

## Anderson localization for the almost Mathieu operator in the exponential regime

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**Abstract.** For the almost Mathieu operator

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n,$$

Jitomirskaya conjectures that for a.e.  $\theta$ ,  $H_{\lambda,\alpha,\theta}$  satisfies Anderson localization if  $|\lambda| > e^\beta$ . Avila and Jitomirskaya verify this for  $|\lambda| > e^{\frac{16}{9}\beta}$ . In the present paper, we extend their result to regime  $|\lambda| > e^{\frac{3}{2}\beta}$ .

**Mathematics Subject Classification (2010).** 34K08, 47A10, 47A35.

**Keywords.** Anderson localization, almost Mathieu operator.

### 1. Introduction

The *almost Mathieu operator* (AMO) is the (discrete) quasi-periodic Schrödinger operator on  $\ell^2(\mathbb{Z})$ :

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha)u_n, \quad \text{with } v(\theta) = 2 \cos 2\pi\theta, \quad (1.1)$$

where  $\lambda$  is the coupling,  $\alpha$  is the frequency, and  $\theta$  is the phase.

$H_{\lambda,\alpha,\theta}$  is a tight binding model for the Hamiltonian of an electron in a one-dimensional lattice or in a two-dimensional lattice, subject to a perpendicular (uniform) magnetic field (through a Landau gauge) [9]. For more applications in physics, we refer the reader to [13] and the references therein.

Besides its relations to some fundamental problems in physics, the AMO itself is also fascinating because of its remarkable richness of the related spectral theory. In B. Simon's list of Schrödinger operator problems for the twenty-first century [14], there are three problems about the AMO. The spectral theory of

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<sup>1</sup> Supported by NNSFC 11271076 and NNSFC 11121101.

AMO has attracted many authors, for example, Avila and Jitomirskaya [1] and [2], Avila and Krikorian [3], Bourgain [5] and [6], Jitomirskaya and Simon [12], and so on.

Anderson localization (i.e., only pure point spectrum with exponentially decaying eigenfunctions) is not only meaningful in physics, but also relates to some problems of the quasi-periodic Schrödinger operator, such as the reducibility of cocycles via Aubry duality [8] and the ten Martini problem (Cantor spectrum conjecture) [1].

For  $\alpha \in \mathbb{Q}$ , it is easy to verify that  $H_{\lambda, \alpha, \theta}$  has no eigenvalues, let alone Anderson localization. Thus, in the present paper, we always assume  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ .

For simplicity, we say  $H_{\lambda, \alpha, \theta}$  satisfies AL if for a.e. phase  $\theta$ ,  $H_{\lambda, \alpha, \theta}$  satisfies Anderson localization.

Jitomirskaya [10] conjectures<sup>1</sup> that  $H_{\lambda, \alpha, \theta}$  satisfies AL for  $|\lambda| > e^\beta$ , where

$$\beta = \beta(\alpha) = \limsup_{n \rightarrow \infty} \frac{\ln q_{n+1}}{q_n}, \quad (1.2)$$

and  $\frac{p_n}{q_n}$  is the continued fraction approximants to  $\alpha$ . One usually calls set

$$\{\alpha \in \mathbb{R} \setminus \mathbb{Q} : \beta(\alpha) > 0\}$$

*exponential regime* and set

$$\{\alpha \in \mathbb{R} \setminus \mathbb{Q} : \beta(\alpha) = 0\}$$

*sub-exponential regime*.

This conjecture is optimal in some way. On the one hand, for every  $\alpha$  there is a generic set of  $\theta$  for which there is no eigenvalues [12]. On the other hand, if  $|\lambda| \leq e^\beta$ , for every  $\theta$ ,  $H_{\lambda, \alpha, \theta}$  is expected to have no localized eigenfunctions (i.e., exponentially decaying eigenfunctions), see footnote 3 in [1].

In [11], Jitomirskaya proves that  $H_{\lambda, \alpha, \theta}$  satisfies AL if  $\alpha \in DC$  and  $|\lambda| > 1$ . In fact Jitomirskaya's arguments also hold for  $\beta(\alpha) = 0$  and  $|\lambda| > 1$ . In order to prove the ten Martini problem, Avila and Jitomirskaya [1] show that  $H_{\lambda, \alpha, \theta}$  satisfies AL if  $|\lambda| > e^{\frac{16}{9}\beta}$ . You and Zhou [15] prove that for almost every phase  $\theta$ , the eigenvalues of operator  $H_{\lambda, \alpha, \theta}$  with exponentially decaying eigenfunctions are dense in the spectrum if  $|\lambda| > Ce^\beta$ , where  $C$  is a large absolute constant. We also should point out that they did not show the Anderson Localization. In the present paper, we verify the conjecture in regime  $|\lambda| > e^{\frac{3}{2}\beta}$ , i.e., the following theorem.

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<sup>1</sup> After submitting the present paper, we learned of that Avila, You, and Zhou claimed they completed the conjecture. (Their preprint is not available yet.)

**Theorem 1.1** (main theorem). *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be such that  $\beta = \beta(\alpha) < \infty$ , then for almost every phase  $\theta$ ,  $H_{\lambda, \alpha, \theta}$  satisfies Anderson localization if  $|\lambda| > e^{\frac{3}{2}\beta}$ .*

We investigate AL following the general scheme of Avila and Jitomirskaya, but some estimates are more subtle.

The present paper is organized as follows. In §2, we give some preliminary notions and facts which are taken from other authors, such as Avila and Jitomirskaya [1], Bourgain [6], and so on. In §3, we set up the regularity of resonant  $y$  if  $|\lambda| > e^{\frac{3}{2}\beta}$ . In §4, we give the proof of main theorem by block resolvent expansion.

## 2. Preliminaries and some known results

It is well known that Anderson localization for a self-adjoint operator  $H$  on  $\ell^2$  is equivalent to the following statements [4].

Assume  $\phi$  is an extended state, i.e.,

$$H\phi = E\phi \quad \text{with } E \in \Sigma(H) \text{ and } |\phi(k)| \leq (1 + |k|)^C,$$

where  $\Sigma(H)$  is the spectrum of self-adjoint operator  $H$ . Then there exists some constant  $c > 0$  such that

$$|\phi(k)| < e^{-c|k|} \quad \text{for } k \rightarrow \infty.$$

We will actually prove a slightly more precise version of Theorem 1.1. Let

$$\mathcal{R}_1 = \{\theta : |\sin \pi(2\theta + k\alpha)| \leq k^{-2} \text{ holds for infinitely many } k, k \in \mathbb{Z}\},$$

and

$$\mathcal{R}_2 = \{\theta : \text{there exists } s \in \mathbb{Z} \text{ such that } 2\theta + s\alpha \in \mathbb{Z}\}.$$

Clearly,

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$$

has zero Lebesgue measure.

