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The Tan 2Θ Theorem for indefinite quadratic forms

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Abstract. A version of the Davis–Kahan Tan 2Θ theorem [3] for not necessarily semibounded linear operators defined by quadratic forms is proven. This theorem generalizes a result by Motovilov and Selin [13].

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1. Introduction

In the 1970 paper [3], Davis and Kahan studied the rotation of spectral subspaces for 2×2 operator matrices under off-diagonal perturbations. In particular, they proved the following result, the celebrated

Tan 2Θ Theorem. *Let $A_\pm$ be strictly positive bounded operators in Hilbert spaces $\mathfrak{H}_\pm$, respectively, and W a bounded operator from $\mathfrak{H}_-$ to $\mathfrak{H}_+$. Denote by*

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & -A_- \end{pmatrix} \quad \textit{and} \quad B = A + V = \begin{pmatrix} A_+ & W \\ W^* & -A_- \end{pmatrix}$$

the block operator matrices with respect to the orthogonal decomposition of the Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

Then

(i) *the open interval* $(\min \operatorname{spec}(A_+), \max \operatorname{spec}(-A_-))$ *belongs to the resolvent set of the operator B;*

(ii) *the operator angle Θ between the subspaces* $\operatorname{Ran} \mathsf{E}_A(\mathbb{R}_+)$ *and* $\operatorname{Ran} \mathsf{E}_B(\mathbb{R}_+)$ *admits the bound*

$$\|\tan 2\Theta\| \le \frac{2\|V\|}{d}, \quad \operatorname{spec}(\Theta) \subset [0, \pi/4), \tag{1.1}$$

where $d = \operatorname{dist}(\operatorname{spec}(A_+), \operatorname{spec}(-A_-))$.

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For the concept of an operator angle and further details we refer to [8] and references therein.

Estimate (1.1) can equivalently be expressed as the following inequality for the norm of the difference of the orthogonal projections $P = \mathsf{E}_A(\mathbb{R}_+)$ and $Q = \mathsf{E}_B(\mathbb{R}_+)$:

$$\|P - Q\| \le \sin\Big(\frac{1}{2}\arctan\frac{2\|V\|}{d}\Big), \tag{1.2}$$

which, in particular, implies the estimate

$$\|P - Q\| < \frac{\sqrt{2}}{2}\,. \tag{1.3}$$

Independent of the work of Davis and Kahan, inequality (1.3) has been proven by Adamyan and Langer in [1], where the operators $A_\pm$ were allowed to be semibounded. The "critical" case $d = 0$ has been considered in the paper [9] by Kostrykin, Makarov, and Motovilov. In particular, it was shown that for any orthogonal (not necessarily spectral) projection P satisfying

$$\mathsf{E}_B((0,\infty)) \le P \le \mathsf{E}_B([0,\infty)),$$

there exists a unique orthogonal projection Q such that

$$\mathsf{E}_B((0,\infty)) \le Q \le \mathsf{E}_B([0,\infty))$$

and

$$\|P - Q\| \le \frac{\sqrt{2}}{2}.$$

It is worth mentioning that a particular case of this result has been obtained earlier by Adamyan, Langer, and Tretter, in [2]. Recently, a version of the Tan 2Θ Theorem for off-diagonal perturbations V that are relatively bounded with respect to the diagonal operator A has been proven by Motovilov and Selin in [13], Theorem 1.

In the present work we are concerned with a sesquilinear form

$$\mathfrak{b} = \mathfrak{a} + \mathfrak{v}, \tag{1.4}$$

where $\mathfrak{a}$ and $\mathfrak{v}$ are densely defined symmetric forms, and obtain several generalizations of the aforementioned results assuming that the perturbation $\mathfrak{v}$ is given by an off-diagonal symmetric form.

To introduce the framework of an off-diagonal form-perturbation theory, we pick up a self-adjoint involution J and assume that the form $\mathfrak{a}$ "commutes" with the involution J,

$$\mathfrak{a}[Jx, y] = \mathfrak{a}[x, Jy]. \tag{1.5}$$

We also assume that the form $\mathfrak{a}_J[x,y] \stackrel{\text{def}}{=} \mathfrak{a}[x, Jy]$ on $\mathrm{Dom}[\mathfrak{a}]$ is a closed positive definite form.

Our further assumption is that the form $\mathfrak{v}$ "anticommutes" with the involution J,

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \tag{1.6}$$

and that $\mathfrak{v}$ satisfies the estimate

$$|\mathfrak{v}[x, x]| \le \beta\, \mathfrak{a}_J[x, x], \quad x \in \mathrm{Dom}[\mathfrak{a}_J] = \mathrm{Dom}[\mathfrak{a}],$$

for some $\beta > 0$.

The "commutation" relations (1.5) and (1.6) suggest to interpret the form $\mathfrak{v}$ as an off-diagonal perturbation of the diagonal form $\mathfrak{a}$ with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ with $\mathfrak{H}_\pm = \mathrm{Ran}(I \pm J)$.

