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On the limit behaviour of second order relative spectra of self-adjoint operators

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Abstract. It is well known that the standard projection methods allow one to recover the whole spectrum of a bounded self-adjoint operator but they often lead to spectral pollution, i.e. to spurious eigenvalues lying in the gaps of the essential spectrum. Methods using second order relative spectra are free from spectral pollution, but they have not been proven to approximate the whole spectrum. L. Boulton ([3] and [4]) has shown that second order relative spectra approximate all isolated eigenvalues of finite multiplicity. The main result of the present paper is that second order relative spectra do not in general approximate the whole of the essential spectrum of a bounded self-adjoint operator.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the space of bounded linear operators on $\mathcal{H}$. Let $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots \subset \mathcal{L}_l \subset \mathcal{L}_{l+1} \subset \cdots$ be an increasing sequence of finite dimensional linear subspaces of $\mathcal{H}$ such that the corresponding orthogonal projections $P_l\colon \mathcal{H} \to \mathcal{L}_l$ converge strongly to the identity operator I. Let $\mathfrak{P}(\mathcal{H})$ be the set of all such sequences of subspaces.

Suppose $T = T^* \in \mathcal{B}(\mathcal{H})$ and denote the spectrum of $P_l T\colon \mathcal{L}_l \to \mathcal{L}_l$ by $\operatorname{Spec}(T, \mathcal{L}_l)$. Then

$$\lim_{l\to\infty} \operatorname{Spec}(T, \mathcal{L}_l) \supseteq \operatorname{Spec}(T), \tag{1}$$

where "lim" is defined in an appropriate way (see, e.g., [1] or [19]). Unfortunately the left-hand side of (1) may be strictly larger than the right-hand side. This is called *spectral pollution* (see, e.g., [3], [4], [10], [14], [15], [17], and [19]) which is a well known phenomenon in numerical analysis: spurious "eigenvalues" may appear in the gaps of the essential spectrum of T and as a result $\lim_{l\to\infty} \operatorname{Spec}(T, \mathcal{L}_l)$ may contain points that do not belong to $\operatorname{Spec}(T)$.

A possible way of dealing with spectral pollution is based on the notion of second order relative spectra which was introduced by E.B. Davies in [9]:

$$\mathrm{Spec}_2(T, \mathcal{L}_l) \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \colon P_l(T - \lambda I)^2 \colon \mathcal{L}_l \to \mathcal{L}_l \text{ is not invertible}\}.$$

Although the spectrum of a self-adjoint operator T is a subset of $\mathbb{R}$, the set $\mathrm{Spec}_2(T, \mathcal{L}_l)$ may and usually does contain points from $\mathbb{C} \setminus \mathbb{R}$. Since $T^* = T$, it is easy to see that $\mathrm{Spec}_2(T, \mathcal{L}_l)$ is symmetric with respect to the real line:

$$\lambda \in \mathrm{Spec}_2(T, \mathcal{L}_l) \iff \bar{\lambda} \in \mathrm{Spec}_2(T, \mathcal{L}_l).$$

If $\lambda \in \mathrm{Spec}_2(T, \mathcal{L}_l)$ then

$$\mathrm{Spec}(T) \cap [\mathrm{Re}\,\lambda - |\,\mathrm{Im}\,\lambda|, \mathrm{Re}\,\lambda + |\,\mathrm{Im}\,\lambda|] \neq \emptyset \tag{2}$$

([14] and [19]; see also [12]). This means that if a point of $\mathrm{Spec}_2(T, \mathcal{L}_l)$ is close to the real line, then it is close to $\mathrm{Spec}(T)$, i.e. that, in a sense, second order relative spectra do not pollute.

A natural question, which was first posed in [19] (see also [14] and [20]), is whether $\mathrm{Spec}_2(T, \mathcal{L}_l)$, $(\mathcal{L}_l)_{l\in\mathbb{N}} \in \mathfrak{P}(\mathcal{H})$ capture the whole spectrum of T, i.e. whether or not

$$\lim_{l\to\infty} \mathrm{Spec}_2(T, \mathcal{L}_l) \supseteq \mathrm{Spec}(T).$$

A partial answer to this question was obtained in [3] and [4]:

$$\lim_{l\to\infty} \mathrm{Spec}_2(T, \mathcal{L}_l) \supseteq \{\text{isolated eigenvalues of } T \text{ of finite multiplicity}\}.$$

The main result of the present paper is that $\mathrm{Spec}_2(T, \mathcal{L}_l)$ do not in general approximate the whole of the essential spectrum $\mathrm{Spec}_e(T)$ of T. In order to state the result, we need the following notation. Let $d_H(F, G)$ denote the Hausdorff distance between two sets $F, G \subset \mathbb{C}$:

$$d_H(F, G) \stackrel{\text{def}}{=} \max\{\sup_{x\in F} \inf_{y\in G} |x - y|, \sup_{y\in G} \inf_{x\in F} |x - y|\}.$$

Let $\Sigma \subset \mathbb{R}$ be a compact set,

$$m \stackrel{\text{def}}{=} \min \Sigma, \; M \stackrel{\text{def}}{=} \max \Sigma, \quad [m, M] \setminus \Sigma = \bigcup_j (m_j, M_j),$$

$$(m_j, M_j) \cap (m_l, M_l) = \emptyset \text{ if } j \neq l.$$

Define

$$\mathcal{Q}(\Sigma) \stackrel{\text{def}}{=} B[m, M] \setminus \cup_j B(m_j, M_j),$$

where $B[c_1, c_2]$ and $B(c_1, c_2)$ denote the closed and the open disk with the diameter $[c_1, c_2]$.