**Theorem 2.1.** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be such that  $\beta = \beta(\alpha) < \infty$ , then  $H_{\lambda, \alpha, \theta}$  satisfies Anderson localization if  $\theta \notin \mathcal{R}$  and  $|\lambda| > e^{\frac{3}{2}\beta}$ .*

If  $\alpha$  satisfies  $\beta(\alpha) = 0$ , Theorem 2.1 has been proved by Jitomirskaya [10]. Thus in the present paper, we fix  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  such that  $0 < \beta(\alpha) < \infty$ . Unless stated otherwise, we always assume  $\lambda > e^{\frac{3}{2}\beta}$  (for  $\lambda < -e^{\frac{3}{2}\beta}$ , note that  $H_{\lambda,\alpha,\theta} = H_{-\lambda,\alpha,\theta+\frac{1}{2}}$ ),  $\theta \notin \mathcal{R}$  and  $E \in \Sigma_{\lambda,\alpha}$  (denote by  $\Sigma_{\lambda,\alpha}$  the spectrum of operator  $H_{\lambda,\alpha,\theta}$  since the spectrum does not depend on  $\theta$ ). Since this does not change any of the statements, sometimes the dependence of parameters  $E, \lambda, \alpha, \theta$  will be ignored in the following.

Given an extended state  $\phi$  of  $H_{\lambda,\alpha,\theta}$ , without loss of generality one can assume  $\phi(0) = 1$ . Our objective is to prove that there exists some  $c > 0$  such that

$$|\phi(k)| < e^{-c|k|} \quad \text{for } k \rightarrow \infty.$$

Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}).$$

It is easy to see that  $P_k(\theta)$  is an even function of  $\theta + \frac{1}{2}(k-1)\alpha$  and can be written as a polynomial of degree  $k$  in  $\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)$ :

$$P_k(\theta) = \sum_{j=0}^k c_j \cos^j 2\pi\left(\theta + \frac{1}{2}(k-1)\alpha\right) \triangleq Q_k\left(\cos 2\pi\left(\theta + \frac{1}{2}(k-1)\alpha\right)\right).$$

Let

$$A_{k,r} = \{\theta \in \mathbb{R} : |Q_k(\cos 2\pi\theta)| \leq e^{(k+1)r}\}$$

with  $k \in \mathbb{N}$  and  $r > 0$ .

**Lemma 2.1** ([1], p. 16). *The following inequality holds*

$$\limsup_{k \rightarrow \infty} \sup_{\theta \in \mathbb{R}} \frac{1}{k} \ln |P_k(\theta)| \leq \ln \lambda.$$

By Cramer's rule (see [6], p. 15, for example) for given  $x_1$  and  $x_2 = x_1 + k - 1$ , with  $y \in I = [x_1, x_2] \subset \mathbb{Z}$ , one has

$$|G_I(x_1, y)| = \left| \frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)} \right|, \quad (2.1)$$

and

$$|G_I(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \quad (2.2)$$

By Lemma 2.1, the numerators in (2.1) and (2.2) can be bounded uniformly with respect to  $\theta$ . Namely, for any  $\varepsilon > 0$ ,

$$|P_n(\theta)| \leq e^{(\ln \lambda + \varepsilon)n} \quad (2.3)$$

for  $n$  large enough.

**Definition 2.1.** Fix  $t > 0$ ,  $0 < \delta < 1/2$ . A point  $y \in \mathbb{Z}$  will be called  $(t, k)$ -regular with  $\delta$  if there exists an interval  $[x_1, x_2]$  containing  $y$ , where

$$x_2 = x_1 + k - 1,$$

such that

$$|G_{[x_1, x_2]}(y, x_i)| < e^{-t|y-x_i|} \quad \text{and} \quad |y - x_i| \geq \delta k, \quad \text{for } i = 1, 2.$$

It is easy to check ([6], p. 61) that

$$\phi(x) = -G_{[x_1, x_2]}(x_1, x)\phi(x_1 - 1) - G_{[x_1, x_2]}(x, x_2)\phi(x_2 + 1), \quad (2.4)$$

where  $x \in I = [x_1, x_2] \subset \mathbb{Z}$ . Our strategy is to establish the  $(t, k(y))$ -regularity for every large  $y$ , then localized property is easy to obtain by (2.4) and the block resolvent expansion.

**Definition 2.2.** We say that the set  $\{\theta_1, \dots, \theta_{k+1}\}$  is  $\epsilon$ -uniform if

$$\max_{x \in [-1, 1]} \max_{i=1, \dots, k+1} \prod_{j=1, j \neq i}^{k+1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} < e^{k\epsilon}. \quad (2.5)$$

**Lemma 2.2** ([1], Lemma 9.3). *Suppose that  $\{\theta_1, \dots, \theta_{k+1}\}$  is  $\epsilon_1$ -uniform. Then there exists some  $\theta_i$  in set  $\{\theta_1, \dots, \theta_{k+1}\}$  such that  $\theta_i \notin A_{k, \ln \lambda - \epsilon}$  if  $\epsilon > \epsilon_1$  and  $k$  is sufficiently large.*

Assume without loss of generality that  $y > 0$ . Fix a sufficient small constant  $\eta$  (that will be determined in Theorem 3.3). Define

$$b_n = \max \{ \eta q_{n-1}, q_n^{8/9} \},$$

where  $q_n$  is given by (1.2), and find  $n$  such that

$$b_n \leq y < b_{n+1}.$$

We will distinguish two cases:

- (i)  $|y - \ell q_n| \leq b_n$  for some  $\ell \geq 1$ , called *resonance*;
- (ii)  $|y - \ell q_n| > b_n$  for all  $\ell \geq 0$ , called *non-resonance*.

For the non-resonant  $y$ , Avila and Jitomirskaya have established the regularity for  $y$ . We give the theorem directly.

