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Point spectrum for quasi-periodic long range operators

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Abstract. We generalize Gordon type argument to quasi-periodic operators with finite range interaction and prove that these operators have no point spectrum when the rational approximation rate of the base frequency is relatively large. We also show that, for any irrational frequency, there are operators with infinite range interaction possessing point spectrum. This is a new phenomenon which can not happen in the finite range interaction case.

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1. Introduction

We consider quasi-periodic operators $L_{V,\phi,\alpha}$ acting on $l^2(\mathbb{Z})$:

$$(L_{V,\phi,\alpha}u)_n = \sum_{k\in\mathbb{Z}} V_k u_{n-k} + 2\cos 2\pi(\phi + n\alpha)u_n, \tag{1}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\phi \in \mathbb{T}$, V_k are Fourier coefficients of a real analytic function $V \in C^\omega(\mathbb{T},\mathbb{R})$ so that

$$V(\theta) = \sum_{|k|\le D} V_k e^{2\pi i k\theta}, \quad D \in \mathbb{N}\cup\{\infty\},\ \theta \in \mathbb{T}. \tag{2}$$

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If $D < \infty$ and $V_D \neq 0$, then we call $L_{V,\phi,\alpha}$ quasi-periodic operators with finite range interactions. If $D = \infty$ and $V_k \neq 0$ for infinitely many values of k, we call $L_{V,\phi,\alpha}$ quasi-periodic operators with infinite range interactions or long range operators.

If $D = 1$ and $V(\theta) = 2\lambda^{-1}\cos 2\pi\theta$, then $L_{V,\phi,\alpha}$ is the well-known almost Mathieu operator (AMO):

$$(H_{\lambda,\phi,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(\phi + n\alpha)u_n, \quad \lambda > 0. \tag{3}$$

The operator $L_{V,\phi,\alpha}$ can also been viewed as the Aubry dual [1](Fourier type transformation) of classical quasi-periodic Schrödinger operator on $l^2(\mathbb{Z})$:

$$(H_{V,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \tag{4}$$

where $V(\theta)$ is defined by (2). Quasi-periodic Schrödinger operators have strong background in physics, and have been extensively studied in recent years. The operator $L_{V,\phi,\alpha}$ is of special interest and importance, since by Aubry duality, lots of spectral information of $H_{V,\theta,\alpha}$ can be obtained by studying $L_{V,\phi,\alpha}$. Readers can consult [6, 10] for more relationship between these two operators.

In this paper, we discuss the absence and presence of point spectrum of the operators $L_{V,\phi,\alpha}$ with respect to parameters (D, α). We first discuss the absence of the point spectrum. We know that Gordon type argument [9] is one of the most efficient ways to deal with this problem. Loosely speaking, if α is well approximated by rational numbers, then the Schrödinger operator doesn't have point spectrum. The approximation can be measured by the number

$$\beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}, \tag{5}$$

where $\frac{p_n}{q_n}$ denotes the $n-$th convergent of α. Apply Gordon's lemma to AMO, it can be shown that if $\beta(\alpha) = \infty$, then $H_{\lambda,\phi,\alpha}$ has no point spectrum for any λ, ϕ [7]. The result remains true for all quasi-periodic Schrödinger operators with analytic potentials. This shows that one can not find point spectrum in Schrödinger operators for some irrational frequencies. Combined with some finer estimate and explicit formula of the Lyapunov exponent of AMO, Gordon type argument can actually show that if $1 < \lambda < e^{\beta/2}$, then $H_{\lambda,\phi,\alpha}$ has no point spectrum for any ϕ. This is not optimal, it was conjectured by Jitomirskaya in [13] (see also in [14]) that the optimal condition for $H_{\lambda,\phi,\alpha}$ having purely singular continuous spectrum is $1 < \lambda < e^{\beta}$.

Next we investigate what happens when we generalize AMO to the operators $L_{V,\phi,\alpha}$. Our first question will be:

Question 1. *Whether or not the operators $L_{V,\phi,\alpha}$ has no point spectrum if the frequency is sufficiently Liouville?*

We shall see below that the answer depends on D. If D is finite, then with the help of Gordon type argument, we have the following result similar to AMO:

Theorem 1.1. *Suppose that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $D < \infty$, denote*

$$\|V\|_1 = \sum_{|k|\leq D} |V_k|.$$

If

$$\beta(\alpha) > 2D \ln \frac{\|V\|_1}{|V_D|}, \tag{6}$$

then for

$$\lambda < \frac{1}{4}(|V_D|e^{\frac{\beta}{2D}} - 2\|V\|_1), \tag{7}$$

$L_{\lambda^{-1}V,\phi,\alpha}$ has no point spectrum for any $\phi \in \mathbb{T}$.

