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Localization of two-dimensional massless Dirac fermions in a magnetic quantum dot

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Abstract. We consider a two-dimensional massless Dirac operator H in the presence of a perturbed homogeneous magnetic field $B = B_0 + b$ and a scalar electric potential V. For $V \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, $p \in (2, \infty]$, and $b \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$, $q \in (1, \infty]$, both decaying at infinity, we show that states in the discrete spectrum of H are superexponentially localized. We establish the existence of such states between the zeroth and the first Landau level assuming that $V = 0$. In addition, under the condition that b is rotationally symmetric and that V satisfies certain analyticity condition on the angular variable, we show that states belonging to the discrete spectrum of H are Gaussian-like localized.

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1. Introduction

Graphene is a two-dimensional lattice of carbon atoms arranged on a honeycomb structure. Due to its unusual properties it has attracted a great deal of attention since its discovery; see [4] and [21]. One of the striking facts about graphene is that the dynamics of its low-energy excitations (the charge carriers) can be described by massless two-dimensional Dirac operators. An interesting feature of such Dirac fermions is the lack of localization under the influence of an external electric potential; see [31] and [15]. This fact, related to Klein's paradox [4], is due to the peculiar cone-like gapless structure of the spectrum of massless free Dirac operators.

It was suggested in [7] that it is possible to confine such massless Dirac fermions in graphene by inhomogeneous magnetic fields of the type $B = B_0 + b$, where $B_0 > 0$ is a constant and b a perturbation with negative flux that decays at infinity. The spectrum of the corresponding Dirac operator in a constant magnetic field B_0 is given by the (relativistic) Landau levels. The idea is that as the perturbation b is turned on eigenvalues will emerge from the Landau levels giving rise to states localized on the bulk of the support of b. In this manner a so-called (magnetic) quantum dot or

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artificial atom can be created. These type of models, also with an external electric potential V, have been further studied in the physics literature, for instance in [8], [22], [32], and [16] for the one particle case and in [13] and [9] for the multiparticle case. The articles [8], [22], [32], and [16] deal with specific electromagnetic fields for which the model is partly solvable or suitable for numerical computations.

In this article we consider a large class of electromagnetic perturbations (b, V) with $V \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, $p \in (2, \infty]$, and $b \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$, $q \in (1, \infty]$, both decaying at infinity. The essential spectrum of the corresponding massless Dirac-operator H describing the quantum dot is given by the Landau levels. We show that eigenfunctions belonging to the discrete spectrum of H are superexponentially localized, i.e., they decay faster than any exponential. In the case when $V = 0$ we verify the existence of eigenvalues between the zeroth and the first Landau-level assuming that $b < 0$. Assuming that a certain analyticity conditions on the angular variable of V is fulfilled and that b is rotationally symmetric we prove that those states are actually Gaussian-like localized. These type of results on superexponential and Gaussian localization, although new for Dirac operators, are known to hold for spinless magnetic Schrödinger operators [6], [10], [19], and [29]. We benefit from this insight to prove our statements. A precise description of our results is given in the next section.

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2. Results

We consider the massless two-dimensional Dirac operator with an external magnetic field $B\colon \mathbb{R}^2 \to \mathbb{R}$, pointing perpendicularly to the plane, and an electric potential $V\colon \mathbb{R}^2 \to \mathbb{R}$. We are interested in the Hamiltonians

$$D_{\mathbf{A}} \stackrel{\text{def}}{=} \boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{A}), \tag{1}$$

$$H \stackrel{\text{def}}{=} D_{\mathbf{A}} + V, \tag{2}$$

a priori defined on $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2) \subset L^2(\mathbb{R}^2; \mathbb{C}^2)$. Here $\mathbf{p} \stackrel{\text{def}}{=} \frac{1}{\mathrm{i}}\nabla$ is the momentum of the particle and $\boldsymbol{\sigma} \stackrel{\text{def}}{=} (\sigma_1, \sigma_2)$ is a vector whose entries

$$\sigma_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

are Pauli matrices. The magnetic field B enters in the definitions (1) and (2) by means of the magnetic vector potential $\mathbf{A} = (A_1, A_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ through the relation

$$B = \partial_1 A_2 - \partial_2 A_1 \stackrel{\text{def}}{=} \operatorname{curl} \mathbf{A}, \tag{3}$$

which is understood in the sense of distributions.

Throughout this article we assume the following on (B, V).

(A1) $B = B_0 + b$ where $B_0 > 0$ is a number and $b \in L^q_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ for some $q \in (1,\infty]$ and $\lim_{n\to\infty} \|\mathbb{1}_{\{|\mathbf{x}|\geq n\}} b\|_\infty = 0$.

(A2) $V \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ for some $p \in (2,\infty]$ and $\lim_{n\to\infty} \|\mathbb{1}_{\{|\mathbf{x}|\geq n\}} V\|_\infty = 0$.

Here $\mathbb{1}_I(\cdot)$ denotes the characteristic function on the set I. Assuming that B fulfills (A1) we can always find $\mathbf{A} \in L^t_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ for some $t \in (2,\infty]$ satisfying (3); see Remark 8. For such magnetic vector potentials and electric potentials V satisfying (A2) we know that the operators defined in (1) and (2) are essentially self-adjoint; see Subsection 3.1. We denote their self-adjoint extensions by the same symbols and their domains by $\mathcal{D}(D_{\mathbf{A}})$ and $\mathcal{D}(H)$ respectively.

To the homogeneous magnetic field B_0 we associate the vector potential

$$\mathbf{A}_0 \overset{\mathrm{def}}{=} \frac{B_0}{2}(-x_2, x_1), \tag{4}$$

satisfying $\operatorname{curl} \mathbf{A}_0 = B_0$. It is well known that the spectrum of $D_{\mathbf{A}_0}$ consists of infinitely degenerated eigenvalues $(l_n)_{n\in\mathbb{Z}}$, called Landau levels, given by

$$l_n \overset{\mathrm{def}}{=} \operatorname{sgn}(n)\sqrt{2|n|B_0}, \quad n \in \mathbb{Z},$$

where $\operatorname{sgn}(n) = n/|n|$ if $n \neq 0$ and equals one if $n = 0$.

Given a self-adjoint operator T we write $\sigma_{\mathrm{pp}}(T)$, $\sigma_{\mathrm{d}}(T)$, and $\sigma_{\mathrm{ess}}(T)$ to denote the pure point, discrete, and essential spectra of T respectively. Our first main result is as follows.

Theorem 1. *Assume that B satisfies (A1) and let $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, $p \in (2,\infty]$, with $\operatorname{curl} \mathbf{A} = B$. Then, the spectrum of $D_{\mathbf{A}}$ is symmetric with respect to zero and*

$$\sigma_{\mathrm{ess}}(D_{\mathbf{A}}) = (l_n)_{n\in\mathbb{Z}}\,.$$

Moreover,

(a) *if $b \leq 0$ and strictly negative on some open set, then the discrete spectrum of $D_{\mathbf{A}}$ on $(0, l_1)$ is non-empty, i.e., $\sigma_{\mathrm{d}}(D_{\mathbf{A}}) \cap (0, l_1) \neq \emptyset$ and*

$$\dim(\operatorname{Ran}(\mathbb{1}_{(0,l_1)}(D_{\mathbf{A}}))) = \infty;$$

(b) *if $b \geq 0$ then*

$$\dim(\operatorname{Ran}(\mathbb{1}_{(0,l_1)}(D_{\mathbf{A}}))) = 0.$$

This theorem is a consequence of Lemmata 2 and 3. That the spectrum of $D_{\mathbf{A}}$ is symmetric with respect to zero is well known; see, however, Proposition 1.

Remark 1. A similar result to Theorem 1 is shown in [3] when b is replaced by λb and λ is assumed to be sufficiently large. Moreover, in [3] stronger regularity assumptions on b are made. In addition, the magnetic vector potential $\mathbf{a}$ associated to b is assumed to decay at infinity. However, the results of [3] hold for more general background magnetic fields than B_0. We note also that our proof differs from the one in [3].

Remark 2. For Schrödinger and Pauli operators the spectral subspaces obtained by splitting the Landau levels with electromagnetic perturbations decaying at infinity have been investigated in the last years; see e.g. [25] and references therein.

Remark 3. Assume that (A1) and (A2) are fulfilled. As a consequence of Lemmata 1 and 2 below,

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(D_{\mathbf{A}}) = \sigma_{\text{ess}}(D_{\mathbf{A}_0}),$$

for any $\mathbf{A} \in L^p_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, $p \in (2, \infty]$, with $\operatorname{curl} \mathbf{A} = B$.

Our next result state that eigenfunctions corresponding to the discrete spectrum of H are super-exponentially localized.

Theorem 2. *Assume that B and V satisfy (A1) and (A2) respectively and let $\mathbf{A} \in L^p_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, $p \in (2, \infty]$, with $\operatorname{curl} \mathbf{A} = B = B_0 + b$. Then, for any eigenfunction Ψ of $H = D_{\mathbf{A}} + V$ with $H\Psi = E\Psi$ and $E \in \mathbb{R} \setminus \sigma(D_{\mathbf{A}_0})$ the following holds: for every $r \in [2, \infty]$ and $\gamma > 0$ there exists an $R > 0$ such that*

$$\| \mathbb{1}_{\{|\mathbf{x}| \geq R\}} e^{\gamma |\mathbf{x}|} \Psi \|_r < \infty. \tag{5}$$

This theorem is proven in Section 5.

Remark 4. This type of results are known to hold for magnetic Schrödinger operators $(\mathbf{p} - \mathbf{A})^2 + B$. Our proof follows the ideas presented in [6]. In fact, since our operator is linear in $\mathbf{A}$, some parts of the argument are more straightforward. For instance, we do not require that $b \in C^1(\mathbb{R}^2; \mathbb{R})$ decays in the C^1-norm as done in [6].

Remark 5. One essential ingredient in the proof of Theorem 2 is the explicit knowledge of the Green function G_0 of $D_{\mathbf{A}_0}$. This is calculated in Appendix A.

In order to obtain Gaussian decay we make further assumptions on (B, V). Let $T = \mathbb{R}/(2\pi\mathbb{Z})$ and let $v = v(r, \theta), (r, \theta) \in \mathbb{R}^+ \times T$ be the potential V written in polar coordinates. We assume the following.

(A3) B is radially symmetric, i.e., $b(\mathbf{x}) = b(r)$, $r = |\mathbf{x}|$.

(A4) For any $(r,\theta) \in \mathbb{R}^+ \times T$ the mapping $\mathbb{R} \ni a \mapsto v(r,\theta + a) \stackrel{\text{def}}{=} v_a(r,\theta)$ has an analytic continuation $\tilde{v}_z(r,\theta)$ to $\mathbb{C}$. Moreover, for any $\tau > 0$ there exist a $p \in (2,\infty]$ and a real-valued function $u_\tau \in L^p_{\text{loc}}(\mathbb{R}^+ \times T, r dr\, d\theta)$ such that $\|\mathbb{1}_{\{r>n\}} u_\tau\|_\infty \to 0$ as $n \to \infty$ and

$$|\tilde{v}_z(r,\theta)| \leq u_\tau(r,\theta),$$

for any $(r,\theta) \in \mathbb{R}^+ \times T$ and $z \in S_\tau \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \tau\}$.

(A5) v is differentiable with respect to r and $\mathbb{R} \ni a \mapsto \partial_r v(r,\theta + a)$ can be analytically continued to $\partial_r \tilde{v}_z(r,\theta)$ on $\mathbb{C}$. Moreover, there exist a $\rho > 0$ such that for any $\tau > 0$ there is $\kappa_\tau > 0$ such that $|\mathbb{1}_{\{r>\rho\}} \partial_r v_z(r,\theta)| \leq \kappa_\tau$ for any $(r,\theta) \in \mathbb{R}^+ \times T$ and $z \in S_\tau$.

Theorem 3. *Assume that B satisfies (A1) and (A3) and V satisfies (A2), (A4), and (A5). Let $\mathbf{A} \in L^p_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, $p \in (2,\infty]$, with $\operatorname{curl} \mathbf{A} = B$. Then, for any eigenfunction Ψ of $H = D_{\mathbf{A}} + V$ with $H\Psi = E\Psi$ and $E \in \mathbb{R} \setminus \sigma(D_{\mathbf{A}_0})$ the following holds: For every $0 < \alpha < 1$, we have*

$$\|e^{\alpha B_0/4|\mathbf{x}|^2} \Psi\|_2 < \infty.$$

This theorem is proven in Section 6.

Remark 6. The analyticity assumption (A4) on the angular variable of V implies, by a Paley–Wiener argument, exponential decay of the Fourier modes of the potential in its angular momentum decomposition; see (40), (41), and (50) below. Assumption (A5) is similar to (A4) but for the radial derivative of the potential.

Remark 7. The first proof of Gaussian localization for magnetic Schrödinger operators using assumptions like (A4), but not (A5), was given in [10]. In addition, an example of a potential decaying at infinity for which the corresponding ground state decays slower than a Gaussian is also given in [10]. The proof in [10] is based on a generalized Feynman–Kac formula. An alternative proof using Agmon-type estimates with localizations in space and angular momentum was given in [19]. A variation of the method in [19] was used in [29] to treat the general n-dimensional case, again for magnetic Schrödinger operators. Our proof follows the ideas developed in [19]. However, it turns out to be more involved since our operator is not bounded from below. To overcome this difficulty we square the Dirac operator (or parts of it). This is the reason why (A5) is used in our setting.