In this setting one can show that the form $\mathfrak{b}$ admits the representation

$$\mathfrak{b}[x, y] = \langle A_J^{1/2} x, H A_J^{1/2} y\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{a}],$$

where A_J is the self-adjoint operator associated with the closed positive definite form $\mathfrak{a}_J$ and H is a bounded operator with a bounded inverse. In spite of the fact that the form $\mathfrak{b}$ may not be semibounded, there exists a unique self-adjoint operator B in $\mathfrak{H}$ associated with the form $\mathfrak{b}$, i.e., $\mathrm{Dom}(B) \subset \mathrm{Dom}[\mathfrak{b}]$ and

$$\mathfrak{b}[x, y] = \langle x, By\rangle \quad x \in \mathrm{Dom}[\mathfrak{b}],\ y \in \mathrm{Dom}(B).$$

This result, proven in [4], is an extension of the First Representation Theorem for closed semi-bounded quadratic forms (see, e.g., [7]). A comprehensive exposition on representation theorems for indefinite quadratic forms can be found in [4]. In particular, we mention pioneering works [11] and [12] by McIntosh, where the relationship of indefinite forms to self-adjoint operators has been considered.

In this paper we follow a different path. Based on the observation that

$$\mathfrak{a}[x, Jy] + \mathrm{i}\mathfrak{v}[x, Jy]$$

is a sectorial closed form (cf. [13] and [15]), we give an alternative proof of the First Representation Theorem for block operator matrices associated with the symmetric forms of the type (1.4) (Theorem 2.4).

We also obtain (i) a relative version of the Tan 2Θ Theorem (Theorem 3.1) (for the pair of the operators $A = JA_J$ and B associated with the forms $\mathfrak{a}$ and $\mathfrak{b}$, respectively) and (ii) its variants (Theorem 4.2) in the case where the form $\mathfrak{a}$ is semibounded, including a generalization of the relative sin Θ Theorem obtained in [6].

We would like to emphasize that in the off-diagonal perturbation theory setting, the First Representation Theorem does not require any assumption on the magnitude of the relative bound of the off-diagonal form $\mathfrak{v}$ with respect to the positive definite form $\mathfrak{a}_J$.

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2. The First Representation Theorem for off-diagonal form perturbations

To introduce the notation, it is convenient to assume the following hypothesis.

Hypothesis 2.1. *Let* $\mathfrak{a}$ *be a symmetric sesquilinear form on* $\mathrm{Dom}[\mathfrak{a}]$ *in a Hilbert space* $\mathfrak{H}$. *Assume that* J *is a self-adjoint involution such that*

$$J\,\mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}[\mathfrak{a}].$$

Suppose that

$$\mathfrak{a}[Jx, y] = \mathfrak{a}[x, Jy] \quad x, y \in \mathrm{Dom}[\mathfrak{a}_J] = \mathrm{Dom}[\mathfrak{a}].$$

Assume, in addition, that the form $\mathfrak{a}_J$ *given by*

$$\mathfrak{a}_J[x, y] = \mathfrak{a}[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}_J] = \mathrm{Dom}[\mathfrak{a}],$$

is a positive definite closed form and denote by $m_\pm$ *the greatest lower bound of the form* $\mathfrak{a}_J$ *restricted to the subspace*

$$\mathfrak{H}_\pm = \mathrm{Ran}(I \pm J).$$

Definition 2.2. Under Hypothesis 2.1, a symmetric sesquilinear form $\mathfrak{v}$ on $\mathrm{Dom}[\mathfrak{v}] \supset \mathrm{Dom}[\mathfrak{a}]$ is said to be *off-diagonal with respect to the orthogonal decomposition*

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$$

if

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}].$$

If, in addition,

$$v_0 \stackrel{\text{def}}{=} \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_J[x]} < \infty, \tag{2.1}$$

the form $\mathfrak{v}$ is said to be an $\mathfrak{a}$-*bounded off-diagonal form.*

Remark 2.3. If $\mathfrak{v}$ is an off-diagonal symmetric form and $x = x_+ + x_-$ is the unique decomposition of an element $x \in \mathrm{Dom}[\mathfrak{a}]$ such that $x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}]$, then

$$\mathfrak{v}[x] = 2\mathrm{Re}\, \mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}]. \tag{2.2}$$

Moreover, if $v_0 < \infty$, then

$$|\mathfrak{v}[x]| \le 2v_0 \sqrt{\mathfrak{a}_J[x_+]\mathfrak{a}_J[x_-]}. \tag{2.3}$$

Proof. To prove (2.2), we use the representation

$$\begin{aligned}\mathfrak{v}[x] &= \mathfrak{v}[x_+ + x_-, x_+ + x_-]\\ &= \mathfrak{v}[x_+] + \mathfrak{v}[x_-] + \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+], \quad x \in \mathrm{Dom}[\mathfrak{a}].\end{aligned}$$

Since $\mathfrak{v}$ is an off-diagonal form, we obtain that

$$\mathfrak{v}[x_+] = \mathfrak{v}[x_+, x_+] = \mathfrak{v}[Jx_+, Jx_+] = -\mathfrak{v}[x_+, x_+] = -\mathfrak{v}[x_+] = 0,$$

and similarly $\mathfrak{v}[x_-] = 0$. Therefore,

$$\mathfrak{v}[x] = \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+] = 2\mathrm{Re}\, \mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}].$$

To prove (2.3), first we observe that

$$\mathfrak{a}_J[x] = \mathfrak{a}_J[x_+] + \mathfrak{a}_J[x_-]$$

and, hence, combining (2.2) and (2.1), we get the estimate

$$|2\mathrm{Re}\, \mathfrak{v}[x_+, x_-]| \le v_0 \mathfrak{a}_J[x] = v_0(\mathfrak{a}_J[x_+] + \mathfrak{a}_J[x_-]) \quad x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}].$$

Hence, for any $t \ge 0$ (and, therefore, for all $t \in \mathbb{R}$) we get that

$$v_0 \mathfrak{a}_J[x_+]\, t^2 - 2|\mathrm{Re}\, \mathfrak{v}[x_+, x_-]|\, t + v_0 \mathfrak{a}_J[x_-] \ge 0,$$

which together with (2.2) implies (2.3). □

In this setting we present an analog of the First Representation Theorem in the off-diagonal perturbation theory.

