p, dp) \to A(C_p^{(1)}) \tag{20}$$

are both proper and surjective, and are étale-locally equivalent over the base $A(C_p^{(1)})$. We remark that Chen–Zhu [4] proved a similar result for stacks (without semistability assertions) for arbitrary reductive groups.

The Hitchin–de Rham fibration (20) for C_p in degree dp,

$$h_p^{(1)} : M_{\mathrm{dR}}(C_p, dp) \to A(C_p^{(1)}), \tag{21}$$

is closely related to the Hitchin fibration for the Dolbeault moduli space for C_p in the same degree dp,

$$h_p : M_{\mathrm{Dol}}(C_p, dp) \to A(C_p), \tag{22}$$

via the Hodge moduli space

$$\tau : M_{\mathrm{Hod}}(C_p, dp) \to A(C_p^{(1)}) \times \mathbb{A}_t^1 \to \mathbb{A}_t^1$$

parameterizing t-connections; this was constructed in [21] in the degree 0 case and was extended to the case of $(n, dp) = 1$ by [11, Proposition 3.1]. More precisely, it was shown in [11] that τ is a smooth family over $\mathbb{A}^1_t$; the fiber of τ over $t = 0 \in \mathbb{A}^1_t$ recovers (21); the fiber over $t = 1$ recovers (22), post-composed with the natural universal homeomorphism between the Hitchin bases $A(C_p^{(1)}) \xrightarrow{\simeq} A(C_p)$; this latter can be identified with the relative Frobenius for the Hitchin base $A(C_p)$ [12, Lemma 4.3].

4.2. Cohomological correspondences

Note that the correspondence (20) is local in nature. Under the coprimality assumption $(n, dp) = 1$ for rank and degree, de Cataldo and Zhang [11] established a series of *global* cohomological correspondences that we now describe.

The global nilpotent cones associated with the Dolbeault and the de Rham moduli spaces

$$N_{\mathrm{Dol}}(C_p^{(1)}, d) := h_p^{-1}(0), \quad N_{\mathrm{dR}}(C_p, dp) := {h_p^{(1)}}^{-1}(0) \tag{23}$$

are isomorphic by the étale equivalence of (20). In fact, there is a distinguished isomorphism given by (33) that we will describe later. The cohomology rings of the global nilpotent cones (23) carry natural filtrations induced by the respective perverse filtrations and the decomposition theorem; see [11, Remark 2.3].

We have the following canonical commutative diagram of canonical ring isomorphisms (coefficients $\overline{\mathbb{Q}}_\ell$ throughout), which are also filtered isomorphisms for the respective perverse filtrations:

$$\begin{array}{ccccc} H^*(M_{\mathrm{Dol}}(C_p, dp)) & \xleftarrow[\text{(d)}]{\simeq} & H^*(M_{\mathrm{dR}}(C_p, dp)) & \xleftarrow[\text{(c)}]{\simeq} & H^*(N_{\mathrm{dR}}(C_p, dp)) \\ \simeq \Big\uparrow \Theta_M & & & & \text{(b)} \Big\uparrow \simeq \\ H^*(M_{\mathrm{Dol}}(C_p^{(1)}, d)) & & \xrightarrow[\text{(a)}]{\simeq} & & H^*(N_{\mathrm{Dol}}(C_p^{(1)}, d)) \end{array} \tag{24}$$

where the perverse filtrations on the cohomology groups of N_{Dol} and of N_{dR} are defined in [11, Section 2.2]; (a) is the filtered isomorphism in [11, (33) combined with Remark 2.3], and it is induced by restriction; (b) is the second filtered isomorphism in [11, (36)] and it is induced (this also uses results of M. Groechenig, as indicated in *loc. cit.*) by the local non-abelian Hodge correspondence over $0 \in A(C_p)$ (see (33)); (c) is the filtered isomorphism in [11, (31)], and it is induced by restriction; (d) is the filtered isomorphism in [11, (25)]. See also the companion diagram (30). The composition is the filtered ring isomorphism

$$\Theta_M : H^*(M_{\mathrm{Dol}}(C_p^{(1)}, d), \overline{\mathbb{Q}}_l) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C_p, dp), \overline{\mathbb{Q}}_l). \tag{25}$$

4.3. Lift to characteristic zero via finite fields

Given a curve C over the complex numbers, for each prime p with $p > n$ and $p \neq \ell$, we can use the canonical isomorphism Θ_M to produce a *non-canonical* ring isomorphism

$$\widetilde{\Theta}_M : H^*(M_{\mathrm{Dol}}(C, d), \mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C, dp), \mathbb{C}), \tag{26}$$

filtered with respect to the respective perverse filtrations.

