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On a rigidity result for the first conformal eigenvalue of the Laplacian

Romain Petrides

Abstract. Given (M, g) a smooth compact Riemannian manifold without boundary of dimension $n \geq 3$, we consider the first conformal eigenvalue which is by definition the supremum of the first eigenvalue of the Laplacian among all metrics conformal to g of volume 1. We prove that it is always greater than $n\omega_n^{\frac{2}{n}}$, the value it takes in the conformal class of the round sphere, except if (M, g) is conformally diffeomorphic to the standard sphere.

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Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 3$ and let us define the first conformal eigenvalue of (M, g) by

$$\Lambda_1(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(M, \tilde{g}) \operatorname{Vol}_{\tilde{g}}(M)^{\frac{2}{n}}$$

where $\lambda_1(M, g)$ is the first nonzero eigenvalue of the Laplacian $\Delta_g = -\operatorname{div}_g(\nabla)$ and $[g]$ is the conformal class of g. In this paper, we aim at proving a rigidity result concerning this first conformal eigenvalue.

The maximisation on conformal classes is natural because the scale invariant quantity supremum is infinite among all metrics [3] (except in dimension 2, [16]), while El Soufi and Ilias [7] proved that it is always bounded among conformal metrics. Generalizing a result by Li and Yau [13] in dimension 2, they gave an explicit upper bound thanks to the m-conformal volume $V_c(m, M, [g])$ of $(M, [g])$

$$\Lambda_1(M, [g]) \leq n V_c(m, M, [g])^{\frac{2}{n}} \tag{1}$$

These conformal invariants on the standard sphere $(\mathbb{S}^n, [\mathrm{can}])$ satisfy (cf. [7])

$$\Lambda_1(\mathbb{S}^n, [\mathrm{can}]) = n\omega_n^{\frac{2}{n}} = nV_c(\mathbb{S}^n, [\mathrm{can}])^{\frac{2}{n}} \tag{2}$$

and this value is achieved if and only if the metric is round. Here, ω_n denotes the volume of the standard n-sphere. Colbois and El Soufi [4] also proved that, for any compact Riemannian manifold (M, g) of dimension $n \geq 3$,

$$\Lambda_1(M, [g]) \geq \Lambda_1(\mathbb{S}^n, [\mathrm{can}]).$$

We prove here that the case of equality characterizes the standard sphere :

Theorem 1. *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$. Then*

$$\Lambda_1(M, [g]) > \Lambda_1(\mathbb{S}^n, [\mathrm{can}])$$

if $(M, [g])$ is not conformally diffeomorphic to $(\mathbb{S}^n, [\mathrm{can}])$.

This theorem answers the question raised in [2] and [11]. Note that a similar result was proved by the author in dimension 2 (see [14]). Note also that thanks to (1) and (2), the theorem implies

$$V_c(m, M, [g]) > \omega_n = V_c(\mathbb{S}^n, [\mathrm{can}])$$

if $(M, [g])$ is not conformally diffeomorphic to $(\mathbb{S}^n, [\mathrm{can}])$. This gives a positive answer to Question 2 in [13].

In the rest of this paper, we prove the theorem. Based on the idea of Ledoux [12] and Druet [5], we start from a sharp Sobolev inequality in dimensions $n \geq 3$ (see [9, 5, 6]) which possesses extremal functions. These extremal functions give natural metrics $\tilde{g} \in [g]$ with

$$\mathrm{Vol}_{\tilde{g}}(M) = 1 \quad \text{and} \quad \lambda_1(\tilde{g}) \geq n\omega_n^{\frac{2}{n}}.$$

