J. Spectr. Theory (Online first)
DOI 10.4171/JST/468

Published by EMS Press

Morse theory for discrete magnetic operators and nodal count distribution for graphs

Lior Alon and Mark Goresky

Abstract. Given a discrete Schrödinger operator h on a finite connected graph G of n vertices, the nodal count $\phi(h,k)$ denotes the number of edges on which the k-th eigenvector changes sign. A *signing* h' of h is any real symmetric matrix constructed by changing the sign of some off-diagonal entries of h, and its nodal count is defined according to the signing. The set of signings of h lie in a naturally defined torus $\mathbb{T}_h$ of "magnetic perturbations" of h. G. Berkolaiko [Anal. PDE 6 (2013), 1213–1233] discovered that every signing h' of h is a critical point of every eigenvalue $\lambda_k \colon \mathbb{T}_h \to \mathbb{R}$, with Morse index equal to the nodal surplus. We add further Morse theoretic information to this result. We show if $h_\alpha \in \mathbb{T}_h$ is a critical point of λ_k and the eigenvector vanishes at a single vertex v of degree d, then the critical point lies in a non-degenerate critical submanifold of dimension $d+n-4$, closely related to the configuration space of a planar linkage. We compute its Morse index in terms of spectral data.

The *average nodal surplus distribution* is the distribution of values of $\phi(h',k)-(k-1)$, averaged over all signings h' of h. If all critical points correspond to simple eigenvalues with nowhere-vanishing eigenvectors, then the average nodal surplus distribution is binomial. In general, we conjecture that the nodal surplus distribution converges to a Gaussian in a CLT fashion as the first Betti number of G goes to infinity.

1. Introduction

In some ways, this paper is both an analogue of [2] for discrete graphs and a continuation and expansion of the papers [7, 13], although it is completely self-contained.

1.1. The setting

Let G be a simple graph on n ordered vertices labeled $1, 2, \dots, n$. Write $r \sim s$ if $r \neq s$ are vertices connected by an edge. A (real or complex) *function* on G is

2020 Mathematics Subject Classification. Primary 05C50; Secondary 05C22, 81Q35, 47A56, 47A10, 57R70, 58K05, 49J52, 14M15, 58A35, 57Z05.
Keywords. Magnetic operators, spectral graph theory, nodal count, Morse theory.

a function on the vertices of G, that is, a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$ and we denote the value of such a function $v = (v_1, v_2, \dots, v_n)$ by $v(r)$ or v_r. An $n \times n$ matrix h is *supported on* G if $h_{rs} \neq 0 \implies r \sim s$ or $r = s$. Let $\mathcal{S}(G)$ and $\mathcal{H}(G)$ denote the vector spaces of real symmetric matrices and complex Hermitian matrices supported on G. A *discrete Schrödinger operator* is a real symmetric matrix $h \in \mathcal{S}(G)$ with $h_{rs} < 0$ for $r \sim s$. The quadratic form associated with $h \in \mathcal{S}(G)$ may be expressed as the quadratic form of $\Delta + V$, that is

$$\langle f, hf \rangle = -\sum_{r \sim s} h_{rs}(f(r) - f(s))^2 + \sum_{r=1}^{n} V(r) f(r)^2 \tag{1.1}$$

where the "potential" is $V(r) = h_{rr} + \sum_{r \sim s} h_{rs}$ and Δ is a *weighted Laplace operator* on G.

A discrete Schrödinger operator h has real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Suppose λ_k is a simple (multiplicity one) eigenvalue of h with a nowhere-vanishing eigenvector v (meaning that $v_r \neq 0$ for all r). A basic problem in graph theory is to understand the behavior of the *nodal count* $\phi(h, k)$, that is, the number of edges $r \sim s$ for which v changes sign: $v(r)v(s) < 0$. It is known that

$$k - 1 \leq \phi(h, k) \leq k - 1 + \beta, \tag{1.2}$$

where β is the first Betti number of G. (See [16] for a review of the many works leading to the upper bound, an analogue of Courant's theorem,[1] and [6] for the lower bound.) This motivates the definition of the *nodal surplus*

$$\phi(h, k) - (k - 1) \in \{0, 1, \dots, \beta\}$$

and its probability distribution $P(h) = (P(h)_0, \dots, P(h)_\beta)$ over the n possible eigenvalues:

$$P(h)_s = \frac{1}{n} \#\{1 \leq k \leq n : \phi(h, k) - (k - 1) = s\}.$$

In numerical simulations for large graphs, this distribution seems to concentrate around $\frac{\beta}{2}$ with variance of the order of β, similar to the observations for metric graphs in [2].

[1] The Courant theorem states, for a domain Ω in Euclidean space with homogeneous boundary conditions, that the nodal set of the k-th eigenfunction of the Laplacian divides Ω into no more than k subdomains, see [15, Chapter 6, Section 6].

1.2. Nodal count for signed graphs

If $h \in \mathcal{S}(G)$ is a discrete Schrödinger operator we may consider other *signings* $h' \in \mathcal{S}(G)$ obtained from h by changing the sign of some collection of off-diagonal entries. Every symmetric matrix $h' \in \mathcal{S}(G)$ is a signing of a uniquely determined Schrödinger operator h. We may consider h' to be an analogue of the discrete Schrödinger operator on the corresponding *signed graph* G' obtained from G by attaching signs to the edges, as originally introduced in [20] and extensively studied, see [11, 28, 35]. In this case, taking the signing into account, the nodal count is defined to be the number of edges $r \sim s$ such that $v(r)h'_{rs}v(s) > 0$.

Denote by $\mathcal{S}(h)$ the collection of all possible signings of h (cf. Section 2.6). The inequality (1.2) continues to hold for any signing of h. The *average nodal surplus distribution* $P(\mathcal{S}(h))$ is the average of $P(h')$ over all signings $h' \in \mathcal{S}(h)$. In Theorem 3.2 we show that if the diagonal entries of h are all equal, then $P(\mathcal{S}(h))$ is symmetric around $\beta/2$. Numerical experiments lead to the following conjecture.

Conjecture. Given a simple connected graph G, there is a generic set (open, dense and full measure) of $h \in \mathcal{S}(G)$ for which the average nodal surplus distribution $P(\mathcal{S}(h))$ is symmetric around $\beta/2$ with variance σ_h^2 of order β. Moreover, the normalized distribution

$$\rho_{G,h} := \sum_{j=0}^{\beta} P(\mathcal{S}(h))_j \delta_{x_j} \quad \text{with } x_j = \frac{j - \beta/2}{\sigma_h},$$

converges in the weak topology to the normal Gaussian distribution $N(0, 1)$ as $\beta \to \infty$, uniformly over all simple connected G with first Betti number β, and generic $h \in \mathcal{S}(G)$.

1.3. Gauge invariance

The *gauge group* $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ acts on the space $\mathcal{H}(G)$ where $(\theta_1, \theta_2, \dots, \theta_n)$ acts by conjugation with $\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$. This action preserves eigenvalues, nodal count, and most other graph properties that are studied in this paper. Elements $h, h' \in \mathcal{H}(G)$ that differ by a gauge transformation are said to be *gauge equivalent*. If $h \in \mathcal{S}(G)$ is a discrete Schrödinger operator, then the signings h' of h for which the corresponding signed graph G' is *balanced* (see [20]) are exactly those h' that are gauge equivalent to h.

1.4. Magnetic operators and nodal count

In [7, 9], G. Berkolaiko suggested that one might better understand the nodal count by considering its variation under *magnetic perturbations of* h. The discrete analogue for the Schrödinger operator associated to a particle in a magnetic field appears in [21,22]. See also [14, 26], [12, Section 2.1], and [13]. It is quickly reviewed in Appendix B.

Given a discrete Schrödinger operator $h \in \mathcal{S}(G)$, a magnetic potential α is a real anti-symmetric matrix supported on G and the associated *magnetic Schrödinger operator* $h_\alpha \in \mathcal{H}(G)$ is the Hermitian matrix $(h_\alpha)_{rs} = e^{i\alpha_{rs}} h_{rs}$. The manifold (2.7) of such magnetic perturbations, $\mathbb{T}_h \subset \mathcal{H}_n$, is a torus containing h, cf. Section 2.4 below. Its quotient, see equation (2.9), modulo gauge transformations, $\mathcal{M}_h$ is a torus of dimension β. In [13] and [7], G. Berkolaiko and Y. Colin de Verdière discovered a remarkable fact: for any real symmetric $h \in \mathcal{S}(G)$ with simple eigenvalue λ_k and nowhere vanishing eigenvector, the nodal surplus $\phi(h,k) - (k-1)$ is equal to the Morse index of λ_k, interpreted as a Morse function on the manifold $\mathcal{M}_h$.

1.5. Morse theory for magnetic perturbations modulo gauge transformations

We wish to apply Morse theory to the function $\lambda_k \colon \mathcal{H}_n \to \mathbb{R}$, restricted to the torus $\mathbb{T}_h$ or its quotient $\mathcal{M}_h$. In principle, Morse theory provides a prescription for building the homology of $\mathcal{M}_h$ from local data at the critical points of λ_k together with some homological information as to how these local data fit together. Since the homology of $\mathcal{M}_h$ is known, Morse theory should provide restrictions on the number and type of critical points of λ_k, and in turn, restrictions on the nodal surplus.

There are several difficulties with this plan, the first being that λ_k is continuous but not smooth: it is analytic on each stratum of a certain stratification of $\mathcal{H}_n$ (see Section 7) [25, 29]. If $\lambda_k(h)$ is simple, then λ_k is analytic near h and one may search for its critical points on $\mathbb{T}_h$. The torus $\mathbb{T}_h$ and its quotient $\mathcal{M}_h$ are preserved under complex conjugation, and the function λ_k is invariant under complex conjugation. The simplest critical points of λ_k are the *symmetry points* (Section 2.6): the points $h' \in \mathbb{T}_h$ (or $[h'] \in \mathcal{M}_h$) fixed by complex conjugation, i.e., the real symmetric matrices in $\mathbb{T}_h$.

The set of symmetry points of $\mathbb{T}_h$ is denoted $\mathcal{S}(h)$. If h is real symmetric, then $\mathcal{S}(h)$ consists precisely of the various signings of h. Following [5], we show the following result.

Theorem 3.2. *Each critical point $h' \in \mathbb{T}_h$ with simple eigenvalue $\lambda_k(h')$ and nowhere vanishing eigenvector is necessarily in the gauge equivalence class of a symmetry point. In other words, its image $[h'] \in \mathcal{M}_h$ is a symmetry point. Suppose that for each k ($1 \le k \le n$) each critical point $h_\alpha \in \mathbb{T}_h$ of λ_k has $\lambda_k(h_\alpha)$ as a simple eigenvalue with nowhere vanishing eigenvector. Then the average nodal count distribution is a binomial distribution*[2] *with mean $\beta/2$ and variance $\beta/4$. Consequently, if the aver-*

[2] See Section 3.3.

age nodal distribution is not binomial, then there must exist critical points (of some eigenvalue) that are not symmetry points.

We give a homological characterization of symmetry points.

Theorem 2.7. *Let $h \in \mathcal{S}(G)$ and $\alpha \in \mathcal{A}(G)$ which we may identify as a 1-form on G. Then h_α is gauge equivalent to a symmetry point if and only if $\int_\xi \alpha \equiv 0 \pmod{\pi}$ for all cycles ξ, i.e., chains $\xi \in C_1(G, \mathbb{Z})$ with $\partial\xi = 0$.*

1.6. Classification of critical gauge-equivalence classes

In general, the nodal surplus distribution $P(\mathcal{S}(h))$ depends on Morse data from all critical points of λ_k (for all k), whether or not they are symmetry points. Following Theorem 3.2, there are two possible types of non-symmetry critical points $[h'] \in \mathcal{M}_h$ of λ_k.

(1) *Exceptional critical points*, for which $\lambda_k(h')$ is simple but its eigenvector vanishes on one or more vertices. In this case $[h']$ is (usually) a degenerate critical point (see Theorem 4.4): it is contained in a larger critical submanifold.

(2) *Incorrigible critical points*, for which the multiplicity of $\lambda_k(h')$ is greater than one. In this case, λ_k fails to be smooth and one must replace the usual Morse theory with stratified Morse theory ([19]).

Concerning the first case, suppose the eigenvector v vanishes only at a single vertex v_0 of the graph G. Suppose that v_0 has degree $\deg(v_0)$.

Theorem 4.4. *Assuming the critical point $[h'] \in \mathcal{M}_h$ is sufficiently generic,*[3] *then it lies in a non-degenerate (Morse–Bott) critical submanifold of $\mathcal{M}_h$, of dimension* $\deg(v_0) - 3$*, which is diffeomorphic to the configuration space of a particular planar linkage. Its Morse index may be expressed in terms of spectral data.*

The configuration spaces of planar linkages are fascinating objects. They have been extensively studied and their homology is completely known, cf. [17, 24, 33].

For the second case, when the multiplicity of $\lambda_k(h')$ is greater than one, G. Berkolaiko and I. Zelenko [10] have determined the *normal Morse data* for λ_k, and its Betti numbers, which forms the central ingredient required for stratified Morse theory. However, in order to apply stratified Morse theory to the mapping $\lambda_k \colon \mathbb{T}_h \to \mathbb{R}$ it is required that the manifold $\mathbb{T}_h \subset \mathcal{H}_n$ should be Whitney stratified. Its stratification comes by intersecting with the natural stratification of $\mathcal{H}_n$ (cf. Section 7), but this requires that $\mathbb{T}_h$ should be transverse to the strata of the stratification of $\mathcal{H}_n$. The challenge is to guarantee transversality of the torus $\mathbb{T}_h$ by a generic choice of the single

[3] Specific conditions on h' are given in Theorem 4.4 in Section 4.3.

element $h \in \mathcal{S}(G)$. The transversality lemma in [10] does not address this situation. The first non-trivial case concerns the stratum $S_2(k)$ where λ_k has multiplicity 2. Suppose h_α is a critical point of λ_k, an eigenvalue of multiplicity 2. In Section 7.8 we define the notion of a splitting of the graph G by the eigenspace of λ_k. (A related condition was considered by L. Lovász in [27, Section 10.5.2].)

Theorem 7.9. *If the eigenspace of $\lambda_k(h_\alpha)$ does not split the graph G, then the space $\mathcal{H}(G)$ is transverse to $S_2(k)$ at given point $h_\alpha \in \mathbb{T}_h$.*

Corollary 7.10. *As above, if the eigenspace of $\lambda_k(h_\alpha)$ does not split G, then for generic choice $h' \in \mathcal{S}(G)$ the torus $\mathbb{T}_{h'}$ is transverse to the stratum $S_2(k)$ near h_α.*

2. Notation and definitions

2.1. Symmetric and Hermitian forms

Let $\mathcal{S}_n$ denote the vector space of $n \times n$ real symmetric matrices, $\mathcal{A}_n$ the space of $n \times n$ real antisymmetric matrices, and $\mathcal{H}_n$ the space of $n \times n$ Hermitian matrices, that is, matrices of linear operators on $\mathbb{C}^n$ expressed in the standard basis and that are self-adjoint with respect to the standard Hermitian form $\langle x, y\rangle = \sum \bar{x}_i y_i$.

