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A priori bounds and existence of non-real eigenvalues of indefinite Sturm–Liouville problems

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Abstract. The present paper gives a priori bounds on the possible non-real eigenvalues of regular indefinite Sturm–Liouville problems and obtains sufficient conditions for such problems to admit non-real eigenvalues.

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1. Introduction

The present paper is concerned with the indefinite spectral problem

$$-y'' + qy = \lambda wy,\ y(-1) = y(1) = 0, \quad \text{in } L^2_{|w|}[-1,1] \tag{1.1}$$

under the standing hypothesis that q and w are real-valued functions satisfying

$$q, w \in L^1[-1,1], \quad w(x) \neq 0 \text{ a.e. on } [-1,1], \tag{1.2}$$

and $w(x)$ changes sign on $[-1,1]$. The indefinite problem (1.1) has discrete, real eigenvalues, unbounded from both below and above, and may also admit non-real eigenvalues. Such problems occur in certain physical models, particularly in transport theory and statistical physics. The indefinite nature of the problem was noticed by Haupt [9] and Richardson [12] at the beginning of the last century. For a review of the early work in this direction, see [11].

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As a simple example of (1.1), the Richardson problem [13]

$$-y'' - \mu y = \lambda \operatorname{sgn}(x) y, \quad x \in [-1, 1], \quad y(-1) = 0 = y(1) \tag{1.3}$$

was studied by many authors, such as Turyn [14], Atkinson and Jabon [1], Fleckinger and Mingarelli [8], and P. Binding and H. Volkmer [6]. For the indefinite problem (1.1), non-real eigenvalues might appear only if the corresponding right-definite problem

$$-y'' + qy = \lambda |w| y, \, y(-1) = y(1) = 0 \text{ in } L^2_{|w|}[-1, 1] \tag{1.4}$$

has negative eigenvalues, namely, here holds the following result.

Proposition 1.1 ([10], Theorem 2, p. 523, and [7], Corollary 1.7). *If problem* (1.4) *has n negative eigenvalues, then problem* (1.1) *has at most* $2n$ *non-real eigenvalues.*

Although the upper bound given in Proposition 1.1 is sharp (see [12] and [2]), determining a priori bounds and the exact number of non-real eigenvalues are still difficult and interesting open problems in Sturm–Liouville theory (see [11] and [15], p. 126). Recently, by means of the operator theory in Krein spaces, Behrndt, Katatbeh, and Trunk [2], Theorem 2.3 and Corollary 2.4, have given sufficient conditions for the existence of non-real eigenvalues of the singular indefinite Sturm–Liouville operator

$$(Af)(x) \stackrel{\text{def}}{=} \operatorname{sgn}(x)(-f''(x) + V(x)f(x)) = \lambda f(x), \quad x \in \mathbb{R}, \tag{1.5}$$

and if $V \in L^\infty(\mathbb{R})$, Behrndt, Philipp and Trunk [3], Theorem 4.2, have obtained explicit bounds on the non-real eigenvalues of (1.5) in terms of V.

In the present paper, we will first obtain a priori bounds for possible non-real eigenvalues and then find sufficient conditions for the existence of non-real eigenvalues of (1.1). These results will answer or partially answer several open problems posed in [11]. We state these results in this section and prove them in Sections 2 and 3.

Denote by $\|\cdot\|_p$ the norm of the space $L^p[-1, 1]$ and by $\|\cdot\|_C$ the maximum norm of $C[-1, 1]$. If $xw(x) > 0$ a.e. on $[-1, 1]$, we set

$$S_1(\varepsilon) = \{x \in [-1, 1] \colon xw(x) < \varepsilon\}, \quad m_1(\varepsilon) = \operatorname{mes} S_1(\varepsilon). \tag{1.6}$$

If $w \in AC_{\text{loc}}[-1, 1]$, $w' \in L^2[-1, 1]$, we set

$$S_2(\varepsilon) = \{x \in [-1, 1] \colon w^2(x) < \varepsilon\}, \quad m_2(\varepsilon) = \operatorname{mes} S_2(\varepsilon). \tag{1.7}$$

A value of x about which $w(x)$ changes its sign will be called a *turning point* [10]. If $w(x)$ has only one turning point, we will obtain the following a priori bounds for possible non-real eigenvalues.

