

A remark on the Mourre theory for two body Schrödinger operators

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Abstract. On this short note, we apply the Mourre theory of the limiting absorption with *difference* type conditions on the potential, instead of conditions on the derivatives. In order that we modify the definition of the conjugate operator, and we apply the standard abstract Mourre theory. We also discuss examples to which the method applies.

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1. Introduction

We consider the Schrödinger operator on \mathbb{R}^d , i.e.,

$$H = H_0 + V(x), \quad H_0 = -\frac{1}{2} \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^d)$$

with $d \geq 1$. $V(x)$ is the potential, and we always suppose $V(x)$ is a real-valued locally L^2 -function.

Let $I \subset \mathbb{R}$ be an open interval. We say *the Mourre theory applies to H on I* , if for any interval $J \Subset I$, there is a self-adjoint operator A on \mathcal{H} such that

- (i) for $z \in \rho(H)$, $t \mapsto e^{itA}(H - z)^{-1}e^{-itA}$ is a $\mathcal{L}(\mathcal{H})$ -valued C^2 -function on \mathbb{R} ;
- (ii) there is $c > 0$ such that

$$E_J(H)[H, iA]E_J(H) \geq cE_J(H) + K, \quad (1)$$

with some compact operator K , where $E_J(H)$ is the spectral projection.

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It is well-known (see e.g., [1, 3]) that under these conditions, the following properties hold.

- (a) $\sigma_p(H) \cap I$ is discrete, and each eigenvalues are of finite rank.
- (b) H is absolutely continuous on $I \setminus \sigma_p(H)$.
- (c) Let $\gamma > 1/2$. Then, for each $\lambda \in I \setminus \sigma_p(H)$,

$$\lim_{\varepsilon \downarrow 0} \langle A \rangle^{-\gamma} (H - \lambda \pm i\varepsilon)^{-1} \langle A \rangle^{-\gamma} \in \mathcal{L}(\mathcal{H})$$

exist and the limits are Hölder continuous in $\lambda \in I \setminus \sigma_p(H)$.

Here we have used the standard notation: $\langle x \rangle = (1 + |x|^2)^{1/2}$.

When we apply the Mourre theory to 2 body Schrödinger operators, we usually use $A = \frac{1}{2i}(x \cdot \partial_x + \partial_x \cdot x)$, and some derivative conditions are imposed on the potential, at least on the long-range part. Instead, we suppose difference type conditions as follows. Let $\beta > 0$, and let $e_j = (\delta_{jk})_{k=1}^d \in \mathbb{R}^d$ ($j = 1, \dots, d$) be the standard basis of \mathbb{R}^d . We set

$$T_j^\beta f(x) = f(x + \beta e_j), \quad T_j^{\beta*} f(x) = f(x - \beta e_j),$$

and also

$$\Delta_j^\beta f(x) = \frac{1}{\beta}(T_j^\beta - 1)f(x) = \frac{1}{\beta}(f(x + \beta e_j) - f(x)),$$

for a function f on \mathbb{R}^d , $x \in \mathbb{R}^d$ and $j = 1, \dots, d$.

Assumption A. Let $\beta > 0$. $V(x)$ and $x_j \Delta_j^\beta V(x)$ ($j = 1, \dots, d$) are H_0 -compact. Moreover, $x_j x_k \Delta_j^\beta \Delta_k^\beta V(x)$ ($j, k = 1, \dots, d$) are H_0 -bounded.

Theorem 1. *Suppose Assumption A with $\beta > 0$, and let $I = (0, \frac{1}{2}(\pi/\beta)^2)$. Then the Mourre theory applies to H on I . Hence, in particular, properties (a)–(c) holds on I .*

Corollary 2. *Suppose Assumption A holds for all $\beta > 0$. Then the Mourre theory applies to H on $(0, \infty)$.*

Remark 1. In Corollary 2, we do not assume Assumption A with uniform bounds in $\beta > 0$. Hence, V is not necessarily differentiable.

Example 1. Suppose $V = V_1 + V_2 + V_3$, where

- (i) $|x|^2 V(x)$ is H_0 -bounded;
- (ii) $V_2 \in C^1(\mathbb{R}^d)$. V_2 and $|x|^2 \partial_{x_j} V_2$ ($j = 1, \dots, d$) are H_0 -compact;
- (iii) $V_3 \in C^2(\mathbb{R}^d)$. V_3 and $x_j \partial_{x_j} V_3$ ($j = 1, \dots, d$) are H_0 -compact, and, for $j, k = 1, \dots, d$, $x_j x_k \partial_{x_j} \partial_{x_k} V_3$ are H_0 -bounded.