The article is organized as follows. In Section 3 we review some essentially well known facts about magnetic Dirac operators. Sections 4, 5, and 6 are devoted to the proofs of theorems 1, 2, and 3 respectively. The article ends with an appendix containing some useful technical results.

3. Preliminaries

3.1. Essential self-adjointness. Throughout this article we consider magnetic potentials $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and electric potentials $V \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$, $p,q \in (2,\infty]$. In order to show essential self-adjointness of the operators H and $D_{\mathbf{A}}$ defined in (1) and (2) it suffices to prove that

$$H_R\varphi \stackrel{\text{def}}{=} \boldsymbol{\sigma}\cdot(\mathbf{p} - \mathbb{1}_{\{|\mathbf{x}|\le R\}}\mathbf{A})\varphi + \mathbb{1}_{\{|\mathbf{x}|\le R\}}V\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^2;\mathbb{C}^2),$$

is essentially self-adjoint for every $R > 0$; see [5]. Using that for $f \in L^p(\mathbb{R}^2;\mathbb{C})$ and $2 < p < \infty$

$$f(\mathbf{x})(\mathbf{p}^2+1)^{-1/2}$$

is a compact operator (see [28], Theorem 4.1) we get that $\mathbb{1}_{\{|\mathbf{x}|\le R\}}(V - \boldsymbol{\sigma}\cdot\mathbf{A})$ is a relative compact perturbation of $D_{\mathbf{0}}$. This shows essential self-adjointness of H_R, since $D_{\mathbf{0}}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$.

3.2. Gauge invariance. Let $\mathbf{A}, \widehat{\mathbf{A}} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, $2 < p < \infty$, be two vector potentials with

$$\operatorname{curl}\mathbf{A} = \operatorname{curl}\widehat{\mathbf{A}}$$

in the sense of distributions. According to [17] there is a gauge function $\widehat{\Phi} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ such that

$$\mathbf{A} = \widehat{\mathbf{A}} + \nabla\widehat{\Phi}.$$

It follows, for any electric potential $V \in L^q_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$, $q \in (2,\infty]$, that

$$(D_{\mathbf{A}} + V) = e^{\mathrm{i}\widehat{\Phi}}(D_{\widehat{\mathbf{A}}} + V)e^{-\mathrm{i}\widehat{\Phi}}.$$

In particular, $e^{\mathrm{i}\widehat{\Phi}}(D_{\widehat{\mathbf{A}}} + V)e^{-\mathrm{i}\widehat{\Phi}}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$. This can be seen as follows. Note that

$$\mathcal{D}(e^{\mathrm{i}\widehat{\Phi}}(D_{\widehat{\mathbf{A}}} + V)e^{-\mathrm{i}\widehat{\Phi}}) = \{f \in L^2(\mathbb{R}^2;\mathbb{C}^2) \colon e^{-\mathrm{i}\widehat{\Phi}}f \in \mathcal{D}(D_{\widehat{\mathbf{A}}} + V)\}.$$

Pick functions $\eta, \eta' \in C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and a sequence $(\widehat{\Phi}_m)_{m\in\mathbb{N}}$ in $C^\infty(\mathbb{R}^2;\mathbb{R})$ with $\widehat{\Phi}_m \to \widehat{\Phi}$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ (and hence in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$) as $m \to \infty$. Then,

$$\begin{aligned}
\langle (D_{\widehat{\mathbf{A}}} + V)\eta', e^{-\mathrm{i}\widehat{\Phi}}\eta\rangle &= \lim_{m\to\infty}\langle (D_{\widehat{\mathbf{A}}} + V)\eta', e^{-\mathrm{i}\widehat{\Phi}_m}\eta\rangle \\
&= \lim_{m\to\infty}\langle e^{\mathrm{i}\widehat{\Phi}_m}\eta', (D_{\widehat{\mathbf{A}}} + V)\eta\rangle - \lim_{m\to\infty}\langle e^{\mathrm{i}\widehat{\Phi}_m}\eta', \boldsymbol{\sigma}\cdot\nabla\widehat{\Phi}_m\eta\rangle \\
&= \langle e^{\mathrm{i}\widehat{\Phi}}\eta', (D_{\widehat{\mathbf{A}}} + V)\eta\rangle - \langle e^{\mathrm{i}\widehat{\Phi}}\eta', \boldsymbol{\sigma}\cdot\nabla\widehat{\Phi}\eta\rangle.
\end{aligned}$$

Since η' is an arbitrary element of a core of $D_{\widehat{\mathbf{A}}} + V$, it follows that

$$e^{-\mathrm{i}\widehat{\Phi}}\eta \in \mathcal{D}(D_{\widehat{\mathbf{A}}} + V)$$

and

$$(D_{\widehat{\mathbf{A}}} + V)e^{-\mathrm{i}\,\widehat{\Phi}}\eta = e^{-\mathrm{i}\,\widehat{\Phi}}(D_{\widehat{\mathbf{A}}} + V - \boldsymbol{\sigma}\cdot\nabla\widehat{\Phi})\eta,$$

which implies that

$$e^{\mathrm{i}\,\widehat{\Phi}}(D_{\widehat{\mathbf{A}}} + V)e^{-\mathrm{i}\,\widehat{\Phi}}\eta = (D_{\mathbf{A}} + V)\eta\,, \quad \eta \in C_0^\infty(\mathbb{R}^2;\mathbb{C}^2).$$

Due to the essential self-adjointness of $D_{\mathbf{A}} + V$ we deduce that $e^{\mathrm{i}\,\widehat{\Phi}}(D_{\widehat{\mathbf{A}}} + V)e^{-\mathrm{i}\,\widehat{\Phi}}$ is also essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and that the two operators coincide.

3.3. Supersymmetry. For $\mathbf{A} = (A_1, A_2)$ with $A_j \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, $p \in (2,\infty]$, $j = 1, 2$, we define the following two operators

$$d_1\varphi = [(p_1 - A_1) + \mathrm{i}\,(p_2 - A_2)]\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^2;\mathbb{C}),$$

$$d_2\varphi = [(p_1 - A_1) - \mathrm{i}\,(p_2 - A_2)]\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^2;\mathbb{C}).$$

Clearly, we have that

$$D_{\mathbf{A}} \restriction_{C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)} = \begin{pmatrix} 0 & d_2 \\ d_1 & 0 \end{pmatrix}.$$

Since $D_{\mathbf{A}} \restriction_{C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)}$ is essentially self-adjoint it follows that d_1 and d_2 are closable; see [30], Section 5.2.2. In addition, setting $d \overset{\mathrm{def}}{=} \overline{d_1}$ one finds that $d^* = \overline{d_2}$ and

$$D_{\mathbf{A}} = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \quad \text{on } \mathcal{D}(D_{\mathbf{A}}) = \mathcal{D}(d) \oplus \mathcal{D}(d^*). \tag{6}$$

It is known that dd^* and d^*d are self-adjoint with domains $\mathcal{D}(dd^*) = \{\varphi \in \mathcal{D}(d^*)\colon d^*\varphi \in \mathcal{D}(d)\}$ and $\mathcal{D}(d^*d) = \{\varphi \in \mathcal{D}(d)\colon d\varphi \in \mathcal{D}(d^*)\}$. Moreover, there is a unitary map S from $\mathrm{Ker(dd^*)}^\perp$ to $\mathrm{Ker(d^*d)}^\perp$, such that

$$dd^* \restriction_{\mathrm{Ker(dd^*)}^\perp} = S^* d^* d \restriction_{\mathrm{Ker(d^*d)}^\perp} S. \tag{7}$$

Let us note that we can block-diagonalize $D_{\mathbf{A}}$ using the Foldy–Wouthuysen transformation. Setting

$$a_+ = \begin{cases} 1/\sqrt{2} & \text{on } \mathrm{Ker}(D_{\mathbf{A}})^\perp, \\ 1 & \text{on } \mathrm{Ker}(D_{\mathbf{A}}), \end{cases}$$

$$a_- = \begin{cases} 1/\sqrt{2} & \text{on } \mathrm{Ker}(D_{\mathbf{A}})^\perp, \\ 0 & \text{on } \mathrm{Ker}(D_{\mathbf{A}}), \end{cases}$$

we define the Foldy–Wouthuysen transformation as

$$U = a_+ + \sigma_3 \mathrm{sgn}(D_{\mathbf{A}})a_-\,,$$

where $\mathrm{sgn}(D_{\mathbf{A}}) = D_{\mathbf{A}}/|D_{\mathbf{A}}|$ on $\mathrm{Ker}(D_{\mathbf{A}})^{\perp}$ and equals zero on $\mathrm{Ker}(D_{\mathbf{A}})$ and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The unitarity of the above transformation can be easily verified observing that $\varphi \in \mathrm{Ker}(D_{\mathbf{A}}) \Leftrightarrow \sigma_3\varphi \in \mathrm{Ker}(D_{\mathbf{A}})$ and that $\sigma_3 \mathrm{sgn}(D_{\mathbf{A}}) = -\mathrm{sgn}(D_{\mathbf{A}})\sigma_3$. The latter relation holds since $\sigma_3 D_{\mathbf{A}} = -D_{\mathbf{A}}\sigma_3$ and $\sigma_3|D_{\mathbf{A}}| = |D_{\mathbf{A}}|\sigma_3$. A direct computation yields

$$UD_{\mathbf{A}}U^* = \begin{pmatrix} \sqrt{d^*d} & 0 \\ 0 & -\sqrt{dd^*} \end{pmatrix}. \tag{8}$$

Equation (7) and (8) imply the following statement.

Proposition 1. *Let* $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ *for some* $p \in (2, \infty]$. *Then, the spectrum of* $D_{\mathbf{A}}$ *is symmetric with respect to zero and*

$$\sigma_\sharp(D_{\mathbf{A}}) \cap (0, \infty) = \sigma_\sharp(\sqrt{d^*d}) \setminus \{0\}, \quad \sharp \in \{\mathrm{pp}, \mathrm{d}, \mathrm{ess}\}.$$

4. The spectrum of $D_{\mathbf{A}}$

The aim of this section is to show Theorem 1. An important ingredient is the study of the essential spectrum of $D_{\mathbf{A}}$. In order to do that we modify an argument from [14] obtaining Lemma 1 below. We combine this with a result from [24] on the infiniteness of zero modes for Pauli operators (see Lemma 2 below). The proof of the theorem is then a consequence of Lemmata 2 and 3.

In the following discussion we assume that $B = B_0 + b$ with $B_0 > 0$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R})$ such that $|b|^{1/2}$ is relative $\sqrt{\mathbf{p}^2 + 1}$-compact. Let $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, $p \in (2, \infty]$, with $\mathrm{curl}\,\mathbf{A} = B$. We start by observing that, for $\varphi \in C_0^\infty(\mathbb{R}^2; \mathbb{C})$,

$$\begin{aligned} \langle d^*\varphi, d^*\varphi \rangle &= \sum_{j=1}^{2} \|(p_j - A_j)\varphi\|^2 + \langle \varphi, B\varphi \rangle, \\ \langle d\varphi, d\varphi \rangle &= \sum_{j=1}^{2} \|(p_j - A_j)\varphi\|^2 - \langle \varphi, B\varphi \rangle \end{aligned} \tag{9}$$

holds. This implies the commutator relation

$$\langle \varphi, [d, d^*]\varphi \rangle \stackrel{\mathrm{def}}{=} \langle d^*\varphi, d^*\varphi \rangle - \langle d\varphi, d\varphi \rangle = 2\langle \varphi, B\varphi \rangle, \quad \varphi \in C_0^\infty(\mathbb{R}^2; \mathbb{C}). \tag{10}$$

The idea in [14] is to use this commutator to study the essential spectrum of dd^* and d^*d. In order to extend this identity we define these operators as quadratic forms and show that $\mathcal{Q}(b) \supset \mathcal{Q}(d^*d) = \mathcal{Q}(dd^*)$ and $|b|^{1/2}(d^*d+1)^{-1/2}$ is a compact operator. Here $\mathcal{Q}(\cdot)$ is used to denote the form domain.