If $V \subset \mathbb{C}^n$ is a complex subspace, then the standard Hermitian form restricts to a Hermitian form on V and we denote by $\mathcal{H}(V)$ the self-adjoint linear operators $V \to V$. If $\xi \in \mathcal{H}_n$, then it may fail to preserve V however its "restriction" to V may be defined by expressing $\xi = \left(\begin{smallmatrix} A & B \\ B* & D \end{smallmatrix}\right)$ with respect to the decomposition $\mathbb{C}^n = V \oplus V^\perp$. The restriction $\xi|V$ is defined to be the operator $A \in \mathcal{H}(V)$. Equivalently, $\xi|V$ is the operator corresponding to the restriction to $x, y \in V$ of the *sesquilinear* form $(x, y)_\xi = \langle x, \xi y\rangle$.

2.2. Laplace and Schrödinger operators

Throughout this section, we fix a graph $G = G([n], E)$. The natural ordering on the set of vertices $[n] := \{1, 2, \ldots, n\}$ determines an orientation for each edge. Write $r \sim s$ if $r \neq s$ and vertices r, s are joined by an edge. Write $r \simeq s$ if $r \sim s$ or $r = s$.

A (real or complex) *matrix supported on* G is an $n \times n$ matrix h such that

$$h_{rs} \neq 0 \implies r \simeq s.$$

Such a matrix is *properly supported* on G if, in addition,

$$r \sim s \implies h_{rs} \neq 0.$$

Symmetric, antisymmetric, and Hermitian matrices supported on G are denoted $\mathcal{S}(G)$, $\mathcal{A}(G)$, and $\mathcal{H}(G)$, respectively. Examples of matrices in $\mathcal{S}(G)$ include the *adjacency* *matrix* for G, (weighted) *Laplace operators* for G and discrete Schrödinger operators, see Section 1.1 above. More generally, any matrix $h \in \mathcal{H}(G)$ may be considered a *magnetic Schrödinger operator* for G (see Section 2.4 below and references [12,13]).

2.3. Graph homology

The space $C_0(G;\mathbb{Z}) \cong \mathbb{Z}^n$ of 0-chains is the vector space of formal linear combinations of vertices, $\sum_{r=1}^{n} c_r[r]$. Each edge rs with $r < s$ is orientated from r to s so that the group $C_1(G;\mathbb{Z})$ of 1-chains is the group of formal linear combinations

$$\xi = \sum_{\substack{r \sim s \\ r<s}} \xi_{rs}[rs], \quad \xi_{rs} \in \mathbb{Z}. \tag{2.1}$$

Then $H_1(G;\mathbb{Z}) = \ker(\partial)$, where $\partial\colon C_1(G;\mathbb{Z}) \to C_0(G;\mathbb{Z})$ with $\partial[rs] = [s] - [r]$. The first Betti number is

$$\beta = \operatorname{rank} H_1(G,\mathbb{Z}) = |E| - n + c,$$

where c is the number of connected components of G.

The vector space $\mathbb{R}^n$ may be viewed as the space of real-valued functions $\Omega^0(G)$ on the vertices of G. If $v = (v_1, v_2, \dots, v_n)$, we sometimes write $v_r = v(r)$. The vector space $\mathcal{A}(G)$ of real, antisymmetric matrices supported on G may be viewed as the space of 1-forms $\Omega^1(G)$ on G with coboundary differential

$$d\colon \Omega^0(G) = \mathbb{R}^n \to \Omega^1(G) = \mathcal{A}(G); \quad (df)_{rs} = \begin{cases} f(s) - f(r) & \text{if } r \sim s, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

There are no 2-forms on a graph so $H^1(G;\mathbb{R}) = \Omega^1(G)/d\Omega^0(G)$ is canonically dual to the homology $H_1(G;\mathbb{R})$ under the natural pairing that is determined by integration $\Omega^1(G) \times C_1(G;\mathbb{R}) \to \mathbb{R}$. If $\alpha \in \mathcal{A}(G) = \Omega^1(G)$ and $\xi \in C_1(G;\mathbb{R})$ as in (2.1), then

$$\int_{\xi} \alpha = \sum_{\substack{r \sim s \\ r<s}} \xi_{rs}\alpha_{rs}$$

2.4. Action of $\mathcal{A}_n$

The vector space $\mathcal{A}_n = \mathcal{A}_n(\mathbb{R})$ of $n \times n$ real antisymmetric matrices acts on the vector space $\mathcal{H}_n$ of $n \times n$ Hermitian matrices by

$$(\alpha * h)_{rs} = e^{i\alpha_{rs}} h_{rs}$$

for all $\alpha \in \mathcal{A}_n(\mathbb{R})$ and $h \in \mathcal{H}_n$ with $(x + y) * h = x * (y * h)$ and with $0 * h = h$. Then $\mathcal{A}(G)$ acts on $\mathcal{H}(G)$.

If h is the discrete Schrödinger operator, then $\alpha * h$ may be interpreted as the corresponding magnetic Schrödinger operator in the presence of a *magnetic field* described by α, whose flux through a cycle ξ is $\int_\xi \alpha$, with a sesquilinear form

$$\langle f, (\alpha * h) f\rangle = -\sum_{r\sim s} h_{rs}|f(s) - e^{i\alpha_{rs}} f(r)|^2 + \sum_{r=1}^{n} V(r)|f(r)|^2, \tag{2.3}$$

instead of the quadratic form of h in (1.1). If $h = (h_{rs}) \in \mathcal{H}_n$ define $|h| \in \mathcal{S}_n$ by $|h|_{rs} = |h_{rs}|$ for $r \neq s$ and $|h|_{rr} = h_{rr}$ (diagonal entries of $|h|$ can be negative). Then there exists $\alpha \in \mathcal{A}_n$ so that $h = \alpha * (|h|)$.

2.5. Gauge invariance

The $*$ action factors through the torus $\mathcal{A}_n(\mathbb{R})/\mathcal{A}_n(2\pi\mathbb{Z})$. So, the subtorus *supported on G*

$$\mathbb{T}(G) := \{\alpha \in \mathcal{A}_n(\mathbb{R})/\mathcal{A}_n(2\pi\mathbb{Z})\colon \alpha_{rs} \neq 0 \implies r \sim s\},$$

acts on $\mathcal{H}(G)$ by the $*$ action. The differential (2.2) also factors

$$\begin{array}{ccc} \Omega^0(G) = \mathbb{R}^n & \xrightarrow{d} & \Omega^1(G) = \mathcal{A}(G) \\ {\scriptstyle (\mathrm{mod}\ 2\pi)}\downarrow & & \downarrow{\scriptstyle (\mathrm{mod}\ 2\pi)} \\ \mathbb{T}^n & \xrightarrow{\quad d \quad} & \mathbb{T}(G) \end{array} \tag{2.4}$$

through the *gauge group* $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. *Gauge invariance* is the statement that the $*$ action by coboundaries is simply given by conjugation: for any $\theta \in \mathbb{T}^n$ and any $h \in \mathcal{H}_n$, direct calculation gives

$$d\theta * h = e^{i\theta} h e^{-i\theta} \tag{2.5}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{T}^n$ and $e^{i\theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$. The $*$ action by $d\theta$ preserves eigenvalues and preserves eigenvectors *up to phase*: if $E_\lambda(h) = \ker(h - \lambda I)$, then

$$E_\lambda(d\theta * h) = e^{i\theta} E_\lambda(h). \tag{2.6}$$

Elements $h, h' \in \mathcal{H}(G)$ that differ by a gauge transformation ($h' = d\theta * h$) are said to be *gauge equivalent*. Gauge equivalence determines an identification, cf. (2.10) of the *quotient torus* (the manifold of magnetic fields modulo gauge transformations) with cohomology

$$\mathbb{T}^{\mathcal{A}/d}(G) := \mathbb{T}(G)/d(\mathbb{T}^n) \cong H^1(G; \mathbb{R}/2\pi\mathbb{Z}).$$

2.6. The embedded torus and its symmetry points

Recall (2.2) that a matrix $h \in \mathcal{H}(G)$ is *properly supported* on G if $h_{rs} \neq 0$ whenever $r \sim s$. (Diagonal entries h_{rr} may vanish.) Such h defines a mapping $\mathbb{T}(G) \to \mathcal{H}_n$ by $\alpha \mapsto \alpha * h$, whose image is an embedding of $\mathbb{T}(G)$ into $\mathcal{H}(G)$,

$$\mathbb{T}_h := \mathbb{T}(G) * h = \{\alpha * h : \alpha \in \mathbb{T}(G)\} = \{\alpha * |h| : \alpha \in \mathbb{T}(G)\}. \tag{2.7}$$

We refer to $\mathbb{T}_h$ as *the embedded torus*. For $h \in \mathcal{H}(G)$ which is not properly supported on G, the dimension of the embedded torus $\mathbb{T}_h$ is the number of non-zero elements h_{rs} with $r < s$. The embedded torus is invariant under complex conjugation and we refer to the set of its fixed points (i.e. the real points)

$$\mathcal{S}(h) := \mathbb{T}_h \cap \mathcal{S}(G) = \{\alpha * |h| : \alpha \equiv 0 \ (\mathrm{mod}\ \pi)\}$$

as *symmetry points*. If $h \in \mathcal{S}_n$, then its symmetry points $\mathcal{S}(h)$ consist of symmetric matrices h' obtained from h by changing the signs in any subset of off-diagonal entries h_{rs} or equivalently

$$h' = \alpha * h \quad \text{where } \alpha \equiv 0 \ (\mathrm{mod}\ \pi).$$

The action of the *integral gauge group* $(\pi\mathbb{Z})^n \subset \mathbb{R}^n$ preserves the set of symmetry points and changes the signs of the components of the corresponding eigenvectors. The set $\mathcal{S}(h)$ decomposes into a union of orbits under the integral gauge group. If h is properly supported on G ($h_{rs} \neq 0$ whenever $r \sim s$), then $\mathcal{S}(h)$ has $2^{|E|}$ elements, partitioned into 2^β orbits (cf. Section 2.8). Each orbit corresponds to a choice of parity of the circulations around a choice of elementary cycles.

2.7 Theorem. *Suppose $h \in \mathcal{H}(G)$ is properly supported on G. Let $\alpha \in \mathcal{A}(G) = \Omega^1(G)$ so that $h = \alpha * |h|$. Then h is gauge-equivalent to a symmetry point $h' \in \mathcal{S}(h)$ if and only if*

$$\int_\xi \alpha \equiv 0 \ (\mathrm{mod}\ \pi) \tag{2.8}$$

for all cycles ξ, i.e. chains $\xi \in C_1(G, \mathbb{Z})$ with $\partial\xi = 0$.

Proof. Since h is properly supported, the element α is uniquely determined modulo $2\pi\mathbb{Z}$. If h is a symmetry point, then $\alpha \equiv 0 \ (\mathrm{mod}\ \pi)$ so (2.8) holds. If h changes by gauge-equivalence, the integral (2.8) is unchanged, by Stokes' theorem.

On the other hand, if (2.8) holds for all cycles, then by duality the cohomology class $[\alpha]$ vanishes in $H^1(G;\mathbb{R})/H^1(G;\pi\mathbb{Z})$, so it lies in $H^1(G;\pi\mathbb{Z}) \subset H^1(G;\mathbb{R})$ and it comes from a 1-form $\alpha' \in \Omega^1(\pi\mathbb{Z})$, that is, an antisymmetric matrix whose entries are multiples of π. Then the cohomology class $[\alpha' - \alpha] \in H^1(G;\mathbb{R})$ vanishes so there exists $\theta \in \Omega^0(G;\mathbb{R})$ with $\alpha' = \alpha + d\theta$. This proves that the symmetry point $\alpha' * |h|$ is gauge-equivalent to $h = \alpha * |h|$. ∎

2.8. Eigenvalues as Morse functions

Eigenvalues of elements $h \in \mathcal{H}_n$ are real and ordered, say

$$\lambda_1(h) \le \lambda_2(h) \le \cdots \le \lambda_n(h).$$

For each k $(1 \le k \le n)$ the mapping $\lambda_k\colon \mathcal{H}_n \to \mathbb{R}$ is well defined, continuous and piecewise real-analytic: there is a stratification of $\mathcal{H}_n$ by analytic subvarieties such that the restriction of λ_k to each stratum is analytic (cf. Section 7.1 and Lemma 7.3).

The restriction of each λ_k to the embedded torus $\mathbb{T}_h$ is invariant under gauge transformations, so it determines a function on the quotient,

$$\mathcal{M}_h = \mathbb{T}_h \mathbin{/\!\!/} \mathbb{T}^n, \tag{2.9}$$

where we use the notation $/\!\!/\mathbb{T}^n$ to denote dividing by gauge equivalence. The torus $\mathcal{M}_h$ has dimension β, and is referred to in [13] as the *manifold of magnetic perturbations modulo gauge transformations*:

$$\begin{tikzcd}
\mathbb{T}(G) \arrow[r, "*h"] \arrow[d, "d(\mathbb{T}^n)"'] & \mathbb{T}_h \arrow[r, hook] \arrow[d, "/\!\!/\mathbb{T}^n"'] & \mathcal{H}_n \arrow[r, "\lambda_k"] & \mathbb{R} \\
\mathbb{T}^{\mathcal{A}/d}(G)\mathcal{M}_h \arrow[r] & {} \arrow[urr, "\lambda_k"'] & &
\end{tikzcd} \tag{2.10}$$

If $\alpha \in \mathbb{T}(G)$, then the equivalence class of $\alpha * h$ in $\mathcal{M}_h$ is denoted $[\alpha * h]$ or $[h_\alpha]$.

If $\theta \in \mathbb{R}^n$, then $\overline{(d\theta) * h} = d(-\theta) * \bar{h}$ so complex conjugation passes to an involution on $\mathcal{M}_h$. *Every fixed point of this involution comes from a symmetry point in* $\mathbb{T}_h$: for if $h \in \mathcal{H}_n$ and $[h] \in \mathcal{M}_h$ is fixed, this means $\bar{h} = (d\theta) * h$ for some $\theta \in \mathbb{R}^n$, so $(d\frac{\theta}{2}) * h$ is a symmetry point. It is therefore reasonable to refer to these fixed points of $\mathcal{M}_h$ as *symmetry points of* $\mathcal{M}_h$.

