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Optimal lower bound of the resonance widths for a Helmholtz tube-shaped resonator

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Abstract. The study of the resonances of the Helmholtz resonator has been broadly described in previous works (see [11] and references therein). Here, for a simple 2-dimensional resonator in the shape of a tube, we analyze the transition zone where oscillations start to appear. Following a careful analysis, we obtain an optimal lower bound of the width of the resonances.

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1. Introduction

The Helmholtz resonator was conceived and built by Hermann von Helmholtz to study resonating cavities and how they are heard by humans. A resonator consists of a bounded cavity (the chamber) connected to the exterior by a thin tube (the neck of the chamber). Blowing air into the aperture of the neck creates an instability leading to pressure oscillations in the chamber. The frequency of the sound that is generated is determined by the shape of the chamber in a non-obvious way.

Mathematically, this phenomenon is described by the resonances of the Dirichlet Laplacian $-\Delta_\Omega$ on the domain Ω consisting in the union of the chamber, the neck and the exterior (see Figure 1).

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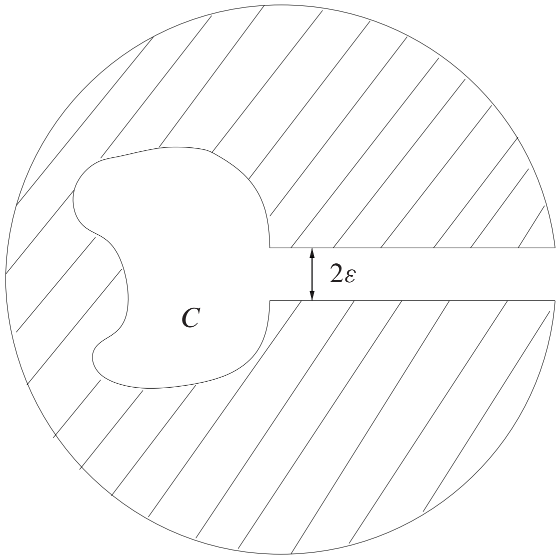

Figure 1. The Helmholtz resonator.

More precisely, the resonances are defined as the eigenvalues of a complex deformation of $-\Delta_\Omega$; their real part corresponds to the frequency, while their imaginary part corresponds to the inverse of the life-time of the vibrational mode. It is therefore of physical interest to compute these two quantities as precisely as possible. A practical way to do this is to study this problem asymptotically when the width ε of the neck is arbitrarily small. In this situation, the frequencies are close to those of the chamber alone (that is, to the real eigenvalues of the Dirichlet Laplacian on the cavity), and it is possible to find an exponentially small upper bound to the absolute values of the imaginary part (the width) of the resonances [11]. However, no lower bound is known in the general case.

Lower bounds have been obtained in a 1-dimensional geometry (see, e.g., [7] and [8]). The few results in higher dimensions concerning exponentially small widths of resonances are those of [6], [3], and [9], and they do not apply to a Helmholtz resonator. The lower bound obtained in [9] is optimal (see also [5] for a generalization).

We derive here a lower bound in the particular case of a 2-dimensional tube-shaped resonator (see Figure 2) which is optimal in the sense that it has the same order of magnitude as the upper bound. We also include calculations indicating that this lower bound should remain true in the 3-dimensional setting. We discuss here only the case where the neck has a square cross-section, but anticipate that the result also holds when the cross-section is circular. The computations would be more delicate then because the eigenfunctions of the disc are less explicit. The general case where the complement of the exterior domain is bounded is still completely open.

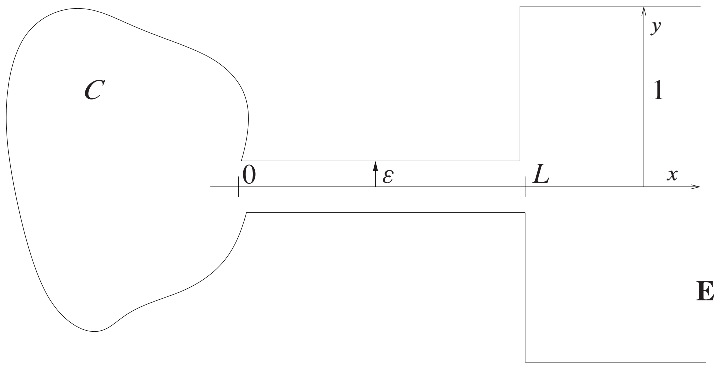

Figure 2. The tube-shaped resonator.

2. Geometrical description and results

Consider, in $\mathbb{R}^2$, a Helmholtz resonator consisting of a regular bounded open set $\mathcal{C}$ (the cavity), connected to an unbounded domain $\mathbf{E}$ through a thin straight tube $\mathcal{T}(\varepsilon)$ (the neck) of radius $\varepsilon > 0$ (see Figure 2). We consider here the case where ε is very small.

More precisely, assume that the Euclidean coordinates (x, y) of $\mathbb{R}^2$ can be chosen in such a way that, for some $L, \delta > 0$ independent of ε, one has:

$$\begin{aligned}
&\mathbf{E} = (L, +\infty) \times (-1, 1);\\
&0 \in \partial\mathcal{C};\\
&\bar{\mathcal{C}} \cap \big(([0, L] \times \{0\}) \cup \bar{\mathbf{E}}\big) = \emptyset;\\
&\mathcal{T}(\varepsilon) = ([-\delta, L] \times (-\varepsilon, \varepsilon)) \cap \big(\mathbb{R}^2 \backslash \mathcal{C}\big).
\end{aligned}$$

Here, $\partial\mathcal{C}$ is the boundary of $\mathcal{C}$. In particular, as $\varepsilon \to 0^+$, the resonator $\Omega(\varepsilon) \stackrel{\text{def}}{=} \mathcal{C} \cup \mathcal{T}(\varepsilon) \cup \mathbf{E}$ collapses to $\Omega_0 \stackrel{\text{def}}{=} \mathcal{C} \cup [0, M_0] \cup \mathbf{E}$, with $M_0 = (L, 0)$.

Let $P_\varepsilon = -\Delta_{\Omega(\varepsilon)}$ be the Dirichlet Laplacian on $\Omega(\varepsilon)$.

Let also $P_{\mathcal{C}(\varepsilon)} = -\Delta_{\mathcal{C}\cup\mathcal{T}(\varepsilon)}$ be the Dirichlet Laplacian on $\mathcal{C}(\varepsilon) \stackrel{\text{def}}{=} \mathcal{C} \cup \mathcal{T}(\varepsilon)$. Finally, let $-\Delta_{\mathcal{C}}$ and $-\Delta_{\mathbf{E}}$ be the Dirichlet Laplacian on $\mathcal{C}$ and $\mathbf{E}$, respectively.

In this situation, the resonances of P_ε are defined as the eigenvalues of the operator obtained by performing a complex dilation with respect to the coordinate x, for $x > L$ large. We are interested in those resonances of P_ε that are close to the eigenvalues of $-\Delta_{\mathcal{C}}$. So, let λ_0 be an eigenvalue of $-\Delta_{\mathcal{C}}$ with u_0 the corresponding (normalized) eigenfunction. Set

$$\alpha_k \stackrel{\text{def}}{=} k\pi/2, \tag{2.1}$$

where $k \in \mathbb{N}$. The quantities α_k^2 ($k \geq 1$) correspond to the thresholds of $-\Delta_{\mathbf{E}}$.

