Minimal partitions for anisotropic tori

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Abstract. We analyze spectral minimal k-partitions for the torus. In continuation with what we have obtained for thin annuli or thin strips on a cylinder (Neumann case), we get similar results for anisotropic tori.

1. Introduction

For $a \geq b > 0$, consider the Laplacian on the $2D$-torus:

$$T(a, b) \stackrel{\text{def}}{=} \mathbb{S}^1\left(\frac{a}{2\pi}\right) \times \mathbb{S}^1\left(\frac{b}{2\pi}\right).$$

Concretely, we can also consider

$$\mathcal{R}(a, b) = (0, a) \times (0, b), \tag{1.1}$$

and the Laplacian on $\mathcal{R}(a, b)$ with periodic boundary conditions but except for the pictures this is not the most convenient point of view. It is indeed better to think of the torus as a compact regular manifold.

We can, following [5], consider k-partitions $\mathcal{D}$ of the torus, i.e. families of disjoint open sets $(D_1, \ldots, D_k)$ of the torus and the sequence of partition energies $\mathfrak{L}_k(T(a, b))$ obtained by minimizing over $\mathcal{D}$ of the torus some energy defined by

$$\Lambda_k(\mathcal{D}) = \max_j \lambda(D_j), \tag{1.2}$$

where $\lambda(D_j)$ is the ground state energy of the Dirichlet Laplacian in D_j. We then define

$$\mathfrak{L}_k(T(a, b)) \stackrel{\text{def}}{=} \inf_{\mathcal{D}} \Lambda_k(\mathcal{D}), \tag{1.3}$$

where the infimum is over all the k-partitions of $T(a, b)$. A minimal k-partition is a partition whose energy is $\mathfrak{L}_k(T(a, b))$. As in the case of an open set in $\mathbb{R}^2$, minimal

k-partitions exist and are strong and regular (see Section 2). Without loss of generality, we consider the case $a = 1$. Note that for the torus, when $b < 1$, $\lambda_1 = 0$ and that $\lambda_2 = \lambda_3 = 4\pi^2$. Hence $\mathfrak{L}_3 > \lambda_3$ and using the results of [5] (extended to the case of the torus) (see Theorem 2.1 in Section 2) the associated minimal 3-partition cannot be nodal, i.e a partition obtained as the nodal domains of an eigenfunction. On the other hand for $k = 4$, we see that $\lambda_4 = 16\pi^2$ for $b < \frac{1}{2}$ and that any corresponding eigenfunction has four nodal domains. So the minimal 4-partition is nodal. Our aim in this paper is to describe what are the minimal k-partitions. Our main result is the following theorem.

Theorem 1.1. *There exists $b_k > 0$ such that, if $b < b_k$, then $\mathfrak{L}_k(T(1,b)) = k^2\pi^2$ and the corresponding minimal k-partition $\mathcal{D}_k = (D_1, \ldots, D_k)$ is represented in $\mathcal{R}(1, b)$ by*

$$D_i = ((i-1)/k, i/k) \times [0, b), \quad for\ i = 1, \ldots, k. \tag{1.4}$$

Moreover we can take $b_k = \frac{2}{k}$ for k even and $b_k = \frac{1}{k}$ for k odd.

Note that the boundaries of the D_i in $T(1, b)$ are just k circles (see Figure 1 where these circles are represented by vertical segments).
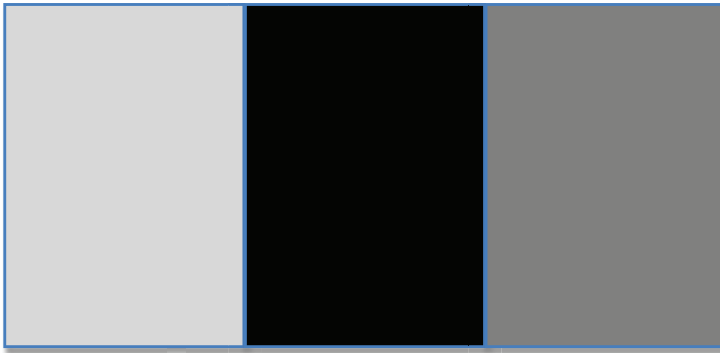

Figure 1. One candidate for the minimal 3-partition represented in $\mathcal{R}(1, b)$.