**Theorem 2.2** ([1], Lemma 9.4). *Assume  $\theta \notin \mathcal{R}$ ,  $\lambda > e^\beta$ , and that  $y$  is non-resonant. Let  $s \in \mathbb{N}$  be the largest number such that*

$$sq_{n-1} \leq \text{dist}(y, \{\ell q_n\}_{\ell \geq 0}).$$

*Then for all  $\varepsilon > 0$  and  $n$  large enough, the following hold:*

(i) *if  $s \geq 1$ , then  $y$  is  $(\ln \lambda + 9 \ln(sq_{n-1}/q_n)/q_{n-1} - \varepsilon, 2sq_{n-1} - 1)$ -regular with*

$$\delta = \frac{1}{8};$$

(ii) *If  $s = 0$ , then  $y$  is either  $(\ln \lambda - \varepsilon, 2[q_{n-1}/2] - 1)$ -, or  $(\ln \lambda - \varepsilon, 2[q_n/2] - 1)$ -, or  $(\ln \lambda - \varepsilon, 2q_{n-1} - 1)$ -regular with*

$$\delta = \frac{\eta}{2}.$$

**Remark 2.1.** Avila and Jitomirskaya let

$$b_n = \max \left\{ \frac{q_{n-1}}{20}, q_n^{\frac{8}{9}} \right\}$$

(i.e.,  $\eta = \frac{1}{20}$ ) in defining resonance and nonresonance, and they obtain  $y$  is regular with  $\delta = \frac{1}{8}$  in case (i), and with  $\delta = \frac{1}{40}$  (i.e.,  $\delta = \frac{\eta}{2}$ ) in case (ii). We give the general definition of resonance and nonresonance, and Theorem 2.2 also holds. The analysis follows from Avila–Jitomirskaya’s arguments, we omit the proof.

**Lemma 2.3** ([1], Lemma 9.8). *Let  $m \in \mathbb{N}$  be such that*

$$m < \frac{q_{r+1}}{10q_n},$$

*where  $r \geq n$ . Given a integer sequence  $|m_k| \leq m - 1$ ,  $k = 1, \dots, q_n$ , let  $1 \leq k_0 \leq q_n$  be such that*

$$|\sin \pi(x + (k_0 + m_{k_0} q_r) \alpha)| = \min_{1 \leq k \leq q_n} |\sin \pi(x + (k + m_k q_r) \alpha)|,$$

*then*

$$\left| \sum_{\substack{k=1 \\ k \neq k_0}}^{q_n} \ln |\sin \pi(x + (k + m_k q_r) \alpha)| + (q_n - 1) \ln 2 \right| < C \ln q_n + C(\Delta_n + (m - 1)\Delta_r)q_n \ln q_n,$$

*where  $\Delta_n = |q_n \alpha - p_n|$ .*

### 3. Regularity for resonant $y$

In this section, we mainly concern the regularity for resonant  $y$ . In this condition  $y > \frac{q_n}{2}$ . Thus by the definition of resonance, there exists some positive integer  $\ell$  with  $1 \leq \ell \leq q_{n+1}^{8/9}/q_n$  such that  $|y - \ell q_n| \leq b_n$ . Fix the positive integer  $\ell$  and set  $I_1, I_2 \subset \mathbb{Z}$  as

$$I_1 = \left[ -\left\lfloor \frac{2}{3}q_n \right\rfloor, \left\lfloor \frac{2}{3}q_n \right\rfloor - 2 \right],$$

and

$$I_2 = \left[ (\ell - 1)q_n + \left\lfloor \frac{2}{3}q_n \right\rfloor - 1, (\ell + 1)q_n - \left\lfloor \frac{2}{3}q_n \right\rfloor - 1 \right],$$

and let

$$\theta_j = \theta + j\alpha \quad \text{for } j \in I_1 \cup I_2.$$

The set  $\{\theta_j\}_{j \in I_1 \cup I_2}$  consists of  $2q_n$  elements.

Note that, below, we replace  $I = [x_1, x_2] \cap \mathbb{Z}$  with  $I = [x_1, x_2]$  for simplicity, and assume  $\varepsilon > 0$  is sufficiently small.

We will use the following three steps to establish regularity for  $y$ .

**STEP 1.** For any  $\varepsilon > 0$ , we set up the  $\frac{\beta}{2} + \varepsilon$ -uniformity of  $\{\theta_j\}$  where  $\theta_j = \theta + j\alpha$  and  $j$  ranges through  $I_1 \cup I_2$ . By Lemma 2.2, there exists some  $j_0$  with  $j_0 \in I_1 \cup I_2$  such that  $\theta_{j_0} \notin A_{2q_n-1, \ln \lambda - \frac{\beta}{2} - C\eta}$ .

**STEP 2.** We show that for all  $j \in I_1$ ,  $\theta_j \in A_{2q_n-1, \ln \lambda - \frac{\beta}{2} - C\eta}$  if  $\lambda > e^{\frac{3}{2}\beta}$ . Thus there exists  $\theta_{j_0} \notin A_{2q_n-1, \ln \lambda - \frac{\beta}{2} - C\eta}$  for some  $j_0 \in I_2$ .

**STEP 3.** We establish the regularity for  $y$ .

**Remark 3.1.** In [1], Avila and Jitomirskaya construct

$$I_1 = \left[ -\left\lfloor \frac{5}{8}q_n \right\rfloor, \left\lfloor \frac{5}{8}q_n \right\rfloor - 1 \right],$$

$$I_2 = \left[ (\ell - 1)q_n + \left\lfloor \frac{5}{8}q_n \right\rfloor, (\ell + 1)q_n - \left\lfloor \frac{5}{8}q_n \right\rfloor - 1 \right],$$

and set

$$\theta_j = \theta + j\alpha \quad \text{for } j \in I_1 \cup I_2.$$

They use the above three steps to establish the regularity of  $y$ . More precisely, firstly, they establish the  $(\frac{\beta}{2} + \varepsilon)$ -uniformity of  $\{\theta_j\}$  and that there exists  $\theta_{j_0} \notin A_{2q_n-1, \ln \lambda - \frac{\beta}{2} - 2\varepsilon}$  for some  $j_0 \in I_1 \cup I_2$ . Secondly, they prove that for all  $j \in I_1$ ,  $\theta_j \in A_{2q_n-1, \ln \lambda - \frac{\beta}{2} - 2\varepsilon}$ , and thus there exists  $\theta_{j_0} \notin A_{2q_n-1, \ln \lambda - \frac{\beta}{2} - 2\varepsilon}$  for some  $j_0 \in I_2$ , if  $\lambda > e^{\frac{16}{9}\beta}$ . Thirdly, they set up the regularity of  $y$ . In the present paper, we reconstruct  $I_1$  and  $I_2$ , and show that the three steps also hold.

Recall that

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \Delta_n, \quad \text{for all } 1 \leq k < q_{n+1}, \quad (3.1)$$

and

$$\frac{1}{2q_{n+1}} \leq \Delta_n \leq \frac{1}{q_{n+1}}, \quad (3.2)$$

where

$$\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{j \in \mathbb{Z}} |x - j|.$$

STEP 1. We establish the  $(\frac{\beta}{2} + \varepsilon)$ -uniformity for  $\{\theta_j\}_{j \in I_1 \cup I_2}$ .