We assume the following conditions:

$$\begin{cases} \lambda_0 \text{ is simple;} \\ \lambda_0 > \alpha_1^2 \text{ and } \lambda_0 \neq \alpha_k^2; \\ u_0 \text{ does not vanish on } \mathscr{C} \text{ near the point } (0,0). \end{cases} \tag{H}$$

Note that the first and last properties are automatically satisfied when λ_0 is the lowest eigenvalue of $-\Delta_{\mathscr{C}}$. When λ_0 is a higher eigenvalues, then the third property means that 0 does not lie on a nodal line of u_0.

By the arguments of [11], we know that there is a resonance $\rho(\varepsilon) \in \mathbb{C}$ of P_ε such that $\rho(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0$, and there is an eigenvalue $\lambda(\varepsilon)$ of $P_{\mathscr{C}(\varepsilon)}$ such that, for all $\delta > 0$, there is $C_\delta > 0$, with,

$$|\rho(\varepsilon) - \lambda(\varepsilon)| \leq C_\delta e^{-\pi(1-\delta)L/\varepsilon}, \tag{2.2}$$

for all $\varepsilon > 0$ small enough. In particular, since $\lambda(\varepsilon)$ is real, we obtain immediately that

$$|\operatorname{Im} \rho(\varepsilon)| \leq C_\delta e^{-\pi(1-\delta)L/\varepsilon}. \tag{2.3}$$

Our main result here is the following.

Theorem 2.1. *Under Assumption* (**H**)*, there exists* $N_0 > 0$ *such that, for all* $\varepsilon > 0$ *small enough, one has*

$$|\operatorname{Im} \rho(\varepsilon)| \geq \varepsilon^{N_0} e^{-\pi L/\varepsilon}.$$

Remark 2.2. Following the proof carefully, one can see that one can take any $N_0 > 10$.

Remark 2.3. An extension to the 3-dimensional case is given in Section 9.

The strategy of the proof is the following one.

- By Green's formula, we reduce the problem to finding a lower-bound estimate on the resonant state u_ε in the exterior domain $\mathbf{E}$;
- We find a representation of u_ε by means of series on both sides of the aperture $\{L\} \times [-\varepsilon, \varepsilon]$;
- By matching the two representations at the aperture, we reduce to finding a lower-bound estimate on u_ε inside the neck $\mathscr{T}(\varepsilon)$;
- Then, using an argument from [2], the required estimate is deduced from an estimate on u_0 near $(0,0)$.

3. Properties of the resonant state

By definition, the resonance $\rho(\varepsilon)$ is an eigenvalue of the complex distorted operator,

$$P_\varepsilon(\mu) \stackrel{\text{def}}{=} U_\mu P_\varepsilon U_\mu^{-1},$$

where $\mu > 0$ is a small parameter, and U_μ is a complex distortion of the form,

$$U_\mu \varphi(x, y) \stackrel{\text{def}}{=} \varphi(x + i\mu f(x), y),$$

with $f \in C^\infty(\mathbb{R})$, $f(x) = 0$ if $x \leq L + 1$, $f(x) = x$ for $x \geq L + 2$. (Observe that, by Ichinose's lemma, the essential spectrum of $P_\varepsilon(\mu)$ consists of the union of the half-lines $e^{-2i\theta}[\alpha_k^2, +\infty)$ $(k \geq 1)$, with $\theta = \arctan \mu$.)

It is well known that such eigenvalues do not depend on μ (see, e.g., [12] and [10]), and that the corresponding eigenfunctions are of the form $U_\mu u_\varepsilon$ with u_ε independent of μ, smooth on $\mathbb{R}^2$ and analytic with respect to x in a complex sector around $(L + 1, +\infty)$. In other words, u_ε is a non trivial analytic solution of the equation $-\Delta u_\varepsilon = \rho(\varepsilon) u_\varepsilon$ in $\Omega(\varepsilon)$, such that $u_\varepsilon|_{\partial\Omega(\varepsilon)} = 0$ and, for all $\mu > 0$ small enough, $U_\mu u_\varepsilon$ is well defined and is in $L^2(\Omega(\varepsilon))$ (in our context, this latter property will be taken as a definition of the fact that u_ε is *outgoing*). Moreover, u_ε can be normalized by setting, for some fixed $\mu > 0$,

$$\|U_\mu u_\varepsilon\|_{L^2(\Omega(\varepsilon))} = 1.$$

In that case, we learn from [11] (in particular Proposition 3.1 and eq. (5.13)), that, for any $\delta > 0$, one has,

$$\|u_\varepsilon\|_{L^2(\mathscr{C} \cup \mathscr{T}(\varepsilon) \cup (L, L+1) \times (-1,1))} \geq 1 - \mathcal{O}(e^{(\delta - \frac{\pi L}{2})/\varepsilon}) \tag{3.1}$$

and

$$\|u_\varepsilon\|_{H^1((L, L+1) \times (-1,1))} = \mathcal{O}(e^{(\delta - \frac{\pi L}{2}))/\varepsilon}). \tag{3.2}$$

Now, using the equation $-\Delta u_\varepsilon = \rho u_\varepsilon$ and Green's formula on the domain $\mathscr{C}(\varepsilon) \cup ([L, L + 1) \times (-1, 1))$, we obtain

$$\operatorname{Im} \int_{-1}^{1} u_\varepsilon(L + 1, y) \frac{\partial \bar{u}_\varepsilon}{\partial x}(L + 1, y) dy = \operatorname{Im} \rho \int_{\mathscr{C}(\varepsilon) \cup ([L, L+1) \times (-1,1))} |u_\varepsilon|^2 dx dy$$

and thus, by (3.1),

$$\operatorname{Im} \rho = (1 + \mathcal{O}(e^{(\delta - \pi/L)/\varepsilon})) \ \operatorname{Im} \int_{-1}^{1} u_\varepsilon(L + 1, y) \frac{\partial \bar{u}_\varepsilon}{\partial x}(L + 1, y) dy. \tag{3.3}$$

Therefore, to prove our result, it is sufficient to obtain a lower bound on

$$\operatorname{Im} \int_{-1}^{1} u_\varepsilon(L + 1, y) \frac{\partial \bar{u}_\varepsilon}{\partial x}(L + 1, y) dy.$$

Note that, by using (3.2), we immediately obtain (2.3).

4. Representation in the thin tube

Let $(\psi_k)_{k\geq 1}$ be an orthonormal basis of eigenvectors of the Dirichlet realization of $-d^2/dy^2$ on $L^2(-\varepsilon,\varepsilon)$, with corresponding eigenvalues α_k^2/ε^2. More precisely, for $k\geq 1$, we set

$$\begin{aligned} \psi_{2k-1}(y) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{\varepsilon}}\cos\Big(\alpha_{2k-1}\frac{y}{\varepsilon}\Big);\\ \psi_{2k}(y) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{\varepsilon}}\sin\Big(\alpha_{2k}\frac{y}{\varepsilon}\Big). \end{aligned} \tag{4.1}$$

We also set

$$\theta_k \stackrel{\text{def}}{=} \sqrt{\alpha_k^2-\varepsilon^2\rho(\varepsilon)},$$

where $\sqrt{\cdot}$ stands for the principal square root, and we denote by $u_\varepsilon = u_\varepsilon(x,y)$ the resonant state of P_ε corresponding to the resonance $\rho(\varepsilon)$, that is, the outgoing solution of the Dirichlet problem,

$$\begin{cases} -\Delta u_\varepsilon = \rho(\varepsilon)u_\varepsilon & \text{in } \Omega(\varepsilon),\\ u|_{\partial\Omega(\varepsilon)} = 0. \end{cases}$$

Then, for any $x\in(0,L)$, and for ε small enough, write

$$u_\varepsilon(x,y) = \sum_{k\geq 1} u_k(x)\psi_k(y),$$

where

$$u_k(x) \stackrel{\text{def}}{=} \int_{-\varepsilon}^{\varepsilon} u_\varepsilon(x,y)\psi_k(y)dy.$$

