J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1394

Published by EMS Press

Abhishek Oswal

A non-archimedean definable Chow theorem

Received July 21, 2021; revised January 24, 2022

Abstract. Peterzil and Starchenko have proved the following surprising generalization of Chow's theorem: A closed analytic subset of a complex algebraic variety that is definable in an o-minimal structure is in fact an algebraic subset. In this paper, we prove a non-archimedean analogue of this result.

Keywords. Non-archimedean, algebraization, definability, Chow, o-minimality

1. Introduction

The theory of o-minimality provides a framework for a 'tame topology' on $\mathbb{R}$, imposing some strict finiteness properties, and in many instances providing a way to interpolate between the algebraic and analytic topologies.

In particular, there have been two influential algebraization theorems arising from the theory of o-minimal structures that have been, in conjunction, a powerful tool in studying problems of 'unlikely intersections'. The first one of these is the counting theorem of Pila and Wilkie [35] that (in imprecise terms) proves that an o-minimally definable subset of $\mathbb{R}^n$ which contains at least polynomially many rational points (counted with respect to height) must contain a positive-dimensional semialgebraic subset.

The second of these algebraization theorems is due to Peterzil and Starchenko. In [33], Peterzil–Starchenko develop a theory of holomorphic functions and analytic manifolds in the category of definable objects in an o-minimal structure. A celebrated result in their work is the definable Chow theorem [32, Theorem 5.1], which states that a closed analytic subset of a complex algebraic variety that is simultaneously definable in an o-minimal structure is in fact an algebraic subset.

These theorems have played a key role in several important developments in Diophantine geometry and Hodge theory in the past decade. The Pila–Wilkie counting theorem was used by Pila and Zannier [36] to provide a new proof of the Manin–Mumford conjecture. The strategy of Pila and Zannier has in turn led to an influential program over the last decade that culminated in the proof of the André–Oort conjecture, first for $\mathcal{A}_g$

Abhishek Oswal: The Division of Physics, Mathematics and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA; abo691@gmail.com

Mathematics Subject Classification (2020): Primary 03C64; Secondary 11U09

by Tsimerman [41], and more recently for general Shimura varieties by Pila–Shankar–Tsimerman–Esnault–Groechenig [34].

A central step in the proofs of these conjectures is establishing a 'functional transcendence' result known as an Ax–Schanuel theorem, which is yet another instance of where algebraization theorems such as the ones mentioned above play a fundamental role. We mention in passing that these Ax–Schanuel theorems (and their generalizations to various other settings) (see [4, 31]) have furthermore had applications to Diophantine results such as the recent proof of the Mordell conjecture by Lawrence–Venkatesh [18] and also in recent work of Lawrence–Sawin [17].

Finally, the definable Chow theorem of Peterzil and Starchenko has been generalized in the work of Bakker–Brunenbarbe–Tsimerman [3] wherein they develop a correspondence between coherent definable analytic sheaves and coherent algebraic sheaves on quasi-projective complex algebraic varieties, in the style of Serre's famous GAGA paper [39]. They further use their powerful generalization to settle a longstanding conjecture of Griffiths on the algebraicity of images of period maps.

In light of the several striking applications the aforementioned algebraization theorems have seen, it is a natural question to ask whether analogues of these algebraization results exist in the non-archimedean setting.

In [9], Cluckers–Comte–Loeser prove a p-adic analogue of the Pila–Wilkie counting theorem for p-adic subanalytic sets. The main results of this paper provide a framework for a statement and proof of a non-archimedean analogue of the algebraization theorem of Peterzil and Starchenko. In future work, we hope to provide applications of this circle of ideas towards certain functional transcendence and bi-algebraicity questions in the p-adic setting.

In the non-archimedean setting, a number of analogues of o-minimality have been studied, all with the broad goal of creating a framework that would isolate a class of subsets satisfying 'tame' topological and finiteness properties. From our perspective, an important class of such subsets is furnished by the so-called rigid subanalytic sets developed by Lipshitz [20] and further studied in a series of influential works by Lipshitz and Robinson [22–24]). The rigid subanalytic sets in a sense form the analogue of the o-minimal structure $\mathbb{R}_{\text{an}}$ consisting of the collection of restricted subanalytic subsets of $\mathbb{R}^n$. We refer the reader to the beginning of Section 2 for an overview of the notion of rigid subanalytic sets.

As a first step towards the definable Chow theorem, in Section 2 we prove the following strong version of the Riemann extension theorem in the context of rigid subanalytic sets.

Theorem (A rigid subanalytic Riemann extension theorem, Theorem 2.29). *Let K be an algebraically closed field that is complete with respect to a non-trivial, non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. Suppose X is a separated and reduced rigid analytic space over K. Let $Y \subseteq X$ be a closed analytic subvariety of X that is everywhere of positive codimension. Then any analytic function $f \in \mathcal{O}_X(X \setminus Y)$ whose graph is a locally subanalytic subset of $X(K) \times K$ extends to a meromorphic function on all of X, i.e. $f \in \mathcal{M}(X)$.*

We also prove an analogue of the definable Chow theorem in the rigid subanalytic setting. In fact, we prove this in the setting of what we refer to as ‘tame structures’. The definition of a tame structure follows very closely the definition of an o-minimal structure. In [21], Lipshitz–Robinson prove that rigid subanalytic subsets of the one-dimensional unit disk K° are none other than the subsets that are Boolean combinations of disks. Thus, it is natural to consider arbitrary structures on K° such that the definable subsets of the closed one-dimensional unit disk K° are the Boolean combinations of (open or closed) disks. This is (in an imprecise sense) what we refer to as a ‘tame structure’. We refer the reader to Section 3 for the precise definitions. It is worth mentioning that our definition of a tame structure is in fact very closely related to the notion of a C-minimal field introduced by Macpherson and Steinhorn [28]. We refer the reader to Remark 3.6 for more details on this relation.

After going through the preliminary definitions of tame structures, we prove some basic results in the dimension theory of tame structures that are needed for the proof of the definable Chow theorem. The two key results are the invariance of dimension under definable bijections and the Theorem of the Boundary.

Proposition (Invariance of dimension under definable bijections, Proposition 3.19). *Let $X \subseteq (\mathcal{O}_{\mathbb{C}_p})^m$ and $Y \subseteq (\mathcal{O}_{\mathbb{C}_p})^n$ be definable sets (in a fixed tame structure) and $f : X \to Y$ a definable bijection. Then* $\dim(X) = \dim(Y)$.

Theorem (Theorem of the Boundary, Theorem 3.21). *Let $X \subseteq (\mathcal{O}_{\mathbb{C}_p})^m$ be a definable set. Then $\dim(\mathrm{Fr}(X)) < \dim(X)$, where $\mathrm{Fr}(X)$ denotes the frontier of X in $(\mathcal{O}_{\mathbb{C}_p})^m$, that is, $\mathrm{Fr}(X) = \mathrm{cl}_{(\mathcal{O}_{\mathbb{C}_p})^m}(X) \setminus X$.*

Next we prove the following theorem which may be viewed as a definable version of a classical theorem of Liouville in complex geometry.

Proposition (A non-archimedean definable Liouville theorem, Theorem 4.6). *Let X be a reduced scheme of finite type over $\mathbb{C}_p$ and denote by X^{an} the rigid analytification of X. Let $f \in H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ be a global rigid analytic function on X^{an} such that the graph of f viewed as a subset of $X(\mathbb{C}_p) \times \mathbb{C}_p$ is definable. Then $f \in H^0(X, \mathcal{O}_X)$.*

Finally, we prove the non-archimedean version of the definable Chow theorem.

Theorem (The non-archimedean definable Chow theorem, Corollary 4.14). *Let V be a reduced algebraic variety over $\mathbb{C}_p$, and let $X \subseteq V^{\mathrm{an}}$ be a closed analytic subvariety of the rigid analytic variety V^{an} associated to V such that $X \subseteq V(\mathbb{C}_p)$ is definable in a tame structure on $\mathbb{C}_p$. Then X is algebraic.*

Outline of the paper

In Section 2, we start with some background on the theory of rigid subanalytic sets as developed by Lipshitz and Robinson in the series of papers [19–24]. In Section 2.3, we prove the strong version of the Riemann extension theorem in the rigid subanalytic category.

In Section 3 we introduce the notion of tame structures, and proceed to develop some preliminary dimension theory in this context. The theorem of the boundary and the invariance of dimensions under definable bijections are proved here. In Section 3.2, we collect some lemmas on the general dimension theory of rigid analytic varieties that will be used in the proof of the definable Chow theorem.

In Section 4, we proceed to the proof of the non-archimedean definable Chow theorem.

2. Rigid subanalytic sets and a Riemann extension theorem

In this section, we provide a brief overview of subanalytic geometry in the non-archimedean setting. Analogous to the real case, one starts by considering sets that are locally described by Boolean combinations of sets of the form $\{\underline{x} : |f(\underline{x})| \leq |g(x)|\}$ where f, g are analytic functions. As in the real case, for such sets to define a reasonable 'tame topology' one must restrict the class of analytic functions. In the non-archimedean setting, such a theory has been developed in a series of works by Leonard Lipshitz and Zachary Robinson based on the analytic functions in the 'ring of separated power series'. In the first part of this section, we summarize some of the main results of their work.

In Section 2.3, we prove a strong version of the Riemann extension theorem for rigid subanalytic sets.

Notations for Section 2. We suppose throughout this section that K is an algebraically closed field complete with respect to a non-trivial, non-archimedean absolute value $|\cdot|_K : K \to \mathbb{R}_{\geq 0}$. We denote by K° the valuation ring consisting of power bounded elements of K, and $K^{\circ\circ}$ denotes the maximal ideal of K° consisting of the topologically nilpotent elements of K. We denote by $\widetilde{K} := K^\circ / K^{\circ\circ}$ the residue field of K, and $\widetilde{\ }: K^\circ \to \widetilde{K}$ will denote the reduction map.

For a complete valued field extension $(F, |\cdot|_F)$ of $(K, |\cdot|_K)$, we denote by $F_{\mathrm{alg}} \supseteq F$ an algebraic closure of F. There is a unique multiplicative norm $|\cdot|' : F_{\mathrm{alg}} \to \mathbb{R}_{\geq 0}$ that extends the norm $|\cdot|_F$ on F. We set $F_{\mathrm{alg}}^\circ := \{x \in F_{\mathrm{alg}} : |x|' \leq 1\}$. We shall often abuse notation by denoting $|\cdot|', |\cdot|_F$, and $|\cdot|_K$ all by the same symbol $|\cdot|$.

2.1. Rings of separated power series

Definition 2.1. A valued subring $C \subseteq K^\circ$ is called a *B-ring* if every $x \in C$ with $|x| = 1$ is a unit in C.

Remark 2.2. Every B-ring C is a local ring with $C \cap K^{\circ\circ}$ being its unique maximal ideal.

Definition 2.3. A B-ring $C \subseteq K^\circ$ is said to be *quasi-Noetherian* if every ideal $\mathfrak{a} \subseteq C$ has a 'quasi-finite generating set', i.e. a zero-sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathfrak{a}$ such that any element $a \in \mathfrak{a}$ can be written in the form $a = \sum_{i \geq 0} b_i x_i$ for some $b_i \in C$. We note that we are not insisting that every infinite sum of the form $\sum_{i \geq 0} b_i x_i$ also lies in $\mathfrak{a}$.

Proposition 2.4 (Properties of quasi-Noetherian rings). *We have the following properties of quasi-Noetherian rings:*

(1) *A Noetherian B-subring of K° is quasi-Noetherian.*

(2) *If B is quasi-Noetherian and $\{a_i\}_{i\in\mathbb{N}} \subseteq K^\circ$ is a zero-sequence* (*i.e.* $\lim a_i = 0$) *then*

$$B[a_0, a_1, \ldots]_{\{a\in B[a_0,a_1,\ldots]:\,|a|=1\}}$$

is also quasi-Noetherian.

(3) *The completion of a quasi-Noetherian subring $B \subseteq K^\circ$* (*with respect to the restriction of the absolute value $|\cdot|$ to B*) *is also a quasi-Noetherian subring of K°.*

(4) *The value semigroup $|B \setminus \{0\}|$ is a discrete subset of $\mathbb{R}_{>0}$.*

Definition 2.5. (a) If R is a complete, Hausdorff topological ring whose topology is defined by a system $\{\mathfrak{a}_i\}_{i\in I}$ of ideals, we define the *ring of convergent power series* with coefficients in R as

$$R\{x_1, \ldots, x_n\} := \Big\{ \sum_{\nu=(\nu_1,\ldots,\nu_n)\in\mathbb{N}^n} a_\nu x_1^{\nu_1} \ldots x_n^{\nu_n} \in R[\![x_1, \ldots, x_n]\!] : \lim_{\nu_1+\cdots+\nu_n\to\infty} a_\nu = 0 \Big\}.$$

The topology on $R\{x_1, \ldots, x_n\}$ is defined by declaring $\{\mathfrak{a}_i \cdot R\{\underline{x}\}\}_{i\in\mathbb{N}}$ to be a fundamental system of neighbourhoods of 0. With this topology $R\{x_1, \ldots, x_n\}$ is also a complete, Hausdorff topological ring.

(b) The *Tate algebra $T_m(K)$ in m variables over K* is defined as $T_m(K) := K \otimes_{K^\circ} K^\circ\{x_1, \ldots, x_m\}$. We equip $T_m(K)$ with the *Gauss norm* which is defined as follows: $\| \sum_{i\geq 0} a_i x^i \|_{\text{Gauss}} := \max_i |a_i|$. It is a multiplicative norm on $T_m(K)$ that makes $T_m(K)$ a Banach K-algebra.

Definition 2.6 (Rings of separated power series). We fix a complete, quasi-Noetherian subring $E \subseteq K^\circ$. Denote by $\mathfrak{B}$ the following family of complete, quasi-Noetherian subrings of K°:

$$\mathfrak{B} := \{E[a_0, a_1, \ldots]^\wedge_{\{x\in E[a_0,\ldots]:\,|x|=1\}} : \{a_i\}_{i\geq 0} \subseteq K^\circ \text{ satisfies } \lim |a_i| = 0\}.$$

In the above definition, $E[a_0, a_1, \ldots]^\wedge_{\{x\in E[a_0,\ldots]:\,|x|=1\}}$ denotes the completion (with respect to the absolute value induced from K) of the localization of $E[a_0, a_1, \ldots]$ at the multiplicative subset $\{x \in E[a_0, a_1, \ldots] : |x| = 1\}$.

Define

$$S_{m,n}(E, K)^\circ := \varinjlim_{B\in\mathfrak{B}} B\{x_1, \ldots, x_m\}[\![\rho_1, \ldots, \rho_n]\!],$$

$$S_{m,n}(E, K) := S_{m,n}(E, K)^\circ \otimes_{K^\circ} K.$$

For an $f \in S_{m,n}(E, K)$ we define its Gauss norm in the usual way: writing

$$f = \sum_{\mu\in\mathbb{N}^m,\nu\in\mathbb{N}^n} b_{\mu,\nu} x_1^{\mu_1} \cdots x_m^{\mu_n} \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$$

we set $\|f\|_{\text{Gauss}} := \sup_{\mu,\nu} |b_{\mu,\nu}| = \max_{\mu,\nu} |b_{\mu,\nu}|$.

Remark 2.7. (a) We call $S_{m,n}(E,K)$ the ring of separated power series over K. When $K=\mathbb{C}_p$ for instance, we may choose E to be the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. We shall often suppress the reference to E and K in notation and write simply $S_{m,n}$.

(b) Note that $S_{m,0}=T_m(K)$ and $S_{m,n}\supseteq T_{m+n}(K)$.

Theorem 2.8 (Lipshitz–Robinson [23]). *The rings $S_{m,n}$ have the following properties:*

(1) *$S_{m,n}$ is Noetherian, a UFD, and a Jacobson ring of Krull dimension $m+n$.*

(2) *For every maximal ideal $\mathfrak{m}$ of $S_{m,n}$ the quotient ring $S_{m,n}/\mathfrak{m}$ is an algebraic extension of K. Furthermore, there is a bijection*

$$\{\mathfrak{n}\in \mathrm{Max}(K[\underline{x},\underline{\rho}]) : |x_i(\mathfrak{n})|\le 1, |\rho_j(\mathfrak{n})|<1\}\leftrightarrow \mathrm{Max}(S_{m,n}),\quad \mathfrak{n}\mapsto \mathfrak{n}\cdot S_{m,n}.$$

We now recall the notion of a generalized ring of fractions $\phi : T_m\to A$ as defined in [24].

Definition 2.9. A *quasi-affinoid algebra* over K is a topological K-algebra that is isomorphic to a quotient of $S_{m,n}(E,K)$ for some $m,n\ge 0$, endowed with the quotient topology.

We remark that since every ideal of $S_{m,n}(E,K)$ is closed, a quasi-affinoid K-algebra A equipped with the quotient topology from some surjection of K-algebras, $S_{m,n}\twoheadrightarrow A$ is naturally a Banach algebra over K. We also remark that this topology on A is independent of the presentation of A as a quotient of $S_{m,n}(E,K)$. We refer the reader to [23, Theorem 5.2.3 and Corollary 5.2.4] for a proof of this fact.

Definition 2.10 ([23, Definition 5.3.1]). Let A be a quasi-affinoid algebra, say $A=S_{m,n}/I$. Suppose $f_1,\dots,f_M,g_1,\dots,g_N,h\in A$. Then we set

$$A\langle \tfrac{f}{h}\rangle[\![\tfrac{g}{h}]\!]_s := S_{m+M,n+N}/J$$

where J is the ideal generated by the elements of I along with the elements $hx_{m+i}-f_i$ for $i=1,\dots,M$ and $h\rho_{n+j}-g_j$ for $j=1,\dots,N$.

Remark 2.11. In the above definition, the quasi-affinoid K-algebra $A\langle \frac{f}{h}\rangle[\![\frac{g}{h}]\!]_s$ is independent of the presentation of A as a quotient of a separated power series ring $S_{m,n}$. We refer the reader to [23, Theorem 5.2.6] for a proof of this.

Definition 2.12 ([24, Definition 2.1]). A *generalized ring of fractions over the Tate algebra $T_m(K)$* is a map of K-algebras $\phi : T_m(K)\to A$, where A is a quasi-affinoid algebra A over K, that is obtained via the following inductive procedure: firstly, the identity map $T_m(K)\to T_m(K)$ is a generalized ring of fractions; secondly, for $\psi : T_m(K)\to A'$ a generalized ring of fractions, and elements $f_1,\dots,f_M,g_1,\dots,g_N,h\in A'$, the composite map $T_m(K)\to A'\to A'\langle \frac{f}{h}\rangle[\![\frac{g}{h}]\!]_s$ is also a generalized ring of fractions over $T_m(K)$.

Definition 2.13. For a generalized ring of fractions $\phi : T_m(K)\to A$ over $T_m(K)$, and a complete valued field extension F of K, we define $\mathrm{Dom}(A)(F)\subseteq (F^\circ_{\mathrm{alg}})^m$ inductively as follows: for the identity map $T_m(K) \to T_m(K)$, we set $\mathrm{Dom}(T_m(K))(F) := (F^{\circ}_{\mathrm{alg}})^m$; for $\psi : T_m(K) \to A'$ a generalized ring of fractions, and elements $f_1, \ldots, f_M, g_1, \ldots, g_N, h$ in A', we set

$$\mathrm{Dom}\big(A'\langle \tfrac{f}{h}\rangle [\![\tfrac{g}{h}]\!]_s\big)(F) := \{x \in \mathrm{Dom}(A')(F) : |f_i(x)| \leq |h(x)| \neq 0 \text{ and } |g_j(x)| < |h(x)|\}.$$

Remark 2.14. We observe that for a generalized ring of fractions $\phi : T_m(K) \to A$, any element $f \in A$ defines in a natural way a locally analytic function $\mathrm{Dom}(A)(F) \to F_{\mathrm{alg}}$ (see [24, Remark 2.3 (iii)]).