Theorem 2.4. *Assume Hypothesis 2.1. Suppose that $\mathfrak{v}$ is an $\mathfrak{a}$-bounded off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ symmetric form. On* $\mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}[\mathfrak{a}]$ *introduce the symmetric form*

$$\mathfrak{b}[x, y] = \mathfrak{a}[x, y] + \mathfrak{v}[x, y], \quad x, y \in \mathrm{Dom}[\mathfrak{b}].$$

Then

(i) *there exists a unique self-adjoint operator B in $\mathfrak{H}$ such that* $\mathrm{Dom}(B) \subset \mathrm{Dom}[\mathfrak{b}]$ *and*

$$\mathfrak{b}[x, y] = \langle x, By\rangle \quad x \in \mathrm{Dom}[\mathfrak{b}],\ y \in \mathrm{Dom}(B),$$

(ii) *the operator B is boundedly invertible and the open interval $(-m_-, m_+) \ni 0$ belongs to its resolvent set.*

Proof. (i) Given $\mu \in (-m_-, m_+)$, on $\mathrm{Dom}[\mathfrak{a}_\mu] = \mathrm{Dom}[\mathfrak{a}]$ we introduce the positive closed form $\mathfrak{a}_\mu$ by

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu\langle x, Jy\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{a}_\mu],$$

and denote by $\mathfrak{H}_{\mathfrak{a}_\mu}$ the Hilbert space $\mathrm{Dom}[\mathfrak{a}_\mu]$ equipped with the inner product $\langle\cdot,\cdot\rangle_\mu = \mathfrak{a}_\mu[\cdot,\cdot]$. We remark that the norms $\|\cdot\|_\mu = \sqrt{\mathfrak{a}_\mu[\cdot]}$ on $\mathfrak{H}_{\mathfrak{a}_\mu} = \mathrm{Dom}[\mathfrak{a}_\mu]$ are obviously equivalent. Since $\mathfrak{v}$ is $\mathfrak{a}$-bounded, we conclude then that

$$v_\mu \stackrel{\text{def}}{=} \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]} < \infty, \quad \mu \in (-m_-, m_+).$$

Along with the off-diagonal form $\mathfrak{v}$, introduce a dual form $\mathfrak{v}'$ by

$$\mathfrak{v}'[x, y] = \mathrm{i}\mathfrak{v}[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}].$$

We claim that $\mathfrak{v}'$ is an $\mathfrak{a}$-bounded off-diagonal symmetric form. It suffices to show that

$$v_\mu = v'_\mu < \infty, \quad \mu \in (-m_-, m_+),$$

where

$$v'_\mu \stackrel{\text{def}}{=} \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}'[x]|}{\mathfrak{a}_\mu[x]}. \tag{2.4}$$

Indeed, let $x = x_+ + x_-$ be the unique decomposition of an element $x \in \mathrm{Dom}[\mathfrak{a}]$ such that $x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}]$. By Remark 2.3,

$$\mathfrak{v}[x] = \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+] = 2\mathrm{Re}\,\mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}].$$

In a similar way (since the form $\mathfrak{v}'$ is obviously off-diagonal) we get that

$$\begin{aligned}
\mathfrak{v}'[x] &= \mathrm{i}\mathfrak{v}[x_+ + x_-, J(x_+ + x_-)] \\
&= \mathrm{i}\mathfrak{v}'[x_+] - \mathrm{i}\mathfrak{v}'[x_-] - \mathrm{i}\mathfrak{v}[x_+, x_-] + \mathrm{i}\mathfrak{v}[x_-, x_+] \\
&= -\mathrm{i}\mathfrak{v}[x_+, x_-] + \mathrm{i}\overline{\mathfrak{v}[x_+, x_-]} = 2\mathrm{Im}\,\mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}].
\end{aligned}$$

Clearly, from (2.4) follows that

$$v'_\mu = 2 \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathrm{Im}\,\mathfrak{v}[x_+, x_-]|}{\mathfrak{a}_\mu[x]} = 2 \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathrm{Re}\,\mathfrak{v}[x_+, x_-]|}{\mathfrak{a}_\mu[x]} = v_\mu,$$

for all $\mu \in (-m_-, m_+)$, which completes the proof of the claim.

Next, on $\mathrm{Dom}[\mathfrak{t}_\mu] = \mathrm{Dom}[\mathfrak{a}]$ introduce the sesquilinear form

$$\mathfrak{t}_\mu \stackrel{\mathrm{def}}{=} \mathfrak{a}_\mu + \mathrm{i}\mathfrak{v}', \quad \mu \in (-m_-, m_+).$$

The real part of $\mathfrak{t}_\mu$,

$$(\mathrm{Re}\,\mathfrak{t}_\mu)[x,y] \stackrel{\mathrm{def}}{=} \frac{1}{2}(\mathfrak{t}_\mu[x,y] + \overline{\mathfrak{t}_\mu[y,x]}),$$

equals $\mathfrak{a}_\mu$. Hence, $\mathfrak{t}_\mu$ is closed. Since the form $\mathfrak{a}_\mu$ is positive definite and the form $\mathfrak{v}'$ is an $\mathfrak{a}_\mu$-bounded symmetric form, the form $\mathfrak{t}_\mu$ is a closed sectorial form with the vertex 0 and semi-angle

$$\theta_\mu = \arctan(v'_\mu) = \arctan(v_\mu). \tag{2.5}$$

Let T_μ be a unique m-sectorial operator associated with the form $\mathfrak{t}_\mu$ (cf., e.g., Theorem VI.2.1 in [7]). Introduce the operator

$$B_\mu = JT_\mu \quad \text{on } \mathrm{Dom}(B_\mu) = \mathrm{Dom}(T_\mu),\ \mu \in (-m_-, m_+).$$