The coefficient function u_k satisfies

$$\varepsilon^2 u_k''(x) = \theta_k^2 u_k$$

and hence

$$u_k(x) = a_{k,+}e^{\theta_k x/\varepsilon} + a_{k,-}e^{-\theta_k x/\varepsilon}$$

for some $a_{k,+}, a_{k,-}\in\mathbb{C}$. This proves that, for $x\in(0,L)$ and ε small enough,

$$u_\varepsilon(x,y) = \sum_{k=1}^{\infty}(a_{k,+}e^{\theta_k x/\varepsilon} + a_{k,-}e^{-\theta_k x/\varepsilon})\psi_k(y), \tag{4.2}$$

where the sum converges in $H^2((\delta, L-\delta)\times(-\varepsilon,\varepsilon))$ for any $\delta>0$. Differentiating this identity with respect to x, we also obtain,

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{1}{\varepsilon}\sum_{k=1}^{\infty}\theta_k(a_{k,+}e^{\theta_k x/\varepsilon} - a_{k,-}e^{-\theta_k x/\varepsilon})\psi_k(y). \tag{4.3}$$

5. Representation in the external tube

As in the previous section, set

$$\begin{aligned} \varphi_{2j-1}(y) &\stackrel{\text{def}}{=} \cos(\alpha_{2j-1} y); \\ \varphi_{2j}(y) &\stackrel{\text{def}}{=} \sin(\alpha_{2j} y). \end{aligned} \tag{5.1}$$

Also, write u_ε in $\mathbf{E}$ as

$$u_\varepsilon(x, y) = \sum_{j \geq 1} v_j(x) \varphi_j(y),$$

where the v_j satisfy

$$v_j'' = (\alpha_j^2 - \rho(\varepsilon)) v_j.$$

Thus we have

$$v_j(x) = b_{j,+} e^{(x-L)\sqrt{\alpha_j^2 - \rho}} + b_{j,-} e^{-(x-L)\sqrt{\alpha_j^2 - \rho}}, \tag{5.2}$$

with $b_{j,\pm} \in \mathbb{C}$. Note that when $\operatorname{Im} \rho < 0$, our choice of the square root implies that $\operatorname{Im} \sqrt{\alpha_j^2 - \rho} > 0$.

By Assumption (**H**), there exists $j_0 \geq 1$ such that

$$\alpha_{j_0}^2 < \lambda_0 < \alpha_{j_0+1}^2. \tag{5.3}$$

In particular, for ε small enough, $\alpha_{j_0}^2 < \operatorname{Re} \rho(\varepsilon) < \alpha_{j_0+1}^2$. Moreover, by definition, u_ε is outgoing, that is, for $\mu > 0$ small (but independent of ε), the distorted function $u_\varepsilon((1+i\mu)x, y)$ is in $L^2([2L, +\infty) \times (-1, 1))$. Therefore, in view of (5.2) and (5.3), we necessarily have

$$\begin{aligned} b_{j,-} &= 0 \quad \text{for } j \leq j_0; \\ b_{j,+} &= 0 \quad \text{for } j > j_0. \end{aligned}$$

In other words, for $x > L$,

$$u_\varepsilon(x, y) = \sum_{j \leq j_0} b_j e^{i(x-L)\sqrt{\rho - \alpha_j^2}} \varphi_j(y) + \sum_{j > j_0} b_j e^{-(x-L)\sqrt{\alpha_j^2 - \rho}} \varphi_j(y), \tag{5.4}$$

where, for any $L' > L$, the series converges in $H^2((L, L') \times (-1, 1))$.

6. Representation at the aperture

Now consider the trace $u_\varepsilon(L, y)$ of u_ε on $\{x = L\}$ (note that u_ε is continuous on $\Omega(\varepsilon)$ so its trace is a well defined continuous function on $(-\varepsilon, \varepsilon)$).

Since we have $u_\varepsilon \in H^2([L, L'] \times [-1, 1])$, with $L' > L$ arbitrary, and since the part $\{(L, y)\colon |y| < 1\}$ of the boundary of $[L, L'] \times [-1, 1]$ is smooth, we see that $u_\varepsilon(L, y)$ is in $H^{3/2}_{\mathrm{loc}}(-1, 1)$. However, since u_ε vanishes identically on $\{\varepsilon < |y| < 1\}$, we conclude that in fact

$$u_\varepsilon(L, y) \in H^{3/2}([-1, 1]). \tag{6.1}$$

On the other hand, on $\{|y| < \varepsilon\}$, $u_\varepsilon(L, y)$ can be decomposed with respect to the basis $(\psi_k)_{k\geq 1}$ as

$$u_\varepsilon(L, y) = \sum_{k\geq 1} C_k \psi_k(y),$$

where $(C_k) \in \ell^2(\mathbb{N})$. Moreover, since $u_\varepsilon(L, \pm\varepsilon) = 0$, if we denote by $\mathcal{L}(D_y)$ the Dirichlet realization of $-d^2/dy^2$ on $(-\varepsilon, \varepsilon)$, then (6.1) implies

$$(\mathcal{L}(D_y) + 1)^{3/4} u_\varepsilon(L, y) \in L^2(-\varepsilon, \varepsilon),$$

and thus, using that $\mathcal{L}(D_y)\psi_k = \alpha_k^2 \psi_k$, and $\alpha_k \sim k$ as $k \to \infty$, we easily conclude

$$\sum_{k=1}^{\infty} k^3 |C_k|^2 < \infty. \tag{6.2}$$

Now, with the notations of (4.2), we prove

Lemma 6.1. *For all $k \geq 1$, one has,*

$$C_k = a_{k,+} e^{\theta_k L/\varepsilon} + a_{k,-} e^{-\theta_k L/\varepsilon}.$$

Proof. By (4.2), it is enough to prove that the quantity

$$\|u_\varepsilon(L, \cdot) - u_\varepsilon(x, \cdot)\|_{L^2(-\varepsilon,\varepsilon)}$$

tends to 0 as $x \to L_-$. This is a probably well-known fact, but let us recall the proof. For $x < L$ and $|y| < \varepsilon$, write

$$u_\varepsilon(L, y) - u_\varepsilon(x, y) = \int_x^L \partial_x u_\varepsilon(t, y)\, dt.$$

By the Cauchy–Schwarz inequality,

$$\|u_\varepsilon(L, \cdot) - u_\varepsilon(x, \cdot)\|_{L^2(-\varepsilon,\varepsilon)} \leq \sqrt{(L - x)}\, \|\partial_x u_\varepsilon\|_{L^2((0,L)\times(\varepsilon,\varepsilon))}.$$

Denoting by $u_\varepsilon^\mu(x, y) \overset{\text{def}}{=} u_\varepsilon(x + i\mu f(x), y)$ the function obtained by distorting u_ε, we also have,

$$\|\partial_x u_\varepsilon\|_{L^2((0,L)\times(\varepsilon,\varepsilon))} = \|\partial_x u_\varepsilon^\mu\|_{L^2((0,L)\times(\varepsilon,\varepsilon))} \leq \|\nabla u_\varepsilon^\mu\|_{L^2(\Omega(\varepsilon))} = \mathcal{O}(1),$$

and the result follows. $\square$

Therefore, for $|y| < \varepsilon$, we have proved that

$$u_\varepsilon(L, y) = \sum_{k\geq 1}(a_{k,+}e^{\theta_k L/\varepsilon} + a_{k,-}e^{-\theta_k L/\varepsilon})\psi_k(y) \tag{6.3}$$

and

$$\sum_{k\geq 1} k^3 |a_{k,+}e^{\theta_k L/\varepsilon} + a_{k,-}e^{-\theta_k L/\varepsilon}|^2 < \infty. \tag{6.4}$$

