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Ground state energy of trimmed discrete Schrödinger operators and localization for trimmed Anderson models

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Abstract. We consider discrete Schrödinger operators of the form $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian and V is a bounded potential. Given $\Gamma \subset \mathbb{Z}^d$, the Γ-trimming of H is the restriction of H to $\ell^2(\mathbb{Z}^d \setminus \Gamma)$, denoted by H_Γ. We investigate the dependence of the ground state energy $E_\Gamma(H) = \inf \sigma(H_\Gamma)$ on Γ. We show that for relatively dense proper subsets Γ of $\mathbb{Z}^d$ we always have $E_\Gamma(H) > E_\emptyset(H)$. We use this lifting of the ground state energy to establish Wegner estimates and localization at the bottom of the spectrum for Γ-trimmed Anderson models, i.e., Anderson models with the random potential supported by the set Γ.

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1. Introduction

We consider discrete Schrödinger operators of the form $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian, defined by

$$(-\Delta\varphi)(x) = \sum_{\substack{y \in \mathbb{Z}^d, \\ \|x-y\|=1}} (\varphi(x) - \varphi(y)) = 2d\varphi(x) - \sum_{\substack{y \in \mathbb{Z}^d, \\ \|x-y\|=1}} \varphi(y),$$

and V is a bounded potential. Given $\Gamma \subsetneq \mathbb{Z}^d$, the Γ-trimming of H is the restriction H_Γ of $\chi_{\Gamma^c} H \chi_{\Gamma^c}$ to $\ell^2(\Gamma^c)$, where χ_A denotes the characteristic function of the set A and $A^c = \mathbb{Z}^d \setminus A$ for $A \subset \mathbb{Z}^d$. We focus our attention on $E_\Gamma(H) = \inf \sigma(H_\Gamma)$, the ground state energy (or bottom of the spectrum) of the trimmed discrete Schrödinger operator H_Γ. (Note that with this notation $H = H_\emptyset$ and $E_\emptyset(H) = \inf \sigma(H)$.) Since

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$E_\Gamma(H)$ is a nondecreasing function of the set Γ, trimming lifts the bottom of the spectrum, that is, $E_\Gamma(H) \geq E_\emptyset(H)$.

We show that for relatively dense proper subsets Γ of $\mathbb{Z}^d$ we always have strict lifting of the bottom of the spectrum, i.e., $E_\Gamma(H) > E_\emptyset(H)$. We use this lifting of the ground state energy to establish Wegner estimates and localization at the bottom of the spectrum for Γ-trimmed Anderson models, i.e., Anderson models with the random potential supported by the set Γ.

1.1. The ground state energy of trimmed discrete Schrödinger operators. Our motivation comes from continuous Schrödinger operators $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, where Δ is the Laplacian operator and V is a bounded potential. Let us consider first the case $H = -\Delta$ and Γ^c an open subset of $\mathbb{R}^d$, and let Δ_Γ be the Laplacian on Γ^c with Dirichlet boundary condition. When $\overline{\Gamma^c}$ is compact, the ground state energy $E_\Gamma(-\Delta)$ of $-\Delta_\Gamma$ is the first eigenvalue λ_Γ of $-\Delta_\Gamma$. The problem of obtaining a lower bound for the first eigenvalue of the Dirichlet Laplacian on a compact Riemannian manifold has been intensively studied in Geometric Analysis, and it is given by Cheeger's inequality [7]: $\lambda_\Gamma \geq \frac{\beta(\Gamma)^2}{4}$, where $\beta(\Gamma)$ is Cheeger's isoperimetric constant for the set Γ^c. It is known that $\beta(\Gamma) > 0$ if $\overline{\Gamma^c}$ is compact, but for noncompact sets $\overline{\Gamma^c}$ the Cheeger isoperimetric constant may be zero.

Cheeger's inequality has been extended to the discrete case; see [12] and [28]. In this context, $\beta(\Gamma) = \inf_{S \subset \Gamma^c : 1 \leq |S| < \infty} \frac{|\partial S|}{|S|}$, where ∂S denotes the boundary of the set S and $|S|$ its cardinality. (See Section 2.3 for notation and details. An equivalent definition for $\beta(\Gamma)$ is given in (2.18)). Clearly $\beta(\Gamma) > 0$ if Γ^c is a finite set. But it is not difficult to see that $\beta(\Gamma) = 0$ if we can find a sequence of boxes in $\mathbb{Z}^d$, $\Lambda_{K_n}(x_n)$ ($\Lambda_K(x)$ is the box of side $K \in \mathbb{N}$ centered at $x \in \mathbb{Z}^d$), such that $\lim_{n\to\infty} \frac{|\Gamma \cap \Lambda_{K_n}(x_n)|}{|\Gamma^c \cap \Lambda_{K_n}(x_n)|} = 0$. This lead us to consider relatively dense subsets Γ of $\mathbb{Z}^d$, for which we show $\beta(\Gamma) > 0$.

The addition of a potential V breaks down Cheeger's argument. Indeed, in general flat functions are no longer good approximants for the low-lying eigenvectors of $H = -\Delta + V$. For example, let $H_\lambda = -\Delta + \lambda V$, where V is a periodic potential whose average over a fundamental cell is equal to zero. Then $E_\emptyset(H_\lambda) < 0$ for all $\lambda > 0$, see Theorem 1 in [22] (the result there is proven for the continuum, but it is easy to see that holds in the discrete case as well), but it can be shown that $\beta_\lambda(\emptyset) = 0$ for λ small, where $\beta_\lambda(\emptyset)$ is the Cheeger constant for H_λ. Another striking counterexample can be constructed by taking $H_\lambda = -\Delta + \lambda V$ with $V = -\chi_{\{0\}}$, a negative rank one perturbation to $-\Delta$, and $\lambda > 0$. It is well known that in this case $E_\emptyset(H_\lambda) < 0$ for all $\lambda > 0$, while it is easy to see that $\beta_\lambda(\emptyset) = 0$ for $\lambda \leq 2d$.

For continuous Schrödinger operators the bound $E_\Gamma(H) > E_\emptyset(H)$ can be established in the presence of an arbitrary bounded potential using the unique continuation principle; see [24] and [32]. Unfortunately, discrete Schrödinger operators do not satisfy a unique continuation principle. It turns out, however, that *the ground state* of a discrete Schrödinger operator H enjoys a similar property, which suffices to establish the desired result.

It is intuitively clear that the Schrödinger operator H_Γ is, in a suitable sense, the limit of the Schrödinger operators $H_\Gamma(t) = H + t\chi_\Gamma$ on $\ell^2(\mathbb{Z}^d)$ as $t \to \infty$. This is the motivation behind Theorem 1.1, where we obtain a lower bound for $E_\Gamma(H) - E_\emptyset(H)$ as the limit of lower bounds for $E_\Gamma(H,t) - E_\emptyset(H)$, where $E_\Gamma(H,t) = E_\emptyset(H(t))$. Note that $E_\Gamma(H,t)$ is nondecreasing in t, so $E_\Gamma(H,\infty) := \lim_{t\to\infty} E_\Gamma(H,t) = \sup_{t\geq 0} E_\Gamma(H,t)$, and it follows from the min-max principle that $E_\Gamma(H,t) \leq E_\Gamma(H)$ for all $t \geq 0$, so

$$E_\emptyset(H) \leq E_\Gamma(H,t) \leq E_\Gamma(H,\infty) \leq E_\Gamma(H). \tag{1.1}$$

Before stating our results, we introduce some additional notation. A bounded potential V is given by multiplication by a a function $V\colon \mathbb{Z}^d \to \mathbb{R}$ with $V_\infty = \|V\|_\infty < \infty$. We set $V_+ = \max\{V,0\}$ and $V_- = -\min\{V,0\}$; note that $V = V_+ - V_-$, $V_\pm \geq 0$, and $V_+V_- = 0$. We define the spread of the bounded potential V by

$$\operatorname{spr}(V) = \sup_{x\in\mathbb{Z}^d} V(x) - \inf_{x\in\mathbb{Z}^d} V(x) \in [0,\infty).$$

We also introduce the notation

$$Y_{d,V} = 2d + 1 + \operatorname{spr}(V),$$

$$\delta_\Gamma(H) = E_\Gamma(H) - E_\emptyset(H),$$

$$\delta_\Gamma(H,t) = E_\Gamma(H,t) - E_\emptyset(H).$$

Theorem 1.1. *Let $H = -\Delta + V$ be a Schrödinger operator on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential, and let $\Gamma \subsetneq \mathbb{Z}^d$. Then*

$$E_\Gamma(H,\infty) = E_\Gamma(H).$$

Moreover, we have

$$2d + \operatorname{spr}(V) \geq \delta_\Gamma(H,t) \geq \frac{t}{t + 6d + 2\operatorname{spr}(V)}\,\delta_\Gamma(H), \quad t \geq 0. \tag{1.2}$$

It follows, using (1.1), that $E_\Gamma(H) > E_\emptyset(H)$ if and only if $E_\Gamma(H,t) > E_\emptyset(H)$ for all $t > 0$.