Theorem 1.1. *Let*

$$-\infty < \rho_-^{(1)} < \rho_+^{(1)} < \rho_-^{(2)} < \rho_+^{(2)} < \cdots < \rho_-^{(n)} < \rho_+^{(n)} < +\infty, \quad n \in \mathbb{N},$$

and let

$$F \subseteq \mathcal{Q}\Big(\bigcup_{j=1}^{n} [\rho_-^{(j)}, \rho_+^{(j)}]\Big)$$

be a compact set symmetric with respect to the real line and such that

$$F \cap (\rho_-^{(j)}, \rho_+^{(j)}) \neq \emptyset, \quad j = 1, \dots, n. \tag{3}$$

Then there exist $T = T^* \in \mathcal{B}(\mathcal{H})$ *and* $(\mathcal{L}_l) \in \mathfrak{P}(\mathcal{H})$ *such that*

$$\operatorname{Spec}(T) = \bigcup_{j=1}^{n} [\rho_-^{(j)}, \rho_+^{(j)}]$$

and

$$d_H(\operatorname{Spec}_2(T, \mathcal{L}_l), F) \longrightarrow 0 \ \text{ as } \ l \to +\infty.$$

Note that

$$\bigcup_{(\mathcal{L}_l) \in \mathfrak{P}(\mathcal{H})} \lim_{l \to +\infty} \operatorname{Spec}_2(T, \mathcal{L}_l) = \operatorname{Spec}(T) \cup \mathcal{Q}(\operatorname{Spec}_e(T)),$$

where $T = T^* \in \mathcal{B}(\mathcal{H})$ and where "lim" is defined in an appropriate way ([19]; see also [8]).

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2. Auxiliary results

Proposition 2.1. *Let* $B, M \in \mathcal{B}(\mathcal{H})$, $B^* = B$, $M^* = M \geq 0$. *There exist a Hilbert space* $\mathcal{H}_0 \supseteq \mathcal{H}$ *and* $T = T^* \in \mathcal{B}(\mathcal{H}_0)$ *such that* $B = PT|_{\mathcal{H}}$, $M = PT^2|_{\mathcal{H}}$, *where* $P : \mathcal{H}_0 \to \mathcal{H}$ *is the orthogonal projection, if and only if*

$$B^2 \leq M. \tag{4}$$

Proof. Suppose such $\mathcal{H}_0$ and T exist. Then

$$(B^2x, x) = \|Bx\|^2 = \|PTx\|^2,$$

$$(Mx, x) = (PT^2x, x) = (T^2x, x) = \|Tx\|^2,$$

for all $x \in \mathcal{H}$. Hence $(B^2x, x) \le (Mx, x)$, for all $x \in \mathcal{H}$, i.e. (4) holds (cf. [18], Appendix).

Suppose now (4) holds. Then $M - B^2 \ge 0$ has a nonnegative square root $(M - B^2)^{1/2}$. Let

$$\mathcal{H}_0 \stackrel{\text{def}}{=} \mathcal{H} \bigoplus \mathcal{H},$$

let

$$P : \mathcal{H}_0 \longrightarrow \mathcal{H}$$

be the projection onto the first component, and let

$$T \stackrel{\text{def}}{=} \begin{pmatrix} B & (M - B^2)^{1/2} \\ (M - B^2)^{1/2} & 0 \end{pmatrix} : \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{matrix} = \mathcal{H}_0. \tag{5}$$

Then $T^* = T$, $PT|_{\mathcal{H}} = B$,

$$T^2 = \begin{pmatrix} M & B(M - B^2)^{1/2} \\ (M - B^2)^{1/2}B & M - B^2 \end{pmatrix}$$

and $PT^2|_{\mathcal{H}} = M$. □

Lemma 2.2. *For any $\rho_- < \rho_+ \in \mathbb{R}$, $r \in (\rho_-, \rho_+)$ and $\delta, \varepsilon > 0$ there exist $N \in \mathbb{N}$ and Hermitian matrices $B, R \in \mathbb{C}^{N\times N}$ such that $\|R\| < \varepsilon$, $\mathrm{Spec}(B) \subset [\rho_-, \rho_+]$, the distance from any point of $[\rho_-, \rho_+]$ to $\mathrm{Spec}(B)$ is less than δ, and all roots of the equation*

$$\det(\lambda^2 I - 2\lambda B + B^2 + R^2) = 0 \tag{6}$$

belong to the vertical interval $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda = r, \ |\mathrm{Im}\,\lambda| < \varepsilon\}$.

Proof. It is sufficient to prove the lemma for $\rho_\pm = \pm\rho$, $\rho > 0$ as the general case can be reduced to this one by dealing with $B - \frac{\rho_- + \rho_+}{2} I$ instead of B.

Let ε_0 be a small positive number to be specified later and let w be a conformal mapping of the unit disk onto the ellipse with the axes

$$[-\rho, \rho] + i\,\frac{\varepsilon_0}{2} \quad \text{and} \quad i\,[0, \varepsilon_0],$$

such that $\mathrm{Re}\, w(0) = r$.

Let $b \in C(\mathbb{T})$ and $a \in C(\mathbb{T})$ be the boundary values of $\mathrm{Re}\, w$ and $\mathrm{Im}\, w$ respectively. Then $b(\mathbb{T}) = [-\rho, \rho]$ and $a(\mathbb{T}) = [0, \varepsilon_0]$.

For any $n \in \mathbb{N}$, the $n \times n$ Toeplitz matrix $T_n(b)$ with the symbol b is Hermitian and

$$\|T_n(b)\| \le \|T(b)\| = \|b\|_\infty = \rho,$$

where $T(b)\colon l^2 \to l^2$ is the corresponding Toeplitz operator. Hence

$$\operatorname{Spec}(T_n(b)) \subset [-\rho, \rho].$$

It follows from Szegö's theorem (see, e.g., Theorem 5.10 in [2]) that the distance from any point of $[-\rho, \rho]$ to $\operatorname{Spec}(T_N(b))$ is less than δ provided N is sufficiently large. Fix such an N and set

$$B \stackrel{\text{def}}{=} T_N(b)$$

and

$$A \stackrel{\text{def}}{=} \sqrt{2\rho\varepsilon_0} I + T_N(a) = A^*.$$

Since $b + ia$ is the boundary value of the function w analytic in the unit disk, $B + iA = i\sqrt{2\rho\varepsilon_0} I + T_N(b + ia)$ is a lower triangular matrix with the diagonal entries equal to $i\sqrt{2\rho\varepsilon_0} + b_0 + ia_0$, where

$$b_0 \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} w(e^{it})dt = \operatorname{Re} w(0) = r,$$

and

$$a_0 \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Im} w(e^{it})dt = \operatorname{Im} w(0) \in (0, \varepsilon_0).$$

Hence

$$\operatorname{Spec}(B + iA) = \{r + i(\sqrt{2\rho\varepsilon_0} + a_0)\}, \tag{7}$$

and

$$\operatorname{Spec}(B - iA) = \operatorname{Spec}((B + iA)^*) = \{r - i(\sqrt{2\rho\varepsilon_0} + a_0)\}.$$

Consider the matrix polynomial

$$(\lambda I - (B + iA))(\lambda I - (B - iA)) = \lambda^2 I - 2\lambda B + B^2 - i[B, A] + A^2,$$

where the square brackets denote the commutator. The Hermitian matrix

$$-i[B, A] + A^2$$

is nonnegative. Indeed,

$$i[B, A] = i[B, T_N(a)] \leq (2\|B\|\|T_N(a)\|)I \leq 2\rho\varepsilon_0 I \leq A^2,$$

where the last inequality follows from the non-negativity of the Toeplitz matrix $T_N(a)$ with the symbol $a \geq 0$.