Theorem 1.1 shows that the operators with finite range interaction has no point spectrum when the frequency is very Liouville. But Theorem 1.1 is not applicable to exclude the point spectrum for operators with infinite range interaction even if $\beta(\alpha) = \infty$.

If α is not very Liouville, the operators with infinite range interaction may have point spectrum. For any $1 \leq D \leq \infty$, Bourgain and Jitomirskaya [8] proved that if α is Diophantine, and λ is large enough, then $L_{\lambda^{-1}V,\phi,\alpha}$ has Anderson localization (pure point spectrum with exponentially decay eigenfunctions) for a.e. ϕ. This result was later generalized by Avila-Jitomirskaya [6], who proved that if $\beta(\alpha) < \infty$, $\lambda > \lambda_0(\beta, V)$, then $L_{\lambda^{-1}V,\phi,\alpha}$ has Anderson localization for a.e. ϕ. Avila and Jitomirskaya in [5] proved that AMO has Anderson localization for a.e. ϕ when $\lambda > e^{16\beta/9}$. This is not optimal, Jitomirskaya [13] conjectured that $H_{\lambda,\phi,\alpha}$ exhibit Anderson localization for a.e. ϕ provided $\lambda > e^{\beta}$.

Unfortunately, the above results on Anderson localization does not tell us any information if $\beta(\alpha) = \infty$. Therefore, one naturally asks the following question:

Question 2. *If $\beta = \infty$ and $D = \infty$, could $L_{V,\phi,\alpha}$ still possesses some point spectrum?*

We will give a positive answer to this question. Denote

$$C_h^\omega(\mathbb{T},\mathbb{R}) := \{\, V \in C^\omega(\mathbb{T},\mathbb{R}) \mid |V\|_h := \sup_{|\Im z|<h} |V(z)| < \infty\},$$

and

$$\mathcal{B}_h(\varepsilon) := \{\, V \in C_h^\omega(\mathbb{T},\mathbb{R}) \mid \|V\|_h < \varepsilon\},$$

then our precise results are as follows.

Theorem 1.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $h > 0$, then we have the following.*

(a) *There exist $\varepsilon_0 = \varepsilon_0(h)$, $0 < h_* < h$ and $\Theta \subseteq \mathbb{T}$ with full Lebesgue measure, such that for any fixed $\phi \in \Theta$, when $\varepsilon < \varepsilon_0$, any $V \in \mathcal{B}_h(\varepsilon)$ is accumulated by $V_k \in \mathcal{B}_{h_*}(\varepsilon)$ such that $L_{V_k,\phi,\alpha}$ has point spectrum with exponentially decay eigenfunctions.*

(b) *If $\beta(\alpha) = \infty$, then there is a dense subset $\mathcal{D}_2 \subseteq C_h^\omega(\mathbb{T},\mathbb{R})$, such that for any $V \in \mathcal{D}_2$, $L_{V,\phi,\alpha}$ has no point spectrum for any $\phi \in \mathbb{T}$.*

Remark 1.1. Gordon's lemma gives a criterion for absence of point spectrum for all phases, this can be generalized to finite range interaction case, see Theorem 1.1, however, Theorem 1.2 (a) shows that this may not work when $D = \infty$.

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2. Preliminaries

2.1. Schrödinger cocycles. A quasi-periodic $\mathrm{SL}(2,\mathbb{R})$ cocycle $(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q} \times C^0(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ is defined as

$$\begin{aligned}(\alpha, A)\colon \mathbb{T}\times\mathbb{R}^2 &\longrightarrow \mathbb{T}\times\mathbb{R}^2,\\ (\theta, v) &\longmapsto (\theta+\alpha, A(\theta)\cdot v).\end{aligned}$$

Note the Schrödinger equation $(H_{V,\theta,\alpha}u)_n = Eu_n$ can be rewritten as

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E^V(\theta+n\alpha)\cdot\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where

$$S_E^V(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

then (α, S_E^V) can be seen as a quasi-periodic $\mathrm{SL}(2, \mathbb{R})$ cocycle introduced above, and we call it Schrödinger cocycle.

