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Uniqueness and stability of Lamé parameters in elastography

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Abstract. This paper concerns an hybrid inverse problem involving elastic measurements called Transient Elastography (TE) which enables detection and characterization of tissue abnormalities. In this paper we assume that the displacements are modeled by linear isotropic elasticity system and the tissue displacement has been obtained by the first step in hybrid methods. We reconstruct Lamé parameters of this system from knowledge of the tissue displacement. More precisely, we show that for a sufficiently large number of solutions of the elasticity system and for an open set of the well-chosen boundary conditions, Lamé parameters can be uniquely and stably reconstructed. The set of well-chosen boundary conditions is characterized in terms of appropriate complex geometrical optics solutions.

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Contents

1. Introduction

Medical imaging is the technique and process used to create images of the human body for clinical purposes or medical science. Each available imagine method has its advantages and disadvantages. Medical imaging modalities such as Computerized Tomography (CT), Magnetic Resonance Imaging (MRI) and Ultrasound Imaging (UI) are examples of modalities providing high resolution. In some situations, these modalities fail to exhibit a sufficient contrast between different types of tissues. For instance, in breast imaging ultrasound provides high resolution, while suffers from a low contrast. Other modalities, based on optical, elastic, or electrical properties of these tissues, display high contrast, such as Optical Tomography (OT) and Electrical Impedance Tomography (EIT). For example, some breast tumors on early stages might have no contrast with the healthy tissues with respect to ultrasound propagation, but a huge contrast in their optical and electric properties.

In order to obtain better image, the natural idea is to try to combine different imaging modalities. These are coupled-physics imaging methods, also called hybrid methods. It is to combine the high resolution modality with another high contrast modality. Examples of possible physical couplings include: Photo-Acoustic Tomography (PAT) and Thermo-Acoustic Tomography (TAT), Ultrasound modulated Optical Tomography (UMOT) and elasticity with ultrasound in Transient Elastography (TE).

Reconstructions in hybrid inverse problems involve two steps. The first step is to solve the inverse problem concerning the high-resolution-low-contrast modality. For instance, in PAT and TAT, this corresponds to the reconstructing the initial condition of a wave equation from boundary measurements. In TE, this is to solving an inverse scattering problem in a time-dependent wave equation. In this paper, we assume that this first step has been performed. We will focus on the second step which is to reconstruct the coefficients that display high contrast from the mappings obtained during the first step. For more details about hybrid inverse problems, please refer to [1].

In this paper, the modality we consider is Transient Elastography. TE is a non-invasive tool for measuring liver stiffness. The device creases high-resolution shear stiffness images of human tissue for diagnostic purposes. Shear stiffness is targeted because shear wave speed is larger in abnormal tissue than in normal tissue. In the experiment tissue initially is excited with pulse at the boundary. This pulse creates the shear wave passing through the liver tissue. Then the tissue displacement is measured by using the ultrasound. The displacement is related to the tissue stiffness because the soft tissue has larger deformation than stiff tissue.

When we have tissue displacement, we want to reconstruct shear modulus μ and the first parameter λ. See [8], [9] and references there for more details. In TE, the high resolution modality is also ultrasound. The tissue displacement data can be obtained by the ultrasound in the first step. The second step is to recover the Lamé parameters from the knowledge of the tissue displacement. In the following paper, we will assume the first step has been performed.

We will formulate now the mathematical problem. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open bounded domain with smooth boundary. Let $u \in H^4(\mathbb{R}^n)$ be the displacement satisfying the linear isotropic elasticity system

$$\begin{cases} \nabla \cdot (\lambda(\nabla \cdot u)I + 2S(\nabla u)\mu) + k^2 u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $S(A) = (A + A^T)/2$ is the symmetric part of the matrix A. Here (λ, μ) are Lamé parameters and $k \in \mathbb{R}$ is the frequency. For the forward problem of elasticity system, we refer the readers to [4]. Assume that k is not the eigenvalue of the elasticity system. The set of internal functions obtained by ultrasound in TE is given by

$$H(x) = (u^{(j)}(x))_{1 \le j \le J} \quad \text{in } \Omega$$

for some positive integer J.

Hybrid inverse problems for the modality TE has less results than PAT and TAT due to its complicated equations. The first result, to the author's best knowledge, is considered by Bal and Uhlmann [2] in a scalar model for the elastic displacement u, in which the parameters are uniquely and stably determined by the internal data. They also gave an explicit reconstruction procedure for those parameters. However, the scalar model is not accurate in many applications. Our main contribution in this paper is to consider a more general setting, that is, the linear isotropic elasticity system and provide uniqueness and stability results. To study the elasticity system we applied the CGO solutions due to [5] and [10]. We used the strategies in [2] to obtain a transport equation for μ or λ, then we can recover the parameters by following the methods of [3].

Set

$$\begin{aligned} \mathcal{P} = \{(\lambda, \mu) \in C^7(\bar{\Omega}) \times C^9(\bar{\Omega}) \colon 0 < m \le \|\lambda\|_{C^7(\bar{\Omega})}, \\ \|\mu\|_{C^9(\bar{\Omega})} \le M, \text{ and} \\ \lambda,\ \mu > 0\}. \end{aligned}$$

Let H and $\widetilde{H}$ be two sets of internal data of the elasticity system with parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively. Below is our main result:

Theorem 1.1. *Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary. Suppose that the Lamé parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu}) \in \mathcal{P}$ and $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$. Let $u^{(j)}$ and $\tilde{u}^{(j)}$ be the solutions of the elasticity system with boundary data $g^{(j)}$ for parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively. Let $H = (u^{(j)})_{1\le j\le J}$ and $\widetilde{H} = (\tilde{u}^{(j)})_{1\le j\le J}$ be the corresponding internal data for (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively for some integer $J \ge 3n + 1$.*

Then there is an open set of the boundary data $(g^{(j)})_{1\le j\le J}$ such that if $H = \widetilde{H}$ implies $(\lambda, \mu) = (\tilde{\lambda}, \tilde{\mu})$ in Ω.
Moreover, we have the stability estimate

$$\|\mu - \tilde{\mu}\|_{C(\Omega)} + \|\lambda - \tilde{\lambda}\|_{C(\Omega)} \le C\|H - \widetilde{H}\|_{C^2(\Omega)}.$$

The remainder of this paper is organized as follows. In Section 2, we introduce the CGO solutions of the elasticity system. Section 3 is devoted to constructing the Lamé parameters in the two-dimensional case. The reconstruction of (λ, μ) in 3D is presented in Section 4.

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2. Complex geometric optics solutions of the elasticity system

In this section, we will briefly introduce the complex geometric optics (CGO) solutions of the elasticity system. Based on [6](or see [5]), we can derive the following reduced system. Let $w, f \in H^5(\mathbb{R}^n)$ and $(w, f)^T$ satisfy

$$\Delta\begin{pmatrix} w \\ f \end{pmatrix} + V_1(x)\begin{pmatrix} \nabla f \\ \nabla\cdot w \end{pmatrix} + V_0(x)\begin{pmatrix} w \\ f \end{pmatrix} = 0. \tag{2}$$

Here V_0 contains the third derivative of μ and

$$V_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2\mu^{-1} + \mu^{-3/2}k^2 & -\nabla\log\mu \\ 0 & \dfrac{\lambda+\mu}{\lambda+2\mu}\mu^{1/2} \end{pmatrix}.$$

Then the solution of the elasticity system (1) is

$$u := \mu^{-1/2}w + \mu^{-1}\nabla f - f\nabla\mu^{-1}.$$

Here $\nabla^2 g$ denotes the Hessian matrix $\partial^2 g/\partial x_i \partial x_j$. Note that we will not need the explicit form of V_0 in the construction of CGO solution.

The construction of CGO solutions of (2) with linear phase was first deduced by Nakamura and Uhlmann in [10] and [11], where they introduced the intertwining property in handling the first order terms. Eskin and Ralston [5] also gave a similar result in 2002. Later Uhlmann and Wang [12] used Carleman estimate to deduce the CGO solutions in the two-dimensional case.

In the following sections, we will use the CGO solutions constructed in Eskin and Ralston's paper [5].

To construct CGO solutions of (2), it is convenient to work on $\mathbb{R}^n$ instead of Ω. Since Ω is bounded, we pick a ball B_R for $R > 0$, such that $\bar{\Omega} \subset B_R$ and extend λ and μ to $\mathbb{R}^n$ by preserving its smoothness and also suppλ, supp$\mu \subset B_R$.

Let α and β be two orthogonal unit vectors in $\mathbb{R}^n$. Denote that $\rho = \tau(\alpha + i\beta) \in \mathbb{C}^n$ and $\theta = \alpha + i\beta$ with $\tau > 0$. Eskin and Ralston [5] showed the following important result in the three-dimensional case. For $n = 2$, it still holds (see reference [10]).

Lemma 2.1. (*Eskin-Ralston*) *Consider the Schrödinger equation with external Yang-Mills potentials*

$$Lu = -\Delta u - 2iA(x) \cdot \nabla u + B(x)u = 0, \quad x \in B_R \subset \mathbb{R}^n \tag{3}$$

where $A(x) = (A_1(x), \ldots, A_n(x)) \in C^{n_0}(B_R)$, $n_0 \geq 6$ *with* $A_j(x)$ *and* $B(x)$ *are* $(n+1) \times (n+1)$ *matrices. Then there are solutions of* (3) *of the form*

$$u = e^{i\rho\cdot x}(C_0(x,\theta)g(\theta \cdot x) + O(\tau^{-1}))$$

where $C_0 \in C^{n_0}(B_R)$ *is the solution of*

$$i\theta \cdot \frac{\partial}{\partial x} C_0(x,\theta) = \theta \cdot A(x) C_0(x,\theta)$$

with $\det C_0$ *is never zero,* $g(z)$ *is an arbitrary polynomial in complex variables* z, *and* $O(\tau^{-1})$ *is bounded by* $C(1/\tau)$ *in* $H^l(B_R)$, $0 \leq l \leq n_0 - 2$.

By the lemma above, the CGO solutions of (2) can be written as follows:

$$\begin{pmatrix} w \\ f \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r \\ s \end{pmatrix} + O(\tau^{-1})\right),$$

where $(r,\ s)^T$ is $C_0(x,\theta)g(\theta \cdot x)$. We can write $w = e^{i\rho\cdot x}(r + O(\tau^{-1}))$ and $f = e^{i\rho\cdot x}(s + O(\tau^{-1}))$. Plug $(w, f)^T$ into $u = \mu^{-1/2}w + \mu^{-1}\nabla f - f\nabla\mu^{-1}$.

Then we have the CGO solutions of the elasticity system. Note that $(r, s)^T$ satisfies the equation

$$-2\theta \cdot \nabla \begin{pmatrix} r \\ s \end{pmatrix} = V_1(x) \begin{pmatrix} 0_{n\times n} & \theta \\ \theta^T & 0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}. \tag{4}$$

Remark 1. Since C_0 is invertible at every point in Ω, we can conclude that $(r, s)^T$ is not zero everywhere in Ω. This does not imply that both r and s never vanish in Ω. However, for any point $y \in \Omega$, there is a small neighborhood B_y of y in Ω and a CGO solution of $(w, f)^T$ such that both r and s do not vanish in B_y.

3. Reconstruction of Lamé parameters in two dimensional case

In the previous section, we already have the CGO solutions of the elasticity system. Now we want to use them to give a reconstruction of μ first. Let $u = (u_1, u_2)^T$ be the displacement which satisfies the elasticity system

$$\nabla \cdot (\lambda(\nabla \cdot u)I + 2S(\nabla u)\mu) + k^2 u = 0. \tag{5}$$

We will recover μ and λ separately in the following two subsections.

3.1. Reconstruction of μ in 2D. From (5), we can deduce the following equation

$$u^\sharp \cdot F + u^\flat \cdot G = -k^2 u^*. \tag{6}$$

Here we denote

$$u^\sharp = \begin{pmatrix} \partial_1(\nabla \cdot u) \\ \partial_2(\nabla \cdot u) \\ \nabla \cdot u \\ \nabla \cdot u \end{pmatrix}, \quad F = \begin{pmatrix} \lambda + \mu \\ \lambda + \mu \\ \partial_1(\lambda + \mu) \\ \partial_2(\lambda + \mu) \end{pmatrix}, \tag{7a}$$

$$u^\flat = \begin{pmatrix} a + b \\ a - b \\ \partial_1(a + b) \\ \partial_2(a - b) \end{pmatrix}, \quad G = \begin{pmatrix} \partial_1 \mu \\ \partial_2 \mu \\ \mu \\ \mu \end{pmatrix}, \tag{7b}$$

with $a = \partial_2 u_1 + \partial_1 u_2$, $b = \partial_1 u_1 - \partial_2 u_2$, and $u^* = (u_1 + u_2)$. The component $u^\sharp$ and $u^\flat$ are known since they are vectors which only depend on the internal data u. In order to recover μ, we want to eliminate the first term of (6) so that μ satisfies the transport equation.

