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The Peierls–Onsager effective Hamiltonian in a complete gauge covariant setting: determining the spectrum

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Abstract. Using the procedures in [4] and [8] and the magnetic pseudodifferential calculus we have developped in [17, 19, 12, 13] we construct an effective Hamiltonian that describes the spectrum in any compact subset of the real axis for a large class of periodic pseudo-differential Hamiltonians in a bounded smooth magnetic field, in a completely gauge covariant setting, without any restrictions on the vector potential and without any adiabaticity hypothesis.

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Contents

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1. Introduction

In this paper we consider once again the construction of an effective Hamiltonian for a particle described by a periodic Hamiltonian and subject also to a magnetic field that will be considered bounded and smooth but neither periodic nor slowly varying. Our aim is to use some of the ideas in [4, 8] in conjunction with the magnetic pseudodifferential calculus developed in [17, 12, 13, 19] and obtain the following improvements:

(1) consider magnetic fields that are neither constant nor slowly varying, to work in a manifestly covariant form and obtain results that clearly depend only on the magnetic field;

(2) give up the adiabatic hypothesis (slowly varying fields) and consider only the intensity of the magnetic field as a small parameter;

(3) cover also the case of pseudodifferential operators, as for example the relativistic Schrödinger operators with principal symbol $\langle\eta\rangle := (1 + |\eta|^2)^{1/2}$.

Let us point out from the beginning, that as in [8] we construct an effective Hamiltonian associated to any compact interval of the energy spectrum but its significance concerns only the description of the real spectrum as a subset of $\mathbb{R}$. In a forthcoming paper our covariant magnetic pseudodifferential calculus will be used in order to construct an effective dynamics associated to any spectral band of the periodic Hamiltonian. Let us mention here that the magnetic pseudodifferential calculus has been used in the Peierls–Onsager problem in [5] where some improvements of the results in [23] are obtained but still in an adiabatic setting.

Finally let us also point out here that an essential ingredient in the method elaborated in [8] is a necessary and sufficient criterion for a tempered distribution to belong to some given Hilbert spaces (Propositions 3.2 and 3.6 in [8]). In our "magnetic" setting some similar criteria have to be proved and this obliges us to some different formulations with respect to those in [8, 6].

Let us very briefly describe the content of our paper. The Introduction contains a very brief formulation of the problem and the main results we obtain together with some notions concerning the Floquet representation and the localized Wannier functions that we shall need further. Section 2 contains the proofs of our main results based on some extensions of the magnetic pseudodifferential calculus ([17, 12, 13]) discussed in the Appendices and on some ideas from [4, 8, 10, 11]. Some particular cases, where more complete results can be obtained, are discussed in Section 3. Let us mention here that a large number of standard pseudodifferential techniques are used without detailed presentations in order to limit the dimension of our paper, but a complete and detailed version of our paper can be found on web ([14]).

1.1. The problem. We use the notation $\mathcal{X} \equiv \mathbb{R}^d$, its dual $\mathcal{X}^*$ being canonically isomorphic to $\mathbb{R}^d$; let

$$\langle \cdot, \cdot \rangle : \mathcal{X}^* \times \mathcal{X} \longrightarrow \mathbb{R}$$

denote the duality relation. We define

$$\Xi := \mathcal{X} \times \mathcal{X}^*,$$

as a symplectic space with the canonical symplectic form

$$\sigma(X, Y) := \langle \xi, y \rangle - \langle \eta, x \rangle,$$

and

$$\bar{\Xi} := \mathcal{X}^* \times \mathcal{X}.$$

We shall consider a discrete subgroup $\Gamma \subset \mathcal{X}$ described as a lattice

$$\Gamma := \bigoplus_{j=1}^{d} \mathbb{Z} e_j,$$

with $\{e_1, \ldots, e_d\}$ an algebraic basis of $\mathbb{R}^d$. We consider the quotient group $\mathbb{R}^d / \Gamma$ that is canonically isomorphic to the d-dimensional torus $\mathbb{T}$. Let us consider an *elementary cell*

$$E = \Big\{ y = \sum_{j=1}^{d} t_j e_j \in \mathbb{R}^d : 0 \leq t_j < 1, \text{ for all } j \in \{1, \ldots, d\} \Big\},$$

having the interior locally homeomorphic to its projection on $\mathbb{T}$. The dual lattice of Γ is defined as

$$\Gamma_* := \{\gamma^* \in \mathcal{X}^* : \langle \gamma^*, \gamma \rangle \in (2\pi)\mathbb{Z}, \text{ for all } \gamma \in \Gamma\}.$$

Considering the dual basis $\{e_1^*, \ldots, e_d^*\} \subset \mathcal{X}^*$ of $\{e_1, \ldots, e_d\}$, defined by

$$\langle e_j^*, e_k \rangle = (2\pi)\delta_{jk},$$

we have evidently that

$$\Gamma_* := \bigoplus_{j=1}^{d} \mathbb{Z} e_j^*.$$

By definition, we have that $\Gamma_* \subset \mathcal{X}^*$ is the polar of $\Gamma \subset \mathcal{X}$. We define

$$\mathbb{T}_* := \mathcal{X}^*/\Gamma_*$$

and E_*, and note that $\mathbb{T}_*$ is isomorphic to the dual group of Γ.

Let us recall the usual periodic Schrödinger Hamiltonian

$$H_{0,V} := -\Delta + V(y), \quad V \in BC^\infty(\mathcal{X}, \mathbb{R}), \Gamma - \text{periodic}, \tag{1.1}$$

that describes the evolution of an electron in a periodic crystal without external fields. The above operator has a self-adjoint extension in $L^2(\mathcal{X})$ that commutes with the translations τ_γ for any $\gamma \in \Gamma$. By the Floquet–Bloch theory for any $\xi \in \mathcal{X}^*$ we can define the operator

$$H_{0,V}(\xi) := (D_y + \xi)^2 + V(y)$$

that has a self-adjoint extension in $L^2(\mathbb{T})$ that has compact resolvent. Thus its spectrum consists in a growing sequence of finite multiplicity eigenvalues

$$\lambda_1(\xi) \leq \lambda_2(\xi) \leq \dots$$

that are continuous and Γ^*-periodic functions of ξ. Thus, if we set

$$J_k := \lambda_k(\mathbb{T}_*),$$

we can write

$$\sigma(H_{0,V}) = \bigcup_{k=1}^{\infty} J_k,$$

and it follows that this spectrum is absolutely continuous. The above analysis implies the following statement that can be considered as *the spectral form of the Onsager–Peierls substitution* in a trivial situation (with 0 magnetic field):

$$\lambda \in \sigma(H_{0,V}) \implies \text{there exists } k \geq 1 \text{ such that } 0 \in \sigma(\lambda - \lambda_k(D)), \tag{1.2}$$

where $\lambda_k(D)$ is the Weyl quantization of the symbol λ_k and thus defines a bounded self-adjoint operator on $L^2(\mathcal{X})$.

The problem we are interested in, consists in superposing a *magnetic field B* in the above situation; let us first consider a constant magnetic field

$$B = (B_{jk})_{1\leq j,k\leq d}, \quad \text{with } B_{jk} = -B_{kj}.$$

Let us recall that using *the transversal gauge* one can define the following *vector potential $A = (A_j)_{1\leq j\leq d}$* given by

$$A_j(x) := -\frac{1}{2} \sum_{1\leq k\leq d} B_{jk} x_k.$$

Then the associated *magnetic Hamiltonian* is defined as

$$H_{A,V} := (D + A)^2 + V(y),$$

that has also a self-adjoint extension in $L^2(\mathcal{X})$. The structure of the spectrum of this operator may be very different of the structure of $\sigma(H_{0,V})$ (for example it may be pure point with infinite multiplicity!), but one expects that modulo some small correction (depending on B), for small $|B|$ the property 1.2 with D replaced by $D + A$ should still be true. More precisely it is conjectured that there exists a symbol $r_k(x, \xi; B, \lambda)$ (in fact a $BC^\infty(\Xi)$ function) such that

$$\lim_{|B|\to 0} r_k(x, \xi; B, \lambda) = 0 \quad \text{in } BC^\infty(\Xi),$$

and for λ in a compact neighborhood of J_k and for small $|B|$ we have that

$$\lambda \in \sigma(H_{A,V}) \iff 0 \in \sigma(\lambda - \lambda_k(D + A(x)) + r_k(x, D + A(x); B, \lambda)), \quad (1.3)$$

where $r_k(x, D + A(x); B, \lambda)$ is the Weyl quantization of $r_k(x, \xi + A(x); B, \lambda)$.

The first rigorous proof of such a result appeared in [21] for a simple spectral band (i.e. $\lambda_k(\xi)$ is a non-degenerated eigenvalue of $H_{0,V}(\xi)$ for any $\xi \in \mathcal{X}^*$ and $J_k \cap J_l = \emptyset$, for all $l \neq k$). In [10] the authors study this case of a simple spectral band but also the general case, by using Wannier functions. In these references the operator appearing on the right hand side of the equivalence (1.3) is considered to act in the Hilbert space $[l^2(\Gamma)]^N$ (with $N = 1$ for the simple spectral band). In fact we shall prove that for a simple spectral band one can replace $l^2(\Gamma)$ with $L^2(\mathcal{X})$.

In [8] the authors consider the evolution of an electron (ignoring the spin) in a periodic crystal under the action of exterior non-constant, slowly varying, magnetic and electric fields. More precisely the magnetic field B is defined as

$$B = dA,$$

with a vector potential $A = (A_1, \dots, A_d)$, $\quad A_j \in C^\infty(\mathcal{X}; \mathbb{R})$, satisfying

$$\partial^\alpha A_j \in BC^\infty(\mathcal{X}), \quad \text{for all } |\alpha| \geq 1,$$

and the electric potential is described by

$$\phi \in BC^\infty(\mathcal{X}; \mathbb{R}).$$

The Hamiltonian is taken to be

$$P_{A,\phi} = \sum_{1\leq j\leq d} (D_{y_j} + A_j(\epsilon y))^2 + V(y) + \phi(\epsilon y),$$

with $\epsilon > 0$ small enough; this defines also a self-adjoint operator in $L^2(\mathcal{X})$.

In this situation, in order to define an effective Hamiltonian, the authors apply an idea of Buslaev [4] (see also [9]); this idea consists in "doubling" the number of variables and separating the periodic part (that is also "rapidly varying") from the non-periodic part (that is also "slowly varying"). In order to define an effective Hamiltonian to describe the spectrum of $P_{A,\phi}$ (in fact of some Hamiltonians generalizing this one), in [8] the authors bring together three important ideas from the literature on the subject.

(1) First, the idea introduced in [4, 9] of "doubling the variables."

(2) Then, following an idea from [4], one uses an operator valued pseudodifferential calculus, as the one developped in [3].

(3) The formulation of a Grushin type problem, as proposed in [10].

Let us note that if one would like to consider also non-constant magnetic fields, then the above Weyl quantization of $A(x)$-dependent symbols (as used in [8]) gives operators that are not gauge covariant and thus unsuitable for a physical interpretation.

1.2. Summary of our results.

Let us briefly comment upon our hypothesis.

Hypothesis H.1. *The magnetic field B_ϵ is a closed 2-form with components depending on a real parameter*

$$[-\epsilon_0, \epsilon_0] \ni \epsilon \longmapsto B_{\epsilon,jk} \in BC^\infty(\mathcal{X};\mathbb{R}), \quad \textit{for some } \epsilon_0 > 0,$$

and verifying

$$\lim_{\epsilon\to 0} B_{\epsilon,jk} = 0 \quad \textit{in } BC^\infty(\mathcal{X};\mathbb{R}).$$

Using the transversal gauge we can define a family of vector potentials A_ϵ,

$$A_{\epsilon,j}(x) := -\sum_{1\leq k\leq d} x_k \int_0^1 B_{\epsilon,jk}(sx)s\, ds. \tag{1.4}$$

The symbols we are considering are also functions of a real parameter

$$[-\epsilon_0, \epsilon_0] \ni \epsilon \longmapsto p_\epsilon \in C^\infty(\mathcal{X} \times \mathcal{X} \times \mathcal{X}^*)$$

satisfying conditions of type S_1^m with $m > 0$ uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$. In fact, the "physical" symbols we are really interested in are just usual symbols on Ξ, but as we shall use the procedure of "doubling" the space variables, as proposed by Buslaev, we prefer to consider from the beginning this larger class of symbols having in view also some other possible applications of our results (see also the comments in [8]).

Hypothesis H.2. *We shall denote by* $S_1^m(\mathcal{X} \times \mathcal{X})$ *the space of* C^∞ *functions on* $\mathcal{X} \times \Xi$ *(with the natural Fréchet topology) such that there exists* $m > 0$*, such that for all* $(\tilde{\alpha}, \beta) \in \mathbb{N}^{2d} \times \mathbb{N}^d$*, there exists* $C_{\tilde{\alpha}\beta} > 0$ *such that*

$$|(\partial_{x,y}^{\tilde{\alpha}} \partial_\eta^\beta p_\epsilon)(x, y, \eta)| \leq C_{\tilde{\alpha}\beta} \langle \eta \rangle^{m-|\beta|},$$

for all $(x, y, \eta) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}^*$*, and all* $\epsilon \in [-\epsilon_0, \epsilon_0]$*.*

Hypothesis H.3. $\lim_{\epsilon \to 0} p_\epsilon = p_0$ *in* $S_1^m(\mathcal{X} \times \mathcal{X})$*.*

Hypothesis H.4. *For all* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \geq 1$*,*

$$\lim_{\epsilon \to 0} (\partial_x^\alpha p_\epsilon) = 0$$

in $S_1^m(\mathcal{X} \times \mathcal{X})$*.*

Hypothesis H.5. p_ϵ *is an elliptic symbol uniformly in* $\epsilon \in [-\epsilon_0, \epsilon_0]$*, i.e. there exist* $C > 0$ *and* $R > 0$ *such that*

$$p_\epsilon(x, y, \eta) \geq C|\eta|^m,$$

for all $(x, y, \eta) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}^*$ *with* $|\eta| \geq R$*, and all* $\epsilon \in [-\epsilon_0, \epsilon_0]$*.*

Hypothesis H.6. p_ϵ *is* Γ*-periodic with respect to the second variable, i.e.*

$$p_\epsilon(x, y + \gamma, \eta) = p_\epsilon(x, y, \eta),$$

for all $\gamma \in \Gamma$*,* $(x, y, \eta) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}^*$*, and* $\epsilon \in [-\epsilon_0, \epsilon_0]$*.*

Let us remark here that Hypotheses H.3 and H.4 imply that the limit p_0 only depends on the second and third variables ($(y, \eta) \in \Xi$) and thus we can write

$$p_\epsilon(x, y, \eta) := p_0(y, \eta) + r_\epsilon(x, y, \eta) \quad \text{and} \quad \lim_{\epsilon \to 0} r_\epsilon(x, y, \eta) = 0$$

in $S_1^m(\mathcal{X} \times \mathcal{X})$.

Let us also note that our Hypothesis H.3 is not satisfied if we consider a perturbation of the form (adiabatic electric field) $\phi(\epsilon y)$ but is verified for a perturbation of the form $\epsilon\phi(y)$. One can consider a weaker hypothesis, allowing also for the adiabatic electric field perturbation, without losing the general construction of the effective Hamiltonian, but some consequences that we shall prove would no longer be true.

We associate to our symbols the two types of symbols proposed in [8]:

$$\mathring{p}_\epsilon(y,\eta) := p_\epsilon(y,y,\eta), \quad \tilde{p}_\epsilon(x,y,\xi,\eta) := p_\epsilon(x,y,\xi+\eta). \tag{1.5}$$

We shall use the magnetic pseudodifferential calculus as developped in [17, 12, 13]; let us just recall the definition of the magnetic 2-cocycle

$$\omega_{A_\epsilon}(x,y) := \exp\Big\{-i\int\limits_{[x,y]} A_\epsilon\Big\}$$

(here $[x,y]$ denotes the closed interval with boundary points x and y) and the magnetic Weyl operators defined by the oscillating integrals

$$[\mathfrak{Op}^{A_\epsilon}(p)u](x) := (2\pi)^{-d}\int\limits_{\Xi} e^{i\langle\eta,x-y\rangle}\omega_{A_\epsilon}(x,y)\,p\Big(\frac{x+y}{2},\eta\Big)u(y)\,dy\,d\eta, \quad \text{for all } x\in\mathcal{X}. \tag{1.6}$$

The operator we want to study is

$$P_\epsilon := \mathfrak{Op}^{A_\epsilon}(\mathring{p}_\epsilon). \tag{1.7}$$

The auxiliary operator is defined as in Appendix 4.2 by

$$\widetilde{P}_\epsilon := \mathfrak{Op}^{A_\epsilon}(\mathfrak{q}_\epsilon), \quad \mathfrak{q}_\epsilon(x,\xi) := \mathfrak{Op}(\tilde{p}_\epsilon(x,\cdot,\xi,\cdot)). \tag{1.8}$$

Let us note that in particular all the above hypothesis are satisfied if we take

$$B_\epsilon := \epsilon B,$$

with B a magnetic field with components of class $BC^\infty(\mathcal{X})$,

$$A_\epsilon = \epsilon A,$$

with A a vector potential associated to B by (1.4), and P_ϵ one of the possible Schrödinger operators

$$P_\epsilon = \sum_{1\le j\le d} (D_{y_j} + \epsilon A_j(y))^2 + V(y) + \epsilon\phi(y), \tag{1.9}$$

$$P_\epsilon = \mathfrak{Op}^{\epsilon A}(\langle\eta\rangle) + V(y) + \epsilon\phi(y), \tag{1.10}$$

$$P_\epsilon = \sqrt{\mathfrak{Op}^{\epsilon A}(|\eta|^2) + 1} + V(y) + \epsilon\phi(y), \tag{1.11}$$

where V satisfies 1.1 and $\phi \in BC^\infty(\mathcal{X};\mathbb{R})$.

In order to define an effective Hamiltonian for P_ϵ we shall apply the same ideas as in [8] with the important remark that the operator valued pseudodifferential calculus we use is not a semi classical calculus but the “magnetic” calculus so that all our constructions are explicitly gauge covariant. This fact obliges us to a lot of new technical lemmas in order to deal with this new calculus. Our main result is the following theorem.

Theorem 1.1. *We assume Hypotheses H.1–H.6. For any compact interval $I \subset \mathbb{R}$ there exist $\epsilon_0 > 0$ and $N \in \mathbb{N}^*$ such that for all $\lambda \in I$ and for all $\epsilon \in [-\epsilon_0, \epsilon_0]$ there exists a bounded self-adjoint operator*

$$\boldsymbol{E}_{-+}(\epsilon,\lambda) := \mathfrak{Op}^{A_\epsilon}(E^{-+}_{\epsilon,\lambda})$$

acting in $[\mathfrak{V}_0]^N$ (see Definition 2.36), where $E^{-+}_{\epsilon,\lambda} \in BC^\infty(\Xi;\mathbb{B}(\mathbb{C}^N))$ uniformly in $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0]\times I$ and is Γ^*-periodic in the variable $\xi \in \mathcal{X}^*$, for which the following equivalence is true:

$$\lambda \in \sigma(P_\epsilon) \iff 0 \in \sigma(\boldsymbol{E}_{-+}(\epsilon,\lambda)). \tag{1.12}$$

The symbol $E^{-+}_{\epsilon,\lambda}$ is a perturbation of the one at zero magnetic field $E^{-+}_{0,\lambda}$ that is defined in terms of a family of quasi-Bloch functions associated to the energy interval I. Moreover, if I is an isolated band with sufficiently regular Bloch eigenvalues and associated eigenvectors, then this principal symbol can be written explicitly in terms of quantizations of the Bloch eigenvalues. The case of a single spectral band is presented in Proposition 1.4 and the general case will be treated in a forthcoming paper.

A direct consequence of the above theorem is a stability property for the spectral gaps of the operator P_ϵ of the same type as that obtained in [2, 20, 1] for the Schrödinger operator.

Corollary 1.2. *Under Hypothesis H.1–H.6, for any compact interval $K \subset \mathbb{R}$ disjoint from $\sigma(P_0)$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in [-\epsilon_0, \epsilon_0]$ the interval K is disjoint from $\sigma(P_\epsilon)$.*

In fact we obtain a much stronger result, giving the optimal regularity property but only for $\epsilon = 0$ i.e. at vanishing magnetic field.

Proposition 1.3. *We denote by $\mathfrak{d}_H(F_1, F_2)$ the Hausdorff distance between the two closed subsets F_1 and F_2 of $\mathbb{R}$. Then, under Hypotheses H.1–H.7 (see lower on this page for Hypotheses H.7) and I.1–I.3 (see the end of Section 2), there exists a strictly positive constant C such that*

$$\mathfrak{d}_H(\sigma(P_\epsilon) \cap I, \sigma(P_0) \cap I) \leq C\epsilon, \quad \textit{for all } \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{1.13}$$

Let us consider now the case of a simple spectral band and study the result we discussed previously in this case. By hypothesis we have that $\tau_\gamma P_0 = P_0 \tau_\gamma$, for all $\gamma \in \Gamma$ and we can apply the Floquet–Bloch theory. We denote by

$$\lambda_1(\xi) \leq \lambda_2(\xi) \leq \dots$$

the eigenvalues of the operators

$$P_{0,\xi} := \mathfrak{Op}(p_0(\cdot, \xi + \cdot))$$

that are self-adjoint in $L^2(\mathbb{T})$; they are continuous functions on the torus

$$\mathbb{T}_* := \mathcal{X}^* / \Gamma_*$$

(and they are even C^∞ in the case of a simple spectral band). Thus

$$\sigma(P_0) = \bigcup_{j=1}^{d} J_j,$$

with

$$J_j := \lambda_j(\mathbb{T}_*).$$

Let us consider now the following new hypothesis.

Hypothesis H.7. *There exists $k \geq 1$ such that J_k is a simple spectral band for P_0, i.e., for all $\xi \in \mathbb{T}_*$, $\lambda_k(\xi)$ is a non-degenerate eigenvalue of $P_{0,\xi}$ and for any $l \neq k$ we have that $J_l \cap J_k = \emptyset$.*

Proposition 1.4. *Assume that Hypotheses H.1–H.7 are true and that moreover we have that* $p_0(y,-\eta) = p_0(y,\eta)$*, for all* $(y,\eta) \in \Xi$*. Let* $I \subset \mathbb{R}$ *be a compact neighborhood of* J_k *disjoint from* $\bigcup_{l\neq k} J_l$*. Then there exists* $\epsilon_0 > 0$ *such that for all* $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$ *in Theorem 1.1 we can take* $N = 1$ *and*

$$E^{-+}_{\epsilon,\lambda}(x,\xi) = \lambda - \lambda_k(\xi) + r_{\epsilon,\lambda}(x,\xi), \tag{1.14}$$

with

$$\lim_{\epsilon\to 0} r_{\epsilon,\lambda} = 0, \quad \text{in } BC^\infty(\Xi), \text{ uniformly in } \lambda \in I.$$

In the case of a constant magnetic field, under some more assumptions on the symbol p_ϵ we can have even more information concerning the operator $\boldsymbol{E}_{-+}(\epsilon,\lambda)$.

Proposition 1.5. *Assume that Hypotheses H.1–H.7 are true and that* B_ϵ *are constant magnetic fields (for any* ϵ*) and that the symbols* p_ϵ *do not depend on the first variable* ($x \in \mathcal{X}$)*. Then we can complete the conclusion of Theorem 1.1 with the following statements:*

(1) $\boldsymbol{E}_{-+}(\epsilon,\lambda)$ *is a bounded self-adjoint operator in* $[L^2(\mathcal{X})]^N$;

(2) *the symbol* $E^{-+}_{\epsilon,\lambda}$ *is independent of the first variable* ($y \in \mathcal{X}$) *and is* Γ_**-periodic in the second variable* ($\xi \in \mathcal{X}^*$)*.*

1.3. Overview of periodic pseudodifferential operators. We shall denote by $\mathcal{S}'_\Gamma(\mathcal{X})$ the space of Γ-periodic distributions on $\mathcal{X}$ and by

$$\mathcal{S}(\mathbb{T}) := C^\infty(\mathbb{T}),$$

with the usual Fréchet topology; $\mathcal{S}'(\mathbb{T})$ is the dual of $\mathcal{S}(\mathbb{T})$ and we denote by $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ the duality relation on $\mathcal{S}'(\mathbb{T})\times\mathcal{S}(\mathbb{T})$ and by $(\cdot,\cdot)_{\mathbb{T}}$ the sesquilinear map obtained by extending the scalar product from $L^2(\mathbb{T})$. It is well known that the spaces $\mathcal{S}'_\Gamma(\mathcal{X})$ and $\mathcal{S}'(\mathbb{T})$ have a natural identification. We shall denote by $S^m_\rho(\mathbb{T})$ the Hörmander type symbols of class $S^m_\rho(\mathcal{X})$ that are Γ-periodic and thus may be considered as symbols on the torus.

For any distribution $u \in \mathcal{S}'_\Gamma(\mathcal{X}) \cong \mathcal{S}'(\mathbb{T})$ we have the Fourier series decomposition

$$u = \sum_{\gamma^*\in\Gamma_*} \hat{u}(\gamma^*)\sigma_{\gamma^*}, \quad \hat{u}(\gamma^*) := |E|^{-1}\langle u,\sigma_{-\gamma^*}\rangle_{\mathbb{T}}, \tag{1.15}$$

where

$$\sigma_{\gamma^*}(y) := e^{i\langle\gamma^*,y\rangle},$$

for all $y \in \mathbb{T}$ and all $\gamma^* \in \Gamma_*$ and the series converges as tempered distribution.