In Lemma 2.3, let  $r = n$  and  $m = \ell \leq q_{n+1}^{8/9}/q_n$ , one has

$$(\Delta_n + (m-1)\Delta_r)q_n = \ell\Delta_n q_n \leq C,$$

since  $\Delta_n \leq \frac{1}{q_{n+1}}$  by (3.2). Moreover, we obtain the following lemma.

**Lemma 3.1.** *Given a integer sequence  $|m_k| \leq \ell - 1$ ,  $k = 1, \dots, q_n$ , let  $1 \leq k_0 \leq q_n$  be such that*

$$|\sin \pi(x + (k_0 + m_{k_0}q_n)\alpha)| = \min_{1 \leq k \leq q_n} |\sin \pi(x + (k + m_k q_n)\alpha)|.$$

Then

$$\begin{aligned} -(q_n - 1) \ln 2 - C \ln q_n &\leq \sum_{\substack{k=1 \\ k \neq k_0}}^{q_n} \ln |\sin \pi(x + (k + m_k q_n)\alpha)| \\ &\leq -(q_n - 1) \ln 2 + C \ln q_n. \end{aligned} \quad (3.3)$$



**Theorem 3.1.** *For all  $\varepsilon > 0$ , the set  $\{\theta_j\}_{j \in I_1 \cup I_2}$  is  $(\frac{\beta}{2} + \varepsilon)$ -uniform for  $\theta \notin \mathcal{R}$  and sufficiently large  $n$ .*

*Proof.* Let

$$I'_1 = \left[ -\left\lfloor \frac{2}{3}q_n \right\rfloor, -\left\lfloor \frac{2}{3}q_n \right\rfloor + q_n - 1 \right]$$

and

$$I'_2 = \left[ -\left\lfloor \frac{2}{3}q_n \right\rfloor + q_n, \left\lfloor \frac{2}{3}q_n \right\rfloor - 2 \right] \cup \left[ (\ell - 1)q_n + \left\lfloor \frac{2}{3}q_n \right\rfloor - 1, (\ell + 1)q_n - \left\lfloor \frac{2}{3}q_n \right\rfloor - 1 \right].$$

Clearly, both  $\{\theta_j\}_{j \in I'_1}$  and  $\{\theta_j\}_{j \in I'_2}$  consist of  $q_n$  elements, and  $I'_1 \cup I'_2 = I_1 \cup I_2$ . In (2.5), let  $x = \cos 2\pi a$ ,  $k = 2q_n - 1$  and take the logarithm. Thus in order to prove the theorem, it suffices to show that for any  $a \in \mathbb{R}$  and  $i \in I'_1 \cup I'_2$ ,

$$\begin{aligned} & \ln \prod_{j \in I'_1 \cup I'_2, j \neq i} \frac{|\cos 2\pi a - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} \\ &= \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| - \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j| \\ &< (2q_n - 1) \left( \frac{\beta}{2} + \varepsilon \right). \end{aligned}$$

Without loss of generality assume  $i \in I'_1$ . We estimate

$$\sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j|$$

first.

Clearly,

$$\begin{aligned} & \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \\ &= \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\sin \pi(a + \theta_j)| \\ & \quad + \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\sin \pi(a - \theta_j)| + (2q_n - 1) \ln 2 \\ &= \Sigma_+ + \Sigma_- + (2q_n - 1) \ln 2, \end{aligned} \tag{3.4}$$

where

$$\Sigma_+ = \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\sin \pi(a + \theta + j\alpha)|,$$

and

$$\Sigma_- = \sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\sin \pi(a - \theta - j\alpha)|.$$

Write  $\Sigma_+$  as the form

$$\Sigma_+ = \sum_{j \in I'_1, j \neq i} \ln |\sin \pi(a + \theta + j\alpha)| + \sum_{j \in I'_2} \ln |\sin \pi(a + \theta + j\alpha)|. \quad (3.5)$$

We will estimate

$$\sum_{j \in I'_1, j \neq i} \ln |\sin \pi(a + \theta + j\alpha)| \quad \text{and} \quad \sum_{j \in I'_2} \ln |\sin \pi(a + \theta + j\alpha)|,$$

respectively.

On the one hand,

$$\begin{aligned} & \sum_{j \in I'_1, j \neq i} \ln |\sin \pi(a + \theta + j\alpha)| \\ &= \sum_{j \in I'_1} \ln |\sin \pi(a + \theta + j\alpha)| - \ln |\sin \pi(a + \theta + i\alpha)| \\ &= \sum_{k=1}^{q_n} \ln |\sin \pi(x + k\alpha)| - \ln |\sin \pi(a + \theta + i\alpha)| \\ &= \sum_{k=1, k \neq k_0}^{q_n} \ln |\sin \pi(x + k\alpha)| + \ln |\sin \pi(x + k_0\alpha)| - \ln |\sin \pi(a + \theta + i\alpha)|, \end{aligned}$$

where

$$x = a + \theta - \left( \left[ \frac{2}{3} q_n \right] + 1 \right) \alpha$$

and  $k_0$  satisfies

$$|\sin \pi(x + k_0\alpha)| = \min_{1 \leq k \leq q_n} |\sin \pi(x + k\alpha)|.$$

In Lemma 3.1, let  $m_k = 0$ ,  $k = 1, 2, \dots, q_n$ , by the second equality of (3.3), one has

$$\sum_{k=1, k \neq k_0}^{q_n} \ln |\sin \pi(x + k\alpha)| \leq -(q_n - 1) \ln 2 + C \ln q_n.$$

Since

$$\ln |\sin \pi(x + k_0 \alpha)| \leq \ln |\sin \pi(a + \theta + i \alpha)|$$

(by the minimality of  $k_0$ ), we have

$$\sum_{j \in I'_1, j \neq i} \ln |\sin \pi(a + \theta + j \alpha)| \leq -(q_n - 1) \ln 2 + C \ln q_n. \quad (3.6)$$