Obverse that the vector $u^\sharp$ has three different entries, we only need to construct three linearly independent vectors on some subdomain in Ω. With these three vectors, we can remove the first term of the left hand side of the equation (6). More precisely, suppose that $u^{(j)}$, for $j = 0, 1, 2$, are three different solutions of (6) which satisfy

$$u^{(j)\sharp} \cdot F + u^{(j)\flat} \cdot G = -k^2 u^{(j)*}. \tag{8}$$

The notations $\Re f$ and $\Im f$ are defined to be the real and imaginary part of f, respectively. Now for $j = 1, 2, 3$ and $\natural = \sharp, \flat, *$, we let

$$\mathrm{u}_1^{(0)\natural} = \Re\chi(x)u^{(0)\natural}, \quad \mathrm{u}_2^{(0)\natural} = \Im\chi(x)u^{(0)\natural}$$

and

$$\begin{aligned}
\mathrm{u}^{(1)\natural} &= \Re\chi_3(x)u^{(1)\natural},\\
\mathrm{u}^{(2)\natural} &= \Im\chi_3(x)u^{(1)\natural},\\
\mathrm{u}^{(3)\natural} &= \Re(\chi_1(x)u^{(1)\natural} + \chi_2(x)u^{(2)\natural}),
\end{aligned}$$

where $\chi_j(x)$ is a nonzero function. Then we get

$$\mathrm{u}_l^{(0)\sharp} \cdot F + \mathrm{u}_l^{(0)\flat} \cdot G = -k^2 \mathrm{u}_l^{(0)*}, \quad l = 1, 2 \tag{9}$$

and

$$\mathrm{u}^{(j)\sharp} \cdot F + \mathrm{u}^{(j)\flat} \cdot G = -k^2 \mathrm{u}^{(j)*}, \quad j = 1, 2, 3. \tag{10}$$

Assume that $\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ are three linearly independent vectors in some subdomain of Ω, say Ω_0. Then there exist three functions Θ_1^l, Θ_2^l, and Θ_3^l such that $\mathrm{u}_l^{(0)\sharp} + \sum_{j=1}^3 \Theta_j^l \mathrm{u}^{(j)\sharp} = 0$. For $l = 1, 2$, multiplying (10) by Θ_j^l and summing over j with equation (9), we have

$$\mathrm{v}_l \cdot G = -k^2\Big(\mathrm{u}_l^{(0)*} + \sum_{j=1}^3 \Theta_j^l \mathrm{u}^{(j)*}\Big),$$

where $\mathrm{v}_l = \mathrm{u}_l^{(0)\flat} + \sum_{j=1}^3 \Theta_j^l \mathrm{u}^{(j)\flat}$. Let $\beta_l = (\mathrm{v}_l \cdot e_1)\tilde{e}_1 + (\mathrm{v}_l \cdot e_2)\tilde{e}_2$ and $\gamma_l = (e_3 + e_4) \cdot \mathrm{v}_l$. Here $e_j \in \mathbb{R}^4$, $\tilde{e}_j \in \mathbb{R}^2$ with the j^{th} entry is 1 and others are zero. Then the above equation can be rewritten as

$$\beta_l \cdot \nabla\mu + \gamma_l \mu = -k^2\Big(\mathrm{u}_l^{(0)*} + \sum_{j=1}^3 \Theta_j^l \mathrm{u}^{(j)*}\Big).$$

Suppose that $\beta_1(x)$ and $\beta_2(x)$ are linearly independent for every $x \in \Omega_0$. Then we can recover μ in $\Omega_0 \subset \Omega$ for each frequency k independently.

Lemma 3.1. *Let $u^{(j)}$ for $0 \leq j \leq 2$ be C^2 solutions of the elasticity system with boundary conditions $u^{(j)} = g^{(j)}$ on $\partial\Omega$. Let us define*

$$u^\sharp = (\partial_1(\nabla \cdot u),\ \partial_2(\nabla \cdot u),\ \nabla \cdot u,\ \nabla \cdot u)^T$$

and assume that

(1) $\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ *are three linearly independent vectors in Ω_0, the neighborhood of x_0 in Ω.*

(2) $\{\beta_1(x), \beta_2(x)\}$ *are linearly independent in Ω_0.*

Then the reconstruction is stable in the sense that

$$\|\mu - \tilde{\mu}\|_{C(\Omega_0)} \leq C(|\mu(x_0^+) - \tilde{\mu}(x_0^+)| + \|H_{x_0} - \widetilde{H}_{x_0}\|_{C^2(\Omega_0)}) \tag{11}$$

where $x_0^+ \in \partial\Omega_0$ and $H_{x_0} = (u^{(j)})_{0 \leq j \leq 2}$.

Proof. Since $\{\beta_1(x), \beta_2(x)\}$ are linearly independent in Ω_0, we can construct two vector-valued functions $\Gamma(x), \Phi(x) \in C(\Omega_0)$ such that

$$\nabla\mu + \Gamma(x)\mu = \Phi(x) \quad \text{in } \Omega_0. \tag{12}$$

Since μ and $\tilde{\mu}$ are solutions of (12) with coefficients (Γ, Φ) and $(\widetilde{\Gamma}, \widetilde{\Phi})$, respectively, we have

$$\nabla(\mu - \tilde{\mu}) + \Gamma(x)(\mu - \tilde{\mu}) = \tilde{\mu}(\widetilde{\Gamma}(x) - \Gamma(x)) + (\Phi(x) - \widetilde{\Phi}(x)).$$

Let $x \in \Omega_0$, there exists a integral curve $\psi(t)$ with $\psi(0) = x_0^+$ and $\psi(1) = x$. Thus

$$\begin{aligned}(\mu - \tilde{\mu})(\psi(t)) =& (\mu - \tilde{\mu})(\psi(0))e^{-\int_0^t \Gamma(\psi(s))\cdot\psi'(s)ds} \\ &+ \int_0^t \tilde{\mu}(\Gamma(\psi(s)) - \widetilde{\Gamma}(\psi(s))) + (\Phi - \widetilde{\Phi})(\psi(s)) \cdot \psi'(s)ds,\end{aligned} \tag{13}$$

for $t \in [0,1]$. Since Γ and Φ only depend on Ω and $u^{(j)}$, $j = 0,1,2$, and $u^{(j)}$ are in the class of $C^2(\overline{\Omega})$, the value of $|\Gamma - \widetilde{\Gamma}|$ and $|\Phi - \widetilde{\Phi}|$ are bounded by the sum of $|\partial^\alpha u^{(j)} - \partial^\alpha \tilde{u}^{(j)}|$ for $|\alpha| \leq 2$. There is a constant C such that

$$|(\mu - \tilde{\mu})(\psi(t))| \leq C|(\mu - \tilde{\mu})(\psi(0))| + C\|H_{x_0} - \widetilde{H}_{x_0}\|_{C^2(\Omega_0)}, \quad t \in [0,1]$$

for $\mu,\ \tilde{\mu} \in \mathcal{P}$. Thus, for any $x \in \Omega_0$, the value $|(\mu - \tilde{\mu})(x)|$ is controlled by the internal data and μ at the boundary point $x_0 \in \partial\Omega_0$. $\square$

3.1.1. Global reconstruction of μ in 2D. Let $(\lambda, \mu) \in \mathcal{P}$, by Lemma 2.1, it implies that r and s are in C^7 and $O(\tau^{-1}) \in H^5$. Now we will show that how we can get three linear independent vectors of the form $\mathrm{u}^\sharp$. We plug the CGO solutions $u_\rho = \mu^{-1/2} w_\rho + \mu^{-1} \nabla f_\rho - f_\rho \nabla \mu^{-1}$ into $u^\sharp$. Then we have the expression

$$u_\rho^\sharp = e^{i\rho\cdot x} \frac{\sqrt{\mu}}{\lambda + 2\mu} \left(\begin{pmatrix} -\rho_1(\rho \cdot r) + O(|\rho|) \\ -\rho_2(\rho \cdot r) + O(|\rho|) \\ i\rho \cdot r \\ i\rho \cdot r \end{pmatrix} + O(1) \right) \tag{14}$$

by using the following equality which is part of the equation (4)

$$-2\rho \cdot \nabla s = \frac{\lambda + \mu}{\lambda + 2\mu} \mu^{1/2} \rho \cdot r. \tag{15}$$

Now we fix any point $x_0 \in \bar{\Omega}$ and let $\rho = \tau(1, i) = -i\tilde{\rho} \in \mathbb{C}^2$ with $\tau > 0$. Since, in Lemma 2.1, the matrix solution $C_0(x, \theta)$ is invertible, we can choose a constant vector $g^{(0)}$ such that $C_0(x, \theta) g^{(0)} = (r^{(0)}, s^{(0)})^T$ with the conditions

$$s^{(0)}(x_0) = 1, \; s^{(0)} \neq 0 \quad \text{and} \quad \rho \cdot r^{(0)} \neq 0$$

in a neighborhood of x_0 in B_R, say U. Then we have the CGO solution of the elasticity system, that is,

$$u_\rho^{(0)} = \mu^{-1/2} w_\rho^{(0)} + \mu^{-1} \nabla f_\rho^{(0)} - f_\rho^{(0)} \nabla \mu^{-1},$$

where

$$\begin{pmatrix} w_\rho^{(0)} \\ f_\rho^{(0)} \end{pmatrix} = e^{i\rho\cdot x} \left(\begin{pmatrix} c r^{(0)} \\ s^{(0)} \end{pmatrix} + O(\tau^{-1}) \right).$$

Let $\theta = \rho/\tau, \tilde{\theta} = \tilde{\rho}/\tau$. Let $C_1(x, \theta)$ and $C_2(x, \tilde{\theta})$ satisfy the following two equations

$$i\theta \cdot \frac{\partial}{\partial x} C_1(x, \theta) = \theta \cdot V_1(x) C_1(x, \theta), \quad i\tilde{\theta} \cdot \frac{\partial}{\partial x} C_2(x, \tilde{\theta}) = \tilde{\theta} \cdot V_1(x) C_2(x, \tilde{\theta}),$$

respectively. Since $\tilde{\rho} = i\rho$, we can choose $C_2(x, \tilde{\theta}) = C_1(x, \theta)$. By Lemma 2.1, the matrix solution $C_1(x, \theta)$ is invertible, we can choose a suitable constant vector g such that $C_1(x, \theta) g = (r^{(1)}, s^{(1)})^T$, $C_2(x, \tilde{\theta}) g = (r^{(2)}, s^{(2)})^T$ and

$$s^{(l)}(x_0) = 0\,, \;\; r^{(1)}(x_0) = (1, -i) = r^{(2)}(x_0).$$

Note that $r^{(2)} = r^{(1)}$ and $s^{(2)} = s^{(1)}$. By continuity of $r^{(l)}$, we have $\rho \cdot r^{(l)} \neq 0$ in a neighborhood U of x_0. Then the CGO solutions are

$$u_\rho^{(l)} = \mu^{-1/2} w_\rho^{(l)} + \mu^{-1} \nabla f_\rho^{(l)} - f_\rho^{(l)} \nabla \mu^{-1},$$

where

$$\begin{pmatrix} w_\rho^{(1)} \\ f_\rho^{(1)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(1)} \\ s^{(1)} \end{pmatrix} + O(\tau^{-1}) \right),$$

and

$$\begin{pmatrix} w_\rho^{(2)} \\ f_\rho^{(2)} \end{pmatrix} = e^{i\tilde{\rho}\cdot x}\left(\begin{pmatrix} r^{(2)} \\ s^{(2)} \end{pmatrix} + O(\tau^{-1}) \right).$$

So far we have three CGO solutions, that is, $u_\rho^{(0)},\ u_\rho^{(1)}$, and $u_\rho^{(2)}$.

We consider

$$e^{-i\rho\cdot x}|\rho|^{-2} u_\rho^{(1)\sharp} = \frac{1}{2} \begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}(r_1^{(1)} + i r_2^{(1)}) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}(i r_1^{(1)} - r_2^{(1)}) \\ 0 \\ 0 \end{pmatrix} + O(\tau^{-1}),$$

$$e^{-i\rho\cdot x}|\rho|^{-1} u_\rho^{(1)\sharp} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}\tau(r_1^{(1)} + i r_2^{(1)}) + O(1) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}\tau(i r_1^{(1)} - r_2^{(1)}) + O(1) \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}(r_1^{(1)} + i r_2^{(1)}) \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}(r_1^{(1)} + i r_2^{(1)}) \end{pmatrix} + O(\tau^{-1})$$

and

$$e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1} u_\rho^{(2)\sharp} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}\tau(-r_1^{(2)} - i r_2^{(2)}) + O(1) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}\tau(-i r_1^{(2)} + r_2^{(2)}) + O(1) \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}(i r_1^{(2)} - r_2^{(2)}) \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}(i r_1^{(2)} - r_2^{(2)}) \end{pmatrix} + O(\tau^{-1}).$$

Since $r^{(2)} = r^{(1)}$ and $s^{(2)} = s^{(1)}$, we have

$$\begin{aligned}
&e^{-i\rho\cdot x}|\rho|^{-1}u_\rho^{(1)\sharp} + e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1}u_\rho^{(2)\sharp} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} O(1) \\ O(1) \\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}((1+i)r_1^{(1)} + (-1+i)r_2^{(1)}) \\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}((1+i)r_1^{(1)} + (-1+i)r_2^{(1)}) \end{pmatrix} + O(\tau^{-1}).
\end{aligned}$$

We define

$$\mathrm{u}_{1\rho}^{(0)\natural} = \Re e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(0)\natural}, \quad \mathrm{u}_{2\rho}^{(0)\natural} = \Im e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(0)\natural}, \quad \natural = \sharp, \flat, *$$
$$\mathrm{u}_{\rho}^{(1)\natural} = \Re e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(1)\natural}, \quad \mathrm{u}_{\rho}^{(2)\natural} = \Im e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(1)\natural},$$

and

$$\mathrm{u}_{\rho}^{(3)\natural} = \Re(e^{-i\rho\cdot x}|\rho|^{-1}u_\rho^{(1)\natural} + e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1}u_\rho^{(2)\natural}).$$

Then $\{\mathrm{u}_\rho^{(1)\sharp}, \mathrm{u}_\rho^{(2)\sharp}, \mathrm{u}_\rho^{(3)\sharp}\}$ are linearly independent in a small neighborhood U of x_0 when τ is sufficiently large.