Theorem 1.1 is proven in Section 2.1. Note that once we have a lower bound for $\delta_\Gamma(H)$, as in Theorem 1.3, (1.2) (we may use the sharper (2.4)) provides lower bounds for $\delta_\Gamma(H,t)$ for all $t > 0$.

Given $x \in \mathbb{Z}^d$ and $L > 0$, we set

$$\Lambda_L(x) = \{y \in \mathbb{Z}^d : \|y - x\|_\infty \leq \tfrac{L}{2}\}$$

and

$$\Lambda_L^{(0)}(x) = \{y \in \mathbb{Z}^d : \|y - x\|_\infty < \tfrac{L}{2}\};$$

note that $\Lambda_L(x) = \Lambda_L^{(0)}(x) \iff L \notin 2\mathbb{N}$.

Given $K \in \mathbb{N}$ we have

$$|\Lambda_K(x)| = K_*^d,$$

where

$$K_* = \begin{cases} K & \text{if } K \text{ is odd,} \\ K+1 & \text{if } K \text{ is even.} \end{cases}$$

Moreover $\Lambda_K(x) = \Lambda_K^{(0)}(x)$ if and only if K is odd.

Definition 1.2. A set $\Gamma \subset \mathbb{Z}^d$ is (K, Q)-*relatively dense*, where $K, Q \in \mathbb{N}$, if

$$|\Gamma \cap \Lambda_K^{(0)}(\zeta)| \geq Q, \quad \zeta \in K\mathbb{Z}^d.$$

By a relatively dense subset $\Gamma \subset \mathbb{Z}^d$ we will always mean a (K, Q)-relatively dense set Γ for some appropriate $K, Q \in \mathbb{N}$. Note that we must have $Q \leq |\Lambda_K^{(0)}| \leq K^d$, and that $K \geq 2$ unless $\Gamma = \mathbb{Z}^d$.

Theorem 1.3. *Let $\Gamma \subsetneq \mathbb{Z}^d$ be (K, Q)-relatively dense, and let $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential. Then*

$$\delta_\Gamma(H, t) \geq \frac{Q}{(2dK-1)Y_{d,V}^{2dK-1}}\left(1 - \left(\frac{Y_{d,V}}{Y_{d,V}+t}\right)^{2dK-1}\right), \quad t \geq 0. \tag{1.3}$$

As a consequence, we get

$$\delta_\Gamma(H) \geq \frac{Q}{(2dK-1)Y_{d,V}^{2dK-1}} > 0. \tag{1.4}$$

In the special case $H = -\Delta$ we can improve the previous bound to

$$\delta_\Gamma(-\Delta) = E_\Gamma(-\Delta) \geq \frac{1}{4dK_*^{2d}}. \tag{1.5}$$

We prove (1.3) from a 'quantitative unique continuation principle for ground states' given in Lemma 2.2. The lower bound given in (1.4) holds for arbitrary bounded potential V; note that it depends on V only through $\mathrm{spr}(V)$.

The special case (1.5) follows from a Cheeger's inequality. We remark that $E_\Gamma(-\Delta)$ can also be estimated by an argument of Bourgain and Kenig Section 4 in [5] (see also [21], Remark 4.4). They treated continuum models, but Rojas-Molina, see Section 1.2.5 in [30] and Lemma 2.1 in [31], noted that the argument applies also to the discrete case. This argument yields the bound $E_\Gamma(-\Delta) \geq \frac{C}{K^{2d+2}}$.

Theorem 1.3 has a continuum counterpart; in particular we can use Cheeger's inequality to obtain the continuum version of (1.5). We did not include it in this paper because the continuous version of the general estimate (1.3) is only marginally better than the estimate in [24], Lemma 4.2, and the continuous version of (1.5) is similarly only marginally better that what we get with the Bourgain–Kenig argument.

Theorem 1.3 follows from Theorems 2.1 and 2.3 in Section 2.2.

1.2. Wegner estimates and localization for trimmed Anderson models. If $\zeta \in \mathbb{Z}^d$, we will use the notation χ_ζ for $\chi_{\{\zeta\}}$ as a multiplication operator in $\ell^2(\mathbb{Z}^d)$, but we will write δ_ζ instead when considering $\chi_{\{\zeta\}}$ as a function in $\ell^2(\mathbb{Z}^d)$.

1.2.1. Trimmed Anderson models.

Definition 1.4. Let $\Gamma \subset \mathbb{Z}^d$ be (K, Q)-relatively dense. A Γ-*trimmed Anderson model* is a discrete random Schrödinger operator on on $\ell^2(\mathbb{Z}^d)$ of the form

$$H_{\boldsymbol{\omega},\lambda} := H_0 + \lambda V_{\boldsymbol{\omega}}, \tag{1.6}$$

where

(i) $H_0 = -\Delta + V^{(0)}$, with $V^{(0)}$ a bounded (background) potential;

(ii) $V_{\boldsymbol{\omega}}$ is the random potential given by

$$V_{\boldsymbol{\omega}} := \sum_{\zeta \in \Gamma} \omega_\zeta \chi_\zeta, \tag{1.7}$$

where $\boldsymbol{\omega} = \{\omega_\zeta\}_{\zeta \in \Gamma}$ is a family of independent random variables whose probability distributions $\{\mu_\zeta\}_{\zeta \in \Gamma}$ are non-degenerate with

$$\operatorname{supp} \mu_\zeta \subset [0, M], \quad \zeta \in \mathbb{Z}^d; \tag{1.8}$$

(iii) $\lambda > 0$ is the disorder parameter.

If $\Gamma = \mathbb{Z}^d$, $V^{(0)} = 0$, and $\mu_\zeta = \mu$ for all $\zeta \in \mathbb{Z}^d$, then $H_{\boldsymbol{\omega},\lambda}$ is the standard Anderson model. This model was introduced by Anderson [4] to study the effect of disorder on electronic states within the suitable energy range. The main phenomenon is localization, which manifests itself as spectral localization (the spectral measure of $H_{\boldsymbol{\omega},\lambda}$ is almost surely pure point with exponential decay of eigenfunctions) and as dynamical localization (non-spreading of wave packets).

Trimmed Anderson models are the discrete analogues of the crooked Anderson Hamiltonians introduced in [24], Definition 1.2. (By a trimmed Anderson model we will always mean a Γ-trimmed Anderson model for some relatively dense subset $\Gamma \subset \mathbb{Z}^d$.)

The standard Anderson model with sufficiently regular single site probability distribution μ was intensively studied during the last two decades; see [1], [2], [3], [11], [13], [14], [15], [16], [27], [34], and [35] and the reviews [20], [23], and [33] for a more exhaustive list of references. (In this paper we consider only results valid in arbitrary dimension d; the $d = 1$ case is special and we will not mention $d = 1$ only results.) It exhibits localization in an interval at the bottom of the spectrum for fixed disorder and on the whole real line for large disorder. On the other hand, until very recently there had been no localization results for ergodic Γ- trimmed Anderson models with $\Gamma \neq \mathbb{Z}^d$, say $\Gamma = K\mathbb{Z}^d$ with $K \geq 2$. The reason is the lack of a covering condition, i.e., that the support of the random potential is all of $\mathbb{Z}^d$ with probability one. Indeed, $\sum_{\zeta \in K\mathbb{Z}^d} \chi_\zeta = \chi_\Gamma$, and hence $\sum_{\zeta \in K\mathbb{Z}^d} \chi_\zeta \geq c > 0$ if and only if $\Gamma = \mathbb{Z}^d$. The covering condition has played a crucial role in the proofs of Wegner estimates (which are bounds on the regularity of the integrated density of states, first proved by Wegner [36] for the standard Anderson model) and localization for the Anderson model.