The construction proceeds as follows. First, notice that, for a given p, it suffices to construct $\widetilde{\Theta}_M$ for a specific curve C and then apply a monodromy argument, as in [9], or [11, Proposition 3.3 (2)], to extend to arbitrary curves. Note that this introduces an ambiguity governed by the monodromy action of $\mathrm{Sp}(V)$.

We will choose C to be the lift of a curve defined over $\mathbb{F}_p$. Let C_p be a smooth curve defined over the finite field $\mathbb{F}_p$; here for notational convenience, we denote the base change of C_p to an algebraic closure $\mathbf{k}$ of $\mathbb{F}_p$ by the same symbol C_p. By our choice, we have $C_p = C_p^{(1)}$. We can lift C_p to a morphism $\mathcal{C} \to \mathrm{Spec}(R)$ with R a complete strictly Henselian DVR of characteristic zero with residue field $\mathbf{k}$, where the geometric generic fiber C is a connected non-singular curve of genus $g \geq 2$.

Specialization morphisms associated with the smooth family $\mathcal{C} \to \mathrm{Spec}(R)$ induce canonical isomorphisms

$$H^*(C, \overline{\mathbb{Q}}_\ell) = H^*(C_p, \overline{\mathbb{Q}}_\ell), \quad H^*(M_{\mathrm{Dol}}(C,d), \overline{\mathbb{Q}}_\ell) = H^*(M_{\mathrm{Dol}}(C_p,d), \overline{\mathbb{Q}}_\ell), \tag{27}$$

where we apply [11, Proposition 3.3 (2)] for the second isomorphism. If we combine these with the canonical isomorphism Θ_M of (24) and tensor with the fixed isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$, we obtain the desired $\widetilde{\Theta}_M$.

Theorem 4.1. *The ring isomorphism $\widetilde{\Theta}_M$ satisfies*

$$\widetilde{\Theta}_M(c(\gamma,k)) = c(p^{-1}\mathrm{Fr}_p^*\gamma, k), \quad \forall \gamma \in H^*(C,\mathbb{C}),\ \forall k \in \mathbb{N}.$$

Here, $\mathrm{Fr}_p^* : H^*(C_p,\mathbb{C}) \xrightarrow{\simeq} H^*(C_p,\mathbb{C})$ is the Frobenius pullback associated with the relative Frobenius $\mathbf{k}$-morphism $\mathrm{Fr}_p : C_p \to C_p^{(1)} = C_p$, and we have used the fixed isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$.

Remark 4.2. If we view the operator $A_p := p^{-1}\mathrm{Fr}_p^* : H^*(C,\mathbb{C}) \to H^*(C,\mathbb{C})$ as an element in $\mathrm{GSp}(\Lambda_{\mathbb{C}})$, then by the peculiar action defined in Section 1.2, it acts as id on $H^2(C,\mathbb{C})$ and as multiplication by $\lambda_{A_p} = p^{-1}$ on $H^0(C,\mathbb{C})$, thus justifying introducing this action.

5. Proof of Theorem 4.1

5.1. Isomorphisms

In the setting of Section 4.3, we consider the relative Dolbeault moduli space for degree d,

$$M_{\mathrm{Dol}}(\mathcal{C}/R, d) \to \mathrm{Spec}(R), \tag{28}$$

that is smooth over R, and the Hodge moduli space that is smooth over $\mathbb{A}^1_t$ [11, Proposition 3.1]:

$$\tau : M_{\mathrm{Hod}}(C, dp) \to \mathbb{A}^1_t. \tag{29}$$

The morphism $\widetilde{\Theta}_M$ for Theorem 4.1 is obtained as a composition, by inserting the Dolbeault moduli spaces for C into (24), by using (27), and by keeping in mind that $C_p = C_p^{(1)}$,

$$\begin{array}{ccccccc}
H^*(M_{\mathrm{Dol}}(C,dp)) & \xleftarrow[(27)\ \mathrm{u}_2]{\simeq} & H^*(M_{\mathrm{Dol}}(C_p,dp)) & \xleftarrow[(\mathrm{d})]{\simeq} & H^*(M_{\mathrm{dR}}(C_p,dp)) & \xleftarrow[(\mathrm{c})]{\simeq} & H^*(N_{\mathrm{d}R}(C_p,dp)) \\
\simeq \uparrow \widetilde{\Theta}_M & & \simeq \uparrow \Theta_M & & & & (\mathrm{b}) \uparrow \simeq \\
H^*(M_{\mathrm{Dol}}(C,d)) & \xrightarrow[(27)\ \mathrm{u}_1]{\simeq} & H^*(M_{\mathrm{Dol}}(C_p,d)) & & \xrightarrow[(\mathrm{a})]{\simeq} & & H^*(N_{\mathrm{Dol}}(C_p,d))
\end{array} \tag{30}$$