Therefore, for $l = 1, 2$, there exist functions Θ_j^l, $j = 1, 2, 3$ such that

$$\mathrm{u}_{l\rho}^{(0)\sharp} + \sum_{j=1}^{3}\Theta_j^l \mathrm{u}_\rho^{(j)\sharp} = 0, \quad l = 1, 2. \tag{16}$$

Since $u_\rho^{(0)}$, $u_\rho^{(1)}$, and $u_\rho^{(2)}$ are solutions of the equation (6), we have the equations

$$\mathrm{u}_{l\rho}^{(0)\sharp}\cdot F + \mathrm{u}_{l\rho}^{(0)\flat}\cdot G = -k^2\mathrm{u}_{l\rho}^{(0)*}, \quad l = 1, 2,$$
$$\mathrm{u}_{\rho}^{(j)\sharp}\cdot F + \mathrm{u}_{\rho}^{(j)\flat}\cdot G = -k^2\mathrm{u}_{\rho}^{(j)*}, \quad j = 1, 2, 3.$$

Summing over the equations above and using (16), we have two equations

$$\beta_{\rho,l}\cdot\nabla\mu + \gamma_{\rho,l}\mu = -k^2\Big(\mathrm{u}_{l\rho}^{(0)*} + \sum_{j=1}^{3}\Theta_j^l\mathrm{u}_\rho^{(j)*}\Big), \quad l = 1, 2, \tag{17}$$

where

$$\beta_{\rho,l} = (\mathrm{v}_l\cdot e_1)\tilde{e}_1 + (\mathrm{v}_l\cdot e_2)\tilde{e}_2$$

and

$$\gamma_{\rho,l} = \mathrm{v}_l\cdot e_3 + \mathrm{v}_l\cdot e_4,$$

Here we define $\mathrm{v}_l = \mathrm{u}_{l\rho}^{(0)\flat} + \sum_{j=1}^{3}\Theta_j^l\mathrm{u}_\rho^{(j)\flat}$.

Remark 2. By choosing suitable $g^{(0)}$, $|\Theta^l_3(x_0)|$ can be as small as we want. To show that, we choose a new constant vector $\hat{g}^{(0)}$, instead of the original $g^{(0)}$, such that $(\hat{r}^{(0)}, \hat{s}^{(0)})^T = C_0(x,\theta)\hat{g}^{(0)}(\theta\cdot x)$ where $\hat{r}^{(0)}(x_0) = r^{(0)}(x_0)/M$, $M > 0$, $\hat{s}^{(0)}(x_0) = s^{(0)}(x_0)$ and $(\hat{r}^{(0)}, \hat{s}^{(0)})^T$ satisfies the original assumption, that is, $\hat{s}^{(0)}(x_0) = 1$ and $\hat{s}^{(0)} \neq 0$ and $\rho\cdot\hat{r}^{(0)} \neq 0$ in a neighborhood of x_0. Then

$$\Theta^l_3(x_0) = -P^{3j}\Re\left(\frac{1}{2}\begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}(1,i)\cdot r^{(0)}(x_0) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}i(1,i)\cdot r^{(0)}(x_0) \\ 0 \\ 0 \end{pmatrix} + O(\tau^{-1})\right)\cdot p_j$$

and

$$\widehat{\Theta}^l_3(x_0) = -P^{3j}\Re\left(\frac{1}{2M}\begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}(1,i)\cdot r^{(0)}(x_0) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}i(1,i)\cdot r^{(0)}(x_0) \\ 0 \\ 0 \end{pmatrix} + O(\tau^{-1})\right)\cdot p_j,$$

where $P = (p_{ij})$ and $P^{-1} = (P^{ij})$ with $p_{ij} = p_i\cdot p_j$. Note that P is a boundedly invertible symmetric matrix. Here $p_1 = \mathrm{u}^{(1)\sharp}_\rho$, $p_2 = \mathrm{u}^{(2)\sharp}_\rho$, $p_3 = \mathrm{u}^{(3)\sharp}_\rho$. From above, we obtain that $\widehat{\Theta}^l_3$ is close to Θ^l_3/M as τ is large. Therefore, the new $|\widehat{\Theta}^l_3(x_0)|$ is small when M and τ is sufficiently large.

Lemma 3.2. *Given any point $x_0 \in \bar{\Omega}$, there exists a small neighborhood V of x_0 such that $\beta_{\rho,j}$ is not zero in V for $j = 1, 2$.*

Note that we denote by $\sim$ equalities up to terms which are asymptotically negligible as τ goes to infinity.

Proof. It is sufficiently to prove that $\beta_{\rho,1}$ does not vanish in some neighborhood of x_0. Recall that

$$\beta_{\rho,1} = (\mathrm{v}_1\cdot e_1)\tilde{e}_1 + (\mathrm{v}_1\cdot e_2)\tilde{e}_2, \tag{18}$$

where

$$\mathrm{v}_1 = \mathrm{u}^{(0)\flat}_{1\rho} + \sum_{j=1}^3 \Theta^1_j \mathrm{u}^{(j)\flat}_\rho.$$

By (16), we have that $\Theta_j^1 \sim \tau^0$ for $j = 1, 2, 3$. Since $s^{(1)}(x_0) = 0$, we get that

$$((\mathrm{u}_\rho^{(1)\flat} \cdot e_1)\tilde{e}_1 + (\mathrm{u}_\rho^{(1)\flat} \cdot e_2)\tilde{e}_2)(x_0) \sim \tau^{-1} \tag{19}$$

and

$$((\mathrm{u}_\rho^{(2)\flat} \cdot e_1)\tilde{e}_1 + (\mathrm{u}_\rho^{(2)\flat} \cdot e_2)\tilde{e}_2)(x_0) \sim \tau^{-1}. \tag{20}$$

Since $s^{(k)}(x_0) = 0$, $r^{(2)} = r^{(1)}$, $s^{(2)} = s^{(1)}$, and $(1, -i) \cdot r^{(1)}(x_0) = 0$, we obtain that

$$\begin{aligned} &((\mathrm{u}_\rho^{(3)\flat} \cdot e_1)\tilde{e}_1 + (\mathrm{u}_\rho^{(3)\flat} \cdot e_2)\tilde{e}_2)(x_0) \\ &\quad = 4\mu^{-1}\Re\frac{1}{\sqrt{2}}\left(\begin{pmatrix} i\partial_1 s^{(1)} + \partial_2 s^{(1)} \\ -\partial_1 s^{(1)} + i\partial_2 s^{(1)} \end{pmatrix} + O(\tau^{-1})\right)(x_0). \end{aligned} \tag{21}$$

Combining from (18) to (21), since $\Theta_3^1(x_0)$ can be taken as small as we want (See Remark 2) in the construction of CGO solutions above, it follows that

$$\beta_{\rho,1}(x_0) \sim \mu^{-1}\left(\begin{pmatrix} 2 \\ -2 \end{pmatrix} + \Theta_3^1\Re\begin{pmatrix} \frac{4}{\sqrt{2}}(i\partial_1 s^{(1)} + \partial_2 s^{(1)}) \\ \frac{4}{\sqrt{2}}(-\partial_1 s^{(1)} + i\partial_2 s^{(1)}) \end{pmatrix}\right)(x_0) \neq 0.$$

Similarly,

$$\beta_{\rho,2}(x_0) \sim \mu^{-1}\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \Theta_3^2\Re\begin{pmatrix} \frac{4}{\sqrt{2}}(i\partial_1 s^{(1)} + \partial_2 s^{(1)}) \\ \frac{4}{\sqrt{2}}(-\partial_1 s^{(1)} + i\partial_2 s^{(1)}) \end{pmatrix}\right)(x_0) \neq 0.$$

By continuity of $\beta_{\rho,j}$, we complete the proof. □

Let $g_\rho^{(j)} = u_\rho^{(j)}|_{\partial\Omega}$ be the given boundary data. By Lemma 2.1, since $(\lambda, \mu) \in \mathcal{P}$, we knew that $u_\rho^{(j)} \in H^4(B_R)$. Let $g^{(j)} \in C^{1,\alpha}(\partial\Omega)$ be the boundary data close to $g_\rho^{(j)}$, that is,

$$\|g^{(j)} - g_\rho^{(j)}\|_{C^{1,\alpha}(\partial\Omega)} < \varepsilon \quad \text{for some } \varepsilon > 0, \tag{22}$$

then we can find solutions $u^{(j)}$ of the elasticity system with boundary data $g^{(j)}$(the existence of such solutions, see e.g. Ch.4 of [7]). By elliptic regularity theorem, we have

$$\|u^{(j)} - u_\rho^{(j)}\|_{C^2(\bar{\Omega})} < C\varepsilon \tag{23}$$

for some constant C which is independent of λ, μ. Then we obtain that

$$\|\mathrm{u}^{(j)\sharp} - \mathrm{u}_\rho^{(j)\sharp}\|_{C^0(\bar{\Omega})} < C\varepsilon.$$

Here the notation $\mathrm{u}^{(j)\sharp}$ is constructed in the same way as $\mathrm{u}_\rho^{(j)\sharp}$ with the CGO solutions $u_\rho^{(j)}$ replaced by the solutions $u^{(j)}$. Therefore, $\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ are also linearly independent when ε is sufficiently small.

We construct β_j as in the equation (17) with $u_\rho^{(j)}$ replaced by $u^{(j)}$, $j = 0, 1, 2$. Therefore, by the definitions of β_j and $\beta_{\rho,j}$ and (23), it follows that

$$\|\beta_j - \beta_{\rho,j}\|_{C^1(\bar{\Omega})}$$

is small when ε is small. Since $\beta_{\rho,j}$ is not zero in V by Lemma 3.2, we can deduce that β_j is also not zero in V if ε is small enough and τ is sufficiently large. Moreover, with the suitable chosen CGO solutions u_ρ, $\{\beta_{\rho,1}, \beta_{\rho,2}\}$ are linearly independent in V as τ is sufficiently large (See the proof of Lemma 3.2). When ε is sufficiently small, it implies that $\{\beta_1, \beta_2\}$ are also linearly independent in V. Then we have the equations

$$\beta_l \cdot \nabla\mu + \gamma_l \mu = -k^2\Big(\mathrm{u}_l^{(0)*} + \sum_{j=1}^{3} \Theta_j^l \mathrm{u}^{(j)*}\Big), \quad l = 1, 2 \tag{24}$$

with $\{\beta_1, \beta_2\}$ a basis in $\mathbb{R}^2$ for every point $x \in \Omega_0$. Here we denote

$$\Omega_0 = U \cap V \cap \Omega. \tag{25}$$

Thus, there exists an invertible matrix $A = (a_{ij})$ such that $\beta_l = \sum a_{lk}\tilde{e}_k$ with inverse of class $C^0(\Omega)$. Thus, we have constructed two vector-valued functions $\Gamma(x)$, $\Phi(x) \in C(\Omega)$ such that (24) can be rewritten as

$$\nabla\mu + \Gamma(x)\mu = \Phi(x) \quad \text{in } \Omega_0. \tag{26}$$

Then we obtain the following result:

Theorem 3.3. *Suppose that (λ, μ) and $(\tilde{\lambda}, \tilde{\mu}) \in \mathcal{P}$. For any fixed $x_0 \in \partial\Omega$, let $u_\rho^{(j)}$ be the corresponding CGO solutions for (λ, μ) and $\mathrm{u}^{(j)}$ constructed as above with internal data $H_{x_0} = (u^{(j)})_{0\le j\le 2}$ and with ε as in (22) sufficiently small. Let $\widetilde{H}_{x_0}$ be constructed similarly with the parameters $(\tilde{\lambda}, \tilde{\mu})$. Assume that $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$.*

Then $H_{x_0} = \widetilde{H}_{x_0}$ implies that $\mu = \tilde{\mu}$ in Ω_0 which is defined in (25), the neighborhood of x_0 in Ω.