Remark 1.2. This result is a complement to what we have obtained for thin annuli or strips on a cylinder (in the case of the Neumann condition); see [4]. Its proof requires new ideas which hopefully can be used for other compact surfaces. We recall that the case of thin annuli with Dirichlet conditions is still open (k odd). For minimal k-partitions of the torus, we will at the end of Section 2 prove that the statement of the theorem holds for k even (the minimal partitions are nodal) and $b_k = \frac{2}{k}$ cannot be improved (see also Section 7 for further discussion). For k even and $\frac{2}{k} < b < \frac{2}{k-2}$, the k-th eigenfunction does not have k nodal domains. Hence it remains to give the proof of our theorem for k odd ($k \geq 3$).

We also recall that in the case $k = 3$, the problem was solved in [6] for the sphere $\mathbb{S}^2$ and is still open for the disk [6] and the square [1].

2. Reminder on the properties of minimal partitions

Let us first recall in more detail the properties of minimal k-partitions. The notion of minimal partition was first introduced for an open set Ω in $\mathbb{R}^2$ in [5] (see references therein). We just present the corresponding definitions for the torus (or more generally on a compact Riemannian manifold). We recall that a k-partition on the torus is simply a family $\mathcal{D}$ of k-disjoint open sets $(D_i)_{i=1,\dots,k}$. Such a partition is called *strong* if $\cup \overline{D_i} = T(1, b)$ and $\text{Int}(\overline{D_i}) = D_i$ for any i. Attached to a strong partition, we associate a closed set in $T(1, b)$, which is called the *boundary set* of the partition:

$$N(\mathcal{D}) = \cup_i \partial D_i. \tag{2.1}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition). We have recalled in the introduction the notion of minimal k-partitions. As in the case of an open set in $\mathbb{R}^2$, minimal k-partitions exist and are strong and *regular* in the following sense. We call a partition $\mathcal{D}$ regular if its associated boundary set $N(\mathcal{D})$, has the following properties.

(i) Except for finitely many distinct $x_i \in N$ in the neighborhood of which N is the union of $\nu_i = \nu(x_i)$ smooth curves ($\nu_i \geq 3$) with one end at x_i, N is locally diffeomorphic to a regular curve.

(ii) N has the *equal angle meeting property*. The x_i are called the critical points and define the set $X(N)$. By *equal angle meeting property*, we mean that the half curves meet with equal angle at each critical point of N.

In the case of an open set we have also points y_j at the boundary and we call this set $Y(N)$.

Moreover, the minimal k-partitions are bipartite, i.e. can be colored by two colors (neighboring domains have different colors), if and only if they are nodal (i.e. corresponding to the nodal domains of an eigenfunction of the Laplace-Beltrami operator).

Another important statement established in [5] is the following theorem.

Theorem 2.1. *A k-partition consisting of the k nodal domains of an eigenfunction corresponding to the k-th eigenvalue λ_k of the Laplacian is a minimal k-partition.*

In general one could just say that by the well known Courant nodal theorem the number of nodal domains of an eigenfunction u_k associated with λ_k is at most k. The eigenpair (u_k, λ_k) is called *Courant sharp* if the number of nodal domains is exactly k. Theorem 2.1 is moreover optimal as has been proven in [5].