Definition 2.15. (a) For a power series $f(x_1, \ldots, x_m) = \sum_{\nu \in \mathbb{N}^m} a_\nu \underline{x}^\nu$, and for every $\nu \in \mathbb{N}^m$, we define the *Hasse derivative* $D_\nu(f)(x_1, \ldots, x_m)$ to be the power series defined by the following equality of power series in $x_1, \ldots, x_m, y_1, \ldots, y_m$:

$$f(x_1 + y_1, \ldots, x_m + y_m) = \sum_\nu D_\nu(f)(\underline{x})\underline{y}^\nu.$$

We note that in characteristic zero, $D_\nu(f) = \frac{1}{\nu_1! \ldots \nu_m!} \frac{\partial^{|\nu|} f}{\partial x_1^{\nu_1} \ldots \partial x_m^{\nu_m}}$. We also remark that if f is given by a power series with coefficients in K that converges on a closed polydisk, then so do all its Hasse derivatives.

(b) For a generalized ring of fractions $\phi : T_m(K) \to A$ over $T_m(K)$ and an element $f \in A$ we denote by $\Delta(f)$ the set of all its Hasse derivatives $D_\nu(f)$ for all $\nu \in \mathbb{N}^m$.

2.2. *Rigid subanalytic sets*

Definition 2.16 (The language L of mutiplicatively valued rings). Denote by

$$L = (+, \cdot, |\cdot|, 0, 1; \bar{\cdot}, \bar{<}, \bar{0}, \bar{1})$$

the language of multiplicatively valued rings. Note that L is a two-sorted language, the operations $+, -, \cdot$ and elements $0, 1$ refer to the corresponding operations and elements of the underlying ring and $\bar{\cdot}, \bar{0}, \bar{1}$ are the underlying operations and elements of the value group along with $\{\bar{0}\}$ (which is supposed to be interpreted as $|0|$, the norm of 0, and is smaller than every element of the value group).

We set $S := \bigcup_{m,n \in \mathbb{N}} S_{m,n}(E, K)$ and $T := \bigcup_{m \geq 0} T_m$. Consider any subset $\mathcal{H} \subseteq S$ such that $\Delta(\mathcal{H}) \subseteq \mathcal{H}$. The main two examples of such $\mathcal{H}$ are provided by $\mathcal{H} = S$ or $\mathcal{H} = T$.

We now define the languages $L_{\mathcal{H}}$ introduced by Lipshitz–Robinson [24] which are used to define subanalytic sets. $L_{\mathcal{H}}$ is a three-sorted language; the first sort for the closed unit disk K°, the second sort for $K^{\circ\circ}$ the open unit disk and the last sort for the totally ordered value group with $\{\bar{0}\}$ added. The sort structure is mostly a bookkeeping device; the first sort helps us to keep track of non-strict inequalities of the form $|f| \leq |g|$, whereas the second sort helps us to keep track of strict inequalities.

Definition 2.17 (The language $L_{\mathcal{H}}$). The language $L_{\mathcal{H}}$ is the language obtained by augmenting the language L defined above by symbols for every function in $\mathcal{H}$; i.e. for every $f \in \mathcal{H}$, if $f \in S_{m,n}$ we add a function symbol to $L_{\mathcal{H}}$ with arity m for the first sort and n for the second sort. Thus,

$$L_{\mathcal{H}} := (+, \cdot, |\cdot|, 0, 1, \{f\}_{f \in \mathcal{H}}; \overline{\cdot}, \overline{<}, \overline{0}, \overline{1}).$$

Definition 2.18 (Globally $\mathcal{H}$-semianalytic and $\mathcal{H}$-subanalytic sets).

(a) For a complete, valued field F over K, a subset $X \subseteq (F_{\text{alg}}^{\circ})^m$ is said to be *globally $\mathcal{H}$-semianalytic* (resp. *$\mathcal{H}$-subanalytic*) if X is definable by a quantifier-free (resp. existential) $L_{\mathcal{H}}$-formula, i.e. there exists a quantifier-free (resp. existential) first-order formula $\phi(x_1, \ldots, x_m)$ such that $(a_1, \ldots, a_m) \in X$ if and only if $F_{\text{alg}} \models \phi(a_1, \ldots, a_m)$.

(b) In the special case that $\mathcal{H} = S$, the $\mathcal{H}$-semianalytic (resp. subanalytic) sets are referred to as the *globally quasi-affinoid semianalytic* (resp. *quasi-affinoid subanalytic*) sets. Similarly, in the case that $\mathcal{H} = T$, the $\mathcal{H}$-semianalytic (resp. subanalytic) sets are referred to as *affinoid semianalytic* (resp. *affinoid subanalytic*) sets.

The globally $\mathcal{H}$-semianalytic sets are in other words Boolean combinations of sets defined by inequalities among analytic functions in $\mathcal{H}$. Similarly, the $\mathcal{H}$-subanalytic sets, being defined by *existential* formulas, are precisely the sets obtained by coordinate projections of $\mathcal{H}$-semianalytic sets from higher dimensions.

Just as in the real subanalytic setting, one would now ask whether subanalytic sets satisfy basic closure properties. For instance, are they closed under taking complements or closures? It turns out that they are. Lipshitz–Robinson [24] prove a quantifier-simplification theorem for the language $L_{\mathcal{H}}$ (recalled below), which would imply that an *arbitrary* $L_{\mathcal{H}}$-definable set is also $\mathcal{H}$-subanalytic. Since complements and closures are all first-order definable in $\mathcal{L}_{\mathcal{H}}$, the required closure properties would then follow.

Lipshitz and Robinson's proof of the quantifier-simplification theorem for $L_{\mathcal{H}}$ is actually obtained as a consequence of a striking quantifier-*elimination* theorem in a slightly expanded language $\mathcal{L}_{\mathcal{E}(\mathcal{H})}$ which we introduce now. The expanded language $\mathcal{L}_{\mathcal{E}(\mathcal{H})}$, roughly speaking, contains function symbols for every function that is *existentially* definable from functions in $\mathcal{H}$ (the precise definitions are given below). The need to expand the language $\mathcal{L}_{\mathcal{H}}$ to include such functions is reflected in the fact that for an $f \in \mathcal{H}$, the Weierstrass data output by the Weierstrass division theorems in the context of the algebras $S_{m,n}$ are only existentially definable over $\mathcal{H}$.

We also note that for a generalized ring of fractions $\varphi : T_m \to A$ over T_m and for an element $f \in A$, the induced analytic function $f : \mathrm{Dom}(A)(F) \to F_{\text{alg}}$ might not necessarily be in $\mathcal{H}$ but is nevertheless existentially definable over $\mathcal{H}$.

Definition 2.19 (Existentially definable analytic functions [24, Definition 2.6]). Given a complete valued field extension F of K, a subset $X \subseteq (F_{\text{alg}}^{\circ})^m$, and a function $f : X \to F_{\text{alg}}$, we say that f is *existentially definable from the functions* $g_1, \ldots, g_l$ if there exists a quantifier-free formula ϕ in the language L of multiplicatively valued rings such that

$$y = f(x) \iff \exists \underline{z},\ \phi(x, y, \underline{z}, g_1(x, y, \underline{z}), \dots, g_l(x, y, \underline{z})).$$

Definition 2.20 (The expanded language $L_{\mathcal{E}(\mathcal{H})}$).

(a) We set $\mathcal{E}(\mathcal{H})$ to consist of all functions $f : \mathrm{Dom}(A)(F) \to F_{\mathrm{alg}}$ for a generalized ring of fractions $\varphi : T_m \to A$ over T_m and $f \in A$ such that all of its partial derivatives, i.e. all the functions in $\Delta(f)$, are existentially definable from functions in $\mathcal{H}$.

(b) The language $L_{\mathcal{E}(\mathcal{H})}$ is the three-sorted language obtained by augmenting $L_{\mathcal{H}}$ with function symbols for every $f \in \mathcal{E}(\mathcal{H})$.

Theorem 2.21 (The uniform quantifier elimination theorem of Lipshitz and Robinson [24]). *Fix a subset $\mathcal{H} \subseteq S$ such that $\Delta(\mathcal{H}) = \mathcal{H}$. Let $\varphi(\underline{x})$ be an $L_{\mathcal{E}(\mathcal{H})}$-formula. Then there exists a quantifier-free $L_{\mathcal{E}(\mathcal{H})}$-formula $\psi(\underline{x})$ such that for every complete valued field extension F of K we have*

$$F_{\mathrm{alg}} \models (\forall \underline{x},\ \varphi(\underline{x}) \Leftrightarrow \psi(\underline{x})).$$

Corollary 2.22 (Quantifier simplification for $L_{\mathcal{H}}$). *For every $L_{\mathcal{H}}$-formula $\varphi(\underline{x})$, there exists an* existential *$L_{\mathcal{H}}$-formula $\psi(\underline{x})$ such that for every complete valued field F extending K we have*

$$F_{\mathrm{alg}} \models (\forall \underline{x},\ \varphi(\underline{x}) \Leftrightarrow \psi(\underline{x})).$$

In other words, every $L_{\mathcal{H}}$-definable subset is in fact $\mathcal{H}$-subanalytic. In particular, the closures and complements of $\mathcal{H}$-subanalytic sets are again $\mathcal{H}$-subanalytic.

2.3. A rigid subanalytic Riemann extension theorem

In this section we prove a version of the Riemann extension theorem in the setting of rigid subanalytic sets.

Throughout this section and in everything that follows, by subanalytic (without further qualification) we shall simply mean quasi-affinoid subanalytic, i.e. $\mathcal{H}$-subanalytic with $\mathcal{H} = S = \bigcup_{m,n} S_{m,n}(E, K)$. We also recall that K is algebraically closed. We shall denote by $\mathbb{B}^d$ the d-dimensional rigid analytic closed unit disk over K, that is, $\mathbb{B}^d = \mathrm{Sp}(T_d(K))$.

It is convenient to extend the notion of subanalytic sets to subsets of K^n. We make the following definition:

Definition 2.23. A subset $\mathcal{S} \subseteq K^n$ is said to be *subanalytic* if the following equivalent conditions are satisfied:

(i) $\pi_n^{-1}(\mathcal{S}) \subseteq (K^\circ)^{n+1}$ is subanalytic, where

$$\pi_n : (K^\circ)^{n+1} \setminus \{0\} \to \mathbb{P}^n(K^\circ) = \mathbb{P}^n(K)$$

is the map $(z_0, z_1, \dots, z_n) \mapsto [z_0 : z_1 : \dots : z_n]$. We view $K^n \subseteq \mathbb{P}^n(K)$ via the map $(z_0, z_1, \dots, z_{n-1}) \mapsto [z_0 : z_1 : \dots : z_{n-1} : 1]$.

(ii) For every map $\epsilon : \{1, \dots, n\} \to \{\pm 1\}$ the set

$$\mathcal{T}_\epsilon := \{(\alpha_1, \dots, \alpha_n) \in (K^\circ)^n : \alpha_r \neq 0 \text{ if } \epsilon(r) = -1, \\ \text{and } (\alpha_1^{\epsilon(1)}, \dots, \alpha_n^{\epsilon(n)}) \in \mathcal{S}\}$$

is a subanalytic subset of $(K^\circ)^n$.

It follows that the collection of subanalytic subsets of K^n forms a Boolean algebra of subsets, closed under projections, and thus forms a structure on K in the sense of [42, Chapter 1, (2.1)].

Before proceeding to the subanalytic Riemann extension theorem, we recall for the reader some of the notations and terminology from rigid analytic geometry, which we shall use henceforth.

Recollections and conventions from rigid geometry

We recall that given an affinoid algebra A over K, i.e. a quotient of some Tate algebra $T_n(K)$, the affinoid space over K attached to A [7] is denoted by $\mathrm{Sp}(A)$. As a set, it is the set of maximal ideals of A. The affinoid space $\mathrm{Sp}(A)$ comes equipped with a Grothendieck topology (which we refer to as the *G-topology*) consisting of a collection of *admissible opens* and *admissible coverings* (see [7, Section 3.3 and Ch. 5]). Furthermore, there is a natural sheaf of K-algebras on the G-topology on $\mathrm{Sp}(A)$. A *rigid analytic variety* or *space* for us is a locally G-ringed K-space that is locally isomorphic to an affinoid space over K [7, Section 5.3]. We refer the reader to [7, Section 6.3, Definitions 1, 2, 6 and 8] for the definitions of *closed immersions and quasi-compact, separated, quasi-separated,* and *proper morphisms* of rigid analytic spaces over K. Furthermore, to every algebraic variety V over K, one may functorially attach a rigid analytic variety (called the *rigid analytification* or the *associated rigid analytic variety*), denoted by V^{an}, which comes equipped with a map of locally G-ringed K-spaces $a_V : V^{\mathrm{an}} \to V$ [7, Section 5.4].

Definition 2.24. Let X be a separated rigid analytic variety over K and let $S \subseteq X$ be a subset. Then we say that S is *locally subanalytic* in X if there exists an admissible cover by admissible affinoid opens $X = \bigcup_i X_i$ and closed immersions $\beta_i : X_i \hookrightarrow \mathbb{B}^{d_i}$ such that for all i, $\beta_i(S \cap X_i)$ is subanalytic in $(K^\circ)^{d_i}$.

It is easy to see that if the above condition is true for one admissible affinoid cover and some choice of embeddings β_i, then it is true for any other such cover and embeddings.

Definition 2.25. Let V/K be a finite-type reduced scheme over K. We say that a subset $S \subseteq V(K)$ is *subanalytic* if there exists a finite affine open cover $V = \bigcup_i U_i = \bigcup_i \mathrm{Spec}(A_i)$ and closed embeddings $U_i(K) \overset{\beta_i}{\hookrightarrow} K^{n_i}$ (arising from a presentation of A_i as a quotient of $K[t_1, \dots, t_{n_i}]$) such that for all i, $\beta_i(S \cap U_i(K))$ is subanalytic.

Remark 2.26. We note that if $S \subseteq V(K)$ is subanalytic, then for *every* finite affine open cover U_i of V and for any choice of presentations $\beta_i : K[t_1, \dots, t_{n_i}] \twoheadrightarrow \mathcal{O}(U_i)$, the subset $\beta_i(S \cap U_i(K)) \subseteq K^{n_i}$ is subanalytic.

Remark 2.27. Suppose V is a separated finite type scheme over K and V^{an} is the associated rigid analytic variety, with analytification map $a_V : V^{\mathrm{an}} \to V$. Then the map a_V need not necessarily take a locally subanalytic set on V^{an} to a subanalytic subset of $V(K)$ in the sense of Definition 2.25. Indeed, consider the affine line $\mathbb{A}^1_{\mathbb{C}_p}$, and the subset $S := \bigcup_{n \geq 0} \{z \in \mathbb{C}_p : |p^{-2n}| \leq |z| \leq |p^{-(2n+1)}|\}$. Then S is not a rigid subanalytic subset of the algebraic affine line $\mathbb{A}^1(\mathbb{C}_p)$, although it is a locally subanalytic subset of the analytification $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$.

But if V is proper then locally subanalytic sets of V^{an} are indeed subanalytic in $V(K)$.

Lemma 2.28. *Suppose V is a variety over K. Let V^{an} denote the associated rigid analytic space over K, with analytification map $a_V : V^{\mathrm{an}} \to V$. If $S \subseteq V(K)$ is subanalytic as in Definition 2.25 then $a_V^{-1}(S) \subseteq V^{\mathrm{an}}$ is locally subanalytic as in Definition 2.24.*

Moreover, if V is proper *over K then the converse holds, i.e. $S \subseteq V(K)$ is subanalytic $\Leftrightarrow a_V^{-1}(S) \subseteq V^{\mathrm{an}}$ is locally subanalytic.*

Proof. This follows from the fact that proper rigid spaces are quasicompact, and in particular, when V is proper, V^{an} has an admissible covering by *finitely many* affinoids. ∎

We now turn to the proof of the following version of the Riemann extension theorem.

Theorem 2.29. *Suppose X is a separated and reduced rigid analytic space over the algebraically closed field K. Let $Y \subseteq X$ be a closed analytic subvariety of X that is everywhere of positive codimension. Then any analytic function $f \in \mathcal{O}_X(X \setminus Y)$ whose graph is a locally subanalytic subset of $X(K) \times K$ extends to a meromorphic function on all of X, i.e. $f \in \mathcal{M}(X)$.*

Outline of the proof

The proof is inspired by Lütkebohmert's proof of the usual non-archimedean Riemann extension theorem [26]. We make a series of reductions in the course of the proof. We summarize the main reduction steps below.

- *Step 1:* The question of extending f meromorphically along X is local for the G-topology of X and thus we may assume that $X = \mathrm{Sp}(A)$ is a reduced affinoid. Further, working over irreducible components of X, we also assume that $X = \mathrm{Sp}(A)$ is irreducible and thus that A is an integral domain.
- *Step 2:* Choose a Noether normalization $\pi : X \to \mathbb{B}^d$. We show in Lemma 2.34 that if we prove our theorem for $\mathbb{B}^d$ and the analytic subset $\pi(Y) \subseteq \mathbb{B}^d$, we can deduce the theorem for X. Thus, we may assume $X = \mathbb{B}^d$ is the d-dimensional rigid unit disk over K.

- *Step 3:* Since $\mathrm{Sing}(Y)$ is of codimension at least 2 in X, by the non-archimedean Levi extension theorem [26, Theorem 4.1] it suffices to extend f meromorphically to an $f^* \in \mathcal{M}(X \setminus \mathrm{Sing}(Y))$. Replace X, Y by $X \setminus \mathrm{Sing}(Y), Y \setminus \mathrm{Sing}(Y)$ respectively. Once more using Step 1, we reduce to the case where Y is regular/smooth and X is an affinoid subdomain of $\mathbb{B}^d$.
- *Step 4:* Since X and Y are smooth over the algebraically closed field K, we are now in a position to use a result of Kiehl (recalled below, Theorem 2.32) which tells us that *locally* $Y \subseteq X$ looks like $Z \times \{0\} \subseteq Z \times \mathbb{B}^n$ for a smooth affinoid space Z. We may even assume that $n = 1$ since if Y is codimension at least 2, the result we seek is a special case of the non-archimedean Levi extension theorem. All in all, we are down to the case where $X = Z \times \mathbb{B}^1$ and $Y = Z \times \{0\}$ for a smooth, reduced affinoid space Z over K.
- *Step 5:* This final case is proved separately in Lemma 2.35.

We first recall Kiehl's tubular neighbourhood result. We need the following definition.

Definition 2.30 ([16, Definition 1.11]). We say that an affinoid algebra A over the non-trivially valued non-archimedean field k is *absolutely regular at a maximal ideal* x of A if for every complete valued field K extending k and for every maximal ideal y of $A \widehat{\otimes}_k K$ above x, the localization $(A \widehat{\otimes}_k K)_y$ is a regular local ring. If the affinoid algebra A over k is absolutely regular at each of its maximal ideals, we say that A is absolutely regular.

Remark 2.31. For a maximal ideal x of an affinoid algebra A over an *algebraically closed* (or more generally perfect) non-archimedean field k, A is absolutely regular at x if and only if the localization A_x is a regular local ring.