Theorem 1.2. *Suppose that* λ *exists and that it is a non-real eigenvalue of* (1.1). *If* $xw(x) > 0$ *a.e. on* $[-1, 1]$, *then*

$$|\operatorname{Re}\lambda| \leq \frac{4}{\varepsilon_1}(\|q_-\|_1 + 4\|q_-\|_1^2), \quad |\operatorname{Im}\lambda| \leq \frac{4}{\varepsilon_1}\|q_-\|_1, \tag{1.8}$$

where $\varepsilon_1 > 0$ *satisfies* $8\|q_-\|_1^2 m_1(\varepsilon_1) < 1$ *and* $q_-(x) = -\min\{0, q(x)\}$.

In the case where $w(x)$ is allowed to have more turning points, we will obtain the following result.

Theorem 1.3. *Suppose that* λ *exists and that it is a non-real eigenvalue of* (1.1). *If* $w \in AC[-1, 1]$ *and* $w' \in L^2[-1, 1]$, *then*

$$|\operatorname{Re}\lambda| \leq \frac{8}{\varepsilon_2}\|q_-\|_1^2(3\|w\|_C + \|w'\|_2), \quad |\operatorname{Im}\lambda| \leq \frac{8}{\varepsilon_2}\|w'\|_2\|q_-\|_1^2, \tag{1.9}$$

where $\varepsilon_2 > 0$ *is chosen such that* $8\|q_-\|_1^2 m_2(\varepsilon_2) < 1$.

In the particular case where $q \geq 0$, we see by Theorems 1.2 and 1.3 that (1.1) has no any non-real eigenvalues, which is in accordance with the conclusion in Proposition 1.1 since now (1.4) does not have any negative eigenvalues.

In what follows, we impose the symmetry conditions on q and w, namely,

$$q(x) = q(-x) \quad \text{and} \quad w(-x) = -w(x). \tag{1.10}$$

In this case, more accurate a priori bounds on imaginary eigenvalues can be found if q is bounded below and w keeps away from zero.

Theorem 1.4. *Suppose that* (1.10) *holds and* $xw(x) > 0$ *a.e. on* $[-1, 1]$. *If, for some* $q_0 < 0$ *and* $w_0 > 0$,

$$q(x) \geq q_0, \; |w(x)| \geq w_0 \quad \text{a.e. } x \in [-1, 1], \tag{1.11}$$

then for any possible pure imaginary eigenvalue λ *of* (1.1), *there holds*

$$|\operatorname{Im}\lambda| \leq \frac{4(-q_0)^{3/2}}{w_0}. \tag{1.12}$$

In view of (1.10), using the spectral theory of operators in Krein spaces, we obtain an existence result for non-real eigenvalues of the indefinite problem (1.1).

Theorem 1.5. *Let* (1.10) *be fulfilled. If the eigenvalue problem*

$$-y'' + q(x)y = \lambda y, \quad y(-1) = y(1) = 0 \tag{1.13}$$

has one negative eigenvalue and the rest eigenvalues are all positive, then (1.1) *has exactly two purely imaginary eigenvalues.*

Immediate consequences of Proposition 1.1, Theorems 1.4, and Theorem 1.5 are the existence and bounds for non-real eigenvalues of Richardson problem.

Corollary 1.6. *For* $\mu \in \left(\frac{\pi^2}{4}, \pi^2\right)$*, the Richardson eigenvalue problem* (1.3) *has exactly two purely imaginary eigenvalues whose moduli are bounded by* $4\mu^{3/2}$.

Remark 1. *Theorems* 1.2 *and* 1.5 *can be generalized to the problem*

$$\begin{cases} -(p(x)y')' + q(x)y = \lambda w(x)y, \\ \alpha_1 y(-1) + \beta_1 y'(-1) = 0, \\ \alpha_2 y(1) + \beta_2 y'(1) = 0, \end{cases}$$

where $p(x) > 0$ *a.e. on* $[-1,1]$, $1/p \in L^1[-1,1]$, $\alpha_j, \beta_j \in \mathbb{R}$ *for* $j = 1,2$ *and* $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$*, but we do not pursue this here.*

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2. A priori bounds of non-real eigenvalues

In this section we will prove Theorems 1.2, 1.3, and 1.4.