On the other hand,

$$\begin{aligned} & \sum_{j \in I'_2} \ln |\sin \pi(a + \theta + j)| \\ &= \sum_{k=1}^{q_n} \ln |\sin \pi(x + (k + m_k) \alpha)| \\ &= \sum_{k=1, k \neq k_0}^{q_n} \ln |\sin \pi(x + (k + m_k) \alpha)| + \ln |\sin \pi(x + (k_0 + m_{k_0}) \alpha)|, \end{aligned}$$

where

$$x = a + \theta + \left( - \left[ \frac{2}{3} q_n \right] + q_n - 1 \right) \alpha,$$

$$m_k = 0, \quad \text{for } 1 \leq k \leq 2 \left[ \frac{2}{3} q_n \right] - q_n - 1,$$

and

$$m_k = \ell - 1, \quad \text{for } 2 \left[ \frac{2}{3} q_n \right] - q_n \leq k \leq q_n,$$

and  $k_0$  satisfies

$$|\sin \pi(x + (k_0 + m_{k_0}) \alpha)| = \min_{1 \leq k \leq q_n} |\sin \pi(x + (k + m_k) \alpha)|.$$

By the second equality of (3.3) again, one has

$$\sum_{k=1, k \neq k_0}^{q_n} \ln |\sin \pi(x + (k + m_k) \alpha)| \leq -(q_n - 1) \ln 2 + C \ln q_n.$$

In addition

$$\ln |\sin \pi(x + (k_0 + m_{k_0}) \alpha)| \leq 0,$$

and one has

$$\sum_{j \in I'_2} \ln |\sin \pi(a + \theta + j \alpha)| \leq -(q_n - 1) \ln 2 + C \ln q_n. \quad (3.7)$$

Putting (3.5), (3.6), and (3.7) together, we have

$$\Sigma_+ \leq -2q_n \ln 2 + C \ln q_n. \quad (3.8)$$

We are now in the position to estimate  $\Sigma_-$ . In order to avoid repetition, we omit some details. Similarly,  $\Sigma_-$  consists of 2 terms of the form as (3.3), plus two terms of the form  $\min_{k=1, \dots, q_n} \ln |\sin \pi(x + (k + m_k q_n)\alpha)|$ , where  $m_k \in \{0, (\ell - 1)\}$ ,  $k = 1, \dots, q_n$ , minus  $\ln |\sin \pi(a - \theta_i)|$ . Following the estimate of  $\Sigma_+$ ,

$$\Sigma_- \leq -2q_n \ln 2 + C \ln q_n. \quad (3.9)$$

Putting (3.8) and (3.9) into (3.4), we obtain

$$\sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \leq -2q_n \ln 2 + C \ln q_n. \quad (3.10)$$

The estimate of

$$\sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j|$$

require a bit more work.

It is easy to see that

$$\sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j| = \Sigma'_+ + \Sigma'_- + (2q_n - 1) \ln 2, \quad (3.11)$$

where

$$\Sigma'_+ = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(2\theta + (i + j)\alpha)|,$$

and

$$\Sigma'_- = \sum_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(i - j)\alpha|.$$

Firstly, we estimate  $\Sigma'_+$ . Similarly,  $\Sigma'_+$  consists of 2 terms of the form as (3.3), plus two terms of the form

$$\min_{k=1, \dots, q_n} \ln |\sin \pi(x + (k + m_k q_n)\alpha)|,$$

where  $m_k \in \{0, (\ell - 1)\}$ ,  $k = 1, \dots, q_n$ , minus  $\ln |\sin 2\pi(\theta + i\alpha)|$ .

Following the above arguments and using the first inequality of (3.3), we obtain

$$\Sigma'_+ > -2q_n \ln 2 - C \ln q_n + 2 \min_{j,i \in I_1 \cup I_2} \ln |\sin \pi(2\theta + (j+i)\alpha)|. \quad (3.12)$$

Thus it is enough to estimate the last term in (3.12). By the hypothesis  $\theta \notin \mathcal{R}$ , one has

$$\min_{j,i \in [-2q_n, 2q_n-1]} |\sin \pi(2\theta + (j+i)\alpha)| > \frac{1}{16q_n^2} \text{ for large } n. \quad (3.13)$$

If  $k \in I_2$ , let

$$\ell_k = \ell - 1 \quad \text{and} \quad k' = k - \ell_k q_n;$$

if  $k \in I_1$ , let

$$\ell_k = 0 \quad \text{and} \quad k' = k.$$

Then  $i', j' \in [-2q_n, 2q_n - 1]$ . If  $q_{n+1} > q_n^{100}$ , it is easy to verify that

$$|\ell_k \Delta_n| < \frac{1}{q_n^5}.$$

Combining with (3.13), we have, for any  $i, j \in I_1 \cup I_2$ ,

$$\begin{aligned} & |\sin \pi(2\theta + (j+i)\alpha)| \\ &= |\sin \pi(2\theta + (j'+i')\alpha) \cos \pi(\ell_i + \ell_j) \Delta_n \\ &\quad \pm \cos \pi(2\theta + (j'+i')\alpha) \sin \pi(\ell_i + \ell_j) \Delta_n| \\ &> \frac{1}{100q_n^2} \end{aligned} \quad (3.14)$$

(the  $\pm$  depending on the sign of  $q_n \alpha - p_n$ ).

If  $q_{n+1} \leq q_n^{100}$ , we also have

$$|\sin \pi(2\theta + (j+i)\alpha)| > \frac{1}{100q_n^{200}}, \quad \text{for any } i, j \in I_1 \cup I_2, \quad (3.15)$$

Thus, by (3.12), (3.14), and (3.15), one has

$$\Sigma'_+ > -2q_n \ln 2 - C \ln q_n. \quad (3.16)$$

Similarly,  $\Sigma'_-$  consists of 2 terms of the form as (3.3) plus the minimum term (because  $\min_{j \in I'_1} |\sin \pi(i-j)\alpha| = 0$ , then  $\sum_{j \in I'_1, j \neq i} \ln |\sin \pi(i-j)\alpha|$  is exactly of the form (3.3)). It follows that

$$\Sigma'_- > -2q_n \ln 2 - C \ln q_n + \min_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi((j-i)\alpha)|. \quad (3.17)$$

We are now in the position to estimate the last term in (3.17). Note that for any  $i \in I_1 \cup I_2$ , there is only one  $\tilde{i} \in I_1 \cup I_2$  such that  $|i - \tilde{i}| = q_n$  or  $\ell q_n$ . It is easy to check

$$\begin{aligned} \ln |\sin \pi(i - \tilde{i})\alpha| &\geq \min\{\ln |\sin \pi q_n \alpha|, \ln |\sin \pi \ell q_n \alpha|\} \\ &> -\ln q_{n+1} - C, \end{aligned} \quad (3.18)$$

since

$$\Delta_n \geq \frac{1}{2q_{n+1}}.$$

If  $j \neq i, \tilde{i}$  and  $j \in I_1 \cup I_2$ , then

$$j - i = r + m'_j q_n, \quad \text{with } 1 \leq |r| < q_n \text{ and } |m'_j| \leq \ell.$$

Thus by (3.1) and (3.2), one has

$$\|r\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \Delta_{n-1} \geq \frac{1}{2q_n}$$

and

$$\begin{aligned} \min_{j \in I_1 \cup I_2, j \neq i, \tilde{i}} \ln |\sin \pi(j-i)\alpha| &> \ln(\|r\alpha\|_{\mathbb{R}/\mathbb{Z}} - \ell \Delta_n) - C \\ &> -\ln q_n - C, \end{aligned} \quad (3.19)$$

since

$$\ell \Delta_n < \frac{1}{10q_n}$$

for  $n$  large enough.