Let us define

$$q_1(\varphi,\varphi) = q_1[\varphi] \stackrel{\text{def}}{=} \|d\varphi\|^2,$$

$$q_2(\varphi,\varphi) = q_2[\varphi] \stackrel{\text{def}}{=} \|d^*\varphi\|^2,$$

with form domains $\mathcal{Q}(q_1) = \mathcal{D}(d)$ and $\mathcal{Q}(q_2) = \mathcal{D}(d^*)$. Since d and d^* are closed (see Subsection 3.3) we have that q_1 and q_2 are closed and positive. Thus, associated to q_j, $j = 1, 2$, there is a unique self-adjoint operator T_j characterized as follows:

$$\begin{aligned} \langle\psi, T_j\varphi\rangle = q_j(\psi,\varphi), \quad \psi \in \mathcal{Q}(q_j), \varphi \in \mathcal{D}(T_j), \\ \mathcal{D}(T_j) = \{\varphi \in Q(q_j) \,|\, \exists \eta \in L^2(\mathbb{R}^2;\mathbb{C}), \forall \psi \in \mathcal{C}, q_j(\psi,\varphi) = \langle\psi,\eta\rangle\}, \end{aligned} \tag{11}$$

where $\mathcal{C}$ is any form core of q_j. It is easy to check using (11) that in fact $T_1 = d^*d$ and $T_2 = dd^*$. Note that since the restrictions of d and d^* to $C_0^\infty(\mathbb{R}^2,\mathbb{C})$ are closable $C_0^\infty(\mathbb{R}^2,\mathbb{C})$ is a form core for q_1 and q_2. We define yet another quadratic form. For $\varphi \in C_0^\infty(\mathbb{R}^2,\mathbb{C})$ we set

$$\tilde{q}_3[\varphi] \stackrel{\text{def}}{=} \sum_{j=1}^{2} \|(p_j - A_j)\varphi\|^2.$$

It is known (see [27]) that $\tilde{q}_3$ is closable and we denote its closure by q_3. Its associated self-adjoint operator $H_S \stackrel{\text{def}}{=} (\mathbf{p} - \mathbf{A})^2$ is the usual magnetic Schrödinger operator. Recall that $|b|^{1/2}$ is relative $\sqrt{\mathbf{p}^2+1}$-compact. Using the diamagnetic inequality for $|\mathbf{p} - \mathbf{A}|$ (see e.g. [11]) and arguing as in [2], Theorem 2.6, we conclude that $\mathcal{Q}(q_3) = \mathcal{D}(H_S^{1/2}) \subset \mathcal{D}(|b|^{1/2})$ and that $|b|^{1/2}$ is relative $H_S^{1/2}$-compact. Thus, the quadratic form

$$\beta[\varphi] \stackrel{\text{def}}{=} B_0\|\varphi\|^2 + \langle \operatorname{sgn}(b)|b|^{1/2}\varphi, |b|^{1/2}\varphi\rangle$$

is in absolute value bounded with respect to q_3 with bound 0. In particular,

$$q_3^{\pm}[\varphi] \stackrel{\text{def}}{=} q_3[\varphi] \pm \beta[\varphi], \quad \varphi \in \mathcal{Q}(q_3),$$

is closed. Observing that by (9) we have that $q_2 \restriction_{C_0^\infty} = q_3^+ \restriction_{C_0^\infty}$ and $q_1 \restriction_{C_0^\infty} = q_3^- \restriction_{C_0^\infty}$ and using that $C_0^\infty(\mathbb{R}^2;\mathbb{C})$ is a form core for q_1, q_2, q_3 and $q_3^\pm$ we conclude that $\mathcal{Q}(q_1) = \mathcal{Q}(q_2) = \mathcal{Q}(q_3) \equiv \mathcal{Q}$ and $q_1 = q_3^-$ and $q_2 = q_3^+$. Moreover,

$$\begin{aligned} dd^* &= (\mathbf{p} - \mathbf{A})^2 + B, \\ d^*d &= (\mathbf{p} - \mathbf{A})^2 - B, \end{aligned} \tag{12}$$

in the sense of quadratic forms on $\mathcal{Q}$ and hence the commutator formula (10) extends to $\mathcal{Q}$.

Lemma 1. *Let* $B = B_0 + b$ *with* $B_0 > 0$ *and* $|b|^{1/2} \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ *be relative* $\sqrt{\mathbf{p}^2+1}$*-compact. Let* $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$*,* $p \in (2,\infty]$*, with* $\operatorname{curl}\mathbf{A} = B$*. Then, either one of the following statements holds:*

(i) $\sigma_{\mathrm{ess}}(d^*d) = \emptyset$*;*

(ii) $\sigma_{\mathrm{ess}}(d^*d) = \{2B_0 n : n \in \mathbb{N}_0\}$ *and* $\sigma_{\mathrm{ess}}(dd^*) = \{2B_0 n : n \in \mathbb{N}\}$*.*

In addition, if V *satisfies* (A2) *then* V *is relative* $D_{\mathbf{A}}$*-compact and in particular* $\sigma_{\mathrm{ess}}(D_{\mathbf{A}}) = \sigma_{\mathrm{ess}}(D_{\mathbf{A}} + V)$*.*

Remark 8. Note that our assumption on B are satisfied if B fulfills (A1). Indeed, in this case $|b|^{1/2}(\mathbf{p}^2+1)^{-1/2}$ is compact by Lemma 12 in Appendix B.

Moreover, note that if $B \in L^q_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ for some $q > 1$ we can always find $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ for some $p \in (2,\infty]$. In order to see this define h to be a solution of

$$\Delta h = B. \tag{13}$$

A local solution to this equation is given by the Newton potential h_N of B. We know that $h_N \in W^{2,q}(\Omega)$ by the Calderón–Zygmund inequality, where $\Omega \subset \mathbb{R}^2$ is a bounded domain (see e.g. [12], Section 9.4). This property extends to any solution h of (13) since $h - h_N$ is harmonic on Ω. Therefore, $h \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^2)$. Now one can define $\mathbf{A} \stackrel{\mathrm{def}}{=} (-\partial_2 h, \partial_1 h)$. Clearly, $A_j \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2)$. By standard Sobolev inequalities one obtains that $A_j \in L^t_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ for some $2 < t < \infty$ if $q \in (1,2]$ and $A_j \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ if $q > 2$.

Proof. First note that for any $\lambda \geq 0$ the operator $(d^*d + 2B_0 + \lambda)^{-1/2}$ maps $L^2(\mathbb{R}^2;\mathbb{C})$ onto $\mathcal{D}(\sqrt{d^*d})$ which equals $\mathcal{Q}$ and $\mathcal{D}(H_S^{1/2})$. Thus, by the closed graph theorem, the operator $(H_S+1)^{1/2}(d^*d + 2B_0 + \lambda)^{-1/2}$ is bounded. In particular,

$$|b|^{1/2}(d^*d + 2B_0 + \lambda)^{-1/2} = |b|^{1/2}(H_S+1)^{-1/2}(H_S+1)^{1/2}(d^*d + 2B_0 + \lambda)^{-1/2}$$

is compact. Hence, the operator

$$T(\lambda) \stackrel{\mathrm{def}}{=} \overline{(d^*d + 2B_0 + \lambda)^{-1/2}\operatorname{sgn}(b)|b|^{1/2}}\,|b|^{1/2}(d^*d + 2B_0 + \lambda)^{-1/2}$$

is also compact. It is easy to see that $\lambda > 0$ can be chosen so large that $\|T(\lambda)\| < 1$. For such λ's we have, according to the resolvent formula for operators defined as quadratic forms (see [26]), that

$$(d^*d + 2B_0 + 2b + \lambda)^{-1} = (d^*d + 2B_0 + \lambda)^{-1/2}(1 + T(\lambda))^{-1}(d^*d + 2B_0 + \lambda)^{-1/2}.$$

Note that the inverse of $1 + T(\lambda)$ is well defined as a geometric expansion. Since $(1+T(\lambda))^{-1} - 1$ is compact, we conclude that the resolvent difference between

$d^*d + 2B_0 + 2b + \lambda$ and $d^*d + 2B_0 + \lambda$ is also compact. Therefore, by Weyl's theorem, the two operators have the same essential spectrum. Using this and (12) we deduce that

$$\sigma_{\text{ess}}(dd^*) = \sigma_{\text{ess}}(d^*d + 2B_0 + 2b) = \sigma_{\text{ess}}(d^*d + 2B_0). \tag{14}$$

The latter equality and Equation (7) imply (here we follow [14])

$$\begin{aligned} S &\overset{\text{def}}{=} \sigma_{\text{ess}}(d^*d), \quad S \subset [0, \infty), \\ S \setminus \{0\} &= S + 2B_0. \end{aligned} \tag{15}$$

Assume now that $S \neq \emptyset$, then it is easy to see from (15) that $0 \in S$ and hence $2B_0 n \in S$, $n \in \mathbb{N}_0$. Note also that no other points can belong to S. Hence, using (14) we get that $\sigma_{\text{ess}}(dd^*) = 2B_0 n$, $n \in \mathbb{N}$.

Now, assume that V fulfills (A2). Then, V is relative $\sqrt{\mathbf{p}^2 + 1}$-compact (see Lemma 12 in Appendix B). It follows by the diamagnetic inequality that V is relative $H_S^{1/2}$-compact and consequently (arguing as before for b) $\mathcal{D}(V) \supset \mathcal{Q}$ and the operators $V(dd^* + \lambda^2)^{-1/2}$ and $V(d^*d + \lambda^2)^{-1/2}$ are compact for any $\lambda \neq 0$. From these considerations follow that $V(D_{\mathbf{A}} - \mathrm{i}\lambda)^{-1}$ is compact, since the identity

$$\begin{aligned} &(D_{\mathbf{A}} - \mathrm{i}\lambda)^{-1} \\ &\quad = (D_{\mathbf{A}}^2 + \lambda^2)^{-1/2}[(D_{\mathbf{A}}^2 + \lambda^2)^{-1/2}(D_{\mathbf{A}} + \mathrm{i}\lambda)] \\ &\quad = \begin{pmatrix} (d^*d + \lambda^2)^{-1/2} & 0 \\ 0 & (dd^* + \lambda^2)^{-1/2} \end{pmatrix} \cdot [(D_{\mathbf{A}}^2 + \lambda^2)^{-1/2}(D_{\mathbf{A}} + \mathrm{i}\lambda)] \end{aligned}$$

holds and the operator in [...] is bounded. Therefore,

$$\sigma_{\text{ess}}(D_{\mathbf{A}} + V) = \sigma_{\text{ess}}(D_{\mathbf{A}}). \qquad \square$$

We note that if b satisfies (A1) then $\text{Ker}(d^*d)$ is infinitely degenerated. Indeed, this follows from the fact that

$$\begin{aligned} \int_{\mathbb{R}^2} [B]_+ d^2x &= \infty, \\ \int_{\mathbb{R}^2} [B]_- d^2x &< \infty, \end{aligned}$$

(where $[f]_+$ and $[f]_-$ are the positive and negative parts of f) which shows that $B = B_0 + b$ fulfills the conditions of [24], Corollary 3.4. In particular, we know that

$$\text{Ker}(d^*d) = \{\omega e^{-h} \,|\, \omega e^{-h} \in L^2(\mathbb{R}^2; \mathbb{C}),\ \omega \text{ is analytic in } x_1 + \mathrm{i}x_2\},$$

where h is a solution of the equation $\Delta h = B$; see [24]. Therefore, we get the following result.

Lemma 2. *Assume that B satisfies* (A1) *and let* $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, $p \in (2,\infty]$, *with* $\operatorname{curl}\mathbf{A} = B$. *Then,*

$$\begin{aligned}\sigma_{\mathrm{ess}}(d^*d) &= \{2B_0 n \,|\, n \in \mathbb{N}_0\},\\ \sigma_{\mathrm{ess}}(dd^*) &= \{2B_0 n \,|\, n \in \mathbb{N}\}.\end{aligned} \tag{16}$$

In particular,

$$\sigma_{\mathrm{ess}}(D_{\mathbf{A}}) = \sigma_{\mathrm{ess}}(D_{\mathbf{A}_0}) = \{l_n \,|\, n \in \mathbb{Z}\}.$$

Moreover, 0 *is an isolated point of* $\sigma(D_{\mathbf{A}})$ *and* $\sigma(d^*d)$.

Proof. Due to our previous discussion we see that $0 \in \sigma_{\mathrm{ess}}(d^*d)$. This combined with Lemma 1 imply (16). That 0 is an isolated point of $\sigma(d^*d)$ follows by noting that, since $0 \notin \sigma_{\mathrm{ess}}(dd^*)$, 0 is neither an accumulation point of $\sigma(dd^*)$ nor of $\sigma(d^*d)$. The statements on $\sigma(D_{\mathbf{A}})$ are now a consequence of Proposition 1. □

Lemma 3. *Assume that B satisfies* (A1) *and let* $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, $p \in (2,\infty]$, *with* $\operatorname{curl}\mathbf{A} = B$. *Then, we have*

(a) *If* $b \le 0$ *and strictly negative on some open set, then*

$$\dim(\operatorname{Ran}(\mathbb{1}_{(0,\sqrt{2B_0})}(D_{\mathbf{A}}))) = \dim(\operatorname{Ran}(\mathbb{1}_{(-\sqrt{2B_0},0)}(D_{\mathbf{A}}))) = \infty.$$

(b) *If* $b \ge 0$ *then*

$$\dim(\operatorname{Ran}(\mathbb{1}_{(0,\sqrt{2B_0})}(D_{\mathbf{A}}))) = \dim(\operatorname{Ran}(\mathbb{1}_{(-\sqrt{2B_0},0)}(D_{\mathbf{A}}))) = 0.$$

Proof. We may choose $\mathbf{A} \stackrel{\text{def}}{=} (-\partial_2 h, \partial_1 h)$ where h is a solution of $\Delta h = B$. Due to Remark 8 we know that $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ for some $p > 2$.