In the same way, taking the limit $x \to L_+$, for $|y| < 1$ we also obtain

$$u_\varepsilon(L, y) = \sum_{j\geq 1} b_j \varphi_j(y) \tag{6.5}$$

and

$$\sum_{j\geq 1} j^3 |b_j|^2 < \infty. \tag{6.6}$$

Similar arguments can be performed for the derivative

$$\partial_x u_\varepsilon \in H^1(\Omega(\varepsilon) \cap \{|y| < L'\})$$

and they lead to

$$\partial_x u_\varepsilon(L, y) = \frac{1}{\varepsilon}\sum_{k\geq 1}\theta_k(a_{k,+}e^{\theta_k L/\varepsilon} - a_{k,-}e^{-\theta_k L/\varepsilon})\psi_k(y) \tag{6.7}$$

in $H^{1/2}(|y| \leq \varepsilon)$ and

$$\partial_x u_\varepsilon(L, y) = i\sum_{j\leq j_0}\left(\sqrt{\rho - \alpha_j^2}\right) b_j\varphi_j(y) - \sum_{j>j_0}\left(\sqrt{\alpha_j^2 - \rho}\right) b_j\varphi_j(y) \tag{6.8}$$

in $H^{1/2}(|y| \leq 1)$.

7. Estimates on the coefficients

In this section, taking advantage of the two previous representations of u_ε at the aperture, we compute in two different ways the three following quantities:

$$\langle u_\varepsilon, \partial_x u_\varepsilon\rangle_{\{L\}\times[-1,1]}, \quad \langle u_\varepsilon, \varphi_1\rangle_{\{L\}\times[-1,1]}, \quad \langle \partial_x u_\varepsilon, \psi_1\rangle_{\{L\}\times[-\varepsilon,\varepsilon]}.$$

The resulting identities will permit us to give a lower bound on $\sum_{j\leq j_0}|b_j|^2$ in terms of $|a_{1,-}|$ and to conclude the proof by using an argument from [2].

From now on, we set

$$A_{k,\pm} \overset{\text{def}}{=} a_{k,\pm}e^{\pm\theta_k L/\varepsilon}.$$

Since $u_\varepsilon(L,y)$ vanishes identically on $\{|y|>\varepsilon\}$, in view of (6.3)–(6.8), the two computations of $\langle u_\varepsilon, \partial_x u_\varepsilon\rangle_{\{L\}\times[-1,1]}$ give the identity

$$\begin{aligned}\frac{1}{\varepsilon}\sum_{k\geq 1}\theta_k(|A_{k,+}|^2-|A_{k,-}|^2+2i\,\mathrm{Im}(A_{k,+}\bar{A}_{k,-}))\\ =\sum_{j\leq j_0} i\left(\sqrt{\rho-\alpha_j^2}\right)|b_j|^2-\sum_{j>j_0}\left(\sqrt{\alpha_j^2-\rho}\right)|b_j|^2.\end{aligned}\tag{7.1}$$

By definition, one has

$$\theta_k=\frac{k\pi}{2}\sqrt{1-\frac{4\varepsilon^2}{k^2}\rho(\varepsilon)}.$$

Therefore, since $|\,\mathrm{Im}\,\rho(\varepsilon)|=\mathcal{O}(e^{-\delta/\varepsilon})$ for some constant $\delta>0$, one finds $\mathrm{Re}\,\theta_k\sim k\pi/2$ as $k\to\infty$, $|\,\mathrm{Im}\,\theta_k|=\mathcal{O}(k^{-1}e^{-\delta/\varepsilon})$ and $\left|\mathrm{Im}\sqrt{\rho-\alpha_j^2}\right|=\mathcal{O}(e^{-\delta/\varepsilon})$ for all $j\leq j_0$. Using these facts and taking the real part in (7.1), we obtain,

$$\begin{aligned}\frac{1}{\varepsilon}\sum_{k\geq 1}(\mathrm{Re}\,\theta_k)(|A_{k,+}|^2-|A_{k,-}|^2)+\frac{1}{\varepsilon}\sum_{k\geq 1}\mathcal{O}(k^{-1}e^{-\delta/\varepsilon})|A_{k,+}A_{k,-}|\\ =\mathcal{O}(e^{-\delta/\varepsilon})\sum_{j\leq j_0}|b_j|^2-\sum_{j>j_0}\left(\mathrm{Re}\sqrt{\alpha_j^2-\rho}\right)|b_j|^2.\end{aligned}$$

In particular, since $\mathrm{Re}\sqrt{\alpha_j^2-\rho}=\frac{\pi j}{2}(1+\mathcal{O}(\varepsilon^2 j^{-2}))$, we see that there exists a constant $C>0$ such that

$$\begin{aligned}&\sum_{k\geq 1}\mathrm{Re}\,\theta_k(|A_{k,+}|^2-|A_{k,-}|^2)\\ &\quad\leq Ce^{-\delta/\varepsilon}\sum_{j\leq j_0}|b_j|^2+C\sum_{k\geq 1}k^{-1}e^{-\delta/\varepsilon}|A_{k,+}A_{k,-}|\\ &\qquad-\frac{\pi}{2}\varepsilon\sum_{j>j_0}j(1-C\varepsilon^2 j^{-2})|b_j|^2.\end{aligned}\tag{7.2}$$

Moreover, by (A.2), we see that

$$\sum_{k\geq 2}k|A_{k,-}|^2=\mathcal{O}(\varepsilon^{-1/2}e^{-2\pi L/\varepsilon}),\tag{7.3}$$

and we also know from (B.1) that $\sum_{j\leq j_0}|b_j|^2=\mathcal{O}(e^{(\delta'-\pi L)/\varepsilon})$ for any $\delta'>0$. Therefore, we deduce from (7.2) (with other positive constants C,δ),

$$\begin{aligned}&\sum_{k\geq 1}(k-Ck^{-1}e^{-\delta/\varepsilon})|A_{k,+}|^2\\ &\quad\leq(1+Ce^{-\delta/\varepsilon})|A_{1,-}|^2+Ce^{-(\pi L+\delta)/\varepsilon}\\ &\qquad-\varepsilon\sum_{j>j_0}j(1-C\varepsilon^2 j^{-2})|b_j|^2.\end{aligned}\tag{7.4}$$

Now, computing the scalar products $\langle u_\varepsilon(L,\cdot),\varphi_1\rangle$ and $\langle \partial_x u_\varepsilon(L,\cdot),\psi_1\rangle_{L^2(|y|<\varepsilon)}$ in two different ways (by using (6.3)–(6.8) and the fact that $u_\varepsilon(L,y)=0$ on $\{\varepsilon<|y|<1\}$), we find

$$\sum_{k\geq 1}\mu_k(A_{k,+}+A_{k,-})=b_1;$$

$$\frac{1}{\varepsilon}\theta_1(A_{1,+}-A_{1,-})=\sum_{j\leq j_0} i\nu_j\Big(\sqrt{\rho-\alpha_j^2}\Big)b_j-\sum_{j>j_0}\nu_j\Big(\sqrt{\alpha_j^2-\rho}\Big)b_j,$$

with

$$\mu_k\stackrel{\text{def}}{=}\int_{-\varepsilon}^{\varepsilon}\psi_k(y)\varphi_1(y)dy=\begin{cases}0 & \text{if } k \text{ is even,}\\ (-1)^{\frac{k-1}{2}}\dfrac{4k\sqrt{\varepsilon}}{\pi(k^2-\varepsilon^2)}\cos\dfrac{\varepsilon\pi}{2} & \text{if } k \text{ is odd;}\end{cases}$$

and

$$\nu_j\stackrel{\text{def}}{=}\int_{-\varepsilon}^{\varepsilon}\varphi_j(y)\psi_1(y)dy=\begin{cases}0 & \text{if } j \text{ is even,}\\ \dfrac{4\sqrt{\varepsilon}\sin((\varepsilon j-1)\pi/2)}{\pi(\varepsilon^2 j^2-1)} & \text{if } j\neq\dfrac{1}{\varepsilon} \text{ is odd;}\\ \sqrt{\varepsilon} & \text{if } j=\dfrac{1}{\varepsilon} \text{ is odd.}\end{cases}$$