Theorem 2.2. *A nodal minimal k-partition corresponds necessarily to a Courant sharp pair.*

First application: proof of Theorem 1.1 in the even case. For the torus $T(c,d)$, the eigenvalues are given by $4\pi^2(\frac{m^2}{c^2} + \frac{n^2}{d^2})$ $((m,n) \in \mathbb{N}^2$ where $\mathbb{N}$ denotes the set of the non-negative integers) with a corresponding basis given by

- $(x,y) \mapsto \cos(2\pi m\frac{x}{c}) \cos(2\pi n\frac{y}{d})$,
- $(x,y) \mapsto \cos(2\pi m\frac{x}{c}) \sin(2\pi n\frac{y}{d})$,
- $(x,y) \mapsto \sin(2\pi m\frac{x}{c}) \cos(2\pi n\frac{y}{d})$,
- $(x,y) \mapsto \sin(2\pi m\frac{x}{c}) \sin(2\pi n\frac{y}{d})$

(with suitable changes when m or n vanishes). For example, for $n = 0$, we get $(x,y) \mapsto 1$ for $m = 0$ and $(x,y) \mapsto \cos(2\pi m\frac{x}{c})$ and $(x,y) \mapsto \sin(2\pi m\frac{x}{c})$ for $m > 0$. These eigenfunctions have $(2m)$ nodal domains on the torus. When k is even and $k < \frac{2c}{d}$, we get the existence of an k-th eigenfunction with exactly k nodal domains (corresponding to $m = \frac{k}{2}$ and $n = 0$). What will be important in our problem is that Theorem 2.1 implies that for k even and $c > d > 0$ the minimal k-partitions of $T(c,d)$) are nodal for the case that $k < 2c/d$. The corresponding energy is $\frac{\pi^2 k^2}{c^2}$. Hence we have completed the proof of Theorem 1.1 for the even case.

We also observe that for k odd $(k > 1)$ the minimal k-partitions cannot be nodal.

We will prove that, when $\frac{c}{d}$ is small enough, the minimal k-partitions can be lifted into a Courant sharp $(2k)$-partition on the covering $T(2c, 2d)$. The k-partition appearing in Theorem 1.1 corresponds actually to a nodal partition on this covering and this implies the result. The existence of this lifting will be proved in Section 5.

3. Necessary conditions

The computation of the energy of the k-partition (1.4) leads immediately to the following upper bound for $\mathfrak{L}_k$.

Proposition 3.1. *We have*

$$k^2\pi^2 \min(1, b^{-2}) \geq \mathfrak{L}_k(T(1,b)). \tag{3.1}$$

Using this upper-bound we can give necessary conditions on k-partitions to be minimal.

Proposition 3.2. *If $b < \frac{1}{k}$, there is no minimal k-partition $\mathcal{D} = (D_1, \ldots, D_k)$ of the torus with one D_i homeomorphic to a disk.*

The proof is by contradiction. Let $\mathcal{D} = (D_1, \ldots, D_k)$ be a minimal k-partition such that, say D_1 is homeomorphic to a disk. Then, the pullback $\hat{D}_1$ of D_1 in the universal covering $\mathbb{R}^2$ is a union of bounded components $\hat{D}_1^{k,\ell}$ (with $(k, \ell) \in \mathbb{Z}^2$) such that $\hat{D}_1^{k,\ell} + (m, nb) = \hat{D}_1^{k+m,\ell+n}$. Moreover $\hat{D}_1^{0,0}$ has same area as D_1 and $\lambda(D_1) = \lambda(\hat{D}_1^{0,0})$.

Looking at a lower bound for $\lambda(\hat{D}_1^{0,0})$, one could first think of using Faber–Krahn's inequality but it is better to come back to the first step of one proof of the Faber–Krahn inequality which is based on the Steiner symmetrization (see for example the book [7], Section 2.2, or the expository talk [9]).

We now observe that each vertical slice has a total length less than b. We now apply the Steiner symmetrization with respect to the horizontal line $y = \frac{b}{2}$. It is immediate to see that the image $S(\hat{D}_1^{0,0})$ of $\hat{D}_1^{0,0}$ is contained in a rectangle $\hat{R}_b$ in the form $(-\ell_b, \ell_b) \times (0, b)$ for some $\ell_b > 0$. Now it is well known that in this symmetrization we have

$$\lambda(\hat{D}_1^{0,0}) \geq \lambda(S(\hat{D}_1^{0,0})),$$

and by monotonicity

$$\lambda(S(\hat{D}_1^{0,0})) \geq \lambda(\hat{R}_b) = \pi^2(b^{-2} + \ell_b^{-2}) > \pi^2 b^{-2}.$$

This leads to

$$\lambda(D_1) > \pi^2 b^{-2}, \tag{3.2}$$

hence, using (3.1), to

$$b > \frac{1}{k}. \tag{3.3}$$

This gives the contradiction.