For any $s \in \mathbb{R}$ and any $\gamma^* \in \Gamma_*$ we have

$$\langle D\rangle^s \sigma_{\gamma^*} = \langle \gamma^*\rangle^s \sigma_{\gamma^*},$$

and we deduce that $\langle D\rangle^s$ induces on $\mathcal{S}'(\mathbb{T}) \cong \mathcal{S}'_\Gamma(\mathcal{X})$ a well-defined operator, denoted by $\langle D_\Gamma\rangle^s$. For any $\xi \in \Xi$,

$$\langle D_\Gamma + \xi\rangle^s u := \sum_{\gamma^* \in \Gamma_*} \langle \gamma^* + \xi\rangle^s \hat{u}(\gamma^*)\sigma_{\gamma^*}, \quad \text{for all } u \in \mathcal{S}'(\mathbb{T}). \tag{1.16}$$

Definition 1.6. Given any $s \in \mathbb{R}$ we define the complex linear space

$$\mathcal{H}^s(\mathbb{T}) := \{u \in \mathcal{S}'(\mathbb{T})\colon \langle D_\Gamma\rangle^s u \in L^2(\mathbb{T})\}$$

endowed with the Hilbertian norm

$$\|u\|_{\mathcal{H}^s(\mathbb{T})} := \|\langle D_\Gamma\rangle^s u\|_{L^2(\mathbb{T})},$$

for which it becomes a Hilbert space.

The following statements are well known and easy to be proven.

Lemma 1.7. *Let $p \in S^m_1(\mathcal{X})$ and let us set*

$$P := \mathfrak{Op}(p).$$

Then, for any $s \in \mathbb{R}$ and for any $u \in \mathcal{H}^{s+m}_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'(\mathcal{X})$,

$$Pu \in \mathcal{H}^s_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'(\mathcal{X}).$$

Corollary 1.8. *The space $\mathcal{H}^s(\mathbb{T})$ can be identified with the usual Sobolev space of order s on the torus that is defined as $\mathcal{H}^s_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$.*

Definition 1.9. We define the complex linear space

$$\mathcal{K}_{s,\xi} := \{u \in \mathcal{S}'(\mathbb{T})\colon \langle D_\Gamma + \xi\rangle^s u \in L^2(\mathbb{T})\}, \tag{1.17}$$

endowed with the norm

$$\|u\|^2_{\mathcal{K}_{s,\xi}} := |\langle D_\Gamma + \xi\rangle^s u\|^2_{L^2(\mathbb{T})} = |E|^{-1} \sum_{\gamma^* \in \Gamma_*} \langle \gamma^* + \xi\rangle^{2s} |\hat{u}(\gamma^*)|^2$$

that defines a structure of Hilbert space on it.

It is clear that $\mathcal{K}_{s,\xi} = \mathcal{H}^s(\mathbb{T})$ as complex vector spaces and for $\xi = 0$ even as Hilbert spaces. Similar arguments to those in Example 4.4 show that the family $\{\mathcal{K}_{s,\xi}\}_{\xi\in\mathcal{X}^*}$ has temperate variation (see Definition 4.3).

Coming back to Corollary 1.8 we can consider the elements of $\mathcal{K}_{s,\xi}$ as distributions from $\mathcal{H}^s_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ and we can define the spaces

$$\mathcal{F}_{s,\xi} := \{u \in \mathcal{S}'(\mathcal{X}) : \sigma_{-\xi} u \in \mathcal{K}_{s,\xi}\}. \tag{1.18}$$

It is a Hilbert space isometrically isomorphic to $\mathcal{K}_{s,\xi}$ with the norm

$$\|u\|_{\mathcal{F}_{s,\xi}} := \|\sigma_{-\xi} u\|_{\mathcal{K}_{s,\xi}}.$$

Remark 1.10. Let us fix some $\xi \in \mathcal{X}^*$.

(1) Let us set

$$\mathcal{S}'_\xi(\mathcal{X}) := \{u \in \mathcal{S}'(\mathcal{X}) : \tau_{-\gamma} u = e^{i\langle \xi,\gamma\rangle} u, \text{for all } \gamma \in \Gamma\}.$$

(2) We can write

$$\mathcal{F}_{0,\xi} = \{u \in \mathcal{S}'(\mathcal{X}) : \sigma_{-\xi} u \in L^2_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})\} = \mathcal{S}'_\xi(\mathcal{X}) \cap L^2_{\mathrm{loc}}(\mathcal{X})$$

and conclude that we can identify $\mathcal{F}_{0,\xi}$ with $L^2(E)$ and note that we have the equality of the norms $\|u\|_{\mathcal{F}_{0,\xi}} = \|u\|_{L^2(E)}$.

(3) We observe

$$\langle D + \xi\rangle^s = \sigma_{-\xi} \langle D\rangle^s \sigma_\xi.$$

Thus

$$\mathcal{F}_{s,\xi} = \{u \in \mathcal{S}'_\xi(\mathcal{X}) : \langle D\rangle^s u \in \mathcal{F}_{0,\xi}\}, \quad \|u\|_{\mathcal{F}_{s,\xi}} = \|\langle D\rangle^s u\|_{\mathcal{F}_{0,\xi}}.$$

1.3.1. Periodic symbols

Lemma 1.11. *Under the hypothesis of Proposition 4.13, for any $a \in \mathcal{X}$ we have the equality*

$$\tau_a \mathfrak{Op}^A(p) = \mathfrak{Op}^{\tau_a A}((\tau_a \otimes \mathrm{id}) p) \tau_a. \tag{1.19}$$

Proof. It is enough to use formula (1.6) and to note that

$$\int_{[x-a,y-a]} A = -\Big\langle (x-y), \int_0^1 A((1-s)x + sy - a)\, ds \Big\rangle. \qquad \square$$

Lemma 1.12. *For any symbol $p \in S^m_1(\mathbb{T})$ (see Definition 4.9) the pseudodifferential operator*

$$P := \mathfrak{Op}(p)$$

induces on $\mathbb{T}$ an operator

$$P_\Gamma \in \mathbb{B}(\mathcal{K}_{s+m,0}; \mathcal{K}_{s,0}) \quad \textit{for any } s \in \mathbb{R},$$

and the application

$$S^m_1(\mathbb{T}) \ni p \longmapsto P_\Gamma \in \mathbb{B}(\mathcal{K}_{s+m,0}; \mathcal{K}_{s,0})$$

is continuous.

Proof. By equality (1.19) with $A = 0$ and observing that $(\tau_\gamma \otimes \mathrm{id})p = p$, for all $\gamma \in \Gamma$ we deduce that P leaves $\mathcal{S}'_\Gamma(\mathcal{X})$ invariant and thus induces a linear and continuous operator

$$P_\Gamma : \mathcal{S}'(\mathbb{T}) \longrightarrow \mathcal{S}'(\mathbb{T}).$$

If $u \in \mathcal{K}_{s+m,0} = \mathcal{H}^{s+m}(\mathbb{T})$, then

$$\begin{aligned} \|P_\Gamma u\|_{\mathcal{K}_{s,0}} &= \|\langle D_\Gamma\rangle^s P_\Gamma u\|_{L^2(\mathbb{T})} \\ &= \|\langle D\rangle^s P u\|_{L^2(E)} \\ &= \|\langle D\rangle^s P \langle D\rangle^{-s-m} \langle D\rangle^{s+m} u\|_{L^2(E)}. \end{aligned}$$

From the Weyl calculus we know that

$$\langle D\rangle^s P \langle D\rangle^{-s-m} = \mathfrak{Op}(q)$$

for a well defined symbol $q \in S^0_1(\mathcal{X})$ and the map

$$S^m_1(\mathbb{T}) \ni p \longmapsto q \in S^0_1(\mathcal{X})$$

is continuous; by Lemma 1.7 we can find a strictly positive constant $C'_0(p)$ (one of the defining seminorms for the topology of $S^m_1(\mathbb{T})$) and a number $N \in \mathbb{N}$ (that does not depend on p) such that

$$\|\langle D\rangle^s P \langle D\rangle^{-s-m} v\|_{L^2(E)} \le C'_0(p)\|v\|_{L^2(F)}, \quad \text{for all } v \in L^2_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'(\mathcal{X}),$$

where

$$F := \bigcup_{\gamma \in \Gamma_N} \tau_\gamma E,$$

and

$$\Gamma_N := \{\gamma \in \Gamma : |\gamma| \leq N\}.$$

Let us consider now

$$v = \langle D \rangle^{s+m} u \in L^2_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X}).$$

We have

$$\begin{aligned} \|v\|^2_{L^2(F)} &= \sum_{|\gamma| \leq N} \int_{\tau_\gamma E} |v(x)|^2 \, dx \leq C_N^2 \|v\|^2_{L^2(E)} \\ &= C_N^2 \|\langle D \rangle^{s+m} u\|^2_{L^2(E)} \\ &= C_N^2 \|u\|^2_{\mathcal{K}_{s+m,0}}, \end{aligned}$$

and we conclude that

$$\|P_\Gamma u\|_{\mathcal{K}_{s,0}} \leq C_N C_0'(p) \|u\|_{\mathcal{K}_{s+m,0}}. \qquad \square$$

Remark 1.13. For any symbol $p \in S^m_1(\mathbb{T})$ and for any point $\xi \in \mathcal{X}^*$ we know that $(\mathrm{id} \otimes \tau_{-\xi}) p \in S^m_1(\mathbb{T})$, and, due to Lemma 1.12, the operator

$$P_\xi := \mathfrak{Op}((\mathrm{id} \otimes \tau_{-\xi}) p)$$

induces on $\mathbb{T}$ a well defined operator

$$P_{\Gamma,\xi} \in \mathbb{B}(\mathcal{K}_{s+m,0}; \mathcal{K}_{s,0})$$

for any $s \in \mathbb{R}$. By the same lemma we deduce that the application

$$\mathcal{X}^* \ni \xi \longmapsto P_{\Gamma,\xi} \in \mathbb{B}(\mathcal{K}_{s+m,0}; \mathcal{K}_{s,0})$$

is continuous and, noticing that

$$\partial^\alpha_\xi P_\xi = \mathfrak{Op}((\mathrm{id} \otimes \tau_{-\gamma})(\mathrm{id} \otimes \partial^\alpha) p),$$

this application is in fact of class C^∞.

From now on we shall consider $P = \mathfrak{Op}(p)$ with $p \in S^m_1(\mathbb{T})$ a real elliptic symbol. We know the P has a self-adjoint realization as operator acting in $L^2(\mathcal{X})$ with the domain $\mathcal{H}^m(\mathcal{X})$ (the usual Sobolev space of order m). By Lemma 1.11, we obtain that $\tau_\gamma P = P \tau_\gamma$, for all $\gamma \in \Gamma$ and thus we can use the Floquet theory in order to study the spectrum of the operator P.

1.3.2. The Floquet transformation. In this subsection we very briefly recall the main results of the Bloch-Floquet theory in order to fix some precise statements to be referred in the arguments that follow. For the long history of this subject and the main references concerning its development we send to [22, 24].

We shall consider the spaces

$$\begin{aligned} \mathcal{S}'_\Gamma(\Xi) := \{v \in \mathcal{S}'(\Xi) \colon v(y+\gamma,\eta) = e^{i\langle \eta,\gamma\rangle} v(y,\eta), \\ \text{for all } \gamma \in \Gamma, \text{ such that } v(y,\eta+\gamma^*) = v(y,\eta), \\ \text{and all } \gamma \in \Gamma_*\}, \end{aligned} \tag{1.20}$$

endowed with the topology induced by $\mathcal{S}'(\Xi)$,

$$\mathcal{F}_0(\Xi) := \{v \in \mathcal{S}'_\Gamma(\Xi) \colon v \in L^2(E \times E_*)\}, \tag{1.21}$$

endowed with the norm

$$\|v\|_{\mathcal{F}_0(\Xi)} := \Big(|E_*|^{-1} \int_E \int_{E_*} |v(x,\xi)|^2 \, dx \, d\xi \Big)^{1/2},$$

and

$$\mathcal{F}_s(\Xi) := \{v \in \mathcal{S}'_\Gamma(\Xi) \colon (\langle D\rangle^s \otimes \mathrm{id}) v \in \mathcal{F}_0(\Xi)\}, \quad \text{for all } s \in \mathbb{R}, \tag{1.22}$$

endowed with the norm

$$\|v\|_{\mathcal{F}_s(\Xi)} := \|(\langle D\rangle^s \otimes \mathrm{id}) v\|_{\mathcal{F}_0(\Xi)}.$$

We have

$$\mathcal{F}_{m,\xi} = \mathcal{F}_{m,\xi+\gamma^*}, \quad \text{for all } \gamma^* \in \Gamma_*,$$

and

$$\mathcal{F}_m(\Xi) = \int_{\mathbb{T}_*}^{\oplus} \mathcal{F}_{m,\xi} \, d\xi$$

(see [7]).

Lemma 1.14. *The operator*

$$\mathcal{U}_\Gamma \colon L^2(\mathcal{X}) \longrightarrow \mathcal{F}_0(\Xi)$$

defined by

$$(\mathcal{U}_\Gamma u)(x,\xi) := \sum_{\gamma \in \Gamma} e^{i\langle \xi,\gamma\rangle} u(x-\gamma), \quad \textit{for all } (x,\xi) \in \Xi,$$

is a unitary operator with inverse denoted $\mathcal{W}_\Gamma$.

Lemma 1.15. *With the above notations the following statements are true.*

(1) *The operator*

$$\widehat{P} := P \otimes \mathrm{id}$$

leaves invariant the subspace $\mathcal{S}'_\Gamma(\Xi)$.

(2) *Considered as an unbounded operator in the Hilbert space* $\mathcal{F}_0(\Xi)$*, the operator* $\widehat{P}$ *is self-adjoint and lower semi-bounded on the domain* $\mathcal{F}_m(\Xi)$ *and is unitarily equivalent to the operator* P.

Taking into account the Remark 4.12 we note that for any $\xi \in \mathcal{X}^*$ the operator P induces on the Hilbert space $\mathcal{F}_{0,\xi}$ a self-adjoint operator with domain $\mathcal{F}_{m,\xi}$ that we shall denote by $\widehat{P}(\xi)$; we evidently have the periodicity $\widehat{P}(\xi+\gamma^*) = \widehat{P}(\xi)$ for any $\gamma^* \in \Gamma^*$. If we identify $\mathcal{K}_0$ with $L^2_{\mathrm{loc}} \cap \mathcal{S}'_\Gamma(\Xi) \equiv L^2(E)$, the same Remark 4.12 implies that the operator $\widehat{P}(\xi)$ is unitarily equivalent with the operator $\check{P}(\xi)$ that is induced by $\mathfrak{Op}((\mathrm{id} \otimes \tau_{-\xi})p)$ on the space $\mathcal{K}_0$; this is a self-adjoint lower semi-bounded operator on the domain $\mathcal{K}_{m,\xi}$ (identified with $\mathcal{H}^m_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$, with the norm $\|\langle D+\xi\rangle^m \cdot \|_{L^2(E)}$). More precisely,

$$\check{P}(\xi) = \sigma_{-\xi}\widehat{P}(\xi)\sigma_\xi, \quad \text{for all } \xi \in \mathcal{X}^*.$$

Lemma 1.16. *For any*

$$\mathfrak{z} \in \mathbb{C} \setminus \overline{\bigcup_{\xi \in \mathcal{X}^*} \sigma(\check{P}(\xi))}.$$

the application

$$\mathcal{X}^* \ni \xi \longmapsto (\check{P}(\xi) - \mathfrak{z})^{-1} \in \mathbb{B}(\mathcal{K}_0)$$

is of class $C^\infty(\mathcal{X}^*)$.

Remark 1.17. Let us note that $\mathcal{K}_{m,\xi}$ is compactly embedded into $\mathcal{K}_0$ and thus, the operator $\check{P}(\xi)$ has compact resolvent for any $\xi \in \mathcal{X}^*$; it is clearly lower semi-bounded uniformly with respect to $\xi \in \mathcal{X}^*$, taking into account that

$$\check{P}(\xi+\gamma^*) = \sigma_{-\gamma^*}\check{P}(\xi)\sigma_{\gamma^*}, \quad \text{for all } \gamma^* \in \Gamma^*.$$

We deduce that

$$\sigma(\widehat{P}(\xi)) = \sigma(\check{P}(\xi)) = \{\lambda_j(\xi)\}_{j\geq 1},$$

where for any $\xi \in \mathcal{X}^*$ and any $j \geq 1$, $\lambda_j(\xi)$ is a real finitely degenerated eigenvalue and

$$\lim_{j\to\infty} \lambda_j(\xi) = \infty, \quad \text{for all } \xi \in \mathcal{X}^*;$$

we can always renumber the eigenvalues and suppose that $\lambda_j(\xi) \leq \lambda_{j+1}(\xi)$ for any $j \geq 1$ and for any $\xi \in \mathcal{X}^*$. Due to the Γ^*-periodicity of $\widehat{P}(\xi)$ we have that $\lambda_j(\xi + \gamma^*) = \lambda_j(\xi)$ for any $j \geq 1$, for any $\xi \in \mathcal{X}^*$ and for any $\gamma^* \in \Gamma^*$. These are the *Bloch bands of the operator* $\widehat{P}$.

The following lemma is a consequence of semi-boundedness and the *min-max principle.*

Lemma 1.18. *For each* $j \geq 1$ *the function*

$$\mathbb{T}_* \ni \xi \longmapsto \lambda_j(\xi) \in \mathbb{R}$$

is continuous on $\mathbb{T}_*$ *uniformly in* $j \geq 1$.

It is obvious that

$$\hat{P} = \int_{\mathbb{T}_*}^{\oplus} \hat{P}_\xi \, d\xi,$$

and then standard arguments allow us to prove the following proposition.

Proposition 1.19. *We have the following spectral decomposition*

$$\sigma(P) = \sigma(\widehat{P}) = \bigcup_{k=1}^{\infty} J_k$$

with

$$J_k := \lambda_k(\mathbb{T}_*)$$

a compact interval in $\mathbb{R}$.

Standard arguments concerning the direct integrals of self-adjoint operators (see [24]) imply the following statement. We shall need this result only in the special case of the constant magnetic field (see subsection 3.2).

Proposition 1.20. *Considering* $\widetilde{P}_\epsilon$ *as operator acting in* $\mathcal{S}'(\mathcal{X}^2)$ *we shall denote by* $\widetilde{P}'_\epsilon$ *the self-adjoint operator that it induces in* $L^2(\mathcal{X}^2)$ *with domain* $\widetilde{H}^m_{A_\epsilon}(\mathcal{X}^2)$*, as in Proposition 2.9, and by* $\widetilde{P}''_\epsilon$ *the self-adjoint operator that it induces in* $L^2(\mathcal{X} \times \mathbb{T})$ *with domain* $\mathcal{K}^m_\epsilon(\mathcal{X}^2)$ *(as in Proposition 2.23). Then we have the equality*

$$\sigma(\widetilde{P}'_\epsilon) = \sigma(\widetilde{P}''_\epsilon). \tag{1.23}$$

1.3.3. The localized quasi-Bloch functions. The operator $P := \mathfrak{Op}(p)$ defined above has also a self-adjoint realization in $L^2(\mathbb{T})$ with domain $\mathcal{K}_{m,0}$ being also lower semibounded. We shall very briefly recall the main steps for the construction of a finite dimensional system of linearly independent localized vectors associated to a given compact energy interval (*the localized quasi-Bloch functions*) following [10, 11, 8]).

Lemma 1.21. *There exist $N \in \mathbb{N}^*$, $C > 0$, and a linear independent family $\{\phi_1, \dots, \phi_N\} \subset \mathcal{S}(\mathbb{T})$, such that*

$$(P_\Gamma u, u)_{L^2(\mathbb{T})} \geq C^{-1} \|u\|^2_{\mathcal{K}_{m/2,0}} - C \sum_{j=1}^{N} |(u, \phi_j)_{L^2(\mathbb{T})}|^2, \quad \textit{for all } u \in \mathcal{K}_{m,0}. \tag{1.24}$$

Remark 1.22. From Remark 4.12 we know that for any $\xi \in \mathcal{X}^*$ the operator $P_{\Gamma,\xi}$ is self-adjoint and lower semibounded in $L^2(\mathbb{T})$ on the domain $\mathcal{K}_{m,\xi}$. If we identify $\mathcal{K}_{m,\xi}$ with $\mathcal{H}^m_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ endowed with the norm $\|\langle D + \xi\rangle^m u\|_{L^2(E)}$, we deduce that the operator P_ξ is self-adjoint in $L^2_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ with the domain $\mathcal{K}_{m,\xi}$. Noticing that $P = \sigma_\xi P_\xi \sigma_{-\xi}$ and $\sigma_\xi : \mathcal{K}_{s,\xi} \to \mathcal{F}_{s,\xi}$ is a unitary operator for any $s \in \mathbb{R}$ and any $\xi \in \mathcal{X}^*$, it follows that P generates in $\mathcal{F}_{0,\xi}$ a self-adjoint lower semibounded operator on the domain $\mathcal{F}_{m,\xi}$.

Lemma 1.23. *Suppose given a compact interval $I \subset \mathbb{R}$. Then, there exist a constant $C > 0$, a natural integer $N \in \mathbb{N}$, and the family of functions $\{\psi_1, \dots, \psi_N\}$ having the following properties.*

a) $\psi_j \in C^\infty(\Xi)$ (*due to the smoothness of our symbols*).

b) *For all $(y, \eta) \in \Xi$, $\gamma^* \in \Gamma_*$, and $1 \leq j \leq N$,*

$$\psi_j(y, \eta + \gamma^*) = \psi_j(y, \eta).$$

c) *$\{\psi_j(\cdot, \xi)\}_{1 \leq j \leq N}$ is an orthonormal system in $\mathcal{F}_{0,\xi}$ for any $\xi \in \mathcal{X}^*$. We denote by $\mathcal{T}_\xi$ the complex linear space generated by the family $\{\psi_j(\cdot, \xi)\}_{1 \leq j \leq N}$ in $\mathcal{F}_{0,\xi}$ and by $\mathcal{T}_\xi^\perp$ its orthogonal complement in the same Hilbert space.*

d) *For all $u \in \mathcal{F}_{m,\xi} \cap \mathcal{T}_\xi^\perp$, $\xi \in \mathcal{X}^*$, and $\lambda \in I$,*

$$((P - \lambda)u, u)_{\mathcal{F}_{0,\xi}} \geq C \|u\|^2_{\mathcal{F}_{0,\xi}}. \tag{1.25}$$

Lemma 1.24. *Under the assumptions of Lemma 1.23 we denote by Π_ξ the orthogonal projection on $\mathfrak{T}_\xi$ in the Hilbert space $\mathcal{F}_{0,\xi}$ and by $S(\xi,\lambda)$ the unbounded operator in $\mathfrak{T}_\xi^\perp$ defined on the domain $\mathcal{F}_{m,\xi}\cap\mathfrak{T}_\xi^\perp$ by the action of $(\mathbb{1}-\Pi_\xi)(P-\lambda)$. Then,*

a) *the operator $S(\xi,\lambda)$ is self-adjoint and invertible and $S(\xi,\lambda)^{-1}\in\mathbb{B}(\mathfrak{T}_\xi^\perp;\mathfrak{T}_\xi^\perp)$ uniformly with respect to $(\xi,\lambda)\in\mathbb{T}_*\times I$;*

b) *the operator $S(\xi,\lambda)^{-1}$ also belongs to $\mathbb{B}(\mathfrak{T}_\xi^\perp;\mathcal{F}_{m,\xi})$ uniformly with respect to $(\xi,\lambda)\in\mathbb{T}_*\times I$.*

We define now the family of N functions

$$\phi_j(x,\xi):=e^{-i\langle\xi,x\rangle}\psi_j(x,\xi),\quad \text{for all } (x,\xi)\in\Xi,\ 1\le j\le N, \tag{1.26}$$

with the family $\{\psi_j\}_{1\le j\le N}$ defined in Lemma 1.23. One can prove that

Lemma 1.25. *The functions $\{\phi_j\}_{1\le j\le N}$ defined in (1.26) have the following properties:*

a) $\phi_j\in C^\infty(\Xi)$;

b) *for all $(x,\xi)\in\Xi$ and all $\gamma\in\Gamma$,*

$$\phi_j(x+\gamma,\xi)=\phi_j(x,\xi);$$

c) *for all $(x,\xi)\in\Xi$ and all $\gamma^*\in\Gamma_*$,*

$$\phi_j(x,\xi+\gamma^*)=e^{-i\langle\gamma^*,x\rangle}\phi_j(x,\xi);$$

d) *For any $\alpha\in\mathbb{N}^d$ and any $s\in\mathbb{R}$ there exists a strictly positive constant $C_{\alpha,s}$ such that, for all $\xi\in\mathcal{X}^*$,*

$$\|(\partial_\xi^\alpha\phi_j)(\cdot,\xi)\|_{\mathcal{K}_{s,\xi}}\le C_{\alpha,s}. \tag{1.27}$$

2. Proof of Theorem 1.1

Let us consider given a family $\{B_\epsilon\}_{\epsilon\in[-\epsilon_0,\epsilon_0]}$ of magnetic fields on $\mathcal{X}$ satisfying Hypothesis H.1 and $\{A_\epsilon\}_{\epsilon\in[-\epsilon_0,\epsilon_0]}$ an associated family of vector potentials (we shall always work with the vector potentials given by formula (1.4)). Let us also consider a given family of symbols $\{p_\epsilon\}_{\epsilon\in[-\epsilon_0,\epsilon_0]}$ that satisfy Hypotheses H.2–H.6.