Here u_1 and u_2 are the specialization maps (27) associated with (28), for degrees d and dp respectively; (a) and (c) are the natural restriction maps for the global nilpotent cones (23); (d) is the specialization map associated with (29); (b) is induced by the local non-abelian Hodge correspondence over $0 \in A(C_p)$ (see (33)); and $\widetilde{\Theta}_M$ is the composition.

For our purpose, we calculate each map with respect to the tautological classes.

Remark 5.1. We note that by [18, Lemma 3.1 and the proof of Corollary 3.2], the fact that the rank and the degree are coprime ensures that each of the Dolbeault, de Rham, and Hodge moduli spaces above carries a *universal family*. In particular, we may define the tautological classes $c(\gamma, k)$ for each cohomology in the chain above via a universal family, compatibly with restriction maps.

5.2. *Restrictions and specializations*

We first note that restrictions (c, a) and specialization maps $(\mathrm{u}_1, \mathrm{d}, \mathrm{u}_2)$ preserve the tautological classes, i.e., they send $c(\gamma, k)$ to $c(\gamma, k)$.

This statement is clear for restriction maps, since the restriction of a universal family on $M_{\mathrm{Dol}}(C_p, d)$ (resp. $M_{\mathrm{dR}}(C_p, d)$) to the corresponding global nilpotent cone $N_{\mathrm{Dol}}(C_p, d)$ (resp. $N_{\mathrm{dR}}(C_p, d)$) is still a universal family.

For the specialization maps $\mathrm{u}_1, \mathrm{u}_2$ and (d), this follows from the existence of universal families over $\mathrm{Spec}(R)$ and $\mathbb{A}^1_t$ respectively; see Remark 5.1.

5.3. *Global nilpotent cones*

Finally, we treat the morphism (b) in (30). Our goal is to prove the identities (42) and (43), to be used in Section 5.4 when proving Theorem 4.1.

In order to carry out the computation, we need the precise description of the non-abelian Hodge correspondence [15, Corollary 3.28, Lemma 3.46] for the global nilpotent cones, which we review briefly as follows.

Let D_{C_p} be the sheaf of crystalline differential operators on the curve $C_p = C_p^{(1)}$. The pushforward of D_{C_p} along the Frobenius map $\mathrm{Fr}_p : C_p \to C_p$ satisfies

$$\mathrm{Fr}_{p*} D_{C_p} = \pi_* \mathcal{D},$$

where π is the projection $T^*C_p \to C_p$ and $\mathcal{D}$ is a uniquely determined $\mathcal{O}_{T^*C_p}$-algebra; see [15, Lemma 2.8]. Moreover, by [15, Theorem 3.20], the restriction of $\mathcal{D}$ to any spectral curve $C_\alpha \subset T^*C_p$ splits, i.e., we have an isomorphism

$$\mathcal{D}|_{C_\alpha} = \mathcal{E}nd_{\mathcal{O}_{C_\alpha}}(V_\alpha) \tag{31}$$

with V_α a rank p vector bundle on C_α, well-defined up to tensoring with a line bundle.

We denote by $C_n \subset T^*C_p$ the spectral curve associated with $0 \in A(C_p)$; it is the n-th thickening of the 0-section $C_p \subset T^*C_p$. By [15, proof of Corollary 3.45], there is a canonical choice of a vector bundle V_n of rank p that splits $\mathcal{D}|_{C_n}$ as in (31). Such a choice induces a canonical isomorphism ν between the global nilpotent cones $N_{\mathrm{Dol}}(C_p, d)$ and $N_{\mathrm{dR}}(C_p, dp)$. For our purposes, we need to describe the interaction of universal families under this isomorphism.