Part (a). Let Ω be an open set with $b \restriction \Omega < 0$. Recall that there are infinitely many functions ω, analytic in $x_1 + \mathrm{i}\,x_2$, with $\psi \stackrel{\text{def}}{=} \omega e^{-h} \in \operatorname{Ker}(d^*d)$. For such ψ we have, using (12),

$$\langle \psi, dd^*\psi\rangle = 2\langle \psi, B\psi\rangle \le 2B_0\|\psi\|^2 + 2\int_\Omega b(\mathbf{x})|\psi(\mathbf{x})|^2 d\mathbf{x} < 2B_0\|\psi\|^2, \tag{17}$$

where in the last inequality we use the fact that ψ can not vanish on Ω. Let $(\psi_n)_{n\in\mathbb{N}}$ be an orthonormal system such that $\psi_n \stackrel{\text{def}}{=} e^{-h}\omega_n \in \operatorname{Ker} d^*d$ with ω_n analytic in $x_1 + \mathrm{i}\,x_2$. For $N \in \mathbb{N}$ define the self-adjoint matrix $M_N \stackrel{\text{def}}{=} (\langle \psi_n, dd^*\psi_m\rangle)_{1\le n,m\le N}$. It follows from (17) that $M_N < 2B_0$. The Rayleigh–Ritz principle implies

$$0 \le \mu_n(dd^*) \le \mu_n(M_N) < 2B_0, \quad n = 1,\dots,N,$$

where we write

$$\mu_n(T) \overset{\text{def}}{=} \sup_{\varphi_1,\dots,\varphi_{n-1}} \inf_{\substack{\psi\in\mathrm{span}\{\varphi_1,\dots,\varphi_{n-1}\}^\perp \\ \|\psi\|=1,\psi\in Q(T)}} \langle\psi, T\psi\rangle$$

for some self-adjoint operator T. Since N is arbitrary the mini-max principle implies that $\dim(\mathrm{Ran}(\mathbb{1}_{[0,\sqrt{2B_0})}(dd^*))) = \infty$. It follows that $\dim(\mathrm{Ran}(\mathbb{1}_{(0,\sqrt{2B_0})}(dd^*))) = \infty$, for $0 \notin \sigma_{\mathrm{ess}}(dd^*)$ by Lemma 2. The claim is now a consequence of Proposition 1 and (7).

Part (b). In this case we have that $dd^* \geq 2B_0$, since $dd^* - d^*d = 2B \geq 2B_0$. Thus, the claim follows now from Proposition 1 and (7). □

5. Super-exponential localization

The proof of Theorem 2 follows the ideas developed in [6]. An essential ingredient is that, by means of suitable local gauge transformations on certain regions outside a big ball of radius n centered at the origin, one can replace the operator $D_{\mathbf{A}}$ by a Dirac operator $D_{\mathbf{A}_n}$ with $\mathbf{A}_n = \mathbf{A}_0 + \mathbf{a}^n$, where $\mathbf{a}^n$ is a magnetic vector potential of a magnetic field b_n satisfying $\lim_{n\to\infty} \|b_n\|_\infty = 0$. The advantage of this is that we can obtain explicit L^p estimates (see Lemma 4 below) for the resolvents of $D_{\mathbf{A}_n}$, conjugated with exponential weights. These estimates can be derived using a certain resolvent expansion, see (35), in combination with an explicit expression for the Green kernel of $D_{\mathbf{A}_0}$ that can be found in Appendix A below.

Before stating these L^p estimates let us fix some notation. For $p, q \in [1, \infty]$ we denote by $\mathcal{B}(p, q)$ the space of bounded operators from $L^p(\mathbb{R}^2; \mathbb{C}^2)$ to $L^q(\mathbb{R}^2; \mathbb{C}^2)$ and write, for $T \in \mathcal{B}(p, q)$,

$$\|T\|_{p,q} \overset{\text{def}}{=} \|T\|_{\mathcal{B}(p,q)}. \tag{18}$$

Let $\gamma \geq 0$ and $\mathbf{u} \in \mathbb{R}^2$ with $|\mathbf{u}| = 1$. We define the exponential weight function as

$$F(\mathbf{x}) \overset{\text{def}}{=} \gamma\, \mathbf{u}\cdot\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2.$$

Let b_n be a magnetic field with $\lim_{n\to\infty} \|b_n\|_\infty = 0$ and $\mathbf{a}^n$ be the associated vector potential in the transversal gauge, i.e.,

$$\mathbf{a}^n(\mathbf{x}) \overset{\text{def}}{=} \int_0^1 b_n(s\mathbf{x}) \wedge \mathbf{x}\, s\, ds, \tag{19}$$

where we write $a \wedge \mathbf{v} \overset{\text{def}}{=} a(-v_2, v_1)$ for $a \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$. The proof of the lemma below can be found at the end of this section.

Lemma 4. *Let $V_n \in L^\infty(\mathbb{R}^2;\mathbb{R})$, $n \in \mathbb{N}$, be a family of electric potentials satisfying*

$$\lim_{n\to\infty} \|V_n\|_\infty = 0.$$

For any $n \in \mathbb{N}$ define the family of self-adjoint operators $D_{\mathbf{A}_n} + V_n$, where $\mathbf{A}_n \stackrel{def}{=} \mathbf{A}_0 + \mathbf{a}^n$ and $\mathbf{a}^n$ is given in (19). Let $z \in \mathbb{R} \setminus \sigma(D_{\mathbf{A}_0})$ and $q, r \in [1,\infty]$ be such that $1 + \frac{1}{r} - \frac{1}{q} = \frac{1}{p}$ for some $p \in [1,2)$. Then, there exists $N > 0$ such that, for all $n > N$, $z \notin \sigma(D_{\mathbf{A}_n} + V_n)$ and

$$e^F (D_{\mathbf{A}_n} + V_n - z)^{-1} e^{-F} \in \mathcal{B}(q,r). \tag{20}$$

In what follows we apply the above result to show Theorem 2.

Proof of Theorem 2. For $n \in \mathbb{N}$ and $\mathbf{u} \in \mathbb{R}^2$ with $|\mathbf{u}| = 1$ set

$$\Omega_n = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{u}\cdot\mathbf{x} > n\}.$$

For $j \in \{1,2,3\}$ define $\chi_j \in C^\infty(\mathbb{R}^2;[0,1])$ with $\chi_j = 0$ on $\mathbb{R}^2 \setminus \Omega_{jn}$ and $\chi_j = 1$ on $\Omega_{(j+1)n}$. We choose n so large that

$$\|b\|_{L^\infty(\Omega_n)} < \infty.$$

Since $b \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$, $q > 1$, we find a vector potential $\mathbf{a} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, $p > 2$, with $\operatorname{curl}\mathbf{a} = b$ (see Remark 8). Define, for $\mathbf{x} \in \mathbb{R}^2$,

$$\mathbf{a}^n(\mathbf{x}) = \int_0^1 b_n(s\mathbf{x}) \wedge \mathbf{x}\, s ds,$$

where $b_n \stackrel{\mathrm{def}}{=} \mathbb{1}_{\Omega_n} b \in L^\infty(\mathbb{R}^2)$. Observe that

$$\operatorname{curl}\mathbf{a} = \operatorname{curl}\mathbf{a}^n \quad \text{on } \Omega_n,$$

that Ω_n is simply connected, and that $\mathbf{a}^n, \mathbf{a} \in L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ for some $p > 2$. Therefore, there exists a gauge function $\tilde\Phi_n \in W^{1,p}_{\mathrm{loc}}(\Omega_n)$ such that (see [17], Lemma 1.1)

$$\nabla\tilde\Phi_n = \mathbf{a} - \mathbf{a}^n \quad \text{on } \Omega_n. \tag{21}$$

By multiplying $\tilde\Phi_n$ with a C^∞- cutoff function we may define a $\Phi_n \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ that coincides with $\tilde\Phi_n$ on Ω_{2n}. In particular, we find that

$$\nabla\Phi_n = \mathbf{a} - \mathbf{a}^n \quad \text{on } \Omega_{2n}. \tag{22}$$

Define now $V_n \stackrel{\mathrm{def}}{=} \chi_1 V$ and observe that $\|V_n\|_\infty \to 0$ as $n \to \infty$. Then we get, for any $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and $j \in \{2,3\}$, using (22) and the identity $\chi_j = \chi_1\chi_j$,

$$\begin{aligned}\chi_j(D_{\mathbf{A}} + V)\eta &= \chi_j(D_{\mathbf{A}_0} - \boldsymbol{\sigma}\cdot\mathbf{a} + V_n)\eta\\ &= \chi_j(D_{\mathbf{A}_0} - \boldsymbol{\sigma}\cdot\nabla\Phi_n - \boldsymbol{\sigma}\cdot\mathbf{a}^n + V_n)\eta\\ &= \chi_j e^{\mathrm{i}\Phi_n}(D_{\mathbf{A}_0} - \boldsymbol{\sigma}\cdot\mathbf{a}^n + V_n)e^{-\mathrm{i}\Phi_n}\eta.\end{aligned}$$

Set $\mathbf{A}_n \stackrel{\text{def}}{=} \mathbf{A}_0 + \mathbf{a}^n$ and let Ψ be an eigenfunction of $D_{\mathbf{A}}$ with eigenvalue $E \notin \sigma(D_{\mathbf{A}_0})$. By the previous computation we obtain, for any $\eta \in \mathscr{C}_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and $j \in \{2,3\}$,

$$\langle\, e^{\mathrm{i}\,\Phi_n}(D_{\mathbf{A}_n} + V_n - E)e^{-\mathrm{i}\,\Phi_n}\eta \,|\, \chi_j\, \Psi\,\rangle = \langle\, (D_{\mathbf{A}} + V - E)\eta \,|\, \chi_j\, \Psi\,\rangle$$
$$= \langle\, \mathrm{i}\,\boldsymbol{\sigma}\cdot\nabla\chi_j\eta \,|\, \Psi\,\rangle.$$

This equality extends to any η in the domain of $e^{\mathrm{i}\,\Phi_n}(D_{\mathbf{A}_n} + V_n - E)e^{-\mathrm{i}\,\Phi_n}$, since $\mathscr{C}_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ is a core for $e^{\mathrm{i}\,\Phi_n}(D_{\mathbf{A}_n} + V_n - E)e^{-\mathrm{i}\,\Phi_n}$; Subsection 3.2. Clearly, $\mathbf{A}_n$ and V_n satisfy the assumptions of Lemma 4. Thus, for n sufficiently large, $E \notin \sigma(D_{\mathbf{A}_n})$ and we may replace η by $e^{\mathrm{i}\,\Phi_n}(D_{\mathbf{A}_n} + V_n - E)^{-1}e^{-\mathrm{i}\,\Phi_n}\eta$ obtaining that

$$\chi_j\Psi = -\mathrm{i}\,e^{\mathrm{i}\,\Phi_n}(D_{\mathbf{A}_n} + V_n - E)^{-1}e^{-\mathrm{i}\,\Phi_n}(\boldsymbol{\sigma}\cdot\nabla\chi_j)\Psi, \quad j \in \{2,3\}. \tag{23}$$

Observe that using (23) for $j = 2$ in combination with Lemma 4, with $q = 2, r = 3$, and $F = 0$, we obtain that

$$\chi_2\Psi \in L^3(\mathbb{R}^2;\mathbb{C}^2). \tag{24}$$

We use again (23), for large n, to get in addition that

$$e^F\chi_3\Psi = -\mathrm{i}\,e^{\mathrm{i}\Phi_n}(e^F(D_{\mathbf{A}_n} + V_n - E)^{-1}e^{-F})(e^{-\mathrm{i}\,\Phi_n}e^F(\boldsymbol{\sigma}\nabla\chi_3)\chi_2\Psi). \tag{25}$$

Since $\mathrm{supp}(\nabla\chi_3) \subset \Omega_{3n}\setminus\Omega_{4n}$ we find thanks to (24) that $e^{-\mathrm{i}\,\Phi_n}e^F(\boldsymbol{\sigma}\nabla\chi_3)\chi_2\Psi \in L^2(\mathbb{R}^2;\mathbb{C}^2)\cap L^3(\mathbb{R}^2;\mathbb{C}^2)$. Thus, we may apply Lemma 4 with $q = 3, r = \infty$ and $q = 2, r = 2$ to obtain the decay in the L^∞ and L^2 norms respectively for $n \geq n_0$ sufficiently large. We obtain the desired bound (5) from (25) by varying F over finitely many vectors $\mathbf{u}$. □

Proof of Lemma 4. Recall that the magnetic vector potential is given by $\mathbf{A}_n = \mathbf{A}_0 + \mathbf{a}^n$ where $\mathbf{a}^n$ is defined in (19).