We obtain that

$$\begin{aligned} \langle x, B_\mu y\rangle = \langle x, JT_\mu\rangle = \langle Jx, T_\mu y\rangle &= \mathfrak{a}_\mu[Jx,y] + \mathrm{i}\mathfrak{v}'[Jx,y] \\ &= \mathfrak{a}[x,y] - \mu\langle Jx, Jy\rangle + \mathrm{i}^2\mathfrak{v}[Jx,Jy] \\ &= \mathfrak{a}[x,y] - \mu\langle x,y\rangle + \mathfrak{v}[x,y], \end{aligned} \tag{2.6}$$

for all $x \in \mathrm{Dom}[\mathfrak{a}]$, $y \in \mathrm{Dom}(B_\mu) = \mathrm{Dom}(T_\mu)$. In particular, B_μ is a symmetric operator on $\mathrm{Dom}(B_\mu)$, since the forms $\mathfrak{a}$ and $\mathfrak{v}$ are symmetric, and $\mathrm{Dom}(B_\mu) = \mathrm{Dom}(T_\mu) \subset \mathrm{Dom}[a]$.

Since the real part of the form $\mathfrak{t}_\mu$ is closed and positive definite with a positive lower bound, the operator T_μ has a bounded inverse. This implies that the operator $B_\mu = JT_\mu$ has a bounded inverse and, therefore, the symmetric operator B_μ is self-adjoint on $\mathrm{Dom}(B_\mu)$.

As an immediate consequence, we conclude (put $\mu = 0$) that the self-adjoint operator $B \stackrel{\mathrm{def}}{=} B_0$ is associated with the symmetric form $\mathfrak{b}$ and that $\mathrm{Dom}(B) \subset \mathrm{Dom}[\mathfrak{a}]$.

To prove uniqueness, assume that B' is an another self-adjoint operator associated with the form $\mathfrak{b}$. Then for all $x \in \mathrm{Dom}(B)$ and all $y \in \mathrm{Dom}(B')$ we get that

$$\langle x, B'y\rangle = \mathfrak{b}[x, y] = \overline{\mathfrak{b}[y, x]} = \overline{\langle y, Bx\rangle} = \langle Bx, y\rangle,$$

which means that $B = (B')^* = B'$.

(ii) From (2.6) we conclude that the self-adjoint operator $B_\mu + \mu I$ is associated with the form $\mathfrak{b}$ and, hence, by the uniqueness

$$B_\mu = B - \mu I \quad \text{on } \mathrm{Dom}(B_\mu) = \mathrm{Dom}(B).$$

Since B_μ has a bounded inverse for all $\mu \in (m_-, m_+)$, so does $B - \mu I$ which means that the interval $(-m_-, m_+)$ belongs to the resolvent set of the operator B_0. □

Remark 2.5. In the particular case $\mathfrak{v} = 0$, from Theorem 2.4 follows that there exists a unique self-adjoint operator A associated with the form $\mathfrak{a}$.

For a different, more constructive proof of Theorem 2.4 as well as for the history of the subject we refer to our work [4].

Remark 2.6. For the part (i) of Theorem 2.4 to hold it is not necessary to require that the form $\mathfrak{a}_J$ in Hypothesis 2.1 is positive definite. It is sufficient to assume that $\mathfrak{a}_J$ is a semi-bounded from below closed form (see, e.g., [14]).

Remark 2.7. We conjecture that in the case of off-diagonal form perturbation theory in question the following domain stability property

$$\mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}(|B|^{1/2}) \tag{2.7}$$

holds. In this case (see, e.g., [4]), the form $\mathfrak{b}$ is represented by the operator B, i.e.,

$$\mathfrak{b}[x, y] = \langle |B|^{1/2}x, \mathrm{sign}(B)|B|^{1/2}y\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{b}],$$

which is the content of the Second Representation Theorem. We refer however to [4] for a simple counterexample of a not off-diagonal relative bounded perturbation for which the domain stability property fails to hold. We also refer to [17], p. 53, where the domain stability problem in a more general context of the perturbation theory is discussed.

3. The Tan 2Θ Theorem

The main result of this work provides a sharp upper bound for the angle between the positive spectral subspaces $\mathrm{Ran}\,\mathsf{E}_A(\mathbb{R}_+)$ and $\mathrm{Ran}\,\mathsf{E}_B(\mathbb{R}_+)$ of the operators A and B respectively. This result is an extension of Theorem 1 in [13].