Theorem 2.32 (Kiehl's tubular neighbourhood theorem, [16, Theorem 1.18]). *Suppose A is an affinoid algebra over a non-trivially valued non-archimedean field k and let $\mathfrak{a}$ be an ideal of A generated by $f_1, \ldots, f_l \in A$. Suppose that the quotient affinoid algebra $A/\mathfrak{a}$ is absolutely regular and that A is absolutely regular at every point of $V(\mathfrak{a})$. Then there exists an $\epsilon \in k^\times$ such that the 'ϵ-tube' around $V(\mathfrak{a})$,*

$$\mathrm{Sp}(B) := \{x \in \mathrm{Sp}(A) : |f_j(x)| \leq |\epsilon|, \forall j = 1, \ldots, l\}$$

has an admissible affinoid covering $(\mathrm{Sp}(B_i) \to \mathrm{Sp}(B))$, $i = 1, \ldots, r$, along with isomorphisms

$$\phi_i : (B_i/\mathfrak{a}B_i)\{x_1, \ldots, x_{n_i}\} \xrightarrow{\cong} B_i$$

from the free affinoid algebra over $B_i/\mathfrak{a}B_i$ in the variables $x_1, \ldots, x_{n_i}$, such that the elements $\phi_i(X_1), \ldots, \phi_i(x_{n_i})$ generate the ideal $\mathfrak{a}B_i$.

We recall a result on the number of zeroes of a convergent power series in one variable that will be used in our proof, and then prove a lemma that allows us to make the reduction mentioned in Step 2.

Lemma 2.33. *Suppose $f(t) \in K\{t\}$ is an element of the one-dimensional Tate algebra over K. Let $\epsilon(f) := \max\{i \geq 0 : |a_i| = \|f\|_{\text{Gauss}}\}$. Then the number of zeroes of f (counting multiplicities) in the closed unit disk K° is at least $\epsilon(f)$.*

Proof. Note that f is "t-distinguished" of degree $\epsilon(f)$ (see [8, Section 5.2.1, Definition 1]). By the Weierstrass preparation theorem for Tate algebras [8, Section 5.2.2, Theorem 1] we may write $f = e \cdot \omega$ where $e \in K\{t\}^\times$ and $\omega \in K[t]$ is a polynomial of degree $\epsilon(f)$, and ω has $\epsilon(f)$ zeroes (counting multiplicities) in K°. ■

Lemma 2.34. *Let $\pi : X \to S$ be a finite morphism of reduced, irreducible affinoids over K of the same dimension. Suppose $Y \subseteq X$ is a closed analytic subvariety of X. Let $T := \pi(Y)$. Suppose that every analytic function $g \in \mathcal{O}_S(S \setminus T)$ extends uniquely to a meromorphic function $g^* \in \mathcal{M}(S)$ on S. Then every analytic function $f \in \mathcal{O}_X(X \setminus Y)$ extends to a meromorphic function on X.*

Proof. Following the proof of [26, Satz 1.7], we see that for any $f \in \mathcal{O}_X(X \setminus Y)$ there is a meromorphic $f^* \in \mathcal{M}(X)$ such that $f^*|_{X \setminus \pi^{-1}(T)} = f|_{X \setminus \pi^{-1}(T)}$. However, two meromorphic functions that agree on the complement of a positive-codimensional analytic subvariety must agree everywhere [26, Lemma 1.1]. It thus follows that $f^*|_{X \setminus Y} = f$. ■

We are now fully equipped to prove Theorem 2.29.

Proof of the rigid subanalytic Riemann extension theorem, Theorem 2.29. We reduce to the case where X is a reduced affinoid space over K. Indeed, to make this reduction consider an admissible covering of X by affinoid subdomains, $X = \bigcup_{i \in I} U_i$. For each $i \in I$, $U_i \cap Y$ is an analytic subvariety of U_i of positive codimension at every point of U_i and furthermore $f|_{U_i \setminus Y}$ has a locally subanalytic graph in $U_i(K) \times K$. Suppose that we are able to find, for every i, meromorphic functions $f_i^* \in \mathcal{M}(U_i)$ such that $f_i^*|_{U_i \setminus Y} = f|_{U_i \setminus Y}$. Then for any $i, j \in I$ the meromorphic functions $f_i^*|_{U_i \cap U_j}$ and $f_j^*|_{U_i \cap U_j}$ agree on the complement of the positive-codimensional subvariety $Y \cap U_i \cap U_j$ and hence by [26, Lemma 1.1] they agree on $U_i \cap U_j$. Thus, the $\{f_i^*\}_{i \in I}$ glue to a global meromorphic function f^* on X extending f. We may thus assume henceforth that $X = \text{Sp}(A)$ is a reduced affinoid space over K.

By working on the irreducible components of X we may assume that X is irreducible, and hence that A is an integral domain. Choose a Noether normalization for X, i.e. a finite surjective morphism $\pi : X \to \mathbb{B}^d$ where $d = \dim(X)$; with the help of Lemma 2.34 we further assume that $X = \mathbb{B}^d$ is the d-dimensional unit disk over K.

Let $f \in \mathcal{O}(\mathbb{B}^d \setminus Y)$ be an analytic function, so that its graph is locally subanalytic. In order to show that f extends meromorphically to X, it suffices to show that f extends to an $f^* \in \mathcal{M}(\mathbb{B}^d \setminus \text{Sing}(Y))$ such that $f^*|_{\mathbb{B}^d \setminus Y} = f$. Indeed, since $\text{Sing}(Y)$ is an analytic subset of codimension at least 2 in $\mathbb{B}^d$, we have an isomorphism $\mathcal{M}(\mathbb{B}^d) \xrightarrow{\cong} \mathcal{M}(\mathbb{B}^d \setminus \text{Sing}(Y))$ by the non-archimedean Levi extension theorem [26, Theorem 4.1]. Consider an admissible affinoid covering $\mathbb{B}^d \setminus \text{Sing}(Y) = \bigcup_i U_i$. We remark that affinoid subdomains that are finite unions of rational subdomains are indeed rigid subanalytic sets and hence $f|_{U_i\setminus Y}$ has a locally subanalytic graph. Using [26, Lemma 1.1], we may work individually over each U_i, i.e. we are reduced to proving the theorem in the situation where $X = \mathrm{Sp}(A)$ is an affinoid subdomain of $\mathbb{B}^d$ and $Y \subseteq X$ is a *regular* analytic subvariety of X of positive codimension everywhere.

Applying the 'tubular neighbourhood' result of Kiehl [16, Theorem 1.18], we obtain an admissible covering $(\mathrm{Sp}(B_i) \to \mathrm{Sp}(B))$, $i = 1, \dots, l$, of some 'ϵ-tube' $\mathrm{Sp}(B)$ around Y in $X = \mathrm{Sp}(A)$. It suffices now to prove that for every $i = 1, \dots, l$, $f|_{\mathrm{Sp}(B_i)\setminus Y}$ extends to a meromorphic function $f_i^* \in \mathcal{M}(\mathrm{Sp}(B_i))$. Indeed, the f_i^* must necessarily glue to a meromorphic function $f^* \in \mathcal{M}(\mathrm{Sp}(B))$ (using [26, Lemma 1.1]) such that $f^*|_{\mathrm{Sp}(B)\setminus Y} = f|_{\mathrm{Sp}(B)\setminus Y}$. Since the functions $f^* \in \mathcal{M}(\mathrm{Sp}(B))$ and $f \in \mathcal{O}(X \setminus Y)$ agree on the intersection $\mathrm{Sp}(B) \setminus Y$ and noting that $\mathrm{Sp}(B) \cup (X \setminus Y) = X$ is an *admissible* open cover of $X = \mathrm{Sp}(A)$, the sections f^* and f glue to a global meromorphic function on X.

We are thus reduced to proving the theorem in the case that $X = \mathrm{Sp}(B_i/\mathfrak{a}B_i) \times \mathbb{B}^{n_i}$ and $Y = \mathrm{Sp}(B_i/\mathfrak{a}B_i) \times \{\underline{0}\}$. If $n_i \geq 2$, then the codimension of Y is at least 2, and in this case the theorem follows as a special case of the non-archimedean Levi extension theorem [26, Theorem 4.1]. Thus we may even assume that $n_i = 1$. All in all, we are reduced to proving the special case of the theorem in Lemma 2.35 below. ■

Lemma 2.35. *Suppose $Z = \mathrm{Sp}(A)$ is a reduced, irreducible affinoid space, $X = Z \times \mathbb{B}^1$, and $Y = Z \times \{0\} \subseteq X$. Then every analytic function $f \in \mathcal{O}(X \setminus Y)$ whose graph is a locally subanalytic subset of $X \times K$ extends to a meromorphic function $f^* \in \mathcal{M}(X)$.*

Proof. We denote the coordinate on $\mathbb{B}^1$ as t. Let $|\cdot|$ represent the supremum norm on the reduced affinoid A. Note that A is a Banach algebra over K when endowed with its supremum norm, and the supremum norm is equivalent to any residue norm on A (see [8, Section 6.2.4, Theorem 1]).

We may expand $f(t) = \sum_{i\geq 0} a_i t^i + \sum_{j>0} b_j t^{-j}$ with $a_i, b_j \in A$ such that $\lim_i |a_i| = 0$ and for every $R > 0$, $\lim_j |b_j| R^j = 0$. Since $\sum_i a_i t^i \in A\{t\}$, we know that $\sum_i a_i t^i$ is a rigid subanalytic function on $Z \times \mathbb{B}^1$, and thus the function $g := \sum_{j>0} b_j t^{-j}$ defined on $X \setminus Y$ has a graph that is a locally subanalytic subset of $X \times K$. In particular, for each $z \in Z$, the function $g(z,t) = \sum_{j>0} b_j(z) t^{-j}$ on the punctured disk $\mathbb{B}^1 \setminus \{0\}$ is also locally subanalytic. Since discrete subanalytic sets must be finite, we find that for each fixed z either $g(z,t)$ is identically zero on $\mathbb{B}^1 \setminus \{0\}$, or $g(z,t)$ has finitely many zeroes in $\mathbb{B}^1 \setminus \{0\}$.

Consider $h_z(y) := g(z, y^{-1}) = \sum_{j>0} b_j(z) y^j$. The growth hypothesis

$$\forall R \in K^\times, \quad \lim_{j\to 0} |b_j|\,|R|^j = 0$$

on the b_j implies that $h_z(y) \in K\{R^{-1}y\}$ for every $R \in K^\times$. The number of zeroes of $g(z,t)$ on the annulus $|R|^{-1} \leq |t| \leq 1$ is the number of zeroes of $h_z(y)$ on $1 \leq |y| \leq |R|$. For each $R \in K^\times$, we set $h_{z,R}(y) := \sum_{j>0} (b_j R^{-j}) y^j$ so that by Lemma 2.33 the number of zeroes of $h_z(y)$ on the closed disk $|y| \leq R$ is given by $\epsilon(h_{z,R}(y))$.

Now for $i < j$ if $b_i(z), b_j(z) \neq 0$, then for R large enough $|b_i(z)|R^i \leq |b_j(z)|R^j$ and thus $\epsilon(h_{z,R}) \geq j$. Thus, if $b_j(z) \neq 0$ for infinitely many j, then $h_z(y)$ has infinitely many zeroes going off to ∞ and therefore also $g(z,t)$ has an infinite discrete zero set in $\mathbb{B}^1 \setminus \{0\}$, which as noted above is not possible. Thus, for each $z \in Z$, $b_j(z)$ is eventually 0. In other words, $Z = \bigcup_{m\geq 0} \bigcap_{j>m} V(b_j)$. If the set $\bigcap_{j>m} V(b_j)$ is not equal to Z then it is a nowhere dense closed subset of Z. By the Baire category theorem, Z cannot be a countable union of nowhere dense closed subsets and therefore for large enough m, $\bigcap_{j>m} V(b_j) = V(\sum_{j>m}(b_j))$ must be equal to Z. Since Z is reduced, this means that the $b_j \in A$ are eventually zero. Thus f has a finite order pole along Y and hence extends meromorphically. This completes the proof of the lemma and thus also of Theorem 2.29.

∎

3. Tame structures

In this section we introduce the notion of a tame structure. The definition of a tame structure closely follows the definition of an o-minimal structure on $\mathbb{R}$ and is suitably adapted as a generalization of the non-archimedean rigid subanalytic sets discussed in the previous section. Let K be a non-trivially valued non-archimedean field with valuation ring R and totally ordered value group $(\Gamma, <)$. A 'structure' on R is going to be a collection of subsets of R^n for every $n \geq 0$. In fact, it turns out to be convenient to keep track of definable subsets of the value group Γ as well. Thus, in this setting a 'structure' on R is actually a collection of subsets of $R^m \times \Gamma^n$ for $m, n \geq 0$ that is closed under the natural first-order operations (see Definition 3.1 for the precise conditions). A 'tame structure' is then defined to be one where the definable subsets of R are precisely the Boolean combinations of disks of R. In Section 3.1, we provide these preliminary definitions and prove some elementary properties of sets definable in tame structures.

In Section 3.1.2, we develop the basic dimension theory of sets definable in tame structures. As in the o-minimal setting, the dimension of a non-empty definable set $X \subseteq R^m$ is defined as the largest $d \leq m$ such that for some coordinate projection $\pi : R^m \to R^d$ the interior of $\pi(X)$ in R^d is non-empty. The two key results we prove in this section are:

- the invariance of dimension under definable bijections (Proposition 3.19),
- the Theorem of the Boundary, Theorem 3.21, which states that for a definable set $X \subseteq R^m$, $\dim(\mathrm{cl}(X) \setminus X) < \dim(X)$.

The purpose of Section 3.2 is to collect together some results in the dimension theory of rigid geometry that we need. Most importantly, we connect the usual notion of dimension in the rigid analytic setting with the concept of definable dimension of the previous section (Lemma 3.26). We also prove, in Lemma 3.28, a result on the dimensions of local rings of equidimensional rigid varieties. This lemma is used in the course of the proof of the definable Chow theorem.

Notations and conventions for this section

For a subset X of a topological space Y endowed with the subspace topology, the interior, closure, and frontier of X inside Y are denoted by $\mathrm{int}_Y(X)$, $\mathrm{cl}_Y(X)$ and $\mathrm{Fr}_Y(X)$

respectively. We often omit writing the subscript Y when the ambient topological space is clear from the context. We recall that the frontier of X in Y is defined as $\mathrm{Fr}_Y(X) := \mathrm{cl}_Y(X) \setminus X$. We shall call a map between two sets $f : X \to Y$ *quasi-finite* if for every $y \in Y$, $f^{-1}(y)$ is a finite set.

K denotes a field complete with respect to a non-trivial non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. R denotes the valuation ring of K, $\mathfrak{m} \subset R$ the unique maximal ideal of R, $k := R/\mathfrak{m}$ the residue field of K, $\Gamma^\times := |K^\times|$ the value group of K, and $\Gamma := \Gamma^\times \cup \{0\}$. We choose a *pseudo-uniformizer* $\varpi \in K^\times$, i.e. a non-zero element $\varpi \in R$ with $|\varpi| < 1$.

For an element $\underline{x} = (x_1, \ldots, x_n) \in K^n$, we set $\|\underline{x}\| := \max_{1 \leq i \leq n} |x_i|$. For $\underline{x} = (x_1, \ldots, x_n) \in K^n$ and $\underline{r} = (r_1, \ldots, r_n) \in \Gamma^n$, denote by $\mathbb{D}(\underline{x}; \underline{r}) := \{\underline{y} = (y_1, \ldots, y_n) \in K^n : |x_i - y_i| < r_i \text{ for all } i\}$ and let $\overline{\mathbb{D}}(\underline{x}; \underline{r}) := \{\underline{y} \in K^n : |x_i - y_i| \leq r_i \text{ for all } i\}$. The set $\mathbb{D}(\underline{x}, \underline{r})$ is referred to as an *open polydisk* (or *simply open disk*) *of polyradius* $\underline{r}$ and $\overline{\mathbb{D}}(\underline{x}, \underline{r})$ as the *closed polydisk/disk of polyradius* $\underline{r}$.

We shall assume from now on that K is second-countable, i.e. K has a countable dense subset. However, when working with the collection of $\mathcal{H}$-subanalytic sets, the hypothesis of second countability can be eliminated from most statements (see Remark 3.23).

3.1. Preliminaries

3.1.1. Tame structures.

Definition 3.1. A *structure* on (R, Γ) is a collection $(\mathfrak{S}_{m,n})_{m,n \geq 0}$ where each $\mathfrak{S}_{m,n}$ is a collection of subsets of $R^m \times \Gamma^n$ with the following properties:

(i) $\mathfrak{S}_{m,n}$ is a Boolean algebra of subsets of $R^m \times \Gamma^n$.

(ii) If $\mathcal{S} \in \mathfrak{S}_{m,n}$ then $R \times \mathcal{S} \in \mathfrak{S}_{m+1,n}$ and $\mathcal{S} \times \Gamma \in \mathfrak{S}_{m,n+1}$.

(iii) The diagonal $\{(x, x) : x \in R\}$ is in $\mathfrak{S}_{2,0}$, and similarly $\{(\alpha, \alpha) \in \Gamma^2 : \alpha \in \Gamma\} \in \mathfrak{S}_{0,2}$.

(iv) If $\mathcal{S} \in \mathfrak{S}_{m,n}$ then $\mathrm{pr}(\mathcal{S}) \in \mathfrak{S}_{m-1,n}$ and $\mathrm{pr}'(\mathcal{S}) \in \mathfrak{S}_{m,n-1}$, where $\mathrm{pr} : R^m \times \Gamma^n \to R^{m-1} \times \Gamma^n$ denotes the projection forgetting the last R factor and similarly $\mathrm{pr}' : R^m \times \Gamma^n \to R^m \times \Gamma^{n-1}$ denotes the projection omitting the last Γ factor.

Definition 3.2. We say that a structure $(\mathfrak{S}_{m,n})_{m,n \geq 0}$ on (R, Γ) is *tame* if

- $+, \cdot : R^2 \to R$ are definable, i.e. their graphs are in $\mathfrak{S}_{3,0}$,
- $|\cdot| : R \to \Gamma$ is definable, i.e. its graph $\{(x, |x|) : x \in R\} \subseteq R \times \Gamma$ is in $\mathfrak{S}_{1,1}$,
- $\mathfrak{S}_{1,0}$ is the collection of subsets of R consisting of the Boolean combinations of disks (open or closed).

Remark 3.3. (a) It follows from the axioms that in a tame structure (R, Γ), the ordering on Γ is also definable, i.e. the set $\{(\lambda, \mu) \in \Gamma^2 : \lambda < \mu\}$ is in $\mathfrak{S}_{0,2}$.

(b) We also remark that $\mathfrak{S}_{0,1}$ is the collection of finite unions of (open) intervals and points in the totally ordered abelian group Γ. This follows from the fact that $\mathfrak{S}_{1,0}$ is the collection of subsets of R consisting of the Boolean combinations of (open or closed) disks, and that $|\cdot| : R \to \Gamma$ is definable.

(c) The axioms for a tame structure on (R, Γ) imply also that the ordered abelian group $\Gamma^\times$ is divisible.

Remark 3.4. Although we have not explicitly assumed in this section that K is algebraically closed, in most cases of interest (see Lemma 3.5 below for the precise conditions), a tame structure on (R, Γ) can only exist when K is algebraically closed. However, the assumption that K be algebraically closed is not needed to carry out the proofs of this section, and so we defer making this assumption to the next section.

