Polynomials of almost normal arguments in C^*-algebras

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Abstract. The functional calculus for normal elements in C^*-algebras is an important tool of analysis. We consider polynomials $p(a, a^*)$ for elements a with small self-commutator norm $\|[a, a^*]\| \leqslant \delta$ and show that many properties of the functional calculus are retained modulo an error of order δ.

1. Introduction

Let a be a *normal* element of a unital C^*-algebra $\mathcal{A}$. It is well known that there exists a unique C^*-algebra homomorphism

$$C(\sigma(a)) \longrightarrow \mathcal{A}, \quad f \longmapsto f(a)$$

from the algebra of continuous functions on the spectrum $\sigma(a)$ into $\mathcal{A}$ such that $f(z) = z$ is mapped into a, $\sigma(f(a)) = f(\sigma(a))$, and

$$\|f(a)\| = \max_{z \in \sigma(a)} |f(z)| \qquad (1.1)$$

(see, for example, [4]). It is called the functional calculus for normal elements and is widely used in analysis.

The aim of the present paper is to introduce an analogue of functional calculus for "almost normal" elements. More precisely, we shall always be assuming that

$$\|a\| \leqslant 1, \quad \|[a, a^*]\| \leqslant \delta \qquad (1.2)$$

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with a small δ. We restrict the considered class of functions to polynomials in z and $\bar{z}$ and show that some important properties of the functional calculus hold up to an error of order δ.

If $aa^* \neq a^*a$ then the polynomials of a and a^* are, in general, not uniquely defined. We fix the following definition. For a polynomial

$$p(z, \bar{z}) = \sum_{k,l} p_{kl} z^k \bar{z}^l \tag{1.3}$$

let

$$p(a, a^*) = \sum_{k,l} p_{kl} a^k (a^*)^l. \tag{1.4}$$

It is clear that the map $p \mapsto p(a, a^*)$ is linear and involutive, that is $\bar{p}(a, a^*) = p(a, a^*)^*$ where $\bar{p}(z, \bar{z}) = \sum \bar{p}_{lk} z^k \bar{z}^l$. Using the inequality $\|[a, b^m]\| \leqslant m\|b\|^{m-1} \|[a, b]\|$ and (1.2), one can easily show that the map $p \mapsto p(a, a^*)$ is "almost multiplicative",

$$\|p(a, a^*)q(a, a^*) - (pq)(a, a^*)\| \leqslant C(p, q)\,\delta \tag{1.5}$$

where

$$C(p, q) = \sum_{k,l,s,t} ls\, |p_{kl}|\, |q_{st}|.$$

It takes much more effort to obtain an estimate of the norm $\|p(a, a^*)\|$. In the case of an analytic polynomial $p(z) = \sum_k p_k z^k$, according to the von Neumann inequality,

$$\|p(a)\| \leqslant \max_{|z| \leqslant 1} |p(z)| \overset{\text{def}}{=} p_{\max}$$

where it is only assumed that $\|a\| \leqslant 1$; see, for example, I.9 in [13].

Our main results are as follows.

Theorem 1.1. *Let p be a polynomial* (1.3). *There exists a constant $C(p)$ such that the estimate*

$$\|p(a, a^*)\| \leqslant p_{\max} + C(p)\delta \tag{1.6}$$

holds for all a satisfying (1.2). *Here $p(a, a^*)$ is defined by* (1.4)*, and $p_{\max} = \max_{|z| \leqslant 1} |p(z, \bar{z})|$.*

If a is normal and f is a continuous function then the functional calculus gives the following more precise estimate,

$$\|f(a)\| = \max_{z \in \sigma(a)} |f(z)|. \tag{1.7}$$

If $a \in \mathcal{A}$ and $\lambda_j \notin \sigma(a)$, $j = 1, \dots, m-1$, then there exists $R_j > 0$ such that

$$\|(a - \lambda_j)^{-1}\| \leqslant R_j^{-1}, \quad j = 1, \dots, m-1. \tag{1.8}$$

The following theorem gives an analogue of (1.7) for an almost normal a.

Theorem 1.2. *Let $a \in \mathcal{A}$ satisfy* (1.2) *and* (1.8)*, and let the set*

$$S = \{z \in \mathbb{C}: |z| \leqslant 1, \ |z - \lambda_j| \geqslant R_j, \ j = 1, \ldots, m - 1\} \qquad (1.9)$$

be nonempty. For each $\varepsilon > 0$ and each polynomial p defined by (1.3) *there exists a constant $C(p, \varepsilon)$ independent of a such that*

$$\|p(a, a^*)\| \leqslant \max_{z \in S} |p(z, \bar{z})| + \varepsilon + C(p, \varepsilon)\delta.$$

Note that, under the conditions of Theorem 1.2, the set S is a unit disk with $m - 1$ "holes" which contains $\sigma(a)$.

Finally, assume again that a is normal and $\mu \notin f(\sigma(a))$. Then the functional calculus implies that the element $(f(a) - \mu)$ is invertible and

$$\left\|(f(a) - \mu)^{-1}\right\| = \frac{1}{\text{dist}\,(\mu, f(\sigma(a)))}\,. \qquad (1.10)$$

The equality (1.10) also admits the following approximate analogue with $\sigma(a)$ replaced by S and $f(\sigma(a))$ by $p(S)$, where $p(S)$ is the image of S under p considered as a map from $\mathbb{C}$ to $\mathbb{C}$.