Using (7.3) again, we obtain

$$|A_{1,+}+A_{1,-}|\leq\frac{C_0}{\sqrt{\varepsilon}}|b_1|+\sum_{k\geq 2}\Big|\frac{\mu_k}{\mu_1}A_{k,+}\Big|+\frac{C_0}{\sqrt{\varepsilon}}e^{-\pi L/\varepsilon}\tag{7.5}$$

and

$$|A_{1,+}-A_{1,-}|\leq C_0\varepsilon^{\frac{3}{2}}\sum_{j\leq j_0}|b_j|+\frac{\varepsilon}{|\theta_1|}\sum_{j>j_0}|\nu_j\alpha_j b_j|,\tag{7.6}$$

with some constant $C_0>0$.

Then, we observe that $|\mu_k/\mu_1|\leq(k-\varepsilon^2)^{-1}$, thus, by (7.4),

$$\begin{aligned}\sum_{k\geq 2}\Big|\frac{\mu_k}{\mu_1}A_{k,+}\Big|&\leq\Big(\sum_{k\geq 2}\frac{1}{k(k-\varepsilon^2)^2}\Big)^{\frac{1}{2}}\Big(\sum_{k\geq 2}k|A_{k,+}|^2\Big)^{\frac{1}{2}}\\&\leq\tau_1\Big(|A_{1,-}|^2-\varepsilon\sum_{j>j_0}j|b_j|^2\Big)^{\frac{1}{2}}+Ce^{-(\pi L+\delta)/2\varepsilon},\end{aligned}\tag{7.7}$$

where τ_1 can be taken arbitrarily close to

$$\Big(\sum_{k\geq 2}k^{-3}\Big)^{\frac{1}{2}}<\frac{1}{2}.$$

Inserting (7.7) into (7.5), we obtain

$$|A_{1,+}+A_{1,-}| \le \tau_1\Big(|A_{1,-}|^2 - \varepsilon \sum_{j>j_0} j|b_j|^2\Big)^{\frac12} + \frac{C_0}{\sqrt{\varepsilon}}|b_1| + 2Ce^{-(\pi L+\delta)/2\varepsilon}. \tag{7.8}$$

In particular, using the fact that $|\sin t| \le \min(|t|,1)$,

$$|\nu_j| \le \frac{4\sqrt{\varepsilon}}{\pi(\varepsilon j+1)} \min\Big(\frac{\pi}{2}, \frac{1}{|\varepsilon j-1|}\Big).$$

On the other hand, setting $\gamma_0 \stackrel{\text{def}}{=} 1+\frac{2}{\pi} \sim 1.637$, and using the fact that $\alpha_j = \frac{j\pi}{2}$, $|\nu_j| \le \frac{4\sqrt{\varepsilon}}{\pi(\varepsilon j+1)}\min(\frac{\pi}{2},\frac{1}{|\varepsilon j-1|})$, and $|\theta_1| = \frac{\pi}{2}+\mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned}&\frac{\varepsilon}{|\theta_1|}\sum_{j>j_0}|\nu_j\alpha_j b_j|\\ &\qquad \le (1+C\varepsilon^2)\sqrt{\varepsilon}\Big(2\sum_{j_0<j\le\gamma_0/\varepsilon}\frac{\varepsilon j}{\varepsilon j+1}|b_j| + \frac{4}{\pi}\sum_{j\ge\gamma_0/\varepsilon}\frac{\varepsilon j}{\varepsilon^2 j^2-1}|b_j|\Big).\end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\frac{\varepsilon}{|\theta_1|}\sum_{j>j_0}|\nu_j\alpha_j b_j| \le (1+C\varepsilon^2)\sqrt{4\Gamma_1+\frac{6}{\pi^2}\Gamma_2}\Big(\varepsilon\sum_{j>j_0} j|b_j|^2\Big)^{\frac12}, \tag{7.9}$$

with

$$\Gamma_1 \stackrel{\text{def}}{=} \sum_{j_0+1\le j\le\gamma_0/\varepsilon}\frac{\varepsilon^2 j}{(\varepsilon j+1)^2}, \quad \text{and} \quad \Gamma_2 \stackrel{\text{def}}{=} \sum_{j\ge\gamma_0/\varepsilon}\frac{\varepsilon^2 j}{(\varepsilon^2 j^2-1)^2}.$$

When $\varepsilon \to 0$, Γ_1 tends to

$$I_1 \stackrel{\text{def}}{=} \int_0^{\gamma_0}\frac{t\,dt}{(t+1)^2} = \ln(1+\gamma_0) - \frac{\gamma_0}{1+\gamma_0} \sim 0.97-0.62 = 0.35$$

and Γ_2 tends to

$$I_2 \stackrel{\text{def}}{=} \int_{\gamma_0}^{\infty}\frac{t\,dt}{(t^2-1)^2} = -\frac12\Big[\frac{1}{t^2-1}\Big]_{\gamma_0}^{\infty} = \frac{1}{2(\gamma_0^2-1)} \sim 0.298.$$

Therefore, we deduce from (7.6) and (7.9),

$$|A_{1,+}-A_{1,-}| \le C_0\varepsilon^{\frac32}\sum_{j\le j_0}|b_j| + \tau_2\Big(\varepsilon\sum_{j>j_0} j|b_{j,-}|^2\Big)^{\frac12} + \frac{C_0}{\sqrt{\varepsilon}}e^{-\pi L/\varepsilon}, \tag{7.10}$$

where τ_2 can be taken arbitrarily close to

$$\sqrt{4\Gamma_1+\frac{6}{\pi^2}\Gamma_2} \le 1.6.$$

Summing (7.8) with (7.10) and using the triangle inequality we finally obtain

$$2|A_{1,-}| \leq \tau_1 \sqrt{|A_{1,-}|^2 - X} + \tau_2 \sqrt{X} + \sum_{j \leq j_0} \frac{2C}{\sqrt{\varepsilon}} |b_j| + 3Ce^{-(\pi L+\delta)/2\varepsilon}, \quad (7.11)$$

where we have set

$$X \stackrel{\text{def}}{=} \varepsilon \sum_{j > j_0} j |b_j|^2.$$

Now, an elementary computation shows that the map

$$[0, |A_{1,-}|^2] \ni X \mapsto \tau_1 \sqrt{|A_{1,-}|^2 - X} + \tau_2 \sqrt{X}$$

reaches its maximum at $X = \frac{\tau_2^2}{\tau_1^2+\tau_2^2}|A_{1,-}|$, and the maximum value is

$$\Big(\sqrt{\tau_1^2 + \tau_2^2}\Big)|A_{1,-}|.$$

Therefore, we deduce from (7.11),

$$2|A_{1,-}| \leq \Big(\sqrt{\tau_1^2 + \tau_2^2}\Big)|A_{1,-}| + \sum_{j \leq j_0} \frac{2C}{\sqrt{\varepsilon}} |b_j| + 3Ce^{-(\pi L+\delta)/2\varepsilon}. \quad (7.12)$$

Since $\tau_1^2 + \tau_2^2 \leq 3 < 4$, we have proved

Proposition 7.1. *There exist two constants $C, \delta > 0$ such that, for any $\varepsilon > 0$ small enough, one has*

$$|A_{1,-}| \leq \frac{C}{\sqrt{\varepsilon}} \sum_{j \leq j_0} |b_j| + Ce^{-(\pi L+\delta)/\varepsilon}.$$

8. End of the proof

We first observe

Proposition 8.1. *There exists a constant $C_0 > 0$, such that*

$$|\operatorname{Im}(\rho)| \geq \frac{1}{C_0}\Big(\sum_{j=1}^{j_0} |b_j|\Big)^2,$$

for all $\varepsilon > 0$ small enough.