We shall use the notations (1.5). It is evident that with the above hypothesis and notations we have that $\mathring{p}_\epsilon \in S^m_1(\mathcal{X})$ and is elliptic, both properties being uniform in $\epsilon \in [-\epsilon_0, \epsilon_0]$. Then our results in [12] imply that the operator $P_\epsilon := \mathfrak{Op}^{A_\epsilon}(\mathring{p}_\epsilon)$, the main operator we are interested in, is self-adjoint and lower semi-bounded in $L^2(\mathcal{X})$ having the domain $\mathcal{H}^m_{A_\epsilon}(\mathcal{X})$ (the magnetic Sobolev space of order m defined in Definition 2.5 below); moreover, with the choice of vector potential that we made, it is essentially self-adjoint on the space of Schwartz test functions $\mathcal{S}(\mathcal{X})$.

We use the notations from the Appendices. From example 1.6 it follows that $\{\tilde{p}_\epsilon\}_{|\epsilon|\leq\epsilon_0} \in S^m_{1,\epsilon}(\mathcal{X}^2)$ so that by defining

$$\mathfrak{q}_\epsilon(X) := \mathfrak{Op}(\tilde{p}_\epsilon(X,\cdot)),$$

for all $s \in \mathbb{R}$,

$$\{\mathfrak{q}_\epsilon\}_{|\epsilon|\leq\epsilon_0} \in S^0_{0,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X}))).$$

We can then define the *auxiliary operator*

$$\widetilde{P}_\epsilon := \mathfrak{Op}^{A_\epsilon}(\mathfrak{q}_\epsilon).$$

Proposition 4.13 and Example 4.14 imply the following statement.

Lemma 2.1. *With the above notations and under Hypotheses H.1–H.6,*

(1) *For any $s \in \mathbb{R}$,*

$$\widetilde{P}_\epsilon \in \mathbb{B}(\mathcal{S}(\mathcal{X}; \mathcal{H}^{s+m}(\mathcal{X})); \mathcal{S}(\mathcal{X}; \mathcal{H}^s(\mathcal{X}))) \\ \cap \mathbb{B}(\mathcal{S}'(\mathcal{X}; \mathcal{H}^{s+m}(\mathcal{X})); \mathcal{S}'(\mathcal{X}; \mathcal{H}^s(\mathcal{X}))),$$

uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$;

(2) $\widetilde{P}_\epsilon \in \mathbb{B}(\mathcal{S}(\mathcal{X}^2); \mathcal{S}(\mathcal{X}^2)) \cap \mathbb{B}(\mathcal{S}'(\mathcal{X}^2); \mathcal{S}'(\mathcal{X}^2))$, *uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$;*

(3) $\widetilde{P}_\epsilon$ *considered as unbounded operator in $L^2(\mathcal{X}^2)$ with domain $\mathcal{S}(\mathcal{X}^2)$ is symmetric for any $\epsilon \in [-\epsilon_0, \epsilon_0]$.*

We consider the isomorphisms

$$\begin{aligned} \psi: \quad & \mathcal{X}^2 \longrightarrow \mathcal{X}^2, \\ & (x, y) \longmapsto (x, x - y), \end{aligned} \tag{2.1}$$

such that

$$\psi^{-1} = \psi,$$

and

$$\begin{aligned} \boldsymbol{\chi}: \quad \mathcal{X}^2 &\longrightarrow \mathcal{X}^2, \\ (x,y) &\longmapsto (x+y,y), \end{aligned} \tag{2.2}$$

such that

$$\boldsymbol{\chi}^{-1}(x,y) = (x-y,y).$$

The operators $\boldsymbol{\psi}^*$ and $\boldsymbol{\chi}^*$ that they induce on $L^2(\mathcal{X}^2)$ ($\boldsymbol{\psi}^* u := u \circ \boldsymbol{\psi}$) are evidently unitary.

Lemma 2.2. *For any $u \in \mathcal{S}(\mathcal{X}^2)$, the image $\widetilde{P}_\epsilon u$ may be written in any of the three equivalent forms*

$$\begin{aligned} (\widetilde{P}_\epsilon u)(x,y) = (2\pi)^{-d} \int_{\mathcal{X}}\int_{\mathcal{X}^*} & e^{i\langle\eta, y-\tilde{y}\rangle}\omega_{A_\epsilon}(x, x+\tilde{y}-y) \\ & p_\epsilon\Big(x-y+\frac{(y+\tilde{y})}{2}, \frac{(y+\tilde{y})}{2}, \eta\Big) \\ & u(x+\tilde{y}-y, \tilde{y})\, d\tilde{y}\, d\eta, \end{aligned} \tag{2.3}$$

$$(\boldsymbol{\psi}^*\widetilde{P}_\epsilon \boldsymbol{\psi}^* u)(x,y) = [\mathfrak{Op}^{A_\epsilon}(((\mathrm{id}\otimes\tau_y\otimes\mathrm{id})p_\epsilon)^\circ)u(\cdot,y)](x), \tag{2.4}$$

and

$$(\boldsymbol{\chi}^*\widetilde{P}_\epsilon(\boldsymbol{\chi}^*)^{-1}u)(x,y) = [\mathfrak{Op}^{(\tau_{-x}A_\epsilon)}(((\tau_x\otimes\mathrm{id}\otimes\mathrm{id})p_\epsilon)^\circ)u(x,\cdot)](y). \tag{2.5}$$

Proof. Let us fix $u \in \mathcal{S}(\mathcal{X}^2)$. Starting from the definitions of $\widetilde{P}_\epsilon$ and $\mathfrak{q}_\epsilon$ and using oscillating integral techniques, we get

$$\begin{aligned} (\widetilde{P}_\epsilon u)(x,y) &= (2\pi)^{-d}\int_{\mathcal{X}}\int_{\mathcal{X}^*} e^{i\langle\xi,x-\tilde{x}\rangle}\omega_{A_\epsilon}(x,\tilde{x})[\mathfrak{q}_\epsilon((x+\tilde{x})/2,\xi)u(\tilde{x},\cdot)](y)\,d\tilde{x}\,d\xi \\ &= (2\pi)^{-2d}\int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}^*} e^{i\langle\eta,y-\tilde{y}\rangle}\omega_{A_\epsilon}(x,\tilde{x})p_\epsilon\Big(\frac{(x+\tilde{x})}{2},\frac{(y+\tilde{y})}{2},\eta\Big) \\ &\qquad \Big[\int_{\mathcal{X}^*} e^{i\langle\xi,x-\tilde{x}-y+\tilde{y}\rangle}\,d\xi\Big]u(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}\,d\eta. \end{aligned}$$

By the Fourier inversion theorem the inner oscillating integral is in fact

$$(2\pi)^d[\tau_{x-y+\tilde{y}}\delta_0](\tilde{x}),$$

and we obtain (2.3). In order to prove (2.4), we apply (2.3) to $\boldsymbol{\psi}^* u$. Formula (2.5) can be easily obtained in a similar way, starting with (2.3) applied to $(\boldsymbol{\chi}^*)^{-1}u$.

We end the proof of (2.5) by observing that

$$\begin{aligned}\omega_{A_\epsilon}(x+y,x+\tilde{y}) &= \exp\Big\{-i\int\limits_{[x+y,x+\tilde{y}]} A_\epsilon\Big\}\\ &= \exp\Big\{i\Big\langle y-\tilde{y},\int_0^1 A_\epsilon((1-s)(x+y)+s(x+\tilde{y}))\,ds\Big\rangle\Big\}\\ &= \exp\Big\{i\Big\langle y-\tilde{y},\int_0^1 (\tau_{-x}A_\epsilon)((1-s)y+s\tilde{y})\,ds\Big\rangle\Big\}\\ &= \exp\Big\{-i\int\limits_{[y,\tilde{y}]}(\tau_{-x}A_\epsilon)\Big\}\\ &= \omega_{(\tau_{-x}A_\epsilon)}(y,\tilde{y}).\end{aligned}$$

□

Corollary 2.3. *We have the following relations between the operators $\widetilde{P}_\epsilon$ and P_ϵ:*

(1) *for any $v \in \mathcal{S}'(\mathcal{X})$,*

$$(\boldsymbol{\chi}^* \widetilde{P}_\epsilon (\boldsymbol{\chi}^*)^{-1}(\delta_0 \otimes v)) = \delta_0 \otimes (P_\epsilon v), \tag{2.6}$$

and

$$(\boldsymbol{\psi}^* \widetilde{P}_\epsilon \boldsymbol{\psi}^*(v \otimes \delta_0)) = (P_\epsilon v) \otimes \delta_0; \tag{2.7}$$

(2) *if p_ϵ does not depend on its second variable $y \in \mathcal{X}$, then*

$$\boldsymbol{\psi}^* \widetilde{P}_\epsilon \boldsymbol{\psi}^* = P_\epsilon \otimes \mathrm{id}\,. \tag{2.8}$$

The idea of the proof of our Theorem 1.1 follows the main lines of the proof in [8] modified in order to fit with the use of the magnetic pseudodifferential calculus. The main steps of the proof are the following. We start with the "auxiliary operator" $\widetilde{P}_\epsilon$ defined in (1.8) and use a Floquet transformation with respect to the second variable (with respect to which it is Γ-periodic). This transformation puts into evidence a family of pseudodifferential operators on a torus and a family of Wannier functions, similar to those above, is obtained. With these Wannier functions we define a Grushin type problem similar to the one defined in [8] but with magnetic pseudodifferential operators. Now come the important technical steps. First one extends the magnetic pseudodifferential operators to some spaces of tempered distributions in two variables and proves that the inversion formula remains true with a magnetic pseudodifferential operator as inverse. In order to make use of Corollary 2.3 that connects our main Hamiltonian with the “auxiliary” operator we have to put into evidence a Hilbert space $\mathfrak{L}_0$. The idea is to restrict our Grushin problem to the direct sum of this Hilbert space with a second Hilbert space $\mathfrak{V}_0$ and prove that the inversion relation still holds true. The control of the continuity of the operators appearing in the Grushin problem restricted to these new Hilbert spaces is achieved through a non-trivial group of results based on the technical results in Lemmata 4.1, 4.2, 2.37, and 2.38 that allow to prove the criteria in Propositions 2.39, 2.42, and 2.44 that replace the criteria proposed in [8]. Once these technical facts clarified, Corollary 2.3 easily allow us to finish the proof of Theorem 1.1.

Let us recall from [12] some facts about magnetic Sobolev spaces.

Remark 2.4. As in [13] one can define a family of symbols $\{q_{s,\epsilon}\}_{(s,\epsilon)\in\mathbb{R}\times[-\epsilon_0,\epsilon_0]}$ such that

(1) $q_{s,\epsilon} \in S^s_1(\mathcal{X})$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$;

(2) $q_{s,\epsilon} \sharp^{B_\epsilon} q_{-s,\epsilon} = 1$;

(3) for all $s > 0$,

$$q_{s,\epsilon}(x, \xi) = \langle\xi\rangle^s + \mu,$$

for some sufficiently large $\mu > 0$ and $q_{0,\epsilon} = 1$.

Evidently that for any Hilbert space $\mathcal{A}$, using the definitions introduced in subsection 4.2 we can identify the symbol $q_{s,\epsilon}$ with the operator-valued symbol $q_{s,\epsilon}\, \mathrm{id}_{\mathcal{A}}$ and thus we may consider that $q_{s,\epsilon} \in S^s_1(\mathcal{X}; \mathbb{B}(\mathcal{A}))$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. We shall use the notations

$$Q_{s,\epsilon} := \mathfrak{Op}^{A_\epsilon}(q_{s,\epsilon})$$

and

$$Q'_{s,\epsilon} := Q_{s,\epsilon} \otimes \mathrm{id}\,.$$

Let us still set

$$\widetilde{Q}_{s,\epsilon} := \psi^* Q'_{s,\epsilon} \psi^*,$$

with ψ from (2.1) and let us note that due to Corollary 2.3 (2) the operators $\widetilde{Q}_{s,\epsilon}$ and $Q_{s,\epsilon}$ are in the same relation as the pair $\widetilde{P}_\epsilon$ and P_ϵ.

Definition 2.5. For magnetic fields $\{B_\epsilon\}_{\epsilon\in[-\epsilon_0,\epsilon_0]}$ verifying Hypothesis H.1 and for choices of vector potentials given by (1.4) we define the following spaces.

(1) The *magnetic Sobolev space of order* $s \in \mathbb{R}$ is the space

$$\mathcal{H}^s_{A_\epsilon}(\mathcal{X}) := \{u \in \mathcal{S}'(\mathcal{X})\colon Q_{s,\epsilon}u \in L^2(\mathcal{X})\}$$

endowed with the following natural quadratic norm

$$\|u\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})} := \|Q_{s,\epsilon}u\|_{L^2(\mathcal{X})},$$

for all $u \in \mathcal{H}^s_{A_\epsilon}(\mathcal{X})$. $\mathcal{H}^s_{A_\epsilon}(\mathcal{X})$ a Hilbert space containing $\mathcal{S}(\mathcal{X})$ as a dense subspace.

(2) We shall define also

$$\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) := \bigcap_{s\in\mathbb{R}} \mathcal{H}^s_{A_\epsilon}(\mathcal{X}),$$

with the projective limit topology.

(3) For $s \in \mathbb{R}$ we consider also the spaces

$$\widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2) := \{u \in \mathcal{S}'(\mathcal{X}^2)\colon \widetilde{Q}_{s,\epsilon}u \in L^2(\mathcal{X}^2)\}$$

endowed with the following natural quadratic norm

$$\|u\|_{\widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2)} := \|\widetilde{Q}_{s,\epsilon}u\|_{L^2(\mathcal{X}^2)},$$

for all $u \in \widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2)$. $\widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2)$ is a Hilbert space containing $\mathcal{S}(\mathcal{X}^2)$ as a dense subspace.

Remark 2.6. ψ^* is a unitary operator from $\widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2)$ onto $\mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathcal{X})$.

Lemma 2.7. *For any* $s \in \mathbb{R}$ *we have that* $\widetilde{P}_\epsilon \in \mathbb{B}(\widetilde{\mathcal{H}}^{s+m}_{A_\epsilon}(\mathcal{X}^2); \widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2))$ *uniformly for* $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Remark 2.8. Suppose given $r \in S^m_1(\mathcal{X}^2)$. Then evidently $r(\cdot, y, \cdot) \in S^m_1(\mathcal{X})$ uniformly for $y \in \mathcal{X}$. If B is a magnetic field on $\mathcal{X}$ with components of class $BC^\infty(\mathcal{X})$ and A an associated vector potential having components of class $C^\infty_{\mathrm{pol}}(\mathcal{X})$ we define the *magnetic pseudodifferential operator with parameter* $y \in \mathcal{X}$

$$(\mathfrak{R}u)(x, y) := (2\pi)^{-d} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle\xi, x-\tilde{x}\rangle} \omega_A(x, \tilde{x}) r((x + \tilde{x})/2, y, \xi) u(\tilde{x}, y)\, d\tilde{x}\, d\xi, \tag{2.9}$$

for all $u \in \mathcal{S}(\mathcal{X}^2)$ and all $(x, y) \in \mathcal{X}^2$.

A straightforward modification of the arguments from [12], and denoting by $\mathfrak{R}_\epsilon$ the operator defined as above in (2.9) with a vector potential A_ϵ, allows to prove that

$$\mathfrak{R}_\epsilon \in \mathbb{B}(\mathcal{S}(\mathcal{X}^2);\mathcal{S}(\mathcal{X}^2)) \cap \mathbb{B}(\mathcal{H}^{s+m}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathcal{X}); \mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathcal{X})), \tag{2.10}$$

for all $s \in \mathbb{R}$. Moreover, if r is elliptic, then for any $u \in L^2(\mathcal{X}^2)$ and any $s \in \mathbb{R}$ we have the equivalence relation

$$u \in \mathcal{H}^{s+m}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathcal{X}) \iff \mathfrak{R}_\epsilon u \in \mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathcal{X}). \tag{2.11}$$

Proposition 2.9. *$\widetilde{P}_\epsilon$ is a self-adjoint operator in $L^2(\mathcal{X}^2)$ with domain $\widetilde{\mathcal{H}}^m_{A_\epsilon}(\mathcal{X}^2)$. It is essentially self-adjoint on $\mathcal{S}(\mathcal{X}^2)$.*

2.1. The Grushin problem. We consider the Floquet transformation acting on the second variable of $L^2(\mathcal{X} \times \mathcal{X})$.

Definition 2.10. We set

$$\begin{aligned}\mathcal{S}'_\Gamma(\mathcal{X}^2 \times \mathcal{X}^*) := \big\{v \in \mathcal{S}'(\mathcal{X}^2 \times \mathcal{X}^*):&\\ v(x, y+\gamma, \theta) = e^{i\langle\theta,\gamma\rangle} v(x,y,\theta) &\text{ for all } \gamma \in \Gamma,\\ v(x,y,\theta+\gamma^*) = v(x,y,\theta) &\text{ for all } \gamma^* \in \Gamma_*\big\},\end{aligned}$$

endowed with the topology induced by $\mathcal{S}'(\mathcal{X}^2 \times \mathcal{X}^*)$.

Definition 2.11. We set

$$\mathcal{F}_0(\mathcal{X}^2 \times \mathcal{X}^*) := \mathcal{S}'_\Gamma(\mathcal{X}^2 \times \mathcal{X}^*) \cap L^2_{\mathrm{loc}}(\mathcal{X}^2 \times \mathcal{X}^*) \cap L^2(\mathcal{X} \times E \times E_*)$$

endowed with the norm

$$\|v\|_{\mathcal{F}_0} := \sqrt{|E_*|^{-1} \int_{\mathcal{X}} \int_E \int_{E_*} |v(x,y,\theta)|^2 \, dx\, dy\, d\theta}, \quad \text{for all } v \in \mathcal{F}_0(\mathcal{X}^2 \times \mathcal{X}^*), \tag{2.12}$$

that makes $\mathcal{F}_0(\mathcal{X}^2 \times \mathcal{X}^*)$ into a Hilbert space.

Lemma 2.12. *The map*

$$(\widetilde{\mathcal{U}}_\Gamma u)(x,y,\theta) := \sum_{\gamma\in\Gamma} e^{i\langle\theta,\gamma\rangle} u(x, y-\gamma)$$

defined on $\mathcal{S}(\mathcal{X}^2)$, extends as a unitary operator

$$\widetilde{\mathcal{U}}_\Gamma : L^2(\mathcal{X}^2) \longrightarrow \mathcal{F}_0(\mathcal{X}^2 \times \mathcal{X}^*).$$

Lemma 2.13. *Let $\widetilde{P}_\epsilon$ be the operator defined in* (1.8) *and*

$$\widetilde{P}_{\epsilon,\Gamma} := \widetilde{P}_\epsilon \otimes \mathrm{id},$$

as a linear continuous operator in $\mathcal{S}'_\Gamma(\mathcal{X}^2 \times \mathcal{X}^*)$. Then

$$\widetilde{\mathcal{U}}_\Gamma \widetilde{P}_\epsilon = \widetilde{P}_{\epsilon,\Gamma} \widetilde{\mathcal{U}}_\Gamma \quad \text{on } \mathcal{S}'(\mathcal{X}^2).$$

Definition 2.14. We define the operator

$$\widetilde{Q}_{s,\epsilon,\Gamma} := \widetilde{Q}_{s,\epsilon} \otimes \mathrm{id} \quad \text{on } \mathcal{S}'_\Gamma(\mathcal{X}^2 \times \mathcal{X}^*).$$

Definition 2.15. For any $s \in \mathbb{R}$ we define

$$\mathcal{F}_{s,\epsilon}(\mathcal{X}^2 \times \mathcal{X}^*) := \{v \in \mathcal{S}'_\Gamma(\mathcal{X}^2 \times \mathcal{X}^*) \colon \widetilde{Q}_{s,\epsilon,\Gamma} v \in \mathcal{F}_0(\mathcal{X}^2 \times \mathcal{X}^*)\},$$

that is evidently a Hilbert space for the norm

$$\|v\|_{\mathcal{F}_{s,\epsilon}} := \|\widetilde{Q}_{s,\epsilon,\Gamma} v\|_{\mathcal{F}_0}.$$

Lemma 2.16. *The operator*

$$\widetilde{\mathcal{U}}_\Gamma \colon \widetilde{\mathcal{H}}^s_{A_\epsilon}(\mathcal{X}^2) \longrightarrow \mathcal{F}_{s,\epsilon}(\mathcal{X}^2 \times \mathcal{X}^*)$$

is unitary.

Lemma 2.17. *The operator $\widetilde{P}_{\epsilon,\Gamma}$ defined on $\mathcal{F}_{m,\epsilon}(\mathcal{X}^2 \times \mathcal{X}^*)$ is self-adjoint in the space $\mathcal{F}_0(\mathcal{X}^2 \times \mathcal{X}^*)$.*

Definition 2.18. Let $\theta \in \mathcal{X}^*$ and $s \in \mathbb{R}$.

(1) We define

$$\mathcal{S}'_\theta(\mathcal{X}^2) := \{u \in \mathcal{S}'(\mathcal{X}^2) \colon (\mathrm{id} \otimes \tau_{-\gamma})u = e^{i\langle\theta,\gamma\rangle} u \text{ for all } \gamma \in \Gamma\},$$

with the topology induced by $\mathcal{S}'(\mathcal{X}^2)$.

(2) Further, we define

$$\mathcal{H}^s_{\theta,\epsilon}(\mathcal{X}^2) := \{u \in \mathcal{S}'_\theta(\mathcal{X}^2) \colon \widetilde{Q}_{s,\epsilon} u \in L^2(\mathcal{X} \times E)\},$$

endowed with the norm

$$\|u\|_{\mathcal{H}^s_{\theta,\epsilon}} := \|\widetilde{Q}_{s,\epsilon} u\|_{L^2(\mathcal{X} \times E)}.$$

(3) Finally, we define

$$\mathcal{K}^s_\epsilon(\mathcal{X}^2) := \mathcal{H}^s_{0,\epsilon}(\mathcal{X}^2).$$

Remark 2.19. As already noted in the proof of Lemma 2.13, for any $u \in \mathcal{S}'(\mathcal{X}^2)$ we have

$$(\mathrm{id} \otimes \tau_{-\gamma}) \widetilde{P}_\epsilon u = \widetilde{P}_\epsilon (\mathrm{id} \otimes \tau_{-\gamma}) u, \quad \text{for all } \gamma \in \Gamma.$$

It follows that the operators $\widetilde{P}_\epsilon$ and $\widetilde{Q}_{s,\epsilon}$ leave the space $\mathcal{S}'_\theta(\mathcal{X}^2)$ invariant. We note that

$$\mathcal{S}'_0(\mathcal{X}^2) \equiv \mathcal{S}'_\Gamma(\mathcal{X}^2).$$

Let us also note that for $s = 0$ the spaces defined in (2) and (3) above do not depend on ϵ and will be denoted by $\mathcal{H}_\theta(\mathcal{X}^2)$ and respectively by $\mathcal{K}(\mathcal{X}^2)$; this last one may be identified with $L^2(\mathcal{X} \times \mathbb{T})$.

Lemma 2.20. *Let us consider the map* $\boldsymbol{\psi}$ *defined by* (2.1)*. Then for any* $s \in \mathbb{R}$ *the adjoint* $\boldsymbol{\psi}^*$ *is a unitary operator*

$$\mathcal{K}^s_\epsilon(\mathcal{X}^2) \longrightarrow \mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T}).$$

In particular $\mathcal{K}^s_\epsilon(\mathcal{X}^2)$ *is a Hilbert space having* $\mathcal{S}(\mathcal{X} \times \mathbb{T})$ *as a dense subspace.*

Lemma 2.21. *For any* $\theta \in \mathcal{X}^*$ *and* $s \in \mathbb{R}$ *the operator*

$$\Upsilon_\theta : \mathcal{S}(\mathcal{X}^2) \longrightarrow \mathcal{S}(\mathcal{X}^2)$$

defined by

$$(\Upsilon_\theta u)(x, y) := e^{i\langle\theta, x-y\rangle} u(x, y),$$

induces a unitary operator

$$\mathcal{H}^s_{\theta,\epsilon}(\mathcal{X}^2) \longrightarrow \mathcal{K}^s_\epsilon(\mathcal{X}^2).$$

In particular, $\mathcal{H}^s_{\theta,\epsilon}(\mathcal{X}^2)$ *is a Hilbert space containing*

$$\mathcal{S}_\theta(\mathcal{X}^2) := \Upsilon_\theta^{-1}[\mathcal{S}(\mathcal{X} \times \mathbb{T})]$$

as a dense subspace.