Lemma 3.5. *Suppose* (R, Γ) *admits a tame structure. Then:*

(a) *The multiplicative group* $K^\times$ *is divisible. In particular, the multiplicative group of the residue field,* $k^\times$*, is divisible. Therefore, if* $|k| \neq 2$*, then* k *cannot be finite.*

(b) *If* K *has positive characteristic* p*, then* K *does not admit any Artin–Schreier extensions. In other words, the map* $\psi : K \to K$ *sending* $x \in K$ *to* $x^p - x$ *is surjective.*

(c) *The residue field* k *is a* minimal field (*in the sense that every subset of* k*, definable with parameters in the first-order language of rings, is either finite or cofinite in* k). *Thus, if* $|k| \neq 2$ *and* k *has positive characteristic, then* k *is algebraically closed.*

(d) *If* $|k| \neq 2$ *and* k *has positive characteristic, then* K *is algebraically closed. If* k *has characteristic zero, and if the conjecture of Podewski* [37] (*that every minimal field of characteristic zero is algebraically closed*) *holds, then* K *is algebraically closed.*

Proof. (a) For $n \geq 1$, let $P_n := \{y^n : y \in R\}$ be the set of nth powers in R. To show that $K^\times$ is divisible, it suffices to prove that for every $x \in R$ with $0 < |x| < 1$ and for every $n \geq 1$, we have $x \in P_n$. Being definable in R in any tame structure, P_n is a finite Boolean combination of (open or closed) disks. On the other hand, P_n contains elements arbitrary close to 0, and thus there is an $\epsilon > 0$ such that $\mathbb{D}(0, \epsilon) \subseteq P_n$. Given any x in R with $0 < |x| < 1$, pick a sufficiently large number D, coprime to n, such that $|x^D| < \epsilon$. Then there is a $y \in R$ such that $y^n = x^D$. Let E be an integer such that $D \cdot E \equiv 1 \bmod n$. Suppose $D \cdot E = 1 + mn$ for some $m \in \mathbb{Z}$. Then $x = (y^E/x^m)^n$. Thus, $x \in P_n$.

(b) Suppose now that $\operatorname{char}(K) = p > 0$. Then from (a) it follows that $K^\times$ is (uniquely) p-divisible. The map $\psi : K \to K$ sending x to $x^p - x$ has definable image, and contains elements of arbitrarily small absolute value and of arbitrarily large absolute value. Thus, there is an $\epsilon > 0$ such that for $x \in K$ if $|x| < \epsilon$ or $|x| > \epsilon^{-1}$, then $x \in \psi(K)$. Now, given any $x \in K$ with $|x| \neq 1$, we see from the above that for all sufficiently large m, $x^{p^m} \in \psi(K)$ and thus by the p-divisibility of $K^\times$ we also have $x \in \psi(K)$. If $|x| = 1$, we note that $|x - (\varpi^{-p} - \varpi^{-1})| = |\varpi^{-p}| > 1$, and therefore $x - (\varpi^{-p} - \varpi^{-1}) = x - \psi(\varpi^{-1}) \in \psi(K)$ and so $x \in \psi(K)$ as well.

(c) Any tame structure on (R, Γ) induces naturally a first-order structure (in the sense of [42, Chapter 1, (2.1)]) on k. Namely, we declare a subset of k^n to be definable if and only if its preimage under $R^n \twoheadrightarrow k^n$ is definable in R^n in the given tame structure. It is easy to check that this indeed defines a structure on k. It follows that the field operations $+, \cdot$ on k are definable with respect to this structure. Furthermore, since in any tame structure the definable sets of R are precisely the Boolean combinations of (open or closed) disks, it follows that the definable subsets of k are necessarily just the subsets of k that are either finite or have finite complement. In other words, this structure on k makes k a minimal field. It is well-known that every infinite, minimal field of positive characteristic is algebraically closed (see [43]). This fact in conjunction with (b) implies that if $|k| \neq 2$ and if k has positive characteristic then k is algebraically closed.

(d) This follows from (a)–(c) above, combined with the divisibility of $\Gamma^\times$ and [27, Lemma 7]. ∎

Remark 3.6 (The relation to C-minimal structures). We point out that the notion of a tame structure is closely related to the definition of a C-minimal field, which is a special case of the notion of a C-minimal structure. The theory of C-minimal structures was introduced by Macpherson and Steinhorn [28] (building upon some work by Adeleke–Neumann [1]), and was further developed by Haskell and Macpherson [14]. A C-relation is a ternary relation $C(x, y, z)$ satisfying certain axioms. We refer the reader to the above papers for the precise definitions. From our point of view, the central examples of C-minimal structures arise in the context of algebraically closed, non-trivially valued fields. Given such a field K with a (multiplicatively written) non-trivial valuation $|\cdot| : K \to \Gamma^\times \cup \{0\}$ into a totally ordered abelian group $(\Gamma^\times, 1, \cdot, <)$, there is a natural C-relation that one may define:

$$C(x, y, z) \iff |x - y| > |y - z|.$$

For an expansion $(K, C, 0, 1, +, -, \cdot, \dots)$ of the C-structure (K, C) to be C-minimal it is necessary then that the definable subsets of K (in the expanded language) are precisely the class of Boolean combinations of disks. However, it appears that this might not be sufficient to claim that the structure is C-minimal, since for C-minimality one requires the same property to hold for *every* structure elementarily equivalent to $(K, C, \dots)$. The expansion of an algebraically closed non-trivially valued field with function symbols for elements of its strictly convergent power series rings (or more generally separated power series rings) is in fact a C-minimal expansion of the valued field. Thus, the rigid subanalytic sets discussed above are in fact examples of C-minimal structures. In the general context of C-minimal structures, Haskell–Macpherson [14, Section 4] also prove some of the dimension theory results that we prove for tame structures in Section 3.1.2. Nevertheless, we have retained the definition of a tame structure and the following results in their dimension theory to keep the exposition self-contained. Secondly, the proofs we provide in this context are geometric and fairly elementary. Lastly, it appears that the invariance of dimensions under definable bijections is not known in the general setting of C-minimal structures or even for general C-minimal fields (see the discussion in [14, p. 159]). On the other hand, the invariance of dimensions is well-known in several other frameworks of tame geometry on Henselian valued fields, such as v-minimality and b-minimality (see for instance [10]).

For the remainder of this section, we fix a tame structure on (R, Γ), and definability of sets and maps will be with reference to this fixed structure.

Example 3.7 (Rigid subanalytic sets). Suppose K is algebraically closed. For such a K, the central example of a tame structure is those of the rigid subanalytic subsets of Lipshitz [24] and the $\mathcal{H}$-subanalytic sets defined in [24]. Indeed, it is proved in [21] that the subanalytic subsets of R are exactly the Boolean combinations of disks.

It will also be convenient to talk about definable subsets of K^n. We make the following definition:

Definition 3.8. We say that a subset $\mathcal{S} \subseteq K^n$ is *definable* if the following equivalent conditions are satisfied:

(i) $\pi_n^{-1}(\mathcal{S}) \subseteq R^{n+1}$ is definable, where

$$\pi_n : R^{n+1} \setminus \{0\} \to \mathbb{P}^n(R) = \mathbb{P}^n(K), \quad (z_0, z_1, \ldots, z_n) \mapsto [z_0 : z_1 : \ldots : z_n].$$

We view $K^n \subseteq \mathbb{P}^n(K)$ via the map $(z_0, z_1, \ldots, z_{n-1}) \mapsto [z_0 : z_1 : \ldots : z_{n-1} : 1]$.

(ii) For every map $\epsilon : \{1, \ldots, n\} \to \{\pm 1\}$ the set

$$\mathcal{T}_\epsilon := \{(\alpha_1, \ldots, \alpha_n) \in R^n : \alpha_r \neq 0 \text{ if } \epsilon(r) = -1, \\ \text{and } (\alpha_1^{\epsilon(1)}, \ldots, \alpha_i^{\epsilon(i)}, \ldots, \alpha_n^{\epsilon(n)}) \in \mathcal{S}\}$$

is a definable subset of R^n.

It follows that the collection of definable subsets of K^n is a Boolean algebra of subsets, closed under projections, and moreover forms a structure on K in the sense of [42, Chapter 1, (2.1)].

Lemma 3.9 (Basic properties of definable sets and functions). (i) *A polynomial map $\phi : K^n \to K^m$ is definable (i.e. its graph is a definable subset of K^{n+m}). In particular, zero sets of polynomials with K-coefficients are definable subsets of K^n.*

(ii) *For definable functions $f, g : K^n \to K$, the set $\{\underline{z} \in K^n : |f(\underline{z})| \leq |g(\underline{z})|\}$ is a definable subset of K^n.*

(iii) *For a definable function $f : S \to K$ on a definable subset $S \subseteq K^m$, the set $|f(S)| \subseteq \Gamma$ is a finite union of open intervals and points.*

(iv) *Suppose $f : K^n \to K$ is a definable function that is given by a convergent power series $f(z_1, \ldots, z_n) = \sum_{i \geq 0} a_i(z_1, \ldots, z_{n-1}) z_n^i$. Then the functions*

$$a_i : K^{n-1} \to K, \quad (z_1, \ldots, z_{n-1}) \mapsto a_i(z_1, \ldots, z_{n-1}),$$

are also definable.

Proof. All of these facts follow from the definition of a tame structure. We note in particular that $+, \cdot : K^2 \to K$ and $|\cdot| : K \to \Gamma$ are definable, and that subsets defined by a first-order formula involving definable sets and definable functions must themselves be definable. ∎

Definition 3.10. Let V/K be a finite-type reduced scheme over K. We say that a subset $S \subseteq V(K)$ is *definable* if there exists a finite affine open cover $V = \bigcup_i U_i = \bigcup_i \operatorname{Spec}(A_i)$ and closed embeddings $U_i(K) \xhookrightarrow{\beta_i} K^{n_i}$ (arising from a presentation of A_i as a quotient of $K[t_1, \ldots, t_{n_i}]$) such that for all i, $\beta_i(S \cap U_i(K))$ is definable.

Remark 3.11. We note that if $S \subseteq V(K)$ is definable, then for *every* finite affine open cover U_i of V and for any choice of presentations $\beta_i : K[t_1, \ldots, t_{n_i}] \twoheadrightarrow \mathcal{O}(U_i)$, the subset $\beta_i(S \cap U_i(K)) \subseteq K^{n_i}$ is definable.

3.1.2. Dimension theory of tame structures. Parallel to the notion of definable dimension in o-minimality, in this section, we shall develop the basic dimension theory in the context of tame structures. In particular, we prove the so-called 'Theorem of the Boundary' (Theorem 3.21), which will be an important input in the proof of the definable Chow theorem.

We remark that in the context of the $\mathcal{H}$-subanalytic sets of [24], the foundational dimension theory results that we need have been proved in [22] and in the Ph.D. thesis of Florent Martin [29].

For this section, we shall retain the notations and conventions introduced in the previous section. We note that our field K is assumed to be second-countable. Throughout this section, we fix a tame structure on (R, Γ) and definability will be with regards to the fixed structure.

We recall the following definition from [22, Definition 2.1]:

Definition 3.12. (a) For any subset $X \subseteq K^m$, we define its *dimension*, denoted $\dim(X)$, as the largest non-negative integer $d \leq m$ such that there exists a collection of d coordinates $I \subseteq \{1, \ldots, m\}$ (with $|I| = d$) such that if $\mathrm{pr}_I : K^m \to K^d$ denotes the projection to these coordinates, the image $\mathrm{pr}_I(X)$ of X is a subset of K^d with non-empty interior.

(b) For a subset $X \subseteq K^m$ and a point $x \in X$, the *local dimension* of X at x, denoted $\dim_x(X)$, is defined by

$$\dim_x(X) := \min\{\dim(U \cap X) : U \subseteq K^m \text{ is an open subset containing } x\}.$$

Lemma 3.13. *If $X \subseteq R^m$ is definable, then one of X or its complement X^c contains a non-empty open disk of R^m.*

Proof. We induct on m. For $m = 0, 1$ this is clear. Let $m \geq 2$ and suppose $X \subseteq R^m$ is definable. Consider the projection to the first coordinate $\mathrm{pr} : R^m \to R$. For a point $s \in R$ and a set $Y \subseteq R^m$, we denote by $Y_s \subseteq R^{m-1}$ the set $\mathrm{pr}^{-1}(s) \cap Y = (\{s\} \times R^{m-1}) \cap Y$ viewed as a subset of R^{m-1}. Consider the sets

$$S_1 := \{s \in R : X_s \text{ contains a non-empty disk of } R^{m-1}\},$$
$$S_2 := \{s \in R : (X^c)_s \text{ contains a non-empty disk of } R^{m-1}\}.$$

Both are definable. Also, for every fixed $s \in R$, $R^{m-1} = X_s \cup (X^c)_s$. Thus, by the inductive hypothesis, for every s one of X_s or X_s^c must contain a non-empty disk of R^{m-1}, i.e. $R = S_1 \cup S_2$. By the $m = 1$ case, one of S_1 or S_2 contains a non-empty disk. Replacing X by X^c if necessary, we may assume without loss of generality that S_1 contains a non-empty open one-dimensional disk $D \subseteq S_1 \subseteq R$. Recall that R is assumed to be second-countable. Let $\{D_i \subseteq R^{m-1} : i \geq 1\}$ be a countable collection of non-empty open disks in R^{m-1}, forming a basis of the metric topology of R^{m-1}. For each i define

$$T_i := \{s \in D : X_s \supseteq D_i\}.$$

We have $\bigcup_i T_i = D \subseteq R$. Since definable subsets of R are Boolean combinations of disks, either T_i is finite or T_i has non-empty interior. Being complete, K is uncountable and hence there is some i such that T_i contains a non-empty open disk of R, say T_1 contains a non-empty open disk D'. Then $D' \times D_1 \subseteq X$, i.e. X contains an m-dimensional disk. ∎

Corollary 3.14. *For a definable set $X \subseteq R^m$, we have*

$$\mathrm{int}(X) = \emptyset \iff \mathrm{int}(\mathrm{cl}(X)) = \emptyset.$$

Proof. Suppose $\mathrm{int}(X) = \emptyset$, but the closure $\mathrm{cl}(X)$ has non-empty interior. Let $\bar{D} \subseteq \mathrm{cl}(X)$ be a closed disk in R^m, with positive radius. Then $\bar{D}$ is definably homeomorphic to R^m (by scaling the coordinates). So we may apply Lemma 3.13 to definable subsets of $\bar{D}$. In particular, since $X \cap \bar{D}$ has empty interior, by Lemma 3.13, $X^c \cap \bar{D}$ contains a non-empty open disk, i.e. $\mathrm{cl}(X) \setminus X$ has non-empty interior, which is impossible. ∎

Corollary 3.15. (1) *Suppose $\bigcup_{i=1}^{\infty} X_i = R^m$ for a countable collection of definable subsets X_i. Then there is some $i \geq 1$ such that $\mathrm{int}(X_i) \neq \emptyset$.*

(2) *For a countable collection $\{X_i\}_{i \geq 1}$ of definable subsets of R^m, we have*

$$\dim\Big(\bigcup_i X_i\Big) = \max \dim(X_i).$$

Proof. Follows from the Baire category theorem and Corollary 3.14. ∎

Corollary 3.16. *For a definable set $X \subseteq R^m$, we have* $\dim(X) = \dim(\mathrm{cl}(X))$.

Proof. Follows from Corollary 3.14. To elaborate, we only need to show the inequality $\dim(\mathrm{cl}(X)) \leq \dim(X)$. For a subset $I \subseteq \{1, \ldots, m\}$ with $|I| = d$, if we let $\mathrm{pr}_I : R^m \to R^d$ denote the projection to the coordinates corresponding to the coordinates in the subset I, we have

$$\mathrm{int}(\mathrm{pr}_I(\mathrm{cl}(X))) \neq \emptyset \implies \mathrm{int}(\mathrm{cl}(\mathrm{pr}_I(X))) \neq \emptyset \implies \mathrm{int}(\mathrm{pr}_I(X)) \neq \emptyset,$$

where the second implication is a consequence of Corollary 3.14 and the first follows from the fact that pr_I is continuous and therefore $\mathrm{pr}_I(\mathrm{cl}(X)) \subseteq \mathrm{cl}(\mathrm{pr}_I(X))$. The above chain of implications proves the sought-after inequality. ∎

Lemma 3.17. *Let $f : R^m \hookrightarrow R^n$ be an injective definable map. Then*

$$\dim(f(R^m)) \geq m.$$

Proof. We induct on m. If $m = 1$, then since $f(R^1)$ is infinite, there must necessarily be some coordinate projection $\mathrm{pr} : R^n \to R$ such that the image $\mathrm{pr}(f(R^1))$ is infinite. By the tameness axiom, infinite definable sets of R contain non-empty disks. So $\dim(f(R)) \geq 1$.

Now suppose $m \geq 2$. For any $y \in R$, denote by L_y the $(m-1)$-dimensional line $R^{m-1} \times \{y\}$. By the inductive hypothesis, $\dim(f(L_y)) \geq m-1$. So there exists a choice of $m-1$ coordinates (depending on y) of R^n such that the projection of $f(L_y)$ to those coordinates has non-empty interior. For each choice of $I = (i(1), \ldots, i(m-1))$ with $1 \leq i(1) < \cdots < i(m-1) \leq n$, let $\pi_I : R^n \to R^{m-1}$ denote the corresponding projection. Let $T_I := \{y \in R : \pi_I(f(L_y))$ contains a non-empty disk$\}$. Then $\bigcup_I T_I = R$. Hence, there is a choice of I such that T_I contains a closed disk, say D, of positive radius. Replacing R^m by $R^{m-1} \times D$ (and rearranging coordinates if necessary) we may assume that for all $y \in R$, $\pi(f(L_y))$ contains a non-empty open disk of R^{m-1} where $\pi : R^n \to R^{m-1}$ is the projection to the first $m-1$ coordinates. Enumerate a countable basis $\{B_i : i \geq 1\}$ of non-empty open disks of R^{m-1}. Let $\Lambda_i := \{y \in R : \pi(f(L_y)) \supseteq B_i\}$. Then Λ_i is definable in R and $\bigcup_{i \geq 1} \Lambda_i = R$. So there is some i such that Λ_i contains a closed disk, say D', of positive radius. Again replacing R^m by $R^{m-1} \times D'$ we may assume that for all $y \in R, \pi(f(L_y)) \supseteq B_0$ where $B_0 \subseteq R^{m-1}$ is some fixed non-empty open disk of R^{m-1}. Let $X := f(R^m) \subseteq R^n$. For every $b \in B_0$ and every $y \in R$, we have $X_b \cap f(L_y) \neq \emptyset$. Thus for every $b \in B_0$, the set X_b is infinite and so some projection of X_b to the remaining $n-(m-1)$-coordinates must be infinite and hence contains a non-empty open one-dimensional disk. For each of these remaining coordinates $j \in \{m, \ldots, n\}$, let $S_j := \{b \in B_0 : \mathrm{pr}_j(X_b)$ contains a non-empty open disk$\}$. Since $\bigcup_j S_j = B_0$, by Corollary 3.15 some S_j contains a non-empty open disk. Shrinking B_0 further to this smaller disk and rearranging the coordinates if necessary, we may assume that for all $b \in B_0, \mathrm{pr}_m(X_b)$ contains a non-empty open disk of R. Enumerate the disks of R, i.e. let $\{C_i : i \geq 1\}$ be a countable basis of non-empty open disks of R. Let $\Gamma_i := \{b \in B_0 : \mathrm{pr}_m(X_b) \supseteq C_i\}$. Then Γ_i are definable and $\bigcup_i \Gamma_i = B_0$. By Corollary 3.15, we have an i such that Γ_i contains a non-empty open disk, say $B' \subseteq \Gamma_i$. Then $\mathrm{pr}_{(1,\ldots,m)}(X) \supseteq B' \times C_i$, and therefore $\dim(f(R^m)) = \dim(X) \geq m$. ■

Lemma 3.18. *Let $X \subseteq R^m$ be definable. Let $d \leq m$, and let $\mathrm{pr}_{(1,\ldots,d)} : R^m \to R^d$ denote the projection to the first d coordinates. Suppose that $\mathrm{pr}_{(1,\ldots,d)}(X) = B$ is a closed polydisc in R^d of positive polyradius such that the restriction of the projection to X, $\mathrm{pr}_{(1,\ldots,d)} : X \to B$, is a quasi-finite surjection, with all the fibers having the same size of say N elements. Then there exists a smaller closed polydisk $B' \subseteq B$ of positive polyradius and N definable maps $s_i : B' \to R^{m-d}$, $1 \leq i \leq N$, such that $X \cap (B' \times R^{m-d})$ is the disjoint union of the graphs of the s_i, $1 \leq i \leq N$.*

Proof. We induct on N. If $N = 1$, then the projection $\mathrm{pr}_{(1,\ldots,d)} : X \to B$ is a definable bijection and the lemma is clear in this case, since X is evidently the graph of the definable inverse of this bijection, composed with the projection to the last $m-d$ coordinates.