Theorem 1.3. *Let S be defined by* (1.9)*, and let p be a polynomial* (1.3)*. Then for each $\varepsilon > 0$ and $\varkappa > 0$ there exist constants $C(p, \varkappa, \varepsilon)$, $\delta_0(p, \varkappa, \varepsilon)$ such that for all $\delta < \delta_0(p, \varkappa, \varepsilon)$ and for all $\mu \in \mathbb{C}$ satisfying $\text{dist}(\mu, p(S)) \geqslant \varkappa$ the estimate*

$$\|(p(a, a^*) - \mu\mathbf{1})^{-1}\| \leqslant \varkappa^{-1} + \varepsilon + C(p, \varkappa, \varepsilon)\delta$$

holds for all $a \in \mathcal{A}$ satisfying (1.2) *and* (1.8)*.*

The authors' interest to the subject was drawn by its relation with Huaxin Lin's theorem; see [6], and [5]. It says that if a is an $n \times n$-matrix satisfying (1.2), then the distance from a to the set of normal matrices is estimated by a function $F(\delta)$ such that $F(\delta) \to 0$ as $\delta \to 0$ uniformly in n. This result implies Theorems 1.1–1.3 with δ replaced by $F(\delta)$ in the right hand side. By homogeneity reasons, $F(\delta)$ can not decay faster than $C\delta^{1/2}$ as $\delta \to 0$. Therefore this approach gives weaker results in terms of power of δ. Also, our results hold in any unital C^*-algebra, while the infinite-dimensional versions of Lin's theorem require additional index type assumptions on a; see, for example, [5].

Our proofs are based on certain representation theorems for positive polynomials. If a real polynomial of x_1, x_2 is non-negative on the unit disk $\{x: x_1^2 + x_2^2 < 1\}$ then, by a result of [11], it admits a representation

$$\sum_j r_j(x)^2 + \left(1 - x_1^2 - x_2^2\right) \sum_j s_j(x)^2 \qquad (1.11)$$

with real polynomials r_j and s_j (see Proposition 3.2 below). Representations similar to (1.11) are usually referred to as *Positivstellensatz*. We also make use of Positivstellensatz for polynomials positive on the sets (1.9). The corresponding results for sets bounded by arbitrary algebraic curves were obtained in [2], [9], [10], and [11].

In order to prove Theorem 1.3, we need uniform with respect to μ estimates for polynomials appearing in Positivstellensatz-type representations. In order to obtain the estimates, we use the scheme introduced in [12] and [7].

2. Proofs of the main results

The proofs of all three theorems consist of two parts. This section is devoted to the "operator-theoretic" part, which is essentially based on Lemma 2.2. The "algebraic" part is the existence of representations (2.2) for the polynomials (2.3), (2.4), (2.7) which is discussed in Section 3.

2.1. Positive elements of C^*-algebras. Recall that a Hermitian element $b \in \mathcal{A}$ is called *positive* ($b \geq 0$) if one of the following two equivalent conditions holds (see, for example, [4], §1.6):

(1) $\sigma(b) \subset [0, +\infty)$.

(2) $b = h^*h$ for some $h \in \mathcal{A}$.

The set of all positive elements in $\mathcal{A}$ is a cone: if $a, b \geq 0$, then $\alpha a + \beta b \geq 0$ for all real $\alpha, \beta \geq 0$. There exists a partial ordering on the set of Hermitian elements of $\mathcal{A}$: $a \leq b$ if and only if $b - a \geq 0$. For a Hermitian b,

$$-\|b\|\mathbf{1} \leq b \leq \|b\|\mathbf{1} \tag{2.1}$$

and, moreover, if $0 \leq b \leq \beta\mathbf{1}$, $\beta \in \mathbb{R}$, then $\|b\| \leq \beta$. The following fact is also well known.

Proposition 2.1. *Let $h \in \mathcal{A}$, $\rho > 0$. Then $h^*h \geq \rho^2\mathbf{1}$ if and only if the element h is invertible and $\|h^{-1}\| \leq \rho^{-1}$.*

Our proofs use the following simple lemma.

Lemma 2.2. *Let $a \in \mathcal{A}$ satisfy (1.2), and let*

$$q = \sum_{j=0}^{N} r_j^2 + \sum_{i=0}^{m-1}\Big(\sum_{j=0}^{N} r_{ij}^2\Big)g_i, \tag{2.2}$$

where r_j, r_{ij}, g_i are real-valued polynomials of the form (1.3). *Assume that*

$$g_i(a, a^*) \geqslant 0, \quad i = 0, \ldots, m-1.$$

Then

$$q(a, a^*) \geqslant -C\delta \mathbf{1}$$

with some non-negative constant C depending on r_j, r_{ij}, g_j.

