

Convergence of Dirichlet eigenvalues for elliptic systems on perturbed domains

Justin L. Taylor

Abstract. We consider the eigenvalues of an elliptic operator

$$(Lu)^\beta = -\frac{\partial}{\partial x_j} \left(a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \right), \quad \beta = 1, \dots, m,$$

where $u = (u^1, \dots, u^m)^t$ is a vector valued function and $a^{\alpha\beta}(x)$ are $(n \times n)$ matrices whose elements $a_{ij}^{\alpha\beta}(x)$ are at least uniformly bounded measurable real-valued functions such that

$$a_{ij}^{\alpha\beta}(x) = a_{ji}^{\beta\alpha}(x)$$

for any combination of α, β, i , and j . We assume we have two non-empty, open, disjoint, and bounded sets, Ω and $\tilde{\Omega}$, in \mathbb{R}^n , and add a set T_ε of small measure to form the domain Ω_ε . Then we show that as $\varepsilon \rightarrow 0^+$, the Dirichlet eigenvalues corresponding to the family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ converge to the Dirichlet eigenvalues corresponding to $\Omega_0 = \Omega \cup \tilde{\Omega}$. Moreover, our rate of convergence is independent of the eigenvalues. In this paper, we consider the Lamé system, systems which satisfy a strong ellipticity condition, and systems which satisfy a Legendre–Hadamard ellipticity condition.

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1. Introduction

There is a great deal of work studying eigenvalues for elliptic equations, but there seems to be less work on eigenvalues for elliptic systems. Much of the work on equations requires estimates for solutions that do not hold for systems. In this paper, we consider the behavior of eigenvalues for elliptic systems in singularly perturbed domains. We give a simple characterization of the families of domains that we can study and this class includes families such as dumbbell domains formed by connecting two domains by a thin tube. We show that as the measure of the perturbation shrinks away, the convergence of the eigenvalues is obtained. We also provide a rate of convergence, which is independent of any eigenvalue. We make no assumption on

the smoothness of the coefficients and only mild assumptions on the boundary of the domain.

Studying solutions of elliptic boundary value problems with Dirichlet or Neumann boundary conditions on domains which can be approximated by solutions on simpler domains has been an interest for many years, and is still ongoing. The motivation to study such problems is that it is easier to study the spectra on sets with a reduced dimensionality. One may approximate the spectra on these “fattened” sets with the spectra on the “thinner” sets. Some applications include studying quantum wires, free-electron theory of conjugated molecules, and photonic crystals. For a complete description, see the work of Kuchment [26]. Recent work by Exner and Post [15] study the Neumann Laplacian on manifolds with thin tubes which is related to the theory of quantum graphs. The Fireman’s Pole problem consists of approximating the resolvents of a bounded set in \mathbb{R}^3 by the resolvents of this set with a cylinder removed. For a complete description, see Rauch and Taylor [29]. A classic paper by Babuska and Výborný [5] shows continuity of Dirichlet eigenvalues for elliptic equations under a regular variation of the domain, but gives no rates of convergence. Dancer [11], [12] considers how perturbing the domain affects the number of positive solutions for nonlinear equations with Dirichlet boundary conditions and includes the case where solutions are eigenfunctions for the Laplacian. Davies [14] and Pang [28] study the approximation of Dirichlet eigenvalues and corresponding eigenfunctions in a domain Ω by eigenvalues and eigenfunctions in sets of the form $R(\varepsilon) = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \varepsilon\}$. They each give rates of convergence and their estimates include the case when the domain is irregularly shaped. The work of Brown, Hislop, and Martinez [6] provides upper and lower bounds on the splitting between the first two Dirichlet eigenvalues in a symmetric dumbbell region with a straight tube. Chavel and Feldman [9] examine eigenvalues on a compact manifold with a small handle and Dirichlet conditions on the ends of the handle. The work of Anné and Colbois [1] examines the behavior of eigenvalues of the Laplacian on p -forms under a singular perturbation obtained by adding a thin handle to a compact manifold, but requires more regularity on the eigenfunctions than holds in our setting.

More recent work for Dirichlet conditions includes work by Daners [13], which shows convergence of solutions for elliptic equations on sequences of domains. These domains Ω_n converge to a limit domain Ω in the sense of sequences $u_n \in H_0^1(\Omega_n)$ converging to a function $u \in H_0^1(\Omega)$. Also, Burenkov and Lamberti [8] prove sharp spectral stability estimates for higher-order elliptic operators on domains in certain Hölder classes in terms of the Lebesgue measure of the symmetric difference of the different domains. Kozlov [25] obtains asymptotics of Dirichlet eigenvalues for domains in \mathbb{R}^n for $n \geq 2$ using Hadamard’s formula. Grieser and Jerison [20] also give asymptotics for Dirichlet eigenvalues and eigenfunctions, but only on plane domains.

We note here that the results for Neumann eigenvalues may be different than those for Dirichlet eigenvalues. In fact, a classic example of Courant and Hilbert [10] shows that the Neumann eigenvalues may not vary continuously as the domain varies. Their

example is constructed by taking the unit square in \mathbb{R}^2 and attaching a thin handle with a proportional square attached to the other end. They show that if $\{\lambda_n^\varepsilon\}$ and $\{\lambda_n^0\}$ are the Neumann eigenvalues of $-\Delta$ in increasing order including multiplicities with respect to the perturbed square and the unit square, then $\lambda_2^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, but $\lambda_2^0 > 0$. This example shows that one needs additional regularity in order to achieve convergence. Furthermore, Arrieta, Hale, and Han [4] show that for this type of domain, $\lambda_m^\varepsilon \rightarrow \lambda_{m-1}^0$, as $\varepsilon \rightarrow 0$ for $m \geq 3$. Another work of Arrieta [3] gives rates of convergence for eigenvalues of the Neumann Laplacian on a dumbbell domain in \mathbb{R}^2 when the tube is more general. Jimbo and Morita [24] study the first N eigenvalues of the Neumann Laplacian in N disjoint domains connected by thin tubes. They show that the first N eigenvalues approach zero and the $(N + 1)$ st eigenvalue is uniformly bounded away from zero. If D_1 and D_2 are two disjoint domains, then for $\{\sigma_k\} = \{\mu_l\} \cup \{\lambda_j\}$, where $\{\mu_l\}$ are the Neumann eigenvalues of $-\Delta$ in $D = D_1 \cup D_2$ and $\{\lambda_j\}$ are the Dirichlet eigenvalues of $\frac{-d^2}{dx^2}$ in $(-1, 1)$, Jimbo [22] gives a rate of convergence on the difference $\sigma_k - \sigma_k^\varepsilon$. This work was generalized to more classes of domains in a more recent work by Jimbo and Kosugi [23]. Also, Brown, Hislop, and Martinez [7] show that if $\sigma_k \in \{\mu_l\} \setminus \{\lambda_j\}$ then

$$|\sigma_k - \sigma_k^\varepsilon| \leq C \left[\log \left(\frac{1}{\varepsilon} \right) \right]^{-\frac{1}{2}}, \quad n = 2,$$

and

$$|\sigma_k - \sigma_k^\varepsilon| \leq C \varepsilon^{\frac{n-2}{2}}, \quad n \geq 3.$$

Here, we aim to provide an outline of the proof. In section 2, we give several definitions and describe the family of domains for which we can prove the convergence of eigenvalues. We also describe the well-known construction of eigenvalues and state our main result. In section 3, we give Theorem 3.1 from Giaquinta and Modica [17] and [18] which uses a technique introduced by Gehring [16]. We also prove a Caccioppoli type estimate for eigenfunctions in Theorem 3.4 and use this along with Theorem 3.1 to obtain a reverse Hölder inequality given in Theorem 3.5. This gives L^p -integrability for the gradient of the eigenfunctions for $p > 2$. In section 4, we are able to bound these L^p norms by a constant in Proposition 4.2. The proof uses the reverse Hölder inequality as the key ingredient. This estimate is then used to prove Lemma 4.2 and Proposition 4.4, which are used to satisfy the first part of a well-known theorem from Anné [2] given in Lemma 4.1. The second part of Lemma 4.1 follows from the first part along with the above estimates, thus giving Corollary 4.1. The main result follows from this corollary. As a by-product of our research, we give a simple proof of Shi and Wright’s [30] L^p -estimates for the gradient of the Lamé system as well as other elliptic systems. Many of the results first appeared in the author’s Ph.D. dissertation [31].