4. Around Euler's formula

4.1. Standard Euler's formula. In the case of an open set Ω in $\mathbb{R}^2$, observing that the Euler characteristic of Ω is 2, we have for a regular minimal k-partition $\mathcal{D}$

$$k = \mathbf{b_1} - \mathbf{b_0} + 1 + \sum_i \left(\frac{\nu(x_i)}{2} - 1\right) + \frac{1}{2}\sum_j \rho(y_j). \tag{4.1}$$

where $\mathbf{b_0}$ is the number of components of $\partial\Omega$, $\mathbf{b_1}$ is the number of components of $\partial\Omega \cup N$, $\nu(x_i)$ and $\rho(y_j)$ the numbers of arcs associated with the singular points x_i of the boundary set $N = N(\mathcal{D})$ in Ω, respectively with the points y_j of the boundary set contained in $\partial\Omega$. We denote by $X(N)$ the set of the x_i's and by $Y(N)$ the set of the y_j's.

4.2. Euler's formula on the torus and applications. In the case of a flat compact surface M without boundary, it is easier to formulate Euler's formula by using the Euler's characteristics $\chi(M)$ of M and of the elements of the partition $\mathcal{D} = (D_1, \ldots, D_k)$. The formula reads

$$\sum_\ell \chi(D_\ell) = \chi(M) + \sum_i \left(\frac{\nu(x_i)}{2} - 1\right), \qquad (4.2)$$

and is a direct consequence[1] of the Gauss–Bonnet formula applied in each D_i (see for example [8]).

   We recall that for the torus we have $\chi(T(a,b)) = 0$; for the disk B we have $\chi(B) = 1$; for the annulus A we have $\chi(A) = 0$; and for the sphere $\mathbb{S}^2$ we have $\chi(\mathbb{S}^2) = 2$. Hence, in the case of the torus, (4.2) becomes

$$\sum_{\ell=1}^{k} \chi(D_\ell) = \sum_i \left(\frac{\nu(x_i)}{2} - 1\right). \qquad (4.3)$$

Proposition 4.1. *A minimal partition $\mathcal{D} = (D_1, \ldots, D_k)$ for which no D_ℓ is homeomorphic to the disk satisfies $X(N(\mathcal{D})) = \emptyset$.*

Proof. The assumption implies that $\chi(D_\ell) \leq 0$, for $\ell = 1, \ldots, k$. Then we immediately get from (4.3) that $\chi(D_\ell) = 0$ and that $X(N) = \emptyset$. $\qquad\square$

5. Lifting argument

Proposition 5.1. *Suppose $\mathcal{D} = (D_1, \ldots, D_k)$ is a minimal k-partition on the torus $T(1,b)$ for which all the D_i are not homeomorphic to the disk and $X(N(\mathcal{D})) = \emptyset$. Then $\mathcal{D}$ can be lifted to a bipartite $(2k)$- partition of $T(2, 2b)$.*

   The initial guess was that a double covering will suffice but this is not always the case. One can construct (see Figure 2) a 3-partition of the torus without critical point, for which it is necessary to construct a covering of order 4, $T(2, 2b)$ of the torus (doubling in each direction) in order to get a bipartite 6-partition (see Figure 3).