By (3.18) and (3.19), one has

$$\min_{j \in I_1 \cup I_2, j \neq i} \ln |\sin \pi(j-i)\alpha| > -\ln q_{n+1} - C \ln q_n.$$

By the definition of

$$\beta = \limsup_{n \rightarrow \infty} \frac{\ln q_{n+1}}{q_n},$$

equation (3.17) becomes

$$\begin{aligned} \Sigma'_- &> -2q_n \ln 2 - \ln q_{n+1} - C \ln q_n \\ &> -2q_n \ln 2 - (\beta + \varepsilon)q_n - C \ln q_n, \end{aligned} \quad (3.20)$$

for large  $n$ .

By (3.11), (3.16), and (3.20),

$$\sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j| > -2q_n \ln 2 - (\beta + \varepsilon)q_n - C \ln q_n.$$

Together with (3.10), we obtain

$$\sum_{j \in I'_1 \cup I'_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| - \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j| < (\beta + \varepsilon)q_n + C \ln q_n.$$

This implies

$$\max_{x \in [-1, 1]} \max_{i=1, \dots, k+1} \prod_{j=1, j \neq i}^{k+1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} < e^{(2q_n - 1)(\frac{\beta}{2} + \varepsilon)}$$

for large enough  $n$ . □

In Lemma 2.2, let  $k = 2q_n - 1$ ,  $\epsilon_1 = \frac{\beta}{2} + \eta$  and  $\epsilon = \frac{\beta}{2} + C\eta$ . Clearly,  $\epsilon_1 < \epsilon$ . Thus there exists some  $j_0 \in I_1 \cup I_2$  such that  $\theta_{j_0} \notin A_{2q_n - 1, \ln \lambda - \frac{\beta}{2} - C\eta}$  for  $n$  large enough.

STEP 2. We will show that  $\theta_j \in A_{2q_n - 1, \ln \lambda - \frac{\beta}{2} - C\eta}$  for all  $j \in I_1$ .

**Lemma 3.2.** *Suppose  $k \in [-2q_n, 2q_n]$  and*

$$d = \text{dist}(k, \{mq_n\}_{m \geq 0}) \geq \frac{q_n}{4},$$

*then, for sufficiently large  $n$ ,*

$$|\phi(k)| < \exp(-(\ln \lambda - C\eta)d).$$

*Proof.* We will use block resolvent expansion to prove this lemma. By hypothesis  $k \in [-2q_n, 2q_n]$ , there exists some  $m \in \{-2, -1, 0, 1\}$  such that

$$mq_n \leq k < (m+1)q_n.$$

For any  $y \in [mq_n + \eta q_n + 1, (m+1)q_n - \eta q_n - 1]$ , apply Theorem 2.2 with  $\varepsilon = \eta$ . Note that in case (i), we have

$$\ln \lambda + 9 \ln(sq_{n-1}/q_n)/q_{n-1} - \eta \geq \ln \lambda - C\eta,$$

for large  $n$ . Thus  $y$  is regular with  $t = \ln \lambda - C\eta$ . Moreover, there exists an interval

$$I(y) = [x_1, x_2] \subset [(m-1)q_n, (m+2)q_n]$$

such that  $y \in I(y)$  and

$$\text{dist}(y, \partial I(y)) \geq \frac{\eta}{2} |I(y)| \geq \frac{\eta}{2} q_{n-1} \quad (3.21)$$

and

$$|G_{I(y)}(y, x_i)| < e^{-(\ln \lambda - C\eta)|y - x_i|}, \quad i = 1, 2, \quad (3.22)$$

where  $\partial I(y)$  is the boundary of the interval  $I(y)$ , i.e.,  $\{x_1, x_2\}$ , and recall that  $|I(y)|$  is the number of  $I(y)$ , i.e.,

$$|I(y)| = x_2 - x_1 + 1.$$

For  $z \in \partial I(y)$ , let  $z'$  be the neighbor of  $z$  (i.e.,  $|z - z'| = 1$ ) not belonging to  $I(y)$ .

If

$$x_2 + 1 < (m+1)q_n - \eta q_n \quad \text{or} \quad x_1 - 1 > mq_n + \eta q_n,$$

then we can expand  $\phi(x_2 + 1)$  or  $\phi(x_1 - 1)$  as (2.4). We can continue this process until we arrive to  $z$  such that

$$z + 1 \geq (m+1)q_n - \eta q_n \quad \text{or} \quad z - 1 \leq mq_n + \eta q_n,$$

or the iterating number reaches

$$\left[ \frac{2d}{\eta q_{n-1}} \right].$$



Thus, by (2.4)

$$\phi(k) = \sum_{s; z_{i+1} \in \partial I(z'_i)} G_{I(k)}(k, z_1) G_{I(z'_1)}(z'_1, z_2) \dots G_{I(z'_s)}(z'_s, z_{s+1}) \phi(z'_{s+1}), \quad (3.23)$$

where in each term of the summation one has

$$mq_n + \eta q_n + 1 < z_i < (m+1)q_n - \eta q_n - 1, \quad i = 1, \dots, s,$$

and either

$$z_{s+1} \notin [mq_n + \eta q_n + 2, (m+1)q_n - \eta q_n - 2], \quad s+1 < \left\lceil \frac{2d}{\eta q_{n-1}} \right\rceil, \quad (*)$$

or

$$s+1 = \left\lceil \frac{2d}{\eta q_{n-1}} \right\rceil. \quad (**)$$

When (\*), then, by (3.22),

$$\begin{aligned} & |G_{I(k)}(k, z_1) G_{I(z'_1)}(z'_1, z_2) \dots G_{I(z'_s)}(z'_s, z_{s+1}) \phi(z'_{s+1})| \\ & < e^{-(\ln \lambda - C\eta)(|k-z_1| + \sum_{i=1}^s |z'_i - z_{i+1}|)} q_n^C \\ & < e^{-(\ln \lambda - C\eta)(|k-z_{s+1}| - (s+1))} q_n^C \\ & < e^{-(\ln \lambda - C\eta)(d - \eta q_n - 4 - \frac{2d}{\eta q_{n-1}})} q_n^C, \end{aligned} \quad (3.24)$$

since

$$|\phi(z'_{s+1})| \leq (1 + |z'_{s+1}|)^C \leq q_n^C.$$

When (\*\*), then, using (3.21) and (3.22), we obtain

$$\begin{aligned} & |G_{I(k)}(k, z_1) G_{I(z'_1)}(z'_1, z_2) \dots G_{I(z'_s)}(z'_s, z_{s+1}) \phi(z'_{s+1})| \\ & < e^{-(\ln \lambda - C\eta) \frac{\eta q_n - 1}{2} \left\lceil \frac{2d}{\eta q_{n-1}} \right\rceil} q_n^C. \end{aligned} \quad (3.25)$$