2.9 Lemma. *Let G be a simple graph with c connected components and let $h \in \mathcal{H}_n(G)$, properly supported on G. Then, each symmetry point $h' \in \mathcal{S}(h)$ has exactly 2^{n-c} gauge-equivalent symmetry points. Thus, the number of symmetry points in $\mathcal{M}_h$ is $2^{|E|-(n-c)} = 2^\beta$.*

Proof. It is enough to consider the case of real symmetric $h \in \mathcal{S}_n$, in which case its gauge-equivalent symmetry points are

$$[h] \cap \mathcal{S}(h) = \{df * h\colon f(r) \in \{0, \pi\} \text{ for all } r\}.$$

There are 2^n choices for f among which 2^c are in the kernel of d (those which are constant on connected components of G). So, there are 2^{n-c} distinct values for df, and therefore 2^{n-c} distinct values of $df * h$ since h is properly supported on G.

Hence, $[h]$ contains exactly 2^{n-c} gauge-equivalent symmetry points. Repeating this argument for any other $h' \in S(h)$ leaves $2^{|E|-(n-c)} = 2^{\beta}$ equivalence classes of symmetry points in $\mathcal{M}_h$. ∎

2.10. Nodal surplus

Generalizing the notions described in the introduction, let h be a Hermitian matrix supported on G, suppose $\lambda_k(h)$ is a simple eigenvalue with nowhere vanishing eigenvector $v = (v_1, v_2, \ldots, v_n)$. Further, assume that $\bar{v}_r h_{rs} v_s \in \mathbb{R}$ for all $r \sim s$ (which is equivalent to h being a critical point of λ_k, see Theorem 3.2 part (3)). Define the *nodal count* $\phi(h,k)$ to be the number of edges $r \sim s$ such that

$$\bar{v}_r h_{rs} v_s > 0. \tag{2.11}$$

The *nodal surplus* is the number $\phi(h,k) - (k-1)$. This number does not change under gauge transformation and it is known (see Theorem 3.2 below) that the nodal surplus is between 0 and β, the first Betti number of G. The *nodal surplus distribution* $P(h) = (P(h)_0, P(h)_1, \ldots, P(h)_\beta)$ is the vector representing the probability distribution of these numbers over the n possible eigenvalues:

$$P(h)_s = \frac{1}{n}\#\{1 \le k \le n \colon \phi(h,k) - (k-1) = s\}.$$

Assuming that $h \in \mathcal{S}_n$ and all its signings $h' \in \mathcal{S}(h)$ have all eigenvalues simple with nowhere-vanishing eigenvectors, the distribution can be averaged over signings to give the *average nodal distribution*

$$P(\mathcal{S}(h)) = 2^{-|E|} \sum_{h' \in \mathcal{S}(h)} P(h').$$

3. Morse theory

3.1. Critical points

Throughout this section, we fix a graph G with vertices $1, \ldots, n$ and edges $r \sim s$. Let $h \in \mathcal{S}(G)$ be a real symmetric matrix properly supported on G, cf. Section 2.2. For $\alpha \in \mathcal{A}(G)$, denote by $h_\alpha = \alpha * h$ the magnetic perturbation of h. Fix k and write $\lambda_k(\alpha) = \lambda_k(h_\alpha)$ for the k-th eigenvalue. Let $\mathcal{M}_h$ be the manifold (2.9) of magnetic perturbations of h modulo gauge transformations. It is a torus of dimension β, the first Betti number of the graphs G. By equation (2.6), the eigenvalue $\lambda_k(\alpha)$ of an element $[h_\alpha] \in \mathcal{M}_h$, and its multiplicity are well defined; and whether or not an eigenvector vanishes at a given vertex is well defined.

We consider $\lambda_k\colon \mathcal{M}_h \to \mathbb{R}$ to be a sort of generalized Morse function. If λ_k is smooth at a point $x = [h_\alpha] \in \mathcal{M}_h$ (in which case it is also analytic), we say that x is a *smooth point* of λ_k. A *critical point* of λ_k is either a non-smooth point or a smooth point where $\nabla\lambda_k(x) = 0$. Consider the following possibilities:

(0) x may be a *smooth, regular* (i.e., not critical) point of λ_k;

(1) x may be a *symmetry* point of $\mathcal{M}_h$;

(2) x may be a *non-symmetry, smooth*, (possibly degenerate) *critical* point of λ_k;

(3) x may be a *non-smooth* point of λ_k.

3.2 Theorem. *Fix properly supported $h \in \mathcal{S}(G)$. Consider $\lambda_k\colon \mathcal{M}_h \to \mathbb{R}$ as above.*

(1) *Every symmetry point of $\mathcal{M}_h$ is a critical point of λ_k.*

(2) *If the only critical points of λ_k on $\mathcal{M}_h$ are the symmetry points and if they are non-degenerate, then the number of such critical points of index s is $\binom{\beta}{s}$.*

(3) *Suppose $h_\alpha \in \mathbb{T}_h$ has a simple eigenvalue $\lambda_k(h_\alpha)$ with eigenvector v. Then $(h_\alpha)_{rs}\bar{v}_r v_s$ is real for all $r \sim s$ if and only if h_α is a critical point of λ_k as a function on $\mathbb{T}_h$, in which case h_α is gauge equivalent to a matrix h' such that $h'_{rs} \notin \mathbb{R} \implies \bar{v}_r v_s = 0$.*

(4) *In particular, if $h_\alpha \in \mathbb{T}_h$ is a critical point of λ_k and $\lambda_k(h_\alpha)$ is* simple *with nowhere vanishing eigenvector, then $[h_\alpha] \in \mathcal{M}_h$ is a symmetry point.*
(Equivalently, there exists $\theta \in \mathbb{T}^n$ such that $h_{\alpha+d\theta} \in \mathcal{S}(h)$.)

(5) *A critical point $[h_\alpha] \in \mathcal{M}_h$ as in* (4) *is non-degenerate and its Morse index is the nodal surplus, $\phi(h_\alpha, k) - (k-1)$.*

(6) *If the diagonal entries of h are all equal, then the average nodal count distribution is symmetric,*

$$P(\mathcal{S}(h))_s = P(\mathcal{S}(h))_{\beta-s}, \quad s \in \{0, 1, \ldots, \beta\}.$$

(7) *Suppose that for each k ($1 \le k \le n$) each critical point $h_\alpha \in \mathbb{T}_h$ of λ_k has $\lambda_k(h_\alpha)$ as a simple eigenvalue with nowhere vanishing eigenvector. Then the average nodal count distribution is binomial:*

$$P(\mathcal{S}(h))_s = 2^{-\beta}\binom{\beta}{s}.$$

Parts (1) and (5) of Theorem 3.2 are due to Berkolaiko and Colin de Verdière[4] [7, 13]. Part (2) is an immediate consequence, also known to both of these authors.

[4]In both works [7, 13] the matrix h was assumed to be real symmetric, but essentially the same proof works in general.

Part (3) was already observed in [5, Theorem A.1 and Lemma A.2]. Part (4) is an immediate consequence known to the authors of [5]. It says that the only simple critical points of $\lambda_k|\mathbb{T}_h$ with non-vanishing eigenvector occur along the intersection of $\mathbb{T}_h$ with the conjugacy classes of symmetry points: the $2^{|E|}$ real elements $\mathcal{S}(h)$. Proofs for Theorems 3.2 and 4.4 below will appear in Section 5 and Section 6.

3.3. Example of matrices with binomial nodal count distribution

Let $h = h_0 + \eta V$ be a Schrödinger operator on the complete graph, with h_0 properly supported (i.e., $(h_0)_{rs} \neq 0$ for all $r \neq s$), $V = \mathrm{diag}(V_1, \ldots, V_n)$ with distinct entries, and $\eta \in \mathbb{R}$. If η is sufficiently large, then all matrices $\alpha * h \in \mathbb{T}_h$ will have simple eigenvalues and nowhere vanishing eigenvectors, so $P(S(h))$ is binomial.

To see that, set $\varepsilon = \frac{1}{\eta}$ and let $h_\varepsilon = \varepsilon h = V + \varepsilon h_0$. We treat $\alpha * h_\varepsilon = V + \varepsilon(\alpha * h_0)$ as a small perturbation of V whose distinct eigenvalues are V_j with eigenvectors e_j for $j = 1, \ldots, n$. The min-max principle gives $|\lambda_j(\alpha * h_\varepsilon) - V_j| \leq \max_{rs} |\varepsilon(\alpha * h_0)_{rs}| = \max_{rs} |\varepsilon(h_0)_{rs}|$, so there is a uniform constant $C > 0$ such that when $0 < \varepsilon < C$, the eigenvalues of $\alpha * h_\varepsilon$ are distinct, for every α. Suppose $0 < \varepsilon < C$ and let v_ε be the j-th eigenvector of $\alpha * h_\varepsilon$. Comparing v_ε to e_j, perturbation theory gives $v_\varepsilon(j) = 1 + O(\varepsilon^2) \neq 0$, and for $i \neq j$,

$$|v_\varepsilon(i)| = \varepsilon \left| \frac{(\alpha * h_0)_{ij}}{V_i - V_j} \right| + O(\varepsilon^2) \geq \varepsilon \min_{r<s} \left| \frac{(h_0)_{rs}}{V_r - V_s} \right| + O(\varepsilon^2) \neq 0,$$

for sufficiently small ε, uniformly in α.

4. Exceptional critical points and the linkage equation

In this section we consider the case where $[h_\alpha] \in \mathcal{M}_h$ is an exceptional critical point of λ_k (cf. Section 1.6). That is, $[h_\alpha]$ is a non-symmetry, smooth, critical point with $\lambda_k(h_\alpha)$ simple. According to Theorem 3.2, the eigenvector v corresponding to $\lambda_k(h_\alpha)$ vanishes somewhere. (By generic choice of h, we can guarantee that every eigenvector of h is nowhere vanishing, cf. [32], but we cannot guarantee the same holds for all $h_\alpha \in \mathbb{T}_h$.) We address the simple case of eigenvector v that vanishes at a single vertex. By possibly replacing h_α with a gauge equivalent $h_{\alpha+d\theta}$ and v with $e^{i\theta}v$, we may assume that v is real with non-negative entries. The setting for Theorem 4.4 is described next.

4.1. The setting

To simplify the notation we assume the graph G has $n + 1$ vertices labeled $0, 1, 2, \ldots, n$, with corresponding properly supported real symmetric matrix $h \in \mathcal{S}(G)$.

Suppose $h_\alpha = \alpha * h$ is a critical point of λ_k with a simple eigenvalue $\lambda := \lambda_k(h_\alpha)$ and a normalized eigenvector $v = (v_0, v_1, \dots, v_n) = (0, v')$ and $v_0 = 0$, $v_r > 0$ for $1 \le r \le n$. Writing h and h_α as block matrices in the $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$ decomposition gives

$$h = \begin{pmatrix} a & b \\ b^* & D \end{pmatrix} \quad \text{and} \quad h_\alpha = \begin{pmatrix} a & b_\alpha \\ b_\alpha^* & D_\alpha \end{pmatrix} \quad \text{with} \begin{pmatrix} a & b_\alpha \\ b_\alpha^* & D_\alpha \end{pmatrix}\begin{pmatrix} 0 \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda v' \end{pmatrix}. \tag{4.1}$$

Let E_0 be the edges connected to vertex 0. For convenience, write $r \in E_0$ if $0r \in E_0$. Let H be the induced subgraph of G on the vertices $r \ge 1$. Thus, H is obtained from G by removing vertex 0 and its edges E_0. Then, $a \in \mathbb{R}$, $b \in \mathbb{R}^{E_0}$, $b_\alpha \in \mathbb{C}^{E_0}$, $D \in \mathcal{S}(H)$, and $D_\alpha \in \mathcal{H}(H)$. In fact, since h_α is critical and v_r is real and non-zero for $r \ge 1$, then D_α is real by part (3) of Theorem 3.2. Hence, $D_\alpha \in \mathcal{S}(H)$ is a signing of D.

The vector b_α has the form $(b_\alpha)_r = e^{i\alpha_{0r}} b_r$ for $r \in E_0$. Let $M_r := |b_r v_r| > 0$ and $\theta_r \in \mathbb{R}/2\pi\mathbb{Z}$ be the polar coordinates of $(b_\alpha)_r v_r = M_r e^{i\theta_r}$ for every $r \in E_0$.

4.2. Configuration space of a planar linkage

Equation (4.1) implies that the following *planar linkage equation* ([17,24,33]) holds:

$$b_\alpha . v' = \sum_{r \in E_0} e^{i\theta_r} M_r = 0. \tag{4.2}$$

This equation (4.2) describes a collection of vectors $M_r e^{i\theta_r} \in \mathbb{C} = \mathbb{R}^2$ in the plane, placed end to tail, that starts and ends at the origin, that is, a planar linkage, depending on a collection of lengths $L = \{M_r\}_{r \in E_0}$. Let $S^1 \subset \mathbb{C}$ be the unit circle. The *configuration space* Θ_L (see [17]) of the planar linkage defined by (4.2) is the set of solutions modulo rotations, that is,

$$\Theta_L = \Big\{(e^{i\theta_r})_{r \in E_0} \colon \sum_{r \in E_0} e^{i\theta_r} M_r = 0\Big\}/S^1 \subset (S^1)^{E_0}/S^1,$$

where the unit circle acts diagonally on $(S^1)^{E_0}$ by multiplication. The planar linkage is said to be *generic* if for any $\varepsilon \in \{-1, 1\}^{E_0}$,

$$\sum_{r \in E_0} \varepsilon_r M_r \ne 0. \tag{4.3}$$

Let M_s be the maximal length, $M_s = \max(M_r)_{r \in E_0}$. If $M_s > \sum_{r \ne s} M_r$, then there are no solutions, $\Theta_L = \emptyset$. If the planar linkage is generic and $M_s < \sum_{r \ne s} M_r$, then Θ_L is a smooth manifold of dimension $|E_0| - 3$ ([17,24,33]) whose Betti numbers have been computed in [17,23]. Let M_t be the second largest length. If $M_s + M_t \le \frac{1}{2}\sum_r M_r$, then Θ_L is connected, otherwise it has two connected components, exchanged by complex conjugation, each diffeomorphic to the torus of dimension $|E_0| - 3$.

4.3. Exceptional points

In the notation of Section 4.1, suppose $h_\alpha = \alpha * h$ is an exceptional critical point of λ_k with real eigenvector $v = (0, v')$ and simple eigenvalue $\lambda = \lambda_k(h_\alpha)$. The complex conjugate point $\bar{h}_\alpha = (-\alpha) * h$ is also a critical point of λ_k, with the same eigenvalue λ and eigenvector v. Moreover, λ is also an eigenvalue of D_α, say $\lambda = \lambda_{k'}(D_\alpha)$ is its k'-th eigenvalue.