Proof. Note that q is real-valued, so $q(a, a^*)$ is self-adjoint. Since $g_i(a, a^*) \geqslant 0$, we have $g_i(a, a^*) = b_i^* b_i$ for some $b_i \in \mathcal{A}$. Then

$$r_{ij}(a, a^*)g_i(a, a^*)r_{ij}(a, a^*) = (b_i r_{ij}(a, a^*))^*(b_i r_{ij}(a, a^*)) \geqslant 0.$$

We also have $r_j(a, a^*)^2 \geqslant 0$. From (1.5), we have

$$\left\| q(a, a^*) - \sum_j r_j(a, a^*)^2 - \sum_{i,j} r_{ij}(a, a^*)g_i(a, a^*)r_{ij}(a, a^*) \right\| \leqslant C'\delta,$$

and now the proof is completed by using (2.1). $\qquad\square$

2.2. Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1. Proposition 3.2 below implies that the polynomial

$$q(z, \bar{z}) = p_{\max}^2 - |p(z, \bar{z})|^2 \tag{2.3}$$

admits a representation (2.2) with $m = 1$, $g_0(z, \bar{z}) = 1 - |z|^2$ because, by the definition of $p_{\max}$, the polynomial q is non-negative on the unit disk.

Let us apply Lemma 2.2 to q. By (1.2), we have $g_0(a, a^*) = \mathbf{1} - aa^* \geqslant 0$. Therefore

$$q(a, a^*) \geqslant -C_1(p)\delta \mathbf{1}$$

from which, using (2.3) and (1.5), we get

$$p_{\max}^2 \mathbf{1} - p(a, a^*)^* p(a, a^*) \geqslant -C_2(p)\delta \mathbf{1},$$

$$p(a, a^*)^* p(a, a^*) \leqslant (p_{\max}^2 + C_2(p)\delta)\mathbf{1}$$

and

$$\|p(a, a^*)\| \leqslant p_{\max} + \frac{C_2(p)\delta}{2 p_{\max}}. \qquad\square$$

Proof of Theorem 1.2. By Theorem 3.1, the polynomial

$$q(z, \bar{z}) = p_{\max}^2 + \varepsilon p_{\max} - |p(z, \bar{z})|^2 \tag{2.4}$$

admits a representation (2.2) with

$$g_0(z, \bar{z}) = 1 - |z|^2, \quad g_i(z, \bar{z}) = |z - \lambda_i|^2 - R_i^2, \quad i = 1, \ldots, m-1, \tag{2.5}$$

because it is strictly positive on the set S. Note that

$$S = \{z \in \mathbb{C} : g_i(z, \bar{z}) \geqslant 0, \ i = 0, \dots, m-1\}. \tag{2.6}$$

Proposition 2.1 and (1.8) imply

$$g_i(a, a^*) = (a - \lambda_i \mathbf{1})(a - \lambda_i \mathbf{1})^* - R_i^2 \mathbf{1} \geqslant 0,$$

so we can again apply Lemma 2.2. Using (1.5), we obtain

$$q(a, a^*) \geqslant -C_1 \delta \mathbf{1}, \quad C_1 > 0,$$

$$p(a, a^*) p(a, a^*)^* \leqslant \left(p_{\max}^2 + \varepsilon p_{\max} + C_2(p, \varepsilon)\delta\right) \mathbf{1},$$

and

$$\|p(a, a^*)\| \leqslant p_{\max} \sqrt{1 + \frac{\varepsilon}{p_{\max}} + \frac{C_2(p, \varepsilon)\delta}{p_{\max}^2}} \leqslant p_{\max} + \varepsilon + \frac{C_2(p, \varepsilon)\delta}{p_{\max}}. \qquad \square$$

Proof of Theorem 1.3. Fix $\gamma > 0$. By Theorem 3.1, the polynomial

$$q(z, \bar{z}) = |p(z, \bar{z}) - \mu|^2 - \varkappa^2 + \gamma. \tag{2.7}$$

also admits a representation (2.2) with the same g_i given by (2.5). This is because, by the definitions of μ and $\varkappa$, we have $q(z, \bar{z}) > 0$ for all $z \in S$. Since $g_i(a, a^*) \geqslant 0$, Lemma 2.2 implies

$$q(a, a^*) \geqslant -C\delta \mathbf{1}, \quad C > 0.$$

Using (2.7) and (1.5), we obtain

$$(p(a, a^*) - \mu \mathbf{1})^*(p(a, a^*) - \mu \mathbf{1}) \geqslant (\varkappa^2 - \gamma - C'\delta)\mathbf{1}. \tag{2.8}$$

Let us choose γ and δ_0 such that $\gamma + C'\delta \leqslant \varkappa^2/2$. Now, (2.8) and Proposition 2.1 give

$$\|(p(a, a^*) - \mu \mathbf{1})^{-1}\| \leqslant (\varkappa^2 - \gamma - C'\delta)^{-1/2} \leqslant \varkappa^{-1} + \frac{\gamma}{\varkappa^2} + \frac{C'\delta}{\varkappa^2}.$$

Choosing $\gamma \leqslant \varepsilon \varkappa^2$, we obtain the required inequality with $\varkappa^{-2} C'$ instead of C.

The constant C', in general, depends on $p, \varkappa, \gamma$, and μ. Let us show that the theorem holds with C independent of μ. For $|\mu| \geqslant \|p(a, a^*)\| + \varkappa$ it is obvious as

$$\|(p(a, a^*) - \mu \mathbf{1})^{-1}\| \leqslant \frac{1}{|\mu| - \|p(a, a^*)\|} \leqslant \varkappa^{-1}.$$

Thus we can restrict the consideration to the compact set

$$M = \{\mu \in \mathbb{C} : |\mu| \leqslant \|p(a, a^*)\| + \varkappa, \ \mathrm{dist}(\mu, p(S)) \geqslant \varkappa\}.$$

The estimate $q(z, \bar{z}) \geqslant \gamma$ holds for all $\mu \in M$. The number N of the polynomials r_j and r_{ij} as well as their powers and coefficients are bounded uniformly on M by Remark 3.8. Since C' depends only on these parameters, C may be chosen independent of μ. $\qquad \square$

2.3. Corollaries and remarks

Remark 2.3. As mentioned in the beginning of the section, the proofs rely on the existence of representations of the form (2.2) for certain polynomials. In addition, we need continuity of such a representation with respect to the parameter μ to establish Theorem 1.3. We are also interested in the possibility of explicitly computing the constants C and δ_0, which may be important in applications. It is clearly possible if we have explicit formulae for the polynomials in (2.2). We show below that this can be done in Theorems 1.2 and 1.3 (see Remark 3.8).