As in dimension 2, see [14], we deal with the degeneracy consequences of the hypothesis $\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}$.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with $\mathrm{Vol}_g(M) = 1$, which is not conformally diffeomorphic to the standard sphere. For an integer $m \geq 1$, let $h \in \mathcal{C}^m(M)$. We let $J_{g,h}$ be the functional defined for $u \in W^{1,2}(M) \setminus \{0\}$ by

$$J_{g,h}(u) = \frac{\displaystyle\int_M |\nabla u|_g^2 \, dv_g + \int_M hu^2 dv_g - K_n^{-2} \left(\int_M |u|^{2^*} dv_g \right)^{\frac{2}{2^*}}}{\displaystyle\int_M u^2 dv_g} \tag{3}$$

where

$$K_n = \frac{2}{\sqrt{n(n-2)}}\omega_n^{-\frac{1}{n}} \tag{4}$$

is the sharp constant for the Sobolev inequality induced by the critical Sobolev embedding $W_0^{1,2} \subset L^{2^*}$ for bounded domains of $\mathbb{R}^n$, with $2^* = \frac{2n}{n-2}$. Hebey and Vaugon proved in [9] that

$$-\alpha(g,h) = \inf_{u\in W^{1,2}(M)\setminus\{0\}} J_{g,h}(u) \tag{5}$$

is finite. Note that $J_{g,h}$ is scale invariant.

We will assume in the following that up to a conformal change, g is a metric in $[g]$ with volume 1 which has a constant scalar curvature S_g. Since M is not conformally diffeomorphic to the standard sphere, by the resolution of the Yamabe problem by Aubin [1] and Schoen [15], it satisfies

$$\mu(M,g) < K_n^{-2} \tag{6}$$

where $\mu(M,g)$ is the Yamabe invariant of $(M,[g])$. Let V be an open neighbourhood of $\frac{n-2}{4(n-1)}S_g$ in $\mathcal{C}^m(M)$ such that

$$\left\| h - \frac{n-2}{4(n-1)}S_g \right\|_\infty \le \frac{1}{2}(K_n^{-2} - \mu(M,g)), \quad \text{for all } h \in V. \tag{7}$$

Let $s \ge 0$ be such that $s+2 > \frac{n}{2}$ and $m \ge s+2$. By the Sobolev embedding

$$W^{s+2,2} \hookrightarrow \mathcal{C}^0,$$

the subset $W_+^{s+2,2}$ of positive functions of $W^{s+2,2}$ is open. We define

$$\begin{aligned} F : W_+^{s+2,2} \times \mathbb{R} \times V &\longrightarrow W^{s,2}, \\ (u,\beta,h) &\longmapsto \Delta_g u + (h+\beta)u - K_n^{-2}u^{2^*-1}, \end{aligned}$$

which is well defined because of the Sobolev algebra property of $W^{s+2,2}$ and F is a $\mathcal{C}^\infty$ map. By a result of Druet [5], thanks to (6) and (7), for any $h \in V$, the functional $J_{g,h}$ attains its infimum. Let $u \in W^{1,2}(M)$ be such that

$$J_{g,h}(u) = -\alpha(g,h).$$

Up to replace u by $|u|$ and up to normalize, we can take

$$u \ge 0 \quad \text{and} \quad \int_M u^{2^*}\,dv_g = 1.$$

Then, u satisfies the Euler–Lagrange equation

$$F(u,\alpha(g,h),h) = \Delta_g u + (h+\alpha(g,h))u - K_n^{-2}u^{2^*-1} = 0 \tag{8}$$

where, by elliptic regularity theory, $u \in \mathcal{C}^{m+2}$ and, by the maximum principle, $u > 0$.

Let $v \in \mathcal{C}^\infty(M)$ and $t \in \mathbb{R}$ such that $|t| < \|v\|_\infty^{-1}$. Since u is a minimum for (5),

$$\begin{aligned}&\int_M |\nabla(u+tuv)|_g^2 dv_g + \int_M (h+\alpha(g,h))(u+tuv)^2 dv_g \\ &\quad - K_n^{-2}\left(\int_M (u+tuv)^{2^\star} dv_g\right)^{\frac{2}{2^\star}} \geq 0.\end{aligned} \tag{9}$$