Proof. Let us prove first that for any $\theta \in \mathcal{X}^*$ we have the equality $\widetilde{P}_\epsilon \Upsilon_\theta = \Upsilon_\theta \widetilde{P}_\epsilon$, on $\mathcal{S}'(\mathcal{X}^2)$. It is clearly enough to prove it on $\mathcal{S}(\mathcal{X}^2)$; this is a direct consequence of (2.3), because $(x+\tilde{y}-y)-\tilde{y} = x-y$. Then we also have that $\widetilde{Q}_{s,\epsilon} \Upsilon_\theta = \Upsilon_\theta \widetilde{Q}_{s,\epsilon}$ on $\mathcal{S}'(\mathcal{X}^2)$. We note further that Υ_θ takes the space $\mathcal{S}'_\theta(\mathcal{X}^2)$ into the space $\mathcal{S}'_\Gamma(\mathcal{X}^2)$, while the operator $\widetilde{Q}_{s,\epsilon}$ leaves invariant both spaces $\mathcal{S}'_\theta(\mathcal{X}^2)$ and $\mathcal{S}'_\Gamma(\mathcal{X}^2)$. It is then easy to see that for $u \in \mathcal{S}'_\theta(\mathcal{X}^2)$ we have the equivalence relation $u \in \mathcal{H}^s_{\theta,\epsilon}(\mathcal{X}^2) \Longleftarrow \Rightarrow \Upsilon_\theta u \in \mathcal{K}^s_\epsilon(\mathcal{X}^2)$ and the equality $\|\Upsilon_\theta u\|_{\mathcal{K}^s_\epsilon} = \|u\|_{\mathcal{H}^s_{\theta,\epsilon}}$. The last statement is obvious by Lemma 2.20. □

Lemma 2.22. *For any* $s \in \mathbb{R}$ *we have that* $\widetilde{P}_\epsilon \in \mathbb{B}(\mathcal{K}^{s+m}_\epsilon(\mathcal{X}^2); \mathcal{K}^s_\epsilon(\mathcal{X}^2))$ *uniformly for* $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. We use the fact that $\boldsymbol{\psi}^* : \mathcal{K}^s_\epsilon(\mathcal{X}^2) \to \mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ is a unitary operator leaving $\mathcal{S}(\mathcal{X} \times \mathbb{T})$ invariant. □

Proposition 2.23. $\widetilde{P}_\epsilon$ *is a self-adjoint operator in* $\mathcal{K}(\mathcal{X}^2) \equiv L^2(\mathcal{X} \times \mathbb{T})$ *with domain* $\mathcal{K}^m_\epsilon(\mathcal{X}^2)$*; it is essentially self-adjoint on* $\mathcal{S}(\mathcal{X} \times \mathbb{T})$.

As we have already noted in Remark 4.8, the symbol $p_0(x, y, \eta)$ at $\epsilon = 0$ does not depend on the first variable $x \in \mathcal{X}$; thus if we set

$$\mathfrak{p}_0(y, \eta) := p_0(0, y, \eta)$$

and

$$r_\epsilon(x, y, \eta) := p_\epsilon(x, y, \eta) - \mathfrak{p}_0(y, \eta),$$

and we note that $\mathfrak{p}_0 \in S^m_1(\mathbb{T})$ is real and elliptic, we can write

$$p_\epsilon = \mathfrak{p}_0 + r_\epsilon, \quad \lim_{\epsilon \to 0} r_\epsilon = 0, \text{ in } S^m_1(\mathcal{X} \times \mathbb{T}). \tag{2.13}$$

We apply the construction of the Wannier functions (Subsection 1.3.3) to the operator $P_0 := \mathfrak{Op}(\mathfrak{p}_0)$. We set

$$\mathcal{K}_0 := \mathcal{K}_{0,0} \equiv L^2(\mathbb{T}) \equiv L^2(E),$$

and, for any $\xi \in \mathcal{X}^*$, we define the linear operators $P_{0,\xi}$ as in Remark 1.13 and

$$R_+(\xi)u := \{(u, \phi_j)_{\mathcal{K}_0}\}_{1 \le j \le N}, \quad \text{for all } u \in \mathcal{K}_0, \tag{2.14}$$

and

$$R_-(\xi)\underline{u} := \sum_{1 \le j \le N} \underline{u}_j \phi_j(\cdot, \xi), \text{ for all } \underline{u} \in \mathbb{C}^N. \tag{2.15}$$

We evidently have that $R_+(\xi) \in \mathbb{B}(\mathcal{K}_0; \mathbb{C}^N)$, $R_-(\xi) \in \mathbb{B}(\mathbb{C}^N; \mathcal{K}_0)$ and, by (1.27), both are BC^∞ functions of $\xi \in \mathcal{X}^*$. With these operators we can now define the *Grushin type operator*

$$\mathcal{P}_0(\xi, \lambda) := \begin{pmatrix} P_{0,\xi} - \lambda & R_-(\xi) \\ R_+(\xi) & 0 \end{pmatrix} \in \mathbb{B}(\mathcal{K}_{m,\xi} \times \mathbb{C}^N; \mathcal{K}_0 \times \mathbb{C}^N), \tag{2.16}$$

for all $(\xi, \lambda) \in \mathcal{X}^* \times I$.

Proposition 2.24. *With the above notations,*

a) *as a function of* $(\xi,\lambda)\in\mathcal{X}^*\times I$,

$$\mathcal{P}_0\in C^\infty(\mathcal{X}^*\times I;\mathbb{B}(\mathcal{K}_{m,0}\times\mathbb{C}^N;\mathcal{K}_0\times\mathbb{C}^N))$$

and for, any $\alpha\in\mathbb{N}^d$ *and any* $k\in\mathbb{N}$,

$$(\partial_\xi^\alpha\partial_\lambda^k\mathcal{P}_0)(\xi,\lambda)\in\mathbb{B}(\mathcal{K}_{m,\xi}\times\mathbb{C}^N;\mathcal{K}_0\times\mathbb{C}^N)$$

uniformly in $(\xi,\lambda)\in\mathcal{X}^*\times I$;

b) *if we consider* $\mathcal{P}_0(\xi,\lambda)$ *as an unbounded operator in* $\mathcal{K}_0\times\mathbb{C}^N$ *with domain* $\mathcal{K}_{m,\xi}\times\mathbb{C}^N$, *then, for any* $(\xi,\lambda)\in\mathcal{X}^*\times I$, $\mathcal{P}_0(\xi,\lambda)$ *is self-adjoint*;

c) *the operator* $\mathcal{P}_0(\xi,\lambda)$ *has an inverse*

$$\mathcal{E}_0(\xi,\lambda):=\begin{pmatrix}E^0(\xi,\lambda) & E^0_+(\xi,\lambda)\\ E^0_-(\xi,\lambda) & E^0_{-,+}(\xi,\lambda)\end{pmatrix}\in\mathbb{B}(\mathcal{K}_0\times\mathbb{C}^N;\mathcal{K}_{m,\xi}\times\mathbb{C}^N),\quad (2.17)$$

uniformly bounded with respect to $(\xi,\lambda)\in\mathcal{X}^*\times I$;

d) *as a function of* $(\xi,\lambda)\in\mathcal{X}^*\times I$,

$$\mathcal{E}_0\in C^\infty(\mathcal{X}^*\times I;\mathbb{B}(\mathcal{K}_0\times\mathbb{C}^N;\mathcal{K}_{m,0}\times\mathbb{C}^N)),$$

and, for any $\alpha\in\mathbb{N}^d$ *and any* $k\in\mathbb{N}$,

$$(\partial_\xi^\alpha\partial_\lambda^k\mathcal{E}_0)(\xi,\lambda)\in\mathbb{B}(\mathcal{K}_0\times\mathbb{C}^N;\mathcal{K}_{m,\xi}\times\mathbb{C}^N)$$

uniformly in $(\xi,\lambda)\in\mathcal{X}^*\times I$.

Proof. We need to make the change of representation

$$U(\xi):=\begin{pmatrix}\sigma_\xi & 0\\ 0 & \mathbb{1}\end{pmatrix}:\mathcal{K}_{s,\xi}\times\mathbb{C}^N\longrightarrow\mathcal{F}_{s,\xi}\times\mathbb{C}^N,\quad (2.18)$$

$$Q(\xi,\lambda)=U(\xi)\mathcal{P}_0(\xi,\lambda)U(\xi)^{-1}:\mathcal{F}_{m,\xi}\times\mathbb{C}^N\longrightarrow\mathcal{F}_{0,\xi}\times\mathbb{C}^N,\quad (2.19)$$

for all $(\xi,\lambda)\in\mathcal{X}^*\times I$, and note that

$$Q(\xi,\lambda):=\begin{pmatrix}P_0-\lambda & \widetilde{R}_-(\xi)\\ \widetilde{R}_+(\xi) & 0\end{pmatrix},\quad (2.20)$$

where, with the family of functions $\{\psi_j\}_{1\leq j\leq N}$ introduced above (see Subsection 1.3.3), we define, for any $\xi \in \mathbb{T}_*$,

$$\widetilde{R}_+(\xi)u := \{(u, \psi_j(\cdot,\xi))_{\mathcal{F}_{0,\xi}}\}_{1\leq j\leq N} \in \mathbb{C}^N, \quad \text{for all } u \in \mathcal{F}_{0,\xi}, \tag{2.21}$$

and

$$\widetilde{R}_-(\xi)\underline{u} := \sum_{j=1}^{N} \underline{u}_j \psi_j(\cdot,\xi) \in \mathcal{F}_{0,\xi}, \quad \text{for all } \underline{u} := \{\underline{u}_1,\dots,\underline{u}_N\} \in \mathbb{C}^N. \tag{2.22}$$

Evidently we have that for all $\xi \in \mathbb{T}_*$, $\widetilde{R}_+(\xi) \in \mathbb{B}(\mathcal{F}_{0,\xi};\mathbb{C}^N)$ and all $\widetilde{R}_-(\xi) \in \mathbb{B}(\mathbb{C}^N;\mathcal{F}_{0,\xi})$. It is then easy to see that for any values of $(\xi,\lambda) \in \mathbb{T}_*\times I$ the operator $Q(\xi,\lambda)$ acting as an unbounded linear operator in the Hilbert space $\mathcal{F}_{0,\xi} \times \mathbb{C}^N$ is self-adjoint on the domain $\mathcal{F}_{m,\xi} \times \mathbb{C}^N$. Moreover, using Lemma 1.24, one can prove that the operator $Q(\xi,\lambda)$ defined in (2.20) is bijective and has an inverse $Q(\xi,\lambda)^{-1} \in \mathbb{B}(\mathcal{F}_{0,\xi}\times\mathbb{C}^N;\mathcal{F}_{m,\xi}\times\mathbb{C}^N)$ uniformly with respect to $(\xi,\lambda) \in \mathbb{T}_*\times I$. The proposition follows then easily. □

In particular, we observe that

$$\mathcal{P}_0(\cdot,\lambda) \in S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_{m,\xi}\times\mathbb{C}^N;\mathcal{K}_0\times\mathbb{C}^N)), \tag{2.23}$$

and

$$\mathcal{E}_0(\cdot,\lambda) \in S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_0\times\mathbb{C}^N;\mathcal{K}_{m,\xi}\times\mathbb{C}^N)), \tag{2.24}$$

uniformly for $\lambda \in I$.

Let us consider now the operator

$$\mathcal{P}_\epsilon(x,\xi,\lambda) := \begin{pmatrix} \mathfrak{q}_\epsilon(x,\xi)-\lambda & R_-(\xi) \\ R_+(\xi) & 0 \end{pmatrix}, \quad \lambda \in I,\ \epsilon \in [-\epsilon_0,\epsilon_0],\ (x,\xi) \in \Xi, \tag{2.25}$$

where we recall that $\mathfrak{q}_\epsilon(x,\xi) := \mathfrak{Op}(\tilde{p}_\epsilon(x,\cdot,\xi,\cdot))$, $\tilde{p}_\epsilon(x,y,\xi,\eta) := p_\epsilon(x,y,\xi+\eta)$. Taking into account Proposition 4.10 from the Appendices we observe that $\mathfrak{q}_\epsilon \in S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_{m,\xi};\mathcal{K}_0))$ uniformly in $\epsilon \in [-\epsilon_0,\epsilon_0]$; thus

$$\mathcal{P}_\epsilon(x,\xi,\lambda) \in S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_{m,\xi}\times\mathbb{C}^N;\mathcal{K}_0\times\mathbb{C}^N)), \tag{2.26}$$

uniformly with respect to $(\lambda,\epsilon) \in I\times[-\epsilon_0,\epsilon_0]$.

Lemma 2.25. *The operator*

$$\mathcal{P}_{\epsilon,\lambda} := \mathfrak{Op}^{A_\epsilon}(\mathcal{P}_\epsilon(\cdot,\cdot,\lambda))$$

belongs to $\mathbb{B}(\mathcal{K}^m_\epsilon(\mathcal{X}^2)\times L^2(\mathcal{X};\mathbb{C}^N);\mathcal{K}(\mathcal{X}^2)\times L^2(\mathcal{X};\mathbb{C}^N))$ *uniformly with respect to* $(\lambda,\epsilon)\in I\times[-\epsilon_0,\epsilon_0]$. *Moreover, considering* $\mathcal{P}_{\epsilon,\lambda}$ *as an unbounded linear operator in the Hilbert space* $\mathcal{K}(\mathcal{X}^2)\times L^2(\mathcal{X};\mathbb{C}^N)$ *it defines a self-adjoint operator on the domain* $\mathcal{K}^m_\epsilon(\mathcal{X}^2)\times L^2(\mathcal{X};\mathbb{C}^N)$.

Proof. If we set

$$\mathfrak{R}_{\mp,\epsilon} := \mathfrak{Op}^{A_\epsilon}(R_\mp(\xi)),$$

then we can write

$$\mathcal{P}_{\epsilon,\lambda} = \begin{pmatrix} \widetilde{P}_\epsilon - \lambda & \mathfrak{R}_{-,\epsilon} \\ \mathfrak{R}_{+,\epsilon} & 0 \end{pmatrix}. \tag{2.27}$$

Taking into account Lemma 2.22 we may conclude that $\widetilde{P}_\epsilon \in \mathbb{B}(\mathcal{K}^m_\epsilon(\mathcal{X}^2);\mathcal{K}(\mathcal{X}^2))$ uniformly with respect to $\epsilon\in[-\epsilon_0,\epsilon_0]$. Noticing that $R_-(\xi) = R_+(\xi)^*$ and belongs to $S^0_0(\mathcal{X};\mathbb{B}(\mathbb{C}^N;\mathcal{K}_0))$, Proposition 4.17 implies that

$$\mathfrak{R}_{-,\epsilon} = \mathfrak{R}^*_{+,\epsilon} \in \mathbb{B}(L^2(\mathcal{X};\mathbb{C}^N);\mathcal{K}(\mathcal{X}^2))$$

uniformly with respect to $\epsilon\in[-\epsilon_0,\epsilon_0]$. This gives us the first part of the statement of the lemma. The self-adjointness follows from the self-adjointness of $\widetilde{P}_\epsilon$ in $\mathcal{K}(\mathcal{X}^2)$ on the domain $\mathcal{K}^m_\epsilon(\mathcal{X}^2)$ and this follows by Proposition 2.23. □

Lemma 2.26. *The operator*

$$\mathcal{E}_{0,\epsilon,\lambda} := \mathfrak{Op}^{A_\epsilon}(\mathcal{E}_0(\cdot,\lambda))$$

belongs to $\mathbb{B}(\mathcal{K}(\mathcal{X}^2)\times L^2(\mathcal{X};\mathbb{C}^N);\mathcal{K}^m_\epsilon(\mathcal{X}^2)\times L^2(\mathcal{X};\mathbb{C}^N))$ *uniformly with respect to* $(\lambda,\epsilon)\in I\times[-\epsilon_0,\epsilon_0]$.

Proof. We can write

$$\mathcal{E}_{0,\epsilon,\lambda} = \begin{pmatrix} \mathfrak{E}^0_{\epsilon,\lambda} & \mathfrak{E}^0_{+,\epsilon,\lambda} \\ \mathfrak{E}^0_{-,\epsilon,\lambda} & \mathfrak{E}^0_{-+,\epsilon,\lambda} \end{pmatrix}, \tag{2.28}$$

with

$$\mathfrak{E}^0_{\epsilon,\lambda} := \mathfrak{Op}^{A_\epsilon}(E^0(\cdot,\lambda)),$$

$$\mathfrak{E}^0_{\pm,\epsilon,\lambda} := \mathfrak{Op}^{A_\epsilon}(E^0_\pm(\cdot,\lambda)),$$

$$\mathfrak{E}^0_{-+,\epsilon,\lambda} := \mathfrak{Op}^{A_\epsilon}(E^0_{-+}(\cdot,\lambda)).$$

By (2.24), it follows that $\mathcal{E}_0(\cdot,\lambda) \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0 \times \mathbb{C}^N; \mathcal{K}_0 \times \mathbb{C}^N))$ uniformly with respect to $\lambda \in I$. In order to prove the boundedness result in the lemma it is enough to show that

$$\begin{pmatrix} \widetilde{Q}_{m,\epsilon} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \mathcal{E}_{0,\epsilon,\lambda} \in \mathbb{B}(\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N); \mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N)), \tag{2.29}$$

uniformly with respect to $(\lambda,\epsilon) \in I \times [-\epsilon_0,\epsilon_0]$; here $\widetilde{Q}_{m,\epsilon}$ is defined before Definition 2.5, with some suitable identifications. In that Definition we also argued that the operator $\widetilde{Q}_{m,\epsilon}$ corresponds to the operator $Q_{m,\epsilon}$ from Remark 2.4 transformed by *doubling the variables* starting from the operator valued symbol $\mathfrak{q}_{m,\epsilon}$. We may thus conclude that $\widetilde{Q}_{m,\epsilon}$ is obtained by the $\mathfrak{Op}^{A_\epsilon}$ quantization of a symbol from $S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_{m,\xi}; \mathcal{K}_0))$. Taking into account that $E^0(\xi,\lambda) \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0; \mathcal{K}_{m,\xi}))$ and $E^0_+(\xi,\lambda) \in S^0_0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N; \mathcal{K}_{m,\xi}))$, the property (2.29) follows by the Composition Theorem 4.15 a) and by Proposition 4.17. □

Theorem 2.27. *For a sufficiently small $\epsilon_0 > 0$, for $(\lambda,\epsilon) \in I \times [-\epsilon_0,\epsilon_0]$ the operator $\mathcal{P}_{\epsilon,\lambda}$ from Lemma 2.25 has an inverse*

$$\mathcal{E}_{\epsilon,\lambda} := \begin{pmatrix} \mathfrak{E}(\epsilon,\lambda) & \mathfrak{E}_+(\epsilon,\lambda) \\ \mathfrak{E}_-(\epsilon,\lambda) & \mathfrak{E}_{-+}(\epsilon,\lambda) \end{pmatrix} \in \mathbb{B}(\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N); \mathcal{K}^m_\epsilon(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N)), \tag{2.30}$$

uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. Moreover,

$$\mathcal{E}_{\epsilon,\lambda} = \mathcal{E}_{0,\epsilon,\lambda} + \mathcal{R}_{\epsilon,\lambda},$$

$$\mathcal{R}_{\epsilon,\lambda} = \mathfrak{Op}^{A_\epsilon}(\rho_{\epsilon,\lambda}),$$

and

$$\lim_{\epsilon\to 0} \rho_{\epsilon,\lambda} = 0 \quad \text{in } S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0 \times \mathbb{C}^N; \mathcal{K}_{m,\xi} \times \mathbb{C}^N)).$$

In particular,

$$\mathfrak{E}_{-+}(\epsilon,\lambda) = \mathfrak{Op}^{A_\epsilon}(E^{-+}_{\epsilon,\lambda}), \quad \lim_{\epsilon\to 0} E^{-+}_{\epsilon,\lambda} = E^0_{-+}(\cdot,\lambda) \text{ in } S^0_0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N;\mathbb{C}^N)), \tag{2.31}$$

uniformly with respect to $\lambda \in I$.

Proof. The symbols $\mathcal{E}_0(\xi,\lambda)$ and $\mathcal{P}_0(\xi,\lambda)$ appearing in Lemma 2.25 and resp. in Lemma 2.26 do not depend on $x \in \mathcal{X}$ and on $\epsilon \in [-\epsilon_0, \epsilon_0]$. We can thus consider that

$$\mathcal{E}_0(\xi,\lambda) \in S^0_{0,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{K}_0 \times \mathbb{C}^N; \mathcal{K}_{m,\xi} \times \mathbb{C}^N))$$

and

$$\mathcal{P}_0(\xi,\lambda) \in S^0_{0,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{K}_{m,\xi} \times \mathbb{C}^N; \mathcal{K}_0 \times \mathbb{C}^N))$$

uniformly for $\lambda \in I$. By (2.13), (2.16), and (2.25),

$$\mathcal{P}_\epsilon(x,\xi,\lambda) - \mathcal{P}_0(\xi,\lambda) = \begin{pmatrix} \mathfrak{q}'_\epsilon(x,\xi) & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\mathfrak{q}'_\epsilon(x,\xi) := \mathfrak{Op}(\tilde{r}_\epsilon(x,\cdot,\xi,\cdot))$$

and

$$\tilde{r}_\epsilon(x,y,\xi,\eta) := r_\epsilon(x,y,\xi+\eta).$$

By Proposition 4.10 we conclude that

$$\lim_{\epsilon\to 0}[\mathcal{P}_\epsilon(x,\xi,\lambda) - \mathcal{P}_0(\xi,\lambda)] = 0$$

in $S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_{m,\xi} \times \mathbb{C}^N; \mathcal{K}_0 \times \mathbb{C}^N))$ uniformly with respect to $\lambda \in I$.

Let us set

$$\mathcal{P}^0_{\epsilon,\lambda} := \mathfrak{Op}^{A_\epsilon}(\mathcal{P}_0(\xi,\lambda)).$$

We can write that

$$\mathcal{P}_{\epsilon,\lambda}\mathcal{E}_{0,\epsilon,\lambda} = \mathcal{P}^0_{\epsilon,\lambda}\mathcal{E}_{0,\epsilon,\lambda} + (\mathcal{P}_{\epsilon,\lambda} - \mathcal{P}^0_{\epsilon,\lambda})\mathcal{E}_{0,\epsilon,\lambda}$$

in $\mathbb{B}(\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X} : \mathbb{C}^N); \mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X}; \mathbb{C}^N))$. Using the Composition Theorem 4.15 and the above remarks, we conclude that

$$\mathcal{P}_{\epsilon,\lambda}\mathcal{E}_{0,\epsilon,\lambda} = \mathbb{1} + \mathfrak{Op}^{A_\epsilon}(s_{\epsilon,\lambda})$$

in $\mathbb{B}(\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X}; \mathbb{C}^N); \mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X}; \mathbb{C}^N))$, where

$$\lim_{\epsilon\to 0} s_{\epsilon,\lambda} = 0,$$

in $S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0 \times \mathbb{C}^N; \mathcal{K}_0 \times \mathbb{C}^N))$ uniformly with respect to $\lambda \in I$. It follows by Proposition 4.18 that for $\epsilon_0 > 0$ small enough, the operator $\mathbb{1} + \mathfrak{Op}^{A_\epsilon}(s_{\epsilon,\lambda})$ is invertible in $\mathbb{B}(\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X}; \mathbb{C}^N); \mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X}; \mathbb{C}^N))$ for any $(\epsilon,\lambda) \in [-\epsilon_0, \epsilon_0] \times I$ and it exists a symbol $t_{\epsilon,\lambda}$ such that

$$\lim_{\epsilon\to 0} t_{\epsilon,\lambda} = 0,$$

in $S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_0\times\mathbb{C}^N;\mathcal{K}_0\times\mathbb{C}^N))$ uniformly with respect to $\lambda\in I$ and

$$[\mathbb{1}+\mathfrak{Op}^{A_\epsilon}(s_{\epsilon,\lambda})]^{-1}=\mathbb{1}+\mathfrak{Op}^{A_\epsilon}(t_{\epsilon,\lambda}). \tag{2.32}$$

Let us define

$$\mathcal{E}_{\epsilon,\lambda}:=\mathcal{E}_{0,\epsilon,\lambda}[\mathbb{1}+\mathfrak{Op}^{A_\epsilon}(t_{\epsilon,\lambda})]$$

and let us note that it is a right inverse for $\mathcal{P}_{\epsilon,\lambda}$. As the operator $\mathcal{P}_{\epsilon,\lambda}$ is self-adjoint, it follows that $\mathcal{E}_{\epsilon,\lambda}$ defined above is also a left inverse for it. The other properties in the statement of the theorem are evident now. □

Remark 2.28. The operator $\mathfrak{E}_{-+}(\epsilon,\lambda)$ defined in (2.31) is the effective Hamiltonian associated to the Hamiltonian P_ϵ and the interval I. Its importance will partially be explained in the following Corollary (proved in [8]).

Corollary 2.29. *Under the assumptions of Theorem 2.27, for any $\lambda\in I$ and any $\epsilon\in[-\epsilon_0,\epsilon_0]$,*

$$\lambda\in\sigma(\widetilde{P}_\epsilon)\iff 0\in\sigma(\mathfrak{E}_{-+}(\epsilon,\lambda)). \tag{2.33}$$

We shall need further the following commutation property.

Lemma 2.30. *Let $\gamma^*\in\Gamma^*$. Then, for all $(\epsilon,\lambda)\in[-\epsilon_0,\epsilon_0]\times I$,*

$$\begin{pmatrix}\Upsilon_{\gamma^*} & 0\\ 0 & \sigma_{\gamma^*}\end{pmatrix}\mathcal{P}_{\epsilon,\lambda}=\mathcal{P}_{\epsilon,\lambda}\begin{pmatrix}\Upsilon_{\gamma^*} & 0\\ 0 & \sigma_{\gamma^*}\end{pmatrix}, \tag{2.34}$$

as operators on $\mathcal{S}(\mathcal{X}\times\mathbb{T})\times\mathcal{S}(\mathcal{X};\mathbb{C}^N)$ (identifying the test functions on the torus with the associated periodic distributions).

Remark 2.31. Of course the inverse of the operator $\mathcal{P}_{\epsilon,\lambda}$ verifies a commutation equation similar to (2.34) on $\mathcal{S}(\mathcal{X}\times\mathbb{T})\times\mathcal{S}(\mathcal{X};\mathbb{C}^N)$ and for any $(\epsilon,\lambda)\in[-\epsilon_0,\epsilon_0]\times I$.