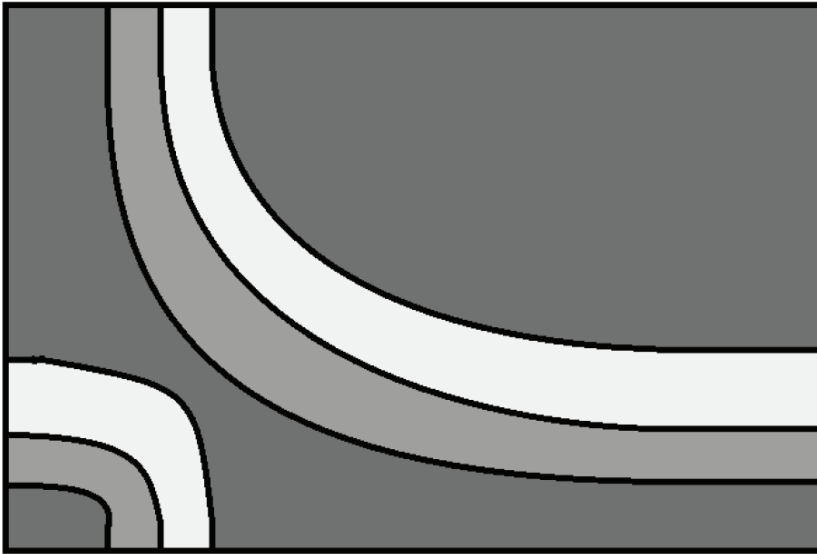[1]Thanks to P. Bérard for giving us the reference.

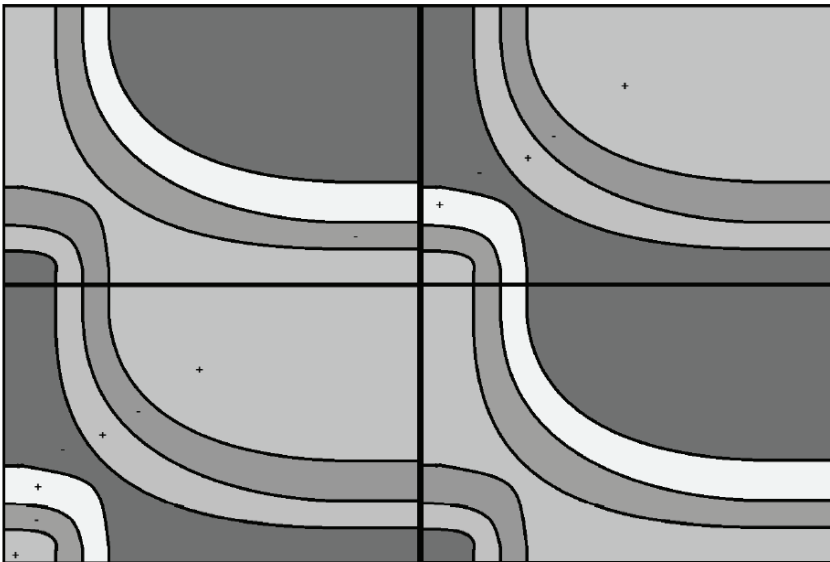Figure 2. A 3-partition of the torus without critical point.

Figure 3. The lifted 3-partition on the four-fold covering of the torus.

Proof of Proposition 5.1. One can classify all the possible topological types of these partitions. The k open sets of the partition have the same topological type. Each open set can be deformed by a retraction onto a simple closed line without self-intersection. Hence the classification corresponds to the classification of closed lines on the torus without self-intersection that are not homotopic to a point (the so-called torus knots). They correspond (see [2], p. 47, Example 1.24) to lines generically denoted by $\ell_{p,q}$ turning p times around one horizontal circle and q times around the vertical one, with p and q mutually prime (except if $q = 0$, $p = 1$ or $p = 1$, $q = 0$). Figure 2 corresponds to $p = 1$, $q = 1$. The candidate for the minimal 3-partition when b is small corresponds to $p = 1$, $q = 0$. Another example is given in the top picture in Figure 4, which represents a closed line on the torus with $p = 3$ and $q = 2$.

We go to a suitable double covering so that either p or q is multiplied by 2; so the greatest common divisor $D(p, q) = 2$. There are two cases:

- pq odd or

- pq even (with p or q odd).