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2. Preliminaries and main result

We give conditions on a family of domains Ω_ε that allow us to prove the convergence of eigenvalues. We let Ω and $\tilde{\Omega}$ in \mathbb{R}^n be two non-empty, open, disjoint, and bounded sets. We let $\varepsilon_1 > 0$ (which will be chosen small later), and then let $\{T_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_1}$ be a family of open sets such that

$$T_{\tilde{\varepsilon}} \subset T_\varepsilon \quad \text{if } \tilde{\varepsilon} \leq \varepsilon.$$

If $|T_\varepsilon|$ denotes the Lebesgue measure of T_ε , then

$$|T_\varepsilon| \leq C\varepsilon^d, \tag{2.1}$$

where C and $0 < d \leq n$ are independent of ε . Fix two points p_1 and p_2 on $\partial\Omega$ and $\partial\tilde{\Omega}$, respectively. For each ε , let B_ε and \tilde{B}_ε be two balls of radius ε in \mathbb{R}^n centered at p_1 and p_2 , respectively. The connections from T_ε to Ω and $\tilde{\Omega}$ will be contained in B_ε and \tilde{B}_ε , so that $T_\varepsilon \cap \Omega = \emptyset$ and $\bar{T}_\varepsilon \cap \bar{\Omega} \subset B_{\frac{\varepsilon}{2}}$ where $B_{\frac{\varepsilon}{2}}$ is the concentric ball to B_ε of radius $\frac{\varepsilon}{2}$. Also, suppose a similar condition for $\tilde{\Omega}$ and \tilde{B}_ε . Then for any ε , define Ω_ε to be the set $\Omega \cup \tilde{\Omega} \cup T_\varepsilon$, which we assume to be open and connected, and $\Omega_0 = \Omega \cup \tilde{\Omega}$. So, if our family is the family of dumbbell domains, you may think of T_ε as a ‘‘tube’’ connecting each of the two domains. We now have the family of domains $\{\Omega_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_1}$.

Next, we give a condition on the boundary of Ω_ε . If B_r is any ball of radius r satisfying $B_r \cap \Omega_\varepsilon^c \neq \emptyset$, then

$$|B_{2r} \cap \Omega_\varepsilon^c| \geq C_0 r^n, \tag{2.2}$$

where C_0 is a constant independent of r and ε . This eliminates domains with ‘‘cracks’’ and ‘‘in-cusps,’’ and will be used to help show the Caccioppoli inequality in Theorem 3.4 for the case when we are close to the boundary.

Throughout this paper we use the convention of summing over repeated indices, where i and j will run from 1 to n and α, β , and γ will run from 1 to m . We let $a_{ij}^{\alpha\beta}(x)$ be bounded, measurable, real-valued functions on \mathbb{R}^n which satisfy the symmetry condition

$$a_{ij}^{\alpha\beta}(x) = a_{ji}^{\beta\alpha}(x), \quad i, j = 1, 2, \dots, n, \quad \alpha, \beta = 1, 2, \dots, m.$$

We let $L^2(\Omega_\varepsilon)$ denote the space of square integrable functions taking values in \mathbb{R}^m and $H_0^1(\Omega_\varepsilon)$ denotes the Sobolev space of vector-valued functions having one derivative in $L^2(\Omega_\varepsilon)$ and which vanish on the boundary. We use u_j^α to denote the partial derivative $\frac{\partial u^\alpha}{\partial x_j}$.

Let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be a cutoff function so that we have $\eta_\varepsilon = 0$ in T_ε , $\eta_\varepsilon = 1$ in $\Omega_0 \setminus (B_\varepsilon \cup \tilde{B}_\varepsilon)$, $|\nabla \eta_\varepsilon| \leq \frac{C_n}{\varepsilon}$, and $0 \leq \eta_\varepsilon \leq 1$, where C_n only depends on n . We emphasize that B_ε , \tilde{B}_ε , and η_ε depend on the parameter ε . With these assumptions and definitions, we have that for any $u \in H_0^1(\Omega_\varepsilon)$, $\eta_\varepsilon u$ will be in $H_0^1(\Omega_0)$.

We now introduce the notion of an eigenvalue and corresponding eigenvector. We say that the number σ is a *Dirichlet eigenvalue of L with Dirichlet eigenfunction* $u \in H_0^1(\Omega)$, if $u \neq 0$ and

$$\int_\Omega a_{ij}^{\alpha\beta}(x) u_i^\alpha(x) \varphi_j^\beta(x) \, dx = \sigma \int_\Omega u^\gamma(x) \varphi^\gamma(x) \, dx, \quad \varphi \in H_0^1(\Omega). \tag{2.3}$$

We say that L satisfies the *Legendre–Hadamard condition* if there exists $\theta > 0$ so that

$$a_{ij}^{\alpha\beta}(x) \xi_\alpha \xi_\beta \psi_i \psi_j \geq \theta |\xi|^2 |\psi|^2, \quad \xi \in \mathbb{R}^m, \psi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\varepsilon. \tag{2.4}$$

If we define the norm on matrices $A = A_{ij} \in \mathbb{R}^{m \times n}$ as

$$|A|^2 = \sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2,$$

and L satisfies the *Legendre–Hadamard condition* with continuous coefficients in $\bar{\Omega}$, then it is well-known that for any $u \in H_0^1(\Omega)$, we have Gårding’s inequality [32], p. 347,

$$C_1 \int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega a_{ij}^{\alpha\beta}(x) u_i^\alpha(x) u_j^\beta(x) \, dx + C_2 \int_\Omega |u|^2 \, dx. \tag{2.5}$$

L is said to satisfy a *strong Legendre condition* or a *strong ellipticity condition* if there exists $\theta > 0$ so that

$$a_{ij}^{\alpha\beta}(x) \xi_i^\alpha \xi_j^\beta \geq \theta |\xi|^2, \quad \xi \in \mathbb{R}^{m \times n}, \text{ a.e. } x \in \Omega_\varepsilon. \tag{2.6}$$

We introduce the Lamé system as $Lu = -\text{div} \zeta(u)$, where $\zeta(u)$ denotes the stress tensor defined by

$$\zeta_j^\beta(u) \stackrel{\text{def}}{=} a_{ij}^{\alpha\beta} u_i^\alpha \tag{2.7}$$

which is defined in terms of the Lamé moduli $\nu(x)$ and $\mu(x)$ by

$$a_{ij}^{\alpha\beta}(x) \stackrel{\text{def}}{=} \nu(x) \delta_{i\alpha} \delta_{j\beta} + \mu(x) \delta_{ij} \delta_{\alpha\beta} + \mu(x) \delta_{i\beta} \delta_{j\alpha}, \tag{2.8}$$

where $\nu(x)$ and $\mu(x)$ are both assumed to be bounded and measurable. Also, define the strain tensor $\kappa(u)$ as

$$\kappa_{ij}(u) \stackrel{\text{def}}{=} \frac{1}{2}(u_j^i + u_i^j). \tag{2.9}$$

Note that for the Lamé system, $m = n$ and the Lamé parameters $\nu(x)$ and $\mu(x)$ given in (2.8) satisfy the conditions

$$\nu(x) \geq 0 \quad \text{and} \quad \mu(x) \geq \delta > 0. \quad (2.10)$$

With these assumptions, the Lamé system satisfies the ellipticity condition

$$a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta \geq \tau |\kappa(u)|^2, \quad u \in H_0^1(\Omega_\varepsilon) \quad (2.11)$$

where $\tau = 2\delta$. With Korn's 1st Inequality, it is easy to see that for the Lamé system, we have

$$\frac{\tau}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 dy \leq \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta dy, \quad u \in H_0^1(\Omega_\varepsilon).$$

Thus, if u satisfies either the ellipticity condition (2.6), (2.11), or (2.4) with continuous coefficients in $\bar{\Omega}$, then we have Gårding's inequality (2.5).