2.2. The auxiliary Hilbert spaces $\mathfrak{V}_0$ and $\mathfrak{L}_0$

Definition 2.32. For any $s\in\mathbb{R}$ and any $\epsilon\in[-\epsilon_0,\epsilon_0]$, we define the subspace of tempered distributions (the map $\boldsymbol{\psi}$ defined in (2.1))

$$\mathfrak{L}_s(\epsilon):=\Big\{w\in\mathcal{S}'(\mathcal{X}^2):\ \text{there exist } v\in\mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \text{ such that } w\equiv w_v=\sum_{\gamma\in\Gamma}\boldsymbol{\psi}^*(v\otimes\delta_{-\gamma})\Big\},$$

endowed with the quadratic norm

$$\|w_v\|_{\mathfrak{L}_s(\epsilon)}:=\|v\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})}.$$

Lemma 2.33. *$\mathfrak{L}_s(\epsilon)$ is a Hilbert space and is embedded continuously into $\mathcal{S}'(\mathcal{X}^2)$. $\mathfrak{L}_0$ does not depend on ϵ.*

It is easy to show that the previous definition is meaningful, the series appearing in the definition of the space $\mathfrak{L}_s(\epsilon)$ being convergent as tempered distributions.

Remark 2.34. For any $w \in \mathfrak{L}_s(\epsilon)$ we have the identity

$$(\mathrm{id} \otimes \tau_\alpha) w = w, \quad \text{for all } \alpha \in \Gamma.$$

Lemma 2.35. *For any $\epsilon \in [-\epsilon_0, \epsilon_0]$,*

(1) $\widetilde{P}_\epsilon \in \mathbb{B}(\mathfrak{L}_m(\epsilon); \mathfrak{L}_0)$ *uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$;*

(2) *the operator $\widetilde{P}_\epsilon$ considered as an unbounded operator in the Hilbert space $\mathfrak{L}_0$ defines a self-adjoint operator $\widetilde{P}_\epsilon'''$ having domain $\mathfrak{L}_m(\epsilon)$ and this self-adjoint operator is unitarily equivalent with P_ϵ.*

Proof. 1. Let us choose two test functions v and φ from $\mathcal{S}(\mathcal{X})$. Using formula (2.4), we obtain that, for all $(x, y) \in \mathcal{X}^2$,

$$[(\boldsymbol{\psi}^* \widetilde{P}_\epsilon \boldsymbol{\psi}^*)(v \otimes \varphi)](x, y) = [\mathfrak{Op}^{A_\epsilon}([(\mathrm{id} \otimes \tau_y \otimes \mathrm{id}) p_\epsilon]^\circ) v](x) \varphi(y).$$

In this equality we insert

$$\varphi(y) \equiv \varphi_\lambda(y) := \lambda^{-d} \theta\Big(\frac{y + \gamma}{\lambda}\Big)$$

for some $(\lambda, \gamma) \in \mathbb{R}_+^* \times \Gamma$ and for any $y \in \mathcal{X}$, where we denoted by θ a test function of class $C_0^\infty(\mathcal{X})$ that satisfies the condition

$$\int_{\mathcal{X}} \theta(y) dy = 1.$$

With this choice we consider the limit for $\lambda \searrow 0$ as tempered distribution on $\mathcal{X}^2$. Taking into account that for $\lambda \searrow 0$ we have that φ_λ converges in $\mathcal{S}'(\mathcal{X})$ to $\delta_{-\gamma}$ and using Hypothesis H.6, we conclude that

$$(\boldsymbol{\psi}^* \widetilde{P}_\epsilon \boldsymbol{\psi}^*)(v \otimes \delta_{-\gamma}) = (P_\epsilon v) \otimes \delta_{-\gamma}, \quad \text{for all } v \in \mathcal{S}(\mathcal{X}), \gamma \in \Gamma.$$

Extending by continuity we can write the equality

$$(\boldsymbol{\psi}^* \widetilde{P}_\epsilon \boldsymbol{\psi}^*)(v \otimes \delta_{-\gamma}) = (P_\epsilon v) \otimes \delta_{-\gamma}, \quad \text{for all } v \in \mathcal{S}'(\mathcal{X}), \gamma \in \Gamma. \tag{2.35}$$

We conclude that for any $u \in \mathfrak{L}_m(\epsilon)$ of the form

$$u \equiv u_v := \sum_{\gamma\in\Gamma} \boldsymbol{\psi}^*(v \otimes \delta_{-\gamma})$$

for some $v \in \mathcal{H}^m_{A_\epsilon}(\mathcal{X})$ we can write

$$\widetilde{P}_\epsilon u = \boldsymbol{\psi}^*\Big(\sum_{\gamma\in\Gamma}(\boldsymbol{\psi}^*\widetilde{P}_\epsilon\boldsymbol{\psi}^*)(v\otimes\delta_{-\gamma})\Big) = \sum_{\gamma\in\Gamma}\boldsymbol{\psi}^*((P_\epsilon v)\otimes\delta_{-\gamma}).$$

The first statement follows now from the fact that $P_\epsilon \in \mathbb{B}(\mathcal{H}^m_{A_\epsilon}(\mathcal{X}); L^2(\mathcal{X}))$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

2. We observe that the linear operator

$$U^s_\epsilon : \mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \longrightarrow \mathfrak{L}_s(\epsilon)$$

defined by

$$U^s_\epsilon v := \sum_{\gamma\in\Gamma}\boldsymbol{\psi}^*(v\otimes\delta_{-\gamma})$$

is in fact a unitary operator for any pair $(s,\epsilon) \in \mathbb{R}\times[-\epsilon_0,\epsilon_0]$. Following the arguments from the proof of the first point of the lemma, $\widetilde{P}_\epsilon U^s_\epsilon = U^s_\epsilon P_\epsilon$ on $\mathcal{H}^m_{A_\epsilon}(\mathcal{X})$ (the domain of self-adjointness of P_ϵ). □

Definition 2.36. We use the notation

$$\delta_\gamma := \tau_\gamma\delta,$$

with δ the Dirac distribution of mass 1 supported in $\{0\}$ and $\gamma \in \Gamma$, and we define

$$\mathfrak{V}_0 := \Big\{w \in \mathcal{S}'(\mathcal{X}) : \text{there exists } f \in l^2(\Gamma) \text{ such that } w = \sum_{\gamma\in\Gamma} f_\gamma\delta_{-\gamma}\Big\},$$

endowed with the norm

$$\|w\|_{\mathfrak{V}_0} := \sqrt{\sum_{\gamma\in\Gamma}|f_\gamma|^2},$$

for all $w \in \mathfrak{V}_0$.

It is evident that $\mathfrak{V}_0$ is a Hilbert space and is canonically unitarily equivalent with $l^2(\Gamma)$. The Hilbert space $\mathfrak{V}_0$ has a "good comparison property" with respect to the scale of magnetic Sobolev spaces. Let us choose vector potentials $\{A_\epsilon\}_{|\epsilon|\le\epsilon_0}$ having components of class $C^\infty_{\mathrm{pol}}(\mathcal{X})$ and defining the magnetic fields $\{B_\epsilon\}_{|\epsilon|\le\epsilon_0}$ satisfying Hypothesis H.1.

Lemma 2.37. *For any $s > d$ and for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ we have the algebraic and topological inclusion*

$$\mathfrak{V}_0 \hookrightarrow \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}),$$

uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. Let

$$u = \sum_{\gamma \in \Gamma} f_\gamma \delta_{-\gamma} \in \mathfrak{V}_0.$$

Then

$$g := Q_{-s,\epsilon} u = \sum_{\gamma \in \Gamma} f_\gamma Q_{-s,\epsilon} \delta_{-\gamma}.$$

Computing in $\mathcal{S}'(\mathcal{X})$ for $s > d$ we see that $Q_{-s,\epsilon}\delta_{-\gamma}$ belongs in fact to $C(\mathcal{X})$ (as Fourier transform of an integrable function) and, moreover,

$$(Q_{-s,\epsilon}\delta_{-\gamma})(x) = (2\pi)^{-d} \int_{\mathcal{X}^*} e^{i\langle \eta, x+\gamma\rangle} \omega_{A_\epsilon}(x, -\gamma) q_{-s,\epsilon}\Big(\frac{x-\gamma}{2}, \eta\Big)\, d\eta.$$

Thus for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and $x \in \mathcal{X}$, we have

$$|(Q_{-s,\epsilon}\delta_{-\gamma})(x)| \leq C_N \langle x + \gamma\rangle^{-N}.$$

Choosing $N > d$ we note that, for any $x \in \mathcal{X}$,

$$\begin{aligned} |g(x)| &\leq C_N \sum_{\gamma \in \Gamma} |f_\gamma| \langle x+\gamma\rangle^{-N} \\ &\leq C_N \Big(\sum_{\gamma \in \Gamma} |f_\gamma|^2 \langle x+\gamma\rangle^{-N}\Big)^{1/2} \Big(\sum_{\gamma \in \Gamma} \langle x+\gamma\rangle^{-N}\Big)^{1/2}. \end{aligned}$$

Thus $g \in L^2(X)$ and $\|g\|_{L^2(\mathcal{X})} \leq C_N \|u\|_{\mathfrak{V}_0}$. Finally this is equivalent with the fact that $Q_{s,\epsilon} g \in \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ and there exists a strictly positive constant C such that

$$\|u\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})} \leq C \|u\|_{\mathfrak{V}_0}, \quad \text{for all } u \in \mathfrak{V}_0, \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.36}$$

□

We shall need a property characterizing the elements from $\mathfrak{V}_0$ (replacing the property proposed in [8] that is not easy to generalize to our situation).

Lemma 2.38. *For any $s > d$ there exists $C_s > 0$ such that*

$$\sum_{\gamma\in\Gamma}|u(\gamma)|^2 \le C_s\|u\|^2_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})}, \quad \textit{for all } u \in \mathcal{S}(\mathcal{X}), \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.37}$$

Proof. For any fixed $u \in \mathcal{S}(\mathcal{X})$ let us define

$$v := Q_{s,\epsilon}u \in \mathcal{S}(\mathcal{X}).$$

Then $u = Q_{-s,\epsilon}v$ and thus for any $N \in \mathbb{N}$ and for any $x \in \mathcal{X}$ we can write that

$$u(x) = \int_{\Xi} e^{i\langle\eta,x-y\rangle}\langle x-y\rangle^{-2N}\omega_{A_\epsilon}(x,y)\Big[(\mathbb{1}-\Delta_\eta)^N q_{-s,\epsilon}\Big(\frac{x+y}{2},\eta\Big)\Big]v(y)\,dy\,đ\eta.$$

Thus there exist C and C' such that for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and for any $x \in \mathcal{X}$ one has

$$\begin{aligned}|u(x)|^2 &\le C^2\Big(\int_{\mathcal{X}}\langle x-y\rangle^{-2N}\,dy\Big)\Big(\int_{\mathcal{X}}\langle x-y\rangle^{-2N}|v(y)|^2\,dy\Big)\\ &\le C'\int_{\mathcal{X}}\langle x-y\rangle^{-2N}|v(y)|^2\,dy.\end{aligned}$$

We choose now $N \in \mathbb{N}$ large enough and note that

$$\begin{aligned}\sum_{\gamma\in\Gamma}|u(\gamma)|^2 &\le C'\int_{\mathcal{X}}\Big(\sum_{\gamma\in\Gamma}\langle\gamma-y\rangle^{-2N}\Big)|v(v)|^2\,dy\\ &\le C''\|v\|^2_{L^2(\mathcal{X})}\\ &\le C_s\|u\|^2_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})}.\end{aligned}$$

□

Proposition 2.39. a) *Given any $u \in \mathfrak{V}_0$ there exists $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$ such that*

$$u = \sum_{\gamma^*\in\Gamma^*}\sigma_{\gamma^*}u_0.$$

Moreover the map

$$\mathfrak{V}_0 \ni u \longmapsto u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

b) *Given any* $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$, *the series*

$$\sum_{\gamma^* \in \Gamma^*} \sigma_{\gamma^*} u_0$$

converges in $\mathcal{S}'(\mathcal{X})$ *and its sum denoted by* u *belongs in fact to* $\mathfrak{V}_0$. *Moreover the map*

$$\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \ni u_0 \longmapsto u \in \mathfrak{V}_0$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. We shall use the notation

$$u_{\gamma^*} := \sigma_{\gamma^*} u_0,$$

for any $\gamma^* \in \Gamma^*$ and for any tempered distribution $u_0 \in \mathcal{S}'(\mathcal{X})$.

a) Lemma 2.37 implies that for any $s > d$ and any $\epsilon \in [-\epsilon_0, \epsilon_0]$ we have that $\mathfrak{V}_0 \subset \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ and there exists a strictly positive constant $C_s > 0$, independent of ϵ, such that

$$\|u\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})} \leq C_s \|u\|_{\mathfrak{V}_0}.$$

Let us choose a real function $\chi \in C_0^\infty(\mathcal{X}^*)$ such that

$$\sum_{\gamma^* \in \Gamma^*} \tau_{\gamma^*} \chi = 1 \quad \text{on } \mathcal{X}^*.$$

For any distribution $u \in \mathfrak{V}_0$ we define

$$u_0 := \mathfrak{Op}^{A_\epsilon}(\chi) u.$$

Due to the fact that $\chi \in S_1^{-\infty}(\mathcal{X})$, it follows by the properties of magnetic Sobolev spaces (see [12]) that $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$ and the map

$$\mathfrak{V}_0 \ni u \longmapsto u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. We define

$$g_{\gamma^*} := Q_{-s,\epsilon} \sigma_{\gamma^*} u_0 = \sigma_{\gamma^*} \mathfrak{Op}^{A_\epsilon}((\mathrm{id} \otimes \tau_{-\gamma^*}) q_{-s,\epsilon}) u_0$$

where we have used the arguments in the proof of Proposition 4.6 for the last equality. We note that the family $\{\langle \gamma^* \rangle^s (\mathrm{id} \otimes \tau_{-\gamma^*}) q_{-s,\epsilon}\}_{|\epsilon| \leq \epsilon_0, \gamma^* \in \Gamma_*}$ is bounded as subset of $S_1^s(\Xi)$ and thus there exists a constant $C > 0$ such that

$$\|g_{\gamma^*}\|_{L^2(\mathcal{X})} \leq C \langle \gamma^* \rangle^{-s} \|u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})}, \quad \text{for all } \epsilon \in [-\epsilon_0, \epsilon_0],\ \gamma^* \in \Gamma^*.$$

We conclude that it exists an element $g \in L^2(\mathcal{X})$ such that

$$\sum_{\gamma^* \in \Gamma^*} g_{\gamma^*} = g \quad \text{in } L^2(\mathcal{X})$$

and we have the estimation

$$\|g\|_{L^2(\mathcal{X})} \leq C' \|u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})} \quad \text{for any } \epsilon \in [-\epsilon_0, \epsilon_0].$$

Due to the properties of the magnetic pseudodifferential calculus (see [12]) it follows that the series $\sum_{\gamma^* \in \Gamma^*} u_{\gamma^*}$ converges in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ to an element $v \in \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ and

$$\|v\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})} \leq C \|u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})}$$

uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. We still have to show that $v = u$ as tempered distributions. Let us fix a test function $\varphi \in \mathcal{S}(\mathcal{X})$ and compute

$$\begin{aligned} \langle v, \varphi \rangle &= \sum_{\gamma^* \in \Gamma^*} \langle \sigma_{\gamma^*} \mathfrak{Op}^{A_\epsilon}(\chi) u, \varphi \rangle \\ &= \sum_{\gamma^* \in \Gamma^*} \langle \mathfrak{Op}^{A_\epsilon}(\tau_{-\gamma^*} \chi) \sigma_{\gamma^*} u, \varphi \rangle \\ &= \sum_{\gamma^* \in \Gamma^*} \langle u, \mathfrak{Op}^{A_\epsilon}(\tau_{-\gamma^*} \chi) \varphi \rangle \end{aligned}$$

where we have used the relation $\sigma_{\gamma^*} u = u$ verified by all the elements from $\mathfrak{V}_0$. Let us also note that, for any $s > d$,

$$\sum_{\gamma^* \in \Gamma^*} \tau_{-\gamma^*} \chi = 1 \quad \text{in } S^s_1(\Xi),$$

so that we can write that

$$\varphi = \sum_{\gamma^* \in \Gamma^*} \mathfrak{Op}^{A_\epsilon}(\tau_{-\gamma^*} \chi) \varphi \quad \text{in } \mathcal{S}(\mathcal{X}).$$

We conclude that

$$\langle v, \varphi \rangle = \langle u, \varphi \rangle \quad \text{for any } \varphi \in \mathcal{S}(\mathcal{X}),$$

and thus $v = u$.

b) During the proof of point (a) we have shown that for any $s > d$ there exists a constant $C_s > 0$ such that for any $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$ and for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ the series $\sum_{\gamma^*\in\Gamma^*} u_{\gamma^*}$ converges in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ to an element $u \in \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ and

$$\|u\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})} \leq C_s \|u_0\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})}, \quad \text{for all } u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}), \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.38}$$

Let us recall the Poisson formula

$$\sum_{\gamma^*\in\Gamma^*} \sigma_{\gamma^*} = \frac{(2\pi)^d}{|E^*|} \sum_{\gamma\in\Gamma} \delta_{-\gamma}, \quad \text{in } \mathcal{S}'(\mathcal{X}). \tag{2.39}$$

Let us first suppose that $u_0 \in \mathcal{S}(\mathcal{X})$. Multiplying in the equality (2.39) with u_0 we obtain

$$u = \frac{(2\pi)^d}{|E^*|} \sum_{\gamma\in\Gamma} u_0(-\gamma)\delta_{-\gamma}, \quad \text{in } \mathcal{S}'(\mathcal{X}). \tag{2.40}$$

Lemma 2.38 implies that $u \in \mathfrak{V}_0$ and, for any $s > d$,

$$\|u\|_{\mathfrak{V}_0} \leq C_s \|u_0\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})}, \quad \text{for all } u_0 \in \mathcal{S}(\mathcal{X}), \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.41}$$

We come now to the general case $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})$. Let us fix some $\epsilon \in [-\epsilon_0, \epsilon_0]$ and some $s > d$. Using the fact that $\mathcal{S}(\mathcal{X})$ is dense in $\mathcal{H}^s_{A_\epsilon}(\mathcal{X})$ we can choose a sequence $\{u_0^k\}_{k\in\mathbb{N}^*} \subset \mathcal{S}(\mathcal{X})$ such that

$$u_0 = \lim_{k\nearrow\infty} u_0^k \quad \text{in } \mathcal{H}^s_{A_\epsilon}(\mathcal{X}).$$

For each element u_0^k we can associate, as we proved above, an element $u^k \in \mathfrak{V}_0$ such that

$$\|u^k\|_{\mathfrak{V}_0} \leq C_s \|u_0^k\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})}, \qquad \text{for all } k \in \mathbb{N}^*,$$

and

$$\|u^k - u^l\|_{\mathfrak{V}_0} \leq C_s \|u_0^k - u_0^l\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})}, \quad \text{for all } (k,l) \in [\mathbb{N}^*]^2.$$

It follows that there exists $v \in \mathfrak{V}_0$ such that

$$v = \lim_{k\nearrow\infty} u^k \quad \text{in } \mathfrak{V}_0$$

and

$$\|v\|_{\mathfrak{V}_0} \leq C_s \|u_0\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})}.$$

We know that

$$u^k = \sum_{\gamma^* \in \Gamma^*} \sigma_{\gamma^*} u_0^k$$

so that by (2.38) we deduce

$$\|u^k - u\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})} \leq C_s \|u_0^k - u_0\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})} \underset{k\nearrow\infty}{\longrightarrow} 0.$$

In conclusion

$$u^k \underset{k\nearrow\infty}{\longrightarrow} u \quad \text{in } \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}),$$

and

$$u^k \underset{k\nearrow\infty}{\longrightarrow} v \quad \text{in } \mathfrak{V}_0.$$

But Lemma 2.37 implies that $\mathfrak{V}_0$ is continuously embedded in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})$ and we conclude that $v = u$. □

The following lemma can be proved similarly to Lemma 2.37

Lemma 2.40. *For any $s > d$ and any $\epsilon \in [-\epsilon_0, \epsilon_0]$ we have a continuous embedding*

$$\mathfrak{L}_0 \hookrightarrow \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$$

uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

We shall obtain a characterization of the space $\mathfrak{L}_0$ that is similar to our Proposition 2.39. We use the notation

$$\mathcal{H}^{\infty}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T}) := \bigcap_{s\in\mathbb{R}} (\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})),$$

with the natural projective limit topology and need the following technical lemma.

Lemma 2.41. *Suppose given some $u_0 \in \mathcal{H}^{\infty}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and for any $\gamma^* \in \Gamma^*$, let us define*

$$u_{\gamma^*} := \Upsilon_{\gamma^*} u_0.$$

For any $s > d$ there exists $C_s > 0$ such that the series $\sum_{\gamma^*\in\Gamma^*} u_{\gamma^*}$ converges in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and the sum denoted by $v \in \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ satisfies

$$\|v\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})} \leq C_s \|u_0\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \tag{2.42}$$

for all $u_0 \in \mathcal{H}^{\infty}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and all $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. From the proof of Proposition 4.6, it follows that on $\mathcal{S}(\mathcal{X})$ we have the equality

$$Q_{-s,\epsilon}\sigma_{\gamma^*} = \sigma_{\gamma^*}\mathfrak{Op}^{A_\epsilon}((\mathrm{id}\otimes\tau_{-\gamma^*})\mathfrak{q}_{-s,\epsilon}),$$

so that finally

$$(Q_{-s,\epsilon}\otimes \mathrm{id})u_{\gamma^*} = \Upsilon_{\gamma^*}[\mathfrak{Op}^{A_\epsilon}((\mathrm{id}\otimes\tau_{-\gamma^*})\mathfrak{q}_{-s,\epsilon})\otimes \mathrm{id}]u_0. \tag{2.43}$$

Taking into account that the family $\{\langle\gamma^*\rangle^s(\mathrm{id}\otimes\tau_{-\gamma^*})\mathfrak{q}_{-s,\epsilon}\}_{(\epsilon,\gamma^*)\in[-\epsilon_0,\epsilon_0]\times\Gamma^*}$ is a bounded subset of $S^s(\mathcal{X})$, it follows the existence of a constant $C > 0$ such that for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ one has

$$\|(Q_{-s,\epsilon}\otimes \mathrm{id})u_{\gamma^*}\|_{L^2(\mathcal{X})\otimes L^2(\mathbb{T})} \leq C\langle\gamma^*\rangle^{-s}\|u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \tag{2.44}$$

for all $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$. It follows that the series

$$\sum_{\gamma^*\in\Gamma^*}(Q_{-s,\epsilon}\otimes \mathrm{id})u_{\gamma^*}$$

converges in $L^2(\mathcal{X}) \otimes L^2(\mathbb{T})$ uniformly for $\epsilon \in [-\epsilon_0, \epsilon_0]$. The stated inequality follows now by summing up the estimation (2.44) over all Γ^*. □

Proposition 2.42. *For any $u \in \mathfrak{L}_0$ there exists a vector $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ such that*

$$u = \sum_{\gamma^*\in\Gamma^*}\Upsilon_{\gamma^*}u_0, \ \textit{in } \mathcal{S}'(\mathcal{X}^2). \tag{2.45}$$

Moreover, the application

$$\mathfrak{L}_0 \ni u \longmapsto u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. We recall the notation $u_{\gamma^*} := \Upsilon^*_\gamma u_0$ and, as in the proof of point (a) of Proposition 2.39 we fix some real function $\chi \in C^\infty_0(\mathcal{X})$ satisfying the identity

$$\sum_{\gamma^*\in\Gamma^*}\tau_{\gamma^*}\chi = 1 \quad \text{on } \mathcal{X}.$$

For any $u \in \mathfrak{L}_0$ let set

$$u_0 := (\mathfrak{Op}^{A_\epsilon}(\chi)\otimes \mathrm{id})u.$$

We note that $\chi \in S_1^{-\infty}(\mathcal{X})$; thus, by Lemma 2.40, $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$, and the continuity property at the end of the proposition is clearly true. We still have to verify (2.45). Following the streamline of the proof of Lemma 2.41, the series $\sum_{\gamma^*\in\Gamma^*} u_{\gamma^*}$ converges in $\mathcal{S}'(\mathcal{X}^2)$. An argument similar to that in the proof of Proposition 2.39 a) proves that on $\mathcal{S}(\mathcal{X}^2)$ we have the equality

$$\sum_{\gamma^*\in\Gamma^*} \mathfrak{Op}^{A_\epsilon}(\tau_{-\gamma^*}\chi) \otimes \mathrm{id} = \mathrm{id}\,. \tag{2.46}$$

From the proof of Proposition 4.6 we have that

$$\mathfrak{Op}^{A_\epsilon}(\tau_{-\gamma^*}\chi) = \sigma_{-\gamma^*}\mathfrak{Op}^{A_\epsilon}(\chi)\sigma_{\gamma^*}.$$

For any $u \in \mathfrak{L}_0$, there exists $v \in L^2(\mathcal{X})$ such that

$$u = \sum_{\gamma\in\Gamma} \boldsymbol{\psi}^*(v \otimes \delta_{-\gamma});$$

thus, for any $\gamma^* \in \Gamma^*$,

$$\Upsilon_{\gamma^*}\boldsymbol{\psi}^*(v \otimes \delta_{-\gamma}) = \boldsymbol{\psi}^*((\mathrm{id}\otimes\sigma_{\gamma^*})(v \otimes \delta_{-\gamma})) = \boldsymbol{\psi}^*(v \otimes \delta_{-\gamma}),$$

and we conclude that $\Upsilon_{\gamma^*}u = u$. Using these results we deduce that, for any $u \in \mathfrak{L}_0$,

$$[\mathfrak{Op}^{A_\epsilon}(\tau_{-\gamma^*}\chi) \otimes \mathrm{id}]u = \Upsilon_{-\gamma^*}(\mathfrak{Op}^{A_\epsilon}(\chi) \otimes \mathrm{id})u = \Upsilon_{-\gamma^*}u_0 = u_{-\gamma^*}.$$

We apply now equality (2.46) to the vector $u \in \mathfrak{L}_0$ in order to obtain that $u = \sum_{\gamma^*\in\Gamma^*} u_{\gamma^*}$, as tempered distributions. □

In order to prove the reciprocal statement of Proposition 2.42 we need a technical lemma similar to Lemma2.38.