Let R be the nonnegative square root of $-i[B,A]+A^2$. Then

$$\begin{aligned}\det(\lambda^2 I - 2\lambda B + B^2 + R^2) &= \det((\lambda I - (B+iA))(\lambda I - (B-iA))) \\ &= \det((\lambda I - (B+iA)))\det((\lambda I - (B-iA))).\end{aligned}$$

Hence it follows from (7) that all roots of (6) belong to the interval $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = r, \ |\operatorname{Im}\lambda| < \varepsilon\}$ provided $\sqrt{2\rho\varepsilon_0} + \varepsilon_0 < \varepsilon$.

It remains to estimate the norm of R:

$$\begin{aligned}\|Rx\|^2 = (R^2x, x) = ((-i[B,A]+A^2)x, x) &\le 2\|B\|\|T_N(a)\| + \|A\|^2 \\ &\le 2\rho\varepsilon_0 + (\sqrt{2\rho\varepsilon_0}+\varepsilon_0)^2,\end{aligned}$$

for all $x \in \mathbb{C}^N$ with $\|x\| = 1$. Choosing $\varepsilon_0 > 0$ such that the right-hand side is less than ε^2 we get $\|R\| < \varepsilon$. □

Remark 2.3. Let

$$T \stackrel{\text{def}}{=} \begin{pmatrix} B & R \\ R & 0 \end{pmatrix} : \mathbb{C}^{2N} \longrightarrow \mathbb{C}^{2N}.$$

Then the set of the roots of (6) is equal to $\operatorname{Spec}_2(T, \mathbb{C}^N)$. Since

$$\|T - \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\| = \|R\| < \varepsilon,$$

$\operatorname{Spec}(T) \subset [\rho_- - \varepsilon, \rho_+ + \varepsilon]$ (see, e.g., Theorem V.4.10 in [13]).

Lemma 2.4. *Let $\varrho_-, \varrho_+ \in \mathbb{R}$ and let $T \in \mathbb{C}^{n\times n}$ be a Hermitian matrix such that* $\operatorname{Spec}(T) \subset [\varrho_-, \varrho_+]$. *Then for any $\rho_- < \varrho_-$, any $\rho_+ > \varrho_+$ and any $r \in (\rho_-, \rho_+)$, $\delta, \varepsilon > 0$, one can choose N and B in Lemma 2.2 in such a way that $N \ge 2n$ and*

$$B = \begin{pmatrix} T & S \\ S^* & K \end{pmatrix}$$

with $\|S\|_{\mathbb{C}^{N-n}\to\mathbb{C}^n} < \delta$.

Proof. Let $\mu_1, \dots, \mu_n$ be the eigenvalues of T repeated according to their multiplicities and let $N \ge 2n$, B', R' satisfy the conditions in Lemma 2.2 with $\delta_0/(2n)$ in place of δ, where $\delta_0 = \min\{\delta, \varrho_- - \rho_-, \rho_+ - \varrho_+\}$. The distance between any two consecutive distinct eigenvalues of B' is less than δ_0/n as otherwise the distance from the centre of the interval between the eigenvalues to $\operatorname{Spec}(B')$ would have been greater than or equal to $\delta_0/(2n)$. Since the multiplicity of each μ_k is at most n, there exist distinct eigenvalues of B' which we denote by $\lambda_{\pm k}$, $k = 1, \dots, n$ and which satisfy the conditions

$$\lambda_{-k} \le \mu_k \le \lambda_k \quad \text{and} \quad \lambda_k - \lambda_{-k} < 2\delta.$$

Then there exist $t_k \in [0, 1]$ such that $\mu_k = (1 - t_k)\lambda_{-k} + t_k\lambda_k$. Let $u_m \in \mathbb{C}^N$, $m = \pm 1, \ldots, \pm n$ be a normalized eigenvector of B' corresponding to λ_m and set

$$v_k \stackrel{\text{def}}{=} \sqrt{1 - t_k}u_{-k} + \sqrt{t_k}u_k \quad \text{and} \quad v_{-k} \stackrel{\text{def}}{=} -\sqrt{t_k}u_{-k} + \sqrt{1 - t_k}u_k.$$

Since $\{u_{\pm k}\}_{k=1}^n$ is an orthonormal set, $\|v_k\| = 1 = \|v_{-k}\|$,

$$(v_k, v_{-k}) = -\sqrt{1 - t_k}\sqrt{t_k} + \sqrt{t_k}\sqrt{1 - t_k} = 0,$$

and $(v_m, v_j) = 0$ if $m, j = \pm 1, \ldots, \pm n, m \neq \pm j$. Hence $\{v_{\pm k}\}_{k=1}^n$ is an orthonormal set. Further,

$$\begin{aligned} (B'v_k, v_k) &= (\sqrt{1 - t_k}\lambda_{-k}u_{-k} + \sqrt{t_k}\lambda_k u_k, \sqrt{1 - t_k}u_{-k} + \sqrt{t_k}u_k) \\ &= (1 - t_k)\lambda_{-k} + t_k\lambda_k = \mu_k, \\ (B'v_k, v_{-k}) &= (\sqrt{1 - t_k}\lambda_{-k}u_{-k} + \sqrt{t_k}\lambda_k u_k, -\sqrt{t_k}u_{-k} + \sqrt{1 - t_k}u_k) \\ &= (\lambda_k - \lambda_{-k})\sqrt{1 - t_k}\sqrt{t_k} \in [0, \delta), \end{aligned} \tag{8}$$

since $0 \leq \sqrt{1 - t_k}\sqrt{t_k} \leq 1/2$. It is also clear that

$$(B'v_k, v_m) = 0, \quad m \neq \pm k. \tag{9}$$

Let $U \in \mathbb{C}^{N \times N}$ be a unitary matrix such that

$$U(\underbrace{0, \ldots, 0, 1}_{k}, 0, \ldots, 0)^T = \begin{cases} v_k, & k = 1, \ldots, n, \\ v_{n-k}, & k = n + 1, \ldots, 2n. \end{cases}$$

Then

$$U^* B' U = \begin{pmatrix} \operatorname{diag}\{\mu_1, \ldots, \mu_n\} & S' \\ (S')^* & K \end{pmatrix},$$

where $S' = (s_{kj})_{n \times (N-n)}$, $|s_{kj}| < \delta$ if $j = n + k$, $s_{kj} = 0$ if $j \neq n + k$, $k = 1, \ldots, n$, $j = n+1, \ldots, N$ (see (8) and (9)). It is easy to see that $\|S'\|_{\mathbb{C}^{N-n} \to \mathbb{C}^n} < \delta$.