A simple calculation shows that the vector potential

$$\mathbf{a}^n_{\mathbf{x}'}(\mathbf{x}) = \int_0^1 b_n(\mathbf{x}' + s(\mathbf{x}-\mathbf{x}')) \wedge (\mathbf{x}-\mathbf{x}')\, s ds, \quad \mathbf{x}' \in \mathbb{R}^2, \tag{26}$$

is also a vector potential of the magnetic field b_n. A crucial property of $\mathbf{a}^n_{\mathbf{x}'}$ is that

$$|\mathbf{a}^n_{\mathbf{x}'}(\mathbf{x})| \leq \|b_n\|_\infty \cdot |\mathbf{x}-\mathbf{x}'|, \quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^2. \tag{27}$$

Since $\mathrm{curl}\,\mathbf{a}^{\mathrm{n}}_{\mathbf{x}'} = \mathrm{curl}\,\mathbf{a}^{\mathrm{n}}$ there exists a function $\varphi_n : \mathbb{R}^2 \to \mathbb{R}$ with

$$\nabla_{\mathbf{x}}\varphi_n(\mathbf{x},\mathbf{x}') = \mathbf{a}^n(\mathbf{x}) - \mathbf{a}^n_{\mathbf{x}'}(\mathbf{x}). \tag{28}$$

We may further require that

$$\varphi_n(\mathbf{x},\mathbf{x}) = 0. \tag{29}$$

The proof of Lemma 4 is based upon L^p estimates for the resolvent expansion (35) below. We start by defining the relevant objects and list their L^p properties. For $z \in \mathbb{R}\setminus\sigma(D_{\mathbf{A}_0})$ let $G_0(\mathbf{x},\mathbf{x}',z)$ be a representation of the Green kernel of $(D_{\mathbf{A}_0}-z)^{-1}$ as 2×2-matrix. Due to (60) from Appendix A and the triangular inequality we obtain that

$$\|e^{F(\mathbf{x})}G_0(\mathbf{x},\mathbf{x}';z)e^{-F(\mathbf{x}')}\|_{\mathbb{C}^2\otimes\mathbb{C}^2} \le e^{-\theta(\mathbf{x}-\mathbf{x}')+\gamma|\mathbf{x}-\mathbf{x}'|}\omega(\mathbf{x}-\mathbf{x}';z) \stackrel{\text{def}}{=} g_F(\mathbf{x}-\mathbf{x}').$$

We observe that thanks to (61) we have that

$$\begin{cases} g_F \in L^t(\mathbb{R}^2) & t\in[1,2), \\ |\mathbf{x}|g_F \in L^t(\mathbb{R}^2) & t\in[1,\infty]. \end{cases} \tag{30}$$

We introduce, for $n\in\mathbb{N}$, the integral operators

$$S_n(z), T_n(z)\colon L^2(\mathbb{R}^2;\mathbb{C}^2) \longrightarrow L^2(\mathbb{R}^2;\mathbb{C}^2)$$

with

$$(S_n(z)f)(\mathbf{x}) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} e^{\mathrm{i}\varphi_n(\mathbf{x},\mathbf{x}')}G_0(\mathbf{x},\mathbf{x}';z)f(\mathbf{x}')d\mathbf{x}', \tag{31}$$

$$(T_n(z)f)(\mathbf{x}) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \boldsymbol{\sigma}\cdot a^n_{\mathbf{x}'}(\mathbf{x})\, e^{\mathrm{i}\varphi_n(\mathbf{x},\mathbf{x}')}G_0(\mathbf{x},\mathbf{x}';z)f(\mathbf{x}')d\mathbf{x}', \tag{32}$$

where φ_n is determined by (28) and (29). Notice that in view of (30), (27), and Young's inequality (see [18], Section 4.2) both operators are well defined and bounded. In fact, since

$$\|(e^F T_n(z)e^{-F})(\mathbf{x},\mathbf{x}')\|_{\mathbb{C}^2\otimes\mathbb{C}^2} \le \|b_n\|_\infty|\mathbf{x}-\mathbf{x}'|g_F(\mathbf{x}-\mathbf{x}'),$$

we find by (30) and Young's inequality that, for $q\in[1,\infty]$,

$$\lim_{n\to\infty}\|e^F T_n(z)e^{-F}\|_{q,q} = 0. \tag{33}$$

Furthermore, a similar argument implies that, for $q,r\in[1,\infty]$ and $t\in[1,2)$ with $\frac{1}{t} = 1+\frac{1}{r}-\frac{1}{q}$,

$$\sup_{n\in\mathbb{N}}\|e^F S_n(z)e^{-F}\|_{q,r} < \infty. \tag{34}$$

Our next task is to show the following resolvent formula in $L^2(\mathbb{R}^2;\mathbb{C}^2)$, for $n\in\mathbb{N}$ so large that $\|T_n(z)\|_{2,2}<1$, see (33),

$$(D_{\mathbf{A}_n}-z)^{-1} = S_n(z)\sum_{k=0}^{\infty}T_n(z)^k. \tag{35}$$

Pick functions $f \in L^2(\mathbb{R}^2;\mathbb{C}^2)$ and $g \in C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and, we find

$$\begin{aligned}
&\langle (D_{\mathbf{A}_n} - z)g,\ S_n(z)f\rangle \\
&\quad = \int_{\mathbb{R}^2} \Big\langle [(D_{\mathbf{A}_n} - z)g](\mathbf{x}),\ \int_{\mathbb{R}^2} e^{\mathrm{i}\varphi_n(\mathbf{x},\mathbf{x}')} G_0(\mathbf{x},\mathbf{x}';z) f(\mathbf{x}')\,d\mathbf{x}'\Big\rangle_{\mathbb{C}^2} d\mathbf{x} \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \langle e^{-\mathrm{i}\varphi_n(\mathbf{x},\mathbf{x}')}[(D_{\mathbf{A}_n} - z)g](\mathbf{x}),\ G_0(\mathbf{x},\mathbf{x}';z) f(\mathbf{x}')\rangle_{\mathbb{C}^2}\, d\mathbf{x}\, d\mathbf{x}',
\end{aligned}$$

where Young's inequality together with Lemma 11 from Appendix A enabled us to use Fubini's theorem in the last equality. Observe that due to (28)

$$e^{-\mathrm{i}\varphi_n(\mathbf{x},\mathbf{x}')}[(D_{\mathbf{A}_n} - z)g](\mathbf{x}) = [(D_{\mathbf{A}_0} - \boldsymbol{\sigma}\cdot\mathbf{a}^n_{\mathbf{x}'} - z)e^{-\mathrm{i}\varphi_n(\cdot,\mathbf{x}')}g](\mathbf{x}).$$

Hence, using again Fubini's theorem,

$$\begin{aligned}
&\langle (D_{\mathbf{A}_n} - z)g,\ S_n(z)f\rangle = \\
&\quad \int_{\mathbb{R}^2} \Big\langle \int_{\mathbb{R}^2} G_0(\mathbf{x}',\mathbf{x};z)[(D_{\mathbf{A}_0} - z)e^{-\mathrm{i}\varphi_n(\cdot,\mathbf{x}')}g](\mathbf{x})\,d\mathbf{x},\ f(\mathbf{x}')\Big\rangle_{\mathbb{C}^2} d\mathbf{x}' \\
&\qquad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \langle \boldsymbol{\sigma}\cdot\mathbf{a}^n_{\mathbf{x}'}(\mathbf{x}) e^{-\mathrm{i}\varphi_n(\mathbf{x},\mathbf{x}')} g(\mathbf{x}),\ G_0(\mathbf{x},\mathbf{x}';z) f(\mathbf{x}')\rangle_{\mathbb{C}^2}\, d\mathbf{x}'\, d\mathbf{x} \\
&= \langle g,\ f\rangle - \langle g,\ T_n(z)f\rangle.
\end{aligned} \tag{36}$$

For the first integral after the first equality above we used (29) together with the fact that G_0 is the Green kernel of $D_{\mathbf{A}_0}$ and thus, for any $\tilde{g} \in C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$,

$$\int_{\mathbb{R}^2} G_0(\mathbf{x}',\mathbf{x};z)[(D_{\mathbf{A}_0} - z)\tilde{g}](\mathbf{x})\,d\mathbf{x} = \tilde{g}(\mathbf{x}') \quad \text{a.e.}$$

Now, since $D_{\mathbf{A}_n}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ we can extend the identity (36) for all $g \in \mathcal{D}(D_{\mathbf{A}_n})$. From this extension follows that $S_n(z)$ maps $L^2(\mathbb{R}^2;\mathbb{C}^2)$ in $\mathcal{D}(D_{\mathbf{A}_n})$ and

$$(D_{\mathbf{A}_n} - z)S_n(z)f = f - T_n(z)f, \quad f \in L^2(\mathbb{R}^2;\mathbb{C}^2).$$

This yields, for n large enough, the operator identity

$$S_n(z) = (D_{\mathbf{A}_n} - z)^{-1}(1 - T_n(z)),$$

from which follows (35).

We observe that (20), for $V_n = 0$, is a consequence of (33), (34), and

$$\|e^F (D_{\mathbf{A}_n} - z)^{-1} e^{-F}\|_{q,r} \le \|e^F S_n(z) e^{-F}\|_{q,r} \cdot \sum_{k=0}^{\infty} (\|e^F T_n(z) e^{-F}\|_{q,q})^k.$$

Note that the last sum above converges for n large enough due to (33).

In order to show (20) for $V_n \neq 0$ we note that by Hölder's inequality

$$\|V_n e^F (D_{\mathbf{A}_n} - z)^{-1} e^{-F}\|_{q,q} \leq \|V_n\|_\infty \|e^F (D_{\mathbf{A}_n} - z)^{-1} e^{-F}\|_{q,q} \longrightarrow 0, \tag{37}$$

as $n \to \infty$. Therefore, the following computation is meaningful for n large enough

$$\begin{aligned}
&\|e^F (D_{\mathbf{A}_n} + V_n - z)^{-1} e^{-F}\|_{q,r} \\
&\quad = \|e^F (D_{\mathbf{A}_n} - z)^{-1} (1 + V_n (D_{\mathbf{A}_n} - z)^{-1})^{-1} e^{-F}\|_{q,r} \\
&\quad \leq \|e^F (D_{\mathbf{A}_n} - z)^{-1} e^{-F}\|_{q,r} \sum_{m=0}^{\infty} \|\{V_n e^F (D_{\mathbf{A}_n} - z)^{-1} e^{-F}\}^m\|_{q,q}.
\end{aligned}$$

This finishes the proof. □

6. Gaussian-localization

In this section we show Theorem 3 on Gaussian localization of eigenfunctions with energies in the discrete spectrum of

$$H = D_{\mathbf{A}} + V,$$

under the assumptions (A1)–(A5) stated in the introduction. We choose the magnetic vector potential to be given by

$$\mathbf{A}(\mathbf{x}) \stackrel{\text{def}}{=} r^{-1} A(r) \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad A(r) = r^{-1} \int_0^r B(s) s \, ds. \tag{38}$$

If $B \in L^q_{\text{loc}}(\mathbb{R}^2, \mathbb{R})$ it is easy to see, using Hölder's inequality, that if $q \in (1, 2]$ then $\mathbf{A} \in L^p_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, for some $p \in (2, \infty)$, and that $\mathbf{A} \in L^\infty_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ whenever $q \in (2, \infty]$.

The proof of Theorem 3, given in Subsection 6.3, follows the ideas of [19] consisting in Agmon-type estimates with localizations in space and in the angular momentum variable. Of course, we have to adapt the method of [19] since our Hamiltonian is not bounded form below.

In Subsection 6.1 we transform the operator H to polar coordinates and we decompose it in the angular momentum variable m_j. The analyticity condition (A4) on V permits us to obtain exponential decay in $|m_j|$ of eigenfunctions of H with eigenvalues $E \in \sigma_{\text{d}}(H)$; see Lemma 5 in Subsection 6.2. In order to obtain the Agmon estimates, in Subsection 6.3, we square the transformed free Dirac operator $K_0^{(2)}$; see (39) for its definition. The Gaussian decay is essentially due to a positive term in $(K_0^{(2)})^2$ that goes like r^2. This term is in competition with a term that behaves like m_j when $m_j \geq 0$. The Gaussian weights in the Agmon estimates are localized in the region where $m_j \lesssim r^2$. The complementary region, on the other hand, is controlled by the exponential decay in $|m_j|$.

