

## On the removal of finite discrete spectrum by coefficient stripping

Barry Simon<sup>1</sup>

**Abstract.** We prove for a large class of operators  $J$ , including block Jacobi matrices, that if  $\sigma(J) \setminus [\alpha, \beta]$  is a finite set, each point of which is an eigenvalue of finite multiplicity, then a finite coefficient stripped,  $J_N$ , has  $\sigma(J_N) \subset [\alpha, \beta]$ . We use an abstract Dirichlet decoupling.

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### 1. Introduction

The work in this note is motivated by the following theorem.

**Theorem 1.1.** *Let  $V \in L^1_{\text{loc}}([0, \infty))$  be such that  $-\frac{d^2}{dx^2} + V(x)$  is limit point at infinity. Suppose  $-\frac{d^2}{dx^2} + V(x)$  with  $u(0) = 0$  boundary conditions on  $L^2((0, \infty); dx)$  has finitely many eigenvalues (and no other spectrum) in  $(-\infty, 0)$ . Then there exists  $A > 0$  so that for all  $a > A$ ,  $-\frac{d^2}{dx^2} + V(x)$ , with  $u(a) = 0$  boundary conditions on  $L^2((a, \infty); dx)$  is a nonnegative operator.*

The proof is an immediate consequence of Sturm oscillation theory: by hypothesis, the function with  $u(0) = 0$ ,  $u'(0) = 1$ ,  $-u'' + Vu = 0$  has finitely many zeros. If  $A$  is the last zero,  $u \upharpoonright [a, \infty)$  solves  $-u'' + Vu = 0$  and has a definite sign on  $[a, \infty)$ . Thus, by Sturm comparison and oscillation arguments, the operator on  $[a, \infty)$  is nonnegative.

While it appears that this argument is well known to some experts, I am aware of only two places that it appears explicitly in print: in Nikishin's lovely paper [2] and in my review article on Sturm oscillation theory for the Sturm 200th Birthday Conference [5].

Going back to Bôcher [1] (and conjectured to be known to Sturm), there are discrete analogs of Sturm oscillation theorems. One looks at zeros of the linear

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interpolation of the solution of the difference equations. This allows one to prove an analog of Theorem 1.1 for Jacobi matrices of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (1)$$

Here and below,  $a_n > 0$ ,  $b_n$  is real, and  $\sup(|a_n| + |b_n|) < \infty$ . (Indeed, it was this case treated in [2] and [5]; Nikishin doesn't use the terminology "oscillation theorem" but he counts sign changes of certain determinants.)

**Theorem 1.2.** *Suppose  $J$  given by (1) has finitely many eigenvalues outside  $[\alpha, \beta]$ . Let  $J_n$  be the Jacobi matrix (1) with the top  $n$  rows and left  $n$  columns removed. Then for some  $N$  and all  $n > N$ ,  $\sigma(J_n) \subset [\alpha, \beta]$ .*

Given the discrete Sturm oscillation and comparison theorems (for the linear interpolation), the proof of Theorem 1.2 is essentially the same as for Theorem 1.1.

The immediate motivation for this work is the question asked me by Rostyslav Kozhan: Does an analog of Theorem 1.2 hold for block Jacobi matrices, that is, objects like (1) but where for some  $\ell$ , the  $b_j$  and  $a_j$  are  $\ell \times \ell$  matrices. Recently, Schulz-Baldes [3] has developed oscillation theorems for some block Jacobi matrices, and it may be possible to use his machinery to answer Kozhan's query. But we decided to seek out another and more direct approach. It has the advantage of working in cases where  $a_n$  and  $b_n$  are operators on an infinite-dimensional space where it is unlikely an oscillation theorem exists. Here is our main result:

**Theorem 1.3.** *Let  $J$  be a bounded selfadjoint operator on a Hilbert space  $\mathcal{H}$ . Suppose  $\{P_n\}_{n=1}^\infty, \{Q_n\}_{n=1}^\infty, \{R_n\}_{n=1}^\infty$  are families of orthogonal projections that obey*

$$P_n Q_n = Q_n P_n = P_n R_n = R_n P_n = Q_n R_n = R_n Q_n = 0, \quad (2)$$

$$P_n + Q_n + R_n = 1, \quad (3)$$

$$\text{s-lim}_{n \rightarrow \infty} Q_n = \text{s-lim}_{n \rightarrow \infty} R_n = 0, \quad (4)$$

$$P_n J Q_n = Q_n J P_n = 0. \quad (5)$$

*Suppose the only spectrum  $J$  has in  $(-\infty, 0)$  is finitely many eigenvalues of finite multiplicity. Then for some  $N$ ,  $Q_n J Q_n \geq 0$  for all  $n \geq N$ .*

In the block diagonal case, we have the following result.