Proof of Theorem 1.2. Let λ be a non-real eigenvalue of (1.1) and $\varphi(x)$ the corresponding eigenfunction with $\|\varphi\|_2 = 1$. Multiplying both sides of $-\varphi'' + q\varphi = \lambda w\varphi$ by $\bar{\varphi}$ and integrating over the interval $[x,1]$ we have

$$(\varphi'\bar{\varphi})(x) + \int_x^1 |\varphi'|^2 + \int_x^1 q|\varphi|^2 = \lambda \int_x^1 w|\varphi|^2. \tag{2.1}$$

Separating the real and imaginary parts of both sides of (2.1) yields

$$\operatorname{Re}\lambda \int_x^1 w|\varphi|^2 = \operatorname{Re}(\varphi'\bar{\varphi})(x) + \int_x^1 |\varphi'|^2 + \int_x^1 q|\varphi|^2, \tag{2.2}$$

$$\operatorname{Im}\lambda \int_x^1 w|\varphi|^2 = \operatorname{Im}(\varphi'\bar{\varphi})(x). \tag{2.3}$$

We will use (2.2) and (2.3) to estimate $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$. To do this, let $x = -1$ in (2.3). From $\operatorname{Im}\lambda \neq 0$ and $\varphi(-1) = 0$, we have $\int_{-1}^1 w|\varphi|^2 = 0$, and hence, by (2.2),

$$\int_{-1}^1 (|\varphi'|^2 + q|\varphi|^2) = 0. \tag{2.4}$$

Set

$$Q(x) = \int_{-1}^{x} q_-(t)\mathrm{d}t.$$

Then $\max |Q(x)| \le \|q_-\|_1$ and

$$\int_{-1}^{1} q_-(x)|\varphi(x)|^2\mathrm{d}x = \int_{-1}^{1} Q'(x)|\varphi(x)|^2\mathrm{d}x - 2\,\mathrm{Re}\left(\int_{-1}^{1} Q(x)\varphi'(x)\overline{\varphi(x)}\mathrm{d}x\right),$$

which, together with $\|\varphi\|_2 = 1$, yields that

$$\begin{aligned}\int_{-1}^{1} q_-|\varphi|^2 &\le 2\|q_-\|_1 \int_{-1}^{1} |\varphi'||\varphi| \\ &\le 2\|q_-\|_1\|\varphi'\|_2 \le 2\|q_-\|_1^2 + \frac{1}{2}\|\varphi'\|_2^2.\end{aligned} \tag{2.5}$$

Then, from (2.4), we get

$$\|\varphi'\|_2^2 \le 4\|q_-\|_1^2, \int_{-1}^{1} q_-|\varphi|^2 \le 4\|q_-\|_1^2. \tag{2.6}$$

From $\varphi(x) = \int_{-1}^{x} \varphi'(t)\mathrm{d}t$, by Cauchy–Schwarz inequality, we have

$$|\varphi(x)|^2 = \left|\int_{-1}^{x} \varphi'(t)\mathrm{d}t\right|^2 \le (x+1)\int_{-1}^{x} |\varphi'(t)|^2\mathrm{d}t \le \int_{-1}^{0} |\varphi'|^2 \le \|\varphi'\|_2^2$$

for $-1 \le x \le 0$. From $\varphi(x) = -\int_{x}^{1} \varphi'(t)\mathrm{d}t$, one similarly proves $|\varphi(x)|^2 \le \|\varphi'\|_2^2$ for $x \in [0, 1]$, and so,

$$|\varphi(x)|^2 \le \|\varphi'\|_2^2, \quad x \in [-1, 1]. \tag{2.7}$$

Since $xw(x) > 0$, a.e. on $[-1, 1]$, one can find $\varepsilon_1 > 0$ such that $8\|q_-\|_1^2 m_1(\varepsilon_1) < 1$, where $m_1(\varepsilon)$ is defined in (1.6). Using $\int_{-1}^{1} w|\varphi|^2 = 0$, from (2.6) and (2.7), we have

$$\begin{aligned}\int_{-1}^{1}\int_{x}^{1} w(t)|\varphi(t)|^2\mathrm{d}t\,\mathrm{d}x &= \int_{-1}^{1} xw(x)|\varphi(x)|^2\mathrm{d}x \\ &\ge \varepsilon_1\left(\int_{-1}^{1} |\varphi(x)|^2\mathrm{d}x - \int_{S_1(\varepsilon_1)} |\varphi(x)|^2\mathrm{d}x\right) \\ &\ge \varepsilon_1[1 - 4\|q_-\|_1^2 m_1(\varepsilon_1)] \\ &\ge \frac{\varepsilon_1}{2}.\end{aligned} \tag{2.8}$$