Proof. Let us compute $\mathrm{Im}(\int_{-1}^{1} u(L+1,y)\frac{\partial \bar{u}(L+1,y)}{\partial x}dy)$ with the help of the expression (5.4). We first obtain

$$\frac{\partial \bar{u}_\varepsilon}{\partial x}(x,y) = \sum_{j\le j_0} -i\,\overline{\sqrt{\rho-\alpha_j^2}}\, b_j e^{\overline{-i(x-L)\sqrt{\rho-\alpha_j^2}}}\, \overline{\varphi_1(y)}$$

$$- \sum_{j>j_0} b_j e^{\overline{-(x-L)\sqrt{\alpha_j^2-\rho}}}\, \overline{\varphi_j(y)} \quad (x>L)$$

and thus

$$\int_{-1}^{1} u(L+1,y)\frac{\partial \bar{u}(L+1,y)}{\partial x}dy$$

$$= -\sum_{j\le j_0} i|b_j|^2 \overline{\sqrt{\rho-\alpha_j^2}} e^{-2\,\mathrm{Im}\,\sqrt{\rho-\alpha_j^2}} - \sum_{j>j_0} |b_j|^2\, \overline{\sqrt{\alpha_j^2-\rho}}\, e^{-2\,\mathrm{Re}\,\sqrt{\alpha_j^2-\rho}}.$$

Taking the imaginary part, this gives

$$\left|\mathrm{Im}\int_{-1}^{1} u(L+1,y)\frac{\partial \bar{u}(L+1,y)}{\partial x}dy\right|$$

$$\ge \sum_{j\le j_0} |b_j|^2\,\mathrm{Re}\,\sqrt{\rho-\alpha_j^2}e^{-2\,\mathrm{Im}\,\sqrt{\rho-\alpha_j^2}} - \sum_{j>j_0} |b_j|^2\,\mathrm{Im}\,\sqrt{\alpha_j^2-\rho}e^{-2\,\mathrm{Re}\,\sqrt{\alpha_j^2-\rho}}.$$

Now, for $j>j_0$, we have $\left|\mathrm{Im}\,\sqrt{\alpha_j^2-\rho}\right| \le C|\,\mathrm{Im}\,\rho|$, while, for $j\le j_0$, there exist $c_0, C_0>0$, such that, $2c_0 \le \mathrm{Re}\,\sqrt{\rho-\alpha_j^2} \le C_0$, and $\left|\mathrm{Im}\,\sqrt{\rho-\alpha_j^2}\right| \le C|\,\mathrm{Im}\,\rho|$. Then, from (B.1), we obtain

$$\left|\mathrm{Im}\int_{-1}^{1} u(L+1,y)\frac{\partial \bar{u}(L+1,y)}{\partial x}dy\right| \ge c_0\sum_{j\le j_0}|b_j|^2 - e^{(\delta'-\pi L)/\varepsilon}|\,\mathrm{Im}\,\rho|.$$

Equation (3.3) combined with the previous estimate gives

$$|\,\mathrm{Im}(\rho)|(1+\mathcal{O}(e^{(\delta'-\pi L)/\varepsilon})) \ge c_0\sum_{j\le j_0}|b_j|^2$$

and, since $\sum_{j\le j_0}|b_j|^2 \ge j_0^{-2}(\sum_{j\le j_0}|b_j|)^2$, the result follows. □

In view of Propositions 7.1 and 8.1, we see that it only remains to find an appropriate lower bound on $|A_{1,-}|$. This will be achieved by using an argument from [2].

Indeed, by Assumption (**H**), we see that the Dirichlet eigenfunction u_0 satisfies the hypothesis of [2], Lemma 3.1. Then, following the arguments of [2] leading to (3) in that paper, and using again [11], Proposition 3.1 and eq. (5.13), we conclude that, for any $\delta > 0$ and any $x \in (0, L)$, there exists C_1 such that the resonant state u_ε verifies

$$\|u_\varepsilon\|_{L^2([x,L]\times[-\varepsilon,\varepsilon])} \geq C_0 \varepsilon^{4.5+\delta} e^{-\pi x/2\varepsilon} \tag{8.1}$$

(see [2], Theorem 1.2). Thanks to this estimate, we can prove the following result.

Proposition 8.2. *For any $\delta > 0$, there exists $C > 0$, such that*

$$|A_{1,-}| \geq C\varepsilon^{4.5+\delta} e^{-\pi L/2\varepsilon}, \tag{8.2}$$

for $\varepsilon > 0$ small enough.

Proof. The estimate (7.4) gives

$$\sum_{k\geq 1} |A_{k,+}|^2 \leq (1 + Ce^{-\delta/\varepsilon})|A_{1,-}|^2 + Ce^{-(\pi L+\delta)/\varepsilon}. \tag{8.3}$$

Let us compute the quantity $\|u_\varepsilon\|_{L^2([x,L]\times[-\varepsilon,\varepsilon])}$ by using the expression (4.2). For any fixed x, we have

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2([x,L]\times[-\varepsilon,\varepsilon])} &= \sum_{k\geq 1} |a_{k,+}|^2 \frac{\varepsilon}{2\operatorname{Re}\theta_k}(e^{L\,2\operatorname{Re}\theta_k/\varepsilon} - e^{x\,2\operatorname{Re}\theta_k/\varepsilon}) \\ &\quad + \sum_{k\geq 1} 2\operatorname{Re}\Big(\frac{\varepsilon a_{k,+}\bar{a}_{k,-}}{2i\operatorname{Im}\theta_k}(e^{iL\,2\operatorname{Im}\theta_k/\varepsilon} - e^{ix\,2\operatorname{Im}\theta_k/\varepsilon})\Big) \\ &\quad + \sum_{k\geq 1} |a_{k,-}|^2 \frac{\varepsilon}{2\operatorname{Re}\theta_k}(e^{-x2\operatorname{Re}\theta_k/\varepsilon} - e^{-L2\operatorname{Re}\theta_k/\varepsilon}).\end{aligned}$$

This leads to the inequality

$$\begin{aligned}&\|u_\varepsilon\|^2_{L^2([x,L]\times[-\varepsilon,\varepsilon])} \\ &\quad \leq 2\sum_{k\geq 1} |A_{k,+}|^2 + \sum_{k\geq 1} 2|A_{k,+}||a_{k,-}|e^{-L\operatorname{Re}\theta_k/\varepsilon} \\ &\qquad + \sum_{k>1} |a_{k,-}|^2 \varepsilon e^{-x\,2\operatorname{Re}\theta_k/\varepsilon} + |a_{1,-}|^2 \frac{\varepsilon}{2\operatorname{Re}\theta_1} e^{-x\,2\operatorname{Re}\theta_1/\varepsilon}\end{aligned}$$

and thus, by the Cauchy–Schwarz inequality

$$\begin{aligned}&\|u_\varepsilon\|^2_{L^2([x,L]\times[-\varepsilon,\varepsilon])} \\ &\quad \leq 4\sum_{k\geq 1} |A_{k,+}|^2 + 4\sum_{k>1} |a_{k,-}|^2 e^{-x\,2\operatorname{Re}\theta_k/\varepsilon} + |a_{1,-}|^2 \varepsilon e^{-x\,2\operatorname{Re}\theta_1/\varepsilon}.\end{aligned}$$