6.1. Unitary transform. In the following we derive an equivalent representation of H. We denote by U the unitary map that represents H in polar coordinates; see e.g. [30], Section 7.3.3.

$$UHU^* = H^{(1)} = K_0^{(1)} + v(r,\theta),$$

$$K_0^{(1)} \stackrel{\text{def}}{=} S_\theta\{-\mathrm{i}\,\partial_r + ir^{-1}\sigma_3 J_3 - \mathrm{i}\,\sigma_3 A(r)\},$$

acting on $\mathcal{H}^{(1)} \stackrel{\text{def}}{=} L^2(\mathbb{R}^+) \otimes L^2(T;\mathbb{C}^2)^2$, where

$$J_3 \stackrel{\text{def}}{=} -\mathrm{i}\,\partial_\theta + 1/2\sigma_3, \quad S_\theta \stackrel{\text{def}}{=} \begin{pmatrix} 0 & e^{-\mathrm{i}\theta} \\ e^{\mathrm{i}\theta} & 0 \end{pmatrix}.$$

Next we identify $L^2(T;\mathbb{C}^2)$ with $\ell^2(\mathbb{Z})^2$ by means of the transformation

$$\mathcal{F} : L^2(T;\mathbb{C}^2) \longrightarrow \ell^2(\mathbb{Z})^2$$

given by

$$\mathcal{F}[f](j) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} M_\theta e^{-\mathrm{i}m_j\theta} f(\theta)\,d\theta,$$

for $f \in L^2(T;d\theta)^2$, where $m_j = (2j+1)/2,\ j \in \mathbb{Z}$, and

$$M_\theta \stackrel{\text{def}}{=} \begin{pmatrix} e^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{i}\,e^{-\mathrm{i}\theta/2} \end{pmatrix}.$$

Under these transformations we find the decomposition

$$L^2(\mathbb{R}^2;\mathbb{C}^2) \cong \mathcal{H}^{(2)} \stackrel{\text{def}}{=} \bigoplus_{j\in\mathbb{Z}} L^2(\mathbb{R}^+;dr)^2$$

and the corresponding operator

$$H \cong H^{(2)} = K_0^{(2)} + W,$$

which is essentially self-adjoint on $\mathcal{D}^{(2)} \stackrel{\text{def}}{=} \mathcal{F}UC_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$. For $h \in \mathcal{D}^{(2)}$, $K_0^{(2)} = \mathcal{F}UD_{\mathbf{A}}U^*\mathcal{F}^*$ acts as

$$(K_0^{(2)}h)(r,j) = (-\mathrm{i}\,\sigma_2\partial_r + \sigma_1(-m_j r^{-1} + A(r)))h(r,j), \tag{39}$$

where we used the fact that $\mathcal{F}S_\theta\mathcal{F}^* = \sigma_2$, that $\mathcal{F}S_\theta\sigma_3\mathcal{F}^* = \mathrm{i}\,\sigma_1$, and that $\mathcal{F}J_3\mathcal{F}^*$ is the multiplication operator by m_j. The electric potential $W = \mathcal{F}v\mathcal{F}^*$ acts as

$$(Wh)(r,l) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \hat{v}(r,l-j)h(r,j), \tag{40}$$

where

$$\hat{v}(r,n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}n\theta} v(r,\theta)d\theta, \quad n \in \mathbb{Z}. \tag{41}$$

Two other quantities play an important role in our analysis, namely $W_1 \stackrel{\text{def}}{=} \mathcal{F}\partial_r v \mathcal{F}^*$ given by

$$(W_1 h)(r,l) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \partial_r \hat{v}(r, l-j)h(r,j), \tag{42}$$

and $W_2 \stackrel{\text{def}}{=} \mathcal{F}\partial_\theta v \mathcal{F}^*$ that acts as

$$(W_2 h)(r,l) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \mathrm{i}(j-l)\hat{v}(r,l-j)h(r,j). \tag{43}$$

6.2. Rotation-analyticity. For $f \in \mathcal{H}^{(1)}$ and $a \in \mathbb{R}$ we set

$$(U_a f)(r,\theta) \stackrel{\text{def}}{=} (e^{\mathrm{i}J_3 a} f)(r,\theta) = e^{\mathrm{i}\sigma_3 a/2} f(r, \theta + a).$$

We call a vector $f \in \mathcal{H}^{(1)}$ rotation-analytic, if and only if the series

$$\sum_{n\in\mathbb{N}} \frac{\|J_3^n f\|}{n!}\rho^n, \quad \rho > 0,$$

has an infinite radius of convergence. We start by presenting a lemma that gives us some a priori decay of some eigenfunctions of $H^{(2)}$ in the angular momentum variable.

Lemma 5. *Assume that (A1)–(A4) hold. Let $\Psi \in \mathcal{H}^{(2)}$ be an eigenfunction of $H^{(2)}$ to the eigenvalue $E \in \sigma_d(H^{(2)})$. Then, for every $\gamma > 0$, we have*

$$\sum_{j\in\mathbb{Z}} \int_0^\infty e^{2\gamma|m_j|}|\Psi(r,j)|^2 dr < \infty.$$

Proof. The proof is analog to the one given in [19], Section 3. We sketch it here for the reader's convenience. Due to Lemma 14 (in Appendix B) $\{H^{(1)}(z)\}_{z\in\mathbb{C}}$ defined on $\mathcal{D}(K_0^{(1)})$ through

$$H^{(1)}(z) = K_0^{(1)} + \tilde{v}_z,$$

is an analytic family of type (A); see [23]. Note that when $a \in \mathbb{R}$ the identity $H^{(1)}(a) = U_a H^{(1)} U_a^*$ holds. Moreover, by Lemma 13 (in Appendix B) we have that

$$\tilde{v}_z(K_0^{(1)} - \mathrm{i})^{-1}$$

is a compact operator in $\mathcal{H}^{(1)}$ for any $z \in \mathbb{C}$. Therefore, $\sigma_{\mathrm{ess}}(H^{(1)}(z)) = \sigma_{\mathrm{ess}}(K_0^{(1)})$ by Weyl's theorem. Arguing with analytic perturbation theory and using the fact that the spectrum of $H^{(1)}(a)$ and $H^{(1)}$ is the same for a real (see e.g. the proof of Theorem XIII.36 in [23] for a similar argument) we find that $E \in \sigma_{\mathrm{d}}(H^{(1)})$ of multiplicity $N \in \mathbb{N}$ is also an eigenvalue of $H^{(1)}(z)$ of the same multiplicity.

Let P_z be the N-dimensional E-eigenprojection of $H^{(1)}(z)$. Since rotation-analytic vectors are dense in $\mathcal{H}^{(1)}$ (see e.g. [20]) we find some rotation-analytic vectors $f_1, \ldots, f_N$ such that $\operatorname{Ran} P_0 = \operatorname{Span}\{P_0 f_1, \ldots, P_0 f_N\}$. Observing that, for $a \in \mathbb{R}$ and $j \in \{1, \ldots, N\}$,

$$U_a P_0 f_j = P_a U_a f_j,$$

we find an analytic continuation of $a \mapsto U_a P_0 f_j \in \mathcal{H}^{(1)}$ to the complex plane. In particular, $e^{\mathrm{i} J_3 z} P_0 f_j$ belongs to $\mathcal{H}^{(1)}$ for any $z \in \mathbb{C}$. Let $\Psi_1 \in \operatorname{Ran} P_0$ be such that $\mathcal{F}\Psi_1 = \Psi$. By the discussion above we get that

$$\mathcal{F} e^{J_3 \gamma} \Psi_1 \in \mathcal{H}^{(2)}, \quad \gamma \in \mathbb{R}.$$

This ends the proof since $(\mathcal{F} e^{J_3 \gamma} \Psi_1)(r, j) = e^{m_j \gamma} \Psi(r, j)$ and

$$\sum_{j \in \mathbb{Z}} \int_0^\infty e^{2\gamma |m_j|} |\Psi(r, j)|^2 dr$$
$$\leq \sum_{j \in \mathbb{Z}} \int_0^\infty e^{-2\gamma m_j} |\Psi(r, j)|^2 dr + \sum_{j \in \mathbb{Z}} \int_0^\infty e^{2\gamma m_j} |\Psi(r, j)|^2 dr < \infty. \qquad \square$$

6.3. Agmon-type Estimate. In this section we deduce the Agmon estimates needed in the proof of Theorem 3. They were obtained in [19] for the case of a magnetic Schrödinger operator. These estimates uses heavily the fact that the Schrödinger operator is bounded from below. As we commented before we will obtain these estimates for the square of the Dirac operator $K_0^{(2)}$.

Fix a number $\tilde{B} > B_0$ and note that, due to (A2), there exists $R_0 > 0$ so large that the estimate (47) is fulfilled and moreover

$$\| \mathbb{1}_{\{r > R_0\}} B \| < \tilde{B}, \quad r > R_0. \tag{44}$$

We set, for $0 < q_2 < q_1 < 1$,

$$r(j) \stackrel{\text{def}}{=} \begin{cases} \sqrt{\dfrac{4\tilde{B}}{(q_1^2 - q_2^2) B_0^2} m_j} & m_j \geq 0 \\ 0 & m_j < 0, \end{cases}$$

and

$$\Omega_{q_1, q_2} \stackrel{\text{def}}{=} \{(r, j) \in \mathbb{R}^+ \times \mathbb{Z} \,|\, r \geq r(j)\}.$$

Moreover, we define

$$\rho(r,j) \overset{\text{def}}{=} \begin{cases} q_2 B_0/4(r^2 - r(j)^2) & m_j \geq 0,\ r \geq r(j), \\ q_2 B_0 r^2/4 & m_j < 0, \\ 0 & m_j \geq 0,\ r < r(j). \end{cases}$$

Eventually we will choose q_2 to be sufficiently close to 1. A direct calculation shows that

$$|\rho(r,j_1) - \rho(r,j_2)| \leq \frac{q_2 \tilde{B}}{(q_1^2 - q_2^2) B_0} |j_1 - j_2|.$$

Let $\rho_\varepsilon \overset{\text{def}}{=} \rho(1+\varepsilon\rho)^{-1}$. It is easy to see that

$$|\rho_\varepsilon(r,j_1) - \rho_\varepsilon(r,j_2)| \leq \frac{q_2 \tilde{B}}{(q_1^2 - q_2^2) B_0} |j_1 - j_2|. \tag{45}$$

Finally, for $R > R_0$, we fix a smooth function f_R in r with bounded derivatives in $\mathbb{R}^+ \times \mathbb{Z}$ satisfying

$$f_R(r,j) = \begin{cases} 1 & r \geq 2R \text{ and } (r,j) \in \Omega_{q_1,q_2}, \\ 0 & r \leq R \text{ or } (r,j) \notin \Omega_{q_1,\mu q_2}, \end{cases}$$

where $\mu \in (0,1)$ is a fixed number that will be chosen sufficiently close to 1. Note that $\Omega_{q_1,q_2} \subset \Omega_{q_1,\mu q_2}$.

Let Ψ be the eigenfunction from Theorem 3 and $\widehat{\Psi} \overset{\text{def}}{=} \mathcal{F} U \Psi$ be a normalized eigenfunction of $H^{(2)}$ with corresponding energy $E \in \sigma_d(H^{(2)})$. We set, for $R > R_0$ and $\delta \in (0,\mu)$,

$$g \overset{\text{def}}{=} e^{\delta\rho_\varepsilon} f_R \widehat{\Psi}. \tag{46}$$

Observe that δ can be chosen arbitrarily close to 1.

Lemma 6. *We find constants $R_1 > R_0$ and $c > 0$ such that, for any $\delta \in (-1,1)$, $\rho > R_1$, and $j \in \{0,1,2\}$,*

$$\sup_{\varepsilon>0} \| \theta_\rho e^{\delta\,\rho_\varepsilon} W_j e^{-\delta\,\rho_\varepsilon} \theta_\rho \| < c, \tag{47}$$

where $W_0 \overset{\text{def}}{=} W$ and $\theta_\rho \overset{\text{def}}{=} \mathbb{1}_{\{r>\rho\}}$. In particular, the commutator

$$[K_0^{(2)}, W_0] = -\mathrm{i}\,\sigma_2 W_1 + \frac{\mathrm{i}\,\sigma_1}{r} W_2 \tag{48}$$

satisfies the estimate

$$\sup_{\varepsilon>0} \| e^{\delta\,\rho_\varepsilon} f_R [K_0^{(2)}, W_0] e^{-\delta\,\rho_\varepsilon} \| < 2c, \quad R > R_1. \tag{49}$$

Proof. We show (47) only for $j = 2$, for the other cases follow analogously. For any $m \in \mathbb{Z}$ and $r > 0$ we define the analytic function $\mathbb{C} \ni z \mapsto h_m(r,z) \overset{\text{def}}{=} e^{-\mathrm{i} mz}\tilde{v}_z(r,0)$. Using (41) and the decay and analyticity assumptions on v (A5) we find for any $r > \rho$ (sufficiently large), $m \in \mathbb{Z}$, and $\gamma \in \mathbb{R}$ that there is a constant $C > 0$ such that

$$\begin{aligned} |\widehat{v}(r,m)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} h_m(r,\theta)d\theta\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} h_m(r,\theta - \mathrm{i}\gamma)d\theta\right| \\ &\leq \frac{e^{-m\gamma}}{2\pi}\int_0^{2\pi} u_{2|\gamma|}(r,\theta)d\theta \leq \|\theta_\rho u_{2|\gamma|}\|_\infty e^{-m\gamma} \leq Ce^{-m\gamma}. \end{aligned} \tag{50}$$

Here we also used Cauchy's integral theorem and the fact that $h_m(r,z)$ is 2π-periodic with respect to $\mathrm{Re}(z)$. In particular, replacing γ by $-\gamma$ in the above estimate we see that, for $\gamma > 0$ and $m \in \mathbb{Z}$, the bound $\theta_\rho|\widehat{v}(r,m)| \leq Ce^{-|m|\gamma}$ holds. Therefore, using (45), (43), and Young's inequality for $\ell^2(\mathbb{Z}^2;\mathbb{C}^2)$ in combination with the Cauchy–Schwarz inequality for $L^2((0,\infty);\mathbb{C})$, we get, for γ sufficiently large and every $f \in \mathcal{H}^{(2)}$, that $|\langle f, \theta_\rho e^{\delta\rho_\varepsilon}W_2 e^{-\delta\rho_\varepsilon}\theta_\rho f\rangle|$ is bounded by

$$\int_0^\infty \sum_{l\in\mathbb{Z}} |f(r,l)| \sum_{j\in\mathbb{Z}} |\theta_\rho\widehat{v}(r,l-j)|\,|l-j|e^{\delta\tau|l-j|}\,|f(r,j)|dr \leq \widetilde{C}\|f\|^2,$$

for some constant $\widetilde{C} > 0$, where $\tau \overset{\text{def}}{=} \frac{q_2\tilde{B}}{(q_1^2-q_2^2)B_0}$.