Then V satisfies Assumption A with any $\beta > 0$. This is a variation of the standard assumption of the Mourre theory for 2 body Schrödinger operators.

Example 2. Suppose $W(x)$ is a β -periodic locally L^p -function, i.e.,

$$W(x + \beta e_j) = W(x), \quad x \in \mathbb{R}^d, \quad j = 1, \dots, d,$$

where $p = 2$ if $d \leq 3$ and $p > d/2$ if $d \geq 4$. Let $\gamma > 0$ and we set

$$V(x) = \langle x \rangle^{-\gamma} W(x).$$

Then $V(x)$ satisfies Assumption A with the above β , and hence H is absolutely continuous except for discrete eigenvalues on $(0, \frac{1}{2}(\pi/\beta)^2)$. This example shows that even the long-range part may be rather singular for the Mourre theory to be applied.

The Mourre theory is one of the most useful methods in the scattering theory [7]. For comprehensive reviews and applications, see, for example, [1], [2], [3], [4], [8], [9], [5], etc. Usually a differential operator (the dilation generator, in particular) is used as the conjugate operator A , and hence some discussion about the differentiability of the potential is necessary, though it is possible to avoid differentiability assumptions using approximation arguments. We employ a difference operator as conjugate operator, and this is partially motivated by the Mourre theory for discrete Schrödinger operators ([2], [6]). Difference operators belong to the Fourier multipliers, and conjugate operators in terms of Fourier multipliers are not new. In fact they appear in the original paper by Mourre [7], and studied extensively, for example, in Amrein, Boutet de Monvel, Georgescu [1]. In fact, our theorem may be considered as an application of Proposition 7.5.6 of [1]. However, by using difference operators, the formulation and the argument become considerably simpler, and we can give an elementary and self-contained proof in this short article. The choice of the conjugate operator and its applications, Example 2 in particular, seem new, as far as the author is aware of.

2. Proof

We fix $\beta > 0$ and suppose Assumption A in the following. We denote the Fourier transform by \mathcal{F} :

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-d/2} \int e^{-ix \cdot \xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^d, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

We write

$$Q_j u(x) = \frac{1}{2i\beta} (T_j^\beta - T_j^{\beta*}) u(x) = \frac{1}{2i\beta} (u(x + \beta e_j) - u(x - \beta e_j)).$$

We note

$$\mathcal{F}Q_j \mathcal{F}^* u(\xi) = \frac{1}{\beta} \sin(\beta \xi_j) u(\xi), \quad \xi \in \mathbb{R}^d, u \in L^2(\mathbb{R}^d).$$

We now define

$$Au = \frac{1}{2} \sum_{j=1}^d (Q_j x_j + x_j Q_j) u \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d).$$

We then note that

$$-i\mathcal{F}A\mathcal{F}^* = \frac{1}{2\beta} \sum_{j=1}^d \left(\sin(\beta \xi_j) \frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial \xi_j} \sin(\beta \xi_j) \right)$$

is a first order differential operator which generates a unitary group through a change of coordinates. This implies, in particular, $\mathcal{F}A\mathcal{F}^*$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^d)$, and hence A is also essentially self-adjoint on $\mathcal{S}(\mathbb{R}^d)$. Moreover, e^{-itA} leaves the domain of H_0 and H invariant.

Now by easy computations, we have

$$\mathcal{F}[H_0, iA]\mathcal{F}^* = \sum_{j=1}^d \frac{1}{\beta} \sin(\beta \xi_j) \xi_j,$$

and it is easy to see

$$\sum_{j=1}^d \frac{1}{\beta} \sin(\beta \xi_j) \xi_j > 0 \quad \text{if } 0 < |\xi| < \frac{\pi}{\beta}.$$

We let $\eta > 0$ sufficiently small and choose $f \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp } f \subset \left[\frac{\eta}{2}, \frac{1}{2} \left(\frac{\pi}{\beta} - \frac{\eta}{2} \right)^2 \right]; \quad f(t) = 1 \text{ if } t \in \left[\eta, \frac{1}{2} \left(\frac{\pi}{\beta} - \eta \right)^2 \right].$$

Then we learn

$$f(H_0)[H_0, iA]f(H_0) \geq \delta f(H_0)^2,$$

with some $\delta > 0$.