Set

$$q_+(x) = \max\{0, q(x)\}.$$

Then $q = q_+ - q_-$ and $|q| = q_+ + q_- = q + 2q_-$. Repeatedly using (2.4), we have

$$\begin{aligned}\left|\int_{-1}^{1}\int_{x}^{1}(|\varphi'|^2 + q|\varphi|^2)\mathrm{d}t\,\mathrm{d}x\right| &= \left|\int_{-1}^{1} x(|\varphi'|^2 + q|\varphi|^2)\mathrm{d}x\right| \\ &\le \int_{-1}^{1}(|\varphi'|^2 + q|\varphi|^2 + 2q_-|\varphi|^2)\mathrm{d}x \\ &= 2\int_{-1}^{1} q_-|\varphi|^2\mathrm{d}x.\end{aligned}$$

Now, by (2.6), the integration of (2.2) gives

$$\begin{aligned}|\operatorname{Re}\lambda|\int_{-1}^{1}\int_{x}^{1} w|\varphi|^2 &= \left|\int_{-1}^{1}\operatorname{Re}(\varphi'\bar{\varphi})\mathrm{d}x + \int_{-1}^{1}\int_{x}^{1}(|\varphi'|^2 + q|\varphi|^2)\mathrm{d}t\,\mathrm{d}x\right| \\ &\le \|\varphi'\|_2 + 2\int_{-1}^{1} q_-|\varphi|^2\mathrm{d}x \le 2\|q_-\|_1 + 8\|q_-\|_1^2.\end{aligned}$$

Therefore, in view of (2.8), we conclude that

$$|\operatorname{Re}\lambda| \le \frac{4}{\varepsilon_1}(\|q_-\|_1 + 4\|q_-\|_1^2). \tag{2.9}$$

Moreover, integrating (2.3) and using (2.8) and (2.6), we have

$$\frac{\varepsilon_1}{2}|\operatorname{Im}\lambda| \le |\operatorname{Im}\lambda|\int_{-1}^{1}\int_{x}^{1} w|\varphi|^2 = \left|\int_{-1}^{1}\operatorname{Im}(\varphi'\bar{\varphi})\right| \le \|\varphi'\|_2 \le 2\|q_-\|_1, \tag{2.10}$$

and (1.8) follows immediately. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let λ be a non-real eigenvalue of (1.1) and φ the corresponding eigenfunction with $\|\varphi\|_2 = 1$. In this case we still can make use of (2.1), (2.2), and (2.3). From (2.3), since $\operatorname{Im}\lambda \ne 0$, one sees that $\int_{-1}^{1} w|\varphi(x)|^2\mathrm{d}x = 0$. Thus, (2.4), (2.6), and (2.7) hold, and, in particular,

$$|\varphi(x)|^2 \le \|\varphi'\|_2^2, \quad x \in [-1, 1], \quad \|\varphi'\|_2^2 \le \int_{-1}^{1} q_-|\varphi|^2 \le 4\|q_-\|_1^2. \tag{2.11}$$

Multiplying $-\varphi'' + q\varphi = \lambda w\varphi$ by $w\bar{\varphi}$ and integrating by parts, we get

$$\int_{-1}^{1} w|\varphi'|^2 + \int_{-1}^{1} w'\varphi'\bar{\varphi} + \int_{-1}^{1} wq|\varphi|^2 = \lambda\int_{-1}^{1} w^2|\varphi|^2. \tag{2.12}$$

Separating the real and imaginary parts of the both sides of (2.12) yields

$$\operatorname{Re}\lambda \int_{-1}^{1} w^2|\varphi|^2 = \operatorname{Re}\left(\int_{-1}^{1} w'\varphi'\bar{\varphi}\right) + \int_{-1}^{1} w(|\varphi'|^2 + q|\varphi|^2), \tag{2.13}$$

$$\operatorname{Im}\lambda \int_{-1}^{1} w^2|\varphi|^2 = \operatorname{Im}\left(\int_{-1}^{1} w'\varphi'\bar{\varphi}\right). \tag{2.14}$$