By the classical BNR correspondence, a Higgs bundle in $N_{\mathrm{Dol}}(C_p, d)$ is given as $(\pi_*\mathcal{F}, \theta)$ where $\mathcal{F}$ is a stable pure 1-dimensional sheaf supported on C_n, and $\pi : C_n \to C_p$ is the projection. It corresponds to a flat connection $(\mathcal{E}, \nabla) \in N_{\mathrm{dR}}(C_p, dp)$ satisfying

$$\mathrm{Fr}_{p*}\mathcal{E} \simeq \pi_*(\mathcal{F} \otimes V_n). \tag{32}$$

To globalize (32) over the moduli spaces, we denote by ν the non-abelian Hodge correspondence for the global nilpotent cones induced by V_n:

$$\nu : N_{\mathrm{dR}}(C_p, dp) \xrightarrow{\simeq} N_{\mathrm{Dol}}(C_p, d), \quad (\mathcal{E}, \nabla) \mapsto (\pi_*\mathcal{F}, \theta), \tag{33}$$

where $\mathcal{E}$ and $\mathcal{F}$ satisfy (32). The isomorphism (b) in (30) is ν^* with inverse ν_*. We consider the degree p finite map

$$F := \mathrm{Fr}_p \times \nu : C_p \times N_{\mathrm{dR}}(C_p, dp) \to C_p \times N_{\mathrm{Dol}}(C_p, d), \tag{34}$$

and the projection

$$\widetilde{\pi} : C_n \times N_{\mathrm{Dol}}(C_p, d) \to C_p \times N_{\mathrm{Dol}}(C_p, d).$$

We also have the following natural projections:

$$\begin{aligned} p_{\mathrm{dR}} &: C_p \times N_{\mathrm{dR}}(C_p, dp) \to C_p, \quad q_{\mathrm{dR}} : C_p \times N_{\mathrm{dR}}(C_p, dp) \to N_{\mathrm{dR}}(C_p, dp), \\ p_{\mathrm{Dol}} &: C_p \times N_{\mathrm{Dol}}(C_p, d) \to C_p, \quad q_{\mathrm{Dol}} : C_p \times N_{\mathrm{Dol}}(C_p, d) \to N_{\mathrm{Dol}}(C_p, d), \\ r &: C_n \times N_{\mathrm{Dol}}(C_p, dp) \to C_n. \end{aligned}$$

Let $\mathcal{V}$ be the pullback r^*V_n of V_n along the projection r; it is a vector bundle of rank p. Let $(\mathcal{U}_{\mathrm{dR}}, \nabla)$ and

$$(\mathcal{U}_{\mathrm{Dol}}, \theta) = (\widetilde{\pi}_*\mathcal{F}_n, \theta) \tag{35}$$

be universal families over the schemes on both sides of (34), where $\mathcal{F}_n$ is a universal family over $C_n \times N_{\mathrm{Dol}}(C_p, d)$. The isomorphism (32) shows that the pushforward $F_*\mathcal{U}_{\mathrm{dR}}$ and $\widetilde{\pi}_*(\mathcal{F}_n \otimes \mathcal{V})$ coincide after restricting over any closed point on $N_{\mathrm{Dol}}(C_p, d)$. Therefore $F_*\mathcal{U}_{\mathrm{dR}}$ is the tensor product of $\widetilde{\pi}_*(\mathcal{F}_n \otimes \mathcal{V})$ and a line bundle pulled back from

$N_{\mathrm{Dol}}(C_p, d)$. Modifying the universal family $\mathcal{F}_n$ by this line bundle, we may assume that the universal families $\mathcal{U}_{\mathrm{dR}}$ and $\mathcal{F}_n$ satisfy

$$F_*\mathcal{U}_{\mathrm{dR}} \simeq \widetilde{\pi}_*(\mathcal{F}_n \otimes \mathcal{V}). \tag{36}$$

Recall the following type of classes (cf. (9)), which play a role in producing normalized and tautological classes. Let $\alpha_0 \in H^2(C \times N_{\mathrm{Dol}}(C_p, d), \overline{\mathbb{Q}}_\ell)$ be a class of the form

$$\alpha_0 = p_{\mathrm{Dol}}^*\alpha_0' + q_{\mathrm{Dol}}^*\alpha_0''. \tag{37}$$

We need the following two classes of the form (37) (for the first, Dol is replaced by dR):

$$\alpha_1 := \frac{p-1}{2} p_{\mathrm{dR}}^* c_1(\omega_{C_p}), \quad \beta := \frac{p-1}{2p} p_{\mathrm{Dol}}^* c_1(\omega_{C_p}). \tag{38}$$

By explicit calculation, we have the identity $\mathrm{td}(T_F) = 1 + \alpha_1$ for the Todd class $\mathrm{td}(T_F)$ of the virtual tangent bundle T_F (cf. [14, Theorem B.7.6]). The class β will appear naturally shortly.