Lemma 2.43. *For any $s > d$ there exists $C_s > 0$ such that*

$$\sqrt{\int_{\mathcal{X}} |u(x,x)|^2 dx} \le C_s \|u\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \tag{2.47}$$

for all $u \in \mathcal{S}(\mathcal{X}\times\mathbb{T})$ and all $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. Let us fix some $u \in \mathcal{S}(\mathcal{X} \times \mathbb{T})$ and $\epsilon \in [-\epsilon_0, \epsilon_0]$, and let us define

$$v := (Q_{s,\epsilon} \otimes \mathrm{id})u \in \mathcal{S}(\mathcal{X} \times \mathbb{T}).$$

It follows that $u = (Q_{-s,\epsilon} \otimes \mathrm{id})v$ and we deduce that for any $N \in \mathbb{N}$ (that we shall choose sufficiently large),

$$u(x,y) = \int_\Xi \langle x-z\rangle^{-2N} e^{i\langle \zeta, x-z\rangle} \omega_{A_\epsilon}(x,z) \Big[((\mathrm{id} - \Delta_\zeta)^N \mathfrak{q}_{-s,\epsilon})\Big(\frac{x+z}{2}, \zeta\Big)\Big] v(z,y)\, dz\, đ\zeta, \quad \text{for all } (x,y) \in \mathcal{X}^2.$$

We deduce that there exist the strictly positive constants $C_N, C'_N, \ldots$, such that

$$|u(x,y)| \leq C_N \int_{\mathcal{X}} \langle x-z\rangle^{-2N} |v(z,y)|\, dz,$$

and

$$|u(x,y)|^2 \leq C'_N \int_{\mathcal{X}} \langle x-z\rangle^{-2N} |v(z,y)|^2\, dz.$$

In conclusion,

$$\begin{aligned} \int_{\mathcal{X}} |u(x,x)|^2 dx &= \sum_{\gamma\in\Gamma} \int_E |u(x+\gamma, x)|^2\, dx \\ &\leq C'_N \sum_{\gamma\in\Gamma} \int_E \int_{\mathcal{X}} \langle z-\gamma\rangle^{-2N} |v(z,x)|^2\, dz\, dx \\ &\leq C''_N \int_E \int_{\mathcal{X}} |v(z,x)|^2\, dz\, dx \\ &= C''_N \|v\|^2_{L^2(\mathcal{X}\times\mathbb{T})} \\ &\leq C_s^2 \|u\|^2_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}. \end{aligned}$$

□

We come now to the reciprocal statement of Proposition 2.42.

Proposition 2.44. *Suppose given $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and for any $\gamma^* \in \Gamma^*$ let us consider*

$$u_{\gamma^*} := \Upsilon_{\gamma^*} u_0.$$

Then the series $\sum_{\gamma^*\in\Gamma^*} u_{\gamma^*}$ converges in $\mathcal{S}'(\mathcal{X}^2)$ to an element $u \in \mathfrak{L}_0$. Moreover, the application

$$\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T}) \ni u_0 \longmapsto u \in \mathfrak{L}_0$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. For any $s > d$ and any $u_0 \in \mathcal{H}^{\infty}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$, Lemma 2.41 implies that $\sum_{\gamma^* \in \Gamma^*} u_{\gamma^*}$ converges in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ to an element $u \in \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and there exists $C_s > 0$ such that

$$\|u\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})} \leq C_s \|u_0\|_{\mathcal{H}^{s}_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \tag{2.48}$$

for all $u_0 \in \mathcal{H}^{s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and all $\epsilon \in [-\epsilon_0, \epsilon_0]$.

We still have to prove that $u \in \mathfrak{L}_0$ and that the continuity property stated above is true. As in the proof of Proposition 2.39 b) we make use of the Poisson formula (2.39). Once we note that

$$\boldsymbol{\psi}^*(\mathrm{id} \otimes \sigma_{\gamma^*}) = \Upsilon_{\gamma^*},$$

we conclude that, for any $u_0 \in \mathcal{S}(\mathcal{X}^2)$,

$$\sum_{\gamma^*} u_{\gamma^*} = \frac{(2\pi)^d}{|E^*|}\Big[\sum_{\gamma \in \Gamma} \boldsymbol{\psi}^*(\mathrm{id} \otimes \delta_{-\gamma})\Big] u_0. \tag{2.49}$$

But we note that $\sum_{\gamma\in\Gamma} \boldsymbol{\psi}^*(\mathrm{id} \otimes \delta_{-\gamma})$ belongs to $\mathcal{S}'(\mathcal{X} \times \mathbb{T})$ and we deduce that the identity (2.49) also holds for $u_0 \in \mathcal{S}(\mathcal{X} \times \mathbb{T})$. In this case, $\boldsymbol{\psi}^*(\mathrm{id} \otimes \delta_{-\gamma}) \cdot u_0$ also belongs to $\mathcal{S}'(\mathcal{X}^2)$ and, for any $\varphi \in \mathcal{S}(\mathcal{X}^2)$,

$$\begin{aligned}
\langle \boldsymbol{\psi}^*(\mathrm{id} \otimes \delta_{-\gamma}) \cdot u_0, \varphi\rangle &= \langle \boldsymbol{\psi}^*(\mathrm{id} \otimes \delta_{-\gamma}), u_0\varphi\rangle \\
&= \langle \mathrm{id} \otimes \delta_{-\gamma}, \boldsymbol{\psi}^*(u_0\varphi)\rangle \\
&= \int_{\mathcal{X}} \varphi(x, x+\gamma) u_0(x,x)\, dx.
\end{aligned}$$

Let us set

$$v_0(x) := u_0(x,x),$$

so that we obtain a test function $v_0 \in \mathcal{S}(\mathcal{X})$ such that

$$\boldsymbol{\psi}^*(\mathrm{id} \otimes \delta_{-\gamma}) \cdot u_0 = \boldsymbol{\psi}^*(v_0 \otimes \delta_{-\gamma}).$$

Let us further set

$$v := ((2\pi)^d / |E_*|) v_0 \in \mathcal{S}(\mathcal{X}) \subset L^2(\mathcal{X}).$$

If we use this equality into (2.49), then we obtain

$$u := \sum_{\gamma^* \in \Gamma^*} u_{\gamma^*} = \sum_{\gamma \in \Gamma} \boldsymbol{\psi}^*(v \otimes \delta_{-\gamma}) \in \mathfrak{L}_0. \tag{2.50}$$

Let us verify now the continuity property. Take both $u_0 \in \mathcal{H}^{\infty}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ and $\{u_{0,k}\}_{k\in\mathbb{N}^*} \subset \mathcal{S}(\mathcal{X}\times\mathbb{T})$, as well as some $s > d$ such that

$$u_0 = \lim_{k\nearrow\infty} u_{0,k} \quad \text{in } \mathcal{H}^{s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T}).$$

We introduce the notations

$$v_k(x) := \frac{(2\pi)^d}{|E^*|} u_{0,k}(x,x), \quad \text{for all } x \in \mathcal{X}$$

and

$$u_k := \sum_{\gamma\in\Gamma} \boldsymbol{\psi}^*(v_k \otimes \delta_{-\gamma}) \in \mathfrak{L}_0.$$

By Lemma 2.43, we deduce that there exists a strictly positive constant C_s such that, for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and for any pair of indices $(k,l) \in [\mathbb{N}^*]^2$,

$$\|u_k - u_l\|_{\mathfrak{L}_0} := \|v_k - v_l\|_{L^2(\mathcal{X})} \le C_s \|u_{0,k} - u_{0,l}\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \tag{2.51a}$$

and

$$\|u_k\|_{\mathfrak{L}_0} \le C_s \|u_{0,k}\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}. \tag{2.51b}$$

By (2.51), we deduce that there exists $v \in L^2(\mathcal{X})$ limit of the sequence $\{v_k\}_{k\in\mathbb{N}^*}$ in $L^2(\mathcal{X})$ such that

$$\|v\|_{L^2(\mathcal{X})} \le C_s \|u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \quad \text{for all } \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.52}$$

Let us define

$$\tilde{u} := \sum_{\gamma\in\Gamma} \boldsymbol{\psi}^*(v \otimes \delta_{-\gamma}) \in \mathfrak{L}_0.$$

By (2.52), we deduce that

$$\|\tilde{u}\|_{\mathfrak{L}_0} \le C_s \|u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}, \quad \text{for all } \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.53}$$

In order to end the proof we have to show that

$$\tilde{u} = u := \sum_{\gamma^*\in\Gamma^*} u_{\gamma^*} \quad \text{in } \mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T}).$$

If we use now inequality (2.48) with u_0 replaced by $u_{0,k} - u_0$, we obtain

$$\|u_k - u\|_{\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})} \le C_s \|u_{0,k} - u_0\|_{\mathcal{H}^s_{A_\epsilon}(\mathcal{X})\otimes L^2(\mathbb{T})}.$$

We deduce that $u = \lim_{k\nearrow\infty} u_k$ in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$. But, by (2.51), $\tilde{u} = \lim_{k\nearrow\infty} u_k$ in $\mathfrak{L}_0$ and thus, due to Lemma 2.40, also in $\mathcal{H}^{-s}_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$. In conclusion $\tilde{u} = u$ and the proof is finished. □

Lemma 2.45. *We have*

$$\mathfrak{L}_s(\epsilon) = \{w \in \mathcal{S}'(\mathcal{X}^2) \colon \widetilde{Q}_{s,\epsilon} w \in \mathfrak{L}_0\}$$

and its definition norm is equivalent with the norm

$$\|w\|'_{\mathfrak{L}_s(\epsilon)} := \|\widetilde{Q}_{s,\epsilon} w\|_{\mathfrak{L}_0}.$$

If $s \geq 0$, then $\mathfrak{L}_s(\epsilon)$ is continuously embedded into $\mathfrak{L}_0$, uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. Let

$$w \equiv w_v = \sum_{\gamma \in \Gamma} \boldsymbol{\psi}^*(v \otimes \delta_{-\gamma}) \quad \text{for some } v \in \mathcal{H}^s_{A_\epsilon}(\mathcal{X}).$$

Then, by definition, $Q_{s,\epsilon} v \in L^2(\mathcal{X})$, so that we deduce that

$$\widetilde{Q}_{s,\epsilon} w_v = \boldsymbol{\psi}^*(Q_{s,\epsilon} \otimes \mathrm{id})\boldsymbol{\psi}^* w_v \in \mathfrak{L}_0.$$

Reciprocally let $w \in \mathcal{S}'(\mathcal{X}^2)$ be such that $\widetilde{Q}_{s,\epsilon} w$ belongs to $\mathfrak{L}_0$. By the definition of this last space it follows that there exists $f \in L^2(\mathcal{X})$ such that

$$\widetilde{Q}_{s,\epsilon} w = \sum_{\gamma \in \Gamma} \boldsymbol{\psi}^*(f \otimes \delta_{-\gamma}).$$

It follows that

$$w = \sum_{\gamma \in \Gamma} \boldsymbol{\psi}^*((Q_{-s,\epsilon} f) \otimes \delta_{-\gamma}).$$

But then we have that

$$v = Q_{-s,\epsilon} f \in \mathcal{H}^s_{A_\epsilon}(\mathcal{X}),$$

and in conclusion w belongs to $\mathfrak{L}_s(\epsilon)$. The result concerning the norm follows from the Closed Graph Theorem and the last statement from the continuous embedding of $\mathcal{H}^s_{A_\epsilon}(\mathcal{X})$ into $L^2(\mathcal{X})$ for any $s \geq 0$ uniformly for $\epsilon \in [-\epsilon_0, \epsilon_0]$. □

Lemma 2.46. *For any $m \in \mathbb{R}_+$ and for any $\epsilon \in [-\epsilon_0, \epsilon_0]$, we have the topological embedding*

$$\mathfrak{L}_m(\epsilon) \hookrightarrow \mathcal{S}'(\mathcal{X}; \mathcal{H}^m(\mathbb{T})), \tag{2.54}$$

uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. By Lemmata 2.45 and 2.40 it follows that we have the topological embedding

$$\mathfrak{L}_m(\epsilon) \hookrightarrow \mathcal{S}'(\mathcal{X}; L^2(\mathbb{T}))$$

uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. From here on we proceed as in the proof of the second inclusion in Lemma 4.2. □

We shall study now the *effective Hamiltonian* $\mathfrak{E}_{-+}(\epsilon, \lambda)$, that we defined in Theorem 2.27. The following two technical results will be used in proving the boundedness and self-adjointness of $\mathfrak{E}_{-+}(\epsilon, \lambda)$ in $\mathfrak{V}_0^N$.

Lemma 2.47. *Suppose given an operator-valued symbol* $\mathfrak{q} \in S_0^0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N))$ *that is Hermitian (i.e.* $\mathfrak{q}(x,\xi)^* = \mathfrak{q}(x,\xi)$*, for all* $(x,\xi) \in \Xi$*) and verifies the invariance property*

$$(\mathrm{id} \otimes \tau_{\gamma^*})\mathfrak{q} = \mathfrak{q}, \quad \textit{for all } \gamma^* \in \Gamma^*.$$

Then, for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ *the operator* $\mathfrak{Op}^{A_\epsilon}(\mathfrak{q})$ *belongs to* $\mathbb{B}(\mathfrak{V}_0^N)$ *uniformly with respect to* $\epsilon \in [-\epsilon_0, \epsilon_0]$ *and is self-adjoint. The application*

$$S_0^0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N)) \ni \mathfrak{q} \longmapsto \mathfrak{Op}^{A_\epsilon}(\mathfrak{q}) \in \mathbb{B}(\mathfrak{V}_0^N)$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. The invariance with respect to translations from Γ^* assumed in the statement implies that the operator-valued symbol $\mathfrak{q}$ is in fact a Γ^*-periodic function with respect to the second variable $\xi \in \mathcal{X}^*$ and thus can be decomposed in a Fourier series (as tempered distributions in $\mathcal{S}'(\Xi; \mathbb{B}(\mathbb{C}^N))$)

$$\mathfrak{q}(x,\xi) = \sum_{\alpha\in\Gamma} \hat{\mathfrak{q}}_\alpha(x) e^{i\langle\xi,\alpha\rangle}, \quad \hat{\mathfrak{q}}_\alpha(x) := |E_*|^{-1} \int_{E_*} e^{-i\langle\xi,\alpha\rangle} \mathfrak{q}(x,\xi)\, d\xi. \tag{2.55}$$

Due to the regularity of the symbol functions we deduce that for any $\beta \in \mathbb{N}^d$ and for any $k \in \mathbb{N}$ there exists a strictly positive constant $C_{\beta,k}$ such that

$$|(\partial_x^\beta \hat{\mathfrak{q}}_\alpha)(x)| \le C_{\beta,k} \langle\alpha\rangle^{-k}, \quad \text{for all } x \in \mathcal{X}, \alpha \in \Gamma, \tag{2.56}$$

and we conclude that the series in (2.55) converges in fact in $BC^\infty(\Xi; \mathbb{B}(\mathbb{C}^N)) \equiv S_0^0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N))$. By (2.55), we deduce that

$$(\mathfrak{Op}^{A_\epsilon}(\mathfrak{q})u)(x) = \sum_{\alpha\in\Gamma} (Q_\alpha u)(x), \quad \text{for all } x \in \mathcal{X}, u \in \mathcal{S}(\mathcal{X}; \mathbb{C}^N), \tag{2.57}$$

where Q_α is the linear operator defined on $\mathcal{S}(\mathcal{X}; \mathbb{C}^N)$ by the oscillating integral

$$\begin{aligned}(Q_\alpha u)(x) &:= \int_\Xi e^{i\langle\eta, x-y+\alpha\rangle} \omega_{A_\epsilon}(x,y) \hat{\mathfrak{q}}_\alpha\Big(\frac{x+y}{2}\Big) u(y)\, dy\, đ\eta \\ &= \omega_{A_\epsilon}(x, x+\alpha) \hat{\mathfrak{q}}_\alpha\Big(x + \frac{\alpha}{2}\Big)(\tau_{-\alpha}u)(x).\end{aligned} \tag{2.58}$$

Both (2.57) and (2.58) may be extended by continuity to any $u \in \mathcal{S}'(\mathcal{X}; \mathbb{C}^N)$. Let us consider

$$u \equiv u_{\underline{f}} = \sum_{\gamma \in \Gamma} \underline{f}_{\gamma} \delta_{-\gamma} \in \mathfrak{V}_0^N,$$

for some $\underline{f} \in [l^2(\Gamma)]^N$. We can write

$$\begin{aligned} Q_\alpha u &= \sum_{\gamma \in \Gamma} \omega_{A_\epsilon}(-\gamma - \alpha, -\gamma) \hat{\mathfrak{q}}_\alpha\Big(-\gamma - \frac{\alpha}{2}\Big) \underline{f}_{\gamma} \delta_{-\alpha-\gamma} \\ &= \sum_{\gamma \in \Gamma} \omega_{A_\epsilon}(-\gamma, \alpha - \gamma) \hat{\mathfrak{q}}_\alpha\Big(-\gamma + \frac{\alpha}{2}\Big) \underline{f}_{\gamma-\alpha} \delta_{-\gamma}. \end{aligned} \tag{2.59}$$

By (2.59) in (2.57),

$$\mathfrak{Op}^{A_\epsilon}(\mathfrak{q}) u = \sum_{\gamma \in \Gamma} \tilde{f}_\gamma \delta_{-\gamma}, \tag{2.60}$$

and

$$\begin{aligned} \tilde{f}_\gamma &:= \sum_{\alpha \in \Gamma} \omega_{A_\epsilon}(-\gamma, \alpha - \gamma) \hat{\mathfrak{q}}_\alpha\Big(-\gamma + \frac{\alpha}{2}\Big) \underline{f}_{\gamma-\alpha} \\ &= \sum_{\alpha \in \Gamma} \omega_{A_\epsilon}(-\gamma, -\alpha) \hat{\mathfrak{q}}_{\gamma-\alpha}\Big(-\frac{\gamma+\alpha}{2}\Big) \underline{f}_{\alpha}. \end{aligned} \tag{2.61}$$

Let us verify that $\tilde{f} \in [l^2(\Gamma)]^N$. By (2.56) and (2.61) it follows that for any $k \in \mathbb{N}$ (sufficiently large) there exists $C_k > 0$ such that

$$\begin{aligned} |\tilde{f}_\gamma| &\le C_k \sum_{\alpha \in \Gamma} \langle \gamma - \alpha \rangle^{-k} |\underline{f}_{\alpha}| \\ &\le C_k \sqrt{\sum_{\alpha \in \Gamma} \langle \gamma - \alpha \rangle^{-k}} \sqrt{\sum_{\alpha \in \Gamma} \langle \gamma - \alpha \rangle^{-k} |\underline{f}_{\alpha}|^2}, \end{aligned}$$

so that we have the estimation

$$\|\tilde{f}\|^2_{[l^2(\Gamma)]^N} = \sum_{\gamma \in \Gamma} |\tilde{f}_\gamma|^2 \le C' \sum_{\alpha \in \Gamma} |\underline{f}_{\alpha}|^2 = C' \|\underline{f}\|^2_{[l^2(\Gamma)]^N}. \tag{2.62}$$

By (2.60) and (2.62),$\mathfrak{Op}^{A_\epsilon}(\mathfrak{q}) \in \mathbb{B}(\mathfrak{V}_0^N)$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ and the continuity of the application

$$S_0^0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N)) \ni \mathfrak{q} \longmapsto \mathfrak{Op}^{A_\epsilon}(\mathfrak{q}) \in \mathbb{B}(\mathfrak{V}_0^N)$$

uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ clearly follows by (2.61) and (2.55).

In order to prove the self-adjointness of $\mathfrak{Op}^{A_\epsilon}(\mathfrak{q})$ we fix $v \in \mathfrak{V}_0^N$ of the form

$$v \equiv v_{\underline{g}} = \sum_{\gamma\in\Gamma} \underline{g}_\gamma \delta_{-\gamma} \quad \text{for some } \underline{g} \in [l^2(\Gamma)]^N.$$

Then we note that

$$\begin{cases} \mathfrak{Op}^{A_\epsilon}(\mathfrak{q})v = \sum_{\gamma\in\Gamma} \tilde{g}_\gamma \delta_{-\gamma}, \\ \tilde{g}_\gamma = \sum_{\alpha\in\Gamma} \omega_{A_\epsilon}(-\gamma,-\alpha)\hat{\mathfrak{q}}_{\gamma-\alpha}\Big(-\frac{\gamma+\alpha}{2}\Big)\underline{g}_\alpha. \end{cases} \tag{2.63}$$

Let us point out the obvious equalities

$$[\hat{\mathfrak{q}}_\alpha(x)]^* = \hat{\mathfrak{q}}_{-\alpha}(x); \quad \overline{\omega_{A_\epsilon}(-\gamma,-\alpha)} = \omega_{A_\epsilon}(-\alpha,-\gamma) \tag{2.64}$$

in order to deduce that

$$\begin{aligned} (\mathfrak{Op}^{A_\epsilon}(\mathfrak{q})u, v)_{\mathfrak{V}_0^N} &= \sum_{\gamma\in\Gamma} (\tilde{f}_\gamma, \underline{g}_\gamma)_{\mathbb{C}^N} \\ &= \sum_{(\alpha,\gamma)\in\Gamma^2} \Big(\omega_{A_\epsilon}(-\gamma,-\alpha)\hat{\mathfrak{q}}_{\gamma-\alpha}\Big(-\frac{\gamma+\alpha}{2}\Big)\underline{f}_\alpha, \underline{g}_\gamma\Big)_{\mathbb{C}^N} \\ &= \sum_{(\alpha,\gamma)\in\Gamma^2} \Big(\underline{f}_\alpha, \omega_{A_\epsilon}(-\alpha,-\gamma)\hat{\mathfrak{q}}_{\alpha-\gamma}\Big(-\frac{\gamma+\alpha}{2}\Big)\underline{g}_\gamma\Big)_{\mathbb{C}^N} \\ &= \sum_{\alpha\in\Gamma} (\underline{f}_\alpha, \tilde{g}_\alpha)_{\mathbb{C}^N} \\ &= (u, \mathfrak{Op}^{A_\epsilon}(\mathfrak{q})v)_{\mathfrak{V}_0^N}. \end{aligned}$$

□

Remark 2.48. Let us point out that a shorter proof of the boundedness of $\mathfrak{Op}^{A_\epsilon}(\mathfrak{q})$ on $\mathfrak{V}_0^N$ may be obtained by using the Proposition 2.39 characterizing the distributions from $\mathfrak{V}_0$. The proof we have given has the advantage of giving the explicit form of the operator $\mathfrak{Op}^{A_\epsilon}(\mathfrak{q})$ when we identify $\mathfrak{V}_0^N$ with $[l^2(\Gamma)]^N$ (see (2.60) and (2.61)). Moreover, the self-adjointness is a very easy consequence of these formulae.

In order to prove that the effective Hamiltonian $\mathfrak{E}_{-+}(\epsilon,\lambda)$ satisfies the hypothesis of the Lemma 2.47 we shall need the commutation properties stated in Lemma 2.30 and Remark 2.31, that we now recall in the following lemma.