The well-known construction of eigenvalues and eigenfunctions for scalar functions (which is the same for vector-valued functions) is taken from Gilbarg and Trudinger [19], p. 212. If we define the bilinear form on $H_0^1(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)$ as

$$B_\varepsilon(u, v) \stackrel{\text{def}}{=} \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha v_j^\beta dx \quad (2.12)$$

and define the Rayleigh quotient R_ε as

$$R_\varepsilon(u) \stackrel{\text{def}}{=} \frac{B_\varepsilon(u, u)}{\|u\|_{L^2(\Omega_\varepsilon)}^2} \quad (2.13)$$

for $u \neq 0$, then we can construct an increasing sequence of eigenvalues, listed according to multiplicity, $\{\sigma_k\}_{k=1}^\infty$ such that for each corresponding eigenfunction $u_k \in H_0^1(\Omega_\varepsilon)$, we have

$$\min_{w \in \{u_1, \dots, u_{k-1}\}^\perp} R_\varepsilon(w) = R_\varepsilon(u_k) = \sigma_k \quad (2.14)$$

and

$$\|u_k\|_{L^2(\Omega_\varepsilon)} = 1 \quad (2.15)$$

for any k . Furthermore, each eigenspace is finite-dimensional and the constructed set of eigenfunctions forms an orthonormal basis in $L^2(\Omega_\varepsilon)$.

We now state the main result.

Theorem 2.1. *Let*

$$(Lu)^\beta = -\frac{\partial}{\partial x_j} \left(a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \right) \quad \beta = 1, \dots, m$$

satisfy one of the following conditions:

(1) L has uniformly bounded coefficients and satisfies either the ellipticity condition (2.6) or the ellipticity condition (2.11);

(2) L has continuous coefficients and satisfies the ellipticity condition (2.4).

Also assume $\{\sigma_k^0\}_{k=1}^\infty$ and $\{\sigma_k^\varepsilon\}_{k=1}^\infty$ are the Dirichlet eigenvalues of L with respect to Ω_0 and Ω_ε in increasing order numbered according to multiplicity. Then for each $J \in \mathbb{N}$, we have the following estimate:

$$|\sigma_J^\varepsilon - \sigma_J^0| \leq C \varepsilon^a$$

for $0 < \varepsilon \leq \varepsilon_0(J)$, where $\varepsilon_0(J)$ depends on the multiplicity of σ_J^0 . Moreover, the rate $a > 0$ is independent of any eigenvalue and C only depends on the eigenvalue σ_J^0 and the distance from σ_J^0 to nearby eigenvalues.

3. A reverse Hölder inequality

If

$$\int_E |f(y)| dy$$

is defined to be the average of f on E , then recall that the maximal function is defined for $f \in L^1_{loc}(\mathbb{R}^n)$ to be

$$M(f)(x) \stackrel{\text{def}}{=} \sup_{r>0} \int_{B_r(x)} |f(y)| dy$$

where $B_r(x)$ is a ball of radius r centered at x . Also, define $M_R(f)(x)$ to be

$$M_R(f)(x) \stackrel{\text{def}}{=} \sup_{R>r>0} \int_{B_r(x)} |f(y)| dy.$$

We will need the following theorem from Giaquinta [17], p. 122, which uses the technique introduced by Gehring [16], and refined by Giaquinta and Modica [18].

Theorem 3.1. *Let $r > q > 1$, and Q_R be a cube in \mathbb{R}^n with sidelength R centered at 0. Define*

$$d(x) = \text{dist}(x, \partial Q_R).$$

If f and g are measurable functions such that $f \in L^r(Q_R)$, $g \in L^q(Q_R)$, $f = g = 0$ outside Q_R , and with the added condition that

$$M_{\frac{d(x)}{2}}(|g|^q)(x) \leq bM^q(g)(x) + M(|f|^q) + aM(|g|^q)(x)$$

for almost every x in Q_R where $b \geq 0$ and $0 \leq a < 1$, then $g \in L^p(Q_{\frac{R}{2}})$, for $p \in [q, q + \varepsilon)$ and

$$\left(\int_{Q_{R/2}} |g|^p(y) dy \right)^{\frac{1}{p}} \leq C \left[\left(\int_{Q_R} |g|^q(y) dy \right)^{\frac{1}{q}} + \left(\int_{Q_R} |f|^p(y) dy \right)^{\frac{1}{p}} \right] \quad (3.1)$$

where ε and C depend on b, q, n, a , and r .

The conclusion of this theorem is known as a reverse Hölder inequality. To show that the gradient of eigenfunctions satisfy this inequality, we will need to prove a Caccioppoli inequality. However, to show this Caccioppoli inequality, we first need the following two well-known inequalities taken from Hebey [21], p. 44, and Oleinik [27], p. 27.

Theorem 3.2 (Sobolev–Poincaré Inequality). *Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Also, let B_r be any ball of radius r with $u \in W^{1,p}(B_r)$. Then, for S contained in B_r with $|S| \geq c_0 r^n$,*

$$\int_{B_r} |u(x) - u_S|^q dx \leq C \left(\int_{B_r} |\nabla u|^p(x) dx \right)^{\frac{q}{p}} \tag{3.2}$$

where $u_S = \frac{1}{|S|} \int_S u dy$, for some constant $C(n, p, c_0)$, independent of u .

Theorem 3.3 (Korn’s Inequality on a Ball). *If $u \in H^1(B_r)$ then*

$$\|\nabla u\|_{L^2(B_r)}^2 \leq C(\|\kappa(u)\|_{L^2(B_r)}^2 + \frac{1}{r^2} \|u\|_{L^2(B_r)}^2) \tag{3.3}$$

where C only depends on n .

We now state and prove a Caccioppoli inequality for eigenfunctions.

Theorem 3.4. *Let u be an eigenfunction with eigenvalue σ associated to the operator L satisfying either (2.6) or (2.11) with uniformly bounded coefficients or associated to (2.4) with continuous coefficients. Extending u to be 0 outside Ω_ε , there exists $r_0 > 0$ so that if $r_0 \geq r > 0$, $x \in \mathbb{R}^n$, we have*

$$\begin{aligned} \int_{B_r} |\nabla u|^2 dy &\leq C_1 \left(\int_{B_{2r}} |\nabla u|^{\frac{2n}{n+2}} dy \right)^{\frac{n+2}{n}} \\ &+ C_2 |\sigma| \int_{B_{2r}} |u|^2 dy + C_3 \int_{B_{2r}} |\nabla u|^2 dy \end{aligned} \tag{3.4}$$

where B_r is a ball with radius r centered at x , $C_3 < 1$, and $C_1 > 0$ only depends on $M = \max_{i,j,\alpha,\beta} \|a_{ij}^{\alpha\beta}\|_{L^\infty(\Omega_\varepsilon)}$, n , m , θ , τ , and C_0 . Furthermore, if L satisfies either (2.6) or (2.11) with uniformly bounded coefficients, then the inequality holds for any $r > 0$.

Proof. First, choose a ball B_r and define a cutoff function $v \in C_c^\infty(\mathbb{R}^n)$ to be so that $v = 1$ in B_r , $v = 0$ outside B_{2r} , $|\nabla v| \leq \frac{C_n}{r}$, and $0 \leq v \leq 1$, where C_n only depends on n . Below, we will find an appropriate constant vector $\rho \in \mathbb{R}^m$, so that $v^2(u - \rho) \in H_0^1(\Omega_\varepsilon)$. By the weak formulation (2.3), we have

$$\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha [v^2(u - \rho)]_j^\beta dy = \sigma \int_{\Omega_\varepsilon} u^\gamma [v^2(u - \rho)]^\gamma dy.$$

Then, performing the differentiations, we get

$$\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} u_i^\alpha [2v v_j (u - \rho)^\beta + v^2 u_j^\beta] dy = \sigma \int_{\Omega_\varepsilon} u^\gamma v^2 (u - \rho)^\gamma dy. \tag{3.5}$$

From this point, the argument depends on the ellipticity condition. We have three cases.

Case 1: L satisfies the strong ellipticity condition (2.6). Using (2.6) and properties of v , we obtain the inequality

$$\int_{B_{2r}} v^2 a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta dy \leq \int_{B_{2r}} 2M \frac{C_n}{r} v |\nabla u| |u - \rho| dy + \int_{B_{2r}} |\sigma| |u| |u - \rho| dy$$

which, for any constant $\omega > 0$, then leads to

$$\begin{aligned} \int_{B_{2r}} v^2 a_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta dy &\leq \int_{B_{2r}} \frac{\omega v^2 |\nabla u|^2}{2} dy + \frac{C}{\omega r^2} \int_{B_{2r}} |u - \rho|^2 dy \\ &\quad + C |\sigma| \int_{B_{2r}} |u|^2 dy \end{aligned} \tag{3.6}$$

where C depends on M and C_n . Then choosing $\omega = \theta$ in (3.6) gives

$$\frac{\theta}{2} \int_{B_{2r}} v^2 |\nabla u|^2 dy \leq \frac{C}{\theta r^2} \int_{B_{2r}} |u - \rho|^2 dy + C |\sigma| \int_{B_{2r}} |u|^2 dy.$$

Then, multiplying both sides by $\frac{2}{\theta}$ and using that $v = 1$ on B_r gives

$$\int_{B_r} |\nabla u|^2 dy \leq \frac{2C}{\theta^2 r^2} \int_{B_{2r}} |u - \rho|^2 dy + \frac{2C |\sigma|}{\theta} \int_{B_{2r}} |u|^2 dy. \tag{3.7}$$

Now, for the term $\int_{B_{2r}} |u - \rho|^2 dy$, we must consider two subcases.