Let $U_0 \in \mathbb{C}^{n \times n}$ be a unitary matrix such that

$$U_0 T U_0^* = \operatorname{diag}\{\mu_1, \ldots, \mu_n\},$$

i.e.

$$U_0^* \operatorname{diag}\{\mu_1, \ldots, \mu_n\} U_0 = T,$$

and let

$$U_1 \stackrel{\text{def}}{=} \begin{pmatrix} U_0 & 0 \\ 0 & I_{N-n} \end{pmatrix}.$$

Then U_1 is a unitary matrix and

$$U_1^* U^* B' U U_1 = \begin{pmatrix} U_0^* \operatorname{diag}\{\mu_1, \dots, \mu_n\} U_0 & U_0^* S' \\ (S')^* U_0 & K \end{pmatrix} = \begin{pmatrix} T & S \\ S^* & K \end{pmatrix},$$

where $S \overset{\text{def}}{=} U_0^* S'$. It is clear that $\|S\|_{\mathbb{C}^{N-n} \to \mathbb{C}^n} = \|S'\|_{\mathbb{C}^{N-n} \to \mathbb{C}^n} < \delta$.

Let

$$B \overset{\text{def}}{=} V^* B' V, \quad R \overset{\text{def}}{=} V^* R' V,$$

where $V \overset{\text{def}}{=} U U_1$ is a unitary matrix. Then $B^* = B$, $R^* = R$, $\operatorname{Spec}(B) = \operatorname{Spec}(B')$, $\|R\| = \|R'\| < \varepsilon$, and all zeros of the polynomial

$$\begin{aligned} \det(\lambda^2 I - 2\lambda B + B^2 + R^2) &= \det(V^*(\lambda^2 I - 2\lambda B' + (B')^2 + (R')^2) V) \\ &= \det(\lambda^2 I - 2\lambda B' + (B')^2 + (R')^2) \end{aligned}$$

belong to the interval $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = r, \ |\operatorname{Im} \lambda| < \varepsilon\}$. □

Let $T \in \mathbb{C}^{n \times n}$ and $m \le n, m \in \mathbb{N}$. Then for any $\varepsilon > 0$ there exists $\Delta(T, m, \varepsilon) > 0$ such that for any $D \in \mathbb{C}^{m \times m}$ with $\|D\| < \Delta(T, m, \varepsilon)$ the Hausdorff distance between the set of roots of the equation

$$\det(P_m (\lambda I - T)^2|_{\mathbb{C}^m} + D) = 0$$

and $\operatorname{Spec}_2(T, \mathbb{C}^m)$ is less than ε (see, e.g., Theorem 4.10c in [11]).

We will use the following notation

$$\ell^2(N) \overset{\text{def}}{=} \{x = (x_k)_{k \in \mathbb{N}} \in \ell^2 : x_k = 0, k > N\} \cong \mathbb{C}^N$$

and will identify vectors $(x_1, \dots, x_N) \in \mathbb{C}^N$ with

$$(x_1, \dots, x_N, 0, 0, \dots) \in \ell^2(N) \subset \ell^2.$$

Lemma 2.5. *For any* $\rho_- < \rho_+ \in \mathbb{R}$, $r \in (\rho_-, \rho_+)$ *and any sequence* $\alpha_l \in (0, 1)$, $l \in \mathbb{N}$ *converging to* 0 *there exist a self-adjoint operator* $T \in \mathcal{B}(\ell^2)$ *and* $N_l \in \mathbb{N}$, $l \in \mathbb{N}$ *such that* $\operatorname{Spec}(T) = [\rho_-, \rho_+]$, $N_l \uparrow +\infty$ *as* $l \uparrow +\infty$, *and*

$$\operatorname{Spec}_2(T, \ell^2(N_l)) \subset \{\lambda \in \mathbb{C} : |\lambda - r| < \alpha_l\}, \quad l \in \mathbb{N}. \tag{10}$$

Proof. Similarly to the proof of Lemma 2.2 we can assume that $[\rho_-, \rho_+] = [-2, 2]$ as the general case can be reduced to this one by dealing with

$$\frac{4}{\rho_+ - \rho_-}\Big(T - \frac{\rho_- + \rho_+}{2} I\Big)$$

instead of T.

Let $\rho_0 = \varrho_0 = 0$, $\alpha_0 = \delta_0 = \varepsilon_0 = 1/4$, $N_0 = 1$, $B_0 = 0$, and $T_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)_{2\times 2}$. The main idea of the proof is to use Lemmas 2.2 and 2.4 and Remark 2.3 to successively construct matrices

$$\begin{array}{llll} \ldots \longrightarrow & T_l & & \\ & \downarrow & & \\ & B_{l+1} = \begin{pmatrix} T_l & S_l \\ S_l^* & K_l \end{pmatrix} & \longrightarrow & T_{l+1} = \begin{pmatrix} B_{l+1} & R_{l+1} \\ R_{l+1} & 0 \end{pmatrix} \\ & & & \downarrow \\ & & & \ldots \end{array}$$

in such a way that the spectra of T_l converge to $[-2, 2]$, the second order relative spectra of T_l converge to r, while S_l and R_l have small norms. A most important feature of the construction is that each of the T_l and B_l matrices goes into the top left corner of the next one: $\cdots \hookrightarrow T_l \hookrightarrow B_{l+1} \hookrightarrow T_{l+1} \hookrightarrow B_{l+2} \hookrightarrow \cdots$. To make the construction work, one needs to choose the numbers $\varepsilon, \delta, \varrho$ and ρ in Lemmas 2.2 and 2.4 at each step in an appropriate way. More precisely, we successively construct N_l, $\delta_l = \varepsilon_l$, B_l, T_l such that $B_l^* = B_l \colon \ell^2(N_l) \to \ell^2(N_l)$, $T_l^* = T_l \colon \ell^2(2N_l) \to \ell^2(2N_l)$,

$$T_l \stackrel{\text{def}}{=} \begin{pmatrix} B_l & R_l \\ R_l & 0 \end{pmatrix},$$