Now, using (2.11), $|q| = q + 2q_-$ and

$$\int_{-1}^{1} q|\varphi|^2 = -\int_{-1}^{1} |\varphi'|^2 < 0,$$

we obtain

$$\begin{aligned} \left|\int_{-1}^{1} w|\varphi'|^2\right| &\le \|w\|_C \|\varphi'\|_2^2 \le 4\|w\|_C \|q_-\|_1^2, \\ \left|\int_{-1}^{1} wq|\varphi|^2\right| &\le \|w\|_C \int_{-1}^{1} |q||\varphi|^2 \le 8\|w\|_C \|q_-\|_1^2, \\ \left|\int_{-1}^{1} w'\varphi'\bar{\varphi}\right| &\le \|\varphi'\|_2 \|w'\|_2 \|\varphi'\|_2 \le 4\|w'\|_2 \|q_-\|_1^2. \end{aligned} \tag{2.15}$$

Recall that $m_2(\varepsilon_2) = \operatorname{mes} S_2(\varepsilon_2)$ defined in (1.7) and $w^2(x) \ge \varepsilon_2$ on the set $\Omega(\varepsilon_2) \stackrel{\text{def}}{=} [-1, 1] \setminus S_2(\varepsilon_2)$. Then $8\|q_-\|_1^2 m(\varepsilon_2) < 1$ yields that

$$\begin{aligned} \int_{-1}^{1} w^2(x)|\varphi(x)|^2 \mathrm{d}x &\ge \varepsilon_2 \int_{\Omega(\varepsilon_2)} |\varphi|^2 \\ &= \varepsilon_2 \left(1 - \int_{S(\varepsilon_2)} |\varphi|^2\right) \\ &\ge \varepsilon_2 \left(1 - 4\|q_-\|_1^2 m(\varepsilon_2)\right) \\ &\ge \frac{\varepsilon_2}{2}, \end{aligned} \tag{2.16}$$

which, together with (2.13), (2.14), and (2.15), gives (1.9) and completes the proof. □

Under conditions (1.2) and (1.10), it is easy to see that if $\lambda \in \mathbb{C}$ is an eigenvalue of (1.1) with an eigenfunction φ, then $-\bar{\lambda}$ is an eigenvalue of (1.1) with the eigenfunction $\overline{\varphi(-\cdot)}$. Thus, if $\lambda = \mathrm{i}\alpha$ with $\alpha \in \mathbb{R}$, then $\overline{\varphi(-x)} = C\varphi(x)$ for some $C \neq 0$ since the geometric multiplicity is one. Then it follows that $|C| = 1$ from $\overline{\varphi(0)} = C\varphi(0)$, $\overline{\varphi'(0)} = -C\varphi'(0)$, and $|\varphi(0)| + |\varphi'(0)| \neq 0$. To sum up, we have the following result.

Lemma 2.1. *Let* (1.2) *and* (1.10) *hold. If* $\lambda \in \mathbb{C}$ *is an eigenvalue of* (1.1) *with an eigenfunction* φ, *then* $-\bar{\lambda}$ *is an eigenvalue of* (1.1) *with the eigenfunction* $\overline{\varphi(-\cdot)}$. *In particular, if* $\lambda = \mathrm{i}\alpha$ *with* $\alpha \in \mathbb{R}$ *and* $\alpha \neq 0$, *then* $\overline{\varphi(-\cdot)} = C\varphi$ *for some* $C \in \mathbb{C}$ *with* $|C| = 1$.

Proof of Theorem 1.4. Let φ be an eigenfunction corresponding to $\lambda = \mathrm{i}\alpha$ with $\|\varphi\|_2 = 1$. It follows from Lemma 2.1 that there exists an $\omega \in [0, 2\pi)$ such that $\overline{\varphi(-x)} = \mathrm{e}^{\mathrm{i}\omega}\varphi(x)$ and $-\overline{\varphi'(-x)} = \mathrm{e}^{\mathrm{i}\omega}\varphi'(x)$. So, $|\varphi(x)|$ and $|\varphi'(x)|$ are even functions. We see that (2.1)–(2.4) hold for this φ. Similarly to (2.7), we have

$$\begin{aligned} |\varphi(x)|^2 &\leq (x+1)\int_{-1}^{x} |\varphi'(t)|^2 \mathrm{d}t \\ &\leq \int_{-1}^{0} |\varphi'(t)|^2 \mathrm{d}t = \frac{1}{2}\|\varphi'\|_2^2, \quad x \in [-1, 0], \end{aligned} \tag{2.17}$$

since $|\varphi'(x)|$ is even. Actually, (2.17) is true for $x \in [-1, 1]$ since $|\varphi(x)|$ is even.