Equation (49) follows from (47) and (48). Equation (48) is a consequence of

$$[K_0^{(2)}, W] = \mathcal{F}[K_0^{(1)}, v]\mathcal{F}^* \tag{51}$$

$$= \mathcal{F}\Big(-\mathrm{i}\,S_\theta\partial_r v + \frac{S_\theta\sigma_3}{r}\partial_\theta v\Big)\mathcal{F}^*, \tag{52}$$

and the fact that $\mathcal{F}S_\theta\mathcal{F}^* = \sigma_2$ and $\mathcal{F}S_\theta\sigma_3\mathcal{F}^* = \mathrm{i}\,\sigma_1$. □

Before continuing let us state a simple technical result.

Lemma 7. *For any $\gamma \in \mathbb{R}$ we have that $e^{\gamma\rho_\varepsilon} f_R\widehat{\Psi} \in \mathcal{D}(K_0^{(2)})$.*

Proof. Let $\lambda > 0$ and $\eta \in \mathcal{F}UC_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$. First observe that a simple computation shows that

$$(\partial_r e^{\gamma\rho_\varepsilon} f_R)e^{-\lambda r}$$

extends to a bounded operator on $\mathcal{H}^{(2)}$. In addition, $e^{\lambda r}\widehat{\Psi} \in \mathcal{H}^{(2)}$ by Theorem 2. Therefore, we get by explicit calculation on $\mathcal{F}UC_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$, that

$$\begin{aligned} \langle K_0^{(2)}\eta, e^{\gamma\rho_\varepsilon} f_R\widehat{\Psi}\rangle &= \langle f_R e^{\gamma\rho_\varepsilon} K_0^{(2)}\eta, \widehat{\Psi}\rangle \\ &= \langle K_0^{(2)} f_R e^{\gamma\rho_\varepsilon}\eta, \widehat{\Psi}\rangle + \langle \mathrm{i}\,\sigma_2(\partial_r e^{\gamma\rho_\varepsilon} f_R)\eta, \widehat{\Psi}\rangle \\ &= \langle \eta, e^{\gamma\rho_\varepsilon} f_R K_0^{(2)}\widehat{\Psi}\rangle - \mathrm{i}\,\langle \eta, \sigma_2(\partial_r e^{\gamma\rho_\varepsilon} f_R)e^{-\lambda r}\,(e^{\lambda r}\widehat{\Psi})\rangle. \end{aligned}$$

Since η can be chosen arbitrarily from the domain of essential self-adjointness of $K_0^{(2)}$ we get the desired result. □

An important role in our analysis is played by the quantity

$$Q \stackrel{\text{def}}{=} \mathrm{Re}\langle K_0^{(2)} e^{\delta\,\rho_\varepsilon} g, K_0^{(2)} e^{-\delta\,\rho_\varepsilon} g\rangle, \tag{53}$$

which is well defined due to Lemma 7. Before we show Theorem 3 we state two preparatory lemmata whose proofs are given in the next subsection.

Lemma 8. *There are* R, ε*-independent constants* $C_1, C_2 > 0$ *such that, for* $R > R_1$ *sufficiently large,*

$$Q \geq (C_1 R^2 - C_2)\|g\|^2.$$

Lemma 9. *There is an* R, ε*-independent constant* C_3 *and an* ε*-independent constant* $C(R)$ *such that, for* $R > R_1$ *sufficiently large,*

$$Q \leq C_3\|g\|^2 + C(R)\|g\|.$$

Proof of Theorem 3. Fix $\delta, q_1, q_2 \in (0,1)$. Combining Lemma 9 and 8 we find, for $R > R_1$ sufficiently large,

$$\|g\| \leq (C_1 R^2 - C_2 - C_3)^{-1} C(R) \tag{54}$$

Since the right hand side of (54) is independent of ε we obtain, by the monotone convergence theorem,

$$\|e^{\delta\rho}\hat{\Psi}\|^2 = \lim_{\varepsilon\to 0} \|e^{\delta\rho_\varepsilon}\hat{\Psi}\|^2 \leq (\sup_{\varepsilon>0}\|g\| + \|e^{\delta\rho}(1 - f_R)\|)^2 < \infty.$$

For $M > 1$ define

$$\tilde{\Omega}_{q_1,q_2,M} = \{(r,j) \in \mathbb{R}^+ \times \mathbb{Z} \mid r^2 \geq M r(j)^2\}.$$

We have that $\tilde{\Omega}_{q_1,q_2,M} \subset \Omega_{q_1,q_2}$. Thus, for any $(r,j) \in \tilde{\Omega}_{q_1,q_2,M}$, we get

$$\rho(r,j) = \frac{q_2 B_0}{4}(r^2 - r(j)^2) \geq \frac{q_2 B_0}{4}\Big(1 - \frac{1}{M}\Big) r^2.$$

Therefore, setting $\alpha \stackrel{\text{def}}{=} \delta q_2(1 - M^{-1})$, we obtain

$$\|e^{\alpha B_0/4 r^2} \mathbb{1}_{\tilde{\Omega}_{q_1,q_2,M}} \hat{\Psi}\| < \infty. \tag{55}$$

If $(r,j) \notin \tilde{\Omega}_{q_1,q_2,M}$ then

$$m_j \geq \frac{(q_1^2 - q_2^2) B_0^2 r^2}{4M\tilde{B}} \stackrel{\text{def}}{=} \beta r^2.$$

Thus, thanks to Lemma 5 we deduce, for any $\gamma > 0$, that

$$\|e^{\beta\gamma r^2}\mathbb{1}_{\widetilde{\Omega}^c_{q_1,q_2,M}}\widehat{\Psi}\| < \infty. \tag{56}$$

Choosing $\gamma = \alpha/\beta \cdot B_0/4$ and combining (56) with (55) we conclude that

$$\|e^{\alpha B_0/4r^2}\widehat{\Psi}\| < \infty.$$

The latter holds for $\alpha > 0$ arbitrarily close to 1, since δ and q_2 can be chosen arbitrarily close to 1 and $M > 1$ can be as large as we want. This proves the theorem. □

6.4. Proof of Lemmata 8 and 9 . Before we give the proof of Lemmata 8 and 9 we need a preparatory result.

Lemma 10. *For $R > R_1$ sufficiently large we have that*

$$\|K_0^{(2)}g\|^2_{\mathcal{H}^{(2)}} \geq \mu^2 q_2^2 B_0^2\|r\, g\|^2/4 - \|r^{-1}\, g\|^2/4 - \tilde{B}\,\|g\|^2.$$

Proof. Let us write $g = (g^+, g^-)^{\mathrm{T}}$ and $g_j^\pm \stackrel{\text{def}}{=} g^\pm(\cdot, j)$. By Equation (39) we have

$$\begin{aligned}&\|K_0^{(2)}g\|^2_{\mathcal{H}^{(2)}}\\ &\quad= \sum_{j\in\mathbb{Z}}(\|(\partial_r - m_j r^{-1} + A(r))g_j^+\|^2 + \|(-\partial_r - m_j r^{-1} + A(r))g_j^-\|^2).\end{aligned}$$

Furthermore, dropping the term $-\partial_r^2$, we get

$$\begin{aligned}&\|(\pm\partial_r - m_j r^{-1} + A(r))g_j^\pm\|^2\\ &\quad\geq \langle g_j^\pm, ((m_j^2 \mp m_j)r^{-2} + A(r)^2)g_j^\pm\rangle + \langle g_j^\pm, \mp\partial_r A(r) - 2m_j r^{-1}A(r))g_j^\pm\rangle.\end{aligned}$$

Observe that (A2) implies that

$$\frac{1}{r^2}\int_0^r b(s)s\,ds = \mathrm{o}(1)\,, \quad \text{as } r \to \infty. \tag{57}$$

This can be seen by splitting the integral above in the regions where $b(s)s$ is integrable and the one where b decays in the L^∞- norm. Hence, given $q_3 \in (q_1, 1)$ we find, using (57), a constant $R_2 > R_1$ such that, for all $r > R_2$,

$$\begin{aligned}B(r) &\geq q_3 B_0\,, \quad A(r) \geq q_1 B_0 r/2,\\ |\partial_r A(r)| &\leq \tilde{B}, \quad A(r) \leq \tilde{B}r/2.\end{aligned} \tag{58}$$

Therefore, for all $r > R > R_2$, we get

$$\begin{aligned}&\|(\pm\partial_r - m_j r^{-1} + A(r))g_j^\pm\|^2\\ &\quad\geq \langle g_j^\pm, (-r^{-2}/4 + q_1^2 B_0^2 r^2/4 - 2m_j r^{-1}A(r) - \tilde{B})g_j^\pm\rangle,\end{aligned}$$

where we also use that $(m_j^2 \pm m_j) \geq -1/4$.

Assume that $m_j < 0$. Since $q_1 > q_2$ and $A(r) > 0$, for $r > R_2$, we find that

$$\|(\pm\partial_r - m_j r^{-1} + A(r))g_j^\pm\|^2 \geq \langle g_j^\pm \,|\, \big(q_2^2 B_0^2 r^2/4 - r^{-2}/4 - \tilde{B}\big) g_j^\pm\rangle.$$

Assume now that $m_j \geq 0$. Recall that $A(r) \leq \tilde{B}r/2$, for $r > R_2$. Using that $m_j \leq r^2(q_1^2 - \mu^2 q_2^2)B_0^2/(4\tilde{B})$ on supp $g \subset \Omega_{q_1,\mu q_2}$ we get

$$\begin{aligned}
&\|(\pm\partial_r - m_j r^{-1} + A(r))g_j^\pm\|^2 \\
&\geq \langle g_j^\pm \,|\, (q_1^2 B_0^2 r^2/4 - r^{-2}/4 - m_j\tilde{B} - \tilde{B})g_j^\pm\rangle \\
&\geq \langle g_j^\pm \,|\, (\mu^2 q_2^2 B_0^2 r^2/4 - r^{-2}/4 - \tilde{B})g_j^\pm\rangle.
\end{aligned}$$

This finishes the proof. □

Proof of Lemma 8. Notice that

$$e^{\pm\delta\,\rho_\varepsilon} K_0^{(2)} e^{\mp\delta\,\rho_\varepsilon} = K_0^{(2)} + Z^{\pm\rho_\varepsilon}, \quad Z^{\pm\rho_\varepsilon} \stackrel{\text{def}}{=} \pm \mathrm{i}\,\delta\partial_r\rho_\varepsilon\sigma_2.$$

Thus, we have

$$\begin{aligned}
Q &= \mathrm{Re}\langle (K_0^{(2)} + Z^{-\rho_\varepsilon})g \,|\, (K_0^{(2)} + Z^{\rho_\varepsilon})g\rangle \\
&= \|K_0^{(2)} g\|^2 - \delta^2\|\partial_r\rho_\varepsilon g\|^2.
\end{aligned}$$

Since $|\partial_r\rho_\varepsilon| \leq |\partial_r\rho| \leq q_2 B_0 r/2$ we find

$$Q \geq \|K_0^{(2)} g\|^2 - (1/4)\delta^2 q_2^2 B_0^2 \|rg\|^2.$$

Combining this with Lemma 10 and that supp $g \subset \{(r, j) \,|\, r \geq R\}$ we obtain (recall that $0 < \delta < \mu < 1$)

$$Q \geq ((\mu^2 - \delta^2)q_2^2 B_0^2 R^2/4 - R^{-2}/4 - \tilde{B})\|g\|^2.$$

This concludes the proof. □

Proof of Lemma 9. We clearly have

$$Q \leq |\langle K_0^{(2)} e^{\delta\rho_\varepsilon} g, f_R(E - W)\widehat{\Psi}\rangle| + |\langle K_0^{(2)} e^{\delta\rho_\varepsilon} g, \sigma_2(\partial_r f_R)\widehat{\Psi}\rangle|. \tag{59}$$

We analyze each of the above terms separately. Using that $(K_0^{(2)} + W)\widehat{\Psi} = E\widehat{\Psi}$ and noting that Wf_R extends trivially to a bounded operator (for $R > R_1$ large enough), we have, for any $\eta \in \mathcal{F}UC_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$,

$$\begin{aligned}
&\langle K_0^{(2)}\eta, f_R(E - W)\widehat{\Psi}\rangle \\
&= \langle (E - W) f_R K_0^{(2)}\eta, \widehat{\Psi}\rangle \\
&= \langle K_0^{(2)} f_R(E - W)\eta, \widehat{\Psi}\rangle + \langle [(E - W) f_R, K_0^{(2)}]\eta, \widehat{\Psi}\rangle \\
&= \langle \eta, (E - W)^2 f_R\widehat{\Psi}\rangle + \langle \eta, [W, K_0^{(2)}] f_R\widehat{\Psi}\rangle + \langle \eta, \mathrm{i}\,\sigma_2(\partial_r f_R)(W - E)\widehat{\Psi}\rangle.
\end{aligned}$$

This identity extends to any $\eta \in \mathcal{D}(K_0^{(2)})$, in particular, we may choose $\eta = e^{\delta\rho_\varepsilon} g$ (see Lemma 7). Thus, using Lemma 6, we find a constant $C > 0$, independent of R and ε, such that

$$\begin{aligned}
&|\langle K_0^{(2)} e^{\delta\rho_\varepsilon} g, f_R(E-W)\widehat{\Psi}\rangle| \\
&\quad \le \|g\|\, \|e^{\delta\rho_\varepsilon}[(E-W)^2 f_R + [W, K_0^{(2)}] f_R + \mathrm{i}\,\sigma_2(\partial_r f_R)(W-E)]\widehat{\Psi}\| \\
&\quad \le C\|g\|\, \|e^{\delta\rho_\varepsilon}\widehat{\Psi}\| \le C\|g\|(\|g\| + \|e^{\delta\rho}(1-f_R)\|).
\end{aligned}$$