$\|R_l\| < \varepsilon_l$, $\operatorname{Spec}(B_l) \subset [-\rho_l, \rho_l]$, the distance from any point of $[-\rho_l, \rho_l]$ to $\operatorname{Spec}(B_l)$ is less than δ_l, $\rho_l = 2 - 2^{-l}$,

$$\delta_l = \varepsilon_l < \frac{1}{2} \min\{\sqrt{\Delta(T_{l-1}, N_{l-1}, \alpha_{l-1}/2)}, \alpha_l, \varepsilon_{l-1}\}, \tag{11}$$

$$\begin{aligned} \operatorname{Spec}_2(T_l, \ell^2(N_l)) &\subset \{\lambda \in \mathbb{C} \colon |\lambda - r| < \varepsilon_l\} \\ &\subset \{\lambda \in \mathbb{C} \colon |\lambda - r| < \alpha_l/2\}, \end{aligned}$$

$$B_{l+1} = \begin{pmatrix} T_l & S_l \\ S_l^* & K_l \end{pmatrix},$$

$\|S_l\| < \delta_{l+1}$, and $\operatorname{Spec}(T_l) \subset [-(\rho_l + \varepsilon_l), \rho_l + \varepsilon_l] \subset [-\varrho_l, \varrho_l]$, $\varrho_l = 2 - 3 \cdot 2^{-l-2} < \rho_{l+1}$. The last inclusion follows from (11) as

$$\varepsilon_l < \frac{\varepsilon_{l-1}}{2} < \cdots < \frac{\varepsilon_0}{2^l} = 2^{-l-2}, \tag{12}$$

and

$$\rho_l + \varepsilon_l < 2 - 2^{-l} + 2^{-l-2} = 2 - 3 \cdot 2^{-l-2} = \varrho_l < 2 - 2^{-l-1} = \rho_{l+1}.$$

Note that the restriction $\delta_l = \varepsilon_l < \frac{1}{2}\sqrt{\Delta(T_{l-1}, N_{l-1}, \alpha_{l-1}/2)}$ (see (11)) is not used at this stage. It will be need later (see (16) below).

Since $T_l^* = T_l$ and $\mathrm{Spec}(T_l) \subset [-\varrho_l, \varrho_l]$,

$$\|T_l\| \le \varrho_l = 2 - 3 \cdot 2^{-l-2} < 2, \quad l \in \mathbb{N}. \tag{13}$$

Let

$$\widehat{B}_l \stackrel{\text{def}}{=} \begin{pmatrix} B_l & 0 \\ 0 & 0 \end{pmatrix} : \ell^2 \longrightarrow \ell^2 \quad \text{and} \quad \widehat{T}_l \stackrel{\text{def}}{=} \begin{pmatrix} T_l & 0 \\ 0 & 0 \end{pmatrix} : \ell^2 \longrightarrow \ell^2.$$

Suppose $x \in \ell^2(2N_j)$, $j \le l$, $\|x\| \le 1$. Then

$$\begin{aligned} \|\widehat{T}_{l+1}x - \widehat{T}_l x\| &\le \|\widehat{T}_{l+1}x - \widehat{B}_{l+1}x\| + \|\widehat{B}_{l+1}x - \widehat{T}_l x\| \\ &= \|R_{l+1}x\| + \|S_l^* x\| < \varepsilon_{l+1} + \delta_{l+1} < 2^{-l-3} + 2^{-l-3} = 2^{-l-2} \end{aligned}$$

(see (12)), and therefore

$$\begin{aligned} \|\widehat{T}_{l+m}x - \widehat{T}_l x\| &\le \textstyle\sum_{p=0}^{m-1} \|\widehat{T}_{l+p+1}x - \widehat{T}_{l+p}x\| < \sum_{p=0}^{m-1} 2^{-l-p-2} \\ &= 2^{-l-1} - 2^{-l-m-1} < 2^{-l-1}, \end{aligned}$$

$m \in \mathbb{N}$. Hence $(\widehat{T}_l x)_{l\in\mathbb{N}}$ is a convergent sequence in ℓ^2 for any $x \in \ell^2(2N_j)$, for all $j \in \mathbb{N}$. Since $\|\widehat{T}_l\| = \|T_l\| < 2$, for all $l \in \mathbb{N}$ (see (13)), the sequence $(\widehat{T}_l)_{l\in\mathbb{N}}$ is strongly convergent. Let $T \in \mathcal{B}(\ell^2)$ be its limit. Then $T^* = T$, $\|T\| \le 2$ and

$$\|Tx - \widehat{T}_l x\| \le 2^{-l-1}, \quad x \in \ell^2(2N_l), \|x\| \le 1. \tag{14}$$

Further, $\mathrm{Spec}(T) = [-2, 2]$. Indeed, take any $\lambda \in [-2, 2]$. The distance from λ to $\mathrm{Spec}(B_l)$ is less than $2^{-l} + \delta_l = 2^{-l} + \varepsilon_l < 2^{-l} + 2^{-l-2}$ (see (12)). Using Theorem V.4.10 in [13] as in Remark 2.3, one can show that the distance from λ to $\mathrm{Spec}(T_l)$ is less than $2^{-l} + 2^{-l-2} + \varepsilon_l < 2^{-l} + 2^{-l-1}$. Hence there exists an eigenvector $x_l \in \ell^2(2N_l)$ of T_l such that $\|x_l\| = 1$ and $\|T_l x_l - \lambda x_l\| < 2^{-l} + 2^{-l-1}$. It follows from (14) that

$$\|Tx_l - \lambda x_l\| < 2^{-l} + 2^{-l-1} + 2^{-l-1} = 2^{-l+1}, \quad l \in \mathbb{N}.$$

Therefore, $\lambda \in \mathrm{Spec}(T)$.