Lemma 2.49. *With the notations introduced in Lemma 2.30 and Remark 2.31, for any $\gamma^* \in \Gamma^*$ and for any $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$,*

$$\begin{cases} \mathfrak{R}_{-,\epsilon}\sigma_{\gamma^*} = \Upsilon_{\gamma^*}\mathfrak{R}_{-,\epsilon}, \\ \mathfrak{R}_{+,\epsilon}\Upsilon_{\gamma^*} = \sigma_{\gamma^*}\mathfrak{R}_{+,\epsilon}, \end{cases} \tag{2.65}$$

and

$$\begin{cases} \mathfrak{E}(\epsilon,\lambda)\Upsilon_{\gamma^*} = \Upsilon_{\gamma^*}\mathfrak{E}(\epsilon,\lambda), \\ \mathfrak{E}_{-+}(\epsilon,\lambda)\sigma_{\gamma^*} = \sigma_{\gamma^*}\mathfrak{E}_{-+}(\epsilon,\lambda), \end{cases} \qquad \begin{cases} \mathfrak{E}_{+}(\epsilon,\lambda)\sigma_{\gamma^*} = \Upsilon_{\gamma^*}\mathfrak{E}_{+}(\epsilon,\lambda), \\ \mathfrak{E}_{-}(\epsilon,\lambda)\Upsilon_{\gamma^*} = \sigma_{\gamma^*}\mathfrak{E}_{-}(\epsilon,\lambda). \end{cases} \tag{2.66}$$

Lemma 2.50. *Under the hypothesis of Theorem 2.27, $\mathfrak{E}_{-+}(\epsilon,\lambda) \in \mathbb{B}(\mathfrak{V}_0^N)$ uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$ and is self-adjoint on the Hilbert space $\mathfrak{V}_0^N$.*

Proof. We recall that $\mathfrak{E}_{-+}(\epsilon,\lambda) := \mathfrak{Op}^{A_\epsilon}(E^{-,+}_{\epsilon,\lambda})$ where $E^{-,+}_{\epsilon,\lambda} \in S^0_0(\mathcal{X};\mathbb{B}(\mathbb{C}^N))$. In order to use Lemma 2.47 we show that $E^{-,+}_{\epsilon,\lambda}$ is Hermitian and Γ^*-periodic in the second variable $\xi \in \mathcal{X}^*$. In order to prove the symmetry we use the fact that the operator $\mathcal{E}_{\epsilon,\lambda}$ is self-adjoint on $\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N)$ and deduce that $\mathfrak{E}_{-+}(\epsilon,\lambda)$ is self-adjoint on the Hilbert space $L^2(\mathcal{X};\mathbb{C}^N)$. Thus we have the equality

$$[\mathfrak{E}_{-+}(\epsilon,\lambda)]^* = \mathfrak{E}_{-+}(\epsilon,\lambda),$$

from which we deduce that

$$\mathfrak{Op}^{A_\epsilon}([E^{-,+}_{\epsilon,\lambda}]^* - E^{-,+}_{\epsilon,\lambda}) = 0.$$

As the application

$$\mathfrak{Op}^{A_\epsilon} : \mathcal{S}'(\Xi) \longrightarrow \mathbb{B}(\mathcal{S}(\mathcal{X});\mathcal{S}'(\mathcal{X}))$$

is an isomorphism (see [17]) it follows the symmetry relation

$$[E^{-,+}_{\epsilon,\lambda}]^* = E^{-,+}_{\epsilon,\lambda}.$$

For the Γ^*-periodicity we use one of the equalities in (2.66) that can also be written as

$$\sigma_{-\gamma^*}\mathfrak{E}_{-+}(\epsilon,\lambda)\sigma_{\gamma^*} = \mathfrak{E}_{-+}(\epsilon,\lambda).$$

Considering now the arguments in the proof of Proposition 4.6 for the $\mathfrak{Op}^{A_\epsilon}$ quantization, we can write

$$\sigma_{-\gamma^*}\mathfrak{E}_{-+}(\epsilon,\lambda)\sigma_{\gamma^*} = \mathfrak{Op}^{A_\epsilon}((\mathrm{id}\otimes\tau_{-\gamma^*})E^{-+}_{\epsilon,\lambda}).$$

We repeat the above argument based on the injectivity of the quantization map (cf. [17]) to get

$$(\mathrm{id}\otimes\tau_{-\gamma^*})E^{-+}_{\epsilon,\lambda} = E^{-+}_{\epsilon,\lambda},$$

for all $\gamma^* \in \Gamma^*$. □

Lemma 2.51. $\mathfrak{R}_{+,\epsilon} \in \mathbb{B}(\mathfrak{L}_0; \mathfrak{V}_0^N)$ *uniformly with respect to* $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. Let us recall that $\mathfrak{R}_{+,\epsilon} = \mathfrak{Op}^{A_\epsilon}(R_+)$ with $R_+ \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0; \mathbb{C}^N))$ so that finally we deduce that $\mathfrak{R}_{+,\epsilon} \in \mathbb{B}(\mathcal{S}'(\mathcal{X}; \mathcal{K}_0); \mathcal{S}'(\mathcal{X}; \mathbb{C}^N))$. By Proposition 4.17, for any $s \in \mathbb{R}$ we get $\mathfrak{R}_{+,\epsilon} \in \mathbb{B}(\mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0; [\mathcal{H}^s_{A_\epsilon}(\mathcal{X})]^N)$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. Suppose fixed some $u \in \mathfrak{L}_0$; by Proposition 2.42 we deduce the existence of $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0 \equiv \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes L^2(\mathbb{T})$ such that

$$u = \sum_{\gamma^*} \Upsilon_{\gamma^*} u_0,$$

with convergence in $\mathcal{S}'(\mathcal{X}^2)$. In fact Lemma 2.41 implies the convergence of the above series in $\mathcal{S}'(\mathcal{X}; \mathcal{K}_0)$. Using now also the second equation in (2.65) we can write that

$$\mathfrak{R}_{+,\epsilon} u = \sum_{\gamma^*} \mathfrak{R}_{+,\epsilon} \Upsilon_{\gamma^*} u_0 = \sum_{\gamma^*} \sigma_{\gamma^*} \mathfrak{R}_{+,\epsilon} u_0.$$

But we have seen that $\mathfrak{R}_{+,\epsilon} u_0 \in [\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})]^N$ and thus Proposition 2.39 b) implies that $\mathfrak{R}_{+,\epsilon} u \in \mathfrak{V}_0^N$. The fact that $\mathfrak{R}_{+,\epsilon} \in \mathbb{B}(\mathfrak{L}_0; \mathfrak{V}_0^N)$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ follows now using the following three facts:

(1) the above mentioned continuity property of $\mathfrak{R}_{+,\epsilon}$ that follows by Proposition 4.17;

(2) the uniform continuity of the application

$$\mathfrak{L}_0 \ni u \longmapsto u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0$$

with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$, that follows by Proposition 2.42;

(3) the uniform continuity of the application

$$\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \ni \mathfrak{R}_{+,\epsilon} u_0 \longmapsto \mathfrak{R}_{+,\epsilon} u \in \mathfrak{V}_0^N$$

with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$, that follows by Proposition 2.39 b).

□

Lemma 2.52. *$\mathfrak{E}_-(\epsilon,\lambda) \in \mathbb{B}(\mathfrak{L}_0;\mathfrak{V}_0^N)$ uniformly with respect to the pair $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$.*

Proof. Let us recall that $\mathfrak{E}_-(\epsilon,\lambda) = \mathfrak{Op}^{A_\epsilon}(E^-_{\epsilon,\lambda})$ with $E^-_{\epsilon,\lambda} \in S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_0;\mathbb{C}^N))$ uniformly with respect to the pair $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. We continue as in the above proof of Lemma 2.51. □

Lemma 2.53. *$\mathfrak{E}_+(\epsilon,\lambda) \in \mathbb{B}(\mathfrak{V}_0^N;\mathfrak{L}_m(\epsilon))$ uniformly with respect to the pair $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$.*

Proof. Let us recall that $\mathfrak{E}_+(\epsilon,\lambda) = \mathfrak{Op}^{A_\epsilon}(E^+_{\epsilon,\lambda})$ with $E^+_{\epsilon,\lambda} \in S^0_0(\mathcal{X};\mathbb{B}(\mathbb{C}^N;\mathcal{K}_{m,\xi}))$ uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. We conclude that $\mathfrak{E}_+(\epsilon,\lambda) \in \mathbb{B}(\mathcal{S}'(\mathcal{X};\mathbb{C}^N);\mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0}))$. Noticing that by Lemma 2.37 the space $\mathfrak{V}_0^N$ embeds continuously into $\mathcal{S}'(\mathcal{X};\mathbb{C}^N)$ we conclude that $\mathfrak{E}_+(\epsilon,\lambda) \in \mathbb{B}(\mathfrak{V}_0^N;\mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0}))$ uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. Fix now $u \in \mathfrak{V}_0^N$. By Proposition 2.39, there exists an element $u_0 \in [\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})]^N$ such that

$$u = \sum_{\gamma^* \in \Gamma^*} \sigma_{\gamma^*} u_0$$

converging as tempered distribution and such that the application

$$\mathfrak{V}_0^N \ni u \longmapsto u_0 \in [\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})]^N$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0,\epsilon_0]$. Using this result and the second equation in (2.66) we obtain that

$$\mathfrak{E}_+(\epsilon,\lambda)u = \sum_{\gamma^* \in \Gamma^*} \mathfrak{E}_+(\epsilon,\lambda)\sigma_{\gamma^*} u_0 = \sum_{\gamma^* \in \Gamma^*} \Upsilon_{\gamma^*}(\mathfrak{E}_+(\epsilon,\lambda)u_0).$$

Using now Lemma 2.45, in order to prove that $\mathfrak{E}_+(\epsilon,\lambda)u \in \mathfrak{L}_m(\epsilon)$ all we have to prove is that $\widetilde{Q}_{m,\epsilon}\mathfrak{E}_+(\epsilon,\lambda)u \in \mathfrak{L}_0$. In order to do that we shall need two of the properties of the operator $\widetilde{Q}_{m,\epsilon}$ that we have proved in the previous sections.

First we know that

$$\widetilde{Q}_{m,\epsilon}\Upsilon_{\gamma^*} = \Upsilon_{\gamma^*}\widetilde{Q}_{m,\epsilon}, \quad \text{for all } \gamma^* \in \Gamma^*.$$

Secondly, at the end of the proof of Lemma 2.26 we have shown that

$$\widetilde{Q}_{m,\epsilon} = \mathfrak{Op}^{A_\epsilon}(\tilde{\mathfrak{q}}_{m,\epsilon})$$

with $\tilde{\mathfrak{q}}_{m,\epsilon} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_{m,\xi}; \mathcal{K}_0))$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. If we use the Composition Theorem 4.15 we note that $\tilde{\mathfrak{q}}_{m,\epsilon} \sharp^{B_\epsilon} E^+_{\epsilon,\lambda} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N; \mathcal{K}_0))$ uniformly with respect to $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times I$. Applying then Proposition 4.17 gives that $\widetilde{Q}_{m,\epsilon} \mathfrak{E}_+(\epsilon, \lambda) \in \mathbb{B}([\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})]^N; \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0)$ uniformly with respect to $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times I$. We conclude that

$$\widetilde{Q}_{m,\epsilon} \mathfrak{E}_+(\epsilon, \lambda) u = \sum_{\gamma^* \in \Gamma^*} \Upsilon_{\gamma^*} \widetilde{Q}_{m,\epsilon} (\mathfrak{E}_+(\epsilon, \lambda) u_0),$$

and this last element belongs to $\mathfrak{L}_0$ as implied by Proposition 2.44. The conclusion of the lemma follows now from the following remarks.

(1) The application

$$\mathfrak{V}^N_0 \ni u \longmapsto u_0 \in [\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X})]^N$$

is continuous uniformly with respect to $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times I$, as proved in Proposition 2.39 a).

(2) The application

$$\mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0 \ni \widetilde{Q}_{m,\epsilon} \mathfrak{E}_+(\epsilon, \lambda) u_0 \longmapsto \widetilde{Q}_{m,\epsilon} \mathfrak{E}_+(\epsilon, \lambda) u \in \mathfrak{L}_0$$

is continuous uniformly with respect to $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times I$, as proved in Proposition 2.44. □

Lemma 2.54. $\mathfrak{R}_{-,\epsilon} \in \mathbb{B}(\mathfrak{V}^N_0; \mathfrak{L}_m(\epsilon))$*uniformly with respect to* $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. Let us recall that $\mathfrak{R}_{-,\epsilon} = \mathfrak{Op}^{A_\epsilon}(R_-)$ with $R_- \in S^0_0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N; \mathcal{K}_{m,\xi}))$ as implied by its definition and (1.27). Using now the first equality in (2.65), we observe that

$$\mathfrak{R}_{-\epsilon} \sigma_{\gamma^*} = \Upsilon_{\gamma^*} \mathfrak{R}_{-,\epsilon}, \quad \text{for all } \gamma^* \in \Gamma^*,$$

and the arguments from the proof of Lemma 2.53 may be repeated and one obtains the desired conclusion of the lemma. □

Lemma 2.55. $\mathfrak{E}(\epsilon, \lambda) \in \mathbb{B}(\mathfrak{L}_0; \mathfrak{L}_m(\epsilon))$ *uniformly with respect to the pair* $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times I$.

Proof. Let us recall that $\mathfrak{E}(\epsilon,\lambda) = \mathfrak{Op}^{A_\epsilon}(E_{\epsilon,\lambda})$ with $E_{\epsilon,\lambda} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0; \mathcal{K}_{m,\xi}))$ uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. As magnetic pseudodifferential operator we can then extend it to $\mathfrak{E}(\epsilon,\lambda) \in \mathfrak{B}(\mathcal{S}'(\mathcal{X};\mathcal{K}_0); \mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0}))$. Recalling that we have a continuous embedding $\mathfrak{L}_0 \hookrightarrow \mathcal{S}'(\mathcal{X};\mathcal{K}_0)$ we deduce that $\mathfrak{E}(\epsilon,\lambda) \in \mathbb{B}(\mathfrak{L}_0; \mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0}))$. We use now Proposition 2.42 and the first equality in (2.66) and write that for any $u \in \mathfrak{L}_0$ there exists $u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0$ such that

$$\mathfrak{E}(\epsilon,\lambda)u = \sum_{\gamma^*\in\Gamma^*} \mathfrak{E}(\epsilon,\lambda)\Upsilon_{\gamma^*}u_0 = \sum_{\gamma^*\in\Gamma^*} \Upsilon_{\gamma^*}(\mathfrak{E}(\epsilon,\lambda)u_0),$$

with convergence in the sense of tempered distributions on $\mathcal{X}^2$. We deduce by Proposition 2.42 that the application

$$\mathfrak{L}_0 \ni u \longmapsto u_0 \in \mathcal{H}^\infty_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{K}_0$$

is continuous uniformly with respect to $\epsilon \in [-\epsilon_0,\epsilon_0]$ and from the Composition Theorem 4.15 we deduce that $\tilde{\mathfrak{q}}_{m,\epsilon}\sharp^{B_\epsilon} E_{\epsilon,\lambda} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_0))$ and the proof of the lemma ends exactly as the proof of Lemma 2.53. □

Now we shall prove a variant of Theorem 2.27 in the frame of the Hilbert spaces $\mathfrak{V}_0$ and $\mathfrak{L}_0$.

Theorem 2.56. *We suppose verified the hypothesis of Theorem 2.27 and use the same notations; then we have that*

$$\mathcal{P}_{\epsilon,\lambda} \in \mathbb{B}(\mathfrak{L}_m(\epsilon) \times \mathfrak{V}_0^N; \mathfrak{L}_0 \times \mathfrak{V}_0^N), \quad \mathcal{E}_{\epsilon,\lambda} \in \mathbb{B}(\mathfrak{L}_0 \times \mathfrak{V}_0^N; \mathfrak{L}_m(\epsilon) \times \mathfrak{V}_0^N), \tag{2.67}$$

uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. Moreover, for any pair $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$ the operator $\mathcal{P}_{\epsilon,\lambda}$ is invertible and its inverse is $\mathcal{E}_{\epsilon,\lambda}$.

Proof. The boundedness properties in (2.67) follow by Lemmata 2.35 (a), 2.50, 2.51, 2.52, 2.53, 2.54, and 2.55. Concerning the invertibility of $\mathcal{P}_{\epsilon,\lambda}$ let us recall that in Theorem 2.27 we have proved that the operator $\mathcal{P}_{\epsilon,\lambda}$ considered as operator in $\mathbb{B}(\mathcal{K}^m_\epsilon(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N); \mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N))$ is invertible and its inverse is $\mathcal{E}_{\epsilon,\lambda} \in \mathbb{B}(\mathcal{K}(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N); \mathcal{K}^m_\epsilon(\mathcal{X}^2) \times L^2(\mathcal{X};\mathbb{C}^N))$. By (2.26) we recall that $\mathcal{P}_{\epsilon,\lambda}$ is a magnetic pseudodifferential operator with symbol $\mathcal{P}_\epsilon$ of class $S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{K}_{m,\xi} \times \mathbb{C}^N; \mathcal{K}_0 \times \mathbb{C}^N))$ uniformly with respect to $(\epsilon,\lambda) \in [-\epsilon_0,\epsilon_0] \times I$. Applying Proposition 4.13 we deduce that

$$\mathcal{P}_{\epsilon,\lambda} \in \mathbb{B}(\mathcal{S}(\mathcal{X};\mathcal{K}_{m,0}) \times \mathcal{S}(\mathcal{X};\mathbb{C}^N); \mathcal{S}(\mathcal{X};\mathcal{K}_0) \times \mathcal{S}(\mathcal{X};\mathbb{C}^N)), \tag{2.68}$$

and extending by continuity we also have that

$$\mathcal{P}_{\epsilon,\lambda} \in \mathbb{B}(\mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0}) \times \mathcal{S}'(\mathcal{X};\mathbb{C}^N); \mathcal{S}'(\mathcal{X};\mathcal{K}_0) \times \mathcal{S}'(\mathcal{X};\mathbb{C}^N)). \tag{2.69}$$

Similarly, the operator $\mathcal{E}_{\epsilon,\lambda}$ appearing in Theorem 2.27 has a symbol of class $S^0_0(\mathcal{X};\mathbb{B}(\mathcal{K}_0\times\mathbb{C}^N;\mathcal{K}_{m,\xi}\times\mathbb{C}^N))$ and thus defines first an operator of the form

$$\mathcal{E}_{\epsilon,\lambda}\in\mathbb{B}(\mathcal{S}(\mathcal{X};\mathcal{K}_0)\times\mathcal{S}(\mathcal{X};\mathbb{C}^N);\mathcal{S}(\mathcal{X};\mathcal{K}_{m,0})\times\mathcal{S}(\mathcal{X};\mathbb{C}^N)), \tag{2.70}$$

and extending by continuity we also have that

$$\mathcal{E}_{\epsilon,\lambda}\in\mathbb{B}(\mathcal{S}'(\mathcal{X};\mathcal{K}_0)\times\mathcal{S}'(\mathcal{X};\mathbb{C}^N);\mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0})\times\mathcal{S}'(\mathcal{X};\mathbb{C}^N)). \tag{2.71}$$

By the first inclusion in Lemma 4.2, $\mathcal{S}(\mathcal{X};\mathcal{K}_{m,0})\hookrightarrow\mathcal{K}^m_\epsilon(\mathcal{X}^2)$, so that from the invertibility implied by Theorem 2.27 (see above in this proof), it also follows that the operator $\mathcal{P}_{\epsilon,\lambda}$ appearing in (2.68) is invertible and its inverse is the operator $\mathcal{E}_{\epsilon,\lambda}$ appearing in (2.70). As both operators $\mathcal{P}_{\epsilon,\lambda}$ and $\mathcal{E}_{\epsilon,\lambda}$ are symmetric, by duality we deduce that also the operators appearing in (2.69) and (2.71) are the inverse of one another. This property, together with the embeddings $\mathfrak{L}_m(\epsilon)\hookrightarrow\mathcal{S}'(\mathcal{X};\mathcal{K}_{m,0})$ given by Lemma 2.46, $\mathfrak{L}_0\hookrightarrow\mathcal{S}'(\mathcal{X};\mathcal{K}_o)$ given by Lemma 2.40 and $\mathfrak{V}_0\hookrightarrow\mathcal{S}'(\mathcal{X})$ given by Lemma 2.37 allow us to end the proof of the theorem. □

We come now to the proof of the main result of this paper.

Proof of Theorem 1.1. We proceed exactly as in the proof of Corollary 2.29. We start from the equality $\mathcal{P}_{\epsilon,\lambda}\mathcal{E}_{\epsilon,\lambda}=\mathrm{id}_{\mathfrak{L}_0\oplus\mathfrak{V}_0^N}$ and use the fact that $\widetilde{P}'''_\epsilon$ is a self-adjoint operator in $\mathfrak{L}_0$ that is unitarily equivalent with P_ϵ (by Lemma 2.35) so that we deduce that $\sigma(\widetilde{P}'''_\epsilon)=\sigma(P_\epsilon)$. Then we can write that

$$\begin{aligned}&0\notin\sigma(\mathfrak{E}_{-+}(\epsilon,\lambda))\implies\\&\qquad\lambda\notin\sigma(\widetilde{P}'''_\epsilon),\ \text{and}\\&\qquad(\widetilde{P}'''_\epsilon-\lambda)^{-1}=\mathfrak{E}(\epsilon,\lambda)-\mathfrak{E}_{+,\epsilon}(\epsilon,\lambda)\mathfrak{E}_{-+}(\epsilon,\lambda)^{-1}\mathfrak{E}_{-,\epsilon}(\epsilon,\lambda),\end{aligned} \tag{2.72}$$

and

$$\begin{aligned}&\lambda\notin\sigma(\widetilde{P}'''_\epsilon)\implies\\&\qquad 0\notin\sigma(\mathfrak{E}_{-+}(\epsilon,\lambda)),\ \text{and}\\&\qquad\mathfrak{E}_{-+}(\epsilon,\lambda)^{-1}=-\mathfrak{R}_{+,\epsilon}(\widetilde{P}'''_\epsilon-\lambda)^{-1}\mathfrak{R}_{-,\epsilon}.\end{aligned} \tag{2.73}$$

In conclusion we have obtained that $\lambda\in\sigma(\widetilde{P}'''_\epsilon)\iff 0\in\sigma(\mathfrak{E}_{-+}(\epsilon,\lambda))$ and this implies that $\lambda\in\sigma(P_\epsilon)\iff 0\in\sigma(\mathfrak{E}_{-+}(\epsilon,\lambda))$. □

Proof of Corollary 1.2. We apply Theorem 1.1 and the arguments from its proof above, taking $I = K$ and $\epsilon_0 > 0$ sufficiently small. Knowing that

$$\operatorname{dist}(K, \sigma(P_0)) > 0,$$

we deduce that we also have

$$\operatorname{dist}(K, \sigma(\widetilde{P}_0''')) > 0$$

and thus

$$\sup_{\lambda \in K} \|(\widetilde{P}_0''' - \lambda)^{-1}\|_{\mathbb{B}(\mathfrak{L}_0)} < \infty.$$

By (2.73),

$$\begin{aligned} \lambda \in K \implies & \\ & 0 \notin \sigma(\mathfrak{E}_{-+}(0, \lambda)) \text{ and} \\ & \mathfrak{E}_{-+}(0, \lambda)^{-1} = -\mathfrak{R}_{+,0}(\widetilde{P}_0''' - \lambda)^{-1}\mathfrak{R}_{-,0}, \end{aligned}$$

and thus

$$\sup_{\lambda \in K} \|\mathfrak{E}_{-+}(0, \lambda)^{-1}\|_{\mathbb{B}(\mathfrak{V}_0^N)} < \infty.$$

By Theorem 2.27, for any $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times K$,

$$\mathfrak{E}_{-+}(\epsilon, \lambda) = \mathfrak{E}_{-+}(0, \lambda) + \mathfrak{S}_{-+}(\epsilon, \lambda), \quad \mathfrak{S}_{-+}(\epsilon, \lambda) := \mathfrak{Op}^{A_\epsilon}(S^{-+}_{\epsilon,\lambda}), \tag{2.74}$$

and

$$\lim_{\epsilon \to 0} S^{-+}_{\epsilon,\lambda} = 0 \quad \text{in } S^0(\mathcal{X}; \mathbb{B}(\mathbb{C}^N)),$$

uniformly with respect to $\lambda \in K$. We note that the symbol $S^{-+}_{\epsilon,\lambda}(x, \xi)$ is Γ^*-periodic in the second variable $\xi \in \mathcal{X}^*$, so that by Lemma (2.47) we deduce that

$$\lim_{\epsilon \to 0} \|\mathfrak{S}_{-+}(\epsilon, \lambda)\|_{\mathbb{B}(\mathfrak{V}_0^N)} = 0,$$

uniformly with respect to $\lambda \in K$. We conclude that for $\epsilon_0 > 0$ sufficiently small, the magnetic pseudodifferential operator $\mathfrak{E}_{-+}(\epsilon, \lambda)$ is invertible in $\mathbb{B}(\mathfrak{V}_0)$ for any $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times K$; in conclusion $0 \notin \sigma(\mathfrak{E}_{-+}(\epsilon, \lambda))$ and thus $\lambda \notin \sigma(P_\epsilon)$ for any $(\epsilon, \lambda) \in [-\epsilon_0, \epsilon_0] \times K$. □

The arguments elaborated in the proof of Corollary 1.2 allow to obtain an interesting relation between the spectra of the operators P_ϵ and P_0, under some stronger hypotheses.

Hypothesis I.1. *Under the conditions of Hypothesis H.1 we suppose further that for any pair (j,k) of indices between 1 and d the families $\{\epsilon^{-1}B_{\epsilon,jk}\}_{0<|\epsilon|\leq\epsilon_0}$ are bounded subsets of $BC^\infty(\mathcal{X})$.*

Hypothesis I.2. *We suppose that*

$$p_\epsilon(x,y,\eta)=p_0(y,\eta)+r_\epsilon(x,y,\eta)$$

where p_0 is a real valued symbol from $S^m_1(\mathbb{T})$ with $m>0$ and $\{\epsilon^{-1}r_\epsilon\}_{0<|\epsilon|\leq\epsilon_0}$ is a bounded subset of $S^m_1(\mathcal{X}\times\mathbb{T})$, each symbol r_ϵ being real valued.