In the first case we choose $T(2, 2b)$ and in the second case the minimal choice is $T(1, 2b)$ or $T(2, b)$ but $T(2, 2b)$ is also suitable, the important point being that $D(2p, 2q) = 2$. On the covering $T(2, 2b)$, the pull-back of our closed line $\ell_{p,q}$ in $T(1, b)$ is the union of two distinct closed lines in $T(2, 2b)$. Coming back to the k-partition, the lifting to $T(2, 2b)$ leads to a $(2k)$-partition. This ends the proof of the proposition. $\qquad\square$

Remark 5.2. When p and q are not mutually prime, our constructions lead, as explained in [2] to $D(p, q)$ connected closed lines, where $D(p, q)$ is the greatest common divisor of p and q. The bottom picture in Figure 4 corresponds to the case $p = 4$ and $q = 2$.

To understand the point, take the closure of $\mathcal{R}(p, q)$ (see (1.1)) and consider the intersection of the lines of equation $y = -x + c$ ($c \in \mathbb{Z}$) with $\overline{\mathcal{R}(p, q)}$. If we project on the corresponding torus and look at the number of connected components obtained on the torus, then we observe that this number is $D(p, q)$ (see the bottom picture in Figure 4 which has two components). When $D(p, q) = 1$, we get a single closed line on the torus. After a suitable dilation, we can then come back to $T(1, b)$.

When $D(p, q) \neq 1$, it is not possible to find a continuous closed line on the torus without self-intersection with winding pair (p, q).

6. End of the proof of Theorem 1.1

We deduce from Propositions 3.2, 4.1, and 5.1 that, if $b < \frac{1}{k}$ (k odd), then any minimal k-partition can be lifted into a $(2k)$-partition of $T(2, 2b)$ with the same
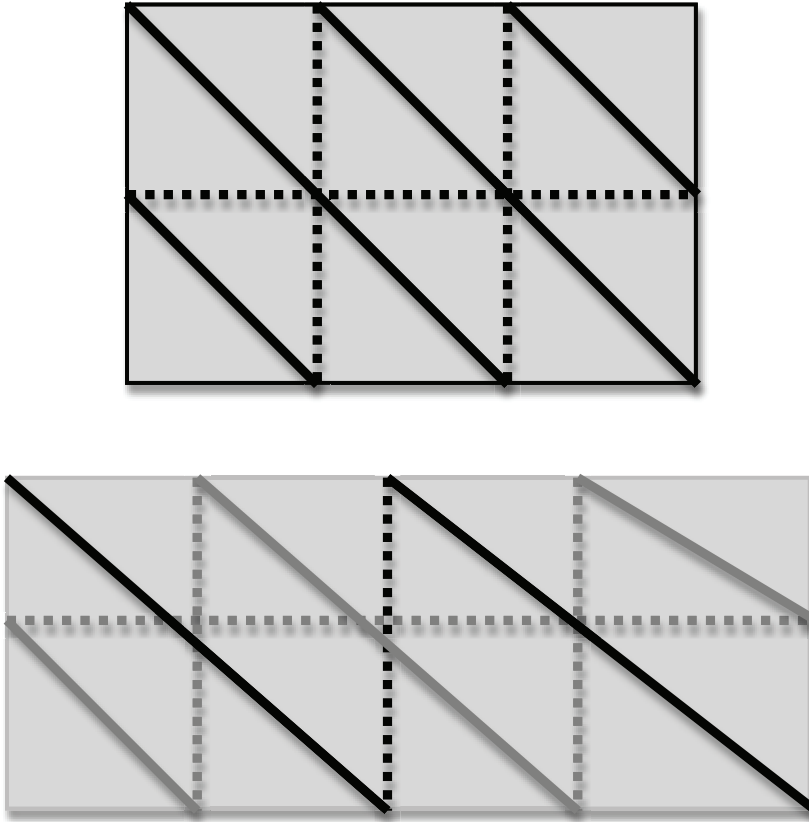
Figure 4. (p=3, q=2) and (p=4, q=2).