This difficulty has been overcome for the continuous analogue of the Anderson model by the use of the unique continuation principle for continuous Schrödinger operators, and localization at the bottom of the spectrum has been proved for continuous Anderson Hamiltonians; see [5], [9], [10], and [21]. These results were further extended to a larger class of continuous random Schrödinger operators with alloy-type random potentials, including non-ergodic random Schrödinger operators such as Delone–Anderson Hamiltonians; see [24], [30], and [32].

Recently, Rojas-Molina [30], Theorem 1.2.6, proved Wegner estimates and localization at the bottom of the spectrum for the special case of $2\mathbb{Z}^d$-trimmed Anderson models with no background potential, i.e., $V^{(0)} = 0$. She circumvented the lack of covering condition using an argument of Bourgain and Kenig [5], Section 4, as described in [21], Remark 4.4. Her approach can be extended for Γ-trimmed Anderson models with Γ an arbitrary relatively dense subset of $\mathbb{Z}^d$, as long as there is no background potential [31], Section 2.1; the Bourgain–Kenig argument does not appear to be able to incorporate a background potential. Cao and Elgart [6] showed localization at small disorder below the bottom of the free spectrum for a class of three-dimensional Anderson-like models without background potential. The 'non overlapping setup' in [6] includes $k\mathbb{Z}^d$-trimmed Anderson models where the random variables are supported by the interval $[-1, 1]$, but it is more general in that it admits finite rank random perturbations with non overlapping supports and no definite sign. Theorem 1 in [6] establishes not only localization at small disorder for this class of Anderson models, but also, using the Feynman diagrammatic technique, gives an explicit expression (as a function of the disorder) for the interval of localization.

Although there is no unique continuation principle for discrete Schrödinger operators, we prove Wegner estimates and localization at the bottom of the spectrum for Γ-trimmed Anderson models with nontrivial background potentials. We are not aware of any previous results on either Wegner estimates or localization for this class of models.

1.2.2. The ground state energy. A trimmed Anderson model $H_{\boldsymbol{\omega},\lambda}$ is a $K\mathbb{Z}^d$-ergodic random Schrödinger operator if and only if $\Gamma = \Gamma + \zeta$ for all $\zeta \in KZ^d$, $V^{(0)}$ is a periodic potential with period K, and $\mu_\zeta = \mu$ for all $\zeta \in \Gamma$. In this case its spectrum $\sigma(H_{\boldsymbol{\omega},\lambda})$ is not random, i.e., it is the same with probability one. In particular, requiring $0 = \inf \operatorname{supp} \mu$, we get

$$E_\emptyset(H_{\boldsymbol{\omega},\lambda}) = E_\emptyset(H_0) \text{ with probability one.} \tag{1.9}$$

Since a trimmed Anderson model $H_{\boldsymbol{\omega},\lambda}$ is not, in general, an ergodic random operator, its spectrum $\sigma(H_{\boldsymbol{\omega},\lambda})$ is a random set. We have $E_\emptyset(H_{\boldsymbol{\omega},\lambda}) \geq E_\emptyset(H_0)$ for all $\boldsymbol{\omega} \in [0, M]^\Gamma$ and $\lambda > 0$. But even after imposing $\mu_\zeta = \mu$ for all $\zeta \in \Gamma$ with $0 = \inf \operatorname{supp} \mu$ we cannot guarantee (1.9). For example, take $V^{(0)} = -6d\chi_{\zeta_0}$ for some $\zeta_0 \in \Gamma$, μ uniformly distributed on $[0, 1]$, and $\lambda > 6d$. Then $E_\emptyset(H_0) \leq \langle \delta_{\zeta_0}, H_0 \delta_{\zeta_0} \rangle = -4d$, but we clearly have $\mathbb{P}\{E_\emptyset(H_{\boldsymbol{\omega},\lambda}) \geq 0\} > 0$, so (1.9) is not true. But if in addition we require $V^{(0)}$ to be a periodic potential with period K, it follows that (1.9) holds by comparison with the ergodic random operator we obtain by removing the Γ-trimming, that is, replacing Γ by $\mathbb{Z}^d$. Actually, (1.9) holds in a broader context as the following proposition will show. (See also [31].)

Given a Schrödinger operator H on $\ell^2(\mathbb{Z}^d)$, we define finite volume operators $H^{(\Lambda)} = H_{\Lambda^c}$, i.e., the restriction $\chi_\Lambda H \chi_\Lambda$ of H to $\ell^2(\Lambda)$, where $\Lambda = \Lambda_L(x)$ is a finite box. In particular, given a trimmed Anderson model $H_{\boldsymbol{\omega},\lambda}$, we define finite volume random operators $H^{(\Lambda)}_{\boldsymbol{\omega},\lambda}$. We also set $S_\Lambda(t) := \max_{\zeta \in \Gamma \cap \Lambda} S_{\mu_\zeta}(t)$ for $t \geq 0$, where $S_\mu(t) := \sup_{a \in \mathbb{R}} \mu([a, a+t])$ denotes the concentration function of the probability measure μ, and let $S(t) := \sup_{\zeta \in \Gamma} S_{\mu_\zeta}(t)$ for $t \geq 0$

Proposition 1.5. *Let $H_{\boldsymbol{\omega},\lambda}$ be a Γ-trimmed Anderson model with $\mu_\zeta = \mu$ for all $\zeta \in \Gamma$ with $0 = \inf \operatorname{supp} \mu$. Suppose for any $\varepsilon > 0$ there is $L = L(\varepsilon) > 0$ such that*

$$|\{x \in \mathbb{Z}^d : E_\emptyset(H_0^{(\Lambda_L(x))}) \leq E_\emptyset(H_0) + \varepsilon\}| = \infty. \tag{1.10}$$

Then $E_\emptyset(H_0)$ is in the essential spectrum of H_0 and $E_\emptyset(H_{\boldsymbol{\omega},\lambda}) = E_\emptyset(H_0)$ with probability one.

The proof is given in Section 3.1.

1.2.3. Wegner estimates and localization. We prove Wegner estimates and localization for Γ-trimmed Anderson models in intervals of the kind $[E_\emptyset(H_0), E_1] \subset$

$[E_\emptyset(H_0), E_\Gamma(H_0))$. Note that we have (1.9), and hence almost sure existence of the spectrum in these intervals, that is,

$$\mathbb{P}\{\sigma(H_{\boldsymbol{\omega},\lambda}) \cap [E_\emptyset(H_0), E_1] \neq \emptyset\} = 1, \quad E_1 > E_\emptyset(H_0),$$

for the class of (generally) non-ergodic trimmed Anderson models given in Proposition 1.5.

Theorem 1.6. *Let $H_{\boldsymbol{\omega},\lambda}$ be a Γ-trimmed Anderson model. Given an energy $E_1 \in (E_\emptyset(H_0), E_\Gamma(H_0))$, set*

$$\kappa = \kappa(H_0, \Gamma, E_1) = \sup_{\substack{s>0 \\ E_\Gamma(H_0,s)>E_1}} \frac{E_\Gamma(H_0, s) - E_1}{s} > 0. \tag{1.11}$$

Then for every box $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L > 0$ we have

$$\chi_{(-\infty,E_1]}(H^{(\Lambda)}_{\boldsymbol{\omega},\lambda})\chi_{\Gamma\cap\Lambda}\chi_{(-\infty,E_1]}(H^{(\Lambda)}_{\boldsymbol{\omega},\lambda}) \geq \kappa\chi_{(-\infty,E_1]}(H^{(\Lambda)}_{\boldsymbol{\omega},\lambda}), \tag{1.12}$$

for $\boldsymbol{\omega} \in [0, M]^\Gamma$, and for any closed interval $I \subset (-\infty, E_1]$ we have

$$\mathbb{E}\{\operatorname{tr} \chi_I(H^{(\Lambda)}_{\boldsymbol{\omega},\lambda})\} \leq 8\kappa^{-1} S_\Lambda(\lambda^{-1}|I|)|\Gamma \cap \Lambda|. \tag{1.13}$$

Remark 1.7. It follows from (1.3) (and its proof) that

$$\kappa \geq \frac{Q}{2dK+1}((1+Z)Y_{d,V^{(0)}})^{-2dK} \tag{1.14}$$

where

$$Z = \frac{2Kd+1}{2Kd}((1-((E_1 - E_\emptyset(H_0))Q^{-1}(2dK-1)Y^{2dK-1}_{d,V^{(0)}}))^{-\frac{1}{2dK-1}} - 1).$$

(See (3.2)–(3.6) for the derivation of (1.14).)