Let F be the connected component of the critical set in $\mathcal{M}_h$ of λ_k that contains h_α, union with the connected component of the critical set of λ_k that contains $\bar{h}_\alpha$, noting that these two sets may be the same.[5]

4.4 Theorem. *Assume the following.*

(1) *The eigenvalue $\lambda = \lambda_{k'}(D_\alpha)$ is simple.*

(2) *The collection $\{M_r = |b_r v_r|\}_{r \in E_0}$ is generic (4.3).*

(3) *For any $[h'] \in F$ the eigenvalue $\lambda = \lambda_k(h')$ is simple, and*

$$c(h') = \sum_{j \neq k} \frac{|\psi_j(0)|^2}{\lambda_k(h') - \lambda_j(h')} \neq 0, \tag{4.4}$$

where $(\psi_j)_{j=1}^{n+1}$ are a choice of orthonormal eigenvectors of h' corresponding to the ordered eigenvalues.

Then the critical set F coincides with the explicitly defined set

$$F' = \{[h'] \in \mathcal{M}_h \colon h'v = \lambda v \text{ and there exists } \alpha'_0 \in \mathbb{T}(E_0) \text{ such that } h' = \alpha'_0 * h_\alpha\}.$$

It is a non-degenerate (Morse–Bott) critical submanifold of dimension $|E_0| - 3$ which is diffeomorphic to the configuration space Θ_L. Moreover, the Morse index of this critical submanifold is equal to

$$\operatorname{ind}(F) = \phi(D_\alpha, k') - (k' - 1) + \begin{cases} 2 & \text{if } c(h') < 0, \\ 0 & \text{if } c(h') > 0. \end{cases}$$

4.5 Remarks. Recall that the pseudo-inverse B^+ of a Hermitian matrix B with kernel V has the same kernel V and acts as $B|_{V^\perp}^{-1}$ on $V^\perp$. If we define the resolvent $(h' - z)^{-1}$ at $z = \lambda_k(h')$ using the pseudo-inverse $A = (h' - \lambda_k(h'))^+$, then $c(h') = A_{0,0}$.

A related observation for periodic metric (quantum) graphs appears in [8, Section 3.4] , where certain graphs are constructed, so that the maximum of their first spectral band is obtained on a critical manifold which is a planar linkage configuration space.

[5] Thus, the set $F \subset \mathcal{M}_h$ has either one or two connected components.

5. Proof of Theorem 3.2

5.1. For part (1), suppose $h' \in \mathbb{T}_h$ is a symmetry point (of $\mathbb{T}_h$), namely $h' = \bar{h}' \in \mathcal{S}(h)$. If h' is not a smooth point of λ_k, then it is a critical point. Suppose h' is smooth, then the directional derivative of λ_k in the direction $\alpha \in \mathcal{A}(G)$ is $\frac{d}{dt}\lambda_k(t\alpha * h')|_{t=0} = 0$ because

$$\lambda_k(t\alpha * h') = \lambda_k(\overline{t\alpha * h'}) = \lambda_k(-t\alpha * h').$$

If $h'' \in [h']$, then it is conjugate to h'. Conjugation takes a neighborhood of h' in $\mathcal{H}(G)$ to a neighborhood of h'', preserving the eigenvalue λ_k, so it also preserves the derivative of λ_k.

Part (2) follows immediately from the Morse inequalities, $C_i(\mathcal{M}_h) \geq b_i(\mathcal{M}_h)$ where C_i denotes the number of critical points of index i and where b_i is the i-th Betti number of $\mathcal{M}_h$. There are 2^β critical points by Lemma 2.9, and the sum of the Betti numbers of $\mathcal{M}_h$, a β-dimensional torus, is also 2^β. So, $C_i = b_i = \binom{\beta}{i}$ for all i.

Assuming parts (4) and (5), the proof of part (7) is a simple computation. In part (7) we assume all the critical points of λ_k correspond to simple eigenvalues with nowhere-vanishing eigenvectors, which means that all critical points are non-degenerate and are symmetry points, by part (4). Part (5) says that in such cases the nodal surplus equals the Morse index. Therefore, the average nodal surplus is

$$\begin{aligned} P(\mathcal{S}(h))_s &= 2^{-|E|} \sum_{h' \in \mathcal{S}(h)} P(h')_s \\ &= \frac{2^{-|E|}}{n} \sum_{h' \in \mathcal{S}(h)} \#\{k \leq n \colon \text{index } (\lambda_k(h')) = s\} \\ &= \frac{2^{-|E|}}{n} \sum_{k=1}^{n} \#\{h' \in \mathcal{S}(h) \colon \text{index } (\lambda_k(h')) = s\}. \end{aligned}$$

Using Lemma 2.9, the number inside the parenthesis can be expressed on the quotient $\mathcal{M}_h$

$$P(\mathcal{S}(h))_s = \frac{2^{-\beta}}{n} \sum_{k=1}^{n} \#\{[h'] \in [\mathcal{S}(h)] \colon \text{index } (\lambda_k([h'])) = s\} = 2^{-\beta}\binom{\beta}{s}$$

because, by part (2), the number in the parentheses is independent of k.

For part (6), by subtracting a multiple of the identity we may assume the diagonal entries of h are all zero. Let $\alpha_\pi \in \mathcal{A}(G)$ be properly supported on G, with $\pm\pi$ on the non-zero entries. For any $h_\alpha = \alpha * h \in \mathbb{T}_h$, the element $-h_\alpha = (\alpha + \alpha_\pi) * h \in \mathbb{T}_h$ is also in the same torus but the order of the eigenvalues is reversed, $\lambda_k(h_\alpha) = \lambda_{n-k}(-h_\alpha)$. This results in an inversion that sends every critical point of λ_k with index s, to a critical point of λ_{n-k} with index $\beta - s$. When averaged it gives the needed symmetry around $\beta/2$.

5.2. In this paragraph we prove parts (3) and (4) of Theorem 3.2. Let $\tilde{h} \in \mathbb{T}_h$ and suppose that $\lambda_k(\tilde{h})$ has multiplicity one. λ_k is analytic in a $\mathbb{T}_h$ neighborhood of $\tilde{h}$, and we ask when is it a critical point. To ease notation, for this paragraph only, we replace $\tilde{h}$ by h so that h is now Hermitian rather than real symmetric. Fix a direction $\alpha \in T_0(\mathbb{T}(G)) = \mathcal{A}(G)$ and consider the one-parameter perturbation of h in that direction $h_t = (t\alpha_0) * h$ for small $t \in (-\varepsilon, \varepsilon)$ so that $\dot{h}_{rs} = i\alpha_{rs}h_{rs}$.

Since λ_k is simple, we get analytic functions $v(t) \in \mathbb{C}^n$ and $\lambda_k(t) \in \mathbb{C}$, such that for all $t \in (-\varepsilon, \varepsilon)$, the vector $v(t)$ is normalized and satisfies $h_t v(t) = \lambda_k(t)v(t)$. Using the Leibniz "dot" notation for derivative with respect to t at $t = 0$ we have

$$\dot{h}v + h\dot{v} = \dot{\lambda}v + \lambda\dot{v}. \tag{5.1}$$

Taking the inner product with $v = v(0)$, using that h is self-adjoint, gives

$$\langle \nabla\lambda(h), \alpha \rangle := \dot{\lambda} = \langle v, \dot{h}v \rangle = \sum_{r \sim s} i\alpha_{rs}(h_{rs}\bar{v}_r v_s - \bar{h}_{rs}v_r\bar{v}_s),$$

so $(\nabla\lambda(h))_{rs} = i(h_{rs}\bar{v}_r v_s - \bar{h}_{rs}v_r\bar{v}_s) = 2\Im(\bar{h}_{rs}v_r\bar{v}_s)$ for all $r \sim s$. Therefore, h is a critical point if and only if $h_{rs}\bar{v}_r v_s \in \mathbb{R}$.

Assume $\nabla\lambda(h) = 0$. If $v_r \neq 0$, set $v_r = R_r e^{i\theta_r}$ with $R_r > 0$; otherwise set $\theta_r = 0$. Then $h_{rs}R_r R_s e^{i(\theta_s - \theta_r)}$ is real for all r, s. Set $e^{i\theta} = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$. Then $h' = e^{-i\theta}he^{i\theta}$ is gauge equivalent to h, and $h'_{rs} = h_{rs}e^{i(\theta_s - \theta_r)}$ is real whenever $\bar{v}_r v_s \neq 0$. In particular, if v is nowhere vanishing then h' is a symmetry point. If $h'_{rs} \notin \mathbb{R}$, then either $v_r = 0$ or $v_s = 0$. This completes the proof.

5.3. In the following paragraphs we prove part (5) of Theorem 3.2. The result was proven by Berkolaiko [7] and Colin de Verdière [13] for real symmetric h with non-positive off-diagonal entries. Both proofs extend to any real symmetric h, as the authors noted, if one defines the nodal count as in equation (2.11). We reorganize the proof of [13] and present it here for completeness and for later use. Now, let $h_\alpha \in \mathbb{T}_h \subset \mathcal{H}(G)$ be an element whose equivalence class is a symmetry point, i.e., h_α is gauge equivalent to a real symmetric matrix. For convenience, we change the notation slightly, using h instead of h_α, so suppose $h \in \mathcal{H}(G)$ is Hermitian properly supported on G, which is a critical point of λ_k, with a simple eigenvalue $\lambda := \lambda_k(h)$ and a nowhere-vanishing eigenvector v. By Theorem 3.2,

$$h_{rs}\bar{v}_r v_s \in \mathbb{R} \quad \text{for all } r \sim s. \tag{5.2}$$

Let $\operatorname{ind}(Q)$ denote the number of negative eigenvalues of a quadratic form Q and use $\operatorname{Hess}(F)$ for the Hessian of a function $F\colon \mathbb{T}(G) \to \mathbb{R}$, evaluated at $\alpha = 0$. It is a quadratic form on the tangent space $T_0\mathbb{T}(G) = \mathcal{A}(G)$.

Define $\mu\colon \mathbb{T}(G) \to \mathbb{R}$ by $\mu(\alpha) = \lambda_k(\alpha * h)$. Since $\mu(\alpha + d\theta) = \mu(\alpha)$ for all $\theta \in \mathbb{R}^n$, it follows that the Morse index of λ_k at the point $h \in \mathbb{T}(G)$ is

$$\operatorname{ind}(\lambda_k)(h) = \operatorname{ind}(\operatorname{Hess}(\mu)) = \operatorname{ind}(\operatorname{Hess}(\mu)|V)$$

for *any* complement $V \oplus d\mathbb{R}^n = \mathcal{A}(G)$. The trick ([13]) is to define $F\colon \mathbb{T}(G) \to \mathbb{R}$ by

$$F(\alpha) = \langle v, (\alpha * h - \lambda)v \rangle = \sum_{r\sim s} \bar{v}_r e^{i\alpha_{rs}} h_{rs} v_s + \sum_r |v_r|^2 h_{rr} - \lambda$$

where $\lambda = \lambda_k(h)$ *and* v *are constant*, and show that

(1) $\alpha = 0$ is a non-degenerate critical point of F with $\operatorname{ind}(\operatorname{Hess}(F)) = \phi(h,k)$;

(2) $\operatorname{ind}(\operatorname{Hess}(F)|d\mathbb{R}^n) = k - 1$;

(3) $\operatorname{ind}(\operatorname{Hess}(\mu)|V) = \operatorname{ind}(\operatorname{Hess}(F)|V)$ where V is now chosen to be the orthogonal complement of $d\mathbb{R}^n$ with respect to $\operatorname{Hess}(F)$:

$$\alpha \in V \iff \langle \alpha, \operatorname{Hess}(F)(d\theta) \rangle = 0 \quad \text{for all } \theta \in \mathbb{R}^n.$$

These three steps complete the proof of Theorem 3.2 (5) because they give

$$\begin{aligned}\operatorname{ind}(\lambda_k)(h) = \operatorname{ind}(\operatorname{Hess}(F)|V) &= \operatorname{ind}(\operatorname{Hess}(F)) - \operatorname{ind}(\operatorname{Hess}(F)|d\mathbb{R}^n)\\ &= \phi(h,k) - (k-1).\end{aligned}$$

5.4. Step 0

To compute $\operatorname{Hess}(\mu)$, namely the Hessian of $\lambda_k(h_\alpha)$ at $\alpha = 0$, let $\gamma, \delta \in \mathcal{A}_n(G)$, set $h(s,t) = (s\gamma + t\delta) * h$ and set $\mu(s,t) = \lambda_k(h(s,t))$ with corresponding normalized eigenvector $v(s,t)$. Using dot and prime to denote derivatives in s, t at $s = 0, t = 0$ respectively, we claim that

$$\langle \gamma, \operatorname{Hess}(\mu)\delta \rangle = 2\Re(\langle v', \dot{h} v \rangle) + \langle \gamma, \operatorname{Hess}(F)\delta \rangle. \tag{5.3}$$

Differentiating $\mu(s,t) - \lambda I = \langle v(s,t), (h(s,t) - \lambda)v(s,t) \rangle$ gives

$$\begin{aligned}\dot{\mu}' &= \langle \dot{v}', (h - \lambda I)v \rangle + \langle v', \dot{h} v \rangle + \langle v', (h - \lambda I)\dot{v} \rangle\\ &\quad + \langle \dot{v}, h' v \rangle + \langle v, \dot{h}' v \rangle + \langle v, h' \dot{v} \rangle\\ &\quad + \langle \dot{v}, (h - \lambda I)v' \rangle + \langle v, \dot{h} v' \rangle + \langle v, (h - \lambda I)\dot{v}' \rangle\\ &= 2\Re[\langle v', \dot{h} v \rangle + \langle \dot{v}, (h - \lambda I)v' \rangle + \langle \dot{v}, h' v \rangle] + \langle v, \dot{h}' v \rangle,\end{aligned}$$

where $\langle \dot{v}', (h - \lambda I)v \rangle$ and $\langle v, (h - \lambda I)\dot{v}' \rangle$ vanish because $v \in \ker(h - \lambda I)$. Furthermore, the criticality condition $\mu' = 0$ applied to (5.1) gives

$$(h - \lambda I)v' = -h' v, \tag{5.4}$$

so the term $\langle \dot{v}, (h - \lambda I)v' \rangle + \langle \dot{v}, h'v \rangle$ vanishes and we are left with

$$\dot{\mu}' = 2\Re(\langle v', \dot{h} v \rangle) + \langle v, \dot{h}' v \rangle.$$

We are done, since $\dot{\mu}' = \langle \gamma, \mathrm{Hess}(\mu)\delta \rangle$ and $\langle v, \dot{h}'v \rangle = \langle \gamma, \mathrm{Hess}(F)\delta \rangle$.