Proof. Based on the construction above, the domain Ω_0 can be taken as a small open ball with center x_0 and $\Omega_0 \subset B_R$. Since $H_{x_0} = \widetilde{H}_{x_0}$, we have that μ and $\tilde{\mu}$ solve the same equation $\nabla\mu + \Gamma(x)\mu = \Phi(x)$ in Ω_0 where the functions Γ and Φ depend on $u^{(j)}$. Let $x \in \Omega_0$ and denote $\psi(t) = (1-t)x_0 + tx,\ t \in [0,1]$. Restricted to this curve, we have

$$\begin{cases} \psi'(t)\cdot\nabla\mu + \Gamma(\psi(t))\cdot\psi'(t)\mu = \Phi(\psi(t))\cdot\psi'(t) & \text{in } \Omega_0, \\ \mu(x_0) = \tilde{\mu}(x_0), \end{cases} \tag{27}$$

The solution of (27) is given by

$$\mu(\psi(t)) = \mu(\psi(0))e^{-\int_0^t \Gamma(\psi(s))\cdot\psi'(s)ds} + \int_0^t \Phi(\psi(s))\cdot\psi'(s)ds, \quad t \in [0,1].$$

The solution $\tilde{\mu}(x)$ is given by the same formula since $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$ so that $\mu = \tilde{\mu}$ in Ω_0. $\square$

We have constructed $\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ are linearly independent and $\{\beta_1, \beta_2\}$ forms a basis in $\mathbb{R}^2$ for every point $x \in \Omega_0$ when ε is sufficiently small and τ is large. Applying Lemma 3.1, we have

Theorem 3.4. *Suppose that (λ,μ) and $(\tilde{\lambda},\tilde{\mu}) \in \mathcal{P}$. For any fixed $x_0 \in \overline{\Omega}$, let $u_\rho^{(j)}$ be the corresponding CGO solutions for (λ,μ) and $u^{(j)}$ constructed as above with internal data $H_{x_0} = (u^{(j)})_{0\le j\le 2}$ and with ε as in (22) sufficiently small. Let $\widetilde{H}_{x_0} = (\tilde{u}^{(j)})_{0\le j\le 2}$ be constructed similarly for the parameters $(\tilde{\lambda},\tilde{\mu})$ with $u^{(j)}|_{\partial\Omega} = \tilde{u}^{(j)}|_{\partial\Omega}$. Then there exists an open neighborhood Ω_0 (which is defined in (25)) of x_0 in Ω such that*

$$\|\mu - \tilde{\mu}\|_{C(\Omega_0)} \le C(|\mu(x_0) - \tilde{\mu}(x_0)| + \|H_{x_0} - \widetilde{H}_{x_0}\|_{C^2(\Omega_0)}), \tag{28}$$

for $x_0 \in \partial\Omega$, and

$$\|\mu - \tilde{\mu}\|_{C(\Omega_0)} \le C(|\mu(x_0^+) - \tilde{\mu}(x_0^+)| + \|H_{x_0} - \widetilde{H}_{x_0}\|_{C^2(\Omega_0)}), \tag{29}$$

for $x_0^+ \in \partial\Omega_0$, and $x_0 \in \Omega$.

The global uniqueness and stability result are stated as follows.

Theorem 3.5 (Global reconstruction of μ). *Let Ω be an open bounded domain of $\mathbb{R}^2$ with smooth boundary. Suppose that the Lamé parameters (λ,μ) and $(\tilde{\lambda},\tilde{\mu}) \in \mathcal{P}$ and $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$. Let $u^{(j)}$ and $\tilde{u}^{(j)}$ be the solutions of the elasticity system with boundary data $g^{(j)}$ for parameters (λ,μ) and $(\tilde{\lambda},\tilde{\mu})$, respectively. Let $H = (u^{(j)})_{1\le j\le J}$ and $\widetilde{H} = (\tilde{u}^{(j)})_{1\le j\le J}$ be the corresponding internal data for (λ,μ) and $(\tilde{\lambda},\tilde{\mu})$, respectively for some integer $J \ge 3$.*

Then there is an open set of the boundary data $(g^{(j)})_{1\leq j\leq J}$ such that if $H = \widetilde{H}$ implies $\mu = \tilde{\mu}$ in Ω.
Moreover, we have the stability estimate

$$\|\mu - \tilde{\mu}\|_{C(\Omega)} \leq C\|H - \widetilde{H}\|_{C^2(\Omega)}.$$

Note that for the uniqueness of μ, we suppose that the two set of internal data are the same, that is, $H = \widetilde{H}$. Since μ is uniquely reconstructed near a fixed point $x_0 \in \partial\Omega$ under the condition $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$, from the stability of μ in Ω, we can obtain that $\mu = \tilde{\mu}$ in Ω.

Proof. In Section 2, we constructed CGO solutions in a ball B_R which contains $\bar{\Omega}$. First, we consider any point x in $\partial\Omega$. Then we can find an open neighborhood $B_x \subset B_R$ of x. By Theorem 3.4, we have the estimate

$$\|\mu - \tilde{\mu}\|_{C(B_x\cap\Omega)} \leq C\|H_x - \widetilde{H}_x\|_{C^2(\Omega)} \tag{30}$$

since $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$.

Second, for any point $y \in \Omega$, by Theorem 3.4, there exists an open neighborhood $B_y \subset \Omega$ of y with $\bar{B}_y \cap \partial\Omega = \emptyset$ such that

$$\|\mu - \tilde{\mu}\|_{C(B_y)} \leq C(|\mu(y^+) - \tilde{\mu}(y^+)| + \|H_y - \widetilde{H}_y\|_{C^2(\Omega)}) \quad \text{for some } y^+ \in \partial B_y. \tag{31}$$

Therefore, the compact set $\bar{\Omega}$ is covered by $\bigcup_{x\in\bar{\Omega}} B_x$. Then there exists finitely many B_x, say, $B_{x_1}, \ldots, B_{x_N}$, such that $\bar{\Omega} \subset \bigcup_{l=1}^{N} B_{x_l}$.

Now for arbitrary point $z \in \Omega$, there is B_{x_j} such that $z \in B_{x_j}$. Suppose that $\bar{B}_{x_j} \cap \partial\Omega \neq \emptyset$, this means that $x_j \in \partial\Omega$. Then, by (30), we have

$$|\mu(z) - \tilde{\mu}(z)| \leq C\|H_{x_j} - \widetilde{H}_{x_j}\|_{C^2(\Omega)}. \tag{32}$$

Otherwise, if $\bar{B}_{x_j} \cap \partial\Omega$ is empty, then, by (31), we get that

$$|\mu(z) - \tilde{\mu}(z)| \leq C(|\mu(x_j^+) - \tilde{\mu}(x_j^+)| + \|H_{x_j} - \widetilde{H}_{x_j}\|_{C^2(\Omega)}) \quad \text{for } x_j^+ \in \partial B_{x_j}. \tag{33}$$

For the point x_j^+, since $\bar{\Omega}$ is covered by finitely many subdomain B_{x_l}, after at most $N-1$ steps, we have

$$|\mu(x_j^+) - \tilde{\mu}(x_j^+)| \leq C\sum_{l\neq j,l=1}^{N} \|H_{x_l} - \widetilde{H}_{x_l}\|_{C^2(\Omega)}. \tag{34}$$

Combining (33) and (34). Then we have

$$|\mu(z) - \tilde{\mu}(z)| \leq C \sum_{l=1}^{N} \|H_{x_l} - \widetilde{H}_{x_l}\|_{C^2(\Omega)}. \tag{35}$$

With (32) and (35), we have the global stability

$$\|\mu - \tilde{\mu}\|_{C(\Omega)} \leq C \sum_{l=1}^{N} \|H_{x_l} - \widetilde{H}_{x_l}\|_{C^2(\Omega)}. \qquad \square$$

3.2. Reconstruction of λ in 2D. The elasticity system can also be written in this form

$$u^{\sharp} \cdot F + u^{\flat} \cdot G = -k^2 u^{*}, \tag{36}$$

where

$$u^{\sharp} = \begin{pmatrix} \nabla \cdot u \\ \nabla \cdot u \\ a+b \\ a-b \end{pmatrix}, \qquad F = \begin{pmatrix} \partial_1(\lambda+\mu) \\ \partial_2(\lambda+\mu) \\ \partial_1 \mu \\ \partial_2 \mu \end{pmatrix},$$

$$u^{\flat} = \begin{pmatrix} \partial_1(\nabla \cdot u) \\ \partial_2(\nabla \cdot u) \\ \partial_1(a+b) \\ \partial_2(a-b) \end{pmatrix}, \quad G = \begin{pmatrix} \lambda+\mu \\ \lambda+\mu \\ \mu \\ \mu \end{pmatrix}.$$

As in the reconstruction of μ, we will construct there linear independent vectors such that the first term of the equation (36) can be eliminated.

Suppose that $u^{(j)}$, for $j = 0, 1, 2, 3$, are three different solutions of (6) which satisfy

$$u^{(j)\sharp} \cdot F + u^{(j)\flat} \cdot G = -k^2 u^{(j)*}. \tag{37}$$

Now for $j = 1, 2, 3$ and $\natural = \sharp, \flat, *$, we let

$$\mathrm{u}^{(0)\natural} = \Re \chi(x) u^{(0)\natural},$$

and

$$\mathrm{u}^{(1)\natural} = \Re \chi_3(x) u^{(1)\natural},$$

$$\mathrm{u}^{(2)\natural} = \Im \chi_3(x) u^{(1)\natural},$$

$$\mathrm{u}^{(3)\natural} = \Re(\chi_1(x) u^{(2)\natural} + \chi_2(x) u^{(3)\natural}),$$

where $\chi_j(x)$ is a nonzero function. Then we get

$$\mathrm{u}^{(0)\sharp} \cdot F + \mathrm{u}^{(0)\flat} \cdot G = -k^2 \mathrm{u}^{(0)*} \tag{38}$$

and

$$\mathrm{u}^{(j)\sharp} \cdot F + \mathrm{u}^{(j)\flat} \cdot G = -k^2 \mathrm{u}^{(j)*} \quad \text{for } j = 1, 2, 3. \tag{39}$$

Assume that $\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ are three linearly independent vectors in some subdomain of Ω, say Ω_0. Then there exist three functions Θ_1, Θ_2, and Θ_3 such that $\mathrm{u}^{(0)\sharp} + \sum_{j=1}^{3} \Theta_j \mathrm{u}^{(j)\sharp} = 0$. Multiplying (39) by Θ_j and summing over j with equation (38), we have

$$\mathrm{v} \cdot G = -k^2 \Big(\mathrm{u}^{(0)*} + \sum_{j=1}^{3} \Theta_j \mathrm{u}^{(j)*} \Big),$$

where $\mathrm{v} = \mathrm{u}^{(0)\flat} + \sum_{j=1}^{3} \Theta_j \mathrm{u}^{(j)\flat}$. Let

$$\kappa = (1,1,0,0)^T \cdot \mathrm{v}, \quad \sigma = -(1,1,1,1)^T \cdot \mathrm{v}.$$

Then the above equation can be rewritten as

$$\kappa\lambda = \sigma\mu - k^2 \Big(\mathrm{u}^{(0)*} + \sum_{j=1}^{3} \Theta_j \mathrm{u}^{(j)*} \Big).$$

Suppose that $\kappa(x)$ does not vanish in Ω_0 . Then we can deduce the following lemma.

Lemma 3.6. *Let $u^{(j)}$ for $0 \le j \le 3$ be C^2 solutions of the elasticity system with boundary conditions $u^{(j)} = g^{(j)}$ on $\partial\Omega$. Let $u = (u_1, u_2)$, $a = \partial_2 u_1 + \partial_1 u_2$ and $b = \partial_1 u_1 - \partial_2 u_2$. We define $u^\sharp = (\nabla \cdot u,\ \nabla \cdot u,\ a + b,\ a - b)$ and assume that*

(1) *$\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ are three linearly independent vectors in Ω_0, the neighborhood of x_0 in Ω.*

(2) *$\kappa(x)$ does not vanish in Ω_0.*

Then the reconstruction is stable in the sense that

$$\|\lambda - \tilde{\lambda}\|_{C(\Omega_0)} \le C(\|\mu - \tilde{\mu}\|_{C(\Omega)} + \|H_{x_0} - \widetilde{H}_{x_0}\|_{C^2(\Omega_0)}), \tag{40}$$

where $H_{x_0} = (u^{(j)})_{0 \le j \le 3}$.

3.2.1. Global reconstruction of λ. We will find three linearly independent vectors $\{\mathrm{u}^{(1)\sharp}, \mathrm{u}^{(2)\sharp}, \mathrm{u}^{(3)\sharp}\}$ first. Then we can deduce stability of λ by using Lemma 3.6 and Theorem 3.5.