Remark 2.4. In general, it is not possible to find a constant C in Theorem 1.1 which would work for all polynomials p. As an example, consider $\mathcal{A} = M_2(\mathbb{C})$,

$$a = \begin{pmatrix} 0 & \sqrt{\delta} \\ 0 & 0 \end{pmatrix}, \quad 0 < \delta < 1.$$

It is clear that a satisfies (1.2). Let $\varepsilon < 1$. There exists a continuous function f such that $f(z) = -1/z$ whenever $|z| \geq \varepsilon$ and $|f(z)| \leq 1/\varepsilon$ for $|z| \leq 1$. There also exists a polynomial $q(z, \bar{z})$ such that $|q(z, \bar{z}) - f(z)| \leq \varepsilon$ for $|z| \leq 1$. Now, let

$$p(z, \bar{z}) = \frac{1}{\varepsilon}(z + z^2 q(z, \bar{z})).$$

Then $p_{\max} \leq 2 + \varepsilon^2$, but $p(a, a^*) = a/\varepsilon$ and $\|p(a, a^*)\| = \sqrt{\delta}/\varepsilon$. Taking ε small, we see that (1.6) can not hold with a C independent of p.

Proposition 2.5. *Under the assumptions of Theorem* 1.2, *there exists a constant* $C(p, \varepsilon)$ *such that*

$$\|\operatorname{Im} p(a, a^*)\| \leq \max_{z \in S} |\operatorname{Im} p(z, \bar{z})| + \varepsilon + C(p, \varepsilon)\delta.$$

Proof. It suffices to apply Theorem 1.2 to the polynomial

$$q(z, \bar{z}) = \frac{p(z, \bar{z}) - \overline{p(z, \bar{z})}}{2i}. \qquad \square$$

In other words, if the values of p on S are almost real, then the element $p(a, a^*)$ itself is almost self-adjoint.

Proposition 2.6. *Under the assumptions of Theorem* 1.2, *there exists a constant* $C(p, \varepsilon)$ *such that*

$$\|p(a, a^*)p(a, a^*)^* - \mathbf{1}\| \leq \max_{z \in S} \left||p(z, \bar{z})|^2 - 1\right| + \varepsilon + C(p, \varepsilon)\delta, \qquad (2.9)$$

$$\|p(a, a^*)^* p(a, a^*) - \mathbf{1}\| \leq \max_{z \in S} \left||p(z, \bar{z})|^2 - 1\right| + \varepsilon + C(p, \varepsilon)\delta. \qquad (2.10)$$

Proof. It is sufficient to apply Theorem 1.2 to the polynomial $q(z, \bar{z}) = |p(z, \bar{z})|^2 - 1$ and use (1.5). □

Remark 2.7. Denote the right hand side of (2.9) and (2.10) by γ. If $\gamma < 1$, then

$$(1 - \gamma)\mathbf{1} \leqslant p(a, a^*)^* p(a, a^*) \leqslant (1 + \gamma)\mathbf{1}$$

and

$$(1 - \gamma)\mathbf{1} \leqslant p(a, a^*) p(a, a^*)^* \leqslant (1 + \gamma)\mathbf{1},$$

which implies that $p(a, a^*)$ and $p(a, a^*)^* p(a, a^*)$ are invertible. The element

$$u = p(a, a^*)(p(a, a^*)^* p(a, a^*))^{-1/2}$$

is unitary (because it is invertible and $uu^* = 1$) and close to u,

$$\|p(a, a^*) - u\| \leqslant \sqrt{1 + \gamma}\left(\frac{1}{\sqrt{1 - \gamma}} - 1\right) \longrightarrow 0 \quad \text{as } \gamma \longrightarrow 0.$$

Thus if the absolute values of p on S are close to 1 then $p(a, a^*)$ is close to a unitary element.

Definition 2.8. The set

$$\sigma_\varepsilon(a) = \{\lambda \in \mathbb{C} : \|(a - \lambda\mathbf{1})^{-1}\| > 1/\varepsilon\} \cup \sigma(a)$$

is called the ε-*pseudospectrum* of the element $a \in \mathcal{A}$.

Its main properties are discussed, for example, in [3], Chapter 9. Note that, under the assumptions of Theorem 1.3, $\sigma_\varepsilon(a) \subset O_\varepsilon(S)$ for all $\varepsilon > 0$, where $O_\varepsilon(S)$ is the ε-neighborhood of S. If a is normal then

$$\sigma_\varkappa(p(a, a^*)) = O_\varkappa\left(p(\sigma(a))\right), \quad \varkappa > 0.$$

The following statement is Theorem 1.3 reformulated in these terms.