**Corollary 1.4.** *Let  $\mathcal{H} = \ell^2(\{1, 2, \dots\}, \mathcal{K})$ , sequences with values in a Hilbert space,  $\mathcal{K}$ , with  $\sum_{n=1}^\infty \|x_n\|^2 < \infty$ . Let  $J$  be a matrix in  $\mathcal{H}$  of the form (1) with entries  $b_j, a_j$  that are bounded operators on  $\mathcal{K}$  with  $\sup_j (\|a_j\| + \|b_j\|) < \infty$ . Suppose the*

only spectrum of  $J$  in  $(-\infty, 0)$  is finitely many points, each an eigenvalue of finite multiplicity. Then for some  $N$  and all  $n \geq N$ ,  $J_n \geq 0$ , where  $J_n$  is  $J$  with the top  $n$  rows and left  $n$  columns dropped.

*Proof.* Let  $P_n$  be the projection onto  $\ell^2(\{1, \dots, n-1\}; \mathcal{K})$ ,  $R_n$  the projection onto  $\ell^2(\{n\}; \mathcal{K})$ , and  $Q_n$  the projection onto  $\ell^2(\{n+1, n+2, \dots\}; \mathcal{K})$ . (2) and (3) are obvious. (4) is simple and (5) is obvious. Thus,  $J_n = Q_n J Q_n \geq 0$  for all large  $n$  by the theorem.  $\square$

We prove Theorem 1.3 in Section 2 and a continuum analog in Section 3.

We end this section with three remarks. (i) As noted in [5], theorems of the form of Theorem 1.3 imply that if  $\sigma_{\text{ess}}(J) = [\alpha, \beta]$ , then for any  $\varepsilon > 0$  and  $n$  large,  $\sigma(Q_n J Q_n) \subset [\alpha - \varepsilon, \beta + \varepsilon]$ . This result is true under much weaker hypothesis (essentially, (5) is not needed), but I believe the stronger result that is Theorem 1.3 requires some kind of decoupling.

(ii) In Corollary 1.4, one can take  $\mathcal{K} = \ell^2(\mathbb{Z}^{\nu-1})$  and so treat discrete Schrödinger operators on  $\mathbb{Z}_+^\nu = \{(n_1, \dots, n_\nu) \in \mathbb{Z}^\nu \mid n_\nu \geq 1\}$ .

(iii) If one uses Neumann boundary conditions (i.e., restriction of a whole-line even operator to even functions), then weak coupling negative  $b_n$ 's and  $a_n \equiv 1$  produce a bound state (see, e.g., Simon [4]) and so the analog of Corollary 1.4, or even Theorem 1.2, fails.

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## 2. The discrete case

In this section, we will prove Theorem 1.3. We begin with the following lemma.

**Lemma 2.1.** *Let  $P_n$  be a family of orthogonal projections on a Hilbert space  $\mathcal{H}$  with*

$$\text{s-lim}_{n \rightarrow \infty} P_n = \mathbf{1}. \tag{6}$$

*Let  $J$  be a bounded selfadjoint operator on  $\mathcal{H}$  so that*

$$\dim P_{(-\infty, 0)}(J) = \ell < \infty, \tag{7}$$

*where  $P_{(-\infty, 0)}(\cdot)$  is a spectral projection. Then, for some  $N$ ,*

$$n > N \implies \dim P_{(-\infty, 0)}(P_n J P_n) = \ell. \tag{8}$$

*Proof.* By a variational argument,

$$\dim P_{(-\infty, 0)}(P_n J P_n) \leq \ell \tag{9}$$

(essentially, if it were dimension  $\geq \ell + 1$ , we could find  $\ell + 1$  unit vectors in  $\text{Ran } P_n$  so that  $\langle \varphi_j, \varphi_m \rangle = \delta_{jm}$  and  $\langle \varphi_j, J\varphi_m \rangle = a_j \delta_{jm}$  with  $a_j < 0$ ).

For any  $\varphi_1, \dots, \varphi_\ell$  in  $\text{Ran } P_n$ , let  $\Delta_{jk} = \langle \varphi_j, \varphi_k \rangle$  and  $A_{jk} = \langle \varphi_j, J\varphi_k \rangle$ . If  $\Delta$  is invertible and  $\Delta^{-1/2} A \Delta^{-1/2} < 0$ , then (9) holds by the variational principle.