energy $\mathfrak{L}_k(T(1,b))$. We need to look at the spectrum of the Laplacian on the 4-covering $T(2,2b)$ and to determine under which condition the $(2k)$-th eigenvalue is Courant sharp. The eigenvalues are given by $\pi^2(\ell^2 + m^2/b^2)$. If $b < \frac{1}{k}$, the $(2k) - th$ eigenvalue corresponds to $m = 0$ and $\ell = k$, and we are in a Courant sharp situation. Theorem 2.1 implies that

$$\pi^2 k^2 = \mathfrak{L}_k(T(2,2b)) \leq \mathfrak{L}_k(T(1,b)).$$

Having in mind (3.1), this ends the proof of the theorem in the odd case.

Remark 6.1. The ideas in the proof might lead to results concerning minimal partitions for other "thin" compact surfaces.

7. More on the Courant sharpness of eigenfunctions
for the case that b^2 is irrational

We recall (see after Theorem 2.2) that on $T(1, b)$, the associated eigenvalues are given by

$$\lambda_{m,n}(1, b) = 4\pi^2\left(m^2 + \frac{n^2}{b^2}\right). \qquad (7.1)$$

If $m, n > 0$ and if b^2 is irrational, then we have multiplicity 4. Following some ideas which we presented already in [5] for rectangles and the disk we have the following result.

Theorem 7.1. *Suppose b^2 is irrational. If $\min(m, n) \geq 1$, then there is no Courant sharp pair $(u, \lambda_{m,n})$.*

The proof is based on the following proposition.

Proposition 7.2. *For $m, n > 0$ any eigenfunction u corresponding to $\lambda_{m,n}$ has at most $4mn$ nodal domains. Moreover the only eigenfunctions with exactly $4mn$ nodal domains have the form $\cos(2\pi mx + \theta_1)\cos(2\pi n\frac{y}{b} + \theta_2)$ for some constants θ_1 and θ_2. The other eigenfunctions have $2D(m, n)$ nodal domains, where $D(m, n)$ is the greatest common divisor of m and n.*

Proof of the proposition. We first observe that a general eigenfunction associated with $\lambda_{m,n}$ can be written in the form

$$u = \mu\left(\cos 2\pi mx \cos(2\pi n\frac{y}{b} + \theta_1) + \lambda \sin 2\pi mx \cos\left(2\pi n\frac{y}{b} + \theta_2\right)\right), \quad (7.2)$$

with $\mu \neq 0$.

Note that it is only here that we use the fact that b^2 is irrational. By rotation, we can reduce to the case when $\theta_2 = 0$ and we write $\theta = \theta_1$.

Then after dilation and rotation, the proof is based on the following lemma. $\square$

Lemma 7.3. *Except when $\lambda = 0$ or $\theta \equiv \frac{\pi}{2} mod(\pi)$, the nodal set of the function*

$$u_{\lambda,\theta} \stackrel{\text{def}}{=} \cos 2\pi x \cos(2\pi y + \theta) + \lambda \sin 2\pi x \sin 2\pi y$$

has no critical zero.

Let us look at the critical zeroes of this functions. They should satisfy

$$\cos 2\pi x \cos(2\pi y + \theta) + \lambda \sin 2\pi x \sin 2\pi y = 0,$$
$$-\sin 2\pi x \cos(2\pi y + \theta) + \lambda \cos 2\pi x \sin 2\pi y = 0,$$
$$-\cos 2\pi x \sin(2\pi y + \theta) + \lambda \sin 2\pi x \cos 2\pi y = 0.$$

We assume $\lambda \neq 0$. Suppose that this system has a solution. The two first equations imply $\cos(2\pi y + \theta) = 0$ and $\sin 2\pi y = 0$. This implies $\cos \theta = 0$.

Hence, when $\cos \theta \neq 0$, our function $u_{\lambda,\theta}$ has no critical zero.