Subcase 1.A. If $B_{2r} \subset \Omega_\varepsilon$, then let

$$\rho^\alpha = \int_{B_{2r}} u^\alpha dy.$$

Our condition on the support of v implies $v^2(u - \rho) \in H_0^1(\Omega_\varepsilon)$. So, setting $q = 2$ and $S = B_{2r}$ in the Sobolev–Poincaré Inequality (3.2), we obtain

$$\int_{B_{2r}} |u - \rho|^2 dy \leq C \left(\int_{B_{2r}} |\nabla u|^{\frac{2n}{n+2}} dy \right)^{\frac{n+2}{n}}.$$

Using this estimate with (3.7) gives

$$\int_{B_r} |\nabla u|^2 dy \leq \frac{C}{r^2} \left(\int_{B_{2r}} |\nabla u|^{\frac{2n}{n+2}} dy \right)^{\frac{n+2}{n}} + C |\sigma| \int_{B_{2r}} |u|^2 dy.$$

Now, dividing through by r^n gives the desired result with $C_3 = 0$.

Subcase 1.B. If $B_{2r} \cap \Omega_\varepsilon^c \neq \emptyset$, then set $\rho = 0$, which, again, guarantees that $v^2(u - \rho) \in H_0^1(\Omega_\varepsilon)$. So setting $q = 2$ and $S = B_{4r} \cap \Omega_\varepsilon$ in the Sobolev–Poincaré Inequality (3.2), we have by our assumption on Ω_ε^c (2.2) that

$$\int_{B_{4r}} |u - \rho|^2 dy \leq C \left(\int_{B_{4r}} |\nabla u|^{\frac{2n}{n+2}} dy \right)^{\frac{n+2}{n}}.$$

From (3.7), we obtain

$$\int_{B_r} |\nabla u|^2 dy \leq \frac{C}{r^2} \left(\int_{B_{4r}} |\nabla u|^{\frac{2n}{n+2}} dy \right)^{\frac{n+2}{n}} + C|\sigma| \int_{B_{4r}} |u|^2 dy.$$

A simple covering argument gives the estimate with B_{4r} replaced with B_{2r} .

Case 2: L satisfies the ellipticity condition (2.11). From (2.11) and (3.6), we have

$$\int_{B_r} \tau |\kappa(u)|^2 dy \leq \int_{B_{2r}} \frac{\omega v^2 |\nabla u|^2}{2} dy + \frac{C}{\omega r^2} \int_{B_{2r}} |u - \rho|^2 dy + C|\sigma| \int_{B_{2r}} |u|^2 dy.$$

Also, by Korn’s inequality (3.3), we have

$$\frac{\tau}{C} \int_{B_r} |\nabla u|^2 dy - \frac{\tau}{r^2} \int_{B_r} |u - \rho|^2 dy \leq \int_{B_r} \tau |\kappa(u)|^2 dy.$$

This implies

$$\begin{aligned} \int_{B_r} |\nabla u|^2 dy &\leq \frac{C\omega}{2\tau} \int_{B_{2r}} |\nabla u|^2 dy + C \left(\frac{1}{\omega\tau r^2} + \frac{1}{r^2} \right) \int_{B_{2r}} |u - \rho|^2 dy \\ &\quad + \frac{C|\sigma|}{\tau} \int_{B_{2r}} |u|^2 dy. \end{aligned}$$

This again leads to two subcases. We must choose ρ appropriately and use the Sobolev–Poincaré inequality (3.2) as in case 1. Then, by taking ω sufficiently small, we obtain the desired result.

Case 3: L satisfies the Legendre–Hadamard condition (2.4) with continuous coefficients in $\bar{\Omega}_\varepsilon$. We note that it suffices to study when $u \in C_c^\infty(\Omega_\varepsilon)$ and first consider when the coefficients are constant. We rewrite the left side of (3.5) as

$$\begin{aligned} &\int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} ((u - \rho)^\alpha v)_i ((u - \rho)^\beta v)_j dy \\ &\quad + \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} [v v_j u_i^\alpha (u - \rho)^\beta - v_i v (u - \rho)^\alpha u_j^\beta - v_i v_j (u - \rho)^\alpha (u - \rho)^\beta] dy. \end{aligned}$$

This implies

$$\begin{aligned} & \int_{B_{2r}} a_{ij}^{\alpha\beta} ((u - \rho)^\alpha v)_i ((u - \rho)^\beta v)_j \, dy \\ & \leq C \int_{B_{2r}} |\nabla v| |\nabla((u - \rho)v)| |u - \rho| + |u - \rho|^2 |\nabla v|^2 + |\sigma| |u| |u - \rho| \, dy. \end{aligned}$$

We note that we may use the Fourier transform to get a lower bound for the left side to achieve the estimate

$$\int_{B_{2r}} |\nabla((u - \rho)v)|^2 \, dy \leq \frac{C}{r^2} \int_{B_{2r}} |u - \rho|^2 \, dy + C |\sigma| \int_{B_{2r}} |u|^2 \, dy.$$

This implies the estimate

$$\int_{B_r} |\nabla u|^2 \, dy \leq \frac{C}{r^2} \int_{B_{2r}} |u - \rho|^2 \, dy + C |\sigma| \int_{B_{2r}} |u|^2 \, dy. \tag{3.8}$$

So, again, if we employ the Sobolev–Poincaré inequality (3.2), we get the desired result in the case of constant coefficients. If the coefficients are continuous and non-constant, then we freeze the coefficients at x . That is, from the weak formulation (2.3), we have

$$\begin{aligned} & \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta}(x) u_i^\alpha ((u - \rho)v^2)_j^\beta \, dy + \int_{\Omega_\varepsilon} (a_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x)) u_i^\alpha ((u - \rho)v^2)_j^\beta \, dy \\ & = \sigma \int_{\Omega_\varepsilon} u^\gamma ((u - \rho)v^2)^\gamma \, dy. \end{aligned} \tag{3.9}$$

So, if we define the modulus of continuity to be

$$M(x_0, R) = \max_{\substack{y \in \overline{B_R}(x_0) \\ i, j, \alpha, \beta}} |a_{ij}^{\alpha\beta}(y) - a_{ij}^{\alpha\beta}(x_0)|$$

then we have that

$$\begin{aligned} & \int_{B_{2r}} (a_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x)) u_i^\alpha ((u - \rho)v^2)_j^\beta \, dy \\ & \leq M(x, 2r) \int_{B_{2r}} v^2 |\nabla u|^2 \, dy + 2M(x, 2r) \int_{B_{2r}} v |\nabla v| |\nabla u| |u - \rho| \, dy \\ & \leq C(M(x, 2r) + M(x, 2r)^2) \int_{B_{2r}} |\nabla u|^2 \, dy + \frac{C}{r^2} \int_{B_{2r}} |u - \rho|^2 \, dy. \end{aligned}$$

Also, by the uniform continuity of the coefficients on $\overline{\Omega_\varepsilon}$, for any $c < 1$, there exists r_0 depending on c , so that if $C(x_0, R) = C(M(x_0, 2R) + M(x_0, 2R)^2)$ and $r \leq r_0$, then

$$C(x_0, r) \leq c$$

for all $x_0 \in \bar{\Omega}_\varepsilon$. So, now moving the second term on the left side of (3.9) to the right and using the constant coefficient case (3.8), we obtain that for any $c < 1$, there exists r_0 so that if $r \leq r_0$,

$$\int_{B_r} |\nabla u|^2 \, dy \leq \frac{C}{r^2} \int_{B_{2r}} |u - \rho|^2 \, dy + C|\sigma| \int_{B_{2r}} |u|^2 \, dy + c \int_{B_{2r}} |\nabla u|^2 \, dy.$$

We again choose ρ appropriately and apply the Sobolev–Poincaré inequality (3.2) to get the desired result. \square

As stated earlier, our proof of Theorem 2.1 relies on the gradient of an eigenfunction satisfying the reverse Hölder inequality, as in our next theorem.