Theorem 1.6 is proved in Section 3.2.

The Wegner type estimate (1.13) allows us to establish localization for Γ-trimmed Anderson models at the bottom of the spectrum. By complete localization on an interval I we mean that for all $E \in I$ there exists $\delta(E) > 0$ such that we can perform the bootstrap multiscale analysis on the interval $(E - \delta(E), E + \delta(E))$, obtaining Anderson and dynamical localization; see [17], [18], and [19]. (Note that by this definition we always have complete localization in $(-\infty, E_\emptyset(H_0))$.)

The following theorem show that we always have localization below $E_\emptyset(H_0)$ at high disorder.

Theorem 1.8. *Let $H_{\boldsymbol{\omega},\lambda}$ be a Γ-trimmed Anderson model, and suppose $S(t) \leq Ct^\theta$ for all $t \geq 0$, where $\theta \in (0, 1]$ and C is a constant. Then, given $E_1 \in (E_\emptyset(H_0), E_\Gamma(H_0))$, there exists $\lambda(E_1) < \infty$ such that $H_{\boldsymbol{\omega},\lambda}$ exhibits complete localization on the interval $(-\infty, E_1)$ for all $\lambda \geq \lambda(E_1)$.*

Theorem 1.8 is proved exactly as [24], Theorem 1.7, using the Wegner estimate (1.13), so we omit the proof.

We also establish localization in an interval at the bottom of the spectrum for fixed disorder.

Theorem 1.9. *Let $H_{\boldsymbol{\omega},\lambda}$ be a Γ-trimmed Anderson model, and suppose $S(t) \leq Ct^{\theta}$ for all $t \geq 0$, where $\theta \in (0,1]$ and C is a constant. Assume in addition that one of the following hypotheses hold.*

(i) *$H_{\boldsymbol{\omega},\lambda}$ is an ergodic Γ- trimmed Anderson model.*

(ii) *There is no background potential, that is, $V^{(0)} = 0$.*

(iii) *The exponent θ satisfies $\theta > \frac{d}{2}$.*

Then for all $\lambda > 0$ there exists $E_\lambda > E_\emptyset(H_0)$ such that $H_{\boldsymbol{\omega},\lambda}$ exhibits complete localization on the interval $(-\infty, E_\lambda)$.

The proof of this theorem is standard once we have the Wegner estimate (1.13). (Thus we will have $E_\lambda < E_\Gamma(H_0)$.) The necessary input for starting the multiscale analysis can be verified as follows.

(i) If $H_{\boldsymbol{\omega},\lambda}$ is ergodic, it has Lifshitz tails [29, 26] (the proofs apply also to the discrete case), and we proceed as in [21], Proposition 4.3.

(ii) If $V^{(0)} = 0$, we proceed as in [21], Remark 4.4; the argument can be adapted to the discrete case as noted in [30], Theorem 1.2.6, and [31].

(iii) If $\theta > \frac{d}{2}$, we employ the same strategy as in (i), replacing the Lipschitz tails with the "classical tails" given by the condition $\theta > \frac{d}{2}$ as in [14], Proof of Theorem 3′, and [25], Proof of Theorem 3.11.

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2. The ground state energy of trimmed Schrödinger operators

In this section we prove Theorems 1.1 and 1.3. Given $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential, we will use the shorthand notation $E_\Gamma = E_\Gamma(H)$, $E_\Gamma(t) = E_\Gamma(H,t)$, $E_\emptyset = E_\emptyset(H)$.

2.1. Equality of the ground state energies.

We start by proving Theorem 1.1.

Proof of Theorem 1.1. We first obtain a simple upper bound on $\delta_\Gamma(H)$ (hence on $\delta_\Gamma(H,t)$ as well), to be used later on. To this end, note that $E_\emptyset \geq \inf_{x\in\mathbb{Z}^d} V(x)$, and hence

$$H_\Gamma - E_\emptyset \leq -\Delta_\Gamma + \operatorname{spr} V \leq 4d + \operatorname{spr} V. \tag{2.1}$$

It follows that

$$E_\Gamma - E_\emptyset \le E_\Gamma(-\Delta) + \operatorname{spr} V \le 2d + \operatorname{spr} V, \tag{2.2}$$

where we used $E_\Gamma(-\Delta) \le \inf_{x\in\Gamma^c}\langle \delta_x, -\Delta\delta_x\rangle = 2d$, that is,

$$\delta_\Gamma(H) \le \delta_\Gamma(-\Delta) + \operatorname{spr}(V) \le 2d + \operatorname{spr}(V).$$

Suppose $E_\Gamma > E_\emptyset$, since otherwise there is nothing to prove. By replacing H by $H - E_\emptyset$, we may assume $E_\emptyset = 0$, so $\delta_\Gamma(H) = E_\Gamma$ and $\delta_\Gamma(H,t) = E_\Gamma(t)$.

Let $\nu > 0$. Then $H + \nu \ge \nu$ (recall $E_\emptyset = 0$), so $(H+\nu)^{-1} \le \frac{1}{\nu}$. It follows that on $\ell^2(\Gamma)$ we have

$$S_\nu = H_{\Gamma^c} + \nu - u(H_\Gamma + \nu)^{-1}u^* \ge \nu, \quad u = \chi_\Gamma\Delta\chi_{\Gamma^c},$$

since S_ν is the the Schur complement of $H_\Gamma + \nu$, and we have

$$S_\nu^{-1} = \chi_\Gamma(H+\nu)^{-1}\chi_\Gamma \le \frac{1}{\nu}.$$

In particular, we conclude that

$$H_{\Gamma^c} \ge u(H_\Gamma + \nu)^{-1}u^* \quad \text{for all} \quad \nu > 0. \tag{2.3}$$

By hypothesis $E_\Gamma > 0$, so we take $\eta \in (0, E_\Gamma)$. Note that for all $\nu > 0$ we have

$$(H_\Gamma - \eta)^{-1} \le \Big(1 + \frac{\eta + \nu}{E_\Gamma - \eta}\Big)(H_\Gamma + \nu)^{-1}.$$

We now consider the Schur complement $S_{-\eta}(t)$ of $(H_\Gamma(t))_\Gamma - \eta$, and use (2.3) and (2.1), getting

$$\begin{aligned} S_{-\eta}(t) &= H_{\Gamma^c} + t - \eta - u(H_\Gamma - \eta)^{-1}u^* \\ &\ge H_{\Gamma^c} + t - \eta - \Big(1 + \frac{\eta+\nu}{E_\Gamma - \eta}\Big)u(H_\Gamma + \nu)^{-1}u^* \\ &\ge H_{\Gamma^c} + t - \eta - \Big(1 + \frac{\eta+\nu}{E_\Gamma - \eta}\Big)H_{\Gamma^c} \\ &\ge t - \eta - \frac{\eta+\nu}{E_\Gamma - \eta}(4d + \operatorname{spr}(V)). \end{aligned}$$

Since $\nu > 0$ is arbitrary, we obtain

$$S_{-\eta}(t) \ge t - \eta - \tfrac{\eta}{E_\Gamma - \eta}(4d + \operatorname{spr}(V)).$$