5.5. Step 1

Calculate

$$\frac{\partial F}{\partial \alpha_{rs}} = i(\bar{v}_r e^{i\alpha_{rs}} h_{rs} v_s - \bar{v}_s e^{-i\alpha_{rs}} \bar{h}_{rs} v_r),$$

which vanishes at $\alpha = 0$ by (5.2), and

$$\frac{\partial^2 F}{\partial^2 \alpha_{rs}}(0) = -(\bar{v}_r h_{rs} v_s + \bar{v}_s \bar{h}_{rs} v_r) = -2\bar{v}_r h_{rs} v_s,$$

which is real and non-zero for $r \sim s$, and all other second derivatives vanish. Therefore, $\mathrm{Hess}(F)$ is non-degenerate with index

$$\mathrm{ind}(\mathrm{Hess}(F)) = \#\{r \sim s, r < s \colon \bar{v}_r h_{rs} v_s > 0\} = \phi(h, k).$$

5.6. Step 2

For (small) $t \in \mathbb{R}$ and $\theta \in \mathbb{R}^n$, we will find the second derivative of

$$F(d(t\theta)) = \langle v, (d(t\theta) * h - \lambda)v \rangle = \langle e^{-it\Theta} v, (h - \lambda) e^{-it\Theta} v \rangle$$

by equation (2.5), where $\Theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_n)$ so $e^{i\Theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ (which was formerly denoted $e^{i\theta}$). Then

$$\frac{d}{dt} F(d(t\theta)) = i\langle \Theta e^{-it\Theta} v, (h - \lambda) e^{-it\Theta} v \rangle - i\langle e^{-it\Theta} v, (h - \lambda)\Theta e^{-it\Theta} v \rangle,$$

$$\frac{d^2}{dt^2} F(d(t\theta))\Big|_{t=0} = -\langle \Theta^2 v, (h - \lambda)v \rangle - \langle v, (h - \lambda)\Theta^2 v \rangle + 2\langle \Theta v, (h - \lambda)\Theta v \rangle.$$

The first two terms vanish. Using $M_v = \mathrm{diag}(v_1, v_2, \ldots, v_n)$, we get

$$\langle d\theta, \mathrm{Hess}(F)(d\theta) \rangle = 2\langle \Theta v, (h - \lambda)\Theta v \rangle = 2\langle \theta, M_v^*(h - \lambda) M_v \theta \rangle,$$

for all $\theta \in \mathbb{R}^n$. According to (5.2), $2M_v^*(h - \lambda)M_v$ is real and is therefore equal to $\mathrm{Hess}(F)|d\mathbb{R}^n$ as these are real symmetric matrices with equal quadratic forms. The matrix M_v is invertible since v is nowhere-vanishing. Two conclusions follow.

(1) Assume $\alpha = d\theta \in V \cap d\mathbb{R}^n$. Then $M_v\theta \in \ker(h - \lambda)$ and so $M_v\theta \propto v$ since λ is simple. Then θ is constant, so $d\theta = 0$. We conclude that

$$V \oplus d\mathbb{R}^n = \mathcal{A}(G).$$

(2) $\mathrm{Hess}(F)|d\mathbb{R}^n$ and $h-\lambda$ have the same number of negative eigenvalues, so

$$\mathrm{ind}(\mathrm{Hess}(F)|d\mathbb{R}^n) = k-1.$$

5.7. Step 3

For any $\theta \in \mathbb{R}^n$, the derivative of v in direction $d\theta$ is $v' = i\Theta v$ and $\mathrm{Hess}(\mu)d\theta = 0$ due to gauge invariance. Let $\partial_\alpha h$ stand for the derivative of h in direction $\alpha \in \mathcal{A}(G)$, so that for any $\delta = d\theta$ equation (5.3) gives

$$\langle \alpha, \mathrm{Hess}(F)d\theta\rangle = -2\Re(\langle v', \partial_\alpha h v\rangle) = 2\Im(\langle \Theta v, \partial_\alpha h v\rangle).$$

It follows from (5.2) that $\langle \Theta v, \partial_\alpha h v\rangle$ is purely imaginary, so $\alpha \in V$ if and only if $\langle \Theta v, \partial_\alpha h v\rangle$ vanish for all real diagonal Θ. Since v is nowhere-vanishing,

$$\alpha \in V \iff \partial_\alpha h v = 0.$$

Consequently, equation (5.3) shows that $\mathrm{Hess}(\mu)$ and $\mathrm{Hess}(F)$ agree on V. ■

6. Proof of Theorem 4.4

6.1. The critical set F'

Recalling the notations of Section 4.1, the graph G has $n+1$ vertices labeled $0, 1, 2, \ldots, n$. The set E_0 is the set of edges connected to 0. The graph H is the induced graph on the non-zero vertices. The torus of perturbations and its tangent space decompose as

$$\mathcal{A}(G) = \mathcal{A}(E_0) \oplus \mathcal{A}(H), \quad \mathbb{T}(G) = \mathbb{T}(E_0) \oplus \mathbb{T}(H),$$

and $h_\alpha = \alpha * h = \left(\begin{smallmatrix} a & b_\alpha \\ b_\alpha^* & D_\alpha \end{smallmatrix}\right)$ is an exceptional critical point of λ_k with simple eigenvalue $\lambda = \lambda_k(h_\alpha)$ and real eigenvector $v = (0, v')$ with $v_0 = 0$ and $v_r > 0$ for $r > 0$. As discussed in Section 4.1, D_α must be real, so it is a signing of D. By replacing $h \in \mathcal{S}(G)$ with a signing of h (if necessary), we may assume $D_\alpha = D$ and $\alpha \in \mathcal{A}(E_0)$, so $h_\alpha v = \lambda v$ becomes $Dv' = \lambda v'$ and

$$b_\alpha . v' = \sum_{r \in E_0} b_r v_r e^{i\alpha_{0r}} = 0.$$

Recall that F denotes the union of the connected components of the critical set of λ_k in $\mathcal{M}_h$ that contain $[h_\alpha]$ and $[\bar{h}_\alpha]$. In Section 6.2 and Section 6.8, we will prove that $F = F'$ where

$$F' := \{[h'] \in \mathcal{M}_h : h'v = \lambda v \text{ and there exists } \gamma \in \mathcal{A}(E_0) \text{ such that } h' = \gamma * h_\alpha\}.$$

Observe that *the set F' is closed under complex conjugation* because λ and v are real, and $\overline{\gamma * h_\alpha} = (-\gamma - 2\alpha) * h_\alpha$ for any $\gamma \in \mathcal{A}(E_0)$. Moreover, *the set F' consists of critical points of λ_k*: since h_α is critical and λ_k is simple, Theorem 3.2 (3) implies $v_s(h_\alpha)_{rs}v_r$ is real for every $r \sim s$. Since $\lambda = \lambda_k(h')$ is simple for any $h' = \gamma * h_\alpha \in F'$, then $v_s h'_{rs} v_r = v_s(h_\alpha)_{rs}v_r$ is real for every $r \sim s$ (since $v_0 = 0$) hence $[h']$ is also a critical point.

6.2. The diffeomorphism between F' and Θ_L

Define $\Phi\colon \mathbb{T}(E_0) * h_\alpha \xrightarrow{\cong} (S^1)^{E_0}$ by

$$\Phi(\gamma * h_\alpha)_r = \begin{cases} e^{i\gamma_{0r}} e^{i\alpha_{0r}} & \text{if } b_r > 0, \\ -e^{i\gamma_{0r}} e^{i\alpha_{0r}} & \text{if } b_r < 0, \end{cases}$$

for any $\gamma \in \mathcal{A}(E_0)$. An element $h' \in \mathbb{T}(E_0) * h_\alpha$ satisfies $h'v = \lambda v$ (with the same λ and $v = (0, v')$) if and only if $\Phi(h') = (e^{i\theta_r})_{r\in E_0}$ is a solution to the planar linkage equation $\sum_{r\in E_0} e^{i\theta_r} M_r = 0$ with $M_r := |b_r v_r|$. The normalizer of $\mathbb{T}(E_0) * h_\alpha$ in the gauge group is the zeroth coordinate $\mathbb{T}^{(0)} := \{(x, 0, 0, \dots, 0) \in \mathbb{T}^{n+1} \colon x \in S^1\}$, cf. equation (2.4). Therefore, the diffeomorphism Φ passes to the quotient,

$$\Phi\colon (\mathbb{T}(E_0) * h_\alpha) /\!\!/ \mathbb{T}^{(0)} \xrightarrow{\cong} (S^1)^{E_0}/S^1$$

with $\Phi(F') = \Theta_L$. By [17, p. 78], the set F' is a smooth manifold, closed under complex conjugation, and either it is connected or it has two connected components that are exchanged by complex conjugation. Moreover, it consists of critical points, so $F' \subset F$. (The reverse inclusion is proven in Section 6.8.)

6.3. Gauge transformations on G, H and E_0

Decompose the space of functions on G, $\mathbb{R}^G = \mathbb{R}^{n+1}$, into $\mathbb{R}^{n+1} = \mathbb{R}^{(0)} \oplus \mathbb{R}^H$, where $\mathbb{R}^H := \{(0, x) \in \mathbb{R}^{n+1} \colon x \in \mathbb{R}^n\} \cong \mathbb{R}^n$. The coboundary differential on H is denoted

$$d_H \colon \mathbb{R}^H \to \mathcal{A}(H) \subset \mathcal{A}(G); \quad (d_H f)_{rs} = \begin{cases} f(s) - f(r) & \text{if } r \sim s \text{ and } r, s \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The image $d_H \mathbb{R}^H$ is the projection of $d\mathbb{R}^H$ into $\mathcal{A}(H)$ and in fact, for any $f \in \mathbb{R}^H$,

$$df = d_H f + \sum_{r\in E_0} f(r) J(r, 0) \tag{6.1}$$

where $J(r,0) \in \mathcal{A}(E_0)$ is the antisymmetric matrix with $J(r,0)_{r0} = 1$, $J(r,0)_{0r} = -1$ and all other entries are 0. Let $\mathbf{1}_H = (0,1,1,\ldots,1) \in \mathbb{R}^H$ denote the constant vector on H and let $\mathbf{1}_{E_0} = \sum_{r\in E_0} J(r,0) \in \mathcal{A}(E_0)$. It is easy to verify that

$$\operatorname{span}_{\mathbb{R}}(\mathbf{1}_{E_0}) = d\mathbb{R}^{(0)} = d(\operatorname{span}_{\mathbb{R}}(\mathbf{1}_H)) = (d\mathbb{R}^H) \cap \mathcal{A}(E_0) = (d\mathbb{R}^{n+1}) \cap \mathcal{A}(E_0). \tag{6.2}$$

Since D is properly supported on H and $\lambda = \lambda'_k(D)$ is a simple eigenvalue of D with a nowhere-vanishing eigenvector v', then H is connected. Theorem 3.2 gives a decomposition

$$\mathcal{A}(H) = V_H \oplus d_H\mathbb{R}^H \tag{6.3}$$

and V_H can be described in terms of directional derivatives, according to Section 5.6,

$$V_H = \{\alpha_H \in \mathcal{A}(H)\colon (\partial_{\alpha_H} h_\alpha)v = 0\}, \quad \partial_{\alpha_H} h_\alpha := \frac{d}{dt}(t\alpha_H * h_\alpha)\Big|_{t=0}. \tag{6.4}$$

Let $\mathcal{A}_0(E_0)$ denote the orthogonal complement to $\operatorname{span}_{\mathbb{R}}(\mathbf{1}_{E_0})$,

$$\mathcal{A}_0(E_0) := \Big\{\gamma \in \mathcal{A}(E_0)\colon \sum_{r\in E_0} \gamma_{0r} = 0\Big\}.$$

Let

$$\pi\colon \mathcal{A}(G) \to \mathcal{A}(G)/d\mathbb{R}^{n+1} \cong H^1(G;\mathbb{R})$$

denote the quotient by

$$d\Omega^0(G) = d\mathbb{R}^{n+1},$$

cf. equation (2.2).

6.4 Lemma. *The space $\mathcal{A}(G)$ decomposes as a direct sum*

$$\mathcal{A}(G) = \mathcal{A}_0(E_0) \oplus V_H \oplus d\mathbb{R}^{n+1}. \tag{6.5}$$

In particular, $\mathcal{A}_0(E_0) \oplus V_H \xrightarrow{\cong} \pi(\mathcal{A}(G))$ and $\mathcal{A}_0(E_0) \xrightarrow{\cong} \pi(\mathcal{A}(E_0))$.

Proof. It follows from (6.1) that $d_H\mathbb{R}^n \subset d\mathbb{R}^{n+1} + \mathcal{A}(E_0)$. Using (6.2) and (6.3),

$$\begin{aligned}\mathcal{A}_0(E_0) + V_H + d\mathbb{R}^{n+1} &= \mathcal{A}(E_0) + V_H + d\mathbb{R}^{n+1}\\ &\supset \mathcal{A}(E_0) + (V_H + d_H\mathbb{R}^n) = \mathcal{A}(G)\end{aligned}$$

so $\mathcal{A}(G)$ is spanned by the sum on the left side. On the other hand, the sum on the left-hand side is a direct sum because the sum of the dimensions of the vector spaces is

$$\begin{aligned}(|E_0| - 1) + (\beta_H) + n &= (|E_0| - 1) + (|E_H| - n + 1) + n\\ &= |E_G| = \dim(\mathcal{A}(G)).\end{aligned}$$

∎

6.5. The tangent space to F'

Consider the preimage of F' in $\mathbb{T}(E_0) * h_\alpha$,

$$\begin{aligned}\widehat{F}' &:= \{\gamma * h_\alpha : \gamma \in \mathbb{T}(E_0),\ [\gamma * h_\alpha] \in F'\} \\ &= \Big\{\gamma * h_\alpha : \gamma \in \mathbb{T}(E_0) \text{ and } \sum_{r \in E_0} e^{i\gamma_{0r}} (b_\alpha)_r v_r = 0\Big\}.\end{aligned}$$

Differentiate and use the identification $T_{h_\alpha}\mathbb{T}_h = \mathcal{A}(G)$ to obtain the tangent space

$$T_{h_\alpha}\widehat{F}' \cong \Big\{\gamma \in \mathcal{A}(E_0) : \sum_{r \in E_0} \gamma_{0r} (b_\alpha)_r v_r = 0\Big\} \subset \mathcal{A}(E_0).$$

By Lemma 6.4, the quotient projection π takes $\mathcal{L} := T_{h_\alpha}\widehat{F}' \cap \mathcal{A}_0(E_0)$ isomorphically to $T_{[h_\alpha]}F'$, that is,

$$\mathcal{L} = \Big\{\gamma \in \mathcal{A}_0(E_0) : \sum_{r \in E_0} \gamma_{0r} (b_\alpha)_r v_r = 0\Big\} \cong \pi(\mathcal{L}) = T_{[h_\alpha]}F'. \tag{6.6}$$

Let $\mathbb{R}_0^{E_0}$ be the space of mean zero elements of $\mathbb{R}^{E_0}$, which we identify with $\mathcal{A}_0(E_0)$. Let $\mathbf{x} \in \mathbb{R}_0^{E_0}$ and $\mathbf{y} \in \mathbb{R}_0^{E_0}$ such that $(b_\alpha)_r v_r = \mathbf{x}_r + i\mathbf{y}_r$ for all $r \in E_0$. Then

$$\mathcal{L} = \{\gamma \in \mathcal{A}_0(E_0) : \gamma \cdot \mathbf{x} = 0 \text{ and } \gamma \cdot \mathbf{y} = 0\}.$$

6.6. The Hessian of λ

Since $d\mathbb{R}^{n+1}$ acts by gauge transformations, the quadratic form $\operatorname{Hess}(\lambda_k)$ at the critical point h_α, expressed with respect to the decomposition (6.5) has the following form:

$$\operatorname{Hess}\lambda = \begin{pmatrix} A & C & 0 \\ C^* & B & 0 \\ 0 & 0 & 0. \end{pmatrix}.$$

We will show that $C = 0$ and $\det(B) \neq 0$ with $\operatorname{ind}(B) = \Phi(D, k') - (k' - 1)$.