Since u satisfies (8), the left term in (9) vanishes until the order 2 in the Taylor development as $t \to 0$. Computing the second-order coefficient as $t \to 0$, one gets

$$\begin{aligned}&\int_M |\nabla(uv)|_g^2 dv_g + \int_M (h+\alpha(g,h))(uv)^2 dv_g \\ &\quad - K_n^{-2}(2^\star-1)\int_M v^2u^{2^\star} dv_g + K_n^{-2}(2^\star-2)\left(\int_M vu^{2^\star} dv_g\right)^2 \geq 0.\end{aligned} \tag{10}$$

We now use the conformal transformation of the conformal Laplacian

$$u^{2^*-1}\Delta_{\tilde{g}} v = \Delta_g(uv) - v\Delta_g u, \quad \text{for all } v \in \mathcal{C}^\infty(M), \tag{11}$$

where

$$\tilde{g} = u^{\frac{4}{n-2}} g.$$

We integrate (11) against uv and with (8),

$$\begin{aligned}&\int_M |\nabla(uv)|_g^2 dv_g \\ &= \int_M |\nabla v|_{\tilde{g}}^2 dv_{\tilde{g}} + \int_M v^2 u\Delta_g u dv_g \\ &= \int_M |\nabla v|_{\tilde{g}} dv_{\tilde{g}}^2 - \int_M (h+\alpha(g,h))v^2u^2 dv_g + K_n^{-2}\int_M v^2u^{2^\star} dv_g\end{aligned}$$

and with (4), (10) becomes

$$\int_M |\nabla v|_{\tilde{g}}^2 \, dv_{\tilde{g}} - n\omega_n^{\frac{2}{n}} \int_M \left(v - \int_M v dv_{\tilde{g}}\right)^2 dv_{\tilde{g}} \geq 0. \tag{12}$$

This gives that $\lambda_1(\tilde{g}) \geq n\omega_n^{\frac{2}{n}}$. Note that if the inequality is strict for one solution (h,u) of $F(u,\alpha(g,h),h) = 0$, the theorem is proved.

We now assume that for any solution (h,u) of $F(u,\alpha(g,h),h) = 0$, we have $\lambda_1(u^{\frac{4}{n-2}}g) = n\omega_n^{\frac{2}{n}}$. We will apply the following theorem ([10],Theorem 5.4, p. 63) of Fredholm theory to F, with $U = W_+^{s+2,2}(M) \times \mathbb{R}$.

Theorem 2. *Let X,Y be two separable Banach spaces, U an open set of X, V a separable $\mathcal{C}^\infty$ Banach manifold and $F \in \mathcal{C}^\infty(U \times V, Y)$ which satisfy:*

- *for all $(u,v) \in F^{-1}(0)$, $DF(u)$ is surjective;*
- *for all $(u,v) \in F^{-1}(0)$, $D_u F(u,v)$ is a Fredholm operator.*

Then there exists a countable intersection of open dense sets (a residual set) $\Sigma \subset V$ such that for all $v \in \Sigma$, and for all $u \in F(.,v)^{-1}(0)$, $D_u F(u,v)$ is surjective.

Using (11) and (4), one gets for $(u,\beta,h) \in F^{-1}(0)$,

$$D_{(u,\beta)}F(u,\beta,h).(\theta,\mu) = u^{2^*-1}\Big(\Delta_{\tilde{g}}\Big(\frac{\theta}{u}\Big) - n\omega_n^{\frac{2}{n}}\frac{\theta}{u}\Big) + \mu u, \tag{13}$$

where $\tilde{g} = u^{\frac{4}{n-2}}g$. Then, $D_{(u,\beta)}F(u,\beta,h)$ is a Fredholm operator. It remains to prove that if $(u,\beta,h) \in F^{-1}(0)$, $DF(u,\beta,h)$ is surjective. We have

$$DF(u,\beta,h).(\theta,\mu,\tau) = u^{2^*-1}\Big(\Delta_{\tilde{g}}\Big(\frac{\theta}{u}\Big) - n\omega_n^{\frac{2}{n}}\frac{\theta}{u}\Big) + \mu u + \tau u. \tag{14}$$

$\mathrm{Im}(D_{(u,\beta)}F(u,\beta,h))$ is a closed space in $W^{s,2}$ of finite codimension. Thus, since $\mathrm{Im}(DF(u,\beta,h))$ contains $\mathrm{Im}(D_{(u,\beta)}F(u,\beta,h))$, it is a closed space in $W^{s,2}$ by the following lemma.