We conclude that

$$S_{-\eta}(t) > 0$$

if

$$\eta < \frac{t + 4d + E_\Gamma + \operatorname{spr}(V) - \sqrt{(t + 4d + E_\Gamma + \operatorname{spr}(V))^2 - 4E_\Gamma t}}{2},$$

so it follows from the Schur complement condition for positive definiteness that

$$\begin{aligned} E_\Gamma(t) &\geq \frac{t + 4d + E_\Gamma + \operatorname{spr}(V) - \sqrt{(t + 4d + E_\Gamma + \operatorname{spr}(V))^2 - 4E_\Gamma t}}{2} \\ &= \frac{2E_\Gamma t}{t + 4d + E_\Gamma + \operatorname{spr}(V) + \sqrt{(t + 4d + E_\Gamma + \operatorname{spr}(V))^2 - 4E_\Gamma t}} \\ &\geq \frac{E_\Gamma t}{t + 4d + E_\Gamma + \operatorname{spr}(V)}, \quad t > 0, \end{aligned} \tag{2.4}$$

Combining with (2.2) we get

$$E_\Gamma(t) \geq \frac{E_\Gamma t}{t + 6d + 2\operatorname{spr}(V)}, \quad t > 0, \tag{2.5}$$

which is (1.2), Letting $t \to \infty$ in (1.2) we get $E_\Gamma(\infty) \geq E_\Gamma$. Since $E_\Gamma(\infty) \leq E_\Gamma$, we get $E_\Gamma(\infty) = E_\Gamma$. □

2.2. Lower bounds on the ground state energy for arbitrary potential. Theorem 1.3 for arbitrary bounded potential V, namely the lower bounds (1.3)-(1.4), follows from the following theorem.

We recall $Y_{d,V} = 2d + 1 + \operatorname{spr}(V)$ for a bounded potential V.

Theorem 2.1. *Let $\Gamma \subsetneq \mathbb{Z}^d$ be (K, Q)-relatively dense, and let $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential. Then*

$$\delta_\Gamma(H, t) \geq \frac{Q}{2dK - 1}\Big(\frac{1}{Y_{d,V}^{2dK-1}} - \frac{1}{(Y_{d,V} + t)^{2dK-1}}\Big), \quad t \geq 0. \tag{2.6}$$

As a consequence, we get

$$\delta_\Gamma(H) \geq \frac{Q}{(2dK - 1)Y_{d,V}^{2dK-1}} > 0. \tag{2.7}$$

The proof of the theorem is based on what may be called a quantitative unique continuation principle for ground states, given in the following lemma.

Given a nonempty connected subset B of $\mathbb{Z}^d$ and $x, y \in B$, we let $d_B(x, y)$ denote the graph distance between x and y in B, i.,e., the minimal length of a path in B connecting x and y. We set $\operatorname{diam} B = \max_{x,y\in B} d_B(x, y)$, the diameter of B in the graph theory sense. Note that we always have $d_B(x, y) \geq \|x - y\|_1$, and $d_B(x, y) = \|x - y\|_1$ for all $x, y \in B$ if $B = \mathbb{Z}^d$ or $B = \Lambda_L(x_0)$. In particular, we have $\operatorname{diam} \Lambda_L(x_0) \leq dL$.

Lemma 2.2. *Let $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential. Let $\Lambda = \Lambda_L(x_0)$ be a box in $\mathbb{Z}^d$. Then $E^{(\Lambda)} = \inf \sigma(H_\Lambda)$ is a simple eigenvalue, and there exists a unique strictly positive ground state $\psi_g^{(\Lambda)}$, i.e., there exists a unique $\psi_g^{(\Lambda)} \in \ell^2(\Lambda)$ such that $H_\Lambda \psi_g^{(\Lambda)} = E^{(\Lambda)} \psi_g^{(\Lambda)}$, $\|\psi_g^{(\Lambda)}\| = 1$, and $\psi_g^{(\Lambda)}(x) > 0$ for all $x \in \Lambda$. Moreover, for all $x \in \Lambda$ and $m \in \mathbb{N}$ we have*

$$\psi_g^{(\Lambda)}(x) \geq Y_{d,V}^{-m} \sum_{\substack{y\in\Lambda \\ \|x-y\|_1\leq m}} \psi_g^{(\Lambda)}(y). \tag{2.8}$$

We also get a uniform lower bound:

$$\psi_g^{(\Lambda)}(x) \geq Y_{d,V}^{-dL}, \quad x \in \Lambda. \tag{2.9}$$

Proof. Without loss of generality we assume $0 = \inf_{x\in\mathbb{Z}^d} V(x)$, so $0 \leq V \leq V_\infty = \operatorname{spr}(V)$ and $E^{(\Lambda)} \geq 0$.

Note that $\ell^2(\Lambda)$ is a finite-dimensional Hilbert space. The existence of the unique strictly positive ground state follows from the Perron-Frobenius Theorem. This can be seen as follows. The self-adjoint operator $T = 2d + 1 + V_\infty - H_\Lambda$ on $\ell^2(\Lambda)$ is positivity preserving, i.e., $\langle \delta_x, T\delta_y\rangle \geq 0$ for all $x, y \in \Lambda$. Moreover,

$$\langle \delta_x, T^m\delta_y\rangle \geq 1 \quad \text{for } m \geq \|x - y\|_1, x, y \in \Lambda.$$

In particular, recalling $\operatorname{diam} \Lambda \leq dL$, we have

$$\langle \delta_x, T^{dL}\delta_y\rangle \geq 1, \quad x, y \in \Lambda.$$

It follows from the Perron–Frobenius Theorem that $\lambda_{\max} = \max \sigma(T)$ is a simple eigenvalue, and there exists a unique $\psi_g^{(\Lambda)} \in \ell^2(\Lambda)$ such that $T\psi_g^{(\Lambda)} = \lambda_{\max}\psi_g^{(\Lambda)}$, $\|\psi_g^{(\Lambda)}\| = 1$, and $\psi_g^{(\Lambda)}(x) > 0$ for all $x \in \Lambda$. Clearly, $H_\Lambda\psi_g^{(\Lambda)} = E^{(\Lambda)}\psi_g^{(\Lambda)}$ and

$$\lambda_{\max} = 2d + 1 + V_\infty - E^{(\Lambda)} \leq 2d + 1 + V_\infty = Y_{d,V}.$$

Moreover, since $T\psi_g^{(\Lambda)} = \lambda_{\max}\psi_g^{(\Lambda)}$ and $\psi_g^{(\Lambda)}(x) > 0$ for all $x \in \Lambda$, we have for all $x \in \Lambda$ and $m \in \mathbb{N}$ ($\psi_g = \psi_g^{(\Lambda)}$)

$$\psi_g(x) \geq \lambda_{\max}^{-m} \sum_{y\in\Lambda;\ \|x-y\|_1\leq m} \psi_g(y),$$

which yields (2.8).

To get (2.9), just notice that $1 = \|\psi_g\|_2 \leq \|\psi_g\|_1 = \sum_{y\in\Lambda} \psi_g(y)$. □

Proof of Theorem 2.1. Given $\zeta \in K\mathbb{Z}^d$, fix $\Gamma_\zeta \subset \Gamma \cap \Lambda_K^{(0)}(\zeta)$ such that $\|\Gamma_\zeta\| = Q$.

Let $R = KJ$ where $J = 1, 3, 5, \ldots$ and consider $\Lambda = \Lambda_R = \Lambda_R(0)$. Then, by Lemma 2.2, for all $t \geq 0$ we have that $E^{(\Lambda)}(t) = \inf \sigma(H_\Gamma(t))$ is a simple isolated eigenvalue with eigenvector $\psi_{g,t}^{(\Lambda)}$ as in Lemma 2.2, so it follows that the orthogonal projection $P_g(t) = \langle \psi_{g,t}^{(\Lambda)}, \cdot\, \psi_{g,t}^{(\Lambda)} \rangle \psi_{g,t}^{(\Lambda)}$ is differentiable in t, and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E^{(\Lambda)}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr} P_g(t) H(t) \\
&= \operatorname{tr} P_g(t) \dot{H}(t) + \operatorname{tr} \dot{P}_g(t) H(t) \\
&= \operatorname{tr} P_g(t) \dot{H}(t) \\
&= \operatorname{tr} P_g(t) \chi_\Gamma \\
&\geq \sum_{\zeta \in K\mathbb{Z}^d \cap \Lambda} \langle \psi_{g,t}^{(\Lambda)}, \chi_{\Gamma_\zeta} \psi_{g,t}^{(\Lambda)} \rangle \\
&= \sum_{\zeta \in K\mathbb{Z}^d \cap \Lambda} \sum_{x \in \Gamma_\zeta} (\psi_{g,t}^{(\Lambda)}(x))^2,
\end{aligned} \tag{2.10}
$$

where on the second line we have used $\dot{P}_g = P_g \dot{P}_g (1 - P_g) + (1 - P_g) \dot{P}_g P_g$, cyclicity of the trace, and $P_g H (1 - P_g) = 0$.