Lemma 5.2. *We have the identities*

$$F_*[\mathrm{ch}^{\alpha_1}(\mathcal{U}_{\mathrm{dR}})] = \mathrm{ch}(F_*\mathcal{U}_{\mathrm{dR}}) = p \cdot \mathrm{ch}^\beta(\mathcal{U}_{\mathrm{Dol}}). \tag{39}$$

Proof. The first identity follows by applying Grothendieck–Riemann–Roch (GRR) (cf. [14, Example 18.3.10]) to the l.c.i. morphism F of (34):

$$\mathrm{ch}(F_*\mathcal{U}_{\mathrm{dR}}) = F_*[\mathrm{ch}(\mathcal{U}_{\mathrm{dR}})\,\mathrm{td}(T_F)] = F_*[\mathrm{ch}^{\alpha_1}(\mathcal{U}_{\mathrm{dR}})]. \tag{40}$$

To argue for the second identity, we need three useful facts: Firstly, since the inclusion $i : C_p \hookrightarrow C_n$ is a section to the projection $\pi : C_n \to C_p$, it follows that the isomorphism $i^* : H^2(C_n) \to H^2(C_p)$ is the inverse to the isomorphism $\pi^* : H^2(C_p) \to H^2(C_n)$. Secondly, the splitting V_n on C_n that is chosen in [15, proof of Corollary 3.45] satisfies the identity of vector bundles $i^*V_n = \mathrm{Fr}_{p*}\mathcal{O}_{C_p}$. Thirdly, we have $c_1(\mathrm{Fr}_{p*}\mathcal{O}_{C_p}) = \frac{p-1}{2}c_1(\omega_{C_p})$, thus $p\beta = p_{\mathrm{Dol}}^* c_1(\mathrm{Fr}_{p*}\mathcal{O}_{C_p})$. With these three facts, we obtain the following equalities:

$$\begin{aligned}
\mathrm{ch}(F_*\mathcal{U}_{\mathrm{dR}}) &= \mathrm{ch}(\widetilde{\pi}_*(\mathcal{F}_n \otimes \mathcal{V})) = \widetilde{\pi}_*[\mathrm{ch}(\mathcal{F}_n)\,\mathrm{td}(T_{\widetilde{\pi}})\,\mathrm{ch}(\mathcal{V})] \\
&= \widetilde{\pi}_*[\mathrm{ch}(\mathcal{F}_n)\,\mathrm{td}(T_{\widetilde{\pi}}) r^*\pi^*(p + i^*c_1(V_n))] \\
&= \widetilde{\pi}_*[\mathrm{ch}(\mathcal{F}_n)\,\mathrm{td}(T_{\widetilde{\pi}})\widetilde{\pi}^* p_{\mathrm{Dol}}^*(p + c_1(\mathrm{Fr}_{p*}\mathcal{O}_{C_p}))] \\
&= \widetilde{\pi}_*[\mathrm{ch}(\mathcal{F}_n)\,\mathrm{td}(T_{\widetilde{\pi}})](p + p\beta) \\
&= p \cdot \mathrm{ch}(\widetilde{\pi}_*\mathcal{F}_n)(1+\beta) = p \cdot \mathrm{ch}^\beta(\mathcal{U}_{\mathrm{Dol}}),
\end{aligned} \tag{41}$$

where the first equality follows from (36); the second follows from GRR; the third follows from the definition $\mathcal{V} := r^*V_n$ and the first useful fact above; the fourth follows from the identity $\pi \circ r = p_{\mathrm{Dol}} \circ \widetilde{\pi} : C_n \times N_{\mathrm{Dol}} \to C_p$ and the second useful fact above; the fifth follows from the projection formula and the third useful fact; the sixth follows from GRR; and the last follows from (35). ■

By twisting (39) with a class α_0 of the form (37), and by using the projection formula, we get the identity

$$F_*[\mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}})] = \mathrm{ch}^{\alpha_0}(F_*\mathcal{U}_{\mathrm{dR}}) = p \cdot \mathrm{ch}^{\alpha_0+\beta}(\mathcal{U}_{\mathrm{Dol}}). \tag{42}$$

By integrating the left-hand side of (42) over a class $\gamma \in H^*(C_p, \overline{\mathbb{Q}}_\ell)$, the projection formula, together with the identities $q_{\mathrm{Dol}} \circ F = \nu \circ q_{\mathrm{dR}}$ and $\mathrm{Fr}_p \circ p_{\mathrm{dR}} = p_{\mathrm{Dol}} \circ F$, yields the identities

$$\begin{aligned}\int_\gamma \mathrm{ch}^{\alpha_0}(F_*\mathcal{U}_{\mathrm{dR}}) &= q_{\mathrm{Dol}*}\big(p_{\mathrm{Dol}}^*\gamma \cup F_*[\mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}})]\big)\\ &= q_{\mathrm{Dol}*}F_*\big(F^*p_{\mathrm{Dol}}^*\gamma \cup \mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}})\big)\\ &= \nu_* q_{\mathrm{dR}*}\big(p_{\mathrm{dR}}^*\mathrm{Fr}_p^*\gamma \cup \mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}})\big)\\ &= \nu_* \int_{\mathrm{Fr}_p^*\gamma} \mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}}).\end{aligned}$$