Lemma. *Let X a Banach space, and $E \subset F \subset X$ some subspaces. If E is a closed finite codimensional subspace of X, then F is a closed subspace of X.*

Proof. Let G a finite dimensional subspace of X such that $X = E \oplus G$. We set $H = G \cap F$. Then, $F = E \oplus H$. Let $x_k \in F$ such that $x_k \to x$ as $k \to +\infty$. We denote $x_k = y_k + z_k$ with $y_k \in E$ and $z_k \in H$.

We suppose that $(z_k)_{k\geq 0}$ is not bounded. Then, up to the extraction of a subsequence, $|z_k| \to +\infty$ as $k \to +\infty$. By Bolzano's theorem, up to the extraction of a subsequence, there exists $z \in H$ such that

$$\frac{z_k}{|z_k|} \longrightarrow z, \quad \text{as } k \to +\infty.$$

Since (x_k) converges as $k \to +\infty$,

$$\frac{y_k}{|z_k|} = \frac{x_k}{|z_k|} - \frac{z_k}{|z_k|} \longrightarrow -z, \quad \text{as } k \to +\infty.$$

Since E is closed, we get $z \in E \cap H = 0$, which contradicts $|z| = 1$.

Then $(z_k)_{k \geq 0}$ is bounded and by Bolzano's theorem, up to the extraction of a subsequence, we can suppose that $z_k \to z \in H$ as $k \to +\infty$. Then,

$$y_k = x_k - z_k \longrightarrow x - z, \quad \text{as } k \to +\infty.$$

and $y = x - z \in E$ since E is closed. Therefore $x = y + z \in E + H = F$ and the proof of the lemma is complete. □

Now, it suffices to prove that $\mathrm{Im}(DF(u, \beta, h))^{\perp} = 0$, where $\perp$ refers to the orthogonal in $W^{s,2}$. Let $\phi \in \mathrm{Im}(DF(u, \beta, h))^{\perp}$. Then, with (14),

$$\langle \phi, u\tau \rangle_{W^{s,2}} = 0, \quad \text{for all } \tau \in \mathcal{C}^m.$$

Since $u \in \mathcal{C}^m$ is positive and $\mathcal{C}^m$ is dense in $W^{s,2}$, we get $\phi = 0$.

By Theorem 2, there exists $h \in V$ such that for all couple (u, β) satisfying $F(u, \beta, h) = 0$, $DF_{(u,\beta)}(u, \beta, h)$ is surjective. We take in particular $\beta = \alpha(g, h)$ and we will deduce that for a minimal function u, $\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}$ is simple with $\tilde{g} = u^{\frac{4}{n-2}} g$. We claim that

$$\int_M u^2 \phi dv_g \neq 0, \quad \text{for all } \phi \in E_1(\tilde{g}) \setminus \{0\}. \tag{15}$$

Indeed, if ϕ is an eigenfunction for $\lambda_1(\tilde{g})$ such that this integral vanishes, one easily checks with (13) that $u\phi$ is orthogonal to the image of $D_{(u,\beta)} F(u, \alpha(h, g), h)$ in $L^2(g)$. It implies $\phi = 0$ and we obtain (15). Since a bounded linear form vanishes on a one-codimensional space, we get that $\lambda_1(\tilde{g})$ is simple. Thus, $\lambda_1(\tilde{g})$ cannot be an extremal eigenvalue in the sense of [8] and as a result, $\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}$ is not locally maximal. The proof of Theorem 1 for $n \geq 3$ is complete.

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Romain Petrides, Université de Lyon, CNRS UMR 5208, Université Lyon 1,
Institut Camille Jordan, 43 bd du 11 novembre 1918, F-69622 Villeurbanne cedex,
France

e-mail: romain.petrides@univ-lyon1.fr