By construction,

$$\begin{aligned} T_l x &= P_{2N_l} B_{l+1} x \\ &= P_{2N_l} P_{N_{l+1}} T_{l+1} x \\ &= P_{2N_l} T_{l+1} x = P_{2N_l} P_{2N_{l+1}} T_{l+2} x \\ &= P_{2N_l} T_{l+2} x \\ &= \dots \\ &= P_{2N_l} T_{l+m} x \\ &= P_{2N_l} \widehat{T}_{l+m} x \\ &= \dots , \end{aligned}$$

for $x \in \ell^2(2N_l)$. So,

$$P_{2N_l} T|_{\ell^2(2N_l)} = T_l, \quad l \in \mathbb{N}. \tag{15}$$

Let us now estimate the difference

$$P_{2N_l} T^2|_{\ell^2(2N_l)} - T_l^2.$$

Since

$$T_{l+1}^2 = \begin{pmatrix} B_{l+1}^2 + R_{l+1}^2 & B_{l+1} R_{l+1} \\ R_{l+1} B_{l+1} & R_{l+1}^2 \end{pmatrix}$$

and

$$B_{l+1}^2 = \begin{pmatrix} T_l^2 + S_l S_l^* & T_l S_l + S_l K_l \\ S_l^* T_l + K_l S_l^* & S_l^* S_l + K_l^2 \end{pmatrix},$$

we get

$$\begin{aligned} \| P_{2N_l} T_{l+1}^2 x - T_l^2 x \| &\le \| P_{2N_l} T_{l+1}^2 x - P_{2N_l} B_{l+1}^2 x \| \\ &\quad + \| P_{2N_l} B_{l+1}^2 x - T_l^2 x \| \\ &= \| P_{2N_l} R_{l+1}^2 x \| + \| S_l S_l^* x \| \\ &< \varepsilon_{l+1}^2 + \delta_{l+1}^2 \\ &= 2\varepsilon_{l+1}^2 , \end{aligned}$$

for $x \in \ell^2(2N_l)$, and $\|x\| \le 1$, and therefore (see (12))

$$\begin{aligned}\|P_{2N_l}T_{l+m}^2x - T_l^2x\| &\le \sum_{p=0}^{m-1}\|P_{2N_l}T_{l+p+1}^2x - P_{2N_l}T_{l+p}^2x\|\\ &\le \sum_{p=0}^{m-1}\|P_{2N_{l+p}}T_{l+p+1}^2x - T_{l+p}^2x\|\\ &< 2\sum_{p=0}^{m-1}\varepsilon_{l+p+1}^2\\ &< 2\varepsilon_{l+1}^2\sum_{p=0}^{m-1}\frac{1}{2^{2p}}\\ &= 2\varepsilon_{l+1}^2\frac{4}{3}\Big(1-\frac{1}{2^{2m}}\Big)\\ &< 4\varepsilon_{l+1}^2,\end{aligned}$$

for $m \in \mathbb{N}$. Hence

$$\|P_{2N_l}T^2|_{\ell^2(2N_l)} - T_l^2\| \le 4\varepsilon_{l+1}^2 < \Delta(T_l, N_l, \alpha_l/2) \tag{16}$$

(see (11)). Finally,

$$\begin{aligned}&P_{N_l}T|_{\ell^2(N_l)} = P_{N_l}T_l|_{\ell^2(N_l)} = B_l \quad\text{and}\\ &\|P_{N_l}T^2|_{\ell^2(N_l)} - P_{N_l}T_l^2|_{\ell^2(N_l)}\| < \Delta(T_l, N_l, \alpha_l/2).\end{aligned}$$

Since $\mathrm{Spec}_2(T_l, \ell^2(N_l)) \subset \{\lambda \in \mathbb{C} \colon |\lambda - r| < \alpha_l/2\}$, (10) follows from the definition of $\Delta(T_l, N_l, \alpha_l/2)$. □

Remark 2.6. The argument in the proof of Lemma 2.5 does not change if one adds the requirement

$$\varepsilon_l < \frac{1}{2}\sqrt{\Delta(T_{l-1}, 2N_{l-1}, \alpha_{l-1})}$$

to (11), although one may get a different sequence of matrices T_l and, correspondingly, a different limit operator T. For these, one has, additionally to the estimates in the proof of Lemma 2.5, the following inequalities

$$\|P_{2N_l}T^2|_{\ell^2(2N_l)} - T_l^2\| \le 4\varepsilon_{l+1}^2 < \Delta(T_l, 2N_l, \alpha_l)$$

(see (16)). Since $\mathrm{Spec}_2(T_l, \ell^2(2N_l)) = \mathrm{Spec}(T_l)$, it follows from the definition of $\Delta(T_l, 2N_l, \alpha_l)$ and from what we know about $\mathrm{Spec}(T_l)$, that $\mathrm{Spec}_2(T, \ell^2(2N_l))$ lies in an α_l-neighbourhood of $[-\varrho_l, \varrho_l]$ and the distance from any point of $[-2, 2]$ to $\mathrm{Spec}_2(T, \ell^2(2N_l))$ is less than $2^{-l} + 2^{-l-1} + \alpha_l$. Hence $\mathrm{Spec}_2(T, \ell^2(2N_l))$ converge to $[-2, 2]$ while $\mathrm{Spec}_2(T, \ell^2(N_l))$ converge to $\{r\}$ as $l \to +\infty$.

3. Proof of Theorem 1.1

Let
$$r_j \in F \cap (\rho_-^{(j)}, \rho_+^{(j)}),$$
for $j = 1, \ldots, n$, and let $T^{(j)}$ and $N_l^{(j)}$, $l \in \mathbb{N}$ be the same as in Lemma 2.5 but with $r_j \in (\rho_-^{(j)}, \rho_+^{(j)})$ in place of $r \in (\rho_-, \rho_+)$. Let $\mathcal{H}_j = \ell^2$, $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$ and $T = \operatorname{diag}\{T^{(1)}, \ldots, T^{(n)}\} \in \mathcal{B}(\mathcal{H})$. It is clear that $T = T^*$ and $\operatorname{Spec}(T) = \bigcup_{j=1}^n [\rho_-^{(j)}, \rho_+^{(j)}]$.