Suppose $N \geq 2$. For every $m \geq 1$, define

$$D_m := \{b \in B : \text{ for all } x_1 \neq x_2 \in X_b,\ \|x_1 - x_2\| \geq |\varpi^m|\}.$$

Note that $\bigcup_{m\geq 1} D_m = B$, and thus by Corollary 3.15 for an m_0, D_{m_0} has non-empty interior. Replacing B with a smaller disk in this interior, and shrinking X too, we assume that for all $b \in B$, and for all $x_1 \neq x_2 \in X_b$, $\|x_1 - x_2\| \geq |\varpi^{m_0}|$. Now, cover R^{m-d} by countably many non-empty open disks $\{\Delta_j\}_{j\geq 1}$ of polyradius strictly less than $|\varpi^{m_0}|$. Since $\bigcup_{j\geq 1} \mathrm{pr}_{(1,\ldots,d)}((B \times \Delta_j) \cap X) = B$, by Corollary 3.15, for some $j \geq 1$, the set $\mathrm{pr}_{(1,\ldots,d)}((B \times \Delta_j) \cap X)$ has non-empty interior in R^d. We replace B by a smaller closed disk of positive radius contained in this interior. Thus, now $\mathrm{pr}_{(1,\ldots,d)} : (B \times \Delta_j) \cap X \to B$ is a *bijection* (since distinct points in the fiber of this projection are at least $|\varpi^{m_0}|$ apart in some coordinate, while the polydisc Δ_j has polyradius $< |\varpi^{m_0}|$ by choice.) Thus the inverse of this bijection provides a definable section $s : B \hookrightarrow X$. And letting $s_1 := \mathrm{pr}_{(d+1,\ldots,m)} \circ s$, we see that the graph of s_1 is exactly $(B \times \Delta_j) \cap X$. Let $Y := X \cap (B \times (R^{m-d} \setminus \Delta_j))$. Then $\mathrm{pr}_{(1,\ldots,d)} : Y \to B$ is a quasi-finite surjection with fibers of constant cardinality $N - 1$. We now apply the induction hypothesis to $\mathrm{pr}_{(1,\ldots,d)} : Y \to B$ to finish the proof. ∎

Proposition 3.19 (Invariance of dimension under definable bijections). *Let $X \subseteq R^m$ and $Y \subseteq R^n$ be definable sets and $f : X \to Y$ a definable bijection. Then* $\dim(X) = \dim(Y)$.

Proof. It suffices to prove that $\dim(X) \leq \dim(Y)$, since the inverse $f^{-1} : Y \to X$ is also definable. Let $d = \dim(X)$. Suppose the projection of X to the first d coordinates contains a d-dimensional non-empty open disk B, i.e. $\mathrm{pr}_{(1,\ldots,d)}(X) \supseteq B$. Replacing X by $X \cap (B \times R^{m-d})$ we may assume that $\mathrm{pr}_{(1,\ldots,d)}(X) = B$.

Claim. *There is a non-empty open disk $B' \subseteq B$ such that the projection map* $\mathrm{pr}_{(1,\ldots,d)}$ *: $X \to B$ is quasi-finite over B' with constant fiber cardinality N.*

For each $j \in \{d + 1, \ldots, m\}$, let

$$T_j := \{b \in B : \mathrm{pr}_j(X_b) \text{ contains a non-empty open disk}\}$$

and for each natural number $k \geq 1$, let $F_k := \{b \in B : |X_b| = k\}$. Then

$$B = \bigcup_{k=1}^{\infty} F_k \cup \bigcup_{j=d+1}^{m} T_j.$$

If for any $k \geq 1$, F_k has non-empty interior, the Claim would be proved. So suppose each F_k has empty interior; then by Corollary 3.15 there is some j such that T_j has non-empty interior, say T_{d+1} contains a non-empty open disk. Replacing B by this smaller disk (and modifying X appropriately), we may assume that $T_{d+1} = B$. Let $\{B_i : i \geq 1\}$ be an enumeration of a countable basis of non-empty open disks of R, and let $\mathcal{K}_i := \{b \in B : \mathrm{pr}_{d+1}(X_b) \supseteq B_i\}$. Then $\bigcup_i \mathcal{K}_i = B$ and so by Corollary 3.15 there is an i_0 such that $\mathcal{K}_{i_0}$ contains a non-empty open disk D. Replacing B by D we assume that $\mathrm{pr}_{d+1}(X_b) \supseteq B_{i_0}$

for all $b \in B$. But then $\mathrm{pr}_{(1,\dots,d+1)}(X) \supseteq B \times B_{i_0}$, contradicting $\dim(X) = d$, and thus proving the claim.

Replacing B with B' obtained from the above claim and replacing X by $X \cap (B' \times R^{m-d})$ we assume $\mathrm{pr}_{(1,\dots,d)} : X \to B$ is quasi-finite with constant fiber cardinality $N \geq 1$. By Lemma 3.18 (after possibly shrinking B further) we can find a definable section $s : B \hookrightarrow X$. By Lemma 3.17 now, $\dim(f(s(B))) \geq d$, and as $Y \supseteq f(s(B))$ we get $\dim(Y) \geq d$ as needed. ∎

Lemma 3.20. *Let $D \subseteq R^d$ be a closed polydisk of positive polyradius. Let $s : D \to R$ be a definable function. Then given any $\epsilon > 0$, there exists a smaller closed polydisk $D' \subseteq D$ of positive polyradius such that $s(D')$ is contained in a disk of diameter $< \epsilon$, i.e. for all $x, y \in D'$, $|s(x) - s(y)| < \epsilon$.*

Proof. Cover R by countably many non-empty open disks $\{B_i\}_{i \geq 1}$ each of diameter $< \epsilon$. Then $D = \bigcup_i s^{-1}(B_i)$. By Corollary 3.15, there is an $i \geq 1$ such that $s^{-1}(B_i)$ has non-empty interior in R^d. For such an i, take $D' \subseteq s^{-1}(B_i)$ to be a closed polydisk of positive polyradius. ∎

Theorem 3.21 (Theorem of the Boundary). *Let $X \subseteq R^m$ be a definable set. Then*

$$\dim(\mathrm{Fr}(X)) < \dim(X).$$

Remark 3.22. We remark that for a subset $X \subseteq R^m$, its boundary is often defined as $\mathrm{bd}_{R^m}(X) := \mathrm{cl}_{R^m}(X) \setminus \mathrm{int}_{R^m}(X)$. With this terminology, it follows from Corollary 3.16 that $\dim(\mathrm{bd}_{R^m}(X)) \leq \dim(X)$. Indeed, being a subset of $\mathrm{cl}_{R^m}(X)$, $\mathrm{bd}_{R^m}(X)$ has dimension at most $\dim(\mathrm{cl}_{R^m}(X)) = \dim(X)$. We also note however that depending on the set X, the inequality might or might not be strict. For instance, if $X \subseteq R^2$ is the diagonal in R^2, its boundary in R^2 is X itself; on the other hand, the boundary (in R) of the set R itself is $\emptyset$.

Proof of Theorem 3.21. Let $d = \dim(X)$. By Corollary 3.16, we first note that $\dim(\mathrm{Fr}(X)) \leq \dim(\mathrm{cl}(X)) = \dim(X) = d$. Suppose for the sake of contradiction that $\dim(\mathrm{Fr}(X)) = d$, and that the projection of $\mathrm{Fr}(X)$ to the first d coordinates has non-empty interior. Thus if $\pi : R^m \to R^d$ denotes the projection to the first d coordinates, there exists a closed polydisk D of positive polyradius in R^d such that $\pi(\mathrm{Fr}(X)) \supseteq D$.

So $D \subseteq \pi(\mathrm{Fr}(X)) \subseteq \pi(\mathrm{cl}(X)) \subseteq \mathrm{cl}(\pi(X))$, and hence in particular $\mathrm{cl}_D(\pi(X) \cap D) = D$. By Corollary 3.16, $\pi(X) \cap D$ contains a smaller closed disk of positive radius, say $D' \subseteq \pi(X) \cap D \subseteq D$. Replacing D with D' and X with $X \cap (D' \times R^{m-d})$, we may assume that $D = \pi(X) = \pi(\mathrm{Fr}(X))$.

We note that in the argument that follows, we shall often replace D with a smaller disk. This is justified, because if $D' \subseteq D$ is a smaller closed disk of positive radius, replacing D by D' and X by $X \cap (D' \times R^{m-d})$ does not change the property that $D = \pi(X) = \pi(\mathrm{Fr}(X))$.

Continuing, for each $j \in \{d+1, \dots, m\}$ we let

$$\Lambda_j := \{b \in D : \pi_j(X_b) \text{ has non-empty interior}\}.$$

If for some j, Λ_j has non-empty interior, then again using the same trick of enumerating a countable basis of disks in R, and following the same line of argument, we would conclude that $\pi_{(1,\ldots,d,j)}(X)$ contains a $(d+1)$-dimensional disk. This is not possible since $\dim(X) = d$. Therefore, for each $j \in \{d+1, \ldots, m\}$, we must have $\mathrm{int}(\Lambda_j) = \emptyset$. Hence, by Corollary 3.15, $D \setminus \bigcup_j \Lambda_j$ contains a closed disk of positive polyradius. Replacing D with this smaller closed disk, we may assume that $\pi : X \to D$ is a quasi-finite surjection. Moreover, using the argument in the proof of the claim in the proof of Proposition 3.19, we may assume the fibers of $\pi : X \to D$ have constant finite cardinality N. Running the same argument for $\mathrm{Fr}(X)$ instead of X and shrinking D if necessary, we also assume that $\pi : \mathrm{Fr}(X) \to D$ is quasi-finite surjection with fibers of constant size, say M.

Further shrinking D to a smaller closed disk, we may assume by Lemma 3.18 that X is the disjoint union of graphs of N definable functions $s_i : D \to R^{m-d}$. If we let T_i denote the graph of s_i, then since $X = \bigcup_{i=1}^{N} T_i$, we have $\mathrm{Fr}(X) \subseteq \bigcup_i \mathrm{Fr}(T_i)$. Furthermore, since $D = \pi(\mathrm{Fr}(X)) = \bigcup_i \pi(\mathrm{Fr}(T_i))$, for some i the set $\pi(\mathrm{Fr}(T_i))$ must have non-empty interior. We may then replace D by a smaller disk in this interior, and X by T_i, and assume that X is the graph of a definable function $s : D \to R^{m-d}$. Furthermore, running the argument in the above paragraphs again, we may ensure that the property that $\pi : \mathrm{Fr}(X) \to D$ is a quasi-finite surjection of constant fiber cardinality still holds. All in all, we have reduced the proof to the following situation:

X is the graph of a definable function $s : D \to R^{m-d}$ such that $\pi(\mathrm{Fr}(X)) = D = \pi(X)$ and $\pi : \mathrm{Fr}(X) \to D$ is a quasi-finite surjection with constant fiber size M.

Applying Lemma 3.18 to $\mathrm{Fr}(X)$, we assume that $\mathrm{Fr}(X)$ is the disjoint union of graphs of M definable functions $g_j : D \to R^{m-d}$, $1 \leq j \leq M$. Let Y_j denote the graph of g_j, so that $\mathrm{Fr}(X) = \bigcup_{j=1}^{M} Y_j$. We note that $X \cap Y_j = \emptyset$ for each j, or in other words for each j and for every $b \in D$, $\|s(b) - g_j(b)\| \neq 0$. For every $m \geq 1$, define $E_m := \{b \in D : \|s(b) - g_j(b)\| > |\varpi^m| \text{ for each } j\}$. We have $\bigcup_{m \geq 1} E_m = D$, and thus by Corollary 3.15 some E_m has non-empty interior. Replacing D by a smaller closed disk contained in this interior, we may assume that there is some m_0 large enough, such that for all $1 \leq j \leq M$ and for all $b \in D$, $\|s(b) - g_j(b)\| > |\varpi^{m_0}|$. Applying Lemma 3.20, and shrinking D to a smaller disk, we may assume that for all $x, y \in D$, $\|s(x) - s(y)\| < |\varpi^{m_0}|$.

Now choose a $b \in D$, and consider $y = (b, g_1(b)) \in Y_1$. Since $y \in \mathrm{Fr}(X)$, in particular y is in the closure of X, and thus there must exist some $b' \in D$ such that $\|s(b') - g_1(b)\| < |\varpi^{m_0}|$. However, by our choice of m_0, $\|s(b) - g_1(b)\| > |\varpi^{m_0}|$. By the non-archimedean triangle inequality, we therefore get $\|s(b') - s(b)\| > |\varpi^{m_0}|$, contradicting the conclusion of the previous paragraph. ∎

Remark 3.23. Note that in all our proofs we have made extensive use of the standing assumption that K is second-countable. However, when the tame structure under consideration is that of the $\mathcal{H}$-subanalytic sets, this assumption can be removed, exploiting the model completeness and *uniform* quantifier-elimination results of [24]. See for example the argument used in the proof of [22, Lemma 2.3]. Running the argument given there with appropriate modifications enables us to reduce the proof of the Theorem of the Boundary for $\mathcal{H}$-subanalytic sets to the case where K is second-countable.

3.2. Miscellaneous lemmas

In this subsection we collect a few auxiliary results relating to the dimension theory of rigid analytic spaces. These results will be used in the following sections. We prove in Lemma 3.26 that the usual notion of dimension in rigid geometry defined via Krull dimensions of associated rings of analytic functions agrees with the notion of definable dimension defined above via coordinate projections. In Lemma 3.28, we show that for any point x of a reduced, equidimensional rigid variety X, every minimal prime ideal of the local ring $\mathcal{O}_{X,x}$ has the same coheight. This result is used in the course of proving the definability of the étale locus of a certain finite map. While the results of this section are fairly standard, we provide their proofs for completeness.

Definition 3.24. If X is a rigid variety over K, we define its *dimension*, denoted $\dim(X)$, by

$$\dim(X) := \max_{x \in X} \dim(\mathcal{O}_{X,x}).$$

Lemma 3.25. *Let $Y = \mathrm{Sp}(A)$ be an affinoid space over K. Let $\{Y_i\}_{1 \le i \le m}$ denote the finitely many irreducible components of Y. Then*

(1) $\dim(Y) = \dim(A)$,

(2) *for any point $y \in Y$, $\dim(\mathcal{O}_{Y,y}) = \max\{\dim(Y_j) : y \in Y_j\}$.*

Proof. These facts are rather standard. Due to lack of an explicit reference, we provide a proof.

(1) For a point $y \in Y$ corresponding to $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, we have $\widehat{A}_{\mathfrak{m}} = \widehat{\mathcal{O}}_{Y,y}$ [8, Section 7.3.2, Proposition 3]. Since the Krull dimension of a Noetherian local ring is preserved under completion (see [40, Tag 07NV]), we get

$$\begin{aligned}\dim(A) &= \max_{\mathfrak{m} \in \mathrm{MaxSpec}(A)} \dim(A_{\mathfrak{m}}) = \max_{\mathfrak{m} \in \mathrm{MaxSpec}(A)} \dim(\widehat{A}_{\mathfrak{m}}) \\ &= \max_{y \in Y} \dim(\widehat{\mathcal{O}}_{Y,y}) = \max_{y \in Y} \dim(\mathcal{O}_{Y,y}) = \dim(Y).\end{aligned}$$

(2) From the argument above, if $\mathfrak{m} \in \mathrm{MaxSpec}(A)$ corresponds to y, we know that $\dim(\mathcal{O}_{Y,y}) = \dim(A_{\mathfrak{m}})$. If the irreducible component Y_i corresponds to the minimal prime $\mathfrak{p}_i \subset A$, then we note that $\dim(A_{\mathfrak{m}}) = \max\{\dim(A_{\mathfrak{m}}/\mathfrak{p}_j A_{\mathfrak{m}}) : \mathfrak{p}_j \subseteq \mathfrak{m}\} = \max\{\dim((A/\mathfrak{p}_j)_{\mathfrak{m}}) : \mathfrak{p}_j \subseteq \mathfrak{m}\}$. Now, $A/\mathfrak{p}_j$ is an affinoid algebra that is an integral domain, and this implies that $\dim(A/\mathfrak{p}_j)_{\mathfrak{m}} = \dim(A/\mathfrak{p}_j)$ – see Lemma 3.27 (1) below. Thus, $\dim(\mathcal{O}_{Y,y}) = \dim(A_{\mathfrak{m}}) = \max\{\dim(A/\mathfrak{p}_j) : \mathfrak{p}_j \subseteq \mathfrak{m}\} = \max\{\dim(Y_j) : y \in Y_j\}$. ∎

Lemma 3.26. *Suppose $Y = \mathrm{Sp}(A)$ is a K-affinoid space. Suppose $\pi : T_n(K) \twoheadrightarrow A$ is a surjective homomorphism of K-algebras. Via π we may view $i : Y \hookrightarrow R^n$ as a subset of the n-dimensional unit ball R^n. Then:*

(1) *The dimension of $i(Y)$ as a subset of R^n (in the sense of Definition 3.12) is the same as the dimension of Y as a rigid analytic space.*

(2) *For a point* $y \in Y$*, the local dimension* $\dim_{i(y)} i(Y)$ *(in the sense of Definition* 3.12*) is equal to* $\dim(\mathcal{O}_{Y,y})$.

(3) *Suppose* X *is a rigid space over* K *and* $i : X \hookrightarrow \mathbb{A}_K^{n,\mathrm{an}}$ *a closed immersion. Then* $\dim(X)$ *equals the dimension of* $i(X)$ *viewed as a subset of* K^n *(as in Definition* 3.12*). For a point* $x \in X$*, the local dimension* $\dim_{i(x)} i(X)$ *(as in Definition* 3.12*) is equal to* $\dim(\mathcal{O}_{X,x})$.

Proof. (1) This is a special case of [22, Lemma 4.2]. Alternatively, we may reduce to the case that K is second-countable (see Remark 3.23). Then, using Noether's normalization for affinoid algebras, if $d = \dim(A)$, we have a quasi-finite subanalytic surjection $i(Y) \twoheadrightarrow R^d$. And then we may use an argument very similar to that of Proposition 3.19. We omit the details.

(2) By definition, we have

$$\dim_{i(y)} i(Y) = \min\{\dim(U \cap i(Y)) : U \subseteq R^n \text{ is open with } i(y) \in U\}.$$

We may take this minimum instead over all closed polydisks $\overline{\mathbb{D}}$ of R^n of positive polyradius containing $i(y)$, i.e.

$$\dim_{i(y)} i(Y) = \min\left\{\dim\bigl(\overline{\mathbb{D}}(i(y), \underline{r}) \cap i(Y)\bigr) : \underline{r} > \underline{0}\right\}.$$

Since $i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))$ is an affinoid subdomain of Y, from (1) we have

$$\dim\bigl(\overline{\mathbb{D}}(i(y), \underline{r}) \cap i(Y)\bigr) = \dim\bigl(i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))\bigr),$$

where the dimension on the right side is the dimension of the affinoid subdomain $i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))$ as an analytic space. Furthermore, note that the affinoid subdomains of the form $i^{-1}(\overline{\mathbb{D}}(i(y), \underline{r}))$ are cofinal in the collection of all affinoid subdomains of Y containing y (use for example [11, Lemma 1.1.4]). Therefore,

$$\dim_{i(y)} i(Y) = \min\{\dim(W) : W \subseteq Y \text{ is an affinoid subdomain containing } y\}.$$

The right-hand side is indeed equal to $\dim(\mathcal{O}_{Y,y})$ (follows from [13, Section 1.17] and Lemma 3.25 (2)).