By integrating the right-hand side of (42), and by using the fact that ν_* and ν^* are mutual inverses, we finally get the following identity for every α_0 and γ:

$$\int_{\mathrm{Fr}_p^*\gamma} \mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}}) = \nu^*\left[p \cdot \int_\gamma \mathrm{ch}^{\alpha_0+\beta}(\mathcal{U}_{\mathrm{Dol}})\right]. \tag{43}$$

5.4. Proof of Theorem 4.1

Recall that the isomorphism (b) in (30) is ν^*, for which we have the identity (43).

We claim that there is a necessarily unique class α_0 of the form (37) such that the two Chern characters in the first and third term of (42) are *simultaneously* normalized.

We pick α_0 to be the unique class of the form (37) so that $\mathrm{ch}_1^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{dR})$ lies in the Künneth component

$$H^1(C_p) \otimes H^1(N_{\mathrm{dR}}(C_p, dp)) \subset H^2(C_p \times N_{\mathrm{dR}}(C_p, dp)),$$

and consequently $\mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{dR})$ is normalized. To do so, we use the fact that F is a product and that F^* is an isomorphism.

Since $F = \mathrm{Fr}_p \times \nu$, we know that F_* respects the Künneth components. Therefore the class $\mathrm{ch}^{\beta+\alpha_0}(\mathcal{U}_{\mathrm{Dol}})$ is also normalized, and our claim is proved.

Since both Chern characters are normalized, it follows from (43) that the isomorphism ν^* respects tautological classes. Moreover, in view of (43), we obtain the following explicit expression for the isomorphism $\widetilde{\Theta}_M$ evaluated on the tautological classes:

$$\begin{aligned}\widetilde{\Theta}_M(c(\gamma,k)) = \Theta_M(c(\gamma,k)) = \nu^*\int_\gamma \mathrm{ch}^{\beta+\alpha_0}(\mathcal{U}_{\mathrm{Dol}}) &= \int_{\frac{1}{p}\cdot\mathrm{Fr}_p^*\gamma} \mathrm{ch}^{F^*\alpha_0+\alpha_1}(\mathcal{U}_{\mathrm{dR}})\\ &= c(p^{-1}\mathrm{Fr}_p^*\gamma, k).\end{aligned}$$

Here we abuse notation by omitting the horizontal isomorphisms in (30), which all respect normalized and tautological classes. ∎

6. Proofs of the main theorems and of Deligne splittings

6.1. *Proof of Theorem* 1.1

In view of Proposition 3.3, it suffices to show that there are infinitely many $A \in \mathcal{G}$ with distinct $\lambda_A \in \mathbb{G}_m$.

We first note that, if for one complex curve C of genus $g \geq 2$ we find an isomorphism of the form G_A as in Section 3.1, then by composing with parallel transport operators we obtain an isomorphism of the same form for any non-singular complex curve of genus g. Therefore, we may choose any curve to construct elements in $\mathcal{G}$.

For any prime $p > n$ with

$$dp = d' \bmod n, \tag{44}$$

we work with C and its reduction C_p as in Section 4.3. By Proposition 2.3 and Theorem 4.1, we obtain

$$A_p = p^{-1}\mathrm{Fr}_p^* \in \mathcal{G} \quad \text{with} \quad \lambda_{A_p} = p^{-1} \in \mathbb{G}_m.$$

The theorem follows since there are infinitely many primes p satisfying (44). ■

6.2. *Proofs of Theorems* 1.2 *and* 1.4

Theorem 1.2 is deduced from Theorem 1.1, since it follows from Proposition 2.2 that the Galois conjugation (5) coincides with the isomorphism $\phi_{d,d'}$ of Theorem 1.1 (b).