We now treat the second term in (59). We define the operators Υ and L acting, for any $h \in \mathcal{H}^{(2)}$ and $(r, j) \in \mathbb{R}^+ \times \mathbb{Z}$, as

$$\begin{aligned}
(\Upsilon h)(r, j) &= e^{-|m_j|} h(r, j), \\
(Lh)(r, j) &= (2\sigma_1\sigma_2(m_j r^{-1} + A(r))(\partial_r f_R \Upsilon h)(r, j).
\end{aligned}$$

Clearly, since $A(r)$ is bounded on the support of $\partial_r f_R$ – for $R > R_1$ large enough; see (58) – L is an anti-symmetric bounded operator on $\mathcal{H}^{(2)}$. With these definitions we have, using again the eigenvalue equation, that for any $\eta \in \mathcal{F}UC_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$

$$\begin{aligned}
&\langle K_0^{(2)}\eta, \sigma_2(\partial_r f_R)\widehat{\Psi}\rangle \\
&\quad = \langle K_0^{(2)}\sigma_2(\partial_r f_R)\eta, \widehat{\Psi}\rangle + \langle \eta, \mathbb{1}_{\operatorname{supp}\partial_r f_R}(\mathrm{i}\,\partial_r^2 f_R \widehat{\Psi} - L\Upsilon^{-1}\widehat{\Psi})\rangle \\
&\quad = \langle \eta, \mathbb{1}_{\operatorname{supp}\partial_r f_R}\big(\sigma_2(\partial_r f_R)(E-W)\widehat{\Psi} + \mathrm{i}\,\partial_r^2 f_R\widehat{\Psi} - L\Upsilon^{-1}\widehat{\Psi}\big)\rangle.
\end{aligned}$$

Note that $\Upsilon^{-1}\widehat{\Psi} \in \mathcal{H}^{(2)}$ by Lemma 5. Next, we extend this identity to $\eta \in \mathcal{D}(K_0^{(2)})$ and replace η by $e^{\delta\rho_\varepsilon} g$. Using that $e^{\delta\rho_\varepsilon}\mathbb{1}_{\operatorname{supp}\partial_r f_R}$ is bounded uniformly in $\varepsilon > 0$, we find ε-independent constants $C(R), C'(R) > 0$ such that

$$\begin{aligned}
&|\langle K_0^{(2)} e^{\delta\rho_\varepsilon} g, \sigma_2(\partial_r f_R)\widehat{\Psi}\rangle| \\
&\quad \le C'(R)\|g\|\, \|e^{\delta\rho}\mathbb{1}_{\operatorname{supp}\partial_r f_R}\|(\|\Upsilon^{-1}\widehat{\Psi}\| + \|\mathbb{1}_{\operatorname{supp}\partial_r f_R} W\widehat{\Psi}\|) \\
&\quad \le C(R)\|g\|,
\end{aligned}$$

where in the last inequality we use again Lemma 5. Therefore, we obtain from (59) and the above bounds that

$$Q \le \|g\|\big(C\|g\| + C\|e^{\delta\rho}(1-f_R)\| + C(R)\big),$$

which concludes the proof. □

A. Bounds for the Green function of $D_{\mathbf{A}_0}$

Let

$$\theta(\mathbf{x}-\mathbf{x}') \stackrel{\text{def}}{=} \frac{B_0|\mathbf{x}-\mathbf{x}'|^2}{4},$$

$$\eta(\mathbf{x},\mathbf{x}') \stackrel{\text{def}}{=} -\frac{B_0}{2}(x_1x_2' - x_2x_1').$$

Lemma 11. *Let $z \in \mathbb{R} \setminus \sigma(D_{\mathbf{A}_0})$ and let $G_0(\mathbf{x},\mathbf{x}',z)$, $\mathbf{x},\mathbf{x}' \in \mathbb{R}^2$, be a representation of the Green kernel of $(D_{\mathbf{A}_0} - z)^{-1}$ as 2×2-matrix. Then we have that*

$$\left\| G_0(\mathbf{x},\mathbf{x}';z)\right\|_{\mathbb{C}^2\otimes\mathbb{C}^2} \leq e^{-\theta(\mathbf{x}-\mathbf{x}')}\omega(\mathbf{x}-\mathbf{x}';z), \tag{60}$$

for some function $\omega(\cdot;z)\colon \mathbb{R}^2 \to \mathbb{R}^+$ that satisfies

$$\sup_{\mathbf{x}\in\mathbb{R}^2} |\mathbf{x}|e^{-\varepsilon|\mathbf{x}|}\omega(\mathbf{x};z) < \infty, \quad \varepsilon > 0. \tag{61}$$

Proof. Recall that by Proposition 1 we have for $E \neq 0$ that $\pm E \in \sigma(D_{\mathbf{A}_0})$ if and only if $E^2 \in \sigma(dd^*) \setminus \{0\} = \sigma(d^*d) \setminus \{0\}$, where

$$\begin{aligned} d^*d &= (\mathbf{p}-\mathbf{A}_0)^2 - B_0, \\ dd^* &= (\mathbf{p}-\mathbf{A}_0)^2 + B_0. \end{aligned} \tag{62}$$

A simple computation using (6) yields, for any $z \in \mathbb{R} \setminus \sigma(D_{\mathbf{A}_0})$,

$$\begin{aligned} (D_{\mathbf{A}_0} - z)^{-1} &= (D_{\mathbf{A}_0} + z)(D_{\mathbf{A}_0}^2 - z^2)^{-1} \\ &= \begin{pmatrix} z(d^*d - z^2)^{-1} & d^*(dd^* - z^2)^{-1} \\ d(d^*d - z^2)^{-1} & z(dd^* - z^2)^{-1} \end{pmatrix}. \end{aligned} \tag{63}$$

It is well-known that the Green function of $(\mathbf{p}-\mathbf{A}_0)^2$ is given by

$$\begin{aligned} &[(\mathbf{p}-\mathbf{A}_0)^2 - \zeta]^{-1}(\mathbf{x},\mathbf{x}') \\ &\quad = (4\pi)^{-1}\Gamma(\alpha)e^{\mathrm{i}\,\eta(\mathbf{x},\mathbf{x}')}e^{-\theta(\mathbf{x}-\mathbf{x}')}U(\alpha,1,2\theta(\mathbf{x}-\mathbf{x}')), \end{aligned} \tag{64}$$

where U is a confluent hypergeometric function and $\alpha = -1/2(\zeta/B_0 - 1) \notin -\mathbb{N}$; see for instance [6], Lemma 2.2.

Combining (62), (63), and (64) we obtain that the Green kernel of $D_{\mathbf{A}_0}$ is given by

$$G_0(\mathbf{x},\mathbf{x}';z) = e^{\mathrm{i}\,\eta(\mathbf{x},\mathbf{x}')-\theta(\mathbf{x}-\mathbf{x}')}\begin{pmatrix} \Omega_{11}(\mathbf{x},\mathbf{x}';z) & \Omega_{12}(\mathbf{x},\mathbf{x}';z) \\ \overline{\Omega_{12}(\mathbf{x},\mathbf{x}';z)} & \Omega_{22}(\mathbf{x},\mathbf{x}';z) \end{pmatrix},$$

where we define $\alpha_{\pm} = -1/2((z^2 \pm B_0)/B_0 - 1)$ and

$$\Omega_{11}(\mathbf{x}, \mathbf{x}'; z) \stackrel{\text{def}}{=} (4\pi)^{-1} z \Gamma(\alpha_+)\, U(\alpha_+, 1, 2\theta(\mathbf{x} - \mathbf{x}')),$$

$$\Omega_{12}(\mathbf{x}, \mathbf{x}'; z) \stackrel{\text{def}}{=} (4\pi)^{-1} B_0 \Gamma(\alpha_- + 1)\, U(\alpha_- + 1, 2, 2\theta(\mathbf{x} - \mathbf{x}'))\{\mathrm{i}\,(x_1 - x_1') + (x_2 - x_2')\},$$

$$\Omega_{22}(\mathbf{x}, \mathbf{x}'; z) \stackrel{\text{def}}{=} (4\pi)^{-1} z \Gamma(\alpha_-)\, U(\alpha_-, 1, 2\theta(\mathbf{x} - \mathbf{x}')).$$

Here we also used that $\frac{d}{dt} U(\alpha, 1, t) = -\alpha_- U(\alpha_- + 1, 2, t)$; see [1], eq. (13.4.22). Since $-\alpha_{\pm} \notin \mathbb{N}_0$, the bounds (60) and (61) follow now from the asymptotic formulae for U; see [1], eq. (13.5.2), eq. (13.5.7), and eq. (13.5.9). □

B. The family $\{H^{(1)}(z)\}_{z \in \mathbb{C}}$

Throughout this section we assume that (A1)–(A4) are satisfied and use that notation introduced in Section 6. Our concern is the family of operators $\{H^{(1)}(z)\}_{z \in \mathbb{C}}$ defined a priori on the dense subspace $UC_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ of $\mathcal{H}^{(1)}$ as

$$H^{(1)}(z) \stackrel{\text{def}}{=} K_0^{(1)} + \tilde{v}_z, \quad z \in \mathbb{C}. \tag{65}$$

We first state a technical lemma.

Lemma 12. *Let T be a (complex-valued) multiplication operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with $T \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}^2)$, $p \in (2, \infty]$ and $\lim_{n \to \infty} \|\mathbb{1}_{\{|\mathbf{x}| > n\}} T\|_\infty = 0$. Then, T is relative $\sqrt{\mathbf{p}^2 + 1}$-compact.*

Proof. For $n \in \mathbb{N}$ write $T = T_1 + T_2$ where T_1 is supported inside the ball $B_n(0) \subset \mathbb{R}^2$ and T_2 on the complement of $B_n(0)$. Then T_1 is relative $\sqrt{\mathbf{p}^2 + 1}$-compact; see [28], Theorem 4.1. Moreover,

$$\|T(\mathbf{p}^2 + 1)^{-1/2} - T_1(\mathbf{p}^2 + 1)^{-1/2}\| \le \|T_2\| \longrightarrow 0,$$

as $n \to \infty$, from which follows the claim. □

Lemma 13. *For any $z \in \mathbb{C}$ the operator $\tilde{v}_z(K_0^{(1)} + \mathrm{i})^{-1}$ is compact in $\mathcal{H}^{(1)}$.*

Proof. Let $z \in \mathbb{C}$ and $\tau > 0$ with $\tau > |z|$. Due to the inequality $|\tilde{v}_z| \le u_\tau$ on $\mathbb{R}^+ \times T$ and the fact that $u_\tau \in L^p(\mathbb{R}^+ \times T, r dr\, d\theta)$ (for some $2 < p \le \infty$) we see that $\tilde{v}_z$ is well defined on the domain of $K_0^{(1)}$. Let $\tilde{u}_\tau = U^* u_\tau U$. It suffices to show that $U^* \tilde{v}_z U(D_{\mathbf{A}} + \mathrm{i})^{-1}$ is compact in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. This is, however, a consequence of Lemma 12 and the discussion at the end of the proof of Lemma 1. □

Lemma 14. $\{H^{(1)}(z)\}_{z\in\mathbb{C}}$ *defined in* (65) *extends to an analytic family of type* (*A*) *with domain* $\mathcal{D}(H^{(1)}(z)) = \mathcal{D}(K_0^{(1)})$.

Proof. Due to Lemma 13 we know that, for any $z \in \mathbb{C}$, $H^{(1)}(z)$ extends to a closed operator with $\mathcal{D}(H^{(1)}(z)) = \mathcal{D}(K_0^{(1)})$. It is enough to show that, for any $\varphi \in \mathcal{D}(K_0^{(1)})$ the mapping $\mathbb{C} \ni z \mapsto H^{(1)}(z)\varphi \in \mathcal{H}^{(1)}$ is analytic.

By the assumption (A5) we have, for any $(r,\theta) \in \mathbb{R}^+ \times T$, that the power series $\tilde{v}_z(r,\theta) = \sum_{n\in\mathbb{N}_0} v^{(n)}(r,\theta)z^n$ with

$$v^{(n)}(r,\theta) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=s} \frac{\tilde{v}_\zeta(r,\theta)}{\zeta^{n+1}} d\zeta, \tag{66}$$

for some $s > 0$, has an infinite convergence radius. In addition, we clearly get from (66) that $|v^{(n)}(r,\theta)| \le u_{2s}(r,\theta)/s^n$ for any $(r,\theta) \in \mathbb{R}^+ \times T$. In particular, we find that

$$\|v^{(n)}\varphi\| \le \frac{1}{s^n}\|u_{2s}\varphi\|, \quad \varphi \in \mathcal{D}(K_0^{(1)}).$$

Therefore, for any $|z| < s$,

$$v_z\varphi = \sum_{n\in\mathbb{N}_0} v^{(n)}z^n\varphi, \quad \varphi \in \mathcal{D}(K_0^{(1)}).$$

This concludes the proof since $s > 0$ can be chosen arbitrarily large. □

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