If $x \in \Gamma_\zeta$, it follows from (2.8) that

$$
\psi_{g,t}^{(\Lambda)}(x) \geq (Y_{d,V} + t)^{-dK} \sum_{y \in \chi_{\Lambda_K(\zeta)}} \psi_{g,t}^{(\Lambda)}(y),
$$

and hence

$$
(\psi_{g,t}^{(\Lambda)}(x))^2 \geq (Y_{d,V} + t)^{-2dK} \sum_{y \in \chi_{\Lambda_K(\zeta)}} (\psi_{g,t}^{(\Lambda)}(y))^2. \tag{2.11}
$$

Combining (2.10) and (2.11) we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E^{(\Lambda)}(t) &\geq Q (Y_{d,V} + t)^{-2dK} \sum_{\zeta \in K\mathbb{Z}^d \cap \Lambda} \sum_{y \in \chi_{\Lambda_K(\zeta)}} (\psi_{g,t}^{(\Lambda)}(y))^2 \\
&= Q (Y_{d,V} + t)^{-2dK}.
\end{aligned} \tag{2.12}
$$

Thus

$$
\begin{aligned}
E^{(\Lambda)}(t) - E^{(\Lambda)}(0) &\geq Q \int_0^t \mathrm{d}s (Y_{d,V} + s)^{-2dK} \\
&= \frac{Q}{2dK - 1} \Big(\frac{1}{Y_{d,V}^{2dK-1}} - \frac{1}{(Y_{d,V} + t)^{2dK-1}} \Big).
\end{aligned}
$$

To conclude the proof of the theorem, just note that $E_\Gamma(t) = \lim_{R \to \infty} E(\Lambda_R)(t)$ for all $t \geq 0$. □

2.3. Cheeger's inequality for the ground state energy. Theorem 1.3 for $H = -\Delta$, namely the lower bound (1.5), follows from the following theorem.

Theorem 2.3. *Let* $\Gamma \subsetneq \mathbb{Z}^d$ *be* (K, Q)*-relatively dense. Then*

$$E_\Gamma(-\Delta) \geq \frac{1}{4dK_*^{2d}}. \tag{2.13}$$

In addition,

$$E_\Gamma(-\Delta, t) \geq \frac{1}{(6d-1)K_*^{2d}}, \quad t \geq 2d-1. \tag{2.14}$$

Remark 2.4. For $H = -\Delta$ the estimate (2.13) in Theorem 2.3 is better than the corresponding estimate from Theorem 2.1. Note that (2.14) only holds for $t \geq 2d-1$, giving a lower bound independent of t. We can get an estimate for all $t \geq 0$ by combining (2.13) and (2.4), getting

$$E_\Gamma(t) \geq \frac{t}{4dK_*^{2d}(t+4d) + 1}. \tag{2.15}$$

This estimate is better than (2.14) for sufficiently large t.

Given $A \subset \mathbb{Z}^d$, let

- $\partial A = \{(x, y) \in A \times A^c; \|x - y\| = 1\}$,
- $\partial_- A = \{x \in A;\ (x, y) \in \partial A \quad \text{for some} \quad y \in A^c\}$,
- $\partial_+ A = \{y \in A^c;\ (x, y) \in \partial A \quad \text{for some} \quad x \in A\}$;
- given $x \in Z^d$, set

$$\eta_A(x) = |\{y \in \mathbb{Z}^d; (x, y) \in \partial A\}| \in \{0, 1, 2, \ldots, 2d\},$$

so $\partial_- A = \{x \in \mathbb{Z}^d;\ \eta_A(x) \geq 1\}$.

Note that

$$\langle \chi_A, (-\Delta)\chi_A \rangle = |\partial A| = \sum_{x \in \mathbb{Z}^d} \eta_A(x) = \sum_{x \in \partial_- A} \eta_A(x).$$

Lemma 2.5. *Let* $\Gamma \subsetneq \mathbb{Z}^d$ *be* (K, Q)*-relatively dense. Then for all* $A \subset \mathbb{Z}^d \setminus \Gamma$ *we have*

$$\langle \chi_A, (-\Delta)\chi_A \rangle = |\partial A| \geq K_*^{-d}|A|. \tag{2.16}$$

Proof. Let $A \subset \mathbb{Z}^d \setminus \Gamma_K$, set $A_\zeta = A \cap \Lambda_K(\zeta)$ for $\zeta \in K\mathbb{Z}^d$, and let $N_A = |\{\zeta \in K\mathbb{Z}^d; A_\zeta \neq \emptyset\}|$. Then

$$|A| \leq K_*^d N_A. \tag{2.17}$$

On the other hand, $A_\zeta \neq \emptyset$ implies $\partial A \cap (\Lambda_K(\zeta) \times \Lambda_K^{(0)}(\zeta)) \neq \emptyset$ since $\Gamma_K \cap \Lambda_K^{(0)}(\zeta) \neq \emptyset$. We conclude that $N_A \leq |\partial A|$, so (2.16) follows from (2.17). ☐

Let $H = -\Delta$ and fix $\Gamma \subsetneq \mathbb{Z}^d$ be (K, Q)-relatively dense. Following [28], we define the Cheeger constants (note $\langle \chi_A, \chi_A \rangle = |A|$)

$$\beta(\Gamma) = \inf_{A \subset \mathbb{Z}^d \setminus \Gamma;\ 1 \leq |A| < \infty} \beta_A(\Gamma), \quad \text{where } \beta_A(\Gamma) = \frac{\langle \chi_A, (-\Delta_\Gamma)\chi_A \rangle}{|A|}, \tag{2.18}$$

and

$$\beta(t) = \inf_{A \subset \mathbb{Z}^d;\ 1 \leq |A| < \infty} \beta_A(t), \qquad \text{where } \beta_A(t) = \frac{\langle \chi_A, H(t)\chi_A \rangle}{|A|}, t \geq 0.$$

Clearly $\beta(\Gamma) \geq E_\Gamma$ and $\beta(t) \geq E(t)$ for all $t \geq 0$.

Lemma 2.6. *We have*

$$K_*^{-d} \leq \beta(\Gamma) \leq 2d.$$

Moreover, $\beta(t)$ is a nondecreasing function of $t \geq 0$, and

$$\beta(t) \geq \beta^{(1)}(\Gamma) \geq K_*^{-d}, \quad t \geq 2d - 1,$$

where $\beta^{(1)}(\Gamma) = \min\{\beta(\Gamma), 1\}$.