Now we prove Theorem 1.4. We consider the action of the 1-dimensional subgroup

$$\mathbb{T} := \{\lambda\, \mathrm{Id}_{2g} \mid \lambda \neq 0\} \subset \mathrm{GSp}(\Lambda_{\mathbb{C}})$$

on the total cohomology $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$ induced by the action of Theorem 1.1 (a). We note that the weight decomposition of this $\mathbb{T}$-action recovers the Hodge–Tate decomposition (8) via non-abelian Hodge (3). More precisely, we have

$$\mathrm{Hdg}^i_j = \{\omega \in H^i(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \mid \lambda\cdot\omega = \lambda^{2j-i}\omega,\ \forall \lambda \in \mathbb{T}\};$$

this follows directly from the explicit description of the $\mathbb{T}$-action on the tautological classes:

$$\lambda \cdot c(\gamma,k) = \lambda^{2-e}c(\gamma,k), \quad \forall \gamma \in H^e(C,\mathbb{C}),\ \forall \lambda \in \mathbb{T}.$$

Furthermore, by Theorem 1.1 (a) each piece of the perverse filtration preserves the $\mathbb{T}$-action. Hence the weight decomposition of $\mathbb{T}$ on each piece $P_kH^i(M_{\mathrm{Dol}}(C,d),\mathbb{C})$ induces the desired decomposition

$$P_kH^i(M_{\mathrm{Dol}}(C,d),\mathbb{C}) = \bigoplus_j \big(P_kH^i(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \cap \mathrm{Hdg}^i_j\big).$$

This completes the proof of Theorem 1.4. ■

6.3. Extension to Deligne splitting

As we will recall briefly, the perverse filtration on the cohomology $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$ admits a natural splitting, known as the first Deligne splitting [6, 13]. Conjecturally this splitting corresponds, via Non-Abelian Hodge Theory, to the Hodge–Tate splitting of the weight filtration on $H^*(M_B(C,d),\mathbb{C})$ in (8).

We first review some background. Suppose we are given a triple (H,F,η) where H is a $\mathbb{C}$-algebra, F an increasing filtration on H concentrated in degrees $[0,2r]$, and η is an element of H. We say that η is an *F-Lefschetz class* if (i) multiplication by η maps F_kH to $F_{k+2}H$ and (ii) multiplication by η^i induces isomorphisms $\eta^i : \mathrm{Gr}^F_{r-i}H \simeq \mathrm{Gr}^F_{r+i}H$. An F-Lefschetz class η on H induces, by means of an explicit linear algebra construction, a natural splitting of F called the first Deligne splitting [6]. In our setting, $H = H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$ is the cohomology of the Dolbeault moduli space equipped with its perverse filtration P, and the P-Lefschetz class η is a degree 2 cohomology class. The first Deligne splitting satisfies the following natural compatibility. Suppose we are given two such triples (H,F,η) and (H',F',η') and a ring isomorphism $f : H \xrightarrow{\simeq} H'$, which is a filtered isomorphism, for which $f(\eta)=\eta'$. It is clear that if η is an F-Lefschetz class then η' is an F'-Lefschetz class and moreover f preserves the corresponding Deligne splittings.

This compatibility gives us the following corollary of our main theorem.

Corollary 6.1. *The operators G_A of Theorem 1.1 preserve the first Deligne splittings for $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$.*

Proof. As explained in [10, Remark 3.5], in the case of $H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C})$, the first Deligne splitting is *independent* of the choice of P-Lefschetz class η. In other words, we have naturally defined subspaces $S_j \subset H$ for which

$$P_k = \bigoplus_{j\le k} S_j.$$

By the compatibility stated before the proposition, given two Dolbeault moduli spaces $M_{\mathrm{Dol}}(C,d)$ and $M_{\mathrm{Dol}}(C,d')$, any ring isomorphism

$$H^*(M_{\mathrm{Dol}}(C,d),\mathbb{C}) \xrightarrow{\simeq} H^*(M_{\mathrm{Dol}}(C,d'),\mathbb{C}),$$

which is a filtered isomorphism, automatically preserves the first Deligne splitting and the graded pieces S_j. In particular, this applies to the operators G_A constructed in this paper. ∎

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References

[1] A'Campo, N.: Tresses, monodromie et le groupe symplectique. Comment. Math. Helv. **54**, 318–327 (1979) Zbl 0441.32004 MR 535062

[2] Beĭlinson, A. A., Bernstein, J., Deligne, P.: Faisceaux pervers. In: Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque **100**, 5–171 (1982) Zbl 0536.14011 MR 751966

[3] Bezrukavnikov, R., Mirković, I., Rumynin, D.: Localization of modules for a semisimple Lie algebra in prime characteristic. Ann. of Math. (2) **167**, 945–991 (2008) Zbl 1220.17009 MR 2415389