Let F_l be a finite subset of the interior of $\mathcal{Q}\big(\bigcup_{j=1}^n [\rho_-^{(j)}, \rho_+^{(j)}]\big)$ symmetric with respect to the real line and such that
$$d_H(F_l, F) < 2^{-l-1}, \tag{17}$$
and let $F_l \cap \{\lambda \in \mathbb{C}\colon \operatorname{Im}\lambda \geq 0\} = \{\mu_1^{(l)}, \ldots, \mu_{n_l}^{(l)}\}$. For any $k = 1, \ldots, n_l$ there exist $\lambda_{1,k}^{(l)}, \lambda_{2,k}^{(l)}, \lambda_{3,k}^{(l)} \in \bigcup_{j=1}^n (\rho_-^{(j)}, \rho_+^{(j)})$ such that the convex hull of $\{(\mu_k^{(l)} - \lambda_{m,k}^{(l)})^2\}_{m=1}^3$ contains 0, i.e. there exists
$$t_{1,k}^{(l)}, t_{2,k}^{(l)}, t_{3,k}^{(l)} \in [0, 1]\colon\ t_{1,k}^{(l)} + t_{2,k}^{(l)} + t_{3,k}^{(l)} = 1,$$
$$\sum_{m=1}^{3} t_{m,k}^{(l)} (\mu_k^{(l)} - \lambda_{m,k}^{(l)})^2 = 0 \tag{18}$$
(see [19]).

Let $\mathcal{L}_0 = \{0\}$, $\widetilde{N}_0 = 1$, and suppose we have constructed $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{l-1} \subset \mathcal{H}$ and $\widetilde{N}_0 < \widetilde{N}_1 < \cdots < \widetilde{N}_{l-1} \in \mathbb{N}$ such that $\mathcal{L}_p \subseteq \bigoplus_{j=1}^n \ell^2(\widetilde{N}_p)$, $p = 1, \ldots, l-1$. Let us construct $\mathcal{L}_l$ and $\widetilde{N}_l$. Let $\widehat{N}_l^{(j)}$ be the smallest number among $N_l^{(j)} < N_{l+1}^{(j)} < N_{l+2}^{(j)} < \ldots$ which is greater than or equal to $\widetilde{N}_{l-1}$. Then $\mathcal{L}_{l-1} \subseteq \mathcal{L}_l^0 \stackrel{\text{def}}{=} \bigoplus_{j=1}^n \ell^2(\widehat{N}_l^{(j)})$ and
$$d_H(\operatorname{Spec}_2(T, \mathcal{L}_l^0), \{r_1, \ldots, r_n\}) < \alpha_l. \tag{19}$$

Let $E(\cdot)$ be the spectral measure of T and let
$$W_{m,k}^{(l)} \subset \bigcup_{j=1}^{n} (\rho_-^{(j)}, \rho_+^{(j)})$$
be the ε_l'-neighbourhood of $\lambda_{m,k}^{(l)}$, where ε_l' is a small positive number to be specified later. Since the subspaces $E(W_{m,k}^{(l)})\mathcal{H} \subset \mathcal{H}$ are infinite dimensional, we can choose vectors $u_{m,k}^{(l)} \in E(W_{m,k}^{(l)})\mathcal{H}$ such that $\|u_{m,k}^{(l)}\| = 1$ and
$$u_{m,k}^{(l)} \perp T^q(\mathcal{L}_l^0) \quad \text{and} \quad u_{m,k}^{(l)} \perp T^q u_{m',k'}^{(l)},$$

for $q = 0, 1, 2, \quad m, m' = 1, 2, 3, \quad k, k' = 1, \dots n_l$, and $(m, k) \neq (m', k')$. Let

$$v_k^{(l)} = \sum_{m=1}^{3} \sqrt{t_{m,k}^{(l)}}\, u_{m,k}^{(l)}, \quad \mathcal{L}_l' = \mathcal{L}_l^0 \oplus \operatorname{span}\{v_k^{(l)}\}_{k=1}^{n_l},$$

and let $\mathcal{P}_l^0 \colon \mathcal{H} \to \mathcal{L}_l^0$ and $\mathcal{P}_l' \colon \mathcal{H} \to \mathcal{L}_l'$ be the corresponding orthogonal projections. Then $\|v_k^{(l)}\| = 1$,

$$v_k^{(l)} \perp T^q(\mathcal{L}_l^0), \quad v_k^{(l)} \perp T^q v_{k'}^{(l)}, \quad q = 0, 1, 2, k, \ \ k' = 1, \dots n_l, \ \ k \neq k',$$

and

$$\mathcal{P}_l'(\lambda I - T)^2|_{\mathcal{L}_l^0} = \mathcal{P}_l^0(\lambda I - T)^2|_{\mathcal{L}_l^0},$$

$$\mathcal{P}_l'(\lambda I - T)^2 v_k^{(l)} = ((\lambda I - T)^2 v_k^{(l)}, v_k^{(l)}) v_k^{(l)} \stackrel{\text{def}}{=} p_k^{(l)}(\lambda) v_k^{(l)}.$$

Hence $\mathcal{P}_l'(\lambda I - T)^2|_{\mathcal{L}_l'}$ is unitarily equivalent to

$$\begin{pmatrix} \mathcal{P}_l^0(\lambda I - T)^2|_{\mathcal{L}_l^0} & 0 \\ 0 & \operatorname{diag}\{p_1^{(l)}(\lambda), \dots, p_{n_l}^{(l)}(\lambda)\} \end{pmatrix}$$

and

$$\operatorname{Spec}_2(T, \mathcal{L}_l') = \operatorname{Spec}_2(T, \mathcal{L}_l^0) \cup \bigcup_{k=1}^{n_l} \{\lambda \in \mathbb{C} \colon p_k^{(l)}(\lambda) = 0\}. \tag{20}$$

By construction, the coefficients of the quadratic polynomial

$$p_k^{(l)}(\lambda) = ((\lambda I - T)^2 v_k^{(l)}, v_k^{(l)}) = \sum_{m=1}^{3} t_{m,k}^{(l)} ((\lambda I - T)^2 u_k^{(l)}, u_k^{(l)})$$

are real and differ by less than $C\varepsilon_l'$ from those of

$$q_k^{(l)}(\lambda) \stackrel{\text{def}}{=} \sum_{m=1}^{3} t_{m,k}^{(l)} (\lambda - \lambda_{m,k}^{(l)})^2.$$