Theorem 3.5. *There exists $\varepsilon_1 > 0$ so that if u is an eigenfunction with eigenvalue σ , then*

$$\int_{\Omega_\varepsilon} |\nabla u|^{\tilde{p}} \, dy \leq C \left[\left(\int_{\Omega_\varepsilon} |\nabla u|^2 \, dy \right)^{\frac{\tilde{p}}{2}} + |\sigma|^{\frac{\tilde{p}}{2}} \int_{\Omega_\varepsilon} |u|^{\tilde{p}} \, dy \right] \tag{3.10}$$

where $2 \leq \tilde{p} < 2 + \varepsilon_1$, and ε_1 and C are independent of ε and any eigenvalue.

Proof. Now if u is an eigenfunction with eigenvalue σ , we have $u \in H_0^1(\Omega_\varepsilon)$, and thus we may employ the Sobolev inequality to get that $|u| \in L^r(\Omega_\varepsilon)$ for some $r > 2$. If L satisfies either (2.6) or (2.11) with uniformly bounded coefficients, then we may choose a cube Q_R , centered at 0, with sidelength R such that $\Omega_\varepsilon \subset Q_{\frac{R}{2}}$, uniformly in ε , and set $g = |\nabla u|^{\frac{2n}{n+2}}$, $f = (C_3|\sigma|)^{\frac{n}{n+2}}|u|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$, and $u = 0$ outside Ω_ε , we may conclude by (3.4) and (3.1) that

$$\left(\int_{\Omega_\varepsilon} |\nabla u|^{\frac{2np}{n+2}} \, dy \right)^{\frac{1}{p}} \leq C \left[\left(\int_{\Omega_\varepsilon} |\nabla u|^2 \, dy \right)^{\frac{n}{n+2}} + |\sigma|^{\frac{n}{n+2}} \left(\int_{\Omega_\varepsilon} |u|^{\frac{2np}{n+2}} \, dy \right)^{\frac{1}{p}} \right]$$

where $\frac{n+2}{n} \leq p \leq \frac{n+2}{n} + \varepsilon$, which, from Theorem 3.4 is independent of ε and any eigenvalue. So, setting $\tilde{p} = \frac{2np}{n+2}$, we have the result. If L satisfies (2.4) with continuous coefficients, then since we only have Theorem 3.4 true for small r , we must cover Ω_ε with a fixed number of cubes and apply (3.1) to each cube to obtain the result. \square

4. Stability of eigenvalues

From this point, let σ_k^ε be the k th eigenvalue with respect to Ω_ε , and φ_k^ε be its corresponding eigenfunction with $\varphi_k^\varepsilon = 0$ outside Ω_ε for $\varepsilon \geq 0$. We also fix an eigenvalue σ_J^0 with multiplicity m_J where $\sigma_{J-1}^0 < \sigma_J^0$ if $J \geq 2$. We will consider the family $\{\sigma_J^\varepsilon\}$ as $\varepsilon > 0$ tends to 0. We begin with the following proposition taken from Anné [2], pp. 2595–2596.

Lemma 4.1. *Let (q, \mathcal{D}) be a closed non-negative quadratic form with form domain \mathcal{D} in the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Define the associated norm $\|f\|_1^2 = \|f\|_{\mathcal{H}}^2 + q(f)$, and the spectral projector Π_I for any interval $I = (\alpha, \beta)$ for which the boundary does not meet the spectrum.*

(1) *Suppose $f \in \mathcal{D}$ and $\lambda \in I$ satisfy*

$$|q(f, g) - \lambda \langle f, g \rangle| \leq \delta \|f\| \|g\|_1, \quad g \in \mathcal{D}.$$

Then there exists a constant $C > 0$, which depends on I , such that if a is less than the distance of α or β to the spectrum of q ,

$$\|\Pi_I(f) - f\|_1 = \|\Pi_{I^c}(f)\|_1 \leq \frac{C\delta}{a} \|f\|.$$

(2) *Suppose the spectral space $E(I)$ has dimension m and f_1, \dots, f_m is an orthonormal family which satisfies*

$$\|\Pi_{I^c}(f_j)\|_1 \leq \delta, \quad j = 1, \dots, m.$$

Also let E be the space spanned by the f_j 's. Then,

$$\text{dist}(E(I), E) \leq C\delta$$

where the distance is measured as the distance between the two orthogonal projectors.

This lemma will give us the results we need for the convergence of eigenvalues. We will prove estimates on eigenfunctions using the reverse Hölder inequality (3.10), which will allow us to use this lemma. We start with the following proposition which follows immediately from the construction of eigenvalues.

Proposition 4.1. *We have for any $\varepsilon > 0$, and any $k \in \mathbb{N}$,*

$$\sigma_k^\varepsilon \leq \sigma_k^0. \tag{4.1}$$

This proposition gives us the easy half of the inequality in our theorem. To prove the second half of the inequality, we will need a few items.

Proposition 4.2. *For any $\varepsilon > 0$, and $k \geq 1$, if $\varphi = \varphi_k^\varepsilon$, then we have*

$$\int_{\Omega_\varepsilon} |\nabla \varphi|^{\tilde{p}} dy \leq C \tag{4.2}$$

where $\tilde{p} > 2$ is from (3.10), and C depends on $|\Omega_0|$ and n , with order, for $n \geq 3$, $\mathcal{O}(|\sigma_k^0|^{\frac{2\tilde{p}+n(\tilde{p}-2)}{4}})$ or, for $n = 2$, $\mathcal{O}(|\sigma_k^0|^{\frac{q\tilde{p}+2(\tilde{p}-q)}{2q}})$, where $2 - \xi < q < 2$ for small ξ . Furthermore, \tilde{p} and C are independent of ε and if $n = 2$, C blows up as $q \rightarrow 2$.

Proof. Now, from (3.10), we have

$$\int_{\Omega_\varepsilon} |\nabla\varphi|^{\tilde{p}} dy \leq C \left[|\Omega_\varepsilon|^{\frac{2-\tilde{p}}{2}} \left(\int_{\Omega_\varepsilon} |\nabla\varphi|^2 dy \right)^{\frac{\tilde{p}}{2}} + |\sigma_k^\varepsilon|^{\frac{\tilde{p}}{2}} \left(\int_{\Omega_\varepsilon} |\varphi|^{\tilde{p}} dy \right) \right] \quad (4.3)$$

where $\tilde{p} > 2$ is from (3.10). Recall that by Gårding's inequality (2.5) and since φ is an eigenfunction, we have

$$\begin{aligned} C_1 \int_{\Omega_\varepsilon} |\nabla\varphi|^2 dy &\leq \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} \varphi_i^\alpha \varphi_j^\beta dy + C_2 \int_{\Omega_\varepsilon} |\varphi|^2 dy \\ &\leq C(1 + |\sigma_k^\varepsilon|) \int_{\Omega_\varepsilon} |\varphi|^2 dy \\ &\leq C(1 + |\sigma_k^\varepsilon|), \end{aligned}$$

the last line owing to the normalization of the eigenfunctions. Next, we will consider $n \geq 3$ and estimate

$$\int_{\Omega_\varepsilon} |\varphi|^{\tilde{p}} dy.$$

Using Sobolev's inequality and (4.4), we have

$$\left(\int_{\Omega_\varepsilon} |\varphi|^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{2n}} \leq C \left(\int_{\Omega_\varepsilon} |\nabla\varphi|^2 dy \right)^{\frac{1}{2}} \leq C(1 + |\sigma_k^\varepsilon|^{\frac{1}{2}}). \quad (4.4)$$

Also, by Hölder's inequality, we have

$$\left(\int_{\Omega_\varepsilon} |\varphi|^{\tilde{p}} dy \right)^{\frac{1}{\tilde{p}}} \leq \left(\int_{\Omega_\varepsilon} |\varphi|^2 dy \right)^{\frac{1-t}{2}} \left(\int_{\Omega_\varepsilon} |\varphi|^{\frac{2n}{n-2}} dy \right)^{\frac{t(n-2)}{2n}}$$

where t satisfies

$$\frac{1}{\tilde{p}} = \frac{1-t}{2} + \frac{t(n-2)}{2n}.$$

From this inequality and (4.4), it follows that

$$\left(\int_{\Omega_\varepsilon} |\varphi|^{\tilde{p}} dy \right)^{\frac{1}{\tilde{p}}} \leq C(1 + |\sigma_k^\varepsilon|^{\frac{t}{2}}) = C(1 + |\sigma_k^\varepsilon|^{\frac{n(\tilde{p}-2)}{4\tilde{p}}}).$$

Now, using this inequality along with (4.3), (4.4), and (4.1), we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla\varphi|^{\tilde{p}} dy &\leq C \left[(1 + |\sigma_k^0|)^{\frac{\tilde{p}}{2}} + |\sigma_k^0|^{\frac{\tilde{p}}{2}} (1 + |\sigma_k^0|^{\frac{n(\tilde{p}-2)}{4\tilde{p}}}) \right] \\ &\leq C \left[|\sigma_k^0|^{\frac{2\tilde{p}+n(\tilde{p}-2)}{4}} + |\sigma_k^0|^{\frac{\tilde{p}}{2}} + 1 \right]. \end{aligned}$$

This completes the proof for $n \geq 3$.