Proof. Given $A \subset \mathbb{Z}^d \setminus \Gamma$, $|A| \geq 1$, it follows from Lemma 2.5, that

$$\beta_A(t) = \beta_A(\Gamma) = \frac{\langle \chi_A, (-\Delta)\chi_A \rangle}{|A|} \geq K_*^{-d}, \quad t \geq 0.$$

It follows that $\beta(\Gamma) \geq K_*^{-d}$. On the other hand, there exists $y_0 \in \mathbb{Z}^d \setminus \Gamma$, since $\Gamma \subsetneq \mathbb{Z}^d$, and we have

$$\beta(\Gamma) \leq \beta_{\{y_0\}}(\Gamma) \leq 2d.$$

Let $A \subset \mathbb{Z}^d;\ 1 \leq |A| < \infty$. Suppose $x \in A \cap \Gamma$, $A_x = A \setminus \{x\}$, and assume $|A_x| \geq 1$. Then $|A| = |A_x| + 1$ and

$$\langle \chi_A, H(t)\chi_A \rangle \geq \langle \chi_{A_x}, H(t)\chi_{A_x} \rangle - 2d + t,$$

so, if $t \geq 2d - 1$,

$$\beta_{A_x}(t) \leq \frac{\langle \chi_A, H(t)\chi_A \rangle - 1}{|A| - 1} \leq \frac{\langle \chi_A, H(t)\chi_A \rangle}{|A|} = \beta_A(t),$$

assuming $\langle \chi_A, H(t)\chi_A \rangle \leq |A|$, i.e., $\beta_A(t) \leq 1$. If $|A \setminus \Gamma| \geq 1$, repeating this procedure until we removed all points of Γ from the set A we obtain

$$\beta_A(t) \geq \beta_{A \setminus \Gamma}(t) = \beta_{A \setminus \Gamma}(\Gamma) \geq \beta(\Gamma).$$

If $A \subset \Gamma$, $|A| \geq 1$, we pick $x_0 \in A$, so we get

$$\beta_A(t) \geq \beta_{\{x_0\}}(t) = 2d + t \geq 2d \geq \beta(\Gamma).$$

We thus conclude that for all $t \geq 2d - 1$ we have $\beta_A(t) \geq \beta^{(1)}(\Gamma)$ for all $A \subset \mathbb{Z}^d$ such that $1 \leq |A| < \infty$. The lemma follows. □

Theorem 2.3 follows from the following theorem.

Theorem 2.7. *Let $\Gamma \subsetneq \mathbb{Z}^d$ be (K, Q)-relatively dense. Then*

$$E_\Gamma(-\Delta) \geq \frac{(\beta(\Gamma))^2}{4d} \geq \frac{1}{4dK_*^{2d}}. \tag{2.19}$$

In addition,

$$E_\Gamma(-\Delta, t) \geq \frac{(\beta^{(1)}(\Gamma))^2}{6d-1} \geq \frac{1}{(6d-1)K_*^{2d}}, \quad t \geq 2d-1. \tag{2.20}$$

Proof. We write

$$H(t) = H_\Gamma(t) = -\Delta + t\chi_\Gamma$$

$$E_\Gamma = E_\Gamma(-\Delta),$$
$$E(t) = E_\Gamma(-\Delta, t).$$

We prove (2.20) first. Following [28], we introduce $\widehat{\mathbb{Z}^d} = \mathbb{Z}^d \cup \{\infty\}$, and for $t > 0$ define the self-adjoint bounded operator $\widehat{H(t)}$ on $\ell^2(\widehat{\mathbb{Z}^d})$ by

$$\widehat{H(t)}\varphi(x) = \sum_{y \in \widehat{\mathbb{Z}^d}} \kappa(x,y)(\varphi(x) - \varphi(y)),$$

where

(i) $\kappa(x,y) = 1$ for $x, y \in \mathbb{Z}^d$, $|x-y| = 1$,
(ii) $\kappa(x,y) = 0$ for $x, y \in \mathbb{Z}^d$, $|x-y| \neq 1$,
(iii) $\kappa(x,\infty) = \kappa(\infty,x) = t\chi_\Gamma(x)$ for $x \in \mathbb{Z}^d$,
(iv) $\kappa(\infty,\infty) = 0$.

Given $\varphi \in \ell^2(\mathbb{Z}^d)$, we extend it to $\hat{\varphi} \in \ell^2(\widehat{\mathbb{Z}^d})$ by setting $\hat{\varphi}(\infty) = 0$. It follows that $\widehat{H(t)\varphi} = \widehat{H(t)}\hat{\varphi}$, and we have

$$\langle \varphi, H(t)\varphi \rangle_{\ell^2(\mathbb{Z}^d)} = \langle \widehat{\varphi}, \widehat{H(t)}\widehat{\varphi} \rangle_{\ell^2(\widehat{\mathbb{Z}^d})} = \frac{1}{2} \sum_{x,y \in \widehat{\mathbb{Z}^d}} \kappa(x,y)|\hat{\varphi}(x) - \hat{\varphi}(y)|^2.$$

Note that

$$\langle \varphi, H(t)\varphi \rangle = \langle \varphi, (-\Delta)\varphi \rangle + t\|\chi_\Gamma \varphi\|^2,$$

so

$$\begin{aligned} E(t) &= \inf\{\langle \varphi, H(t)\varphi \rangle : \varphi \in \ell^2(\mathbb{Z}^d), \|\varphi\| = 1\} \\ &= \inf\{\langle \varphi, H(t)\varphi \rangle : \varphi \in \ell^2(\mathbb{Z}^d;\mathbb{R}), \|\varphi\| = 1, |\operatorname{supp}\varphi| < \infty\} \end{aligned}$$

Now let φ be a real-valued function on $\mathbb{Z}^d$ with finite support. We have, using the Cauchy-Schwarz inequality,

$$\begin{aligned} 2\langle \varphi, H(t)\varphi\rangle_{\ell^2(\mathbb{Z}^d)} &= \sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)(\hat{\varphi}(x)-\hat{\varphi}(y))^2 \\ &\geq \frac{\left(\sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)|\hat{\varphi}(x)^2-\hat{\varphi}(y)^2|\right)^2}{\sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)(\hat{\varphi}(x)+\hat{\varphi}(y))^2}. \end{aligned}$$

For the denominator, we have

$$\begin{aligned} &\sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)(\hat{\varphi}(x)+\hat{\varphi}(y))^2 \\ &\quad= \sum_{x,y\in\mathbb{Z}^d;\,|x-y|=1} (\varphi(x)+\varphi(y))^2 + 2t\langle \varphi, \chi_\Gamma\varphi\rangle \\ &\quad\leq \sum_{x,y\in\mathbb{Z}^d;\,|x-y|=1} (2\varphi(x)^2+2\varphi(y)^2) + 2t\langle \varphi, \chi_\Gamma\varphi\rangle \leq 8d\|\varphi\|^2 + 2t\|\chi_\Gamma\varphi\|^2. \end{aligned}$$

For the numerator, since κ is symmetric, we have, setting

$$A_s = \{\hat{\varphi}^2 > s\} = \{\varphi^2 > s\}$$

for $s \geq 0$,

$$\begin{aligned} &\sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)|\hat{\varphi}(x)^2-\hat{\varphi}(y)^2| \\ &\quad= 2\sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)\chi(\{\hat{\varphi}(x)^2>\hat{\varphi}(y)^2\})|\hat{\varphi}(x)^2-\hat{\varphi}(y)^2| \\ &\quad= 2\int_0^\infty \mathrm{d}s \sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)\chi(\{\hat{\varphi}(x)^2 > s \geq \hat{\varphi}(y)^2\}) \\ &\quad= 2\int_0^\infty \mathrm{d}s \sum_{x,y\in\widehat{\mathbb{Z}^d}} \kappa(x,y)\chi_{A_s}(x)(\chi_{A_s}(x)-\chi_{A_s}(y)) \\ &\quad= 2\int_0^\infty \mathrm{d}s\langle \chi_{A_s}, H(t)\chi_{A_s}\rangle \\ &\quad\geq 2\beta(t)\int_0^\infty \mathrm{d}s|A_s| \\ &\quad= 2\beta(t)\|\varphi\|^2. \end{aligned}$$

We conclude that for a real-valued function φ on $\mathbb{Z}^d$ with finite support and $\|\varphi\| = 1$ we have, for all $t \geq 2d - 1$, using Lemma 2.6,

$$\langle \varphi, H(t)\varphi \rangle \geq \tfrac{1}{2} \frac{(2\beta(t)\|\varphi\|^2)^2}{8d\|\varphi\|^2 + 2t\|\chi_\Gamma \varphi\|^2} \geq \frac{(\beta(t))^2}{4d + t} \geq \frac{(\beta^{(1)}(\Gamma))^2}{4d + t}.$$