Using (8.3) and (A.2), we deduce

$$\begin{aligned}&\|u_\varepsilon\|^2_{L^2([x,L]\times[-\varepsilon,\varepsilon])} \\ &\quad \leq C|a_{1,-}|^2 e^{-x\,2\operatorname{Re}\theta_1/\varepsilon} + C\varepsilon e^{-(\pi L+\delta)/\varepsilon} + C\varepsilon^{-C} e^{-x\,2\operatorname{Re}\theta_1/\varepsilon} e^{-x2C_0/\varepsilon}.\end{aligned} \tag{8.4}$$

Now using (8.1), we get

$$\varepsilon^c(1-\varepsilon^{-c}e^{-x2C_0/\varepsilon}-C\varepsilon^{-c}e^{-(\pi(L-x)+\delta)}) \le |a_{1,-}|^2, \tag{8.5}$$

with $c \stackrel{\text{def}}{=} 9+2\delta$. Thus, for ε small enough, we obtain

$$C\varepsilon^c \le |a_{1,-}|^2. \qquad \square$$

Combining the results of Propositions 7.1, 8.1, and 8.2, our main result Theorem 2.1 follows.

9. An extension to dimension 3

Here, we consider the similar problem in dimension 3, obtained by taking tubes with square sections. That is, $\mathscr{C}$ is a regular bounded open subset of $\mathbb{R}^3$, and we have, in Euclidean coordinates (x, y, z),

$$\begin{aligned}
&\mathbf{E} = (L, +\infty) \times Q_1,\\
&0 \in \partial\mathscr{C},\\
&\bar{\mathscr{C}} \cap (([0, L] \times \{0\}) \cup \bar{\mathbf{E}}) = \emptyset,\\
&\mathscr{T}(\varepsilon) = ([-\delta, L] \times Q_\varepsilon) \cap (\mathbb{R}^3 \backslash \mathscr{C}),
\end{aligned}$$

where $Q_1 \stackrel{\text{def}}{=} \{(y, z)\,;\, |y| < 1, |z| < 1\}$, and $Q_\varepsilon \stackrel{\text{def}}{=} \varepsilon Q_1$.

Again, we consider the resonances of the resonator $\Omega(\varepsilon) \stackrel{\text{def}}{=} \mathscr{C} \cup \mathscr{T}(\varepsilon) \cup \mathbf{E}$. Now, the thresholds of $-\Delta_{\mathbf{E}}$ are given by the quantities $\alpha_j^2 + \alpha_k^2$ $(j, k \ge 1)$.

As before, let λ_0 be an eigenvalue of $-\Delta_{\mathscr{C}}$, and let u_0 be the corresponding normalized eigenfunction.

In this situation, the lower estimate of [11] becomes

$$\operatorname{Im} \rho(\varepsilon) = \mathcal{O}(e^{-(1-\delta)\pi L\sqrt{2}/\varepsilon}),$$

where $\rho(\varepsilon)$ stands for any resonance that tends to λ_0 as $\varepsilon \to 0_+$, and $\delta > 0$ is arbitrary.

We assume the following conditions:

$$\begin{cases}
\lambda_0 \text{ is simple;}\\
\lambda_0 > 2\alpha_1^2 \text{ and } \lambda_0 \ne \alpha_j^2 + \alpha_k^2 \text{ for all } j, k \ge 1;\\
u_0 \text{ does not vanish on } \mathscr{C} \text{ near the point } (0, 0).
\end{cases} \tag{$\mathbf{H'}$}$$

Then, we have the following result.

Theorem 9.1. *Under Assumption* ($\mathbf{H'}$), *there exists $N_1 > 0$ such that, for all $\varepsilon > 0$ small enough, the only resonance $\rho(\varepsilon)$ close to λ_0 satisfies*

$$|\operatorname{Im} \rho(\varepsilon)| \ge \varepsilon^{N_1} e^{-\pi L\sqrt{2}/\varepsilon}.$$

Remark 9.2. It follows from the proof that any $N_1 > 12$ can be taken.

Proof. The computations are very similar to those in dimension 2, and we highlight here only what is specific to dimension 3. The notations are similar, but their meaning is modified as follows. For $k = (k_1, k_2) \in \mathbb{N}^2$, we set

$$\alpha_k \stackrel{\text{def}}{=} \Big(\frac{k_1\pi}{2}, \frac{k_2\pi}{2}\Big) \in \mathbb{R}^2;$$

$$\theta_k \stackrel{\text{def}}{=} \sqrt{|\alpha_k|^2 - \varepsilon^2\rho(\varepsilon)};$$

$$\psi_k(y,z) \stackrel{\text{def}}{=} \psi_{k_1}(y)\psi_{k_2}(z);$$

$$\varphi_k(y,z) \stackrel{\text{def}}{=} \varphi_{k_1}(y)\varphi_{k_2}(z).$$

We also define

$$j_0 \stackrel{\text{def}}{=} \max\{|k| \colon k \in \mathbb{N}^2,\ |\alpha_k| < \lambda_0\} = \max\{|k| \colon k \in \mathbb{N}^2,\ |k| < 2\lambda_0/\pi\}.$$

(Here, $|k|$ stands for the Euclidean norm of k, so in particular, $j_0 \geq \sqrt{2}$.) With these notations, the representation formulas (4.2) and (5.4) remain valid with the following changes:

- $\sum_{k=0}^{\infty}$ must be replaced by $\sum_{k\in\mathbb{N}^2}$;
- $j \leq j_0$ and $j > j_0$ must be respectively replaced by $|j| \leq j_0$ and $|j| > j_0$;
- y must be replaced by (y, z);
- $(-\varepsilon, \varepsilon)$ and $(-1, 1)$ must be respectively replaced by Q_ε and Q_1.

Computing in two ways the quantities $\langle u_\varepsilon, \partial_x u_\varepsilon\rangle_{\{L\}\times Q_1}$, $\langle u_\varepsilon, \varphi_{1,1}\rangle_{\{L\}\times Q_1}$, and $\langle \partial_x u_\varepsilon, \psi_{1,1}\rangle_{\{L\}\times Q_\varepsilon}$, we find the following analogs of (7.4), (7.5), and (7.6):

$$\sum_{k\in\mathbb{N}^2} (|k| - C|k|^{-1}e^{-\delta/\varepsilon})|A_{k,+}|^2$$
$$\leq (1 + Ce^{-\delta/\varepsilon})|A_{1,1,-}|^2 + Ce^{-(\pi L\sqrt{2}+\delta)/\varepsilon} - \varepsilon \sum_{|j|>j_0} |j|(1 - C\varepsilon^2|j|^{-2})|b_j|^2,$$

$$|A_{1,1,+} + A_{1,1,-}| \leq \frac{C_0}{\varepsilon}|b_{1,1}| + \sum_{|k|>\sqrt{2}} \Big|\frac{\mu_k}{\mu_{1,1}} A_{k,+}\Big| + \frac{C_0}{\sqrt{\varepsilon}} e^{-\pi L\sqrt{5}/2\varepsilon},$$

$$|A_{1,1,+} - A_{1,1,-}| \leq C_0\varepsilon^2 \sum_{|j|\leq j_0} |b_j| + \frac{\varepsilon}{|\theta_{1,1}|} \sum_{|j|>j_0} |\nu_j\alpha_j b_j|,$$

where we have set

$$\nu_{j_1,j_2} \stackrel{\text{def}}{=} \nu_{j_1}\nu_{j_2} = \frac{16\varepsilon}{\pi^2}\frac{\sin[(\varepsilon j_1 - 1)\pi/2]}{(\varepsilon^2 j_1^2 - 1)}\frac{\sin[(\varepsilon j_2 - 1)\pi/2]}{(\varepsilon^2 j_2^2 - 1)},$$

$$\mu_{k_1,k_2} \stackrel{\text{def}}{=} \mu_{k_1}\mu_{k_2} = \Big(\int_{-\varepsilon}^{\varepsilon}\psi_{k_1}(y)\varphi_1(y)dy\Big)\Big(\int_{-\varepsilon}^{\varepsilon}\psi_{k_2}(z)\varphi_1(z)dz\Big).$$