Proposition 2.9. *Under the assumptions of Theorem 1.3, for all $\varepsilon > 0$ and $\varkappa > 0$ there exist $C(p, \varkappa, \varepsilon)$ and $\delta_0(p, \varkappa, \varepsilon)$ such that*

$$\sigma_{\varkappa'}(p(a, a^*)) \subset O_\varkappa(p(S)), \quad \delta < \delta_0(p, \varkappa, \varepsilon),$$

where $(\varkappa')^{-1} = \varkappa^{-1} + \varepsilon + C(p, \varkappa, \varepsilon)\delta$.

Proof. Assume that $\text{dist}(\mu, p(S)) \geqslant \varkappa$. By Theorem 1.3, $\|(p(a, a^*) - \mu\mathbf{1})^{-1}\| \leqslant (\varkappa')^{-1}$ and, consequently, $\mu \notin \sigma_{\varkappa'}(p(a, a^*))$. □

3. Representations of non-negative polynomials

This section is devoted to a special case of the following theorem, which is often called *Putinar's Positivstellensatz.* As usual, we denote the ring of real polynomials in n variables by $\mathbb{R}[x_1, \ldots, x_n]$.

Theorem 3.1 ([9]). *Let $g_0, \ldots, g_{m-1} \in \mathbb{R}[x_1, \ldots, x_n]$. Let the set*

$$S = \{x \in \mathbb{R}^n : g_i(x) \geqslant 0, \ i = 0, \ldots, m-1\}$$

be compact and nonempty. If a polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is positive on S then there exist an integer N and polynomials

$$r_i, r_{ij} \in \mathbb{R}[x_1, \ldots, x_n], \quad i = 0, \ldots, m-1, \ j = 0, \ldots, N,$$

such that

$$p = \sum_{j=0}^{N} r_j^2 + \sum_{i=0}^{m-1} \left(\sum_{j=0}^{N} r_{ij}^2 \right) g_i. \tag{3.1}$$

The first result of this type was proved in [2] for the case $m = 1$ with S being a disk. The proof was not constructive and involved Zorn's Lemma. In [9], Theorem 3.1 was proved in a similar way. In [12] and [7], an alternative proof of Theorem 3.1 was presented with its major part being constructive and based on the results of [8].

In Section 2, we have used Theorem 3.1 with the polynomials

$$g_0(x) = 1 - |x|^2, \quad g_i(x) = |x - \lambda_i|^2 - R_i^2, \quad i = 1, \ldots, m-1, \tag{3.2}$$

where $x = (x_1, x_2)$, $|x|^2 = x_1^2 + x_2^2$, $\lambda_i \in \mathbb{R}^2$, and $R_i \in \mathbb{R}$. Let

$$S = \{x \in \mathbb{R}^2 : g_i(x) \geqslant 0, \ i = 0, \ldots, m-1\}. \tag{3.3}$$

As before, the set S is a unit disk with several "holes" centered at λ_i and of radii R_i.

In this section, we give a constructive proof of Theorem 3.1 for the polynomials (3.2). It turns out that in this case the proof simplifies and can be made completely explicit.

If we replace positivity of p with non-negativity, then for $m = 1$ the result still holds.

Proposition 3.2. *Let $p \in \mathbb{R}[x_1, x_2]$ be non-negative on the unit disk $\{x \in \mathbb{R}^2 : |x| \leqslant 1\}$. Then for some N it admits a representation*

$$p = \sum_{j=0}^{N} r_j^2 + \left(\sum_{j=0}^{N} s_j^2 \right)(1 - |x|^2),$$

where $r_j, s_j \in \mathbb{R}[x_1, x_2]$, $j = 0, \ldots, N$.

Proposition 3.2 is a particular case of [11], Corollary 3.3. We have used it to obtain the representation (2.2) for the polynomial (2.3) in Theorem 1.1. Note that, in contrast with Proposition 3.2, the condition $p > 0$ on S in Theorem 3.1 cannot be replaced by $p \geqslant 0$; see Remark 3.9 below.

3.1. Constructive proof for the polynomials (3.2).

The proposed proof relies on the general scheme introduced in [12] and [7] for Theorem 3.1. We have made all the steps constructive and also added a slight variation, the possibility of which was mentioned in [7]. Namely, instead of referring to results of [12] which use [8], we directly apply the results from [8]; see Proposition 3.5 and Lemma 3.7 below.

We need the following explicit version of the Lojasiewicz inequality; see, e.g., [1]. Recall that the angle between intersecting circles is the minimal angle between their tangents in the intersection points.

Lemma 3.3. *Let $g_0, \ldots, g_{m-1}$ be the polynomials (3.2). Assume that $S \neq \varnothing$ and none of the disks $\{x \colon g_i(x) > 0\}$ with $i > 0$ is contained in the union of the others. Then for any $x \in [-1, 1]^2 \setminus S$ the following estimate holds:*

$$\operatorname{dist}(x, S) \leqslant -c_0 \min\{g_0(x), \ldots, g_{m-1}(x)\}.$$

If the circles $S_i = \{x \colon g_i(x) = 0\}$ are pairwise disjoint or tangent, then $c_0 = R_{\min}^{-1}$ where $R_{\min} = \min\limits_{i=0,\ldots,m-1} R_i$ with $R_0 = 1$. Otherwise, c_0 can be chosen as

$$c_0 = \frac{\sqrt{2} + 1}{R_{\min}^2 \sin(\varphi_{\min}/2)},$$

where $\varphi_{\min}$ is the minimal angle between the pairs of intersecting non-tangent circles S_i.