Since $q(x) \geq q_0$ on $[-1, 1]$, it follows from (2.4) and $\|\varphi\|_2 = 1$ that

$$\|\varphi'\|_2^2 = -\int_{-1}^{1} q|\varphi|^2 \leq -q_0,$$

and then the integration of (2.3) produces

$$|\operatorname{Im}\lambda| \left| \int_{-1}^{1}\int_{x}^{1} w|\varphi|^2 \right| = \left| \int_{-1}^{1} \operatorname{Im}(\varphi'\bar{\varphi}) \right| \leq \|\varphi'\|_2 \leq (-q_0)^{1/2}. \tag{2.18}$$

Let $\delta = 1/(-2q_0)$. By (2.17), we have $1 = \int_{-1}^{1} |\varphi|^2 \leq \|\varphi'\|_2^2 \leq -q_0$ and

$$\begin{aligned} \left| \int_{-1}^{1}\int_{x}^{1} w|\varphi|^2 \right| &= \int_{-1}^{1} xw(x)|\varphi|^2 \mathrm{d}x \\ &\geq w_0 \int_{-1}^{1} |x||\varphi|^2 \mathrm{d}x \\ &\geq w_0\delta \int_{|x|\geq\delta} |\varphi|^2 = w_0\delta\left(1 - \int_{-\delta}^{\delta} |\varphi|^2\right) \\ &\geq w_0\delta(1 - \delta(-q_0)) \\ &= -\frac{w_0}{4q_0}. \end{aligned} \tag{2.19}$$

Now, (1.12) follows from (2.18) and (2.19). The proof is complete. □

3. Existence of non-real eigenvalues

In this section we prove Theorem 1.5 and in the proof we will use the following result which was proved, e.g., in [7] and [5].

Lemma 3.1 ([7], Proposition 2.6). *If* $w_j \in L^1[-1,1]$ *and* $w_j(x) > 0$ *a.e. on* $[-1,1]$ *for* $j = 1, 2$*, then the two eigenvalue problems*

$$-y'' + q(x)y = \lambda w_j(x)y, \quad y(-1) = y(1) = 0, \quad j = 1, 2 \tag{3.1}$$

have the same number of negative eigenvalues.

Let K be the Krein space $L^2_{|w|}[-1,1]$, equipped with the indefinite inner product

$$[f, g] = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\mathrm{d}x, \; f, g \in L^2_{|w|}[-1,1] \tag{3.2}$$

and T a self-adjoint operator in K with domain $\mathcal{D}(T)$; see [4], [2], and [7]. We say that the operator T has k *negative squares,* $k \in \mathbb{N}_0$, if there exists a k-dimensional subspace X of K in $\mathcal{D}(T)$ such that $[Tf, f] < 0$ if $f \in X$ and $f \neq 0$, but no $(k+1)$-dimensional subspace with this property.

Proof of Theorem 1.5. Let A and B be the operators associated with $-y'' + q(x)y = \lambda w(x)y$ and $-y'' + q(x)y = \lambda|w(x)|y$ with the Dirichlet boundary conditions, respectively. Then B is self-adjoint with respect to the definite inner product

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}|w(x)|\mathrm{d}x, \; f, g \in L^2_{|w|}[-1,1]$$

and A is self-adjoint with respect to the indefinite inner product (3.2).

It follows from Lemma 3.1 and the assumption in Theorem 1.5 that B has one negative eigenvalue and the rest are positive, and hence, A has exactly one negative square since $[Af, f] = (Bf, f)$ and 0 is a resolvent point of A. It is well known (see, e.g., [7], Proposition 1.5, or [4], Theorem 3.1) that this implies the existence of exactly one eigenvalue λ of (1.1) in $\mathbb{R}$ or the upper half-plane $\mathbb{C}^+$ and that if $\lambda \in \mathbb{R}$ with eigenfunction φ then $[A\varphi, \varphi] = \lambda[\varphi, \varphi] \leq 0$. Let λ be such an eigenvalue with eigenfunction φ. If λ is real, then $-\lambda = -\bar{\lambda}$ is also an eigenvalue with the eigenfunction $\overline{\varphi(-\cdot)}$ by Lemma 2.1 and