2.2. Fibered rotation number. Assume that $A(\cdot) \in C^0(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ is furthermore homotopic to the identity (which includes all Schrödinger cocycles and all cocycles close to constant). In the following, we identity $\mathbb{R}^2$ with $\mathbb{C}$ and denote by $\mathbb{S}^1$ be the set of vectors of $\mathbb{R}^2$ of Euclidian norm 1. Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the projection $\pi(x) = e^{2\pi i x}$. Then the map

$$\begin{aligned} F : \mathbb{T} \times \mathbb{S}^1 &\longrightarrow \mathbb{T} \times \mathbb{S}^1, \\ (\theta, v) &\longmapsto \Big(\theta + \alpha, \frac{A(\theta)v}{\|A(\theta)v\|}\Big), \end{aligned}$$

admits a continuous lift $\widetilde{F} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ of the form

$$\widetilde{F}(\theta, x) = (\theta + \alpha, x + f(\theta, x))$$

such that $f(\theta, x+1) = f(\theta, x)$ and $\pi(x + f(\theta, x)) = A(\theta)\pi(x)/\|A(\theta)\pi(x)\|$. We say that $\widetilde{F}$ is a lift for the cocycle (α, A). f is independent of the choice of the lift up to the addition of a constant integer. Since $\theta \mapsto \theta + \alpha$ is uniquely ergodic on $\mathbb{T}$, then for every $(\theta, x) \in \mathbb{T} \times \mathbb{R}$ the limit

$$\mathrm{rot}_f(\alpha, A) := \lim_{n \to \pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\widetilde{F}^k(\theta, x)) \qquad \mod \mathbb{Z},$$

exists, is independent of (θ, x), the chosen lift $\widetilde{F}$, and the convergence is uniform in (θ, x) [11, 15]. We call this limit $\mathrm{rot}_f(\alpha, A)$ the fibered rotation number of (α, A). It is continuous with respect to A and takes value in $[0, 1/2]$. The fibered rotation number is called Diophantine w.r.t α, if there exist $\tau > 1, \gamma > 0$, such that

$$\|2\mathrm{rot}_f(\alpha, A) - m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m| + 1)^\tau}, \quad m \in \mathbb{Z}.$$

Let

$$R_\phi := \begin{pmatrix} \cos 2\pi\phi & -\sin 2\pi\phi \\ \sin 2\pi\phi & \cos 2\pi\phi \end{pmatrix}, \tag{8}$$

then for any $B \in C^0(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ which is homotopic to $\theta \mapsto R_{k\theta}$ for some $k \in \mathbb{Z}$, we say the degree of B is k, and denote $\deg B = k$.

If there exists $B \in C^0(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ with $\deg B = k$, such that

$$B(\theta+\alpha)^{-1} A_1(\theta) B(\theta) = A_2(\theta),$$

then we have

$$\mathrm{rot}_f(\alpha, A_2) = \mathrm{rot}_f(\alpha, A_1) - k\alpha. \tag{9}$$

2.3. Reducibility. We say that a cocycle (α, A) is (*analytically*) *reducible*, if there exist analytic

$$B \colon \mathbb{T} \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

and a constant $A_* \in \mathrm{SL}(2,\mathbb{R})$ such that

$$B(\theta+\alpha)^{-1} A(\theta) B(\theta) = A_*.$$

However, since what we are interested in this paper is extremely Liouvillean frequency, reducibility is not a suitable concept [18]. A better concept is *rotations reducible*, a cocycle (α, A) is *rotations reducible*, if there exist $B \in C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ and $R \in C^\omega(\mathbb{T}, SO(2,\mathbb{R}))$ such that

$$B(\theta+\alpha)^{-1} A(\theta) B(\theta) = R(\theta).$$

Readers can consult [18] and the reference therein for recent progress on reducibility and rotations reducibility of analytic quasi-periodic $\mathrm{SL}(2,\mathbb{R})$ cocycles.