Theorem 3.1. *Assume Hypothesis 2.1 and suppose that* $\mathfrak{v}$ *is off-diagonal with respect to the decomposition* $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. *Let* A *be a unique self-adjoint operator associated with the form* $\mathfrak{a}$ *and* B *the self-adjoint operator associated with the form* $\mathfrak{b} = \mathfrak{a} + \mathfrak{v}$ *referred to in Theorem 2.4.*

Then the norm of the difference of the spectral projections $P = \mathsf{E}_A(\mathbb{R}_+)$ *and* $Q = \mathsf{E}_B(\mathbb{R}_+)$ *satisfies the estimate*

$$\|P - Q\| \leq \sin\Big(\frac{1}{2}\arctan v\Big) < \frac{\sqrt{2}}{2},$$

where

$$v = \inf_{\mu \in (-m_-, m_+)} v_\mu = \inf_{\mu \in (-m_-, m_+)} \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]},$$

with

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu\langle x, Jy\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{a}_\mu] = \mathrm{Dom}[\mathfrak{a}].$$

The proof of Theorem 3.1 uses the following result borrowed from [16].

Proposition 3.2. *Let* T *be an m-sectorial operator of semi-angle* $\theta < \pi/2$. *Let* $T = U|T|$ *be its polar decomposition. If* U *is unitary, then the unitary operator* U *is sectorial with semi-angle* θ.

Remark 3.3. We note that for a bounded sectorial operator T with a bounded inverse the statement is quite simple. Due to the equality

$$\langle x, Tx\rangle = \langle |T|^{-1/2}y, U|T|^{1/2}y\rangle = \langle y, |T|^{-1/2}U|T|^{1/2}y\rangle, \quad y = |T|^{1/2}x,$$

the operators T and $|T|^{-1/2}U|T|^{1/2}$ are sectorial with the semi-angle θ. The resolvent sets of the operators $|T|^{-1/2}U|T|^{1/2}$ and U coincide. Therefore, since U is unitary, it follows that U is sectorial with semi-angle θ.

Proof of Theorem 3.1. Given $\mu \in (-m_-, m_+)$, let $T_\mu = U_\mu|T_\mu|$ be the polar decomposition of the sectorial operator T_μ with vertex 0 and semi-angle θ_μ, with

$$\theta_\mu = \arctan(v_\mu) \tag{3.1}$$

(as in the proof of Theorem 2.4 (cf. (2.5)). Since $B_\mu = JT_\mu$, we conclude that

$$|T_\mu| = |B_\mu| \quad \text{and} \quad U_\mu = J^{-1}\operatorname{sign}(B_\mu).$$

Since T_μ is a sectorial operator with semi-angle θ_μ, by a result in [16] (see Proposition 3.2), the unitary operator U_μ is sectorial with vertex 0 and semi-angle θ_μ as well. Therefore, applying the spectral theorem for the unitary operator U_μ from (3.1) we obtain the estimate

$$\|J - \operatorname{sign}(B_\mu)\| = \|I - J^{-1}\operatorname{sign}(B_\mu)\| = \|I - U_\mu\| \leq 2\sin\Big(\frac{1}{2}\arctan v_\mu\Big).$$

Since the open interval $(-m_-, m_+)$ belongs to the resolvent set of the operator $B = B_0$, the involution $\operatorname{sign}(B_\mu)$ does not depend on $\mu \in (-m_-, m_+)$ and, hence, we conclude that

$$\operatorname{sign}(B_\mu) = \operatorname{sign}(B_0) = \operatorname{sign}(B), \quad \mu \in (-m_-, m_+).$$

Therefore,

$$\|P - Q\| = \frac{1}{2}\|J - \operatorname{sign}(B)\| = \frac{1}{2}\|J - \operatorname{sign}(B_\mu)\| \le \sin\left(\frac{1}{2}\arctan v_\mu\right) \tag{3.2}$$

and, hence, since $\mu \in (-m_-, m_+)$ has been chosen arbitrarily, from (3.2) follows that

$$\|P - Q\| \le \inf_{\mu \in (-m_-, m_+)} \sin\left(\frac{1}{2}\arctan v_\mu\right) \le \sin\left(\frac{1}{2}\arctan v\right).$$

The proof is complete. □

As a consequence, we have the following result that can be considered a geometric variant of the Birman–Schwinger principle for the off-diagonal form-perturbations.

Corollary 3.4. *Assume Hypothesis 2.1 and suppose that $\mathfrak{v}$ is off-diagonal. Then the form $\mathfrak{a}_J + \mathfrak{v}$ is positive definite if and only if the $\mathfrak{a}_J$-relative bound (2.1) of $\mathfrak{v}$ does not exceed one. In this case*

$$\|P - Q\| \le \sin\left(\frac{\pi}{8}\right),$$

where P and Q are the spectral projections referred to in Theorem 3.1.

Proof. Since $\mathfrak{v}$ is an $\mathfrak{a}$-bounded form, we conclude that there exists a self-adjoint bounded operator $\mathcal{V}$ in the Hilbert space $\operatorname{Dom}[\mathfrak{a}]$ such that

$$v[x, y] = \mathfrak{a}_J[x, \mathcal{V}y], \quad x, y \in \operatorname{Dom}[\mathfrak{a}].$$

Since $\mathfrak{v}$ is off-diagonal, the numerical range of $\mathcal{V}$ coincides with the symmetric about the origin interval $[-\|\mathcal{V}\|, \|\mathcal{V}\|]$. Therefore, we can find a sequence $\{x_n\}_{n=1}^\infty$ in $\operatorname{Dom}[\mathfrak{a}]$ such that

$$\lim_{n\to\infty} \frac{\mathfrak{v}[x_n]}{\mathfrak{a}_J[x_n]} = -\|\mathcal{V}\|,$$

which proves that $\|\mathcal{V}\| \le 1$ if and only if the form $\mathfrak{a}_J + \mathfrak{v}$ is positive definite. If it is the case, applying Theorem 3.1, we obtain the inequality

$$\|P - Q\| \le \sin\left(\frac{1}{2}\arctan(\|\mathcal{V}\|)\right) \le \sin\left(\frac{\pi}{8}\right)$$

which completes the proof. □

Remark 3.5. We remark that in accordance with the Birman–Schwinger principle, for the form $\mathfrak{a}_J + \mathfrak{v}$ to have the negative spectrum it is necessary that the $\mathfrak{a}_J$-relative bound $\|\mathcal{V}\|$ of the perturbation $\mathfrak{v}$ is greater than one. As Corollary 3.4 shows, in the off-diagonal perturbation theory this condition is also sufficient.