(3) follows immediately from (1) and (2). ■

Lemma 3.27. (1) *Suppose* A *is a* K*-affinoid algebra that is an integral domain. Then every maximal ideal of* A *has the same height.*

(2) *Suppose* Y *is an irreducible rigid analytic variety. Then* Y *is equidimensional, i.e. for all* $y \in Y$, $\dim(\mathcal{O}_{Y,y}) = \dim(Y)$.

Proof. For (1), we use the Noether normalization for affinoid algebras, the Going-Down theorem [40, Tag 00H8] and [7, Chapter 2, Proposition 17]. To elaborate, let $\mathfrak{m} \subset A$ be a maximal ideal of A and suppose that the Krull dimension $\dim(A)$ is equal to d. The height of $\mathfrak{m}$ is evidently at most d. We show below that the height of $\mathfrak{m}$ is at least d, thereby proving (1). By the Noether normalization for affinoid algebras [8, Corollary 2,

p. 228], we may find an injective homomorphism of K-algebras $i : T_d(K) \hookrightarrow A$ making A a finite module over $T_d(K)$. The pullback of $\mathfrak{m}$, namely $i^{-1}(\mathfrak{m}) =: \mathfrak{n}$, is a maximal ideal of $T_d(K)$. By [7, Chapter 2, Proposition 17], the height of $\mathfrak{n}$ is d. Therefore, there is a chain of prime ideals $\mathfrak{n} = \mathfrak{q}_d \supsetneq \mathfrak{q}_{d-1} \supsetneq \cdots \supsetneq \mathfrak{q}_0 = (0)$, of length d.

Since $T_d(K)$ is normal (see [8, Section 5.2.6, Theorem 2, p. 208]) and A is an integral domain, the Going-Down property [40, Tag 00H8] holds for the injection i. Thus, the chain of prime ideals $\mathfrak{n} = \mathfrak{q}_d \supsetneq \mathfrak{q}_{d-1} \supsetneq \cdots \supsetneq \mathfrak{q}_0 = (0)$ can be lifted to a chain $\mathfrak{m} = \mathfrak{p}_d \supsetneq \mathfrak{p}_{d-1} \supsetneq \cdots \supsetneq \mathfrak{p}_0 = (0)$. This proves that the height of $\mathfrak{m}$ is at least d.

For (2), we refer the reader to the paragraph preceding [11, Lemma 2.2.3]. ∎

Lemma 3.28. *Let X be a reduced equidimensional rigid space over K, i.e.* $\dim(\mathcal{O}_{X,x}) = \dim(X)$ *for all $x \in X$. Then for every $x \in X$ and every minimal prime ideal $\mathfrak{q}$ of $\mathcal{O}_{X,x}$ we have* $\dim(\mathcal{O}_{X,x}/\mathfrak{q}) = \dim(X)$.

Proof. Evidently for every $x \in X$, $\dim(\mathcal{O}_{X,x}/\mathfrak{q}) \leq \dim(\mathcal{O}_{X,x}) = \dim(X)$. Suppose the lemma were false. Then for some x, we would have

$$\dim(\mathcal{O}_{X,x}/\mathfrak{q}) < \dim(X).$$

Since $\mathcal{O}_{X,x}$ is Noetherian [8, Section 7.3.2 Proposition 7], $\mathfrak{q}$ is finitely generated, say $\mathfrak{q} = (h_1, \ldots, h_m)$ for some $h_i \in \mathcal{O}_{X,x}$. We may choose an open affinoid domain $\mathrm{Sp}(B)$ in X containing x such that the h_i are (images of elements) in B. Let $\mathfrak{n} \in \mathrm{MaxSpec}(B)$ be the maximal ideal corresponding to the point x, and let $J := (h_1, \ldots, h_m)B$ be the ideal in B generated by the h_i.

We claim first that $JB_{\mathfrak{n}}$ is a minimal prime ideal of $B_{\mathfrak{n}}$. To see this, note that since $B_{\mathfrak{n}} \hookrightarrow \mathcal{O}_{X,x}$ is a faithfully flat map (as these local rings have the same completions), $JB_{\mathfrak{n}}$ is the contraction of $J\mathcal{O}_{X,x} = \mathfrak{q}$ (see [40, Tag 05CK]) and is therefore a prime ideal. Moreover, since $B_{\mathfrak{n}} \hookrightarrow \mathcal{O}_{X,x}$ is faithfully flat, it has the Going-Down property [40, Tag 00HS]. Therefore, as $\mathfrak{q}$ is a minimal prime ideal of $\mathcal{O}_{X,x}$, its contraction $JB_{\mathfrak{n}}$ must also be minimal.

We have $(B_{\mathfrak{n}}/JB_{\mathfrak{n}})\widehat{\ } = \widehat{B}_{\mathfrak{n}}/J\widehat{B}_{\mathfrak{n}} = \widehat{\mathcal{O}}_{X,x}/\mathfrak{q}\widehat{\mathcal{O}}_{X,x} = (\mathcal{O}_{X,x}/\mathfrak{q})\widehat{\ }$. Hence,

$$\dim(B_{\mathfrak{n}}/JB_{\mathfrak{n}}) = \dim(\mathcal{O}_{X,x}/\mathfrak{q}).$$

Now let $\mathfrak{p} \subseteq B$ denote the contraction of $JB_{\mathfrak{n}}$ to B, so $\mathfrak{p}$ is a minimal prime of B contained in $\mathfrak{n}$. Then $\dim(B_{\mathfrak{n}}/JB_{\mathfrak{n}}) = \dim((B/\mathfrak{p}B)_{\mathfrak{n}}) = \dim(B/\mathfrak{p}B)$, where the last equality follows from the fact that since $B/\mathfrak{p}$ is an affinoid algebra that is an integral domain, all its maximal ideals have the same height (see Lemma 3.27).

Therefore, we have shown that $\dim(B/\mathfrak{p}) = \dim(\mathcal{O}_{X,x}/\mathfrak{q})$ for a minimal prime $\mathfrak{p}$ of B. And since we are assuming that $\dim(\mathcal{O}_{X,x}/\mathfrak{q}) < \dim(X)$, this means that $\dim(B/\mathfrak{p}) < \dim(X)$. However, we can find a closed point $\mathfrak{n}' \in \mathrm{MaxSpec}(B)$ containing $\mathfrak{p}$ but not containing any other minimal prime of B (this is possible since B is Jacobson and so closed points are dense). If $\mathfrak{n}'$ corresponds to the point $x' \in X$, we have $\dim(X) > \dim(B/\mathfrak{p}) \geq \dim(B_{\mathfrak{n}'}) = \dim(\mathcal{O}_{X,x'})$. This contradicts the equidimensionality of X. ∎

4. The non-archimedean definable Chow theorem

The goal of this section is to prove a version of the definable Chow theorem in the non-archimedean setting. Let K be as in the previous section. Namely, K is an algebraically closed field, complete with respect to a non-trivial non-archimedean absolute value. Moreover, we assume that K is second-countable. The main goal of this section is to prove the following result:

Theorem 4.1. *Let X be a closed analytic subset of $\mathbb{A}_K^{n,\mathrm{an}}$. Suppose that for some tame structure on K, X is definable as a subset of $\mathbb{A}^n(K) = K^n$. Then X is algebraic i.e. X is the vanishing locus of a finite collection of polynomials in $K[t_1, \dots, t_n]$.*

We outline the major steps of the proof below:

- *Step 0:* Our first step is to show that for a reduced variety X over K, a global analytic function $f \in H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ whose graph is definable must be algebraic. This is the content of Theorem 4.6, which may be seen as a non-archimedean definable analogue of Liouville's theorem from complex analysis. The proof proceeds by a devissage argument:
  - First, when $X = \mathbb{A}_K^n$ – Lemma 4.2.
  - Second, when X is a smooth affine variety over K – Lemma 4.4, using Noether normalization to reduce to the first case.
  - And lastly, for a general reduced variety X – Theorem 4.6, using Lemma 4.5 to reduce to the smooth case.
- *Step 1:* Now suppose $X \subseteq \mathbb{A}_K^{n,\mathrm{an}}$ is as in the statement of Theorem 4.1. We shall induct on $\dim(X) + n$. By the Theorem of the Boundary, $\dim(\mathrm{Fr}(X)) < \dim(X)$ and so we can find a point $q \in \mathbb{P}^n(K) \setminus \mathrm{cl}(X)$.
- *Step 2:* The projection from q onto a hyperplane $\mathbb{H} \subseteq K^n$ not containing it, $\pi|_X : X \to \mathbb{H}$, is *finite*. The image $Y = \pi(X)$ is an analytic subset of $\mathbb{A}_K^{n-1,\mathrm{an}}$, and therefore algebraic by induction.
- *Step 3:* The étale locus $U \subseteq Y$ of $\pi|_X : X \to Y$ is definable (thanks to Lemma 4.10), and of smaller dimension, therefore algebraic.
- *Step 4:* The characteristic polynomial of the finite étale map
$$\pi : \pi^{-1}(U^{\mathrm{an}}) \to U^{\mathrm{an}}$$
has coefficients in $H^0(U^{\mathrm{an}})$ that are *definable*. By Step 0, we shall then conclude $\pi^{-1}(U^{\mathrm{an}}) \subseteq X$ is algebraic. The complement in X is of smaller dimension, and thus algebraic by induction.

4.1. A non-archimedean definable Liouville theorem

Lemma 4.2. *Let $(X, \mathcal{O}_X) = \mathbb{A}_K^{n,\mathrm{an}}$ be the rigid n-dimensional affine plane over K and let $f \in H^0(X, \mathcal{O}_X)$ be a global analytic function. Suppose f viewed as a function $f : K^n \to K$ is definable. Then f is a polynomial function.*

Proof. We prove this by induction on n.

Case $n = 1$: A function $f \in H^0(\mathbb{A}_K^{1,\mathrm{an}}, \mathcal{O}_{\mathbb{A}_K^{1,\mathrm{an}}})$ is given by a globally convergent power series $f(z) = \sum_{i\geq 0} a_i z^i$. Thus, $\lim_{i\to\infty}(p^{ir} \cdot |a_i|) = 0$ for every $r \geq 0$. For a given $r \geq 0$, the number of zeroes of $f(z)$ on the disk $\{z \in K : |z| \leq p^r\}$ is the number of zeroes (with the same multiplicities) of $g_r(t) := f(p^{-r}t) = \sum_{\geq 0} a_i p^{-ir} t^i$ in the unit disk $|t| \leq 1$, which by Lemma 2.33 is at least $\epsilon(g_r)$. Now given any $i < j$ with $a_i, a_j \neq 0$, we note that for r large enough, $p^{rj}|a_j| \geq p^{ri}|a_i|$ and thus $\epsilon(g_r) \geq j$. Thus, if $a_i \neq 0$ for infinitely many i, then f must have infinitely many zeroes. However, as f is definable, $f^{-1}(0)$ is a definable subset of K that is discrete, and must therefore be finite. Hence, it cannot be the case that $a_i \neq 0$ for infinitely many i, i.e. f is a polynomial.

Proof for general $n \geq 1$: The global analytic function $f \in H^0(\mathbb{A}_K^{n,\mathrm{an}})$ is again given by a globally convergent power series on K^n. Thus, we write

$$f(z_1, \ldots, z_n) = \sum_{i\geq 0} a_i(z_1, \ldots, z_{n-1}) z_n^i, \quad \text{where } a_i \in H^0(\mathbb{A}_K^{n-1,\mathrm{an}}).$$

Moreover, each $a_i(z_1, \ldots, z_{n-1})$ is also definable viewed as a function on K^{n-1} (by Lemma 3.9 (iv)). By induction, the $a_i(z_1, \ldots, z_{n-1})$ are polynomials in $K[z_1, \ldots, z_{n-1}]$. From the $n = 1$ case, for every $\underline{\boldsymbol{\lambda}} \in K^{n-1}$, the sequence $a_i(\underline{\boldsymbol{\lambda}})$ must be eventually 0. In other words, $K^{n-1} = \bigcup_{j\geq 0} \bigcap_{i\geq j} V(a_i)$, a countable union of closed subsets of K^{n-1}. By the Baire category theorem, this is only possible if for some $j \geq 0$, $K^{n-1} = \bigcap_{i\geq j} V(a_i)$, i.e. $a_i = 0$ for all $i \geq j$ and hence f is a polynomial. ∎

Lemma 4.3. *Suppose that $P(t_1, \ldots, t_d) \in K[t_1, \ldots, t_d]$ is a non-zero element of the polynomial ring in d variables with coefficients in K. Let $A = K[t_1, \ldots, t_d][1/P(t_1, \ldots, t_d)]$ be the localization of the polynomial ring at the multiplicative set of powers of $P(t_1, \ldots, t_d)$. Then there exist d elements $z_1, \ldots, z_d \in A$ that are algebraically independent over K such that A is a finite free module over the subring $K[z_1, \ldots, z_d]$ and furthermore such that A is generically étale over $K[z_1, \ldots, z_d]$. That is, the fraction field of A is a finite separable extension of $K(z_1, \ldots, z_d)$.*

Proof. When the characteristic of K is zero, this is just a consequence of the usual Noether normalization lemma applied to A, since in that case generic étaleness is automatic. So we suppose that the characteristic of K is $p > 0$. The proof we give below follows the lines of a standard proof of the Noether normalization lemma. We have a presentation for A as $A = K[t_1, \ldots, t_d, y]/(P(t_1, \ldots, t_d)y - 1)$. Write $P(t_1, \ldots, t_d) = \sum_{j=0}^{n} P_j(t_1, \ldots, t_d)$ where $P_n \neq 0$ and each P_j is homogeneous of degree j.

We may replace $P(t_1, \ldots, t_d)$ with $P(t_1, \ldots, t_d)^2$ if necessary since this does not change the ring A, and thus we may assume that $p \nmid (n + 1)$. Choose an element $(c_1, \ldots, c_d) \in K^d$ such that $P_n(c_1, \ldots, c_d) \neq 0$, which is possible since K is algebraically closed. Let $t_i' := t_i - c_i y$. Then in A we have

$$\begin{aligned} 0 &= P(t_1' + c_1 y, \ldots, t_d' + c_d y, y) \cdot y - 1 \\ &= P_n(c_1, \ldots, c_d) y^{n+1} + (\text{terms of lower order in } y). \end{aligned}$$

Thus y is integral over $K[t_1', \dots, t_d']$. Furthermore, $P(t_1' + c_1 y, \dots, t_d' + c_d y, y) \cdot y - 1$ viewed as a polynomial in y with coefficients in $K(t_1', \dots, t_d')$ is the minimal polynomial of y over $K(t_1', \dots, t_d')$. Since the degree of this polynomial in y is $n + 1$, which is not divisible by p, we see that this minimal polynomial is separable and therefore the fraction field $Q(A) = K(t_1', \dots, t_d')(y)$ is separable over $K(t_1', \dots, t_d')$. It is also clear that A is finite free over $K[t_1', \dots, t_d']$ of rank n. ■

Lemma 4.4. *Let X be an integral smooth scheme of finite type over K and denote by X^{an} the rigid analytification of X. Let $f \in H^0(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$ be a global rigid analytic function on X^{an} such that the graph of f viewed as a subset of $X(K) \times K$ is definable. Then $f \in H^0(X, \mathcal{O}_X)$.*

Proof. By passing to a *finite* affine cover of X we may assume $X = \operatorname{Spec}(A)$ for a domain A that is regular and a K-algebra of finite type. Choose a generically étale, Noether normalization of A, i.e. a finite inclusion $i : K[t_1, \dots, , t_d] \hookrightarrow A$ such that the induced injection $K(t_1, \dots, t_d) \hookrightarrow A \otimes_{K[t_1,\dots,t_d]} K(t_1, \dots, t_d)$ is a finite étale map or equivalently $A \otimes_{K[t_1,\dots,t_d]} K(t_1, \dots, t_d)$ is a finite separable field extension of $K(t_1, \dots, t_d)$. (By [38, Theorem 3.2.1] this is also equivalent to the existence of a dense Zariski open $U \subseteq \mathbb{A}_K^d$ such that $i^{-1}(U) \to U$ is finite and étale.) If K has characteristic zero, generic étaleness comes for free, and when K has positive characteristic, that there does exist such a Noether normalization (up to further passing to a finite affine cover) follows for instance from [15, Theorem 2].

Since A is regular (in particular Cohen–Macaulay), the inclusion i makes A into a finite, flat (hence locally free) module over $K[t_1, \dots, t_d]$ by Hironaka's Miracle Flatness criterion [40, Tag 00R4]. There is a finite set of polynomials $p_i(\underline{t}), 1 \le i \le m$, generating the unit ideal in $K[\underline{t}]$ such that $A[p_i^{-1}]$ is free over $K[\underline{t}][p_i^{-1}]$ for each i. Moreover, by Lemma 4.3 we see that $K[\underline{t}][p_i^{-1}]$ is finite, free and generically étale over another pure polynomial subring in d variables (after an appropriate change of variables). Thus, by replacing A with $A[p_i^{-1}]$ and modifying the Noether normalization map as above, we are in the case where A is finite, free and generically étale over the polynomial ring $K[t_1, \dots, t_d]$, say of rank r.

Let $a_1, \dots, a_r \in A$ be a module basis over $K[t_1, \dots, t_d]$. It follows that $H^0(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$ is a free module over $H^0(\mathbb{A}^{d,\text{an}})$ again with basis $a_1, \dots, a_r$. Thus, f can be written uniquely as $f = \sum_{k=1}^r a_k \cdot g_k(\underline{t})$ with $g_k(\underline{t}) \in H^0(\mathbb{A}^{d,\text{an}})$. To finish the proof, it suffices to show that the $g_k(\underline{t})$ have definable graphs in K^{d+1}, since then we may appeal to Lemma 4.2 to conclude that the g_k are polynomials. By continuity, it in fact suffices to show that $g_k(\underline{t})|_U$ has a definable graph for some Zariski dense open subset U of K^d. Since the Noether normalization map $i : \operatorname{Spec}(A) \to \mathbb{A}_K^d$ is generically étale, we may take $U \subseteq \mathbb{A}_K^d$ to be a Zariski dense open subset such that the induced map $i^{-1}(U) \to U$ is finite and étale. For any point $\underline{u} \in U$, letting $i^{-1}(\underline{u}) = \{P_1, \dots, P_r\}$, we have r linear equations in r variables:

$$f(P_j) = \sum_{1 \le k \le r} a_k(P_j) g_k(\underline{u})$$

for each $j \in \{1, \ldots, r\}$. Over the open subset U, the matrix $(a_k(P_j))_{1 \le k, j \le r}$ is invertible and thus we may write, for each k, the function $g_k(u)$ as an explicit linear combination of $\{f(P_j) : 1 \le j \le r\}$, with coefficients being rational functions in $a_k(P_j)$. Note that permuting the ordering of the P_j leaves the specific linear combination invariant. Thus, the graph of the function $g_k : U \to K$ can be expressed as a first-order formula with all its terms using definable functions and sets – indeed, we note that f is definable by assumption, and that $U, X(K), i, a_j$, being algebraic, are also definable. We thus find that the $g_k(\underline{u})$ are definable over U, concluding the proof. ∎

Lemma 4.5. *Let A be a reduced finite-type K-algebra and let $X = \mathrm{Spec}(A)$. Let $\{X_i\}$ denote the set of irreducible components of X, given their reduced induced structures. Suppose $f \in H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is a global rigid analytic function such that for every i, there is a non-empty Zariski open subset $U_i \subseteq X_i$ such that $f|_{U_i^{\mathrm{an}}} \in H^0(U_i, \mathcal{O}_{U_i})$. Then $f \in H^0(X, \mathcal{O}_X)$.*

Proof. By our assumptions on f, we may view f as an element of the total ring of fractions $Q(A)$ of A (see [40, Tag 02C5] for the definition). Indeed, if $\eta_i \in X_i$ denotes the generic point of X_i, we have (using for instance [40, Tag 02LX]) that $Q(A) = \prod_i \mathcal{O}_{X_i, \eta_i}$, where $(X_i, \mathcal{O}_{X_i})$ denotes the reduced induced subscheme structure on $X_i \subseteq X$ and $\mathcal{O}_{X_i, \eta_i}$ denotes the generic stalk of $\mathcal{O}_{X_i}$. Thus, the tuple $(f|_{U_i})_i \in \prod_i H^0(U_i, \mathcal{O}_{U_i}) \subseteq \prod_i \mathcal{O}_{X_i, \eta_i} = Q(A)$ defines an element of $Q(A)$.