3. Finite range interaction case

In this section, we generalize Gordon type argument to quasi-periodic operators $L_{\lambda^{-1}V,\phi,\alpha}$ with finite range interaction, i.e., V is a real trigonometric polynomial as in (2) with $D < \infty$. First we assume that $\alpha \in \mathbb{R}$, and consider the eigenvalue equations of the corresponding operators:

$$\sum_{|k|\le D} \lambda^{-1} V_k u_{n-k} + 2\cos 2\pi(\phi + n\alpha) u_n = E u_n.$$

Since $V_{-D} \neq 0$, this equation can be rewritten as a skew-product system of order $2D$:

$$X(n,\alpha) = A(\phi + (n-1)\alpha, E, \alpha) X(n-1,\alpha),$$

where $X(n,\alpha) = (u_{n+D}, u_{n+D-1}, \cdots u_{n-D+1})^T \in \mathbb{C}^{2D}$,

$$A(\phi, E, \alpha) = \begin{pmatrix} -\frac{V_{-D+1}}{V_{-D}} & \cdots & \frac{E - 2\cos 2\pi(\phi) - \lambda^{-1}V_0}{\lambda^{-1}V_{-D}} & \cdots & -\frac{V_{D-1}}{V_{-D}} & -\frac{V_D}{V_{-D}} \\ 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

and we also define the products

$$M_n(\phi, E, \alpha) = A(\phi + (n-1)\alpha, E, \alpha) \cdots A(\phi + \alpha, E, \alpha) A(\phi, E, \alpha)$$

and $M_{-n}(\phi, E, \alpha) = M_n^{-1}(\phi - n\alpha, E, \alpha)$.

Since the proof of Theorem 1.1 can be seen as direct generalization of original Gordon's lemma [9], here we just sketch the proof, similar details can be found in [17]. We approximate quasi-periodic operator by periodic operators, suppose that $\frac{p_k}{q_k}$ is the $k-$th convergent of α, for any nonzero $X(0) \in \mathbb{C}^{2D}$, and for each $k \in \mathbb{N}$, we use the following notations for short:

$$A_n = A(\phi + (n-1)\alpha, E, \alpha), \qquad A_n^{(k)} = A\Big(\phi + (n-1)\frac{p_k}{q_k}, E, \frac{p_k}{q_k}\Big), \tag{11}$$

$$M_n = M_n(\phi, E, \alpha), \qquad M_n^{(k)} = M_n\Big(\phi, E, \frac{p_k}{q_k}\Big), \tag{12}$$

$$X(n) = M_n X(0), \qquad X^{(k)}(n) = M_n^{(k)} X(0), \tag{13}$$

then using Cayley-Hamilton Theorem, it is easy to extend Lemma 7.5, 7.6 in [17] to finite range case which we state as follows without proof:

Lemma 3.1. *The following estimate holds:*

$$\max_{0<|i|\le 2D, i\in\mathbb{Z}} \{\|X^{(k)}(iq_k)\|\} \ge \frac{1}{2D}\|X(0)\|.$$

Denote the spectrum of $L_{\lambda^{-1}V,\phi,\alpha}$ by $\sigma(\lambda^{-1}V, \alpha)$, which is independent of ϕ since $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Now if $E \in \sigma(\lambda^{-1}V, \alpha)$, we can estimate the approximation error $\|M_n - M_n^{(k)}\|$ as follows.

Lemma 3.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, for any $k \in \mathbb{N}$, $E \in \sigma(\lambda^{-1}V, \alpha)$, the following conclusion holds:*

$$\sup_{|n|\leq 2Dq_k} \sup_{\phi\in\mathbb{T}} \|M_n - M_n^{(k)}\| \leq 16\pi D^2 \cdot \frac{q_k}{q_{k+1}} \Big(\frac{4\lambda + 2\|V\|_1}{|V_D|}\Big)^{2Dq_k}.$$

Proof. Since $A(\phi, E, \alpha)$ takes the form as in (10), we have for any $k \in \mathbb{N}$,

$$\sup_{|i|\leq 2Dq_k} \sup_{\phi\in\mathbb{T}} \|A_i - A_i^{(k)}\| < \frac{8\pi\lambda D}{|V_D|q_{k+1}}. \tag{14}$$

Meanwhile, since $E \in \sigma(\lambda^{-1}V, \alpha)$, then $|E| \leq 2 + \lambda^{-1}\|V\|_1$, as a consequence, we have

$$\|A(\phi, E, \alpha)\| \leq \frac{4\lambda + 2\|V\|_1}{|V_D|}, \tag{15}$$

for any $\phi \in \mathbb{T}$.