4. Two sharp estimates in the semibounded case

In this section we will be dealing with the case of off-diagonal form-perturbations of a semi-bounded operator.

Hypothesis 4.1. *Assume that A is a self-adjoint semi-bounded from below operator. Suppose that A has a bounded inverse. Assume, in addition, that the following conditions hold.*

(i) **The spectral condition.** *An open finite interval (α, β) belongs to the resolvent set of the operator A. We set*

$$\Sigma_- = \operatorname{spec}(A) \cap (-\infty, \alpha] \quad \textit{and} \quad \Sigma_+ = \operatorname{spec}(A) \cap [\beta, \infty].$$

(ii) **Boundedness.** *The sesquilinear form $\mathfrak{v}$ is symmetric on* $\operatorname{Dom}[\mathfrak{v}] \supset \operatorname{Dom}(|A|^{1/2})$ *and*

$$v \overset{\text{def}}{=} \sup_{0 \neq x \in \operatorname{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\||A|^{1/2}x\|^2} < \infty. \tag{4.1}$$

(iii) **Off-diagonality.** *The sesquilinear form $\mathfrak{v}$ is off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, with*

$$\mathfrak{H}_+ = \operatorname{Ran} \mathsf{E}_A((\beta, \infty)) \quad \textit{and} \quad \mathfrak{H}_- = \operatorname{Ran} \mathsf{E}_A((-\infty, \alpha)).$$

That is,

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \quad x, y \in \operatorname{Dom}[\mathfrak{a}],$$

where the self-adjoint involution J is given by

$$J = \mathsf{E}_A((\beta, \infty)) - \mathsf{E}_A((-\infty, \alpha)). \tag{4.2}$$

Let $\mathfrak{a}$ be the closed form represented by the operator A. A direct application of Theorem 2.4 shows that under Hypothesis 4.1 there is a unique self-adjoint boundedly invertible operator B associated with the form

$$\mathfrak{b} = \mathfrak{a} + \mathfrak{v}.$$

Under Hypothesis 4.1 we distinguish two cases (see Fig. 1 and 2).

Case I. Assume that $\alpha < 0$ and $\beta > 0$. Set

$$d_+ = \operatorname{dist}(\min(\Sigma_+), 0) \quad \text{and} \quad d_- = \operatorname{dist}(\min(\Sigma_-), 0)$$

and suppose that $d_+ > d_-$.

Case II. Assume that $\alpha, \beta > 0$. Set

$$d_+ = \operatorname{dist}(\min(\Sigma_+), 0) \quad \text{and} \quad d_- = \operatorname{dist}(\max(\Sigma_-), 0).$$

As it follows from the definition of the quantities $d_\pm$, the sum $d_- + d_+$ coincides with the distance between the lower edges of the spectral components Σ_+ and Σ_- in Case I, while in Case II the difference $d_+ - d_-$ is the distance from the lower edge of Σ_+ to the upper edge of the spectral component Σ_-. Therefore, $d_+ - d_-$ coincides with the length of the spectral gap (α, β) of the operator A in the latter case.

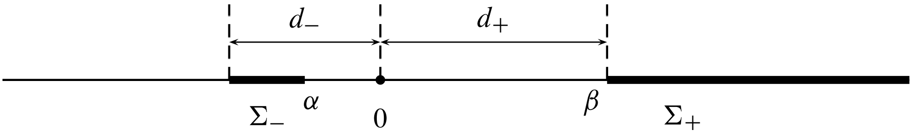

Figure 1. The spectrum of the unperturbed sign-indefinite semibounded invertible operator A in Case I.

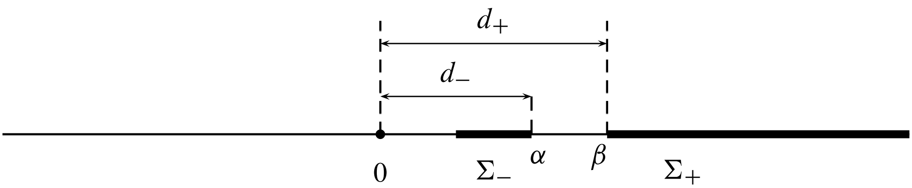

Figure 2. The spectrum of the unperturbed strictly positive operator A with a gap in its spectrum in Case II.

We remark that the condition $d_+ > d_-$ required in Case I holds only if the length of the convex hull of the negative spectrum Σ_- of A does not exceed the one of the spectral gap $(\alpha, \beta) = (\max(\Sigma_-), \min(\Sigma_+))$.

Now we are prepared to state a relative version of the Tan 2Θ Theorem in the case where the unperturbed operator is semi-bounded or positive.

Theorem 4.2. *In either of Cases I or II, introduce the spectral projections*

$$P = \mathsf{E}_A((-\infty, \alpha]) \quad \textit{and} \quad Q = \mathsf{E}_B((-\infty, \alpha]) \tag{4.3}$$

of the operators A and B respectively.