It is well-known that $f(H) - f(H_0)$ is compact if V is H_0 -compact, and by using the standard argument, we have

$$f(H)[H_0, iA]f(H) \geq \delta f(H)^2 + K_1, \tag{2}$$

with a compact operator K_1 .

Next we consider $[V, iA]$. By straightforward computation, we have

$$A = \frac{1}{4i\beta} \sum_{j=1}^d (x_j T_j^\beta - x_j T_j^{\beta*}) + \frac{1}{4i} \sum_{j=1}^d (T_j^\beta + T_j^{\beta*}). \tag{3}$$

The second sum in the right hand side is bounded and commutes with H_0 , and hence its commutator with V is H_0 -compact by the assumption. We also have

$$\begin{aligned} [T_j^\beta, V]u(x) &= T_j^\beta V u(x) - V T_j^\beta u(x) \\ &= V(x + \beta e_j)u(x + \beta e_j) - V(x)u(x + \beta e_j) \\ &= (V(x + \beta e_j) - V(x))u(x + \beta e_j) \\ &= (\beta \Delta_j^\beta V) T_j^\beta u(x). \end{aligned}$$

This implies

$$[x_j T_j^\beta, V] = x_j [T_j^\beta, V] = \beta (x_j \Delta_j^\beta V) T_j^\beta$$

is H_0 -compact again by the assumption. Similarly, we can show $[x_j T_j^*, V]$ is H_0 -compact:

$$[x_j T_j^{\beta*}, V] = x_j T_j^{\beta*} (\beta \Delta_j^\beta V) = \beta T_j^{\beta*} (x_j \Delta_j^\beta V) + \beta T_j^{\beta*} (\Delta_j^\beta V).$$

Thus we learn $[V, iA]$ is H_0 -compact. Combining this with (2), we obtain the Mourre inequality (1).

It remains to show $e^{itA}(H - z)^{-1}e^{-itA}$ is a C^2 -class function in t . Since e^{-itA} leaves $H^2(\mathbb{R}^d) = D(H) = D(H_0)$ invariant, it suffices to show $[H, iA]$ and $[[H, iA], iA]$ are H_0 -bounded. By the above expressions of $[H_0, iA]$ and $[V, iA]$, $[H, iA]$ is obviously H_0 -bounded. $[[H_0, iA], iA]$ is computed as

$$\mathcal{F}[[H_0, iA], iA]\mathcal{F}^* = \frac{1}{\beta^2} \sum_{j=1}^d \left(\sin(\beta \xi_j) \frac{\partial}{\partial \xi_j} [\sin(\beta \xi_j) \xi_j] \right).$$

and hence it is H_0 -bounded. $[[V, iA], iA]$ can be computed and estimated using (3) as above. For example, we have

$$\begin{aligned}
 [[V, x_j T_j^\beta], x_k T_k^\beta] &= \beta [x_k T_k^\beta, (x_j \Delta_j^\beta V) T_j^\beta] \\
 &= \beta x_k [T_k^\beta, (x_j \Delta_j^\beta V)] T_j^\beta + \beta x_j (\Delta_j^\beta V) [x_k, T_j^\beta] T_k^\beta \\
 &= \beta^2 x_j x_k (\Delta_j^\beta \Delta_k^\beta V) T_j^\beta T_k^\beta \\
 &\quad + \beta \delta_{jk} x_k T_k^\beta (\Delta_j^\beta V) T_j^\beta - \beta \delta_{jk} x_j (\Delta_j^\beta V) T_j^\beta T_k^\beta \\
 &= \beta^2 [x_j x_k (\Delta_j^\beta \Delta_k^\beta V)] T_j^\beta T_k^\beta \\
 &\quad + \beta \delta_{jk} T_k^\beta [x_j (\Delta_j^\beta V)] T_j^\beta - \beta \delta_{jk} T_k^\beta (\Delta_j^\beta V) T_j^\beta \\
 &\quad - \beta \delta_{jk} [x_j (\Delta_j^\beta V)] T_j^\beta T_k^\beta
 \end{aligned}$$

and each term is H_0 -bounded by the assumption. Other terms in the expansion of $[[V, iA], iA]$ can be computed similarly. \square

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