Therefore, when $0 < n \leq 2Dq_k$, notice that by the telescoping argument, we have

$$\|M_n - M_n^{(k)}\| = \Big\| \sum_{i=1}^{n} A_n \cdots A_{i+1}(A_i - A_i^{(k)}) A_{i-1}^{(k)} \cdots A_1^{(k)} \Big\|$$

it is easy to show the desired result according to (14) and (15). The above proof holds true for $-2Dq_k \leq n < 0$. □

Proof of Theorem 1.1. Take

$$\varepsilon = \beta - 2D \ln\Big(\frac{4\lambda + 2\|V\|_1}{|V_D|}\Big) > 0. \tag{16}$$

If q_k is sufficiently large and $\ln q_{k+1} > (\beta - \frac{\varepsilon}{2})q_k$, then for any $\phi \in \mathbb{T}$, $E \in \sigma(\lambda^{-1}V, \alpha)$ and $|n| \leq 2Dq_k$, we have

$$\|M_n - M_n^{(k)}\| \leq 16\pi D^2 q_k \exp\Big\{-\frac{1}{2}q_k\varepsilon\Big\} < \frac{1}{4D}.$$

By Lemma 3.1, we know there exists $0 < |i| \leq 2D$, $i \in \mathbb{Z}$, such that $\|X(iq_k)\| \geq \|X^{(k)}(iq_k)\| - \|M_{iq_k} - M_{iq_k}^{(k)}\|\|X(0)\| \geq \frac{1}{4D}\|X(0)\|$, then the absence of point spectrum for $L_{\lambda^{-1}V,\phi,\alpha}$ follows immediately.

Clearly, (16) gives the estimate

$$\beta > 2D \ln \frac{4\lambda + 2\|V\|_1}{|V_D|},$$

which immediately implies (6) and (7). □

4. Long range case

4.1. Proof of Theorem 1.2 (*a*). According to Aubry duality [1], the dual model of the eigenvalue equation of the long range operator (1) $L_{V,\phi,\alpha}u = Eu$ is the eigenvalue equation of the Schrödinger operator: $H_{V,\theta,\alpha}u = Eu$, which is related to Schrödinger cocycle (α, S_E^V). The spectrum of $H_{V,\theta,\alpha}$ is independent of the phase θ since $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and we denote it as $\Sigma_{V,\alpha}$. The basic observation is that if the Schrödinger cocycle (α, S_E^V) is reducible for some $E \in \Sigma_{V,\alpha}$, then the long range operator $L_{V,\phi,\alpha}$ has point spectrum for some ϕ [6, 16].

First, we need the following rotations reducible result which will be the basis of our proof. The positive measure results are obtained in [4, 12], and the full measure rotations reducible results are proved in [2, 18].

Theorem 4.1. [2, 18] *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $h > 0$, $A \in C_h^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, and $R \in \mathrm{SL}(2,\mathbb{R}))$ be some constant matrix, there exist $\varepsilon = \varepsilon(h, R)$ such that if $\|A - R\|_h < \varepsilon$ and the fibered rotation number $\mathrm{rot}_f(\alpha, A)$ is Diophantine with respect to α, then there exist $0 < h_* < h$, $B \in C_{h_*}^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ and $\varphi \in C_{h_*}^\omega(\mathbb{T}, \mathbb{R})$ such that*

$$B(\theta + \alpha)^{-1}A(\theta)B(\theta) = R_{\varphi(\theta)}. \tag{17}$$

For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and any given $\tau > 1, \gamma > 0$, define

$$\Theta_\gamma = \left\{\phi \in \mathbb{T} \,\middle|\, \|2\phi - m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m| + 1)^\tau}, m \in \mathbb{Z}\right\},$$

then $\Theta = \bigcup_{\gamma>0} \Theta_\gamma$ is of full Lebesgue measure. And for $h > 0$, $V \in \mathcal{B}_h(1)$, $\cup_{V \in \mathcal{B}_h(1)} \Sigma_{V,\alpha} \subseteq [-3, 3]$. For any $E \in [-3, 3]$, let $R = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$, according to Theorem 4.1, there is $\varepsilon = \varepsilon(h, E)$ such that if $\|A - R\|_h < \varepsilon$ and $\mathrm{rot}_f(\alpha, A)$ is Diophantine with respect to α, then (17) holds. In fact, in the proof of Theorem 4.1 (see also Remark 1.2 of [12]), we know $\varepsilon = \varepsilon(h, E)$ depends on R only when R is parabolic, i.e., $E = \pm 2$, therefore, we can actually take $\varepsilon_0 = \varepsilon_0(h)$ for which (17) holds, then we have the following proposition.