If $n = 2$, then from Sobolev’s inequality, Hölder’s inequality, and (4.4), we have

$$\begin{aligned} \left(\int_{\Omega_\varepsilon} |\varphi|^{q^*} dy\right)^{\frac{1}{q^*}} &\leq \frac{C}{(2-q)^{\frac{1}{2}}} \left(\int_{\Omega_\varepsilon} |\nabla\varphi|^q dy\right)^{\frac{1}{q}} \\ &\leq \frac{C}{(2-q)^{\frac{1}{2}}} \left(\int_{\Omega_\varepsilon} |\nabla\varphi|^2 dy\right)^{\frac{1}{2}} |\Omega_\varepsilon|^{\frac{1}{q^*}} \\ &\leq \frac{C}{(2-q)^{\frac{1}{2}}} (1 + |\sigma_k^\varepsilon|^{\frac{1}{2}}), \end{aligned}$$

where $q^* = \frac{2q}{q-2}$ is the Sobolev conjugate of q . Then, again applying Hölder’s inequality, we obtain

$$\left(\int_{\Omega_\varepsilon} |\varphi|^{\tilde{p}} dy\right)^{\frac{1}{\tilde{p}}} \leq \frac{C}{(2-q)^{\frac{1}{2}}} (1 + |\sigma_k^\varepsilon|^{\frac{1}{2}}) = \frac{C}{(2-q)^{\frac{(\tilde{p}-q)}{pq}}} (1 + |\sigma_k^\varepsilon|^{\frac{(\tilde{p}-q)}{pq}}).$$

Now using (4.3), (4.4), and (4.1), we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla\varphi|^{\tilde{p}} dy &\leq \frac{C}{(2-q)^{\frac{(\tilde{p}-q)}{q}}} \left[(1 + |\sigma_k^0|)^{\frac{\tilde{p}}{2}} + |\sigma_k^0|^{\frac{\tilde{p}}{2}} (1 + |\sigma_k^0|^{\frac{(\tilde{p}-q)}{q}}) \right] \\ &\leq \frac{C}{(2-q)^{\frac{(\tilde{p}-q)}{q}}} \left[|\sigma_k^0|^{\frac{q\tilde{p}+2(\tilde{p}-q)}{2q}} + |\sigma_k^0|^{\frac{\tilde{p}}{2}} + 1 \right]. \quad \square \end{aligned}$$

Lemma 4.2. *For the eigenfunction φ_k^ε , $J \leq k \leq J + m_J - 1$, and any $w \in H_0^1(\Omega_0)$, we have the following estimate:*

$$\left| \int_{\Omega_0} a_{ij}^{\alpha\beta} (\eta_\varepsilon \varphi_k^\varepsilon)_i^\alpha w_j^\beta dy - \sigma_k^\varepsilon \int_{\Omega_0} (\eta_\varepsilon \varphi_k^\varepsilon)^\alpha w^\alpha dy \right| \leq C \varepsilon^{\frac{n(\tilde{p}-2)}{2\tilde{p}}} \|w\|_1 \quad (4.5)$$

where $\|w\|_1$ is from Lemma 4.1 with

$$q(f, g) = \int_{\Omega_0} a_{ij}^{\alpha\beta} f_i^\alpha g_j^\beta dy,$$

and C only depends on $|\Omega_0|$, n , σ_J^0 , and is independent of ε .

Proof. First, recall that w is extended to be 0 outside Ω_0 and φ_k^ε is extended to be 0 in $(B_\varepsilon \cup \tilde{B}_\varepsilon) \cap \Omega_\varepsilon^c$. We have

$$\begin{aligned} & \left| \int_{\Omega_0} a_{ij}^{\alpha\beta} (\eta_\varepsilon \varphi_k^\varepsilon)_i^\alpha w_j^\beta dy - \sigma_k^\varepsilon \int_{\Omega_0} (\eta_\varepsilon \varphi_k^\varepsilon)^\alpha w^\alpha dy \right| \\ & \leq \left| \int_{\Omega_0} a_{ij}^{\alpha\beta} [(\eta_\varepsilon)_i (\varphi_k^\varepsilon)^\alpha w_j^\beta - (\eta_\varepsilon)_j (\varphi_k^\varepsilon)^\alpha w_i^\beta] dy \right| \\ & \quad + \left| \int_{\Omega_\varepsilon} a_{ij}^{\alpha\beta} (\varphi_k^\varepsilon)_i^\alpha (\eta_\varepsilon w)_j^\beta dy - \sigma_k^\varepsilon \int_{\Omega_\varepsilon} (\varphi_k^\varepsilon)^\alpha (\eta_\varepsilon w)^\alpha dy \right| \\ & = |\text{I} + \text{II}| + |\text{III} + \text{IV}|. \end{aligned}$$

First, since φ_k^ε is an eigenfunction with eigenvalue σ_k^ε , we have that $\text{III} + \text{IV} = 0$. Also, by Hölder's inequality and Poincaré's inequality, we have

$$\begin{aligned} |\text{I} + \text{II}| & \leq \frac{C}{\varepsilon} \|\varphi_k^\varepsilon\|_{L^2(B_\varepsilon \cup \tilde{B}_\varepsilon)} (\|\nabla w\|_{L^2(B_\varepsilon \cup \tilde{B}_\varepsilon)} + \|w\|_{L^2(B_\varepsilon \cup \tilde{B}_\varepsilon)}) \\ & \leq C \|\nabla \varphi_k^\varepsilon\|_{L^2(B_\varepsilon \cup \tilde{B}_\varepsilon)} \|w\|_1 \end{aligned}$$

where we have used Gårding's inequality (2.5) on the last line for w . Thus, from Hölder's inequality and Proposition 4.2,

$$\begin{aligned} |\text{I} + \text{II}| & \leq C \varepsilon^{\frac{n(\bar{p}-2)}{2\bar{p}}} \|\nabla \varphi_k^\varepsilon\|_{L^{\bar{p}}(\Omega_\varepsilon)} \|w\|_1 \\ & \leq C \varepsilon^{\frac{n(\bar{p}-2)}{2\bar{p}}} \|w\|_1. \end{aligned}$$

Since $\sigma_k^0 = \sigma_j^0$, the proof of the lemma is concluded. \square

If we choose an interval I around σ_k^0 such that $\sigma_k^\varepsilon \in I$, and let

$$q(f, g) = \int_{\Omega_0} a_{ij}^{\alpha\beta} f_i^\alpha g_j^\beta dy$$

and $f = \eta_\varepsilon \varphi_k^\varepsilon$, we aim to satisfy the hypotheses for Part 1 of Lemma 4.1. In order to do this, we need $\|\eta_\varepsilon \varphi_k^\varepsilon\|_{L^2(\Omega_0)}$ to be bounded away from 0. To achieve this, we start with the following well-known proposition.

Proposition 4.3. *If A is an $N \times N$ matrix and v is a $N \times 1$ vector such that $Av = 0$ and*

$$\sum_{i \neq l}^N |A_{li}| < |A_{ll}|, \quad l = 1, \dots, N,$$

then $v = 0$.

The next proposition shows that the functions $\{\eta_\varepsilon \varphi_k^\varepsilon\}_{k=J}^{J+m_J-1}$ are almost orthonormal.