Let  $\psi_1, \dots, \psi_\ell$  span  $\text{Ran } P_{(-\infty, 0)}(J)$  and be a set of orthonormal eigenvectors for  $J$ . Let  $\varphi_j^{(n)} = P_n \psi_j$  and  $\Delta_{jk}^{(n)} = \langle \varphi_j^{(n)}, \varphi_k^{(n)} \rangle$ ,  $A_{jk}^{(n)} = \langle \varphi_j^{(n)}, J\varphi_k^{(n)} \rangle$ . By (6),  $\Delta_{jk}^{(n)} \rightarrow \mathbf{1}$ ,  $A_{jk}^{(n)} \rightarrow \alpha_j \delta_{jk}$  with  $\alpha_j < 0$ . It follows for all large  $n$  that  $\Delta^{(n)}$  is invertible and  $(\Delta^{(n)})^{-1/2} A^{(n)} (\Delta^{(n)})^{-1/2} < 0$ .  $\square$

*Proof of Theorem 1.3.* Let  $S_n = P_n + Q_n$ . Then, by a variational argument,

$$\dim P_{(-\infty, 0)}(S_n J S_n) \leq \dim P_{(-\infty, 0)}(J) \equiv \ell < \infty. \quad (10)$$

By (5), on  $\text{Ran } S_n = \text{Ran } P_n \oplus \text{Ran } Q_n$ , we have that

$$S_n J S_n = P_n J P_n \oplus Q_n J Q_n \quad (11)$$

so

$$\dim P_{(-\infty, 0)}(S_n J S_n) = \dim P_{(-\infty, 0)}(P_n J P_n) + \dim P_{(-\infty, 0)}(Q_n J Q_n). \quad (12)$$

By the lemma for some  $N$  and  $n > N$ ,

$$\dim P_{(-\infty, 0)}(P_n J P_n) = \ell. \quad (13)$$

By (10), (12), and (13) for  $n > N$ ,

$$\dim P_{(-\infty, 0)}(Q_n J Q_n) = 0$$

that is,  $Q_n J Q_n \geq 0$ .  $\square$

### 3. The continuum case

Here we will state and prove the analog of Theorem 1.3 for ODEs. For simplicity of exposition, we assume the potential is bounded, but it is easy to extend to  $V$ 's with  $\|V(x)\|$  in  $L^1_{\text{loc}}$  with some kind of selfadjointness criterion at  $\infty$  or to positive  $V$ 's using quadratic forms.

**Theorem 3.1.** *Let  $\mathcal{K}$  be a Hilbert space and  $V(x)$ ,  $x \in [0, \infty)$ , a weakly measurable function on  $[0, \infty)$  with values in the bounded operators on  $\mathcal{K}$  with  $\sup_{x \in [0, \infty)} \|V(x)\| < \infty$ . Let  $H_a$ ,  $a \geq 0$ , be the operator on  $\mathcal{H}_a \equiv L^2((a, \infty), \mathcal{K}; dx)$  of the form  $-\frac{d^2}{dx^2} + V(x)$  with  $u(a) = 0$  boundary conditions. Suppose  $H \equiv H_{a=0}$  has only finitely many eigenvalues in  $(-\infty, 0)$ , each of finite multiplicity and no other spectrum there. Then for some  $A \in [0, \infty)$ ,  $H_a \geq 0$  for  $a \geq A$ .*

**Remark 3.2.** By allowing  $V(x)$  to be a selfadjoint operator bounded from below (to handle the Laplacian in the other directions), this theorem can be extended to treat  $-\Delta + V(x_1, \dots, x_\nu)$  on  $\{x \mid x_\nu > 0\}$ .

*Proof.* For each  $a$ , by adding a Dirichlet boundary condition at  $a$ , we can find  $\tilde{H}_a$  on  $\mathcal{H}_{a=0}$  of the form  $K_a \oplus H_a$  on  $L^2((0, a), \mathcal{K}; dx) \oplus L^2((a, \infty), \mathcal{K}; dx)$  and

$$\tilde{H}_a \geq H_{a=0}. \quad (14)$$

As in the proof of Lemma 2.1,

$$\dim P_{(-\infty, 0)}(K_a) = \dim P_{(-\infty, 0)}(H_{a=0})$$

for all  $a > A$  for some  $A$ .

For such  $a$ , by (14),  $\dim P_{(-\infty, 0)}(H_a) = 0$ . □

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Barry Simon, Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, U.S.A.

E-mail: [bsimon@caltech.edu](mailto:bsimon@caltech.edu)