Using the notation ∂_γ for the directional derivative in direction γ and $\partial^2_{\delta,\gamma}$ for the second derivative in direction γ and then in direction δ, equation (5.3) states that

$$\langle \gamma, \operatorname{Hess}\lambda\,\delta \rangle = 2\Re[\langle \partial_\gamma v, (\partial_\delta h_\alpha) v \rangle] + \langle v, (\partial^2_{\gamma,\delta} h_\alpha) v \rangle.$$

If γ is supported on E_0 and δ is supported on H, then $\partial^2_{\gamma,\delta} h_\alpha = 0$. If $\delta \in V_H$, then $(\partial_\delta h_\alpha) v = 0$ according to (6.4). We conclude that $\langle \gamma, \operatorname{Hess}\lambda\,\delta \rangle = 0$ when $\delta \in V_h$ and $\gamma \in \mathcal{A}_0(E_0)$. Namely, $C = 0$.

Now, consider the block $B = \operatorname{Hess}\lambda|_{V_H}$. Let $\operatorname{Hess} F$ be the Hessian of the function $F(\delta) := \langle v', (\delta * D) v' \rangle$ for $\delta \in \mathbb{T}(H)$ evaluated at $\delta = 0$. Since D is real, then it is a critical point of $\lambda_{k'}$. Since $\lambda = \lambda_{k'}(D)$ is simple with a nowhere-vanishing eigenvector v', Theorem 3.2 implies the restriction $\mathrm{Hess}F|_{V_H}$ is non-degenerate and has $\Phi(D,k') - (k'-1)$ negative eigenvalues. Suppose $\gamma, \delta \in V_H$, then $(\partial_\delta h_\alpha)v = 0$ and $(\partial_\gamma h_\alpha)v = 0$ due to (6.4), and equation (5.3) gives

$$\langle \gamma, B\delta\rangle = \langle v, (\partial^2_{\gamma,\delta} h_\alpha)v\rangle = \langle \gamma, \mathrm{Hess}F\delta\rangle.$$

Therefore, $B = \mathrm{Hess}F|_{V_H}$. That is,

$$\det(B) \neq 0 \quad \text{and} \quad \mathrm{ind}(B) = \Phi(D,k') - (k'-1).$$

6.7. The block $A = \mathrm{Hess}\lambda|_{\mathcal{A}_0(E_0)}$

In this case, for $\gamma, \delta \in \mathcal{A}_0(E_0)$ the matrix $\partial^2_{\gamma,\delta} h_\alpha$ is supported on E_0 so the second term in equation (5.3) vanishes and we get

$$\langle \gamma, \mathrm{Hess}\lambda\delta\rangle = 2\Re[\langle \partial_\gamma v, (\partial_\delta h_\alpha)v\rangle].$$

The vector $(\partial_\delta h_\alpha)v$ is only non-zero at the first coordinate,

$$((\partial_\delta h_\alpha)v)_0 = i\sum_{r\in E_0} \delta_{0r}(b_\alpha)_r v_r = i\delta\cdot\mathbf{x} - \delta\cdot\mathbf{y}.$$

To calculate $\partial_\gamma v$, use (5.4), which states

$$(h_\alpha - \lambda I)\partial_\gamma v = -(\partial_\gamma h_\alpha)v. \tag{6.7}$$

Let ψ_j for $j = 1, 2, \ldots, n+1$ be a choice of orthonormal eigenvectors of h_α corresponding to the ordered eigenvalues. The Moore–Penrose Pseudo-inverse of $(h_\alpha - \lambda I)$ is the matrix

$$(h_\alpha - \lambda I)^+ := \sum_{j:\lambda_j(h_\alpha)\neq\lambda} \frac{1}{\lambda_j(h_\alpha) - \lambda}\psi_j\psi_j^* = \sum_{j\neq k} \frac{1}{\lambda_j(h_\alpha) - \lambda}\psi_j\psi_j^*,$$

where in the last equality we used that $\lambda = \lambda_k(h_\alpha)$ is simple. By left multiplying (6.7) with the matrix $(h_\alpha - \lambda I)^+$ (whose kernel is spanned by v), we get

$$\partial_\gamma v = -(h_\alpha - \lambda I)^+(\partial_\gamma h_\alpha)v + \tilde{c}v,$$

for some constant $\tilde{c}$. Having $v_0 = 0$ yields

$$\begin{aligned}\langle \partial_\gamma v, (\partial_\delta h_\alpha)v\rangle &= -\overline{(h_\alpha - \lambda I)^+_{00}((\partial_\gamma h_\alpha)v)_0}((\partial_\delta h_\alpha)v)_0\\ &= c(h_\alpha)\overline{(i\gamma\cdot\mathbf{x} - \gamma\cdot\mathbf{y})}(i\delta\cdot\mathbf{x} - \delta\cdot\mathbf{y}),\end{aligned}$$

where, from (4.4),

$$c(h_\alpha) = \sum_{j \neq k} \frac{|\psi_j(0)|^2}{\lambda - \lambda_j(h_\alpha)} = -(h_\alpha - \lambda I)^+_{00}.$$

We conclude that

$$\langle \gamma, A\delta \rangle = 2c(h_\alpha)((\gamma \cdot \mathbf{x})^2 + (\delta \cdot \mathbf{y})^2) = \langle \gamma, 2c(h_\alpha)(\mathbf{x}\mathbf{x}^* + \mathbf{y}\mathbf{y}^*)\delta \rangle.$$

Since $c(h_\alpha) \neq 0$ by assumption, then A has rank two over $\mathcal{A}_0(E_0)$. In particular,

$$\ker(A) = \mathcal{L}, \quad \text{and} \quad \operatorname{ind}(A) = \begin{cases} 2 & \text{if } c(h_\alpha) < 0, \\ 0 & \text{if } c(h_\alpha) > 0. \end{cases}$$

Since $\mathrm{Hess}\lambda|_{\mathcal{A}_0(E_0) \oplus V_H} = A \oplus B$, then we conclude that $\ker(\mathrm{Hess}\lambda) = \mathcal{L} \oplus d\mathbb{R}^{n+1}$ and

$$\operatorname{ind}(\mathrm{Hess}\lambda) = \phi(D_\alpha, k') - (k' - 1) + \begin{cases} 2 & \text{if } c(h_\alpha) < 0, \\ 0 & \text{if } c(h_\alpha) > 0. \end{cases}$$

6.8. F is Morse–Bott

Recall that the submanifold of critical points $F' \subset \mathcal{M}_h$ is a *Morse–Bott* critical submanifold of λ_k if, at every point $[h'] \in F'$, the kernel of $\mathrm{Hess}\lambda_k([h'])$ is exactly the tangent space $T_{[h']}F'$ and the number of negative eigenvalues of $\mathrm{Hess}\lambda_k([h'])$ is constant for all $[h'] \in F'$. By (6.6) the kernel condition holds at $[h_\alpha]$. Since $c(h_\alpha)$ is non-zero and continuous, it does not change sign, so the Morse–Bott condition holds at every point $[h'] \in F'$. As the kernel of the Hessian at a point $[h'] \in F' \subset F$ is $T_{[h']}F'$ and λ_k is constant on F, the tangent spaces agree, $T_{[h']}F' = T_{[h']}F$. Since F' is closed it is a union of connected components of F. It contains both $[h_\alpha]$ and its complex conjugate, so $F = F'$ is Morse–Bott and

$$\operatorname{ind}(F) = \phi(D_\alpha, k') - (k' - 1) + \begin{cases} 2 & \text{if } c(h_\alpha) < 0, \\ 0 & \text{if } c(h_\alpha) > 0. \end{cases}$$

7. Transversality to the strata of $\mathcal{H}_n$

7.1. The strata

The vector space $\mathcal{H}_n$ of Hermitian $n \times n$ matrices is stratified according to the multiplicities of the eigenvalues, as described in [3]. (See also [1, 4, 30].) Suppose $h \in \mathcal{H}_n$ has k *distinct* eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_k$. Specifying a multiplicity $r(i)$ for the eigenvalue μ_i determines a stratum $T(r)$, consisting of Hermitian matrices with eigenvalues μ_i and multiplicities $r(i)$. The multiplicity vector r is an ordered partition of n, meaning that $n = \sum_{i=1}^{k} r(i)$, and every ordered partition of n determines a stratum. The set of possible eigenvalues for h forms an open set

$$\mathbb{R}^k_< = \{x \in \mathbb{R}^k : x_1 < x_2 < \cdots < x_k\}$$

in $\mathbb{R}^k$. The eigenspaces V_i determine a partial flag $V_1 \subset V_1 \oplus V_2 \subset \cdots \subset \mathbb{C}^n$. Therefore, the stratum $T(r)$ may be canonically identified with the product

$$P(r) = \mathcal{F}l(r) \times \mathbb{R}^k_<$$

where $\mathcal{F}l(r)$ denotes the partial flag manifold of subspaces $0 \subset W_1 \subset W_2 \subset \cdots \subset \mathbb{C}^n$ with $\dim(W_k) = \sum_{i=1}^{k} r(i)$. This identification endows the stratum $T(r)$ with the canonical structure of an analytic manifold, and each eigenvalue $\mu_i : T(r) \to \mathbb{R}$ is an analytic function.

It is well known [34] that $\mathcal{F}l(r)$ is isomorphic to the quotient $U(n)/\prod_{i=1}^{k} U(r(i))$ of unitary groups, so it has dimension $n^2 - \sum_{i=1}^{k} r(i)^2$ from which it follows that the stratum $T(r)$ has codimension $\sum_{i=1}^{k}(r(i)^2 - 1)$ in $\mathcal{H}_n$.

7.2. The manifold $S_m(k)$

For any $h \in \mathcal{H}_n$, we may label the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Fix m, k. Let $V_k(h) = \ker(h - \lambda_k(h).I)$ be the eigenspace with eigenvalue λ_k. Define[6]

$$S_m(k) = \{h \in \mathcal{H}_n : \dim(V_k(h)) = m \text{ and } \lambda_{k-1}(h) < \lambda_k(h)\}. \tag{7.1}$$

Each $h \in S_m(k)$ has exactly $k-1$ eigenvalues less than λ_k and $n - m - k + 1$ eigenvalues greater than λ_k. It is foliated with leaves indexed by $\lambda \in \mathbb{R}$,

$$S_m(k, \lambda) = \{h \in S_m(k) : \lambda_k(h) = \lambda\}. \tag{7.2}$$

7.3 Lemma. *The set $S_m(k)$ (resp. $S_m(k,\lambda)$) is an analytic manifold of codimension $m^2 - 1$ (resp. codimension m^2) in $\mathcal{H}_n$. The eigenvalue $\lambda_k : S_m(k) \to \mathbb{R}$ is analytic. If r is an ordered partition of n, then stratum $T(r)$ is the transverse intersection*

$$S_{r(1)}(1) \cap S_{r(2)}(1 + r(1)) \cap S_{r(3)}(1 + r(1) + r(2)) \cap \cdots \cap S_{r(k)}(n - r(k) + 1). \tag{7.3}$$

[6] We are grateful to the referee for pointing out an error in our earlier definition of $S_m(k)$.

Proof. If $h \in S_m(k)$, then the eigenspace $V = V_k(h)$ is an element of the Grassmann manifold $G_m(\mathbb{C}^n)$ of m-dimensional subspaces of $\mathbb{C}^n$. Set $t = \lambda_k(h)$. The restriction $h|V^\perp$ determines an orthogonal decomposition $V_k(h)^\perp = W_- \oplus W_+$ as the sum of the $< t$ (resp. $> t$) eigenspaces of $h|V_k(h)^\perp$ of dimension $k-1$ and dimension $n-m-k+1$ respectively. The further restriction $h|W_-$ lies in the set $\mathcal{H}(W_-)^{<t}$ of Hermitian operators all of whose eigenvalues are $< t$, and similarly for $h|W_+$. Therefore, we have parametrized $S_m(k)$ by a double fibration

$$
\begin{tikzcd}
\mathcal{H}(W_-)^{<t} \times \mathcal{H}(W_+)^{>t} \arrow[rr, hookrightarrow] \arrow[dr] & & S_m(k) \arrow[d] \\
& W_- \in G_{k-1}(V^\perp) \arrow[r, hookrightarrow] \arrow[dr] & E \arrow[d] \\
& & (V,t) \in G_m(\mathbb{C}^n) \times \mathbb{R}
\end{tikzcd}
$$

where E is the bundle whose fiber over (V,t) is the Grassmannian of $(k-1)$-dimensional complex subspaces $W_- \subset V^\perp$. From this, we see that $S_m(k)$ is an analytic manifold and $t = \lambda_k$ is an analytic function on $S_m(k)$, as a coordinate in the parametrization.

The dimension of $S_m(k)$ may be calculated from the above diagram,

$$\dim(S_m(k)) = \dim_{\mathbb{R}}(G_m(\mathbb{C}^n)) + \dim_{\mathbb{R}}(G_{k-1}(V^\perp)) + \dim(\mathcal{H}(W_-)^{<t} \times \mathcal{H}(W_+)^{>t}),$$

and a miraculous cancellation of terms gives $\operatorname{codim}(S_m(k)) = m^2 - 1$. It is a direct consequence of the definitions that $T(r)$ is the intersection (7.3). The intersection is transversal because the different factors in (7.3) involve independent conditions. ∎

7.4 Proposition. *Fix $h \in \mathcal{H}_n$ with eigenvalue $\lambda = \lambda_k(h)$ and eigenspace $V = V_k$ of dimension m.*

(A) *The tangent space $T_h S_m(k,\lambda)$ (resp. $T_h S_m(k)$) consists of all tangent vectors $\xi \in T_h\mathcal{H}_n = \mathcal{H}_n$ such that, as sesquilinear forms,*[7] *the restriction $\xi|V = 0$ (resp. such that $\xi|V$ is a scalar*[8]*). With respect to the decomposition*

$$\mathbb{C}^n = V_k \oplus V_k^\perp,$$

[7] Cf. Section 2.1.