$$-\lambda[\overline{\varphi(-\cdot)}, \overline{\varphi(-\cdot)}] = \lambda[\varphi, \varphi] \leq 0$$

by the odd symmetry of w. Thus, we get that λ and $-\lambda$ are two such eigenvalues, which is a contradiction. Since $\lambda \in \mathbb{C}^+$ implies $-\bar{\lambda} \in \mathbb{C}^+$, we see that $\lambda = -\bar{\lambda}$, i.e., λ is purely imaginary. The proof of Theorem 1.5 is complete. □

References

[1] F. Atkinson and D. Jabon, Indefinite Sturm–Liouville problems. In H. Kaper, M.-K. Kwong, and A. Zettl (eds.), *Proceedings of the Focused Research Program on Spectral Theory and Boundary Value Problems.* Vol. I. Argonne National Laboratory, Mathematics and Computer Science Division, Argonne, IL, 1987, 31–45.

[2] J. Behrndt, Q. Katatbeh and C. Trunk, Non-real eigenvalues of singular indefinite Sturm–Liouville operators. *Proc. Amer. Math. Soc.* **137** (2009), 3797–3806. MR 2529889 Zbl 1182.47036

[3] J. Behrndt, F. Philipp and C. Trunk, Bounds on the non-real spectrum of differential operators with indefinite weights. *Math. Ann.* **357** (2013), 185–213. MR 3084346 Zbl 06210502

[4] J. Behrndt and C. Trunk, On the negative squares of indefinite Sturm–Liouville operators. *J. Diff. Equations* **238** (2007), 491–519. MR 2341434 Zbl 1123.47033

[5] P. Binding and M. Möller, Negativity indices for definite and indefinite Sturm–Liouville problems. *Math. Nachr.* **283** (2010), 180–192. MR 2604116 Zbl 1193.34054

[6] P. Binding and H. Volkmer, Eigencurves for two-parameter Sturm–Liouville equations. *SIAM Review* **38** (1996), 27–48. MR 1379040 Zbl 0869.34020

[7] B. Čurgus and H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weigth function. *J. Diff. Equations* **79** (1989), 31–61. MR 0997608 Zbl 0693.34020

[8] J. Fleckinger and A. B. Mingarelli, On the eigenfunctions of non-definite elliptic operators. In I. W. Knowles and R. T. Lewis (eds.), *Differential Equations.* North-Holland, Amsterdam, 1984, 229–228.

[9] O. Haupt, Über eine methode zum beweis von oszillations theoremen. *Math. Ann.* **76** (1915), 67–104.

[10] A. B. Mingarelli, Indefinite Sturm–Liouville problems. In W. N. Everitt and B. D. Sleeman (eds.), *Ordinary and partial differential equations.* Proceedings of the Seventh Conference held at the University of Dundee, Dundee, March 29–April 2, 1982. Lecture Notes in Mathematics 964. Springer Verlag, Berlin etc., 1982, 519-528. MR 0693136 Zbl 0488.00008 (collection)

[11] A. B. Mingarelli, A survey of the regular weighted Sturm–Liouville problem – The non-definite case. In S. T. Xiao and F. Q. Pu (eds.), *International workshop on applied differential equations.* Proceedings of the workshop held at Tsinghua University, Beijing, June 4–7, 1985. World Scientific, Singapore, 1986, 109–137. MR 0901329 Zbl 0624.34021

[12] R. G. D. Richardson, Theorems of oscillation for two linear differential equations of second order with two parameters. *Trans. Amer. Math. Soc.* **13** (1912), 22–34. MR 1500902 JFM 43.0400.03

[13] R. G. D. Richardson, Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order. *Amer. J. Math.* **40** (1918), 283–316. MR 1506360 JFM 46.0698.03

[14] L. Turyn, Sturm–Liouville problems with several parameters. *J. Differential Equations* **38** (1980), 239–259. MR 0597803 Zbl 0421.34023

[15] A. Zettl, *Sturm–Liouville Theory.* Mathematical Surveys and Monographs 121. American Mathematical Society, Providence, RI, 2005. MR 2170950 Zbl 1103.34001

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