Plugging the CGO solution $u_\rho = \mu^{-1/2} w_\rho + \mu^{-1}\nabla f_\rho - f_\rho \nabla \mu^{-1}$ into $u^\sharp$, we get

$$u_\rho^\sharp = i e^{i\rho\cdot x}\left(\begin{pmatrix} \dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ \dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ 2\mu^{-1}(is(\rho_1^2+\rho_1\rho_2)+\omega_1\cdot\nabla s)+\mu^{-1/2}\omega_1\cdot r \\ 2\mu^{-1}(is(\rho_1\rho_2-\rho_1^2)+\omega_2\cdot\nabla s)+\mu^{-1/2}\omega_2\cdot r \end{pmatrix} + O(1)\right),$$

where $\omega_1 = (\rho_1+\rho_2, \rho_1-\rho_2)$ and $\omega_2 = (\rho_2-\rho_1, \rho_1+\rho_2)$.

For the same fixed point $x_0 \in \overline{\Omega}$. Denote that $\rho = \tau(e_1 + ie_2)$ and $\tilde{\rho} = i\rho$. We choose a constant vector $g^{(0)}$ such that $C_0(x,\theta)g^{(0)} = (r^{(0)}, s^{(0)})$ with

$$s^{(0)} \neq 0, \quad \rho\cdot r^{(0)} \neq 0, \quad \rho\cdot r^{(0)}(x_0) = 1$$

in a neighborhood of x_0, U. Then we get the CGO solution of the elasticity system

$$u_\rho^{(0)} = \mu^{-1/2} w_\rho^{(0)} + \mu^{-1}\nabla f_\rho^{(0)} - f_\rho^{(0)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(0)} \\ f_\rho^{(0)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(0)} \\ s^{(0)} \end{pmatrix} + O(\tau^{-1})\right).$$

We choose another constant vector $g^{(1)}$ such that $C_1(x,\theta)g^{(1)} = (r^{(1)}, s^{(1)})$ with

$$s^{(1)} \neq 0 \text{ in } U \text{ and } \rho\cdot r^{(1)}(x_0) = 0.$$

Then we get the CGO solution of the elasticity system

$$u_\rho^{(1)} = \mu^{-1/2} w_\rho^{(1)} + \mu^{-1}\nabla f_\rho^{(1)} - f_\rho^{(1)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(1)} \\ f_\rho^{(1)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(1)} \\ s^{(1)} \end{pmatrix} + O(\tau^{-1})\right).$$

For $l = 2, 3$, we choose a constant vector $g^{(l)}$ such that

$$C_l(x,\theta)g^{(l)} = (r^{(l)}, s^{(l)})$$

with

$$\rho \cdot r^{(l)} \neq 0$$

in U. Here we can choose

$$r^{(2)} = r^{(3)},\ s^{(2)} = s^{(3)}$$

by taking $g^{(2)} = g^{(3)}$ and $C_2(x,\theta) = C_3(x,\tilde{\theta})$. Then we get the CGO solution of the elasticity system

$$u_\rho^{(l)} = \mu^{-1/2} w_\rho^{(l)} + \mu^{-1}\nabla f_\rho^{(l)} - f_\rho^{(l)}\nabla\mu^{-1},$$

with

$$\begin{pmatrix} w_\rho^{(2)} \\ f_\rho^{(2)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(2)} \\ s^{(2)} \end{pmatrix} + O(\tau^{-1})\right),$$

and

$$\begin{pmatrix} w_\rho^{(3)} \\ f_\rho^{(3)} \end{pmatrix} = e^{i\tilde{\rho}\cdot x}\left(\begin{pmatrix} r^{(3)} \\ s^{(3)} \end{pmatrix} + O(\tau^{-1})\right).$$

We let

$$e^{-i\rho\cdot x} i^{-1}|\rho|^{-2} u_\rho^{(1)\sharp} = 2i\mu^{-1} s \begin{pmatrix} 0 \\ 0 \\ 1+i \\ i-1 \end{pmatrix} + O(\tau^{-1}),$$

$$\mathrm{u}_\rho^{2,1} := e^{-i\rho\cdot x} i^{-1}|\rho|^{-1} u_\rho^{(2)\sharp} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{\mu}}{\lambda+2\mu}(1,i)\cdot r^{(2)} \\ \frac{\sqrt{\mu}}{\lambda+2\mu}(1,i)\cdot r^{(2)} \\ 2\mu^{-1}\tau(1+i)is^{(2)} + \nu^{(2)} \\ 2\mu^{-1}\tau(i-1)is^{(2)} + i\nu^{(2)} \end{pmatrix} + O(\tau^{-1})$$

and

$$\mathrm{u}_\rho^{2,2} := e^{-i\tilde{\rho}\cdot x} i^{-1}|\tilde{\rho}|^{-1} u_\rho^{(3)\sharp} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{\mu}}{\lambda+2\mu}(i,-1)\cdot r^{(3)} \\ \frac{\sqrt{\mu}}{\lambda+2\mu}(i,-1)\cdot r^{(3)} \\ 2\mu^{-1}\tau(-1-i)is^{(3)} + \nu^{(3)} \\ 2\mu^{-1}\tau(1-i)is^{(3)} + i\nu^{(3)} \end{pmatrix} + O(\tau^{-1}),$$

where

$$\nu^{(2)} = 2\mu^{-1}(1+i, 1-i)\cdot\nabla s^{(2)} + \mu^{-1/2}(1+i, 1-i)\cdot r^{(2)}$$

and

$$\nu^{(3)} = 2\mu^{-1}(-1+i, i+1)\cdot\nabla s^{(3)} + \mu^{-1/2}(-1+i, i+1)\cdot r^{(3)}.$$

To eliminate the higher order term of $\mathfrak{u}_\rho^{2,j}$, we consider the summation of two vectors:

$$\begin{aligned} &e^{-i\rho\cdot x} i^{-1}|\rho|^{-1} u_\rho^{(2)\sharp} + e^{-i\tilde{\rho}\cdot x} i^{-1}|\tilde{\rho}|^{-1} u_\rho^{(3)\sharp} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{\sqrt{\mu}}{\lambda+2\mu}(1+i, i-1)\cdot r^{(3)} \\ \dfrac{\sqrt{\mu}}{\lambda+2\mu}(1+i, i-1)\cdot r^{(3)} \\ 2\mu^{-1}(2i,2)\cdot\nabla s^{(2)} + \mu^{-1/2}(2i,2)\cdot r^{(3)} \\ 2\mu^{-1}(-2,2i)\cdot\nabla s^{(3)} + \mu^{-1/2}(-2,2i)\cdot r^{(3)} \end{pmatrix} + O(\tau^{-1}). \end{aligned}$$

We define

$$\mathfrak{u}_\rho^{(0)\natural} = \Re e^{-i\rho\cdot x} i^{-1}|\rho|^{-2} u_\rho^{(0)\natural}, \quad \natural = \sharp, \flat, *,$$

$$\mathfrak{u}_\rho^{(1)\natural} = \Re e^{-i\rho\cdot x} i^{-1}|\rho|^{-2} u_\rho^{(1)\natural}; \quad \mathfrak{u}_\rho^{(2)\natural} = \Im e^{-i\rho\cdot x} i^{-1}|\rho|^{-2} u_\rho^{(1)\natural},$$

and

$$\mathfrak{u}_\rho^{(3)\natural} = \Re(e^{-i\rho\cdot x} i^{-1}|\rho|^{-1} u_\rho^{(2)\natural} + e^{-i\tilde{\rho}\cdot x} i^{-1}|\tilde{\rho}|^{-1} u_\rho^{(3)\natural}).$$

Thus we have constructed three linear independent vectors $\{\mathfrak{u}_\rho^{(1)\sharp}, \mathfrak{u}_\rho^{(2)\sharp}, \mathfrak{u}_\rho^{(3)\sharp}\}$ in U as τ is sufficiently large. Therefore, there are three functions Θ_j, $j = 1, 2, 3$, such that

$$\mathfrak{u}_\rho^{(0)\sharp} + \sum_{j=1}^{3} \Theta_j \mathfrak{u}_\rho^{(j)\sharp} = 0. \tag{41}$$

They also satisfy the equations

$$\mathfrak{u}_\rho^{(j)\sharp}\cdot F + \mathfrak{u}_\rho^{(j)\flat}\cdot G = -k^2 \mathfrak{u}_\rho^{(j)*}, \quad 0 \le j \le 3.$$

Summing over j and using (41), we get the equation

$$\mathrm{v}\cdot G = -k^2\Big(\mathfrak{u}_\rho^{(0)*} + \sum_{j=1}^{3}\Theta_j \mathfrak{u}_\rho^{(j)*}\Big), \tag{42}$$

where

$$\mathrm{v} = \mathfrak{u}_\rho^{(0)\flat} + \sum_{j=1}^{3} \Theta_j \mathfrak{u}_\rho^{(j)\flat}.$$

Then we obtain that

$$\kappa\lambda = \sigma\mu - k^2\Big(\mathfrak{u}_\rho^{(0)*} + \sum_{j=1}^{3} \Theta_j \mathfrak{u}_\rho^{(j)*}\Big), \tag{43}$$

where

$$\kappa = (1,1,0,0)^T \cdot \mathrm{v}, \quad \sigma = -(1,1,1,1)^T \cdot \mathrm{v}.$$

Lemma 3.7. *κ does not vanish in some neighborhood of x_0.*

Proof. Since $\Theta_j \sim \tau^0$ and $\rho \cdot r^{(1)}(x_0) = 0$, by using similar argument as in the proof of Lemma 3.2,

$$(1,1,0,0)^T \cdot \Theta_1 \mathfrak{u}_\rho^{(1)\flat} \sim \tau^{-1}, \quad (1,1,0,0)^T \cdot \Theta_2 \mathfrak{u}_\rho^{(2)\flat} \sim \tau^{-1}. \tag{44}$$

Observe that

$$\begin{aligned}\kappa_4 &:= (1,1,0,0)^T \cdot \mathfrak{u}_\rho^{(3)\flat} \\ &= \Re\left(\begin{pmatrix}1\\1\end{pmatrix} \cdot \left(\frac{e^{-i\rho\cdot x}}{i|\rho|}\begin{pmatrix}\partial_1(\nabla\cdot u^{(2)})\\ \partial_2(\nabla\cdot u^{(2)})\end{pmatrix} + \frac{e^{-i\tilde\rho\cdot x}}{i|\tilde\rho|}\begin{pmatrix}\partial_1(\nabla\cdot u^{(3)})\\ \partial_2(\nabla\cdot u^{(3)})\end{pmatrix}\right)\right).\end{aligned}$$

The $O(\tau)$ term of κ_4 is

$$\frac{1}{i\sqrt{2}\tau}\Big(-\frac{\sqrt{\mu}}{\lambda+2\mu}(\rho_1^{(2)}+\rho_2^{(2)})\rho^{(2)}\cdot r^{(2)} - \frac{\sqrt{\mu}}{\lambda+2\mu}(\rho_1^{(3)}+\rho_2^{(3)})\rho^{(3)}\cdot r^{(3)}\Big) = 0$$

since $r^{(2)} = r^{(3)}$. Thus the leading order term of κ_4 is $O(1)$, that is,

$$\varphi = \varphi^{(2)} + \varphi^{(3)},$$

where

$$\begin{aligned}\varphi^{(l)} = \frac{1}{i\sqrt{2}\tau}\Big(&\mu^{-1/2}(i\rho_2^{(l)}\partial_1 r_1^{(l)} + i\rho_1^{(l)}\partial_2 r_2^{(l)} + i\rho_1^{(l)}\partial_2 r_1^{(l)} + i\rho_2^{(l)}\partial_1 r_2^{(l)} \\ &\qquad + 2i\rho\cdot\nabla r^{(l)}) \\ &+ \mu^{-1}(2i\rho\cdot(\nabla(\partial_1 s^{(l)}) + \nabla(\partial_2 s^{(l)})) + i(\rho_1+\rho_2)\Delta s^{(l)}) \\ &+ 2i\partial_1\mu^{-1}\rho^{(l)}\cdot\nabla s^{(l)} - \partial_{11}\mu^{-1} i(\rho_1^{(l)}+\rho_2^{(l)})s^{(l)} \\ &+ 2i\partial_2\mu^{-1}\rho^{(l)}\cdot\nabla s^{(l)} - \partial_{22}\mu^{-1} i(\rho_1^{(l)}+\rho_2^{(l)})s^{(l)} \\ &+ \partial_1\mu^{-1/2}(i\rho_2^{(l)} r_1^{(l)} + i\rho_2^{(l)} r_2^{(l)} + 2i\rho_1^{(l)} r_1^{(l)}) \\ &+ \partial_2\mu^{-1/2}(i\rho_1^{(l)} r_1^{(l)} + i\rho_1^{(l)} r_2^{(l)} + 2i\rho_2^{(l)} r_2^{(l)})\Big).\end{aligned}$$