[8] In fact, it is multiplication by the directional derivative $\partial_\xi(\lambda_k)$.

it is the subspace of matrices

$$\xi = \begin{pmatrix} 0 & B \\ B^* & D \end{pmatrix}, \quad \textit{resp. } \xi = \begin{pmatrix} c.I_V & B \\ B^* & D \end{pmatrix}, \tag{7.4}$$

where $D \in \mathcal{H}_{n-k}$, $B \in M_{k\times(n-k)}(\mathbb{C})$, $c \in \mathbb{R}$, and I_V the identity on V.

(B) *A submanifold $Q \subset \mathcal{H}_n$ is transverse to $S_m(k,\lambda)$ (resp. $S_m(k)$) at $h \in Q$ if and only if the elements $\xi|V$ (resp. the elements $c.I_V + \xi|V$) account for all the Hermitian operators in $\mathcal{H}(V)$, as ξ varies within T_hQ and c within $\mathbb{R}$.*

(C) *The tangent space $T_hS_m(k,\lambda)$ (respectively, $T_hS_m(k)$) can also be expressed as the set of all $\xi \in \mathcal{H}_n$ of the form*

$$\xi = (h-\lambda.I)U + U^*(h-\lambda.I) \quad \textit{with } U \in M_{n\times n}(\mathbb{C})$$

(respectively, $\xi = (h-\lambda.I)U + U^*.(h-\lambda.I) + c.I_V$ with $U \in M_{n\times n}(\mathbb{C})$ and $c \in \mathbb{R}$).

Proof. Let ξ be a tangent vector to $S_m(k,\lambda)$ at the point $h = h_0$. Let $h_t \in S_m(k,\lambda)$ be a smooth one parameter family with $\xi = \dot{h} = \frac{d}{dt}h(0)$. Suppose $u_t \in V \subset \mathbb{C}^n$ is an eigenvector of h_t with eigenvalue λ. Differentiating the eigenvalue equation $hu = \lambda u$ gives $\dot{h}u + h\dot{u} = \lambda\dot{u}$. Taking the inner product with any $w \in V$ gives $\langle w, \xi u\rangle = 0$ which shows that ξ has the form of equation (7.4) above. On the other hand, the codimension of the space of matrices (7.4) is m^2 which equals the codimension of $S_m(k,\lambda)$ so (7.4) describes the full tangent space. A similar procedure works for the tangent space to $S_m(k)$.

Part (B) of the proposition is an immediate consequence.

For part (C), using the decomposition $\mathbb{C}^n = V_k \oplus V_k^{\perp}$, the matrix of $T = h - \lambda.I$ is

$$h - \lambda.I = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \tag{7.5}$$

where A is non-singular. Given $U \in M_{n\times n}(\mathbb{C})$, we have

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \implies TU + U^*T = \begin{pmatrix} 0 & U_3^*A \\ AU_3 & AU_4 + U_4^*A \end{pmatrix}.$$

Hence, $TU + U^*T \in T_hR_m(\lambda)$ as it has the form (7.4). Conversely, since A is invertible, AU_3 and AU_4 account for all matrices in $M_{(n-k)\times k}(\mathbb{C})$ and $M_{k\times k}(\mathbb{C})$ as U varies within $M_{n\times n}(\mathbb{C})$. This proves the case $\xi = \left(\begin{smallmatrix} 0 & B \\ B^* & D \end{smallmatrix}\right)$, and the case of $\xi = \left(\begin{smallmatrix} c.I_V & B \\ B^* & D \end{smallmatrix}\right)$ follows. ∎

A closely related result, concerning transversality with respect to the manifold of matrices with constant rank, appears in [27, Chapter 10.5]. Although Lovász considers only real symmetric matrices, his proof applies also to Hermitian matrices.

7.5. Application to graphs

Fix $k, m, n \geq 1$ and let $\lambda \in \mathbb{R}$. Recall $S_m(k), S_m(k,\lambda) \subset \mathcal{H}_n$ from (7.1), (7.2). Let G be a graph on n vertices with a set of edges E and associated spaces $\mathcal{S}(G) \subset \mathcal{H}(G) \subset \mathcal{H}_n$. In this section we determine when the inclusion $\mathcal{H}(G) \to \mathcal{H}_n$ is transverse to the manifold $S_m(k)$ at a point h of their intersection. The results are used in Corollary 7.10, the case of multiplicity 2, to provide sufficient conditions which guarantee that the mapping $\mathbb{T}_h \to \mathcal{H}_n$ is transverse to $S_2(k)$ locally near h.

Let $\mathcal{H}(\overline{G})$ denote those Hermitian matrices that are supported on the complement of E. That is $h \in \mathcal{H}(\overline{G})$ if $h^* = h$, $h_{rr} = 0$, and $h_{rs} = 0$ for all r and for any edge rs. It is the orthogonal complement[9] in $\mathcal{H}_n$ to $\mathcal{H}(G)$.

7.6 Proposition. *Fix $h \in \mathcal{H}_n$. Suppose $\lambda_{k-1}(h) < \lambda_k(h)$ and the eigenvalue $\lambda := \lambda_k(h)$ has multiplicity m and eigenspace $V = V_k$. The following statements are equivalent.*

(1) *The inclusion $\mathcal{H}(G) \to \mathcal{H}_n$ is transverse to $S_m(k)$ at h.*

(2) *The inclusion $\mathcal{H}(G) \to \mathcal{H}_n$ is transverse to $S_m(k,\lambda)$ at h.*

(3) *$(h - \lambda.I)X \neq 0$ for every non-zero $X \in \mathcal{H}(\overline{G})$.*

(4) *There exist $\xi_1, \xi_2, \dots, \xi_N \in \mathcal{H}(G)$ whose restrictions $\{\xi_1|V, \xi_2|V, \dots, \xi_N|V\}$ span (over $\mathbb{R}$) the space $\mathcal{H}(V)$ of Hermitian operators on V.*

Proof. Let V denote the m-dimensional eigenspace of h with eigenvalue λ. Parts (1) and (2) are equivalent by Proposition 7.4 because the tangent space $T_h\mathcal{H}(G) = \mathcal{H}(G)$ contains the identity matrix $\xi = I$, and $\xi|V = I_V$.

For part (3), as in [27, Section 10.5.2], the manifolds $\mathcal{H}(G)$ and $S_m(k,\lambda)$ are transverse at h if and only if the orthogonal complements of their tangent spaces intersect trivially. The orthogonal complement to $T_h S_m(k,\lambda)$ is

$$T_h S_m(k,\lambda)^\perp = \{X \in \mathcal{H}_n : (h - \lambda.I)X = 0\}$$

by equations (7.5) and (7.4). The orthogonal complement of $T_h\mathcal{H}(G) = \mathcal{H}(G)$ is $\mathcal{H}(\overline{G})$, so transversality to $S_m(k,\lambda)$ fails if and only if there exists $0 \neq X \in \mathcal{H}(\overline{G})$ such that $(h - \lambda.I)X = 0$.

Part (4) is a restatement of part (B) of Proposition 7.4. ■

7.7. Example – Graph splitting

The following example provides some intuition for the definitions in Section 7.8. Given graphs H_1, H_2 of size n_1, n_2 respectively. Suppose that both $h_1 \in \mathcal{H}(H_1)$

[9] $\mathcal{H}_n$ is equipped with the standard inner product $\langle A, B\rangle := \mathrm{trace}(A^* B) = \mathrm{trace}(AB)$.

and $h_2 \in \mathcal{H}(H_2)$ have the same simple eigenvalue λ. Let ϕ_1, ϕ_2 be corresponding eigenfunctions. Suppose there is a vertex v_1 of H_1 with $\phi_1(v_1) = 0$ and a vertex v_2 of H_2 with $\phi_2(v_2) = 0$. Let G be the graph of size $n = n_1 + n_2 - 1$ obtained from joining H_1, H_2 by identifying the vertices v_1 and v_2. Define $h \in \mathcal{H}(G)$ such that its restrictions to H_1, H_2 agree with h_1, h_2. Then λ is an eigenvalue of h with 2-dimensional eigenspace V and eigenfunctions $\Phi_1 = \phi_1 \times \{0\}$ and $\Phi_2 = \{0\} \times \phi_2$ such that $\langle \Phi_1, \Phi_2 \rangle = 0$. For any $\xi \in \mathcal{H}(G)$, we have $\langle \Phi_1, \xi\Phi_2 \rangle = 0$ so the elements $\xi|V$ fail to account for all quadratic forms on V, that is, $\mathcal{H}(G)$ is not transverse to $S_2(k)$ at the point h. In the graph G, both eigenfunctions vanish at the vertex $v_1 = v_2$ so the graph G is "split" into two pieces by this eigenvalue λ.

7.8. Graph theoretic conditions

Maintain the notation of Section 7.1 and Section 7.5. If $u \in \mathbb{R}^n$ the *support* of u is the set $\mathrm{spt}(u)$ of vertices j such that $u_j \neq 0$. If $V \subset \mathbb{R}^n$ is a subspace, its support is

$$\mathrm{spt}(V) = \bigcup_{u \in V} \mathrm{spt}(u).$$

If λ is an eigenvalue of $h \in \mathcal{H}(G)$ with eigenspace $V = \ker(h - \lambda I)$, we say that *the eigenspace V splits G* if the induced subgraph $G|\mathrm{spt}(V)$ of G on the vertices in $\mathrm{spt}(V)$ is *not* connected. Given an edge (rs), we say that V *projects surjectively onto* (rs) if $\{(u_r, u_s) : u \in V\} = \mathbb{C}^2$.

7.9 Theorem. *Suppose $h \in \mathcal{H}(G)$ has eigenvalue $\lambda = \lambda_k(h)$ and eigenspace V.*

(A) *Suppose the multiplicity of λ_k is 2 and either*

   (a) *there exists an edge (rs) on which V projects surjectively, or*

   (b) *the eigenspace V does not split G.*

   Then $\mathcal{H}(G)$ is transverse to $S_2(k)$ at the point h.

(B) *For arbitrary multiplicity m, suppose there exist nonzero vectors $u, v \in V$ whose supports are edge-separated, meaning that* $\mathrm{spt}(u) \cap \mathrm{spt}(v) = \emptyset$ *and there are no edges between* $\mathrm{spt}(u)$ *and* $\mathrm{spt}(v)$. *Then $\mathcal{H}(G)$ is not transverse to $S_m(k)$ at h.*

Proof. Assume there exists an edge (rs) on which V projects surjectively. In this case, we may choose u and v in V such that $(u_r, u_s) = (1, 0)$ and $(v_r, v_s) = (0, 1)$, so that in the (not necessarily orthonormal) basis $\{u, v\}$ of V, for any $\xi \in \mathcal{H}(G)$ with $\xi_{ij} = 0$ for all $ij \notin \{rr, rs, sr, ss\}$,

$$\xi|_V := \begin{pmatrix} \langle u, \xi u \rangle & \langle u, \xi v \rangle \\ \langle v, \xi u \rangle & \langle v, \xi v \rangle \end{pmatrix} = \begin{pmatrix} \xi_{rr} & \xi_{rs} \\ \xi_{sr} & \xi_{ss} \end{pmatrix}.$$

So, these vectors span $\mathcal{H}(V)$ verifying part (4) of Proposition 7.4.

For condition (b), we will show that if V does not split G, then there must be an edge (rs) on which V projects surjectively. Let H be the induced subgraph on $\mathrm{spt}(V)$ and assume it is connected. Choose a generic basis $\{u, v\}$ of V, so that $u(r) \neq 0$ and $v(r) \neq 0$ for all $r \in H$. Assume by contradiction that V does not project surjectively on any edge (rs). That is, $\frac{u(s)}{u(r)} = \frac{v(s)}{v(r)}$ for every edge (rs) in H. Fix an initial vertex $r_0 \in H$. Any other vertex $s \in H$ is connected to r_0 by a path in H, say $(r_0, r_1, r_2, \ldots, r_m, s)$, and so

$$\frac{v(s)}{v(r_0)} = \frac{v(r_1)}{v(r_0)} \cdot \frac{v(r_2)}{v(r_1)} \cdots \frac{v(s)}{v(r_m)} = \frac{u(r_1)}{u(r_0)} \cdot \frac{u(r_2)}{u(r_1)} \cdots \frac{u(s)}{u(r_m)} = \frac{u(s)}{u(r_0)}.$$

Thus, u and v are linearly dependent.

For part (B), given $u, v \in V$ as described in part (B), the matrix $X = uv^* + vu^*$ is in $\mathcal{H}_n(\overline{G})$ and satisfies $(h - \lambda.I)X = 0$, which contradicts Proposition 7.6 (3). ■

7.10 Corollary. *Let $h \in \mathcal{S}(G)$ be properly supported, and let $\alpha \in \mathcal{A}(G)$. Let $\mathbb{T}_h = \mathbb{T}(G) * h \subset \mathcal{H}(G)$ be the embedded torus. Suppose the eigenvalue λ_k of $h_\alpha = \alpha * h$ has multiplicity 2 with eigenspace that does not split G. Then there is a neighborhood $V \subset \mathcal{A}(G)$ of α and a neighborhood $U \subset \mathcal{S}(G)$ of h such that for a generic[10] set of $h' \in U$, the embedding map $\mathbb{T}_{h'} \hookrightarrow \mathcal{H}_n$ takes the open subset*

$$V * h' = \{\alpha' * h' : \alpha' \in V\} \subset \mathbb{T}_{h'}$$

transversally to the stratum $S_2(k)$.

Proof. Consider the composition

$$\Phi\colon \mathbb{T}(G) \times \mathcal{S}(G) \xrightarrow{*} \mathcal{H}(G) \xhookrightarrow{j} \mathcal{H}_n$$

given by $\Phi(\alpha', h') = \Phi_{h'}(\alpha') = \alpha' * h'$. The map $*$ above is surjective, since h is properly supported, with finite fibers, it is an open mapping and a submersion. The non-splitting assumption implies the embedding map j takes $\mathcal{H}(G)$ transversally to $S_2(k)$ at the point h_α, so it takes a neighborhood $W \subset \mathcal{H}(G)$ of h_α transversally to $S_2(k)$. Choose the neighborhoods $V \subset \mathcal{A}(G)$ and $U \subset \mathcal{S}(G)$ so that $V * U \subset W$. Then $\Phi\colon V \times U \to \mathcal{H}_n$ is transverse to $S_2(k)$. Lemma A.1 implies there exists a dense set of values $h' \in U$ so that the resulting map $\Phi_{h'}\colon V \to \mathcal{H}_n$ is transverse to $S_2(k)$. But this map is the composition

$$V \xrightarrow{\cong} V * h' \xrightarrow{j} \mathcal{H}_n.$$

■

[10]An open, dense and full measure set.