We omit the proof of Lemma 3.3 because it is elementary and involves nothing but school geometry.

For the polynomials

$$q(x) = \sum_{|\alpha| \leqslant d} q_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n],$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, consider the norm

$$\|q\| = \max_\alpha |q_\alpha| \frac{\alpha_1! \ldots \alpha_n!}{(\alpha_1 + \ldots + \alpha_n)!}. \tag{3.4}$$

The following proposition is also elementary and is proved in [7].

Proposition 3.4. *Let $x, y \in [-1, 1]^n$, $q \in \mathbb{R}[x_1, \ldots, x_n]$, and $\deg q = d$. Then*

$$|q(x) - q(y)| \leqslant d^2 n^{d-1/2} \|q\| |x - y|.$$

The next proposition, which is a quantitative version of Pòlya's inequality, is proved in [8].

Proposition 3.5. *Let $f \in \mathbb{R}[y_1, \ldots, y_n]$ be a homogeneous polynomial of degree d. Assume that f is strictly positive on the simplex*

$$\Delta_n = \Big\{ y \in \mathbb{R}^n : y_i \geqslant 0, \ \sum_i y_i = 1 \Big\}. \tag{3.5}$$

Let $f_* = \min\limits_{y \in \Delta_n} f(y) > 0$. Then, for any $N > \frac{d(d-1)\|f\|}{2f_*} - d$, all the coefficients of the polynomial $(y_1 + \ldots + y_n)^N f(y_1, \ldots, y_n)$ are positive.

Further on, without loss of generality, we shall be assuming that $0 \leqslant g_i(x) \leqslant 1$ for all $x \in S$ (if not, we normalize g_i multiplying them by positive constants).

Lemma 3.6. *Under the conditions of Theorem 3.1 with g given by (3.2), let $p^* = \min\limits_{x \in S} p(x) > 0$. Then*

$$p(x) - c_0 d^2 2^{d-1/2} \|p\| \sum_{i=0}^{m-1} (1 - g_i(x))^{2k} g_i(x) \geqslant \frac{p^*}{2}, \quad x \in [-1, 1]^2, \tag{3.6}$$

where an integer k is chosen in such a way that

$$(2k + 1)p^* \geqslant mc_0 d^2 2^{d+1/2} \|p\|,$$

and c_0 is the constant from Lemma 3.3.

Proof. Let $x \in S$. Then $p(x) \geqslant p^*$. Due to our choice of k, the elementary inequality

$$(1 - t)^{2k} t < \frac{1}{2k + 1}, \quad 0 \leqslant t \leqslant 1, k \geqslant 0, \tag{3.7}$$

implies that the absolute value of the second term in the left hand side of (3.6) does not exceed $\frac{p^*}{2}$.

Assume now that $x \in [-1, 1]^2 \setminus S$. Let $y \in S$ be such that $\mathrm{dist}(x, y) = \mathrm{dist}(x, S)$. Then Proposition (3.4) and Lemma 3.3 yield

$$
\begin{aligned}
p(x) &\geqslant p(y) - |p(x) - p(y)| \\
&\geqslant p^* - d^2 2^{d-1/2} \|p\| \, \mathrm{dist}(x, S) \\
&\geqslant p^* + c_0 d^2 2^{d-1/2} \|p\| g_{\min}(x),
\end{aligned} \tag{3.8}
$$

where $g_{\min}(x)$ is the (negative) minimum of the values of $g_i(x)$. Note that we have $(1 - g_{\min}(x))^{2k} > 1$. From (3.8), we get

$$p(x) - c_0 d^2 2^{d-1/2} \|p\| (1 - g_{\min}(x))^{2k} g_{\min}(x)$$

$$\geqslant p(x) - c_0 d^2 2^{d-1/2} \|p\| g_{\min}(x) \geqslant p^*.$$

On the other hand, (3.7) and the choice of k imply that the terms with $g_i(x) > 0$ contribute no more than

$$\frac{(m-1)c_0 d^2 2^{d-1/2}\|p\|}{2k+1} \leqslant \frac{p^*}{2}$$

to the sum (3.6). The remaining terms in (3.6) with $g_i(x) < 0$ may only increase the left hand side. $\qquad\square$

Lemma 3.7. *Let* $p \in \mathbb{R}[x_1, x_2]$ *and* $p_* = \min\limits_{x \in [-1;1]^2} p(x) > 0.$ *Then, for some* $M \in \mathbb{N},$

$$p = \sum_{|\alpha| \leqslant M} b_\alpha \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \gamma_3^{\alpha_3} \gamma_4^{\alpha_4} \qquad (3.9)$$

where $b_\alpha \geqslant 0$ *and*

$$\gamma_1(x) = \frac{1+x_1}{4}, \quad \gamma_2(x) = \frac{1-x_1}{4},$$
$$\gamma_3(x) = \frac{1+x_2}{4}, \quad \gamma_4(x) = \frac{1-x_2}{4}. \qquad (3.10)$$

This lemma was obtained in [8] for arbitrary convex polyhedra and associated linear functions γ_k. Below we prove it for the square $[-1, 1]^2$, because in this particular case the formulae are considerably simpler.