Proposition 4.1. *For any $\phi \in \Theta$ and $V \in \mathcal{B}_h(\varepsilon_0)$, there exists $E \in \Sigma_{V,\alpha}$, $0 < h_* < h$ and a sequence of $\widetilde{V}_k \in C_{h_*}^\omega(\mathbb{T}, \mathbb{R})$ with $\|\widetilde{V}_k - V\|_{h_*} \to 0$, such that $(\alpha, S_E^{\widetilde{V}_k})$ is reduced to (α, R_ϕ).*

Proof of Proposition 4.1. Fix $\phi \in \Theta$. For any $V \in \mathcal{B}_h(\varepsilon_0)$, there exists $E \in \Sigma_{V,\alpha}$, such that $\mathrm{rot}_f(\alpha, S_E^V) - \phi \in \alpha\mathbb{Z} \bigoplus \mathbb{Z}$, i.e., there is $l \in \mathbb{Z}$ such that $\mathrm{rot}_f(\alpha, S_E^V) = \phi - l\alpha \mod \mathbb{Z}$, then $\mathrm{rot}_f(\alpha, S_E^V)$ is also Diophantine with respect to α. By Theorem 4.1, there is $0 < h_* < h$, $B \in C_{h_*}^{\omega}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ and $\varphi \in C_{h_*}^{\omega}(\mathbb{T}, \mathbb{R})$ such that $B(\theta+\alpha)^{-1} S_E^V(\theta) B(\theta) = R_{\varphi(\theta)}$. Furthermore, by (9), we have

$$[\varphi] = \phi - (\deg B + l)\alpha.$$

Next we are going to show that although (α, S_E^V) is not necessary reducible, it is accumulated by reducible cocycles. To see this, we only need to consider the truncated systems $(\alpha, R_{\mathcal{T}_{q_k}\varphi(\theta)})$, where $\mathcal{T}_{q_k}\varphi(\theta) = \sum_{|m|\le q_k} \hat{\varphi}(m) e^{2\pi i m\theta}$. The following equation

$$\psi_k(\theta+\alpha) - \psi_k(\theta) = \mathcal{T}_{q_k}\varphi(\theta) - [\varphi]$$

always has a solution $\psi_k \in C_{h_*}^{\omega}(\mathbb{T}, \mathbb{R})$ since $\mathcal{T}_{q_k}\varphi(\theta)$ are polynomials, therefore

$$R_{\psi_k(\theta+\alpha)}^{-1} R_{\mathcal{T}_{q_k}\varphi(\theta)} R_{\psi_k(\theta)} = R_{[\varphi]},$$

which means $(\alpha, R_{\mathcal{T}_{q_k}\varphi(\theta)})$ are reducible. As a consequence, let

$$T_k(\theta) = B(\theta+\alpha) R_{\mathcal{T}_{q_k}\varphi(\theta)} B^{-1}(\theta),$$

the cocycles (α, T_k) are also reducible. Furthermore, if we denote

$$\mathcal{R}_{q_k}\varphi(\theta) = \sum_{|m|>q_k} \hat{\varphi}(m) e^{2\pi i m\theta},$$

then when $k \to \infty$, we have

$$\begin{aligned} \|S_E^V - T_k\|_{h_*} &= \|B(\theta+\alpha)(R_{\varphi(\theta)} - R_{\mathcal{T}_{q_k}\varphi(\theta)})B^{-1}(\theta)\|_{h_*} \\ &\le \|B\|_{h_*}^2 \|\mathcal{R}_{q_k}\varphi(\theta)\|_{h_*} \longrightarrow 0. \end{aligned}$$

To this stage, we need the following observation, which states that any non-Schrödinger perturbation of Schrödinger cocycle can be converted to a Schrödinger cocycle. The proof can be found in [3].