Finally, note that the total number of terms in (3.23) is at most  $2 \left\lceil \frac{2d}{\eta q_{n-1}} \right\rceil$  and  $d \geq \frac{qn}{4}$ . Combining with (3.24) and (3.25), we obtain

$$|\phi(k)| < e^{-(\ln \lambda - C\eta)d}$$

for large  $n$ . □

**Remark 3.2.** Under the hypothesis of Lemma 3.2, Avila and Jitomirskaya only prove that

$$|\phi(k)| < \exp\left(-(\ln \lambda - \varepsilon)\frac{d}{2}\right).$$

We give the refined version.

**Theorem 3.2.** For any  $b \in [-\frac{5}{3}q_n, -\frac{1}{3}q_n]$ , we have

$$\theta + (b + q_n - 1)\alpha \in A_{2q_n-1, 2\ln \lambda/3 + C\eta}$$

if  $n$  is large enough, i.e., for all  $j \in I_1$ ,  $\theta_j \in A_{2q_n-1, 2\ln \lambda/3 + C\eta}$ .

*Proof.* Let

$$b_1 = b - 1 \quad \text{and} \quad b_2 = b + 2q_n - 1.$$

Applying Lemma 3.2, one obtains that for  $i = 1, 2$ ,

$$|\phi(b_i)| \leq \begin{cases} e^{-(\ln \lambda - C\eta)(2q_n + b)} & \text{if } -\frac{5q_n}{3} \leq b \leq -\frac{3q_n}{2}, \\ e^{-(\ln \lambda - C\eta)|q_n + b|} & \text{if } -\frac{3q_n}{2} < b < -\frac{q_n}{2} \text{ and } |b + q_n| > \frac{1}{4}q_n, \\ e^{(\ln \lambda - C\eta)b} & \text{if } -\frac{q_n}{2} \leq b \leq -\frac{q_n}{3}. \end{cases}$$

In (2.4), let

$$I = [b, b + 2q_n - 2] \quad \text{and} \quad x = 0;$$

we get, for  $n$  large enough,

$$\begin{aligned} & \max(|G_I(0, b)|, |G_I(0, b + 2q_n - 2)|) \\ & \geq \begin{cases} e^{(\ln \lambda - C\eta)(2q_n + b)} & \text{if } -\frac{5q_n}{3} \leq b \leq -\frac{3q_n}{2}, \\ e^{(\ln \lambda - C\eta)|q_n + b|} & \text{if } -\frac{3q_n}{2} < b < -\frac{q_n}{2} \text{ and } |b + q_n| > \frac{1}{4}q_n; \\ e^{-(\ln \lambda - C\eta)b}, & \text{if } -\frac{q_n}{2} \leq b \leq -\frac{q_n}{3}, \\ e^{-C\eta}, & \text{if } |b + q_n| \leq \frac{1}{4}q_n, \end{cases} \end{aligned}$$

since  $\phi(0) = 1$  and  $|\phi(k)| \leq (1 + |k|)^C$ .

Let

$$\varepsilon = \eta$$

in (2.3), and let

$$I = [b, b + 2q_n - 2], \quad y = 0, \quad k = 2q_n - 1$$

in (2.1) and (2.2). After careful computation, we obtain

$$\begin{aligned} & |Q_{2q_n-1}(\cos 2\pi(\theta + (b + q_n - 1)\alpha))| \\ &= |P_{2q_n-1}(\theta + b\alpha)| \\ &\leq \min\{|G_I(0, b)|^{-1}e^{(\ln\lambda+\eta)(b+2q_n-2)}, |G_I(0, b+2q_n-2)|^{-1}e^{-(\ln\lambda+\eta)b}\} \\ &\leq e^{(2q_n-1)(2\ln\lambda/3+C\eta)}. \end{aligned}$$

Since  $\ln \lambda > \frac{3\beta}{2}$ , thus

$$\frac{2\ln\lambda}{3} + C\eta < \ln\lambda - \frac{\beta}{2} - C\eta$$

for small enough  $\eta$ . By STEP 1 and STEP 2, we have

$$\theta_j \in A_{2q_n-1, \ln\lambda - \frac{\beta}{2} - C\eta} \quad \text{for all } j \in I_1.$$

This implies there exists some  $j_0 \in I_2$  such that  $\theta_{j_0} \notin A_{2q_n-1, \ln\lambda - \frac{\beta}{2} - C\eta}$ .  $\square$

STEP 3. Establish the regularity for  $y$ .

**Theorem 3.3.** *For some  $t > 0$ ,  $y$  is  $(t, 2q_n - 1)$ -regular with  $\delta = 1/5$  for large enough  $n$ .*

*Proof.* According to the previous two steps, there exists some

$$\theta_{j_0} \notin A_{2q_n-1, \ln\lambda - \frac{\beta}{2} - C\eta}, \quad \text{for } j_0 \in I_2.$$

Set

$$I = [j_0 - q_n + 1, j_0 + q_n - 1] = [x_1, x_2].$$

In (2.3), let  $\varepsilon = \eta$ ; combining with (2.1) and (2.2), it is easy to verify that

$$|G_I(y, x_i)| < e^{(\ln\lambda+\eta)(2q_n-2-|y-x_i|)-2q_n(\ln\lambda-\frac{\beta}{2}-C\eta)}.$$

By a simple computation  $|y - x_i| \geq (\frac{2}{3} - C\eta)q_n$ ; then

$$|G_I(y, x_i)| < e^{-|y-x_i|(\ln\lambda-\frac{3}{2}\beta-C\eta)},$$

for large enough  $n$ . Select  $\eta$  small enough such that  $t = \ln\lambda - \frac{3\beta}{2} - C\eta > 0$ , then  $y$  is  $(\ln\lambda - \frac{3\beta}{2} - C\eta, 2q_n - 1)$ -regular with  $\delta = 1/5$ .  $\square$

### 4. The proof of Theorem 2.1

Now that the regularity for  $y$  is established, we will use block resolvent expansion again to prove Theorem 2.1.