Proof. Consider the following $\mathbb{R}$-algebra homomorphism

$$\varphi \colon \mathbb{R}[y_1, y_2, y_3, y_4] \to \mathbb{R}[x_1, x_2], \quad y_i \mapsto \gamma_i(x).$$

It suffices to find a polynomial $\tilde{p} \in \mathbb{R}[y_1, y_2, y_3, y_4]$ with positive coefficients such that $\varphi(\tilde{p}) = p$. If

$$p = \sum_{i+j \leqslant d} p_{ij} x_1^i x_2^j$$

and

$$\tilde{p}_1(y) = \sum_{i+j \leqslant d} 2^{i+j} p_{ij} (y_1 - y_2)^i (y_3 - y_4)^j (y_1 + y_2 + y_3 + y_4)^{d-i-j},$$

then $\varphi(\tilde{p}_1) = p$ because

$$\varphi(y_1 + y_2 + y_3 + y_4) = 1, \quad 2\varphi(y_1 - y_2) = x_1, \quad 2\varphi(y_3 - y_4) = x_2.$$

Let

$$V = \{y \in \Delta_4 : 2y_1 + 2y_2 = 2y_3 + 2y_4 = 1\},$$

where Δ_4 is the simplex (3.5). If $y \in V$ then $\tilde{p}_1(y) = p(4y_1 - 1, 4y_3 - 1) \geqslant p_*$, as $(4y_1 - 1, 4y_3 - 1) \in [-1, 1]^2$. For an arbitrary y, let $y_0 \in V$ be such that $\text{dist}(y, y_0) = \text{dist}(y, V)$. Then, from Proposition 3.4,

$$\tilde{p}_1(y) \geqslant \tilde{p}_1(y_0) - |\tilde{p}_1(y) - \tilde{p}_1(y_0)| \geqslant p_* - d^2 2^{2d-1} \|\tilde{p}_1\| \, \text{dist}(y, V). \quad (3.11)$$

Let

$$r(y) = 2(y_1 + y_2 - y_3 - y_4)^2.$$

It is easy to see that $\varphi(r) = 0$ and

$$r(y) = (2y_1 + 2y_2 - 1)^2 + (2y_3 + 2y_4 - 1)^2, \quad \forall y \in \Delta_4.$$

If we rewrite the last expression in the coordinates $(y_1 + y_2)/\sqrt{2}$, $(y_1 - y_2)/\sqrt{2}$, $(y_3 + y_4)/\sqrt{2}$, $(y_3 - y_4)/\sqrt{2}$ (obtained by two rotations by the angle $\pi/4$), then we get

$$r(y) \geqslant 8 \, \text{dist}(y, V)^2, \quad y \in \Delta_4. \quad (3.12)$$

Let

$$\tilde{p}_2(y) = \tilde{p}_1(y) + \frac{2^{4d-6} d^4 \|\tilde{p}_1\|^2}{p_*} (y_1 + y_2 + y_3 + y_4)^{d-2} r(y).$$

We still have $\varphi(\tilde{p}_2) = p$. Inequalities (3.11) and (3.12) imply that

$$\tilde{p}_2(y) \geqslant p_* - d^2 2^{2d-1} \|\tilde{p}_1\| \, \text{dist}(y, V) + \frac{2^{4d-3} d^4 \|\tilde{p}_1\|^2}{p_*} \text{dist}(y, V)^2$$

$$= \frac{2^{4d-3} d^4 \|\tilde{p}_1\|^2}{p_*} \left(\text{dist}(y, V) - \frac{p_*}{d^2 2^{2d-1} \|\tilde{p}_1\|} \right)^2 + \frac{p_*}{2}$$

$$\geqslant \frac{p_*}{2},$$

for all $y \in \Delta_4$.

Finally, since $\tilde{p}_2$ is homogeneous, Proposition 3.5 with $N > \frac{d(d-1)\|\tilde{p}_2\|}{p_*} - d$ shows that all the coefficients of

$$\tilde{p}(y) = (y_1 + y_2 + y_3 + y_4)^N \tilde{p}_2(y)$$

are positive. Applying the homomorphism φ to $\tilde{p}$, we obtain the desired representation of p. $\qquad\square$

End of the proof of Theorem 3.1. Let us apply Lemma 3.6 to p. It is sufficient to find a representation of the left hand side of (3.6), because the second term is already of the form (3.1). By Lemma 3.7, the left hand side of (3.6) can be represented in the form (3.9). Note that γ_i can be rewritten as

$$\frac{1}{4}(1 \pm x_{1,2}) = \frac{1}{8}((1 \pm x_{1,2})^2 + g_0(x) + x_{2,1}^2). \quad (3.13)$$

Substituting the last equality into (3.9), we obtain the desired representation for (3.6) and, therefore, for p. $\qquad\square$