Theorem 4.2. *Let $V \in C_h^{\omega}(\mathbb{T}, \mathbb{R})$ be non-identically zero. There exists $\varepsilon > 0$, such that if $A \in C_h^{\omega}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ satisfies $\|A - S_E^V\|_h < \varepsilon$, then there exist $\widetilde{V} \in C_h^{\omega}(\mathbb{T}, \mathbb{R})$, and $\widetilde{B} \in C_h^{\omega}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, such that*

$$\widetilde{B}(\theta+\alpha)^{-1} A(\theta) \widetilde{B}(\theta) = S_E^{\widetilde{V}}(\theta),$$

with $\|\widetilde{V} - V\|_h < \varepsilon$.

When k is large enough, by Theorem 4.2, there exist $\widetilde{B}_k \in C^{\omega}_{h_*}(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, $\widetilde{V}_k \in C^{\omega}_{h_*}(\mathbb{T}, \mathbb{R})$ with $\|\widetilde{V}_k - V\|_{h_*} \to 0$, such that

$$\widetilde{B}_k(\theta+\alpha)^{-1} T_k(\theta) \widetilde{B}_k(\theta) = S_E^{\widetilde{V}_k}(\theta),$$

let $\overline{B}_k(\theta) = \widetilde{B}_k(\theta)^{-1} B(\theta) R_{\psi_k(\theta)}$, then

$$\overline{B}_k(\theta+\alpha)^{-1} S_E^{\widetilde{V}_k}(\theta) \overline{B}_k(\theta) = R_{[\varphi]},$$

since $[\varphi] = \phi - (\deg B + l)\alpha$, let $B_k(\theta) = \overline{B}_k(\theta) R_{-(\deg B + l)\theta}$, then we have

$$B_k(\theta+\alpha)^{-1} S_E^{\widetilde{V}_k}(\theta) B_k(\theta) = R_{\phi}, \tag{18}$$

which means $(\alpha, S_E^{\widetilde{V}_k})$ is reduced to (α, R_ϕ). □

When we have Proposition 4.1, it is well known that for such ϕ, E and $\widetilde{V}_k$, E is an eigenvalue of $L_{\widetilde{V}_k,\phi,\alpha}$ with exponentially decay eigenfunction (consult [6, 16] for example), for the sake of self-completeness, we sketch the proof.

Just complexify the system, rewrite (18) as

$$B_k(\theta+\alpha)^{-1} S_E^{\widetilde{V}_k}(\theta) B_k(\theta) = \begin{pmatrix} e^{2\pi i\phi} & 0 \\ 0 & e^{-2\pi i\phi} \end{pmatrix},$$

and write

$$B_k(\theta) = \begin{pmatrix} z_{11}(\theta) & z_{12}(\theta) \\ z_{21}(\theta) & z_{22}(\theta) \end{pmatrix};$$

then we have

$$(E - \widetilde{V}_k(\theta)) z_{11}(\theta) = z_{11}(\theta-\alpha) e^{-2\pi i\phi} + z_{11}(\theta+\alpha) e^{2\pi i\phi}. \tag{19}$$

Look at the Fourier transformation of (19), we have

$$\sum_{m\in\mathbb{Z}} \widetilde{V}_k(m) \hat{z}_{11}(n-m) + 2\cos 2\pi(\phi + n\alpha) \hat{z}_{11}(n) = E \hat{z}_{11}(n),$$

i.e., $L_{\widetilde{V}_k,\phi,\alpha} \hat{z}_{11} = E \hat{z}_{11}$, moreover, the eigenfunction $\{\hat{z}_{11}(n)\}_{n\in\mathbb{Z}}$ is exponentially decay since $z_{11} \in C^{\omega}_{h_*}(\mathbb{T}, \mathbb{C})$.

4.2. Proof of Theorem 1.2 (*b*). According to Theorem 1.1, if $\beta(\alpha) = \infty$, then for any finite degree trigonometric polynomials V, the finite range operator $L_{V,\phi,\alpha}$ has no point spectrum for any ϕ. Since trigonometric polynomials are dense in $C^{\omega}_h(\mathbb{T}, \mathbb{R})$, we conclude that if $\beta = \infty$, then there is a dense set of $V \in C^{\omega}_h(\mathbb{T}, \mathbb{R})$, such that the operator $L_{V,\phi,\alpha}$ has no point spectrum for any $\phi \in \mathbb{T}$.

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