Proposition 4.4. *For any $\varepsilon > 0$ and $l, k \in \mathbb{N}$, ($J \leq l, k \leq J + m_J - 1$), if $\varphi_k = \varphi_k^\varepsilon$, we have the following estimates:*

$$\int_{\Omega_\varepsilon} \eta_\varepsilon^2 |\varphi_k|^2 dy \geq 1 - C \varepsilon^{\frac{d(\tilde{p}-2)}{p}} \tag{4.6}$$

and

$$\left| \int_{\Omega_\varepsilon} \eta_\varepsilon^2 \varphi_k \cdot \varphi_l dy \right| \leq C \varepsilon^{\frac{d(\tilde{p}-2)}{p}} \quad \text{if } k \neq l, \tag{4.7}$$

where C only depends on $|\Omega_0|$, n , and σ_J^0 , and is independent of ε .

Proof. We start by showing (4.6). Since the eigenfunctions are normalized, we obtain for each k ,

$$\begin{aligned} 1 - \int_{\Omega_\varepsilon} \eta_\varepsilon^2 |\varphi_k|^2 dy &= \int_{\Omega_\varepsilon} (1 - \eta_\varepsilon^2) |\varphi_k|^2 dy \\ &= \int_{T_\varepsilon \cup B_\varepsilon \cup \tilde{B}_\varepsilon} (1 - \eta_\varepsilon^2) |\varphi_k|^2 dy \\ &\leq \|\nabla \varphi_k\|_{L^{\tilde{p}}(\Omega_\varepsilon)}^2 |T_\varepsilon \cup B_\varepsilon \cup \tilde{B}_\varepsilon|^{\frac{\tilde{p}-2}{\tilde{p}}} \\ &\leq C_k \varepsilon^{\frac{d(\tilde{p}-2)}{p}} \end{aligned}$$

where, from (4.2), C_k depends on σ_k^0 . Again, since $\sigma_k^0 = \sigma_J^0$, we have (4.6). Next, to show (4.7), we have

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} \eta_\varepsilon^2 \varphi_k \cdot \varphi_l dy \right| \\ &\leq \left| \int_{B_\varepsilon \cup \tilde{B}_\varepsilon} \eta_\varepsilon^2 \varphi_k \cdot \varphi_l dy \right| + \left| \int_{\Omega_0 \setminus (B_\varepsilon \cup \tilde{B}_\varepsilon)} \eta_\varepsilon^2 \varphi_k \cdot \varphi_l dy \right| \\ &= \left| \int_{B_\varepsilon \cup \tilde{B}_\varepsilon} \eta_\varepsilon^2 \varphi_k \cdot \varphi_l dy \right| + \left| \int_{\Omega_0 \setminus (B_\varepsilon \cup \tilde{B}_\varepsilon)} \varphi_k \cdot \varphi_l dy - \int_{\Omega_\varepsilon} \varphi_k \cdot \varphi_l dy \right| \\ &\leq \int_{B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_k \cdot \varphi_l| dy + \int_{T_\varepsilon \cup B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_k \cdot \varphi_l| dy, \end{aligned}$$

the second inequality following since the set of eigenfunctions form an orthogonal set in $L^2(\Omega_\varepsilon)$. So, next by Hölder’s inequality, we get

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} \eta_\varepsilon^2 \varphi_k \cdot \varphi_l \, dy \right| &\leq \left(\int_{B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_k|^2 \, dy \right)^{\frac{1}{2}} \left(\int_{B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_l|^2 \, dy \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{T_\varepsilon \cup B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_k|^2 \, dy \right)^{\frac{1}{2}} \left(\int_{T_\varepsilon \cup B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_l|^2 \, dy \right)^{\frac{1}{2}} \\ &= \text{I} + \text{II}. \end{aligned}$$

Now, from Poincaré’s inequality and (4.2), we get

$$\begin{aligned} \text{I} &\leq \left[\left(\int_{B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_k|^{\tilde{p}} \, dy \right)^{\frac{2}{\tilde{p}}} |B_\varepsilon \cup \tilde{B}_\varepsilon|^{\frac{\tilde{p}-2}{\tilde{p}}} \right]^{\frac{1}{2}} \left[\left(\int_{B_\varepsilon \cup \tilde{B}_\varepsilon} |\varphi_l|^{\tilde{p}} \, dy \right)^{\frac{2}{\tilde{p}}} |B_\varepsilon \cup \tilde{B}_\varepsilon|^{\frac{\tilde{p}-2}{\tilde{p}}} \right]^{\frac{1}{2}} \\ &\leq \|\nabla \varphi_k\|_{L^{\tilde{p}}(\Omega_\varepsilon)} \varepsilon^{\frac{n(\tilde{p}-2)}{2\tilde{p}}} \|\nabla \varphi_l\|_{L^{\tilde{p}}(\Omega_\varepsilon)} \varepsilon^{\frac{n(\tilde{p}-2)}{2\tilde{p}}} \\ &\leq C_k \varepsilon^{\frac{n(\tilde{p}-2)}{2\tilde{p}}} C_l \varepsilon^{\frac{n(\tilde{p}-2)}{2\tilde{p}}} \end{aligned}$$

where C_k again depends on σ_k^0 and C_l depends on σ_l^0 . Thus, we have

$$\text{I} \leq C \varepsilon^{\frac{n(\tilde{p}-2)}{\tilde{p}}} \tag{4.8}$$

where C depends only on $|\Omega_0|$, n , and σ_J^0 . Similarly,

$$\text{II} \leq C \varepsilon^{\frac{d(\tilde{p}-2)}{\tilde{p}}}, \tag{4.9}$$

so that the proposition is proved. □

Note that with the aid of Lemma 4.2 and Proposition 4.4, if ε is small enough, we have satisfied the hypotheses for Part 1 of Lemma 4.1 with

$$q(f, g) = \int_{\Omega_0} a_{ij}^{\alpha\beta} f_i^\alpha g_j^\beta \, dy$$

and $f = \eta_\varepsilon \varphi_k^\varepsilon$. Here, we relabel ε_1 to be small enough to achieve this for any $\varepsilon \leq \varepsilon_1$, and note that ε_1 only depends on fixed parameters. To satisfy the hypotheses for Part 2 of Lemma 4.1, we need an orthonormal basis. The next proposition shows that for small ε , we have a basis.

Proposition 4.5. *The set $\{\eta_\varepsilon \varphi_k^\varepsilon\}_{k=J}^N$ forms a linearly independent set for any $N \geq J$, for $0 < \varepsilon \leq \varepsilon_0(N)$, where $\varepsilon_0(N)$ depends on N .*

Proof. Assume $C_J \eta_\varepsilon \varphi_J^\varepsilon + \dots + C_N \eta_\varepsilon \varphi_N^\varepsilon = 0$. Then, multiplying this equation by $\eta_\varepsilon \varphi_l^\varepsilon$, we obtain

$$\sum_{k=J}^N C_k \langle \eta_\varepsilon \varphi_k^\varepsilon, \eta_\varepsilon \varphi_l^\varepsilon \rangle_{L^2(\Omega_\varepsilon)} = 0, \quad l = J, \dots, N.$$

So, if $A_{kl} = \langle \eta_\varepsilon \varphi_k^\varepsilon, \eta_\varepsilon \varphi_l^\varepsilon \rangle_{L^2(\Omega_\varepsilon)}$, we obtain by (4.6) and (4.7) that

$$|A_{kk}| \geq 1 - C \varepsilon^{\frac{d(\bar{p}-2)}{p}} > C \varepsilon^{\frac{d(\bar{p}-2)}{p}} \geq \sum_{\substack{k=J \\ k \neq l}}^N |A_{kl}|$$

if $\varepsilon \leq \varepsilon(N)$, where $\varepsilon(N)$ depends on N due to applying (4.7) $N - J$ times. Thus, we may use Proposition 4.3 to see that by setting $C = (C_J, \dots, C_N)^t$, we have $C = 0$, so that the proposition is proved. \square

Now we define

$$J_0: L^2(\Omega_\varepsilon) \longrightarrow L^2(\Omega_0)$$

to be given by

$$J_0 f = \eta_\varepsilon f$$

and, similarly, we define

$$J_\varepsilon: L^2(\Omega_0) \longrightarrow L^2(\Omega_\varepsilon)$$

to be such that

$$J_\varepsilon f(x) = \begin{cases} f(x) & \text{if } x \in \Omega_0, \\ 0 & \text{if } x \in \Omega_\varepsilon \setminus \Omega_0. \end{cases}$$

Let

$$I = \left(\sigma_J^0 - M \varepsilon^{\frac{n(\bar{p}-2)}{4\bar{p}}}, \frac{\sigma_J^0 + \sigma_{J+m_J}^0}{2} \right)$$

for $M > 0$ to be chosen later. Also, let Π be the projector onto the space spanned by the eigenfunctions corresponding to the eigenvalues, $\{\sigma_k^\varepsilon\}_{k=J}^N$, in I . We first consider $\varepsilon = \varepsilon_1$. By Proposition 4.1, we may choose $M = M(\varepsilon_1)$ so that σ_k^ε is in I for $J \leq k \leq N$, where $N \geq J + m_J - 1$, and where N depends on ε_1 . We next note that as ε gets smaller, we may choose $M = M(\varepsilon)$ so that the set of eigenvalues in I , $\{\sigma_k^\varepsilon\}_{k=J}^{N_0}$, will have index N_0 in the range $J + m_J - 1 \leq N_0 \leq N$ since our family $\{\Omega_\varepsilon\}$ is nested. Our aim is to show that for ε small, $N_0 = J + m_J - 1$.