To show that $f \in A$, it suffices to show that for every maximal ideal $\mathfrak{m}$ of A, the image of f in $Q(A)_{\mathfrak{m}}$ (the localization of $Q(A)$ at the multiplicative set $A \setminus \mathfrak{m}$) is also in $A_{\mathfrak{m}}$. Indeed, writing $f = a/s$ with $a, s \in A$ and s a non-zerodivisor, if $f \notin A$, then $a \notin sA$. So we may choose a maximal ideal $\mathfrak{m}$ containing $(sA : a) = \{b \in A : ba \in sA\}$. However, for this choice of $\mathfrak{m}$, $f \notin A_{\mathfrak{m}}$.

So let us now fix a maximal ideal $\mathfrak{m}$ of A. We note that $Q(A)_{\mathfrak{m}}$ is in fact the total ring of fractions $Q(A_{\mathfrak{m}})$ of $A_{\mathfrak{m}}$. We also note that since $f \in H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$, we have $f \in \widehat{A}_{\mathfrak{m}}$, since $\widehat{A}_{\mathfrak{m}} = \widehat{\mathcal{O}}_{X^{\mathrm{an}}, \mathfrak{m}}$ [11, Lemma 5.1.2 (2)]. For notational simplicity let $B := A_{\mathfrak{m}}$, $L := Q(B)$ and $\widehat{L} := L \otimes_B \widehat{B}$. We have *inclusions* $L \subseteq \widehat{L}$ and $\widehat{B} \subseteq \widehat{L}$ and $f \in L \cap \widehat{B}$. We must show that $f \in B$. Since $B \subseteq \widehat{B}$ is faithfully flat it suffices to show that $f \otimes 1 = 1 \otimes f$ in $\widehat{B} \otimes_B \widehat{B}$ [40, Tag 023M]. Since $f \in L$, the equality $f \otimes 1 = 1 \otimes f$ evidently holds in $\widehat{L} \otimes_L \widehat{L}$ and we further note that $\widehat{B} \otimes_B \widehat{B}$ *injects* into $(\widehat{B} \otimes_B \widehat{B}) \otimes_B L = \widehat{L} \otimes_L \widehat{L}$ since B injects into L and $\widehat{B} \otimes_B \widehat{B}$ is B-flat. Hence $f \otimes 1 = 1 \otimes f$ in $\widehat{B} \otimes_B \widehat{B}$, as was to be shown. ∎

Theorem 4.6 (A non-archimedean definable Liouville theorem). *Let X be a reduced scheme of finite type over K and denote by X^{an} the rigid analytification of X. Let $f \in H^0(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ be a global rigid analytic function on X^{an} whose graph viewed as a subset of $X(K) \times K$ is definable. Then $f \in H^0(X, \mathcal{O}_X)$.*

Proof. Again, by passing to a finite affine open cover we may assume that X is affine. For each irreducible component X_i of X, let $U_i \subseteq X_i$ be a dense open subset of X_i that is *smooth* over K. The restriction $f|_{U_i^{\mathrm{an}}} \in H^0(U_i^{\mathrm{an}}, \mathcal{O}_{U_i^{\mathrm{an}}})$ is definable and hence by Lemma 4.4 we know that $f|_{U_i^{\mathrm{an}}} \in H^0(U_i, \mathcal{O}_{U_i})$. From Lemma 4.5 we conclude that $f \in H^0(X, \mathcal{O}_X)$. ■

Remark 4.7. It is clear that reducedness of the underlying variety X is necessary in the above Theorem 4.6 since the graph of a function on the underlying K-points does not record the nilpotent structure. For example, take $X = \mathbb{A}^1_K[\epsilon] = \mathrm{Spec}(K[t, \epsilon]/(\epsilon^2))$. Choose any function $g \in H^0(X^{\mathrm{an}})$ which is not in $H^0(X)$ and take $f = \epsilon \cdot g$.

4.2. *Proof of the non-archimedean definable Chow theorem*

We now turn towards proving our main Theorem 4.1:

Theorem 4.8. *Let X be a closed analytic subset of $\mathbb{A}^{n,\mathrm{an}}_K$ that is also definable as a subset of $\mathbb{A}^n_K = K^n$. Then X is an algebraic subset, i.e. X is defined as the vanishing locus of a finite collection of polynomials in $K[t_1, \ldots, t_n]$.*

Remark 4.9. Recall that if $\mathscr{A}$ is a rigid analytic space over K, then by a *closed analytic subset* $X \subseteq \mathscr{A}$ we mean that there is a closed immersion of rigid spaces $\mathscr{X} \overset{i}{\hookrightarrow} \mathscr{A}$ such that $i(\mathscr{X}) = X$. Equivalently, X is cut out by the vanishing locus of a coherent $\mathcal{O}_{\mathscr{A}}$-ideal, or more concretely, there is an admissible affinoid covering $\mathscr{A} = \bigcup_{i \in I} U_i$, and for each $i \in I$, finitely many functions $f_1^{(i)}, \ldots, f_{n(i)}^{(i)}$ in $\mathcal{O}_{\mathscr{A}}(U_i)$ such that $X \cap U_i$ is the vanishing locus of $\{f_1^{(i)}, \ldots, f_{n(i)}^{(i)}\}$. Moreover, we note that given a closed analytic subset $X \subseteq \mathscr{A}$ as above, there is a canonical structure of a reduced rigid analytic space that can be put on X, with a canonical closed immersion $X \hookrightarrow \mathscr{A}$ (see [8, Section 9.5.3, Proposition 4]). We shall refer to this reduced structure as the *reduced induced structure* on X.

As outlined earlier, the proof of the theorem will proceed by induction on the dimension of the definable set $X \subseteq K^n$ (which agrees with the dimension of X as an analytic space – Lemma 3.26). First, we prove a preparatory lemma concerning the étale locus of a finite morphism of rigid varieties that will be used in the proof.

Lemma 4.10. *Suppose $\pi : X \to Y$ is a finite surjective morphism of reduced rigid analytic varieties over K. Suppose X is equidimensional at every point* (*i.e. for all $x \in X$,* $\dim(\mathcal{O}_{X,x}) = \dim(X)$) *and suppose Y is irreducible and normal* (*i.e. for all $y \in Y$, $\mathcal{O}_{Y,y}$ is a normal domain*)*. Let N be the generic fiber cardinality of π* (*i.e.* $N = \mathrm{rank}_{\mathcal{O}_Y}(\pi_* \mathcal{O}_X)$)*. Then for $y \in Y$, π is étale at every point in the fiber of y if and only if $|\pi^{-1}(y)| = N$.*

Remark 4.11. (a) We recall that a morphism of rigid spaces $\pi : X \to Y$ is said to be étale at a point $x \in X$ if the induced map of local rings $\mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$ is flat and unramified (see [12, Section 3]).

(b) Saying that the generic fiber cardinality is N we mean that for every $y \in Y$, we have

$$N = \dim_{Q(\mathcal{O}_{Y,y})}\big((\pi_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} Q(\mathcal{O}_{Y,y})\big).$$

Here $Q(\mathcal{O}_{Y,y})$ denotes the fraction field of the domain $\mathcal{O}_{Y,y}$. Since Y is connected, the dimension on the right-hand side is indeed independent of the point y. To see this, it suffices to work over a connected affinoid open $\mathrm{Sp}(A)$ of Y. Then A must be a domain, and since π is a finite map, $\pi^{-1}(\mathrm{Sp}(A))$ is an affinoid open $\mathrm{Sp}(B)$ of X with the induced map $A \to B$ making B a finite A-module. For a point $y \in \mathrm{Sp}(A)$ corresponding to the maximal ideal $\mathfrak{m}$ of A, we have that $\dim_{Q(\mathcal{O}_{Y,y})}((\pi_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} Q(\mathcal{O}_{Y,y})) = \dim_{Q(\mathcal{O}_{Y,y})}(B \otimes_A Q(\mathcal{O}_{Y,y})) = \dim_{Q(A)} B \otimes_A Q(A)$.

Proof of Lemma 4.10. By working locally over connected affinoid opens of Y, we may assume that $Y = \mathrm{Sp}(A)$ is affinoid. Since π is a finite morphism, X is also affinoid, and $X = \mathrm{Sp}(B)$ with the induced map $A \to B$ making B a finite A-module. The assumptions on Y imply that A is a normal integral domain. Let F denote its fraction field and let $N = \dim_F(B \otimes_A F)$ be the generic fiber cardinality of π.

For a point $x \in X$, if we denote the maximal ideals corresponding to $x, \pi(x)$ by $\mathfrak{n} \subseteq B$, $\mathfrak{m} \subseteq A$ respectively, then we note that since $\widehat{A}_{\mathfrak{m}} = \widehat{\mathcal{O}}_{Y,\pi(x)}$ and $\widehat{B}_{\mathfrak{n}} = \widehat{\mathcal{O}}_{X,x}$ the map $\mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$ is flat and unramified if and only if the same holds for the map $A_{\mathfrak{m}} \to B_{\mathfrak{n}}$ (the fact that both maps are unramified simultaneously is easy to see, whereas for flatness one may use the local flatness criterion [30, Theorems 22.1 and 22.4]).

Suppose now $y \in Y$ is a point corresponding to the maximal ideal $\mathfrak{m}$ of A such that π is étale at every point of $\pi^{-1}(y)$. Then from the above, $B/\mathfrak{m}B$ must be unramified over $A/\mathfrak{m}$ and thus $|\pi^{-1}(y)| = \dim_{A/\mathfrak{m}} B/\mathfrak{m}B$. Similarly, it follows that $B \otimes_A A_{\mathfrak{m}}$ is finite flat (hence free) over $A_{\mathfrak{m}}$ and hence $\mathrm{rank}_{A_{\mathfrak{m}}}(B \otimes_A A_{\mathfrak{m}}) = \dim_{A/\mathfrak{m}} B/\mathfrak{m}B = \dim_K(B \otimes_A F) = N$. Therefore, we see that $|\pi^{-1}(y)| = N$.

Before turning to prove the converse direction, we first show that $\dim(X) = \dim(Y)$. By Lemma 3.25, $\dim(X) = \dim(B)$ and $\dim(Y) = \dim(A)$. Since $\pi : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$ is surjective, the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains all the closed points of $\mathrm{Spec}(A)$. If I denotes the kernel of $A \to B$, then $A/I \hookrightarrow B$ is a finite *inclusion* of rings, and so by [2, Theorem 5.10], the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the set $V(I)$ of all primes containing I. Thus, we must have $V(I) \supseteq \mathrm{MaxSpec}(A)$. However, since A is Jacobson (by [8, Section 5.2.6, Theorem 3]), this implies $V(I) = \mathrm{Spec}(A)$ and hence $I = 0$. Thus, $A \hookrightarrow B$ is a finite inclusion of rings and therefore $\dim(A) = \dim(B)$.

Now suppose $y \in Y$ is a point such that $|\pi^{-1}(y)| = N$. Let $\pi^{-1}(y) = \{x_1, \ldots, x_N\}$. We would like to show that π is étale at each x_i. We have a canonical isomorphism $B \otimes_A \mathcal{O}_{Y,y} = (\pi_* \mathcal{O}_X)_y \cong \prod_{i=1}^N \mathcal{O}_{X,x_i}$ (see [11, pp. 481–482]). If L denotes the fraction field of $\mathcal{O}_{Y,y}$, we have $B \otimes_A L \cong \prod_{i=1}^N \mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$.

Subclaim. *For each i, the natural map $\mathcal{O}_{X,x_i} \to \mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ is injective.*

Proof of Subclaim. Note that $\mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ is the localization of $\mathcal{O}_{X,x_i}$ at the (image inside $\mathcal{O}_{X,x_i}$ of the) multiplicative set $\mathcal{O}_{Y,y} \setminus \{0\}$. Thus, the claim is equivalent to showing that $\mathcal{O}_{X,x_i}$ is a torsion-free $\mathcal{O}_{Y,y}$-module. Equivalently, we must show that the image of $\mathcal{O}_{Y,y} \setminus \{0\}$ inside $\mathcal{O}_{X,x_i}$ does not contain any zero-divisors of the ring $\mathcal{O}_{X,x_i}$. Since $\mathcal{O}_{X,x_i}$ is a reduced ring, the set of zero-divisors of $\mathcal{O}_{X,x_i}$ is the union of the minimal prime ideals of $\mathcal{O}_{X,x_i}$ [40, Tag 00EW]. Therefore, it suffices to prove that every minimal prime ideal $\mathfrak{q}$ of $\mathcal{O}_{X,x_i}$ contracts to the (0)-ideal of $\mathcal{O}_{Y,y}$. If we set $\mathfrak{q} \cap \mathcal{O}_{Y,y} =: \mathfrak{p}$, then $\mathcal{O}_{Y.y}/\mathfrak{p} \hookrightarrow$

$\mathcal{O}_{X,x_i}/\mathfrak{q}$ a finite *inclusion* of domains, and hence $\dim(\mathcal{O}_{Y.y}/\mathfrak{p}) = \dim(\mathcal{O}_{X,x_i}/\mathfrak{q})$. We now have the chain of equalities

$$\dim(\mathcal{O}_{Y.y}/\mathfrak{p}) = \dim(\mathcal{O}_{X,x_i}/\mathfrak{q}) = \dim(X) = \dim(Y) = \dim(\mathcal{O}_{Y,y}),$$

where the second equality is from Lemma 3.28, the third from the previous paragraphs and the last from Lemma 3.27. But since $\mathcal{O}_{Y,y}$ is an integral domain, the equality $\dim(\mathcal{O}_{Y.y}/\mathfrak{p}) = \dim(\mathcal{O}_{Y.y})$ is only possible if $\mathfrak{p} = (0)$. This completes the proof of the subclaim. ■

The subclaim shows in particular that $\mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ must be non-zero for each i. But since $\dim_L(B \otimes_A L) = N$, this is only possible if $L = \mathcal{O}_{X,x_i} \otimes_{\mathcal{O}_{Y,y}} L$ for each i. In particular, $\mathcal{O}_{X,x_i} \subseteq L$. However, since $\mathcal{O}_{Y,y}$ is a normal domain, and since $\mathcal{O}_{X,x_i}$ is finite over $\mathcal{O}_{Y,y}$, we get $\mathcal{O}_{Y,y} = \mathcal{O}_{X,x_i}$, and so π is evidently étale at x_i. ■

Proof of Theorem 4.1. We induct on $d + n$ where $d := \dim(X)$. If $d = 0$, then X is finite, hence algebraic. And if $d = n$, then $X = K^n$ and we are done.

Suppose that $n > d \geq 1$. It follows from Lemma 3.26 that the set $S = \{x \in X : \dim(\mathcal{O}_{X,x}) < d\}$ is a *definable* subset of K^n. Moreover, the closure of S in K^n, $\mathrm{cl}(S)$, is the union of the irreducible components of X of dimension $< d$, and so by the induction hypothesis, $\mathrm{cl}(S)$ is an algebraic subset of K^n. It then suffices to show that $X \setminus S$ is an algebraic subset of K^n. Note that $X \setminus S$ is the union of the irreducible components of dimension d and therefore $X \setminus S$ is a closed, analytic subset of K^n. Thus, replacing X by $X \setminus S$ we may assume that X is *equidimensional* of dimension d.

Finding a point $q \in \mathbb{P}^n(K) \setminus \mathrm{cl}(X)$: Embed $K^n \subseteq \mathbb{P}^n(K)$ inside projective n-space and denote the homogeneous coordinates of $\mathbb{P}^n(K)$ by $Z_1, \dots, Z_{n+1}$. Let μ denote the point $[1 : 0 : \dots : 0] \in \mathbb{P}^n(K) \setminus K^n$, and consider the neighbourhood $\Delta := \{|Z_1| \geq |Z_2|, \dots, |Z_1| \geq |Z_{n+1}|\} \subseteq \mathbb{P}^n(K)$ of the point μ. The neighbourhood Δ is naturally homeomorphic to the closed unit n-dimensional disk, $(K^\circ)^n$, via the map $\varphi : \Delta \to (K^\circ)^n$ that sends $[Z_1 : \dots : Z_{n+1}]$ to $(Z_2/Z_1, \dots, Z_{n+1}/Z_1)$, and $S := \varphi(X \cap \Delta)$ is a definable subset of $(K^\circ)^n$ of dimension $\leq d$ contained in $(K^\circ)^{n-1} \times K^\circ \setminus \{0\}$. We note that since $\mathrm{cl}(S) \cap (K^\circ)^{n-1} \times \{0\} \subseteq \mathrm{Fr}(S)$, and from the Theorem of the Boundary (Theorem 3.21), since $\dim(\mathrm{Fr}(S)) < d \leq n - 1$ we can find a point $q \in (K^\circ)^{n-1} \times \{0\}$ such that $q \notin \mathrm{cl}(S)$, and pulling back via φ to Δ, we find a point $q \in \mathbb{P}^n(K) \setminus K^n$ such that $q \notin \mathrm{cl}_{\mathbb{P}^n(K)}(X)$. The point $q \in \mathbb{P}^n(K) \setminus K^n = \mathbb{P}^{n-1}(K)$ defines a line in K^n. Now let $T \subseteq X(K)$ be a countable subset of $X(K)$ consisting of *smooth* points of X such that the only closed analytic subset of X containing T is X itself. By Corollary 3.15, the countable union $\bigcup_{t \in T} \mathbb{P}(T_t(X)) \subseteq \mathbb{P}^{n-1}(K)$ must have dimension $\dim(X) - 1 < n - 1$, and therefore cannot completely contain $\mathbb{P}^{n-1}(K) \setminus \mathrm{Fr}(X)$. We may thus further assume that for every $t \in T$, the line defined by the point q is not contained in the tangent space $T_t(X)$.

Consider any $(n-1)$-dimensional linear subspace $\mathbb{H} \subseteq K^n$ not containing the line defined by q, and let $\pi : K^n \to \mathbb{H}$ denote the projection onto $\mathbb{H}$ with kernel being the line defined by q. We are free to make a linear change of coordinates on K^n, and so we may even assume for simplicity that $q = [0 : \dots : 0 : 1] \in \mathbb{P}^{n-1}(K)$ and that $\pi : K^n \to K^{n-1}$ is the projection to the first $n - 1$ coordinates.