Since

$$\kappa_1 := (1,1,0,0)^T \cdot \mathrm{u}_\rho^{(0)\flat}(x_0)$$
$$= \Re\begin{pmatrix}1\\1\end{pmatrix} \cdot \frac{-ie^{-i\rho\cdot x}}{|\rho|^2}\begin{pmatrix}\partial_1(\nabla\cdot u)\\ \partial_2(\nabla\cdot u)\end{pmatrix}(x_0) = \frac{-\sqrt{\mu}}{2(\lambda+2\mu)} + O(\tau^{-1})$$

and (44), we get that

$$\kappa(x_0) = (1,1,0,0)^T \cdot \mathrm{v}(x_0)$$
$$= \kappa_1(x_0) + \Theta_3\kappa_4(x_0) + O(\tau^{-1})$$
$$\sim \frac{-\sqrt{\mu}}{2(\lambda+2\mu)} + \Theta_3(x_0)\varphi(x_0).$$

Since $\mu \in \mathcal{P}$, $\varphi(x_0)$ is some fixed number. Moreover, we can take $s^{(0)}$ small such that $|\Theta_3(x_0)|$ to be sufficiently small (It can be done by following similar argument in Remark 2), then we can obtain that $\frac{-\sqrt{\mu}}{2(\lambda+2\mu)} + \Theta_3(x_0)\varphi(x_0) \neq 0$. By continuity of κ, there exists a neighborhood V such that κ never vanishes in V. $\square$

Let

$$\Omega_0 = U \cap V \cap \Omega. \tag{45}$$

We have $\{\mathrm{u}_\rho^{(1)\sharp}, \mathrm{u}_\rho^{(2)\sharp}, \mathrm{u}_\rho^{(3)\sharp}\}$ are linearly independent in Ω_0 as τ is sufficiently large and also κ does not vanish in Ω_0. Then it follows that

$$\lambda = \frac{\sigma\mu}{\kappa} - \frac{k^2}{\kappa}\Big(\mathrm{u}_\rho^{(0)*} + \sum_{j=1}^{3}\Theta_j \mathrm{u}_\rho^{(j)*}\Big) \quad \text{in } \Omega_0. \tag{46}$$

As in Section 3.1, we also can find the solutions $u^{(j)}$ of the elasticity system such that

$$\|u^{(j)} - u_\rho^{(j)}\|_{C^2(\Omega)} < C\varepsilon, \quad j = 0,1,2,3. \tag{47}$$

Now we let the internal data H_{x_0} contains the three solutions we constructed in Theorem 3.3 and the four solutions $u^{(j)}$, $j = 0,1,2,3$ in this section. Then we have the following result.

Theorem 3.8. *Suppose that (λ,μ) and $(\tilde{\lambda},\tilde{\mu}) \in \mathcal{P}$. For any fixed $x_0 \in \partial\Omega$, let $u_\rho^{(j)}$ be the corresponding CGO solutions for (λ,μ) and $u^{(j)}$ constructed in Sections 3.1 and 3.2 with internal data H_{x_0} and with ε as in (22) sufficiently small. Let $\widetilde{H}_{x_0}$ be constructed similarly with the parameters $(\tilde{\lambda},\tilde{\mu})$. Assume that $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$.*

Then $H_{x_0} = \widetilde{H}_{x_0}$ implies that $\lambda = \tilde{\lambda}$ in Ω_0 which is defined in (45).

Proof. Applying Theorem 3.3 and equation (46), we have the uniqueness of λ near the point x_0. $\square$

We deduce the following result by applying Lemma 3.6 and Theorem 3.4.

Theorem 3.9. *Suppose that (λ,μ) and $(\tilde\lambda,\tilde\mu)\in\mathcal{P}$. For any fixed $x_0\in\overline{\Omega}$, let $u^{(j)}_\rho$ be the corresponding CGO solutions for (λ,μ) and $u^{(j)}$ constructed in Sections 3.1 and 3.2 with internal data H_{x_0} and with ε as in (22) sufficiently small. Let $\widetilde{H}_{x_0}$ be constructed similarly for the parameters $(\tilde\lambda,\tilde\mu)$ with $u^{(j)}|_{\partial\Omega}=\tilde u^{(j)}|_{\partial\Omega}$. Then we have the estimates*

$$\|\lambda-\tilde\lambda\|_{C(\Omega_0)}\le C(|\mu(x_0)-\tilde\mu(x_0)|+\|H_\rho-\widetilde{H}_\rho\|_{C^2(\Omega_0)}), \tag{48}$$

for $x_0\in\partial\Omega$, and

$$\|\lambda-\tilde\lambda\|_{C(\Omega_0)}\le C(|\mu(x_0^+)-\tilde\mu(x_0^+)|+\|H_\rho-\widetilde{H}_\rho\|_{C^2(\Omega_0)}), \tag{49}$$

for $x_0^+\in\partial\Omega_0$ and $x_0\in\Omega$. Here Ω_0 is defined in (45).

With (46) and Theorem 3.9, we follow the same proof as in Theorem 3.5, then we can get the following global reconstruction of λ.

Theorem 3.10 (Global reconstruction of λ). *Let Ω be an open bounded domain of $\mathbb{R}^2$ with smooth boundary. Suppose that the Lamé parameters (λ,μ) and that $(\tilde\lambda,\tilde\mu)\in\mathcal{P}$ and $\mu|_{\partial\Omega}=\tilde\mu|_{\partial\Omega}$. Let $u^{(j)}$ and $\tilde u^{(j)}$ be the solutions of the elasticity system with boundary data $g^{(j)}$ for parameters (λ,μ) and $(\tilde\lambda,\tilde\mu)$, respectively. Let $H=(u^{(j)})_{1\le j\le J}$ and $\widetilde{H}=(\tilde u^{(j)})_{1\le j\le J}$ be the corresponding internal data for (λ,μ) and $(\tilde\lambda,\tilde\mu)$, respectively for some integer $J\ge 7$.*

Then there is an open set of the boundary data $(g^{(j)})_{1\le j\le J}$ such that if $H=\widetilde{H}$ implies $\lambda=\tilde\lambda$ in Ω. Moreover, we have the stability estimate

$$\|\lambda-\tilde\lambda\|_{C(\Omega)}\le C\|H-\widetilde{H}\|_{C^2(\Omega)}.$$

4. Reconstruction of Lamé parameter in three dimensional case

The reconstruction of λ and μ in $\mathbb{R}^3$ basically follows the similar argument as in Section 3. The major difference in proving the stability of Lamé parameters between dimensions two and three is that more CGO solutions are needed to get linearly independent vectors locally in $\mathbb{R}^3$ than in $\mathbb{R}^2$.

4.1. Global reconstruction of μ in 3D. Let $u = (u_1, u_2, u_3)^T$ be the displacement which satisfies the elasticity system

$$\nabla \cdot (\lambda(\nabla \cdot u)I + 2S(\nabla u)\mu) + k^2 u = 0. \tag{50}$$

Denote

$$u^\sharp = \begin{pmatrix} \partial_1(\nabla\cdot u) \\ \partial_2(\nabla\cdot u) \\ \partial_3(\nabla\cdot u) \\ \nabla\cdot u \\ \nabla\cdot u \\ \nabla\cdot u \end{pmatrix}, \quad F = \begin{pmatrix} \lambda+\mu \\ \lambda+\mu \\ \lambda+\mu \\ \partial_1(\lambda+\mu) \\ \partial_2(\lambda+\mu) \\ \partial_3(\lambda+\mu) \end{pmatrix},$$

$$u^\flat = \begin{pmatrix} b_{23} \\ b_{13} \\ b_{12} \\ \partial_1(b_{23}) \\ \partial_2(b_{13}) \\ \partial_3(b_{12}) \end{pmatrix}, \quad G = \begin{pmatrix} \partial_1\mu \\ \partial_2\mu \\ \partial_3\mu \\ \mu \\ \mu \\ \mu \end{pmatrix},$$

where

$$b_{ij} = \partial_l u_l - \partial_i u_i - \partial_j u_j + \partial_i u_l + \partial_l u_i + \partial_j u_l + \partial_l u_j, \quad l, i, j \in \{1,2,3\}$$

are distinct numbers. From (50), we can deduce the equation

$$u^\sharp \cdot F + u^\flat \cdot G = -k^2 u^*.$$

Here $u^* = (u_1 + u_2 + u_3)$.

In the following we will show that how we can get four linearly independent vectors of the form $u^\sharp$ on some subdomain of Ω. The key thing is to observe the behavior of $u^\sharp$. We plug the CGO solutions $u_\rho = \mu^{-1/2} w_\rho + \mu^{-1}\nabla f_\rho - f_\rho \nabla\mu^{-1}$ into $u^\sharp$. Then we get

$$u_\rho^\sharp = e^{i\rho\cdot x}\left(\begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}\rho_1(\rho\cdot r) + O(|\rho|) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}\rho_2(\rho\cdot r) + O(|\rho|) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}\rho_3(\rho\cdot r) + O(|\rho|) \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ i\frac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \end{pmatrix} + O(1)\right). \tag{51}$$

Note that $r = (r_1, r_2, r_3)^T$ and $\rho = (\rho_1, \rho_2, \rho_3)^T$.

Now we fix any point $x_0 \in \overline{\Omega}$. Let $\rho = \tau(1, i, 0)$ and $\tilde{\tilde{\rho}} = \tau(1, 0, i)$ with $\tau > 0$. Let $\theta = \rho/\tau$, $\tilde{\tilde{\theta}} = \tilde{\tilde{\rho}}/\tau$. Since, in Lemma 2.1, the matrix solutions $C_0(x, \theta)$ and $\tilde{\tilde{C}}_0(x, \tilde{\tilde{\theta}})$ are invertible, we can choose two constant vectors $g^{(0)}$ and $\tilde{\tilde{g}}^{(0)}$ such that $C_0(x, \theta)g^{(0)} = (r^{(0)}, s^{(0)})^T$ and $\tilde{\tilde{C}}_0(x, \tilde{\tilde{\theta}})\tilde{\tilde{g}}^{(0)} = (\tilde{\tilde{r}}^{(0)}, \tilde{\tilde{s}}^{(0)})^T$ with $s^{(0)}(x_0) = 1 = \tilde{\tilde{s}}^{(0)}(x_0)$ and $s^{(0)}$, $\tilde{\tilde{s}}^{(0)} \neq 0$ and $\rho \cdot r^{(0)}$, $\tilde{\tilde{\rho}} \cdot \tilde{\tilde{r}}^{(0)} \neq 0$ in a neighborhood of x_0, say U_0. Then we have the CGO solutions of the elasticity system, that is,

$$u_\rho^{(0)} = \mu^{-1/2} w_\rho^{(0)} + \mu^{-1}\nabla f_\rho^{(0)} - f_\rho^{(0)}\nabla\mu^{-1},$$

and

$$\tilde{\tilde{u}}_\rho^{(0)} = \mu^{-1/2} \tilde{\tilde{w}}_\rho^{(0)} + \mu^{-1}\nabla \tilde{\tilde{f}}_\rho^{(0)} - \tilde{\tilde{f}}_\rho^{(0)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(0)} \\ f_\rho^{(0)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(0)} \\ s^{(0)} \end{pmatrix} + O(\tau^{-1})\right),$$

and

$$\begin{pmatrix} \tilde{\tilde{w}}_\rho^{(0)} \\ \tilde{\tilde{f}}_\rho^{(0)} \end{pmatrix} = e^{i\tilde{\tilde{\rho}}\cdot x}\left(\begin{pmatrix} \tilde{\tilde{r}}^{(0)} \\ \tilde{\tilde{s}}^{(0)} \end{pmatrix} + O(\tau^{-1})\right).$$

Let $\tilde{\rho} = \tau(i, -1, 0)$ and $\tilde{\theta} = \tilde{\rho}/\tau$. Let $C_1(x, \theta)$ and $C_2(x, \tilde{\theta})$ satisfy that

$$i\theta \cdot \frac{\partial}{\partial x} C_1(x, \theta) = \theta \cdot V_1(x) C_1(x, \theta), \quad i\tilde{\theta} \cdot \frac{\partial}{\partial x} C_2(x, \tilde{\theta}) = \tilde{\theta} \cdot V_1(x) C_2(x, \tilde{\theta}),$$

respectively. Since $\tilde{\rho} = i\rho$, we can choose $C_2(x, \tilde{\theta}) = C_1(x, \theta)$. Moreover, $r^{(2)} = r^{(1)}$ and $s^{(2)} = s^{(1)}$. With suitable constant vector g, we can get that $s^{(l)}$ is zero at point x_0 and $r^{(1)}(x_0) = (1, -i, 0) = r^{(2)}(x_0)$. By continuity of $r^{(l)}$, we have $\rho \cdot r^{(l)} \neq 0$ in a neighborhood U_1 of x_0. Then the CGO solution is

$$u_\rho^{(l)} = \mu^{-1/2} w_\rho^{(l)} + \mu^{-1}\nabla f_\rho^{(l)} - f_\rho^{(l)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(1)} \\ f_\rho^{(1)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(1)} \\ s^{(1)} \end{pmatrix} + O(\tau^{-1})\right),$$

and

$$\begin{pmatrix} w_\rho^{(2)} \\ f_\rho^{(2)} \end{pmatrix} = e^{i\tilde{\rho}\cdot x}\left(\begin{pmatrix} r^{(2)} \\ s^{(2)} \end{pmatrix} + O(\tau^{-1})\right).$$

For $\tilde{\tilde{\rho}}$, with a suitable constant vector $\tilde{g}$, we can get that $s^{(3)}$ is zero at point x_0 and $r^{(3)}(x_0) = (1, 0, -i)$. By continuity of $r^{(3)}$, we have $\tilde{\tilde{\rho}} \cdot r^{(3)} \neq 0$ in a neighborhood U_2 of x_0. Then the CGO solution is

$$u_\rho^{(3)} = \mu^{-1/2} w_\rho^{(3)} + \mu^{-1} \nabla f_\rho^{(3)} - f_\rho^{(3)} \nabla \mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(3)} \\ f_\rho^{(3)} \end{pmatrix} = e^{i\tilde{\tilde{\rho}}\cdot x} \left(\begin{pmatrix} r^{(3)} \\ s^{(3)} \end{pmatrix} + O(\tau^{-1}) \right).$$

Let $U = \cap_{l=0}^{2} U_l$. So far we have five CGO solutions, that is, $u_\rho^{(0)}, \tilde{\tilde{u}}_\rho^{(0)}, u_\rho^{(1)}$, $u_\rho^{(2)}$, and $u_\rho^{(3)}$.