Then the norm of the difference of P and Q satisfies the estimate

$$\|P - Q\| \leq \sin\Big(\frac{1}{2}\arctan\Big[2\frac{v}{\delta}\Big]\Big) < \frac{\sqrt{2}}{2}, \tag{4.4}$$

where

$$\delta = \frac{1}{\sqrt{d_+ d_-}} \begin{cases} d_+ + d_- & \textit{in Case I}, \\ d_+ - d_- & \textit{in Case II}, \end{cases} \tag{4.5}$$

and v stands for the relative bound of the off-diagonal form $\mathfrak{v}$ (with respect to $\mathfrak{a}$) given by (4.1).

Proof. We start with the remark that the form $\mathfrak{a}-\mu$, where $\mathfrak{a}$ is the form of A, satisfies Hypothesis 2.1 with J given by (4.2). Set

$$\mathfrak{a}_\mu = (\mathfrak{a}-\mu)_J, \quad \mu \in (\alpha, \beta),$$

that is,

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}].$$

Notice that $\mathfrak{a}_\mu$ is a strictly positive closed form represented by the operators $JA - J\mu = |A| - \mu J$ and $JA - \mu J = |A - \mu I|$ in Cases I and II, respectively.

Since $\mathfrak{v}$ is off-diagonal, from Theorem 3.1 follows that

$$\|\mathsf{E}_{A-\mu I}(\mathbb{R}_+) - \mathsf{E}_{B-\mu I}(\mathbb{R}_+)\| \le \sin\Big(\frac{1}{2}\arctan v_\mu\Big), \quad \mu \in (\alpha, \beta), \tag{4.6}$$

with

$$v_\mu \stackrel{\mathrm{def}}{=} \sup_{0 \ne x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]}. \tag{4.7}$$

Since $\mathfrak{v}$ is off-diagonal, by Remark 2.3 we get the estimate

$$|\mathfrak{v}[x]| \le 2v_0\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}, \quad x \in \mathrm{Dom}[\mathfrak{a}],$$

where $x = x_+ + x_-$ is a unique decomposition of the element $x \in \mathrm{Dom}[\mathfrak{a}]$ with

$$x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}].$$

Thus, in these notations, taking into account that

$$v_0 = v,$$

where v is given by (4.1), we get the bound

$$v_\mu \le 2v \sup_{0 \ne x \in \mathrm{Dom}[\mathfrak{a}]} \frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_\mu[x]}. \tag{4.8}$$

Since $\mathfrak{a}_\mu$ is represented by $JA - J\mu = |A| - \mu J$ and $JA - \mu J = |A - \mu I|$ in Cases I and II, respectively, we observe that

$$\mathfrak{a}_\mu[x] = \begin{cases} \mathfrak{a}_0[x_+] - \mu\|x_+\|^2 + \mathfrak{a}_0[x_-] + \mu\|x_-\|^2 & \text{in Case I,} \\ \mathfrak{a}_0[x_+] - \mu\|x_+\|^2 - \mathfrak{a}_0[x_-] + \mu\|x_-\|^2 & \text{in Case II.} \end{cases} \tag{4.9}$$

Introducing the elements $y_\pm \in \mathfrak{H}_\pm$,

$$y_\pm \stackrel{\mathrm{def}}{=} \begin{cases} (|A| \mp \mu I)^{1/2}x_\pm & \text{in Case I,} \\ \pm(A - \mu I)^{1/2}x_\pm & \text{in Case II,} \end{cases}$$

and taking into account (4.9), we obtain the representation

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_\mu[x]} = \frac{\||A|^{1/2}(|A|-\mu I)^{-1/2}y_+\| \, \||A|^{1/2}(-A+\mu I)^{-1/2}y_-\|}{\|y_+\|^2+\|y_-\|^2},$$

valid in both Cases I and II. Using the elementary inequality

$$\|y_+\| \, \|y_-\| \le \frac{1}{2}(\|y_+\|^2+\|y_-\|^2),$$

we arrive at the following bound

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_\mu[x]} \le \frac{1}{2}\||A|^{1/2}(|A|-\mu I)^{-1/2}|_{\mathfrak{H}_+}\| \cdot \||A|^{1/2}(-A+\mu I)^{-1/2}|_{\mathfrak{H}_-}\|. \tag{4.10}$$

It is easy to see that

$$\||A|^{1/2}(|A|-\mu I)^{-1/2}|_{\mathfrak{H}_+}\| \le \frac{\sqrt{d_+}}{\sqrt{d_+-\mu}}, \quad \mu \in (\alpha,\beta), \quad \text{in Cases I and II}, \tag{4.11}$$

while

$$\||A|^{1/2}(-A+\mu I)^{-1/2}|_{\mathfrak{H}_-}\| \le \begin{cases} \dfrac{\sqrt{d_-}}{\sqrt{d_-+\mu}}, & \mu \in (0,\beta), \quad \text{in Case I}, \\ \dfrac{\sqrt{d_-}}{\sqrt{\mu-d_-}}, & \mu \in (\alpha,\beta), \quad \text{in Case II}. \end{cases} \tag{4.12}$$

Choosing $\mu = \frac{d_+-d_-}{2} > 0$ in Case I (recall that $d_+ > d_-$ by the hypothesis) and $\mu = \frac{d_++d_-}{2}$ in Case II, and combining (4.10), (4.11), and (4.12), we get the estimates