*Proof of Theorem 2.1.* Give some  $k$  with  $k > q_n$  and  $n$  large enough. Using Theorems 2.2 and 3.3, for all  $y \in [b_n, 2k]$ , then there exists an interval

$$I(y) = [x_1, x_2] \subset [-4k, 4k], \quad \text{with } y \in I(y),$$

such that

$$\text{dist}(y, \partial I(y)) > \frac{\eta}{2} q_{n-1} \tag{4.1}$$

and

$$|G_{I(y)}(y, x_i)| < e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)|y-x_i|}, \quad i = 1, 2. \tag{4.2}$$

As in the proof of Lemma 3.2, denote by  $\partial I(y)$  the boundary of the interval  $I(y)$ . For  $z \in \partial I(y)$ , let  $z'$  be the neighbor of  $z$ , (i.e.,  $|z - z'| = 1$ ) not belonging to  $I(y)$ .

If

$$x_2 + 1 < 2k \quad \text{or} \quad x_1 - 1 > b_n,$$

then we can expand  $\phi(x_2 + 1)$  or  $\phi(x_1 - 1)$  as (2.4). We can continue this process until we arrive to  $z$  such that

$$z + 1 \geq 2k \quad \text{or} \quad z - 1 \leq b_n,$$

or the iterating number reaches

$$\left[ \frac{2k}{\eta q_{n-1}} \right].$$

By (2.4),

$$\phi(k) = \sum_{s; z_{i+1} \in \partial I(z'_i)} G_{I(k)}(k, z_1) G_{I(z'_1)}(z'_1, z_2) \dots G_{I(z'_s)}(z'_s, z_{s+1}) \phi(z'_{s+1}), \tag{4.3}$$

where in each term of the summation we have

$$b_n + 1 < z_i < 2k - 1, \quad i = 1, \dots, s,$$

and either

$$z_{s+1} \notin [b_n + 2, 2k - 2], \quad s + 1 < \left[ \frac{2k}{\eta q_{n-1}} \right], \tag{\star}$$

or

$$s + 1 = \left[ \frac{2k}{\eta q_{n-1}} \right]. \tag{\star\star}$$

When  $(\star)$ , then, by (4.2), one has

$$\begin{aligned}
& |G_{I(k)}(k, z_1)G_{I(z'_1)}(z'_1, z_2) \dots G_{I(z'_s)}(z'_s, z_{s+1})\phi(z'_{s+1})| \\
& \leq e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)(|k - z_1| + \sum_{i=1}^s |z'_i - z_{i+1}|)} k^C \\
& \leq e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)(|k - z_{s+1}| - (s+1))} k^C \\
& \leq \max\{e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)(k - b_n - 4 - \frac{2k}{nq_{n-1}})} k^C, \\
& \quad e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)(2k - k - 4 - \frac{2k}{nq_{n-1}})} k^C\}.
\end{aligned} \tag{4.4}$$

When  $(\star\star)$ , then, using (4.1) and (4.2), we obtain

$$\begin{aligned}
& |G_{I(k)}(k, z_1)G_{I(z'_1)}(z'_1, z_2) \dots G_{I(z'_s)}(z'_s, z_{s+1})\phi(z'_{s+1})| \\
& \leq e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)\frac{nq_{n-1}}{2} \lfloor \frac{2k}{nq_{n-1}} \rfloor} k^C.
\end{aligned} \tag{4.5}$$

Finally, note that the total number of terms in (4.3) is at most  $2^{\lfloor \frac{2k}{nq_{n-1}} \rfloor}$ . Combining with (4.4) and (4.5), we obtain

$$|\phi(k)| \leq e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)k} \tag{4.6}$$

for large enough  $n$  (or equivalently large enough  $k$ ).

For  $k < 0$ , the proof is similar. Thus

$$|\phi(k)| \leq e^{-(\ln \lambda - \frac{3}{2}\beta - C\eta)|k|} \quad \text{if } |k| \text{ is large enough.} \tag{4.7}$$

This ends the proof of Theorem 2.1.  $\square$

**Corollary 4.1.** *Suppose  $\lambda > e^{\frac{3}{2}\beta}$  and  $\theta \notin \mathcal{R}$ . If a solution  $\Psi_E(k)$  satisfies*

$$H_{\lambda, \alpha, \theta} \Psi_E = E \Psi_E, \quad \text{with } \Psi_E(k) \leq (1 + |k|)^C \text{ and } E \in \Sigma_{\lambda, \alpha},$$

then

$$\limsup_{|k| \rightarrow \infty} \frac{\ln(\Psi_E^2(k) + \Psi_E^2(k+1))}{2|k|} \leq -\left(\ln \lambda - \frac{3}{2}\beta\right). \tag{4.8}$$

In particular, for  $\beta(\alpha) = 0$

$$\lim_{|k| \rightarrow \infty} \frac{\ln(\Psi_E^2(k) + \Psi_E^2(k+1))}{2|k|} = -\ln \lambda. \tag{4.9}$$

*Proof.* If  $\beta(\alpha) > 0$ , in fact (4.7) holds for any  $\eta > 0$ , this implies

$$\limsup_{|k| \rightarrow \infty} \frac{\ln(\Psi_E^2(k) + \Psi_E^2(k+1))}{2|k|} \leq -\left(\ln \lambda - \frac{3}{2}\beta\right) \quad \text{if } \beta > 0. \quad (4.10)$$

If  $\beta(\alpha) = 0$ , following [1] or [2],  $k$  is  $(t, \ell(k))$ -regular for large  $|k|$ , with

$$t = \ln \lambda - \varepsilon.$$

By the method of block resolvent expansion as above, we can obtain

$$|\Psi_E(k)| < e^{-(\ln \lambda - \varepsilon)|k|} \quad \text{if } k \text{ is large enough,}$$

thus

$$\limsup_{|k| \rightarrow \infty} \frac{\ln(\Psi_E^2(k) + \Psi_E^2(k+1))}{2|k|} \leq -\ln \lambda. \quad (4.11)$$

By (4.10) and (4.11), we obtain (4.8).

By Furman's uniquely ergodic theorem (Corollary 2 in [7]),

$$\liminf_{|k| \rightarrow \infty} \frac{\ln(\Psi_E^2(k) + \Psi_E^2(k+1))}{2|k|} \geq -\ln \lambda.$$

The last two inequalities imply (4.9). □

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Received October 1, 2013; revised October 21, 2014

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