Let $r^{(1)} = (r_1^{(1)}, r_2^{(1)}, r_3^{(1)})$. We define

$$\mathrm{u}_\rho^1 := e^{-i\rho\cdot x} |\rho|^{-2} u_\rho^{(1)\sharp} = \frac{1}{2} \begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda + 2\mu}(r_1^{(1)} + i r_2^{(1)}) \\ -\frac{\sqrt{\mu}}{\lambda + 2\mu}(i r_1^{(1)} - r_2^{(1)}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + O(\tau^{-1})$$

and

$$\begin{aligned} \mathrm{u}_\rho^{2,1} &:= e^{-i\rho\cdot x} |\rho|^{-1} u_\rho^{(1)\sharp} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda + 2\mu}\tau(r_1^{(1)} + i r_2^{(1)}) + O(1) \\ -\frac{\sqrt{\mu}}{\lambda + 2\mu}\tau(i r_1^{(1)} - r_2^{(1)}) + O(1) \\ 0 \\ i\frac{\sqrt{\mu}}{\lambda + 2\mu}(r_1^{(1)} + i r_2^{(1)}) \\ i\frac{\sqrt{\mu}}{\lambda + 2\mu}(r_1^{(1)} + i r_2^{(1)}) \\ i\frac{\sqrt{\mu}}{\lambda + 2\mu}(r_1^{(1)} + i r_2^{(1)}) \end{pmatrix} + O(\tau^{-1}) \end{aligned}$$

and

$$
\begin{aligned}
\mathfrak{u}_\rho^{2,2} &:= e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1}u_\rho^{(2)\sharp}\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}
-\dfrac{\sqrt{\mu}}{\lambda+2\mu}\tau(-r_1^{(2)}-ir_2^{(2)})+O(1)\\
-\dfrac{\sqrt{\mu}}{\lambda+2\mu}\tau(-ir_1^{(2)}+r_2^{(2)})+O(1)\\
0\\
i\dfrac{\sqrt{\mu}}{\lambda+2\mu}(ir_1^{(2)}-r_2^{(2)})\\
i\dfrac{\sqrt{\mu}}{\lambda+2\mu}(ir_1^{(2)}-r_2^{(2)})\\
i\dfrac{\sqrt{\mu}}{\lambda+2\mu}(ir_1^{(2)}-r_2^{(2)})
\end{pmatrix}+O(\tau^{-1}).
\end{aligned}
$$

Let $\mathfrak{u}_\rho^2=\mathfrak{u}_\rho^{2,1}+\mathfrak{u}_\rho^{2,2}$. Since $r^{(2)}=r^{(1)}$ and $s^{(2)}=s^{(1)}$, we have

$$
\mathfrak{u}_\rho^2=\frac{1}{\sqrt{2}}\begin{pmatrix}
O(1)\\
O(1)\\
0\\
i\dfrac{\sqrt{\mu}}{\lambda+2\mu}((1+i)r_1^{(1)}+(-1+i)r_2^{(1)})\\
i\dfrac{\sqrt{\mu}}{\lambda+2\mu}((1+i)r_1^{(1)}+(-1+i)r_2^{(1)})\\
i\dfrac{\sqrt{\mu}}{\lambda+2\mu}((1+i)r_1^{(1)}+(-1+i)r_2^{(1)})
\end{pmatrix}+O(\tau^{-1}).
$$

We also define

$$
\begin{aligned}
\mathfrak{u}_\rho^3 &:= e^{-i\tilde{\tilde{\rho}}\cdot x}|\rho|^{-2}u_\rho^{(3)\sharp}\\
&= \frac{1}{2}\begin{pmatrix}
-\dfrac{\sqrt{\mu}}{\lambda+2\mu}(r_1^{(3)}+ir_3^{(3)})\\
0\\
-\dfrac{\sqrt{\mu}}{\lambda+2\mu}(ir_1^{(3)}-r_3^{(3)})\\
0\\
0\\
0
\end{pmatrix}+O(\tau^{-1})
\end{aligned}
$$

We denote

$$\mathrm{u}_{1\rho}^{(0)\natural} = \Re e^{-i\rho\cdot x}|\rho|^{-2}u_{\rho}^{(0)\natural}, \quad \mathrm{u}_{2\rho}^{(0)\natural} = \Im e^{-i\rho\cdot x}|\rho|^{-2}u_{\rho}^{(0)\natural},$$
$$\mathrm{u}_{3\rho}^{(0)\natural} = \Re e^{-i\tilde{\tilde{\rho}}\cdot x}|\rho|^{-2}\tilde{\tilde{u}}_{\rho}^{(0)\natural}; \quad \natural = \sharp, \flat, *,$$

and

$$\begin{aligned}
\mathrm{u}_{\rho}^{(1)\natural} &= \Re e^{-i\rho\cdot x}|\rho|^{-2}u_{\rho}^{(1)\natural},\\
\mathrm{u}_{\rho}^{(2)\natural} &= \Im e^{-i\rho\cdot x}|\rho|^{-2}u_{\rho}^{(1)\natural},\\
\mathrm{u}_{\rho}^{(3)\natural} &= \Re(e^{-i\rho\cdot x}|\rho|^{-1}u_{\rho}^{(1)\natural} + e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1}u_{\rho}^{(2)\natural}),\\
\mathrm{u}_{\rho}^{(4)\natural} &= \Re e^{-i\tilde{\tilde{\rho}}\cdot x}|\rho|^{-2}u_{\rho}^{(3)\natural}.
\end{aligned}$$

Then $\{\mathrm{u}_{\rho}^{(j)\sharp} : 1 \le j \le 4\}$ are linearly independent in the neighborhood U of x_0 as τ is sufficiently large. Therefore, for fixed $l = 1, 2, 3$ there exist functions $\Theta_j^l,\ j = 1, 2, 3, 4$, such that

$$\mathrm{u}_{l\rho}^{(0)\sharp} + \sum_{j=1}^{4} \Theta_j^l \mathrm{u}_{\rho}^{(j)\sharp} = 0.$$

As in Section 3.1, we summing over equations, then we have

$$\beta_{\rho,l} \cdot \nabla\mu + \gamma_{\rho,l}\mu = -k^2\Big(\mathrm{u}_{l\rho}^{(0)*} + \sum_{j=1}^{4} \Theta_j^l \mathrm{u}_{\rho}^{(j)*}\Big) \quad l = 1, 2, 3,$$

where $\beta_{\rho,l}$ and $\gamma_{\rho,l}$ are functions which depend on ρ, Ω and CGO solutions $\tilde{\tilde{u}}_{\rho}^{(0)}$, $u_{\rho}^{(j)}$ for $j = 0, \ldots, 3$.

Lemma 4.1. *Given any point $x_0 \in \overline{\Omega}$, there exists an open neighborhood V of x_0 such that $\beta_{\rho,l}$ is not zero in V, for $l = 1, 2, 3$.*

Proof. Following the similar proof as in Lemma 3.2, we can prove this Lemma. □

Let $\Omega_0 = U \cap V \cap \Omega$. Based on the lemma above, we may suppose that $\beta_{\rho,j}$ are linearly independent in Ω_0 as τ sufficiently large.

Let $g_\rho^{(j)} = u_\rho^{(j)}|_{\partial\Omega}$ and $\tilde{\tilde{g}}_\rho^{(0)} = \tilde{\tilde{u}}_\rho^{(0)}|_{\partial\Omega}$ be the given boundary data for $j = 0, 1, 2, 3$. Let $\tilde{\tilde{g}}^{(0)},\ g^{(j)} \in C^{1,\alpha}(\partial\Omega)$ be the boundary data close to $\tilde{\tilde{g}}_\rho^{(0)},\ g_\rho^{(j)}$, respectively, that is,

$$\|g^{(j)} - g_\rho^{(j)}\|_{C^{1,\alpha}(\partial\Omega)} < \varepsilon, \quad \|\tilde{\tilde{g}}^{(0)} - \tilde{\tilde{g}}_\rho^{(0)}\|_{C^{1,\alpha}(\partial\Omega)} < \varepsilon.$$

Then we can find solutions $\tilde{\tilde{u}}^{(0)},\ u^{(j)}$ of the elasticity system with boundary data $\tilde{\tilde{g}}^{(0)}, g^{(j)}$, respectively. By regularity theorem, it follows that

$$\|u^{(j)} - u_\rho^{(j)}\|_{C^2(\Omega)} < C\varepsilon, \quad \|\tilde{\tilde{u}}^{(0)} - \tilde{\tilde{u}}_\rho^{(0)}\|_{C^2(\Omega)} < C\varepsilon. \tag{52}$$

From (52), we have

$$\|\mathrm{u}^{(j)\sharp} - \mathrm{u}_\rho^{(j)\sharp}\|_{C(\Omega)} \le C\varepsilon, \quad 1 \le j \le 4.$$

Therefore, $\{\mathrm{u}^{(j)\sharp} : 1 \le j \le 4\}$ are also linearly independent when ε is sufficiently small. We construct β_l by replacing $u_\rho^{(j)}$ by $u^{(j)}$. Then from (52), we can deduce that

$$\|\beta_l - \beta_{\rho,l}\|_{C^1(\Omega)}$$

is small when ε is sufficiently small. Then we have the equation

$$\beta_l \cdot \nabla\mu + \gamma_l \mu = -k^2\Big(\mathrm{u}_l^{(0)*} + \sum_{j=1}^{4} \Theta_j^l \mathrm{u}^{(j)*}\Big), \quad l = 1, 2, 3, \tag{53}$$

with $\{\beta_l(x)\}_{l=1,2,3}$ a basis in $\mathbb{R}^3$ for every point $x \in \Omega_0$. There exists an invertible matrix $A = (a_{ij})$ such that $\beta_l = \sum a_{lk} e_k$ with inverse of class $C(\Omega)$. Thus, we have constructed two vector-valued functions $\Gamma(x),\ \Phi(x) \in C(\Omega)$ such that (53) can be rewritten as

$$\nabla\mu + \Gamma(x)\mu = \Phi(x) \quad \text{in } \Omega_0. \tag{54}$$

Then we have the following uniqueness and stability theorem.

Theorem 4.2 (Global reconstruction of μ). *Let Ω be an open bounded domain of $\mathbb{R}^3$ with smooth boundary. Suppose that the Lamé parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu}) \in \mathcal{P}$ and $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$. Let $u^{(j)}$ and $\tilde{u}^{(j)}$ be the solutions of the elasticity system with boundary data $g^{(j)}$ for parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively. Let $H = (u^{(j)})_{1 \le j \le J}$ and $\widetilde{H} = (\tilde{u}^{(j)})_{1 \le j \le J}$ be the corresponding internal data for (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively for some integer $J \ge 5$.*

Then there is an open set of the boundary data $(g^{(j)})_{1 \le j \le J}$ such that if $H = \widetilde{H}$ implies $\mu = \tilde{\mu}$ in Ω. Moreover, we have the stability estimate

$$\|\mu - \tilde{\mu}\|_{C(\Omega)} \le C\|H - \widetilde{H}\|_{C^2(\Omega)}.$$

Proof. The proof is similar to Theorem 3.5. ☐

4.2. Global reconstruction of λ in 3D. The elasticity system can also be written in this form

$$u^\sharp \cdot F + u^\flat \cdot G = -k^2 u^*, \tag{55}$$

where

$$u^\sharp = \begin{pmatrix} \nabla \cdot u \\ \nabla \cdot u \\ \nabla \cdot u \\ b_{23} \\ b_{13} \\ b_{12} \end{pmatrix}, \qquad F = \begin{pmatrix} \partial_1(\lambda+\mu) \\ \partial_2(\lambda+\mu) \\ \partial_3(\lambda+\mu) \\ \partial_1\mu \\ \partial_2\mu \\ \partial_3\mu \end{pmatrix},$$

$$u^\flat = \begin{pmatrix} \partial_1(\nabla \cdot u) \\ \partial_2(\nabla \cdot u) \\ \partial_3(\nabla \cdot u) \\ \partial_1(b_{23}) \\ \partial_2(b_{13}) \\ \partial_3(b_{12}) \end{pmatrix}, \quad G = \begin{pmatrix} \lambda+\mu \\ \lambda+\mu \\ \lambda+\mu \\ \mu \\ \mu \\ \mu \end{pmatrix}.$$

As in the reconstruction of μ, we will construct CGO solutions such that the first term of (55) can be eliminated.