7.11. Example – Graphs for which $\mathbb{T}_h$ is generically transverse to S_2

Suppose G is a graph obtained by removing a set of disjoint edges from the complete graph, say (r_j, s_j) for $j = 1, \ldots, m$ such that the vertices $\{r_1, s_1, r_2, s_2, \ldots\}$ are all distinct. If $h \in \mathcal{H}(G)$ has distinct diagonal elements (a generic assumption), then the embedded torus $\mathbb{T}_h$ intersects $S_2(k)$ transversally for every k. To prove this, it suffices by Corollary 7.10 to show, for any $h_\alpha \in \mathbb{T}_h$, that no multiplicity-two eigenvalue of h_α splits G.

Assume by contradiction that some $h_\alpha \in \mathbb{T}_h$ has a multiplicity two eigenvalue $\lambda = \lambda_k(h_\alpha)$ with eigenspace V such that the induced graph $G|\mathrm{spt}(V)$ is disconnected. By the construction of G, this means that $\mathrm{spt}(V) = \{r_j, s_j\}$ for one of the missing edges (r_j, s_j). So, λ is a multiplicity-two eigenvalue of the restriction

$$h_\alpha|\mathrm{spt}(V) = \begin{pmatrix} h_{r_j,r_j} & 0 \\ 0 & h_{s_j,s_j} \end{pmatrix}.$$

This contradicts the assumption that diagonal elements of h are distinct.

A. Transversality

A.1 Transversality Lemma. *Let $\Phi\colon \mathbb{T} \times B \to \mathcal{H}$ be a smooth map between smooth manifolds and suppose this map is transverse to a submanifold $S \subset \mathcal{H}$. Then there is a dense set of values $b \in B$ such that the partial map*

$$\phi_b\colon \mathbb{T} \to \mathcal{H} \quad \text{given by } \phi_b(x) = \Phi(x, b)$$

is transverse to S. If Φ is proper and $S \subset \mathcal{H}$ is closed, then this set of values is open in B. If $\Phi, \mathbb{T}, B, \mathcal{H}$ and S are analytic then the set of values $b \in B$ for which transversality of ϕ_b fails is a subanalytic subset of B of positive codimension.

Remarks. Here, $\mathbb{T}$ is any finite-dimensional smooth manifold. The symbol $\mathbb{T}$ is being used to indicate that for our application, $\mathbb{T}$ is an open subset of the torus $\mathbb{T}(G)$.

This result says, for example, that two submanifolds of Euclidean space may be made transverse by an arbitrarily small *translation*. The transversality lemma is due originally to R. Thom ([31]). The proof described here may be found in ([18]).

Proof. It suffices to consider the case when B is open in some Euclidean space. By assumption, the set $P = \Phi^{-1}(S)$ is a smooth submanifold of $\mathbb{T} \times B$ and it is easy to check that $b \in B$ is a regular value of the projection $\pi\colon P \to B$ if and only if the partial map $\phi_b\colon \mathbb{T} \to \mathcal{H}$ is transverse to S. But Sard's theorem says that the set of non-regular values of π has Lebesgue measure zero.

Now, assume S is closed and Φ is proper (i.e., the preimage of a compact set is compact). To show the set of "transversal" elements $b \in B$ is open, we show its complement is closed. Let $b_i \in B$ be a convergent sequence of points, say $b_i \to b \in B$ for which there exists points $t_i \in \mathbb{T}$ such that ϕ_{b_i} fails to take the tangent space $T_{t_i}\mathbb{T}$ transversally to $T_{s_i}S$ where $s_i = \Phi(t_i, b_i)$. Since Φ is proper, by taking a subsequence if necessary we may assume the sequence converge, say $t_i \to t \in \mathbb{T}$ and therefore $s_i \to s$ for some $s \in \mathcal{H}$. Since S is closed, we also have $s \in S$. The failure of transversality is a closed condition so ϕ_b fails to take $T_t\mathbb{T}$ transversally to $T_s S$.

Finally, if $\Phi, \mathbb{T}, B, \mathcal{H}, S$ are analytic then the set of points $(t, b) \in \mathbb{T} \times B$ for which ϕ_b fails to be transverse at t is again analytic so its image $Z \subset B$ is a subanalytic subset of B. It has positive codimension, for if Z contains an open set in B, then this contradicts the assumption that Φ is transverse to S. ∎

B. Heuristics for discretization of magnetic Schrödinger operators

The definition of discrete magnetic operators can be found in [14, 26] for example, however, we will give here a heuristic explanation for why this is the right discretization for magnetic Schrödinger operators. For simplicity, we consider domains in $\mathbb{R}^3$, so that magnetism can be described using vector fields: a magnetic field B and magnetic potential A such that $B = \nabla \times A$. (The modern approach would consider A and B as a 1-form and 2-forms).

The quadratic form of a Schrödinger operator $H = \Delta + V$ on a domain $\Omega \subset \mathbb{R}^n$ is

$$\langle f, Hf\rangle = \int_\Omega \sum_{j=1}^{n} \Big(\frac{\partial f(x)}{\partial x_j}\Big)^2 + V(x) f(x)^2 \, dx,$$

for the relevant class of functions f on Ω. If we approximate the quotient $\frac{\partial f(x)}{\partial x_j}$ with $\frac{f(x+\varepsilon e_j) - f(x)}{\varepsilon}$, the quadratic form can be written as in (1.1)

$$\sum_{x,y\in\Lambda_\varepsilon} h_{xy}(f(y) - f(x))^2 + V(x) f(x)^2 dx, \qquad h_{xy} = \begin{cases} \frac{1}{\varepsilon^2} & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Lambda_\varepsilon \subset \Omega$ is a grid of side length ε. Introducing a magnetic field B, the operator H is changed to a magnetic Schrödinger operator H_A by the rule $\frac{\partial f(x)}{\partial x_j} \mapsto \frac{\partial f(x)}{\partial x_j} + iA_j(x) f(x)$, where $A = (A_1, A_2, A_3)$ is the *magnetic potential* defined (uniquely up to gauge transformations $A \sim A' = A + \nabla g$) by the relation $\nabla \times A = B$. Notice that

$$\frac{\partial f(x)}{\partial x_j} + iA_j(x) f(x) = \lim_{t\to 0} \frac{(e^{i\int_x^{x+te_j} A(s)ds} f(x + te_j)) - f(x)}{t},$$

and the ε discretization of the quadratic form can be written as in (2.3)

$$\int_{\Omega} \sum_{j=1}^{3} \left| \frac{\partial f(x)}{\partial x_j} + i A_j(x) f(x) \right|^2 + V(x) |f(x)|^2 \, dx$$
$$\approx \sum_{x,y \in \Lambda_\varepsilon} h_{xy} \left| f(y) - e^{i\alpha_{xy}} f(x) \right|^2 + V(x) |f(x)|^2 \, dx,$$

using $\alpha_{xy} = \int_x^{x+\varepsilon e_j} A(s) ds$ when $y = x + \varepsilon e_j$ and extending it antisymmetrically.

Acknowledgments. The authors would like to thank Gregory Berkolaiko for enlightening discussions and for his comments on earlier versions of this paper. L. Alon would like to thank Nikhil Srivastava and Theo McKenzie for useful discussions. The idea of Theorem 3.2 (7) originated in joint discussions with Ram Band and Gregory Berkolaiko regarding metric graphs. The authors are very grateful to an anonymous referee for many thoughtful comments, which have considerably improved the paper. L. Alon would like to thank the Institute for Advanced Study, as this work began when he was a member there.

Funding. L. Alon author was supported by the Ambrose Monell Foundation and the Simons Foundation Grant 601948, DJ.

References

[1] A. A. Agrachev, Spaces of symmetric operators with multiple ground states. *Funktsional. Anal. i Prilozhen.* **45** (2011), no. 4, 1–15; English transl., *Funct. Anal. Appl.* **45** (2011) no. 4, 241–251 Zbl 1271.46056 MR 2961476

[2] L. Alon, R. Band, and G. Berkolaiko, Universality of nodal count distribution in large metric graphs. *Exp. Math.* **2022**, 1–35.

[3] V. I. Arnold, Modes and quasimodes. *Funkcional. Anal. i Priložen.* **6** (1972), no. 2, 12–20; English transl., *Funct. Anal. Appl* **6** (1972), 94–10 Zbl 0251.70012 MR 297274

[4] V. I. Arnold, Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect. *Selecta Math. (N.S.)* **1** (1995), no. 1, 1–19 Zbl 0841.58008 MR 1327227

[5] R. Band, G. Berkolaiko, and T. Weyand, Anomalous nodal count and singularities in the dispersion relation of honeycomb graphs. *J. Math. Phys.* **56** (2015), no. 12, article no. 122111 Zbl 1335.81078 MR 3434868

[6] G. Berkolaiko, A lower bound for nodal count on discrete and metric graphs. *Comm. Math. Phys.* **278** (2008), no. 3, 803–819 Zbl 1171.05356 MR 2373444

[7] G. Berkolaiko, Nodal count of graph eigenfunctions via magnetic perturbation. *Anal. PDE* **6** (2013), no. 5, 1213–1233 Zbl 1281.35089 MR 3125554

[8] G. Berkolaiko and M. Kha, Degenerate band edges in periodic quantum graphs. *Lett. Math. Phys.* **110** (2020), no. 11, 2965–2982 Zbl 1456.81194 MR 4160926

[9] G. Berkolaiko and T. Weyand, Stability of eigenvalues of quantum graphs with respect to magnetic perturbation and the nodal count of the eigenfunctions. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** (2014), no. 2007, article no. 20120522 Zbl 1322.81042 MR 3151081

[10] G. Berkolaiko and I. Zelenko, Morse inequalities for ordered eigenvalues of generic families of self-adjoint matrices. 2023, arXiv:2304.04331

[11] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica* **26** (2006), no. 5, 495–519 Zbl 1121.05054 MR 2279667

[12] Y. Colin de Verdière, *Spectres de graphes*. Cours Spéc. 4, Société Mathématique de France, Paris, 1998 Zbl 0913.05071 MR 1652692

[13] Y. Colin de Verdière, Magnetic interpretation of the nodal defect on graphs. *Anal. PDE* **6** (2013), no. 5, 1235–1242 Zbl 1281.35090 MR 3125555

[14] Y. Colin De Verdière and N. Torki, Opérateur de Schrödinger avec champ magnétique. In *Séminaire de Théorie Spectrale et Géométrie,* No. 11, Année 1992–1993, pp. 9–18, Sémin. Théor. Spectr. Géom. 11, Université Grenoble I, Saint-Martin-d'Hères, 1993 Zbl 0937.35510 MR 1715941

[15] R. Courant and D. Hilbert, *Methods of mathematical physics.* Vol. I. Interscience Publishers, New York, 1953 Zbl 0051.28802 MR 65391

[16] E. B. Davies, G. M. L. Gladwell, J. Leydold, and P. F. Stadler, Discrete nodal domain theorems. *Linear Algebra Appl.* **336** (2001), 51–60 Zbl 0990.05093 MR 1855391

[17] M. Farber and D. Schütz, Homology of planar polygon spaces. *Geom. Dedicata* **125** (2007), 75–92 Zbl 1132.52024 MR 2322541

[18] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*. Grad. Texts in Math. 14, Springer, Berlin etc., 1973 Zbl 0294.58004 MR 341518

[19] M. Goresky and R. MacPherson, *Stratified Morse theory*. Ergeb. Math. Grenzgeb. (3) 14, Springer, Berlin, 1988 Zbl 0639.14012 MR 932724

[20] F. Harary, On the notion of balance of a signed graph. *Michigan Math. J.* **2** (1953/54), 143–146 (1955) Zbl 0056.42103 MR 67468

[21] P. G. Harper, Single band motion of conduction electrons in a uniform magnetic field. *Proc. Phys. Soc. Lond. Sect. A* **68** (1955), 874–878 Zbl 0065.23708

[22] P. G. Harper, The general motion of conduction electrons in a uniform magnetic field, with application to the diamagnetism of metals. *Proc. Phys. Soc. Lond. Sect. A* **68** (1955), 879–892. Zbl 0065.23707

[23] J.-C. Hausmann and A. Knutson, The cohomology ring of polygon spaces. *Ann. Inst. Fourier (Grenoble)* **48** (1998), no. 1, 281–321 Zbl 0903.14019 MR 1614965

[24] M. Kapovich and J. J. Millson, Universality theorems for configuration spaces of planar linkages. *Topology* **41** (2002), no. 6, 1051–1107 Zbl 1056.14077 MR 1923214

[25] T. Kato, Perturbation theory for linear operators. Second edn., Grundlehren Math. Wiss. 132, Springer, Berlin, 1976 Zbl 0342.47009 MR 407617

[26] E. H. Lieb and M. Loss, Fluxes, Laplacians, and Kasteleyn's theorem. *Duke Math. J.* **71** (1993), no. 2, 337–363 Zbl 0787.05083 MR 1233440

[27] L. Lovász, *Graphs and geometry*. Amer. Math. Soc. Colloq. Publ. 65, American Mathematical Society, Providence, RI, 2019 Zbl 1425.05001
[28] A. W. Marcus, D. A. Spielman, and N. Srivastava, Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Ann. of Math. (2)* **182** (2015), no. 1, 307–325 Zbl 1316.05066 MR 3374962
[29] F. Rellich, *Perturbation theory of eigenvalue problems*. Gordon and Breach Science Publishers, New York etc., 1969 MR 240668
[30] M. Shapiro and A. Vainshtein, Stratification of Hermitian matrices and the Alexander mapping. *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), no. 12, 1599–1604 Zbl 0856.57017 MR 1367814
[31] R. Thom, Un lemme sur les applications différentiables. *Bol. Soc. Mat. Mexicana (2)* **1** (1956), 59–71 Zbl 0075.32201 MR 102115
[32] J. C. Urschel, Nodal decompositions of graphs. *Linear Algebra Appl.* **539** (2018), 60–71 Zbl 1377.05154 MR 3739397
[33] K. Walker, *Configuration spaces of linkages.* Undergraduate thesis, Princeton University, Princeton, NY, 1985
[34] Wikipedia, Generalized flag variety
[35] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas. *Electron. J. Combin.* **5** (1998), Dynamic Surveys 8 Zbl 0898.05001 MR 1744869

Received 21 December 2022; revised 12 May 2023.

Lior Alon
Department of Mathematics, Massachusetts Institute of Technology, Simons Building (Building 2), 77 Massachusetts Avenue, Cambridge, MA 02139, USA; lioralon@mit.edu

Mark Goresky
School of Mathematics, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA; goresky@ias.edu