We apply Proposition 4.5 to get the existence of $\varepsilon_0(N) \leq \varepsilon_1$ so that $\{\eta_\varepsilon \varphi_k^\varepsilon\}_{k=J}^{N_0}$ is a linearly independent set for $\varepsilon \leq \varepsilon_0(N)$ and for any N_0 in the range $J + m_J - 1 \leq N_0 \leq N$. Then, we choose $M = M(\varepsilon(N))$ so that $\{\eta_\varepsilon \varphi_k^\varepsilon\}_{k=J}^{N_0}$ is also a basis for the

range of $J_0 \Pi J_\varepsilon$. Thus, we may apply the Gram–Schmidt process to this basis. That is, define

$$\begin{aligned} f_J &= \eta_\varepsilon \varphi_J^\varepsilon, \\ &\vdots \\ f_k &= \eta_\varepsilon \varphi_k^\varepsilon - \frac{\langle \eta_\varepsilon \varphi_k^\varepsilon, f_J \rangle}{\|f_J\|^2} f_J - \cdots - \frac{\langle \eta_\varepsilon \varphi_k^\varepsilon, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}, \\ &\vdots \end{aligned}$$

We have the following lemma.

Lemma 4.3. *Let I be as defined above. For each k , $J \leq k \leq J + m_J - 1$, we have*

$$\|\Pi_{I^c}(f_k)\|_1 \leq \frac{C \varepsilon^{\frac{d(\bar{p}-2)}{4\bar{p}}}}{M},$$

for $\varepsilon \leq \varepsilon(N)$, and where M only depends on σ_J^0 , σ_{J-1}^0 , and $\varepsilon(N)$.

Proof. Following the previous arguments, when $\varepsilon = \varepsilon(N)$, we find $M = M(\varepsilon(N))$ so that $\{\eta_\varepsilon \varphi_k^\varepsilon\}_{k=J}^{N_0}$ is a basis for the range of $J_0 \Pi J_\varepsilon$, and then apply the Gram–Schmidt process to this basis. We note the dependence on σ_J^0 and σ_{J-1}^0 is so that we only have 1 eigenvalue (with respect to Ω_0) in I . So, defining

$$q(f, g) = \int_{\Omega_0} a_{ij}^{\alpha\beta} f_i^\alpha g_j^\beta dy,$$

we may apply Lemma 4.2, Proposition 4.4, and then Lemma 4.1 (Part 1) to obtain

$$\|\Pi_{I^c}(f_J)\|_1 \leq \frac{C \varepsilon(N)^{\frac{d(\bar{p}-2)}{4\bar{p}}}}{M(\varepsilon(N))}$$

where C depends on $|\Omega_0|$, n , σ_J^0 , and $\sigma_{J+m_J}^0$. Then, from Proposition 4.4, Lemma 4.2, and properties of the norm, we get the result for $\varepsilon(N)$ and $J \leq k \leq J + m_J - 1$. Then, for $\varepsilon \leq \varepsilon(N)$, we may repeat this argument to get the result with $\varepsilon(N)$ replaced with ε and $M(\varepsilon(N))$ replaced with $M(\varepsilon)$. But, since $M(\varepsilon(N)) \leq M(\varepsilon)$, we obtain the desired result for $\varepsilon \leq \varepsilon(N)$. \square

We now let $E = \text{span}\{\varphi_k^\varepsilon\}_{k=J}^{J+m_J-1}$. Also, let Π_I be the spectral projector corresponding to the eigenvalue σ_J^0 and Π_E be the spectral projector onto E .

Corollary 4.1. *We have*

$$\|\Pi_I - J_0 \Pi_E J_\varepsilon\|_{\mathcal{X}\{L^2(\Omega_0)\}} \leq \frac{C \varepsilon^{\frac{d(\bar{p}-2)}{4\bar{p}}}}{M},$$

for $\varepsilon \leq \varepsilon(N)$, where M only depends on σ_J^0 , σ_{J-1}^0 , and $\varepsilon(N)$. Consequently, for some $\varepsilon(J)$, $N_0 = J + m_J - 1$ when $\varepsilon \leq \varepsilon(J)$.

Proof. Again, we first show for $\varepsilon = \varepsilon(N)$. Normalize the f_k 's and observe that

$$\frac{1}{\|f_k\|} \leq \frac{1}{1 - C\varepsilon(N)^{\frac{d(\tilde{p}-2)}{2\tilde{p}}}}.$$

Then apply Lemma 4.1 (Part 2) to the normalized functions. Then for general $\varepsilon \leq \varepsilon(N)$, we note that since Lemma 4.3 is true with a uniform M , we obtain

$$\|\Pi_I - J_0 \Pi_E J_\varepsilon\|_{\mathcal{X}\{L^2(\Omega_0)\}} \leq \frac{C\varepsilon^{\frac{d(\tilde{p}-2)}{4\tilde{p}}}}{M}.$$

We next note that if $N_0 > J + m_J - 1$ for all $\varepsilon \leq \varepsilon(N)$, then we may find another projector Π_A so that

$$\|\Pi_I - J_0 \Pi_A J_\varepsilon\|_{\mathcal{X}\{L^2(\Omega_0)\}} \leq \frac{C\varepsilon^{\frac{d(\tilde{p}-2)}{4\tilde{p}}}}{M}.$$

But this would mean

$$\|J_0 \Pi_E J_\varepsilon - J_0 \Pi_A J_\varepsilon\|_{\mathcal{X}\{L^2(\Omega_0)\}} \leq \frac{C\varepsilon^{\frac{d(\tilde{p}-2)}{4\tilde{p}}}}{M}.$$

Therefore, for some $\varepsilon(J)$, $N_0 = J + m_J - 1$, when $\varepsilon \leq \varepsilon(J)$. □

Proof of Theorem 2.1. We first prove for $J = 1$. By Corollary 4.1, for $\varepsilon \leq \varepsilon(1)$, we obtain $m_1 = N_0$. This implies that $|\sigma_k^\varepsilon - \sigma_k^0| \leq C\varepsilon^{\frac{d(\tilde{p}-2)}{4\tilde{p}}}$ only for $k, 1 \leq k \leq m_1$, and hence, the result for $J = 1$. The result for $J = 1$ implies that not only may we choose M so that all eigenvalues $\{\sigma_k^\varepsilon\}_{k=m_1+1}^{m_1+m_2}$ are in the interval corresponding to the next highest eigenvalue $\sigma_{m_1+1}^0$, but also that σ_1^0 is not in this interval. Thus, we apply the same reasoning here to get the result for $\sigma_{m_1+1}^0$. Then, by an induction argument, we get the result for each $J \in \mathbb{N}$, satisfying $\sigma_J^0 > \sigma_{J-1}^0$. We note here that since C depends on $\varepsilon(J)$, it depends on the multiplicity J . □

We note that this paper introduces the use of L^p -estimates obtained by the reverse Hölder technique to the study of spectral problems for elliptic operators. Thus, this technique may be useful in studying spectral problems in situations where we do not know if higher regularity of solutions is true. We close by listing some open problems.

- If we have some additional regularity on the domain, can we use the methods from this work to get convergence of Neumann eigenvalues for general elliptic systems?

- For elliptic systems on a symmetric dumbbell region with a straight tube, can we achieve upper and lower bounds on the splitting between the smallest eigenvalues?
- Can we investigate this problem further to see if a better rate of convergence exists?

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Justin L. Taylor, Department of Mathematics and Statistics, Murray State University,
Murray, Kentucky, U.S.A.

E-mail: jtaylor52@murraystate.edu