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\frac{d_+-d_-}{2}}[x]} \le \frac{\sqrt{d_+d_-}}{d_++d_-} \quad \text{in Case I}$$

and

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\frac{d_++d_-}{2}}[x]} \le \frac{\sqrt{d_+d_-}}{d_+-d_-} \quad \text{in Case II}.$$

Hence, from (4.8) it follows that

$$v_{\frac{d_+-d_-}{2}} \le 2v\frac{\sqrt{d_+d_-}}{d_++d_-} \quad \text{in Case I}$$

and

$$v_{\frac{d_++d_-}{2}} \le 2v\frac{\sqrt{d_+d_-}}{d_+-d_-} \quad \text{in Case II}.$$

Applying (4.6), we get the norm estimates

$$\|\mathsf{E}_{A-\frac{d_+-d_-}{2}I}(\mathbb{R}_+) - \mathsf{E}_{B-\frac{d_+-d_-}{2}I}(\mathbb{R}_+)\| \le \sin\Big(\frac{1}{2}\arctan\Big[2\frac{\sqrt{d_+d_-}}{d_+ + d_-}v\Big]\Big) \quad (4.13)$$

in Case I and

$$\|\mathsf{E}_{A-\frac{d_++d_-}{2}I}(\mathbb{R}_+) - \mathsf{E}_{B-\frac{d_++d_-}{2}I}(\mathbb{R}_+)\| \le \sin\Big(\frac{1}{2}\arctan\Big[2\frac{\sqrt{d_+d_-}}{d_+ - d_-}v\Big]\Big) \quad (4.14)$$

in Case II. It remains to observe that $\|P - Q\|$, where the spectral projections P and Q are given by (4.3), coincides with the left hand side of (4.13) and (4.14) in Case I and Case II, respectively.

The proof is complete. □

Remark 4.3. We remark that the quantity δ given by (4.5) coincides with the *relative distance* (with respect to the origin) between the lower edges of the spectral components Σ_+ and Σ_- in Case I and it has the meaning of the *relative length* (with respect to the origin) of the spectral gap (d_-, d_+) in Case II.

For the further properties of the relative distance and various relative perturbation bounds we refer to the paper [10] and references quoted therein.

We also remark that in Case II, i.e., in the case of a positive operator A, the bound (4.4) directly improves a result obtained in [6], *the relative* SinΘ *Theorem*, that in the present notations is of the form

$$\|P - Q\| \le \frac{v}{\delta}.$$

We conclude our exposition with considering an example of a 2×2 numerical matrix that shows that the main results obtained above are sharp.

Example 4.4. Let $\mathfrak{H}$ be the two-dimensional Hilbert space $\mathfrak{H} = \mathbb{C}^2$, $\alpha < \beta$ and $w \in \mathbb{C}$.

We set

$$A = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \quad V = \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathfrak{v}$ be the symmetric form represented by (the operator) V.

Clearly, the form $\mathfrak{v}$ satisfy Hypothesis 4.1 with the relative bound v given by

$$v = \frac{|w|}{\sqrt{|\alpha\beta|}},$$

provided that $\alpha, \beta \ne 0$. Since $VJ = -JV$, the form $\mathfrak{v}$ is off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

In order to illustrate our results, denote by B the self-adjoint matrix associated with the form $\mathfrak{a} + \mathfrak{v}$, that is,

$$B = A + V = \begin{pmatrix} \beta & w \\ w^* & \alpha \end{pmatrix}.$$

Denote by P the orthogonal projection associated with the eigenvalue α of the matrix A, and by Q the one associated with the lower eigenvalue of the matrix B.

It is well known (and easy to see) that the classical Davis–Kahan Tan 2Θ Theorem (cf. (1.2)) is exact in the case of 2×2 numerical matrices. In particular, the norm of the difference of P and Q can be computed explicitly

$$\|P - Q\| = \sin\Big(\frac{1}{2}\arctan\Big[\frac{2|w|}{\beta - \alpha}\Big]\Big). \tag{4.15}$$

Since, in the case in question,

$$v_\mu = \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]} = \frac{|w|}{\sqrt{(\beta - \mu)(\mu - \alpha)}}, \quad \mu \in (\alpha, \beta), \tag{4.16}$$

from (4.16) follows that

$$\inf_{\mu \in (\alpha, \beta)} v_\mu = \frac{2|w|}{\beta - \alpha}$$

(with the infimum attained at the point $\mu = \frac{\alpha+\beta}{2}$).

Therefore, the result of the relative tan 2Θ Theorem 3.1 is sharp.

It is easy to see that if $\alpha < 0 < \beta$ (Case I), then the equality (4.15) can also be rewritten in the form

$$\|P - Q\| = \sin\Big(\frac{1}{2}\arctan\Big[2\frac{\sqrt{d_+ d_-}}{d_+ + d_-}v\Big]\Big), \tag{4.17}$$

where $d_+ = \beta$, $d_- = -\alpha$ and $v = \frac{|w|}{\sqrt{|\alpha|\beta}}$.

If $0 < \alpha < \beta$ (Case II), the equality (4.15) states that

$$\|P - Q\| = \sin\Big(\frac{1}{2}\arctan\Big[2\frac{\sqrt{d_+ d_-}}{d_+ - d_-}v\Big]\Big), \tag{4.18}$$

with $d_+ = \beta$, $d_- = \alpha$, and $v = \frac{|w|}{\sqrt{\alpha\beta}}$.

The representations (4.17) and (4.18) show that the estimate (4.4) becomes equality in the case of 2×2 numerical matrices and, therefore, the results of Theorem 4.2 are sharp.

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