Lemma 7.4. *For $\lambda \neq 0$, $\theta_2 = 0$, and $\cos \theta \neq 0$, the nodal partition of the function u of (7.2) has $2D(m,n)$ components.*

In each connected component of the set $\mathcal{A} \overset{\text{def}}{=} \{(\lambda, \theta) \mid \lambda \neq 0, \cos \theta \neq 0\}$ in $\mathbb{R}^2$ the number of nodal domains is constant. Hence it is enough to determine this number for one specific pair (λ, θ) in each component of $\mathcal{A}$. It is enough to consider $\lambda = \pm 1$ and $\theta \equiv 0 (\mathrm{mod}\, \pi)$, where the computation of the number of nodal domains is immediate (see Remark 5.2) and equal to $2D(m,n)$.

Note that when $\cos \theta = 0$, we get a product

$$u_{\lambda,\theta} \overset{\text{def}}{=} \sin 2\pi n y (\lambda \sin 2\pi m x \pm \cos 2\pi m x)$$

which has $4mn$ nodal domains.

Remark 7.5. This is not clear for the case that b^2 is rational, since then higher multiplicities could occur and we do not know how to exclude the possibility of a higher number of nodal domains in higher dimensional eigenspaces.

Proof of Theorem 7.1. We give two alternative proofs (the second is geometric and inspired by arguments developed in [5]).

1. If $\inf(n,m) \geq 1$, then $\lambda_{m,n} = \lambda_{k(n,m)}$ with $k(m,n) \geq 4mn + 2m + 2n - 2$. This is obtained by just adding the multiplicities of the eigenvalues $\lambda_{m',n'}$ with $m' \leq m$, $n' \leq n$, $(m',n') \neq (m,n)$. On the other hand, Proposition 7.2 says that any eigenfunction has at most $4mn$ domains (if $\inf(m,n) \geq 1$). Hence it cannot be Courant sharp.

2. According to Proposition 7.2 it is enough to consider eigenfunctions in the form $\sin(2\pi m x + \theta_1) \sin(2\pi n \frac{y}{b} + \theta_2)$ and to show that it cannot correspond to a Courant sharp case. Consider for simplicity the situation that $m = 1 = n$ and $\theta_2 = \theta_1 = 0$. Then (up to a rotation) the eigenfunction is given by $u_{1,1} = \sin 2\pi x \sin(2\pi y/b)$. The zeros are given by the zeros of the sines. In particular we can for instance consider the zero given by $y = b/2$ and $y = 0$. Consider the $P_1 = \{(x,y) \in T(1,b) \mid 0 < y < b/2\}$ and $P_2 = \{(x,y) \in T(1,b) \mid b/2 < y < b\}$ and consider $N_i = \{(x,y) \in \bar{P}_i \mid (x,y) \in N(u_{1,1}(x,y))\}$, where $N(u)$ denotes the zeroset of u. Suppose we have a minimal partition corresponding to this eigenfunction. Then we can rotate for instance N_1, so that the zeros $x = 0$, $x = 1/2$ are shifted but keep N_2 fixed. The associated partition will still have the same energy. But this cannot correspond to a minimal partition since the equal angle property does not hold; see also [5]. This argument extends to arbitrary $m, n > 0$ and (θ_1, θ_2). $\qquad \square$

Remark 7.6. There exists $0 < b_0 < 1$ sufficiently close to 1, so that, for each irrational b^2 satisfying $b_0 < b < 1$, only the first and the second eigenvalue together with their eigenfunctions are Courant sharp pairs. This follows by counting. Remember $b < 1$. The eigenvalues all have multiplicity 2 or 4. Suppose $n = 0$ then $u_{m,0}$ has $2m$ nodal domains. So Courant sharpness can occur only for $\lambda_{m,0} = \lambda_{2m}$. This will not be the case if $|1 - b|$ is small since then $\lambda_{0,n}$ will be eventually be below $\lambda_{m,0}$ hence $\lambda_{m,0} > \lambda_{2m}$. The case $m, n \geq 1$ has been treated above.

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