Lemma 4.12. *The projection $\pi|_X : X \to K^{n-1}$ is a finite morphism of rigid analytic spaces (endowing X with the reduced induced structure).*

Proof. *$\pi|_X$ is quasi-finite:* Indeed, for $\underline{z} \in K^{n-1}$, $\pi^{-1}(\underline{z}) \cap X$ is a closed analytic subset of the one-dimensional line $\pi^{-1}(\underline{z})$ and is in addition definable. If $\dim(\pi^{-1}(\underline{z}) \cap X) = 1$, then $\pi^{-1}(\underline{z}) \subseteq X$, which would imply that $q = [0 : \ldots : 0 : 1] \in \mathrm{cl}_{\mathbb{P}^n(K)}(X)$, contradicting our choice of q. Thus, $\dim(\pi^{-1}(\underline{z}) \cap X) = 0$, i.e. $\pi^{-1}(\underline{z}) \cap X$ is finite.

To show that $\pi|_X$ is a finite morphism, it thus remains to show that $\pi|_X$ is a proper morphism of rigid spaces [8, Section 9.6.3, Corollary 6]. In order to prove this, we consider the map $\pi|_X$ on the level of the associated Berkovich spaces. Note that X, being a closed analytic subvariety of rigid affine n-space, is a quasi-separated rigid space and has an admissible affinoid covering of finite type, and moroeover its associated Berkovich analytic space is 'good' in the sense of [6, Remark 1.2.16 and Section 1.5]. Recall that the morphism $\pi|_X : X^{\mathrm{Berk}} \to \mathbb{A}_K^{n-1,\mathrm{Berk}}$ of good K-analytic spaces is proper if it is topologically proper and boundaryless (or 'compact and closed' in the terminology of [5, p. 50]).

$\pi|_X$ is separated and topologically proper: $\pi|_X$ is indeed separated. If $E(0, \underline{r})$ denotes the closed polydisc of polyradius $\underline{r}$ in $\mathbb{A}_K^{n-1,\mathrm{Berk}}$, i.e.

$$E(0, \underline{r}) = \mathcal{M}(K\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}\}),$$

then we claim that $\pi^{-1}(E(0, \underline{r})) \cap X^{\mathrm{Berk}}$ is bounded in $\mathbb{A}_K^{n,\mathrm{Berk}}$. If it were not, there would be a sequence of points $x_i \in \pi^{-1}(E(0, \underline{r})) \cap X^{\mathrm{Berk}}$ with $|T_n(x_i)| \to \infty$ as $i \to \infty$. We may even find a sequence $x_i \in X$ since by [5, Proposition 2.1.15], the set of rigid points is everywhere dense. But this would again imply that $q = [0 : \ldots : 0 : 1] \in \mathrm{cl}_{\mathbb{P}^n(K)}(X)$, contradicting the choice of q. Since every compact subset of $\mathbb{A}_K^{n-1,\mathrm{Berk}}$ is contained in some $E(0, \underline{r})$, it follows that the inverse images of compact sets under the map $\pi|_X : X^{\mathrm{Berk}} \to \mathbb{A}_K^{n-1,\mathrm{Berk}}$ are compact. Thus, $\pi|_X$ is topologically proper.

$\pi|_X$ is boundaryless: Since $X^{\mathrm{Berk}} \hookrightarrow \mathbb{A}_K^{n,\mathrm{Berk}}$ is a closed immersion, it follows that $\mathrm{Int}(X^{\mathrm{Berk}}/\mathbb{A}_K^{n,\mathrm{Berk}}) = X^{\mathrm{Berk}}$. By [5, Proposition 3.1.3 (ii)] it suffices to note that $\mathrm{Int}(\mathbb{A}_K^{n,\mathrm{Berk}}/\mathbb{A}_K^{n-1,\mathrm{Berk}}) = \mathbb{A}_K^{n,\mathrm{Berk}}$. To see this last equality, for any $x \in \mathbb{A}_K^{n,\mathrm{Berk}}$ let $y = \pi(x)$, and choose an affinoid neighbourhood $E(0, \underline{r}) \subseteq \mathbb{A}_K^{n-1,\mathrm{Berk}}$ containing y in its interior. Choosing an $R \in |K^\times|$ with $R > |T_n(x)|$, we see that $\chi_x : K\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}, R^{-1}T_n\} \to \mathcal{H}(x)$ is inner over $K\{r_1^{-1}T_1, \ldots, r_{n-1}^{-1}T_{n-1}\}$, i.e. $x \in \mathrm{Int}(\mathbb{A}_K^{n,\mathrm{Berk}}/\mathbb{A}_K^{n-1,\mathrm{Berk}})$.

Therefore, the map $\pi|_X : X^{\mathrm{Berk}} \to \mathbb{A}_K^{n-1,\mathrm{Berk}}$ is proper and hence by [5, Proposition 3.3.2], so is $\pi|_X : X \to \mathbb{A}_K^{n-1,\mathrm{an}}$. ∎

Since $\pi|_X : X \to \mathbb{A}_K^{n-1,\mathrm{an}}$ is finite, the image $Y := \pi(X)$ is a closed analytic subvariety of K^{n-1}, by [8, Section 9.6.3, Proposition 3]. In addition, as Y is a definable subset, by the induction hypothesis Y is an algebraic subset of K^{n-1}. Endowing Y with its structure as a reduced closed affine algebraic subvariety of $\mathbb{A}_K^{n-1}$, the morphism $\pi|_X$ gives rise to a finite, surjective morphism of rigid analytic spaces $\pi|_X : X \to Y^{\mathrm{an}}$.

Lemma 4.13. *There is a Zariski dense open $U \subseteq Y$ such that $\pi|_X^{-1}(U^{\mathrm{an}}) \to U^{\mathrm{an}}$ is a finite étale surjection of rigid varieties.*

Proof. We first claim that the map $\pi|_X : X \to Y^{\mathrm{an}}$ is étale at at least one point of X. If the characteristic of K is zero this is immediate. In the general case, note that $\pi(T) \subseteq Y$ is not contained in any analytic subset of Y^{an}, and in particular there is a $t_0 \in T$ such that $\pi(t_0)$ is a smooth point of Y. Then note that by our choice of q earlier, the map $\pi|_X : X \to Y^{\mathrm{an}}$ is injective on the level of tangent spaces $T_{t_0}(X) \hookrightarrow T_{\pi(t_0)}(Y^{\mathrm{an}})$. Therefore, applying [25, Chapter 4, Corollary 3.27] we see that $\pi|_X$ is étale at t_0.

Let $\{Y_i\}_{1\le i\le r}$ denote the finitely many irreducible components of Y, thus $\{Y_i^{\mathrm{an}}\}_{1\le i\le r}$ are those of Y^{an}. Let $U_i \subseteq Y_i \setminus \bigcup_{j\neq i} Y_j$ be a non-empty, *principal* open subset of Y (that is, the complement in Y of the vanishing set of a single regular function on Y) so that each U_i is an integral (reduced and irreducible) open subvariety of Y (hence U_i^{an} is a reduced and irreducible admissible open subset of Y^{an} [11, Theorem 5.1.3 (2)]). Being a principal open subset of $Y \subseteq K^{n-1}$, U_i may be viewed as a closed affine subvariety of K^n. By Lemma 4.10, the étale locus $E_i \subseteq U_i^{\mathrm{an}}$ of $\pi : \pi^{-1}(U_i^{\mathrm{an}}) \to U_i^{\mathrm{an}}$ is definable as it may be defined using a first-order formula expressing E_i as the subset of points in U_i^{an} whose fiber under π has cardinality equal to the generic fiber cardinality over U_i^{an}. Moreover, the complement $U_i^{\mathrm{an}} \setminus E_i$ is a closed analytic subvariety of $U_i^{\mathrm{an}} \subseteq K^n$ of dimension $< \dim(U_i) = d$. By the induction hypothesis, $U_i^{\mathrm{an}} \setminus E_i$ is a Zariski closed algebraic subset of U_i and hence E_i is a Zariski dense open subset of U_i. Now setting $U = \bigcup_i E_i$ completes the proof of the lemma. ∎

Let U be as in Lemma 4.13 above, let $\{U_j\}$ be the finitely many open connected components of U and let $V_j := \pi|_X^{-1}(U_j)$. Suppose the fiber cardinality of $\pi|_{V_j} : \pi|_X^{-1}(U_j) \to U_j$ is N_j. The characteristic polynomial of $T_n|_{V_j}$ over U_j (here T_n is the last coordinate function of $\mathbb{C}_p^n$) is a polynomial of degree N_j with coefficients in $\mathcal{O}_{Y^{\mathrm{an}}}(U_j^{\mathrm{an}})$. Moreover, since the U_j are Zariski opens and since X is definable, it follows that the coefficients are also definable since they may be defined as symmetric polynomials in the fibers of $\pi|_X$. Hence, by Theorem 4.6 the characteristic polynomial in fact has coefficients in $\mathcal{O}_Y(U_j)$. If $W \subseteq K^{n-1}$ is a Zariski open subset such that $W \cap Y = U$, then it follows from the above that $X \cap (W \times K)$ is a closed algebraic subset of $W \times K$.

If we let Z denote the Zariski closure of $X \cap (W \times K)$ in K^n, then Z is also the closure of $X \cap (W \times K)$ in the metric topology of K^n [11, Theorem 5.1.3 (2)], and hence $Z \subseteq X$. Moreover, $X \setminus (W \times K) = \pi|_X^{-1}(Y \setminus U)$ and so $\dim(X \setminus (W \times K)) < \dim(Y) \le d$. By the induction hypothesis, $X \setminus (W \times K)$ is thus a closed algebraic subset of K^n and since $X = Z \cup (X \setminus (W \times K))$, we conclude that X is algebraic, finishing the proof of Theorem 4.1. ∎

We obtain as a corollary:

Corollary 4.14. *Let V be a reduced algebraic variety over K, and let $X \subseteq V^{\mathrm{an}}$ be a closed analytic subvariety of the rigid analytic variety V^{an} associated to V such that $X \subseteq V(K)$ is definable in a tame structure on K. Then X is algebraic.*

For a *proper* algebraic variety V over K, V^{an} is quasi-compact and thus V^{an} has an admissible cover by *finitely* many (definable) affinoid subdomains. Since affinoid algebras are quotients of Tate algebras, the rigid analytic functions on these affinoids are in fact rigid-subanalytic. A closed analytic subvariety of an affinoid subdomain, being described as the zero locus of finitely many such functions, is also therefore rigid-subanalytic. Thus, when V is proper over K, every closed analytic subvariety of V^{an} is definable in the tame structure of the rigid-subanalytic sets. Thus the familiar version of Chow's theorem for proper varieties follows from Theorem 4.1:

Corollary 4.15 (Chow's theorem for proper varieties). *Every closed analytic subset of the rigid analytic variety associated to a proper algebraic variety over K is algebraic.*

Acknowledgments. It is a great pleasure to thank Jacob Tsimerman for several helpful conversations and suggestions and for asking me the question of whether a non-archimedean version of Peterzil and Starchenko's definable Chow theorem could be proved.

The proof of the non-archimedean definable Chow theorem is based on an alternative proof of Peterzil and Starchenko's theorem. I learned of this proof at a lecture series given by Benjamin Bakker. I am very grateful to him for the inspiring set of lectures that went a long way in helping me understand the work of Peterzil and Starchenko.

I would also like to thank Angelo Vistoli for the suggestion to use faithfully flat descent in order to prove Lemma 4.5 below. It is also a pleasure to thank François Loeser for pointing me to Haskell and Macpherson's work on C-minimality and for a careful reading of an earlier version of the manuscript. Finally, I am grateful to the anonymous referees for their helpful suggestions, which have greatly improved the exposition of this paper.

References

[1] Adeleke, S. A., Neumann, P. M.: Primitive permutation groups with primitive Jordan sets. J. London Math. Soc. (2) **53**, 209–229 (1996) Zbl 0865.20005 MR 1373056

[2] Atiyah, M. F., Macdonald, I. G.: Introduction to commutative algebra. Addison-Wesley Series in Mathematics, Westview Press, Boulder, CO (2016) Zbl 1351.13002 MR 3525784

[3] Bakker, B., Brunebarbe, Y., Tsimerman, J.: o-minimal GAGA and a conjecture of Griffiths. arXiv:1811.12230 (2018)

[4] Bakker, B., Tsimerman, J.: The Ax–Schanuel conjecture for variations of Hodge structures. Invent. Math. **217**, 77–94 (2019) Zbl 1420.14021 MR 3958791

[5] Berkovich, V. G.: Spectral theory and analytic geometry over non-Archimedean fields. Mathematical Surveys and Monographs 33, American Mathematical Society, Providence, RI (1990) Zbl 0715.14013 MR 1070709

[6] Berkovich, V. G.: Étale cohomology for non-Archimedean analytic spaces. Inst. Hautes Études Sci. Publ. Math. **78**, 5–161 (1994) (1993) Zbl 0804.32019 MR 1259429

[7] Bosch, S.: Lectures on formal and rigid geometry. Lecture Notes in Mathematics 2105, Springer, Cham (2014) Zbl 1314.14002 MR 3309387

[8] Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean analysis. Grundlehren der mathematischen Wissenschaften 261, Springer, Berlin (1984) Zbl 0539.14017 MR 746961

[9] Cluckers, R., Comte, G., Loeser, F.: Non-Archimedean Yomdin–Gromov parametrizations and points of bounded height. Forum Math. Pi **3**, art. e5, 60 pp. (2015) Zbl 1393.11032 MR 3406825

[10] Cluckers, R., Loeser, F.: b-minimality. J. Math. Logic **7**, 195–227 (2007) Zbl 1146.03021 MR 2423950

[11] Conrad, B.: Irreducible components of rigid spaces. Ann. Inst. Fourier (Grenoble) **49**, 473–541 (1999) Zbl 0928.32011 MR 1697371

[12] de Jong, J., van der Put, M.: Étale cohomology of rigid analytic spaces. Doc. Math. **1**, 1–56 (1996) Zbl 0922.14012 MR 1386046

[13] Ducros, A.: Variation de la dimension relative en géométrie analytique p-adique. Compos. Math. **143**, 1511–1532 (2007) Zbl 1161.14018 MR 2371379

[14] Haskell, D., Macpherson, D.: Cell decompositions of C-minimal structures. Ann. Pure Appl. Logic **66**, 113–162 (1994) Zbl 0790.03039 MR 1262433

[15] Kedlaya, K. S.: More étale covers of affine spaces in positive characteristic. J. Algebraic Geom. **14**, 187–192 (2005) Zbl 1065.14020 MR 2092132

[16] Kiehl, R.: Die de Rham Kohomologie algebraischer Mannigfaltigkeiten über einem bewerteten Körper. Inst. Hautes Études Sci. Publ. Math. **33**, 5–20 (1967) Zbl 0159.22404 MR 229644

[17] Lawrence, B., Sawin, W.: The Shafarevich conjecture for hypersurfaces in abelian varieties. arXiv:2004.09046 (2020)

[18] Lawrence, B., Venkatesh, A.: Diophantine problems and p-adic period mappings. Invent. Math. **221**, 893–999 (2020) Zbl 1455.11093 MR 4132959

[19] Lipshitz, L.: Isolated points on fibers of affinoid varieties. J. Reine Angew. Math. **384**, 208–220 (1988) Zbl 0636.14006 MR 929984

[20] Lipshitz, L.: Rigid subanalytic sets. Amer. J. Math. **115**, 77–108 (1993) Zbl 0792.14010 MR 1209235

[21] Lipshitz, L., Robinson, Z.: Rigid subanalytic subsets of the line and the plane. Amer. J. Math. **118**, 493–527 (1996) Zbl 0935.14035 MR 1393258

[22] Lipshitz, L., Robinson, Z.: Dimension theory and smooth stratification of rigid subanalytic sets. In: Logic Colloquium ’98 (Prague), Lecture Notes in Logic 13, Association for Symbolic Logic, Urbana, IL, 302–315 (2000) Zbl 0978.14046 MR 1743267

[23] Lipshitz, L., Robinson, Z.: Rings of separated power series and quasi-affinoid geometry. Astérisque **264**, vi+171 pp. (2000) Zbl 0957.32011 MR 1758887

[24] Lipshitz, L., Robinson, Z.: Model completeness and subanalytic sets. Astérisque **264**, 109–126 (2000) MR 1758887 Zbl 0978.14046

[25] Liu, Q.: Algebraic geometry and arithmetic curves. Oxford Graduate Texts in Mathematics 6, Oxford University Press, Oxford (2002) Zbl 0996.14005 MR 1917232

[26] Lütkebohmert, W.: Der Satz von Remmert–Stein in der nichtarchimedischen Funktionentheorie. Math. Z. **139**, 69–84 (1974) Zbl 0283.32022 MR 352527

[27] Macintyre, A., McKenna, K., van den Dries, L.: Elimination of quantifiers in algebraic structures. Adv. Math. **47**, 74–87 (1983) Zbl 0531.03016 MR 689765

[28] Macpherson, D., Steinhorn, C.: On variants of o-minimality. Ann. Pure Appl. Logic **79**, 165–209 (1996) Zbl 0858.03039 MR 1396850

[29] Martin, F.: Constructibilité dans les espaces de Berkovich. PhD thesis, Université Pierre et Marie Curie, Paris (2013)

[30] Matsumura, H.: Commutative ring theory. 2nd ed., Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge (1989) Zbl 0666.13002 MR 1011461

[31] Mok, N., Pila, J., Tsimerman, J.: Ax–Schanuel for Shimura varieties. Ann. of Math. (2) **189**, 945–978 (2019) Zbl 1481.14048 MR 3961087

[32] Peterzil, Y., Starchenko, S.: Complex analytic geometry in a nonstandard setting. In: Model theory with applications to algebra and analysis, Vol. 1, London Mathematical Society Lecture Note Series 349, Cambridge University Press, Cambridge, 117–165 (2008) Zbl 1160.03017 MR 2441378

[33] Peterzil, Y., Starchenko, S.: Tame complex analysis and o-minimality. In: Proceedings of the International Congress of Mathematicians (Hyderabad, 2000), Volume II, Hindustan Book Agency, New Delhi, 58–81 (2010) Zbl 1246.03061 MR 2827785

[34] Pila, J., Shankar, A. N., Tsimerman, J., Esnault, H., Groechenig, M.: Canonical heights on Shimura varieties and the André–Oort conjecture. arXiv:2109.08788 (2021)

[35] Pila, J., Wilkie, A. J.: The rational points of a definable set. Duke Math. J. **133**, 591–616 (2006) Zbl 1217.11066 MR 2228464

[36] Pila, J., Zannier, U.: Rational points in periodic analytic sets and the Manin–Mumford conjecture. Rend. Lincei Mat. Appl. **19**, 149–162 (2008) Zbl 1164.11029 MR 2411018

[37] Podewski, K.-P.: Minimale Ringe. Math.-Phys. Semesterber. **22**, 193–197 (1975) Zbl 0316.16001 MR 392962

[38] Poonen, B.: Rational points on varieties. Graduate Studies in Mathematics 186, American Mathematical Society, Providence, RI (2017) Zbl 1387.14004 MR 3729254

[39] Serre, J.-P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier (Grenoble) **6**, 1–42 (1955/56) Zbl 0075.30401 MR 82175

[40] Stacks project authors: The stacks project. https://stacks.math.columbia.edu (2020)

[41] Tsimerman, J.: The André–Oort conjecture for A_g. Ann. of Math. (2) **187**, 379–390 (2018) Zbl 1415.11086 MR 3744855

[42] van den Dries, L.: Tame topology and o-minimal structures. London Mathematical Society Lecture Note Series 248, Cambridge University Press, Cambridge (1998) Zbl 0953.03045 MR 1633348

[43] Wagner, F. O.: Minimal fields. J. Symbolic Logic **65**, 1833–1835 (2000) Zbl 1039.03030 MR 1812183