Using the fact that $\mu_{k_1,k_2}/\mu_{1,1} \le (k_1-\varepsilon^2)^{-1}(k_2-\varepsilon^2)^{-1}$, this also gives

$$|A_{1,1,+} + A_{1,1,-}| \le \tilde{\tau}_1\Big(|A_{1,1,-}|^2 - \varepsilon\sum_{|j|>j_0}|j||b_j|^2\Big)^{\frac{1}{2}} + \frac{C_0}{\varepsilon}|b_{1,1}| + Ce^{-(\pi L\sqrt{2}+\delta)/2\varepsilon}, \tag{9.1}$$

where $\tilde{\tau}_1$ can be taken arbitrarily close to

$$\Big(\sum_{|(k_1,k_2)|>\sqrt{2}}|(k_1,k_2)|^{-1}k_1^{-2}k_2^{-2}\Big)^{\frac{1}{2}} < 7/10. \tag{9.2}$$

On the other hand, the estimate on ν_j used for proving (7.9) is too rough here, but, keeping ν_j in its actual form, we obtain in a similar way

$$|A_{1,1,+} - A_{1,1,-}| \le C_0\varepsilon^2\sum_{|j|\le j_0}|b_j| + \tilde{\tau}_2\Big(\varepsilon\sum_{|j|>j_0}|j||b_j|^2\Big)^{\frac{1}{2}}, \tag{9.3}$$

where $\tilde{\tau}_2$ can be taken arbitrarily close to the quantity

$$J = \frac{16}{\pi^2\sqrt{2}}\Big(\int_0^\infty\int_0^\infty\frac{\sqrt{x^2+y^2}\sin^2((x-1)\pi/2)\sin^2((y-1)\pi/2)}{(x^2-1)^2(y^2-1)^2}dxdy\Big)^{\frac{1}{2}}. \tag{9.4}$$

Now, a numerical computation gives

$$J \approx 1.56 < 16/10.$$

In particular, for ε small enough, we have

$$\tilde{\tau}_1^2 + \tilde{\tau}_2^2 < \frac{7^2+16^2}{100} = 3.05 < 4. \tag{9.5}$$

(We remark that the numerical computation of J may create some trouble near infinity because of the oscillations of the sine function. However, a sufficiently good upper bound can be obtained by substituting 1 for sin when the argument becomes larger than 15, that is essentially when x or y are larger than 10.)

At this point, we can complete the proof as in the 2-dimensional case, first by deducing from (9.1), (9.3), and (9.5) that we have

$$|A_{1,1,-}| \leq \frac{C}{\varepsilon} \sum_{|j|\leq j_0} |b_j| + Ce^{-(\pi L\sqrt{2}+\delta)/2\varepsilon}, \tag{9.6}$$

then, using

$$\operatorname{Im} \rho(\varepsilon) = (1 + \mathcal{O}(e^{-\delta/h})) \operatorname{Im} \int_{Q_1} u(L+1, y, z) \frac{\partial \bar{u}}{\partial x}(L+1, y, z) dy dz,$$

this leads to

$$|\operatorname{Im} \rho(\varepsilon)| \geq \frac{1}{C} \Big(\sum_{|j|\leq j_0} |b_j| \Big)^2. \tag{9.7}$$

Finally, using again [2], eq. (3), we obtain

$$|A_{1,1,-}| \geq C\varepsilon^{5+\delta} e^{-\pi L\sqrt{2}/2\varepsilon} \tag{9.8}$$

and the result follows from (9.6)–(9.8). □

Appendix A

Using the equation $P_\varepsilon u_\varepsilon = \rho(\varepsilon) u_\varepsilon$, we obtain

$$\|u_\varepsilon\|_{L^2(\mathcal{C}\cup\mathcal{T}(\varepsilon))} + \|\Delta u_\varepsilon\|_{L^2(\mathcal{C}\cup\mathcal{T}(\varepsilon))} = \mathcal{O}(1),$$

uniformly in ε.

Since $u_\varepsilon \in H^2(\mathcal{C} \cup \mathcal{T}(\varepsilon))$ the trace theorem applies at $x = c\varepsilon$ with $c > 0$ sufficiently large, and a scaling proves that

$$\|u_\varepsilon(c\varepsilon, y)\|_{H^{1/2}(-\varepsilon,\varepsilon)} = \mathcal{O}(\varepsilon^{-1/2}),$$

$$\|\frac{\partial u_\varepsilon}{\partial x}(c\varepsilon, y)\|_{L^2(-\varepsilon,\varepsilon)} = \mathcal{O}(1).$$

In particular, we have

$$\|(\mathcal{L}(D_y) + 1)^{1/4} u_\varepsilon(c\varepsilon, y)\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$$

and, using the same argument as for (6.2), we easily conclude

$$\sum_{k\geq 1} k |a_{k,+} e^{c\theta_k} + a_{k,-} e^{-c\theta_k}|^2 = \mathcal{O}(\varepsilon^{-1/2})$$

and

$$\sum_{k\geq 1} k^2|a_{k,+}e^{c\theta_k} - a_{k,-}e^{-c\theta_k}|^2 = \mathcal{O}(1).$$

We deduce

$$\sum_{k\geq 1} k|a_{k,+}e^{c\theta_k}|^2 = \mathcal{O}(\varepsilon^{-1/2}) \tag{A.1}$$

and

$$\sum_{k\geq 1} k|a_{k,-}e^{-c\theta_k}|^2 = \mathcal{O}(\varepsilon^{-1/2}). \tag{A.2}$$

Appendix B

This appendix is devoted to the proof of the estimate

$$\sum_{j\leq j_0} |b_j|^2 + \sum_{j>j_0} |b_j|^2 e^{-\operatorname{Re}\left(\sqrt{\alpha_j^2-\rho}\right)} = \mathcal{O}(e^{(\delta-\pi L)/\varepsilon}). \tag{B.1}$$

Using (5.4), we compute

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2((L,L+1)\times(-1,1))} &= \sum_{j\leq j_0} |b_j|^2 \int_{(L,L+1)} e^{-2(x-L)\operatorname{Im}\left(\sqrt{\rho-\alpha_j^2}\right)}dx \\ &\quad + \sum_{j>j_0} |b_j|^2 \int_{(L,L+1)} e^{-2(x-L)\operatorname{Re}\left(\sqrt{\alpha_j^2-\rho}\right)}dx\end{aligned}$$

and thus

$$\|u_\varepsilon\|^2_{L^2((L,L+1)\times(-1,1))} \geq \frac{1}{C}\sum_{j\leq j_0} |b_j|^2 + \frac{1}{C}\sum_{j>j_0} |b_j|^2 e^{-\operatorname{Re}\left(\sqrt{\alpha_j^2-\rho}\right)},$$

for some positive constant C. With the inequality (3.2), this gives (B.1).

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