3.2. Some remarks

Remark 3.8. If g_i are given by (3.2) then, in principle, it is possible to write down explicit formulae for the polynomials appearing in (3.1). Indeed, assume that we have a polynomial p such that $p(x) \geqslant p^* > 0$ for all $x \in S$. Then

$$p(x) = \hat{p}(x) + c_0 d^2 2^{d-1/2} \|p\| \sum_{i=0}^{m-1} (1 - g_i(x))^{2k} g_i(x), \qquad (3.14)$$

where k is chosen in such a way that $(2k + 1)p^* \geqslant mc_0 d^2 2^{d+1/2} \|p\|$. The second term in the right hand side of (3.14) is an explicit expression of the form (3.1), and the coefficients of $\hat{p}$ can be found from (3.14). From Lemma 3.6, we know that $\hat{p}(x) \geqslant p^*/2$ for all $x \in [-1; 1]^2$. Now it suffices to represent

$$\hat{p}(x) = \sum_{k+l \leqslant \hat{d}} \hat{p}_{kl} \, x_1^k x_2^l$$

in the form (3.1). Consider the following polynomials

$$\tilde{p}_1(y) = \sum_{i+j \leqslant \hat{d}} 2^{i+j} \hat{p}_{ij} (y_1 - y_2)^i (y_3 - y_4)^j (y_1 + y_2 + y_3 + y_4)^{\hat{d}-i-j},$$

$$\tilde{p}_2(y) = \tilde{p}_1(y) + \frac{2^{4\hat{d}-4} \hat{d}^4 \|\tilde{p}_1\|^2}{p^*} (y_1 + y_2 + y_3 + y_4)^{\hat{d}-2} (y_1 + y_2 - y_3 - y_4)^2,$$

and

$$\tilde{p}(y) = (y_1 + y_2 + y_3 + y_4)^N \tilde{p}_2(y)$$

where

$$N > \frac{2\hat{d}(\hat{d} - 1)\|\tilde{p}_2\|}{p^*} - \hat{d}.$$

If we replace y_i, $i = 1, 2, 3, 4$, with $\gamma_i(x)$ given by (3.10) in the definition of $\tilde{p}$, then we get $\hat{p}(x)$. The coefficients of $\tilde{p}$ are positive. Therefore, if we substitute y_i with γ_i and then apply (3.13), we obtain an expression of the form (3.1) for $\hat{p}(x)$. Combining it with (3.14), we get the desired expression for p. As a consequence, if we have a continuous family of positive polynomials with a uniform lower bound on S and uniformly bounded degrees, then the polynomials in the representation (3.1) may also be chosen to be continuously depending on this parameter, and also with uniformly bounded degrees.

Remark 3.9. In [10], an analogue of Theorem 3.1 for a non-negative polynomial p and $m > 1$ was established under some additional assumptions on the zeros of p. The next theorem shows that, in general, Theorem 3.1 may not be true if $p \geqslant 0$.

Theorem 3.10. *Let g_i be defined by (3.2), and assume that $\lambda_i \neq \lambda_j$ for some i and j. Then the polynomial $g_i g_j$ can not be represented in the form (3.1).*

This result is probably well known to specialists, although we could not find it in the literature. For reader's convenience, we prove it below.

Let g_i be defined by (3.2), and let

$$S_i = \{x \in \mathbb{R}^2 : g_i(x) = 0\},$$

$$S_i(\mathbb{C}) = \{x \in \mathbb{C}^2 : g_i(x) = 0\}.$$

Lemma 3.11. *Let $q \in \mathbb{R}[x_1, x_2]$ be a polynomial such that $q(x) = 0$ on an open arc of S_i. Then $g_i \mid q$ (that is, q is divisible by g_i).*

Proof. Consider q as an analytic function on $S_i(\mathbb{C})$. Since the set $S_i(\mathbb{C})$ is connected, $q \equiv 0$ on the whole $S_i(\mathbb{C})$. Hilbert's Nullstellensatz (see, for example, [14], Section 16.3) gives that $g_i \mid q^k$ for some integer k (in $\mathbb{C}[x_1, x_2]$ and, consequently, in $\mathbb{R}[x_1, x_2]$). As the polynomial g_i is irreducible, we have $g_i \mid q$.   □

Lemma 3.12. *Let $\lambda_i \neq \lambda_j$. Then $S_i(\mathbb{C}) \cap S_j(\mathbb{C}) \neq \varnothing$.*

Proof. Let the circles S_i and S_j be given by the equations

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 = R_1^2,$$

$$(x_1 - b_1)^2 + (x_2 - b_2)^2 = R_2^2.$$

Subtracting one from the other, we get a system of a linear and a quadratic equation. The linear one is solvable because $\lambda_i \neq \lambda_j$. Substituting the solution into the quadratic equation, we reduce it to a non-degenerate quadratic equation in one complex variable, which also has a solution.   □

Proof of Theorem 3.10. Assume that $p = g_i g_j$ satisfies (3.1). The left hand side of (3.1) vanishes on the set $S_i \cap \partial S$. All the terms r_k^2 and $r_{kl}^2 g_k$ in the right hand side of (3.1) are non-negative on $S_i \cap \partial S$, and therefore are equal to zero on this set. By Lemma 3.11, they all are multiples of g_i. Similarly, all the terms in the right hand side are multiples of g_j. Therefore, $g_i \mid r_k$, $g_j \mid r_k$, and $g_i^2 g_j^2 \mid r_k^2$.

Since the polynomials g_k and g_i are coprime for all $k \neq i$, we have $g_i^2 \mid r_{kl}^2$ for $k \neq i$ and $g_j^2 \mid r_{kl}^2$ for $k \neq j$. Thus any term in the right hand side of (3.1) is a multiple of either $g_i^2 g_j$ or $g_i g_j^2$. Dividing (3.1) by $g_i g_j$, we see that the left hand side is identically equal to 1, and the right hand side vanishes on the intersection $S_i(\mathbb{C}) \cap S_j(\mathbb{C})$ which is nonempty by Lemma 3.12. This contradiction proves the theorem.   □

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