(It follows from the spectral theorem that one can take

$$C = 2 \max\{1, \|T\|\}$$

here.) Taking ε_l' sufficiently small we can ensure that the zeros of $p_k^{(l)}(\lambda)$ differ from those of $q_k^{(l)}(\lambda)$ by less than 2^{-l-1}. According to (18), $\mu_k^{(l)}$ and its complex conjugate are the zeros of $q_k^{(l)}(\lambda)$. Hence it follows from (17), (19), and (20) that

$$d_H(\operatorname{Spec}_2(T, \mathcal{L}_l'), F) < \max\{\alpha_l, 2^{-l}\}.$$

Let $\widetilde{N}_l > \widetilde{N}_{l-1}$, $\widetilde{N}_l > \widehat{N}_l^{(j)}$, $j = 1, \dots, n$, $P_{(l)} \colon \mathcal{H} \to \bigoplus_{j=1}^n \ell^2(\widetilde{N}_l)$ be the orthogonal projection,

$$\mathcal{L}_l = \mathcal{L}_l^0 \oplus \operatorname{span}\{P_{(l)} v_k^{(l)}\}_{k=1}^{n_l}$$

and let $\mathcal{P}_l \colon \mathcal{H} \to \mathcal{L}_l$ be the corresponding orthogonal projection. $\operatorname{Spec}_2(T, \mathcal{L}_l')$ is the set of zeros of the determinant of a matrix representation of $\mathcal{P}_l'(\lambda I - T)^2|_{\mathcal{L}_l'}$ which is a polynomial in λ. If $\widetilde{N}_l$ is large, then $P_{(l)} v_k^{(l)}$ is close to $v_k^{(l)}$, and the coefficients of the polynomial corresponding to $\mathcal{P}_l(\lambda I - T)^2|_{\mathcal{L}_l}$ are close to their counterparts corresponding to $\mathcal{P}_l'(\lambda I - T)^2|_{\mathcal{L}_l'}$. Hence taking $\widetilde{N}_l$ sufficiently large we get

$$d_H(\operatorname{Spec}_2(T, \mathcal{L}_l), F) < \max\{\alpha_l, 2^{-l}\}$$

(see Theorem 4.10c in [11]). Note that $(\mathcal{L}_l) \in \mathfrak{P}(\mathcal{H})$ because $\mathcal{L}_l \supset \mathcal{L}_l^0 = \bigoplus_{j=1}^n \ell^2(\widehat{N}_l^{(j)})$ and $\widehat{N}_l^{(j)} \geq N_l^{(j)} \to +\infty$ as $l \to +\infty$, $j = 1, \dots n$.

Remark 3.1. $\operatorname{Spec}_2(T, \mathcal{L}_l)$ constructed in the above proof converge to F. The limit behaviour of a sequence of second order relative spectra of T may be considerably more complicated than that. Let, for example, $F_0 \subseteq \mathcal{Q}\big(\bigcup_{j=1}^n [\rho_-^{(j)}, \rho_+^{(j)}]\big)$ be another compact set symmetric with respect to the real line and such that

$$F_0 \cap F \cap (\rho_-^{(j)}, \rho_+^{(j)}) \neq \emptyset, \quad j = 1, \dots, n.$$

Acting as in the proof above one can construct a sequence $(\mathcal{L}_{0,l})$ similar to $(\mathcal{L}_l)$ and such that

$$d_H(\operatorname{Spec}_2(T, \mathcal{L}_{0,l}), F_0) \longrightarrow 0, \quad \text{as } l \to +\infty.$$

Then it is easy to extract subsequences from $(\mathcal{L}_l)$ and to $(\mathcal{L}_{0,l})$ and to combine them into a new sequence $(\mathcal{M}_l) \in \mathfrak{P}(\mathcal{H})$ in such a way that

$$d_H(\operatorname{Spec}_2(T, \mathcal{M}_{2l}), F) \longrightarrow 0 \quad \text{and} \quad d_H(\operatorname{Spec}_2(T, \mathcal{M}_{2l+1}), F_0) \longrightarrow 0,$$

as $l \to +\infty$. One can of course carry out a similar procedure with more than just two limit sets F and F_0.

4. Concluding remarks

The sequence (N_l) in the proof of Lemma 2.5 and $(\dim \mathcal{L}_l)$ in the proof of Theorem 1.1 are very rapidly increasing and it is not clear whether the above results have serious implications for “real life” computations involving second order relative spectra. In all numerical examples studied so far (see, e.g., [3], [4], [5], [6], [7], [8], [14], and [21]), second order relative spectra seemed to approximate the whole spectrum quite well.

Question 1. *Can the phenomenon described by Lemma 2.5 and Theorem 1.1 still happen if one restricts the rate of growth of* $\dim \mathcal{L}_l$*?*

Note that

$$\lim_{N\to+\infty}{}^{*}\, \mathrm{Spec}_2(T, \ell^2(N)) \cap \mathbb{R} = [-2, 2] = \mathrm{Spec}(T)$$

in Remark 2.6. Here

$$\lim_{l\to+\infty}{}^{*} G_l \overset{\text{def}}{=} \{z \in \mathbb{C} \colon \text{there are } l_m \in \mathbb{N}, \text{ and } z_{l_m} \in G_{l_m} \text{ such that } l_m \to +\infty \text{ and } z_{l_m} \to z, \text{ as } m \to +\infty\}$$

for $G_l \subset \mathbb{C}$, and $l \in \mathbb{N}$.

It is well known that

$$\lim_{l\to+\infty}{}_{*}\, \mathrm{Spec}(T, \mathcal{L}_l) \supseteq \mathrm{Spec}(T), \quad (\mathcal{L}_l) \in \mathfrak{P}(\mathcal{H}),$$

where

$$\lim_{l\to+\infty}{}_{*} G_l \overset{\text{def}}{=} \{z \in \mathbb{C} \mid \exists z_l \in G_l \colon \lim_{l\to+\infty} z_l = z\}$$

(see, e.g., [1] or [19]). It is reasonable therefore to use $\lim_*$ when approximating $\mathrm{Spec}(T)$ with the help of $\mathrm{Spec}(T, \mathcal{L}_l)$. On the other hand, the non-pollution result (2) shows it is more natural to use $\lim^*$ when approximating $\mathrm{Spec}(T)$ with the help of $\mathrm{Spec}_2(T, \mathcal{L}_l)$.

Another natural question is whether or not one can drop condition (3) in Theorem 1.1.

Question 2. *Can the limit set of a sequence of second order relative spectra be disjoint from the* (*essential*) *spectrum of* $T = T^* \in \mathcal{B}(\mathcal{H})$*?*

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