Thus

$$E(t) \geq \frac{(\beta^{(1)}(\Gamma))^2}{4d + t}, \quad t \geq 2d - 1.$$

Since $E(t)$ is nondecreasing in t, we get

$$E(t) \geq \frac{(\beta^{(1)}(\Gamma))^2}{6d - 1}, \quad t \geq 2d - 1.$$

To prove (2.19), we repeat the above procedure with $-\Delta_\Gamma$, $\mathbb{Z}^d \setminus \Gamma$, $\mathbb{Z}^d$ and $\widehat{-\Delta_\Gamma}$ instead of $H(t)$, $\mathbb{Z}^d$, $\widehat{\mathbb{Z}^d}$ and $\widehat{H(t)}$, where $\widehat{-\Delta_\Gamma} = (-\Delta_\Gamma) \oplus 0$ on $\ell^2(\mathbb{Z}^d) = \ell^2(\mathbb{Z}^d \setminus \Gamma) \oplus \ell^2(\Gamma)$, and $\kappa(x, y) = 1$ for $x, y \in \mathbb{Z}^d$, $|x - y| = 1$, $\kappa(x, y) = 0$ for $x, y \in \mathbb{Z}^d$, $|x - y| \neq 1$, and, given $\varphi \in \ell^2(\mathbb{Z}^d \setminus \Gamma)$, extending it to $\hat{\varphi} \in \ell^2(\mathbb{Z}^d)$ by setting $\hat{\varphi}(x) = 0$ for $x \in \Gamma$. The proof goes through in exactly the same way, and we get (2.19). □

3. Trimmed Anderson models

In this section we prove Proposition 1.5 and Theorem 1.6.

3.1. The ground state energy

Proof of Proposition 1.5. Let $H_{\omega,\lambda}$ be a Γ-trimmed Anderson model with $\mu_\zeta = \mu$ for all $\zeta \in \Gamma$ with $0 = \inf \operatorname{supp} \mu$. To show that $E_\emptyset = E_\emptyset(H_0) \in \sigma_{\text{ess}}(H_0)$, it suffices to exhibit an orthonormal sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\ell^2(\mathbb{Z}^d)$ such that

$$\|(H_0 - E_\emptyset)\varphi_n\| \leq 1/n, \quad n \in \mathbb{N}.$$

The existence of such a sequence follows from (1.10). Hence $E_\emptyset \in \sigma_{\text{ess}}(H_0)$ by Weyl's criterion.

To show that (1.9) holds, for each $\varepsilon > 0$ we use (1.10) to construct an orthonormal sequence $\{\psi_n^{(\varepsilon)}\}_{n \in \mathbb{N}}$ in $\ell^2(\mathbb{Z}^d)$ such that $\operatorname{supp} \psi_n^{(\varepsilon)} \subset \Lambda_L(x_n)$ with $L = L(\varepsilon)$ for all $n \in N$, with $\|x_n - x_m\|_\infty > L$ for $n \neq m$, and

$$\|(H_0 - E_\emptyset)\psi_n^{(\varepsilon)}\| \leq \varepsilon, \quad n \in \mathbb{N}. \tag{3.1}$$

We then have

$$E_\emptyset(H_{\omega,\lambda}) \le \langle \psi_n^{(\varepsilon)}, H_{\omega,\lambda}\psi_n^{(\varepsilon)}\rangle \le \varepsilon + \sum_{\zeta\in\Gamma\cap\Lambda_L(x_n)} \omega_\zeta |\psi_n^{(\varepsilon)}(\zeta)|^2 \text{ for all } n\in\mathbb{N}.$$

But

$$\mathbb{P}\{\inf_{n\in\mathbb{N}} \max_{\zeta\in\Lambda_L(x_n)} \omega_\zeta < \varepsilon L^{-d}\} = 1,$$

from which it follows that

$$\mathbb{P}\{\sigma(H_{\omega,\lambda}) \cap [E_\emptyset, E_\emptyset + 2\varepsilon] \ne \emptyset)\} = 1.$$

Since ε is arbitrary, the result follows. □

3.2. The Wegner estimate

Proof of Theorem 1.6. Let $H_{\omega,\lambda}$ be a Γ-trimmed Anderson model, fix $E_1 \in (E_\emptyset(H_0), E_\Gamma(H_0))$, and let $\kappa = \kappa(H_0,\Gamma,E_1)$ be as in (1.11). We clearly have $\kappa > 0$. To derive the explicit bound stated in Remark 1.7, namely (1.14), note that the estimate (1.3) yields

$$E_\Gamma(H_0,s) - E_\emptyset(H_0) \ge \frac{Q}{(2dK-1)Y_{d,V^{(0)}}^{2dK-1}}\Big(1-\Big(\frac{Y_{d,V^{(0)}}}{Y_{d,V^{(0)}}+s}\Big)^{2dK-1}\Big) \tag{3.2}$$

for all $s > 0$, which implies $E_\Gamma(H_0,s) > E_1$ for

$$s > s_0 = Y_{d,V^{(0)}}((1-(E_1-E_\emptyset(H_0))Q^{-1}(2dK-1)Y_{d,V^{(0)}}^{2dK-1})^{-\frac{1}{2dK-1}}-1). \tag{3.3}$$

Using (2.12), we get

$$\kappa \ge \sup_{s>s_0} \frac{E_\Gamma(H_0,s)-E_\Gamma(H_0,s_0)}{s} \ge \sup_{s>s_0}\frac{s-s_0}{s} Q(Y_{d,V^{(0)}}+s)^{-2dK}. \tag{3.4}$$

The supremum is attained at

$$s = \frac{2Kd+1}{4Kd}\Bigg(1+\sqrt{1+\frac{8KdY_{d,V^{(0)}}}{(2Kd+1)^2 s_0}}\Bigg)s_0; \tag{3.5}$$

to get a simpler lower bound we take $s = s = \frac{2Kd+1}{2Kd}s_0$, getting

$$\kappa \ge \frac{Q}{2dK+1}\Big(Y_{d,V^{(0)}} + \frac{2Kd+1}{2Kd}s_0\Big)^{-2dK}, \tag{3.6}$$

which is (1.14).

We now proceed as in [24], Proof of Theorem 1.7. Let $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L > 0$, and note that $(H_{0,\Gamma}^{(\Lambda)}(t) = ((H_0)_\Gamma(t))^{(\Lambda)})$

$$E_\Gamma(H_0^\Lambda, t) = E_\emptyset(H_{0,\Gamma}^{(\Lambda)}(t)) \geq E_\emptyset(H_{0,\Gamma}(t)) = E_\Gamma(H_0, t),$$

so

$$\kappa(H_0^\Lambda, \Gamma, E_1) = \sup_{s>0;\ E_\Gamma(H_0^\Lambda, s) > E_1} \frac{E_\Gamma(H_0^\Lambda, s) - E_1}{s} \geq \kappa(H_0, \Gamma, E_1) = \kappa > 0.$$

As a consequence, (1.12) follows immediately from [24], Lemma 4.1.

The Wegner estimate (1.13) follows using (1.12). For any closed interval $I \subset (-\infty, E_1]$ we have

$$\begin{aligned} \operatorname{tr} \chi_I(H_{\omega,\lambda}^{(\Lambda)}) &\leq \kappa^{-1} \operatorname{tr} \chi_I(H_{\omega,\lambda}^{(\Lambda)}) \chi_{\Gamma\cap\Lambda} \chi_I(H_{\omega,\lambda}^{(\Lambda)}) \\ &= \kappa^{-1} \operatorname{tr} \chi_{\Gamma\cap\Lambda} \chi_I(H_{\omega,\lambda}^{(\Lambda)}) \chi_{\Gamma\cap\Lambda} \\ &= \kappa^{-1} \sum_{\zeta\in\Gamma\cap\Lambda} \langle \delta_\zeta, \chi_I(H_{\omega,\lambda}^{(\Lambda)}) \delta_\zeta \rangle. \end{aligned}$$

Since by spectral averaging, [10], eq. (3.16) (see also [8], Appendix A),

$$\int \mathrm{d}\mu_\zeta(\omega_\zeta) \langle \delta_\zeta, \chi_I(H_{\omega,\lambda}^{(\Lambda)}) \delta_\zeta \rangle \leq 8 S_{\mu_\zeta}(\lambda^{-1}|I|),$$

we get (1.13). □

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