[4] Chen, T.-H., Zhu, X.: Non-abelian Hodge theory for algebraic curves in characteristic p. Geom. Funct. Anal. **25**, 1706–1733 (2015) Zbl 1330.14015 MR 3432156

[5] Chuang, W.-Y., Diaconescu, D.-E., Pan, G.: BPS states and the $P = W$ conjecture. In: Moduli spaces, London Mathematical Society Lecture Note Series 411, Cambridge University Press, Cambridge, 132–150 (2014) Zbl 1320.14057 MR 3221294

[6] de Cataldo, M. A. A.: Hodge-theoretic splitting mechanisms for projective maps. J. Singul. **7**, 134–156 (2013) Zbl 1317.14045 MR 3077721

[7] de Cataldo, M. A.: A support theorem for the Hitchin fibration: the case of SL_n. Compos. Math. **153**, 1316–1347 (2017) Zbl 1453.14029 MR 3705257

[8] de Cataldo, M. A. A., Hausel, T., Migliorini, L.: Topology of Hitchin systems and Hodge theory of character varieties: the case A_1. Ann. of Math. (2) **175**, 1329–1407 (2012) Zbl 1375.14047 MR 2912707

[9] de Cataldo, M. A. A., Maulik, D.: The perverse filtration for the Hitchin fibration is locally constant. Pure Appl. Math. Q. **16**, 1441–1464 (2020) Zbl 1469.14018 MR 4221002

[10] de Cataldo, M. A., Maulik, D., Shen, J.: Hitchin fibrations, abelian surfaces, and the $P = W$ conjecture. J. Amer. Math. Soc. **35**, 911–953 (2022) Zbl 07537728 MR 4433080

[11] de Cataldo, M. A., Zhang, S.: A cohomological nonabelian Hodge theorem in positive characteristic. Algebr. Geom. **9**, 606–632 (2022) Zbl 1507.14049 MR 4490708

[12] de Cataldo, M. A. A., Zhang, S.: Projective completion of moduli of t-connections on curves in positive and mixed characteristic. Adv. Math. **401**, art. 108329, 43 pp. (2022) Zbl 1493.14012 MR 4395955

[13] Deligne, P.: Décompositions dans la catégorie dérivée. In: Motives (Seattle, WA, 1991), Proceedings of Symposia in Pure Mathematics 55, American Mathematical Society, Providence, RI, 115–128 (1994) Zbl 0809.18008 MR 1265526

[14] Fulton, W.: Intersection theory. Princeton University Press (2016) Zbl 0541.14005 MR 1644323

[15] Groechenig, M.: Moduli of flat connections in positive characteristic. Math. Res. Lett. **23**, 989–1047 (2016) Zbl 1368.14021 MR 3554499

[16] Hausel, T.: Global topology of the Hitchin system. In: Handbook of moduli. Vol. II, Advanced Lectures in Mathematics 25, International Press, Somerville, MA, 29–69 (2013) Zbl 1322.14027 MR 3184173

[17] Hausel, T., Thaddeus, M.: Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles. Proc. London Math. Soc. (3) **88**, 632–658 (2004) Zbl 1060.14048 MR 2044052

[18] Heinloth, J.: Lectures on the moduli stack of vector bundles on a curve. In: Affine flag manifolds and principal bundles, Trends in Mathematics, Birkhäuser/Springer Basel, Basel, 123–153 (2010) Zbl 1211.14018 MR 3013029

[19] Hitchin, N.: Stable bundles and integrable systems. Duke Math. J. **54**, 91–114 (1987) Zbl 0627.14024 MR 885778

[20] Hitchin, N. J.: The self-duality equations on a Riemann surface. Proc. London Math. Soc. (3) **55**, 59–126 (1987) Zbl 0634.53045 MR 887284

[21] Laszlo, Y., Pauly, C.: On the Hitchin morphism in positive characteristic. Int. Math. Res. Notices **2001**, 129–143 Zbl 0983.14004 MR 1810690

[22] Markman, E.: Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. J. Reine Angew. Math. **544**, 61–82 (2002) Zbl 0988.14019 MR 1887889

[23] Mellit, A.: Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures). Invent. Math. **221**, 301–327 (2020) Zbl 1455.14022 MR 4105090

[24] Shende, V.: The weights of the tautological classes of character varieties. Int. Math. Res. Notices **2017**, 6832–6840 Zbl 1405.14037 MR 3737322

[25] Simpson, C. T.: Moduli of representations of the fundamental group of a smooth projective variety. II. Inst. Hautes Études Sci. Publ. Math. **80**, 5–79 (1994) Zbl 0891.14006 MR 1320603