Plug the CGO solution $u_\rho = \mu^{-1/2} w_\rho + \mu^{-1}\nabla f_\rho - f_\rho \nabla \mu^{-1}$ into $u^\sharp$. Then we get

$$u^\sharp_\rho = e^{i\rho\cdot x}\left(\begin{pmatrix} i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r \\ -2\mu^{-1}\rho_1(\rho_1+\rho_2+\rho_3)s + O(|\rho|) \\ -2\mu^{-1}\rho_2(\rho_1+\rho_2+\rho_3)s + O(|\rho|) \\ -2\mu^{-1}\rho_3(\rho_1+\rho_2+\rho_3)s + O(|\rho|) \end{pmatrix} + O(1)\right).$$

For the same fixed point $x_0 \in \bar{\Omega}$. We choose a constant vector $g^{(0)}$ such that $C_0(x,\theta)g^{(0)} = (r^{(0)}, s^{(0)})$ with $s^{(0)} \neq 0$ and $\rho\cdot r^{(0)} \neq 0$ in a neighborhood of x_0, say U_0, and $\rho\cdot r^{(0)}(x_0) = 1$. Then we get the CGO solution of the elasticity system

$$u^{(0)}_\rho = \mu^{-1/2} w^{(0)}_\rho + \mu^{-1}\nabla f^{(0)}_\rho - f^{(0)}_\rho \nabla \mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(0)} \\ f_\rho^{(0)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(0)} \\ s^{(0)} \end{pmatrix} + O(\tau^{-1})\right).$$

We choose another constant vector $g^{(1)}$ such that $C_1(x,\theta)g^{(1)} = (r^{(1)}, s^{(1)})$ with $s^{(1)} \neq 0$ in a neighborhood of x_0, say U_1, and $\rho \cdot r^{(1)}(x_0) = 0$. Then we get the CGO solution of the elasticity system

$$u_\rho^{(1)} = \mu^{-1/2} w_\rho^{(1)} + \mu^{-1}\nabla f_\rho^{(1)} - f_\rho^{(1)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(1)} \\ f_\rho^{(1)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(1)} \\ s^{(1)} \end{pmatrix} + O(\tau^{-1})\right).$$

For $l = 2, 3$, we choose a constant vector $g^{(l)}$ such that

$$C_l(x,\theta)g^{(l)} = (r^{(l)}, s^{(l)})$$

with $\rho \cdot r^{(l)} \neq 0$ in a neighborhood of x_0, say U_2. Here we can choose $r^{(2)} = r^{(3)}$, $s^{(2)} = s^{(3)}$ by taking $g^{(2)} = g^{(3)}$ and $C_2(x,\theta) = C_3(x,\tilde{\theta})$. Then we get the CGO solution of the elasticity system

$$u_\rho^{(l)} = \mu^{-1/2} w_\rho^{(l)} + \mu^{-1}\nabla f_\rho^{(l)} - f_\rho^{(l)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(2)} \\ f_\rho^{(2)} \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} r^{(2)} \\ s^{(2)} \end{pmatrix} + O(\tau^{-1})\right),$$

and

$$\begin{pmatrix} w_\rho^{(3)} \\ f_\rho^{(3)} \end{pmatrix} = e^{i\tilde{\rho}\cdot x}\left(\begin{pmatrix} r^{(3)} \\ s^{(3)} \end{pmatrix} + O(\tau^{-1})\right).$$

For $\tilde{\tilde{\rho}}$, we choose another constant vector $g^{(4)}$ such that $C_4(x,\theta)g^{(4)} = (r^{(4)}, s^{(4)})$ with $s^{(4)} \neq 0$ in a neighborhood of x_0, say U_3, and $\tilde{\tilde{\rho}} \cdot r^{(4)}(x_0) = 0$. Then we get the CGO solution of the elasticity system

$$u_\rho^{(4)} = \mu^{-1/2} w_\rho^{(4)} + \mu^{-1}\nabla f_\rho^{(4)} - f_\rho^{(4)}\nabla\mu^{-1}$$

with

$$\begin{pmatrix} w_\rho^{(4)} \\ f_\rho^{(4)} \end{pmatrix} = e^{i\tilde{\tilde{\rho}}\cdot x}\left(\begin{pmatrix} r^{(4)} \\ s^{(4)} \end{pmatrix} + O(\tau^{-1})\right).$$

We define

$$\mathfrak{u}^1_\rho := e^{-i\rho\cdot x}|\rho|^{-2}u^{(1)\sharp}_\rho = -2\mu^{-1}s\begin{pmatrix}0\\0\\0\\1+i\\i-1\\0\end{pmatrix} + O(\tau^{-1})$$

and

$$\begin{aligned}\mathfrak{u}^{2,1}_\rho :=& e^{-i\rho\cdot x}|\rho|^{-1}u^{(2)\sharp}_\rho\\ =&\frac{1}{\sqrt{2}}\begin{pmatrix} i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r^{(2)}\\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r^{(2)}\\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\rho\cdot r^{(2)}\\ -2\mu^{-1}\tau(1+i)s^{(2)}+O(1)\\ -2\mu^{-1}\tau(i-1)s^{(2)}+O(1)\\ O(1)\end{pmatrix} + O(\tau^{-1})\end{aligned}$$

and

$$\begin{aligned}\mathfrak{u}^{2,2}_\rho :=& e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1}u^{(3)\sharp}_\rho\\ =&\frac{1}{\sqrt{2}}\begin{pmatrix} i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\tilde{\rho}\cdot r^{(3)}\\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\tilde{\rho}\cdot r^{(3)}\\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}\tilde{\rho}\cdot r^{(3)}\\ -2\mu^{-1}\tau(-1-i)s^{(3)}+O(1)\\ -2\mu^{-1}\tau(1-i)s^{(3)}+O(1)\\ O(1)\end{pmatrix} + O(\tau^{-1}).\end{aligned}$$

Let $\mathrm{u}_\rho^2 = \mathrm{u}_\rho^{2,1} + \mathrm{u}_\rho^{2,2}$, then the higher order is eliminated. Thus we have

$$\mathrm{u}_\rho^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i\dfrac{\sqrt{\mu}}{\lambda+2\mu}(\rho+\tilde{\rho})\cdot r^{(3)} \\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}(\rho+\tilde{\rho})\cdot r^{(3)} \\ i\dfrac{\sqrt{\mu}}{\lambda+2\mu}(\rho+\tilde{\rho})\cdot r^{(3)} \\ O(1) \\ O(1) \\ O(1) \end{pmatrix} + O(\tau^{-1}).$$

Also, we define that

$$\mathrm{u}_\rho^3 := e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(4)\sharp} = -2\mu^{-1}s \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1+i \\ 0 \\ i-1 \end{pmatrix} + O(\tau^{-1}).$$

We denote

$$\mathrm{u}_\rho^{(0)\natural} = \Re e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(0)\natural}, \quad \natural = \sharp, \flat, *,$$

and

$$\begin{aligned} \mathrm{u}_\rho^{(1)\natural} &= \Re e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(1)\natural}, \\ \mathrm{u}_\rho^{(2)\natural} &= \Im e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(1)\natural}, \\ \mathrm{u}_\rho^{(3)\natural} &= \Re(e^{-i\rho\cdot x}|\rho|^{-1}u_\rho^{(2)\natural} + e^{-i\tilde{\rho}\cdot x}|\tilde{\rho}|^{-1}u_\rho^{(3)\natural}), \\ \mathrm{u}_\rho^{(4)\natural} &= \Re e^{-i\rho\cdot x}|\rho|^{-2}u_\rho^{(4)\natural}. \end{aligned}$$

Then $\{\mathrm{u}_\rho^{(j)\sharp} : 1 \le j \le 4\}$ are linearly independent in the neighborhood $U = \cap_{l=0}^3 U_l$ of x_0 as τ is sufficiently large. Therefore, there exist functions Θ_j, $j = 1, 2, 3, 4$, such that

$$\mathrm{u}_\rho^{(0)\sharp} + \sum_{j=1}^4 \Theta_j \mathrm{u}_\rho^{(j)\sharp} = 0.$$

Summing over, we get the following equation as in Section 3.2:

$$\mathrm{v} \cdot G = -k^2\Big(\mathrm{u}_\rho^{(0)*} + \sum_{j=1}^{4} \Theta_j \mathrm{u}_\rho^{(j)*}\Big), \tag{56}$$

with $\mathrm{v} = \mathrm{u}_\rho^{(0)\flat} + \sum_{j=1}^4 \Theta_j \mathrm{u}_\rho^{(j)\flat}$. We obtain that

$$\kappa\lambda = \sigma\mu - k^2\Big(\mathrm{u}_\rho^{(0)*} + \sum_{j=1}^{4} \Theta_j \mathrm{u}_\rho^{(j)*}\Big), \tag{57}$$

where

$$\kappa = (1,1,1,0,0,0)^T \cdot \mathrm{v}, \quad \sigma = -(1,1,1,1,1,1)^T \cdot \mathrm{v}.$$

Lemma 4.3. *κ does not vanish in some neighborhood of x_0.*

Proof. Similar argument as Lemma 3.7. Since

$$\begin{pmatrix} \partial_1(\nabla \cdot u_\rho) \\ \partial_2(\nabla \cdot u_\rho) \\ \partial_3(\nabla \cdot u_\rho) \end{pmatrix} = e^{i\rho\cdot x}\left(\begin{pmatrix} -\frac{\sqrt{\mu}}{\lambda+2\mu}\rho_1(\rho\cdot r) + O(|\rho|) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}\rho_2(\rho\cdot r) + O(|\rho|) \\ -\frac{\sqrt{\mu}}{\lambda+2\mu}\rho_2(\rho\cdot r) + O(|\rho|) \end{pmatrix} + O(1)\right)$$

and $\rho \cdot r^{(1)}(x_0) = 0 = \tilde{\tilde{\rho}} \cdot r^{(4)}(x_0)$, we have

$$(1,1,1,0,0,0)^T \cdot (\Theta_1 \mathrm{u}_\rho^{(1)\flat} + \Theta_2 \mathrm{u}_\rho^{(2)\flat} + \Theta_4 \mathrm{u}_\rho^{(4)\flat})(x_0) \sim \tau^{-1}.$$

Hence, we obtain that

$$\begin{aligned} \kappa(x_0) &= (1,1,1,0,0,0)^T \cdot (\mathrm{u}_\rho^{(0)\flat} + \Theta_3 \mathrm{u}_\rho^{(3)\flat})(x_0) + O(\tau^{-1}) \\ &\sim -\frac{\sqrt{\mu}}{2(\lambda+2\mu)} + \Theta_3 \mathrm{u}_\rho^{(3)\flat}(x_0). \end{aligned}$$

We can take $s^{(0)}$ small enough such that $\Theta_3(x_0)$ is small. Thus, $\kappa(x_0) \neq 0$. By continuity of κ, κ does not vanish in some neighborhood V of x_0. □

Let $\Omega_0 = U \cap V \cap \Omega$. Since κ does not vanish in Ω_0, we have

$$\lambda = \frac{\sigma\mu}{\kappa} - \frac{k^2}{\kappa}\Big(\mathrm{u}_\rho^{(0)*} + \sum_{j=1}^{4} \Theta_j \mathrm{u}_\rho^{(j)*}\Big) \quad \text{in } \Omega_0. \tag{58}$$

Applying the similar proof as in Theorem 3.5, we can deduce the following result.

Theorem 4.4 (Global reconstruction of λ). *Let Ω be an open bounded domain of $\mathbb{R}^3$ with smooth boundary. Suppose that the Lamé parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu}) \in \mathcal{P}$ and $\mu|_{\partial\Omega} = \tilde{\mu}|_{\partial\Omega}$. Let $u^{(j)}$ and $\tilde{u}^{(j)}$ be the solutions of the elasticity system with boundary data $g^{(j)}$ for parameters (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively. Let $H = (u^{(j)})_{1\leq j\leq J}$ and $\widetilde{H} = (\tilde{u}^{(j)})_{1\leq j\leq J}$ be the corresponding internal data for (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$, respectively for some integer $J \geq 10$.*

Then there is an open set of the boundary data $(g^{(j)})_{1\leq j\leq J}$ such that if $H = \widetilde{H}$ implies $\lambda = \tilde{\lambda}$ in Ω. Moreover, we have the stability estimate

$$\|\lambda - \tilde{\lambda}\|_{C(\Omega)} \leq C\|H - \widetilde{H}\|_{C^2(\Omega)}.$$

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