Hypothesis I.3. *The symbol p_0 is elliptic; i.e. there exist $C>0$ and $R>0$ such that*

$$p_0(y,\eta)\geq C|\eta|^m \quad \text{for any } (y,\eta)\in\Xi \text{ with } |\eta|\geq R.$$

Remark 2.57. If we come back to the proofs of Theorem 2.27, Theorem 4.15 and Proposition 4.18 and suppose Hypotheses I.1–I.3 to be true, we can prove the following fact that extends our property (2.31):

$$\text{for all } I\subset\mathbb{R} \text{ compact interval, there exist } \epsilon_0>0,\ N\in\mathbb{N},\ \text{such that}$$
$$\begin{cases}\mathfrak{E}_{-+}(\epsilon,\lambda)=\mathfrak{E}_{-+}(0,\lambda)+\mathfrak{S}_{-+}(\epsilon,\lambda),\\ \mathfrak{S}_{-+}(\epsilon,\lambda):=\mathfrak{Op}^{A_\epsilon}(S^{-+}_{\epsilon,\lambda}), \qquad \text{for all } (\epsilon,\lambda)\in[-\epsilon_0,\epsilon_0]\times I,\\ \text{the family } \{\epsilon^{-1}S^{-+}_{\epsilon,\lambda}\}_{(|\epsilon|,\lambda)\in(0,\epsilon_0]\times I} \text{ is a bounded subset of } S^0(\mathcal{X};\mathbb{B}(\mathbb{C}^N)).\end{cases} \tag{2.75}$$

Once again we note the Γ^*-periodicity of the symbol $S^{-+}_{\epsilon,\lambda}(x,\xi)$ with respect to the variable $\xi\in\mathcal{X}^*$ and by Lemma 2.47 we deduce that there exists a strictly positive constant C_1 such that

$$\|\mathfrak{S}_{-+}(\epsilon,\lambda)\|_{\mathbb{B}(\mathfrak{V}^N_0)}\leq C_1\epsilon, \quad \text{for all } (\epsilon,\lambda)\in[-\epsilon_0,\epsilon_0]\times I. \tag{2.76}$$

Using Lemmata 2.51 and 2.54 we conclude that there exists a strictly positive constant C_2 such that

$$\|\mathfrak{R}_{+,\epsilon}\|_{\mathbb{B}(\mathfrak{L}_0;\mathfrak{V}^N_0)}+\|\mathfrak{R}_{-,\epsilon}\|_{\mathbb{B}(\mathfrak{V}^N_0;\mathfrak{L}_m(\epsilon))}\leq C_2, \quad \text{for all } \epsilon\in[-\epsilon_0,\epsilon_0]. \tag{2.77}$$

Proof of Proposition 1.3. For $M \subset \mathbb{R}$ and $\delta > 0$ we use the notation

$$M_\delta := \{t \in \mathbb{R} \colon \mathrm{dist}(t, M) \le \delta\}.$$

Then we have to prove the inclusions

$$\sigma(P_\epsilon) \cap I \subset \sigma(P_0)_{C\epsilon} \cap I, \quad \text{for all } \epsilon \in [0, \epsilon_0]. \tag{2.78}$$

and

$$\sigma(P_0) \cap I \subset \sigma(P_\epsilon)_{C\epsilon} \cap I, \quad \text{for all } \epsilon \in [0, \epsilon_0]. \tag{2.79}$$

Suppose there exists $\lambda \in I$ such that $\mathrm{dist}(\lambda, \sigma(P_0)) > C\epsilon$. By Lemma 2.35 we know that $\sigma(P_0) = \sigma(\widetilde{P}_0''')$ so that we deduce that $\mathrm{dist}(\lambda, \sigma(\widetilde{P}_0''')) > C\epsilon$ and conclude that

$$\|(\widetilde{P}_0''' - \lambda)^{-1}\|_{\mathbb{B}(\mathfrak{L}_0)} \le (C\epsilon)^{-1}. \tag{2.80}$$

By (2.73),

$$0 \notin \sigma(\mathfrak{E}_{-+}(0, \lambda)) \quad \text{and} \quad \mathfrak{E}_{-+}(0, \lambda)^{-1} = -\mathfrak{R}_{+,0}(\widetilde{P}_0''' - \lambda)^{-1}\mathfrak{R}_{-,0}.$$

Using these facts together with (2.77) and (2.80) we obtain the estimation

$$\|\mathfrak{E}_{-+}(0, \lambda)^{-1}\|_{\mathbb{B}(\mathfrak{V}_0^N)} \le C_2^2 (C\epsilon)^{-1}. \tag{2.81}$$

Using (2.76) and (2.81) we also obtain

$$\|\mathfrak{E}_{-+}(0, \lambda)^{-1}\|_{\mathbb{B}(\mathfrak{V}_0^N)} \cdot \|\mathfrak{S}_{-+}(\epsilon, \lambda)\|_{\mathbb{B}(\mathfrak{V}_0^N)} \le C_1 C_2^2 C^{-1}, \quad \text{for all } \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{2.82}$$

If we choose now $C > 0$ such that $C > C_1 C_2^2$, we note that the operator

$$\mathfrak{E}_{-+}(\epsilon, \lambda) = \mathfrak{E}_{-+}(0, \lambda) + \mathfrak{S}_{-+}(\epsilon, \lambda)$$

is invertible in $\mathbb{B}(\mathfrak{V}_0^N)$ and thus we deduce that $0 \notin \sigma(\mathfrak{E}_{-+}(\epsilon, \lambda))$. It follows then that $\lambda \notin \sigma(P_\epsilon)$ for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and inclusion (2.78) follows.

Let us suppose that for some ϵ with $|\epsilon| \in (0, \epsilon_0]$ there exists $\lambda \in I$ such that

$$\mathrm{dist}(\lambda, \sigma(P_\epsilon)) > C\epsilon.$$

Recalling that $\sigma(P_\epsilon) = \sigma(\widetilde{P}_\epsilon''')$ we deduce that

$$\mathrm{dist}(\lambda, \sigma(\widetilde{P}_\epsilon''')) > C\epsilon$$

and thus

$$\|(\widetilde{P}_\epsilon''' - \lambda)^{-1}\|_{\mathbb{B}(\mathfrak{L}_0)} \le (C\epsilon)^{-1}. \tag{2.83}$$

We also deduce that

$$0 \notin \sigma(\mathfrak{E}_{-+}(\epsilon,\lambda)) \quad \text{and} \quad \mathfrak{E}_{-+}(\epsilon,\lambda)^{-1} = -\mathfrak{R}_{+,\epsilon}(\widetilde{P}_\epsilon''' - \lambda)^{-1}\mathfrak{R}_{-,\epsilon}.$$

Using these facts together with (2.77) and (2.80) we obtain

$$\|\mathfrak{E}_{-+}(\epsilon,\lambda)^{-1}\|_{\mathbb{B}(\mathfrak{V}_0^N)} \leq C_2^2(C\epsilon)^{-1}. \tag{2.84}$$

It follows like above that the operator $\mathfrak{E}_{-+}(0,\lambda) = \mathfrak{E}_{-+}(\epsilon,\lambda) - \mathfrak{S}_{-+}(\epsilon,\lambda)$ is invertible in $\mathbb{B}(\mathfrak{V}_0^N)$ and thus we deduce that $0 \notin \sigma(\mathfrak{E}_{-+}(0,\lambda))$. It follows then that $\lambda \notin \sigma(P_0)$ and the inclusion (2.79) follows. □

Remark 2.58. The relations (2.78) and (2.79) clearly imply that the boundaries of the spectral gaps of the operator P_ϵ are Lipschitz functions of ϵ in $\epsilon = 0$.

3. Some particular situations

3.1. The simple spectral band. In this subsection we shall find some explicit forms for the principal part of the effective Hamiltonian $\mathfrak{E}_{-+}(\epsilon,\lambda)$. We shall suppose Hypotheses H.1–H.6 to be satisfied. If we suppose that Hypothesis H.7 is satisfied, i.e. there exists $k \geq 1$ such that J_k is a simple spectral band for P_0, then we have some more regularity for the Floquet eigenvalue $\lambda_k(\xi)$.

Lemma 3.1. *Under Hypothesis H.7, if J_k is a simple spectral band for P_0, then the function $\lambda_k(\xi)$ is of class $C^\infty(\mathbb{T}_*)$.*

Proof. Let us fix a circle $\mathcal{C}$ in the complex plane having its center on the real axis and such that J_k is contained in the open interior domain delimited by $\mathcal{C}$ and all the other spectral bands J_l with $l \neq k$ are contained in the exterior open domain delimited by $\mathcal{C}$ (that is unbounded). Then $d(\mathcal{C}, \sigma(\check{P}_0)) > 0$ and we define

$$\Pi_k(\xi) := \frac{i}{2\pi}\oint_{\mathcal{C}} (\check{P}_0(\xi) - z)^{-1} dz, \quad \text{for all } \xi \in \mathcal{X}^*. \tag{3.1}$$

It defines a function in $C^\infty(\mathcal{X}^*; \mathbb{B}(\mathcal{K}_0))$ with values one-dimensional orthogonal projections. Let us fix some point $\xi_0 \in \mathcal{X}^*$ and some vector $\phi(\xi_0)$ in the range of $\Pi_k(\xi_0)$ having norm 1. We can find a sufficiently small open neighborhood V_0 of ξ_0 in $\mathcal{X}^*$ such that

$$\|\Pi_k(\xi)\phi(\xi_0)\|_{\mathcal{K}_0} \geq (1/2), \quad \text{for all } \xi \in V_0.$$

We define

$$\phi(\xi) := \|\Pi_k(\xi)\phi(\xi_0)\|_{\mathcal{K}_0}^{-1}\Pi_k(\xi)\phi(\xi_0), \quad \text{for all } \xi \in V_0,$$

and we note that

$$(\lambda_k(\xi) - c)^{-1} = ((\breve{P}_0(\xi) - c\,\mathrm{id})^{-1}\phi(\xi), \phi(\xi))_{\mathcal{K}_0}.$$

Lemma 1.16 implies that $\lambda_k \in C^\infty(V_0)$ and also there exists $C > 0$ such that $C \le \lambda_1(\xi) - 1$, for all $\xi \in \mathcal{X}^*$. □

Lemma 3.2. *With the above definitions and notations,*

(1) *for any $(s,\xi) \in \mathbb{R} \times \mathcal{X}^*$ the Hilbert spaces $\mathcal{K}_{s,\xi}$ and $\mathcal{F}_{s,\xi}$ are stable under complex conjugation*;

(2) *for all $\xi \in \mathcal{X}^*$ and all $\gamma^* \in \Gamma^*$,*

$$\breve{P}_0(\xi + \gamma^*) = \sigma_{-\gamma^*}\breve{P}_0(\xi)\sigma_{\gamma^*}$$

and

$$\lambda_j(\xi + \gamma^*) = \lambda_j(\xi), \quad \textit{for any } j \ge 1;$$

(3) *if the symbol p_0 verifies the property*

$$p_0(x, -\xi) = p_0(x, \xi), \tag{3.2}$$

then

$$\overline{\breve{P}_0(\xi)u} = \breve{P}_0(-\xi)\bar{u}, \quad \textit{for all } u \in \mathcal{K}_{m,\xi}, \xi \in \mathcal{X}^*,$$

$$\lambda_j(-\xi) = \lambda_j(\xi), \quad \textit{for all } j \ge 1,$$

$$\overline{\Pi_k(\xi)u} = \Pi_k(-\xi)\bar{u}, \quad \textit{for all } u \in \mathcal{K}_0, \xi \in \mathcal{X}^*,$$

for any simple spectral band J_k of P_0.

Proof. The first statement follows by Definitions (1.17) and (1.18), while the second follows by Remark 1.17. As we know that $\breve{P}_0(\xi)$ is induced by

$$P_{0,\xi} := \mathfrak{Op}((\mathrm{id} \otimes \tau_{-\xi})p)$$

on the Hilbert space $\mathcal{K}_0$, it is enough to prove that

$$\overline{P_{0,\xi}u} = P_{0,-\xi}\bar{u} \quad \text{for all } u \in \mathcal{S}(\mathcal{X}),$$

but this follows easily by the pseudodifferential calculus. Let us fix now some point $\xi \in \mathcal{X}^*$ and some vector $u \in \mathcal{K}_{m,\xi}$; it follows that the vector u is an eigenvector of $\check{P}_0(\xi)$ for the eigenvalue $\lambda_j(\xi)$ if and only if $\bar{u}$ is eigenvector of $\check{P}_0(-\xi)$ for the eigenvalue $\lambda_j(\xi)$. We deduce that $\{\lambda_j(-\xi)\}_{j\geq 1} = \{\lambda_j(\xi)\}_{j\geq 1}$; as both sequences are monotonous we conclude that $\lambda_j(-\xi) = \lambda_j(\xi)$, for all $j \geq 1$. □

The next lemma (see [10]) is important for the construction in the Grushin problem under Hypothesis H.7.

Lemma 3.3. *Supposing that Hypothesis H.7 is also satisfied and supposing that $p_0(y,-\eta) = p_0(y,\eta)$ for any $(y,\eta) \in \Xi$, we can construct a function ϕ having the following properties*:

(1) $\phi \in C^\infty(\mathcal{X}^*; \mathcal{K}_{lm,0})$, *for any $l \in \mathbb{N}$;*
(2) $\phi(y+\gamma,\eta) = \phi(y,\eta)$, *for all $(y,\eta) \in \Xi$, and all $\gamma \in \Gamma$;*
(3) $\phi(y,\eta+\gamma^*) = e^{-i\langle\gamma^*,y\rangle}\phi(y,\eta)$, *for all $(y,\eta) \in \Xi$, and all $\gamma^* \in \Gamma^*$;*
(4) $\|\phi(\cdot,\eta)\|_{\mathcal{K}_0} = 1$, *for all $\eta \in \mathcal{X}^*$;*
(5) $\overline{\phi(y,\eta)} = \phi(y,-\eta)$, *for all $(y,\eta) \in \Xi$;*
(6) $\phi(\cdot,\eta) \in \mathcal{N}_k(\eta) = \ker(\check{P}_0(\eta) - \lambda_k(\eta))$, *for all $\eta \in \mathcal{X}^*$.*

Remark 3.4. By the argument used in the proof of Lemma 1.25 and properties (1)–(3) of Lemma 3.3, we deduce that for any $\alpha \in \mathbb{N}^d$ and for any $s \in \mathbb{R}$ there exists a constant $C_{\alpha,s} > 0$ such that

$$\|(\partial_\xi^\alpha \phi)(\cdot,\xi)\|_{\mathcal{K}_{s,\xi}} \leq C_{\alpha,s}, \quad \text{for all } \xi \in \mathcal{X}^*. \tag{3.3}$$

Proof of Proposition 1.4. We repeat the construction of the Grushin operator defined in (2.16) under the Hypothesis of Proposition 1.4. We prove that in this case we can take $N = 1$ and $\phi_1(x,\xi) = \phi(x,\xi)$ the function obtained in Lemma 3.3. Due to Lemma 3.3 and Remark 3.4 this function has all the properties needed in Lemma 1.25. It is thus possible to obtain the operator $\mathcal{P}_0(\xi,\lambda)$ and the essential problem is to prove its invertibility in order to obtain a result similar to Proposition 2.24. From that point the proof of Proposition 1.4 just repeats the arguments of Subsection 2.1. □

3.2. The constant magnetic field. In this subsection we prove Proposition 1.5. Thus we suppose that the symbols p_ϵ do not depend on the first argument and the magnetic field has constant components:

$$B_\epsilon = \frac{1}{2}\sum_{1\leq j,k\leq d} B_{jk}(\epsilon)\,dx_j \wedge dx_k, \quad B_{jk}(\epsilon) = -B_{kj}(\epsilon) \in \mathbb{R}, \quad \lim_{\epsilon\to 0} B_{jk}(\epsilon) = 0. \tag{3.4}$$

Using the transversal gauge (1.4) we associate some vector potentials

$$A_\epsilon := (A_{\epsilon,1}, \dots, A_{\epsilon,d})$$

satisfying

$$A_{\epsilon,j}(x) = \frac{1}{2} \sum_{1 \le k \le d} B_{jk}(\epsilon) x_k. \tag{3.5}$$

Proof of Proposition 1.5 (*I*). We use formula (2.5) by Lemma 2.2 noticing that the linearity of the functions $A_{\epsilon,j}$ and the definition of ω_A imply that

$$\omega_{\tau_{-x} A_\epsilon}(y, \tilde{y}) = \omega_{A_\epsilon}(y, \tilde{y}) e^{i\langle A_\epsilon(x), y - \tilde{y}\rangle} = \omega_{A_\epsilon + A_\epsilon(x)}(y, \tilde{y}).$$

We deduce that for any $u \in \mathcal{S}(\mathcal{X}^2)$ and for any $(x, y) \in \mathcal{X}^2$ we have that

$$(\boldsymbol{\chi}^* \widetilde{P}_\epsilon (\boldsymbol{\chi}^*)^{-1} u)(x, y) = [(\mathrm{id} \otimes \sigma_{A(x)})(\mathrm{id} \otimes P_\epsilon)(\mathrm{id} \otimes \sigma_{-A(x)}) u](x, y). \tag{3.6}$$

It follows that the operator $\widetilde{P}_\epsilon$, that is an unbounded self-adjoint operator in $L^2(\mathcal{X}^2)$ denoted in Proposition 1.20 by $\widetilde{P}'_\epsilon$, is unitarily equivalent with the operator $\mathrm{id} \otimes P_\epsilon$ with P_ϵ self-adjoint unbounded operator in $L^2(\mathcal{X})$. It follows that $\sigma(\widetilde{P}'_\epsilon) = \sigma(P_\epsilon)$. By Proposition 1.20, $\sigma(\widetilde{P}'_\epsilon) = \sigma(\widetilde{P}''_\epsilon)$ where $\widetilde{P}''_\epsilon$ is the self-adjoint realization of $\widetilde{P}_\epsilon$ in the space $L^2(\mathcal{X} \times \mathbb{T})$. Finally, by Corollary 2.29, we deduce that for any $(\lambda, \epsilon) \in I \times [-\epsilon_0, \epsilon_0]$ we have the equivalence relation

$$\lambda \in \sigma(\widetilde{P}''_\epsilon) \iff 0 \in \sigma(\mathfrak{E}_{-+}(\epsilon, \lambda)),$$

where $\mathfrak{E}_{-+}(\epsilon, \lambda)$ is considered as a bounded self-adjoint operator on $[L^2(\mathcal{X})]^N$. □

In order to prove the second point of Proposition 1.5 we shall use the *magnetic translations* $T_{\epsilon,a} := \sigma_{A_\epsilon(a)} \tau_a$ for any $a \in \mathcal{X}$, that define a family of unitary operators in $L^2(\mathcal{X})$.

Lemma 3.5. *For any two families of Hilbert spaces with temperate variation* $\{\mathcal{A}_\xi\}_{\xi \in \mathcal{X}^*}$ *and* $\{\mathcal{B}_\xi\}_{\xi \in \mathcal{X}^*}$ *and any operator-valued symbol* $q \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{B}_\bullet))$,

$$T_{\epsilon,a} \mathfrak{Op}^{A_\epsilon}(q) = \mathfrak{Op}^{A_\epsilon}((\tau_a \otimes \mathrm{id}) q) T_{\epsilon,a}, \quad \textit{for all } a \in \mathcal{X}. \tag{3.7}$$

Proof. By Lemma 1.11, it follows that

$$\tau_a \mathfrak{Op}^{A_\epsilon}(q) = \mathfrak{Op}^{\tau_a A_\epsilon}((\tau_a \otimes \mathrm{id}) q) \tau_a,$$

while $\tau_a A_\epsilon = A_\epsilon - A_\epsilon(a)$, implies that

$$\tau_a \mathfrak{Op}^{A_\epsilon}(q) = \mathfrak{Op}^{(A_\epsilon - A_\epsilon(a))}((\tau_a \otimes \mathrm{id}) q) \tau_a.$$

Then, for any $u \in \mathcal{S}(\mathcal{X};\mathcal{A}_0)$ and for any $x \in \mathcal{X}$,

$$\begin{aligned}&(\sigma_{-A_\epsilon(a)}\mathfrak{Op}^{A_\epsilon}(q)\sigma_{A_\epsilon(a)}u)(x)\\ &\quad= (2\pi)^{-d}\int\limits_{\Xi} e^{i\langle\eta,x-y\rangle}e^{-i\langle A_\epsilon(a),x-y\rangle}\omega_{A_\epsilon}(x,y)q\Big(\frac{x+y}{2},\eta\Big)u(y)\,dy\,d\eta.\end{aligned}$$

Noticing that

$$\langle A_\epsilon(a), x-y\rangle = -\int\limits_{[x,y]} A_\epsilon(a),$$

the last formula implies that

$$\mathfrak{Op}^{A_\epsilon}(q)\sigma_{A_\epsilon(a)} = \sigma_{A_\epsilon(a)}\mathfrak{Op}^{(A_\epsilon - A_\epsilon(a))}(q).$$

We conclude that

$$\begin{aligned}T_{\epsilon,a}\mathfrak{Op}^{A_\epsilon}(q) &= \sigma_{A_\epsilon(a)}\mathfrak{Op}^{(A_\epsilon-A_\epsilon(a))}((\tau_a\otimes \mathrm{id})q)\tau_a\\ &= \mathfrak{Op}^{A_\epsilon}((\tau_a\otimes\mathrm{id})q)\sigma_{A_\epsilon(a)}\tau_a\\ &= \mathfrak{Op}^{A_\epsilon}((\tau_a\otimes\mathrm{id})q)T_{\epsilon,a}.\end{aligned}$$ □

Proof of Proposition 1.5 (*2*). The operator

$$\mathcal{P}_{\epsilon,\lambda} := \mathfrak{Op}(\mathcal{P}_\epsilon(\cdot,\cdot,\lambda))$$

from Theorem 2.27 has its symbol defined in (2.25). Under our hypothesis neither the operator-valued symbol $\mathcal{P}_\epsilon$ will not depend on the first variable. By Lemma 3.5, the operator

$$\mathcal{P}_{\epsilon,\lambda}\colon \mathcal{S}(\mathcal{X};\mathcal{K}_{m,0}\times\mathbb{C}^N) \longrightarrow \mathcal{S}(\mathcal{X};\mathcal{K}_0\times\mathbb{C}^N)$$

commutes with the family $\{T_{\epsilon,a}\otimes \mathrm{id}_{\mathcal{K}_0\times\mathbb{C}^N}\}_{a\in\mathcal{X}}$. Then its inverse $\mathcal{E}_{\epsilon,\lambda}$ appearing in Theorem 2.27 also commutes with the family $\{T_{\epsilon,a}\otimes \mathrm{id}_{\mathcal{K}_0\times\mathbb{C}^N}\}_{a\in\mathcal{X}}$. By this property we deduce that also the operator

$$\mathfrak{E}_{-+}(\epsilon,\lambda)\colon L^2(\mathcal{X};\mathbb{C}^N)\longrightarrow L^2(\mathcal{X};\mathbb{C}^N)$$

commutes with the family $\{T_{\epsilon,a}\otimes\mathrm{id}_{\mathbb{C}^N}\}_{a\in\mathcal{X}}$. Using Lemma 3.5 once again we deduce that

$$\begin{aligned}\mathfrak{Op}^{A_\epsilon}(E^{-+}_{\epsilon,\lambda}) &= \mathfrak{E}_{-+}(\epsilon,\lambda)\\ &= [T_{\epsilon,a}\otimes\mathrm{id}_{\mathbb{C}^N}]\mathfrak{E}_{-+}(\epsilon,\lambda)[T_{\epsilon,a}\otimes\mathrm{id}_{\mathbb{C}^N}]^{-1}\\ &= \mathfrak{Op}^{A_\epsilon}((\tau_a\otimes\mathrm{id})E^{-+}_{\epsilon,\lambda}),\quad \text{for all } a\in\mathcal{X}.\end{aligned}$$

We conclude that

$$E^{-+}_{\epsilon,\lambda}(x,\xi) = E^{-+}_{\epsilon,\lambda}(x-a,\xi) \quad \text{for any } (x,\xi) \in \Xi,\, a \in \mathcal{X}.$$

It follows that

$$E^{-+}_{\epsilon,\lambda}(x,\xi) = E^{-+}_{\epsilon,\lambda}(0,\xi) \quad \text{for any } (x,\xi) \in \Xi.$$

The Γ^*-periodicity follows as in the general case (see the proof of Lemma 2.50). □

4. Appendices

4.1. Study of the distributions in $\mathcal{K}^s_\epsilon(\mathcal{X}^2)$. We shall prove a result giving a connection between the spaces: $\mathcal{K}^s_\epsilon(\mathcal{X}^2)$, $\mathcal{S}(\mathcal{X};\mathcal{H}^s(\mathbb{T}))$ and $\mathcal{S}'(\mathcal{X};\mathcal{H}^s(\mathbb{T}))$.

Lemma 4.1. *Let B be a magnetic field with components of class $BC^\infty(\mathcal{X})$ and A an associated vector potential with components of class C^∞_{pol}. Let us consider a symbol $q \in S^s_1(\mathcal{X})$ for some $s \in \mathbb{R}$. We set*

$$Q := \mathfrak{Op}^A(q),$$

$$Q' := Q \otimes \mathrm{id},$$

and

$$\widetilde{Q} := \boldsymbol{\psi}^* Q' \boldsymbol{\psi}^*,$$

where $\boldsymbol{\psi}$ is defined by (2.1). Then we have that

$$\widetilde{Q} \in \mathbb{B}(\mathcal{S}(\mathcal{X};\mathcal{H}^s(\mathbb{T}));\mathcal{S}(\mathcal{X};L^2(\mathbb{T})))$$

uniformly for q varying in bounded subsets of $S^s_1(\mathcal{X})$ and for B varying in bounded subsets of $BC^\infty(\mathcal{X})$.

Proof. On $\mathcal{S}(\mathcal{X};\mathcal{H}^s(\mathbb{T}))$ we shall use the family of seminorms

$$|u|_{s,l} := \sup_{|\alpha|\le l}\left[\int_{\mathcal{X}} \langle x\rangle^{2l}\,\|(\partial^\alpha_x u)(x,\cdot)\|^2_{\mathcal{H}^s(\mathbb{T})}\,dx\right]^{1/2}, \quad l \in \mathbb{N},\, u \in \mathcal{S}(\mathcal{X};\mathcal{H}^s(\mathbb{T})). \tag{4.1}$$

Using (2.3) and (2.8), or a straightforward computation, we obtain that, for any $u \in \mathcal{S}(\mathcal{X}\times\mathbb{T})$,

$$(\widetilde{Q}u)(x,y) = (2\pi)^{-d}\int_{\mathcal{X}}\int_{\mathcal{X}^*} e^{i\langle\eta,y-\tilde{y}\rangle}\omega_A(x,x-y+\tilde{y}) \; q\Big(x+\frac{\tilde{y}-y}{2},\eta\Big)u(x-y+\tilde{y},\tilde{y})\,d\tilde{y}\,d\eta. \tag{4.2}$$

In particular we obtain that $\tilde{Q}u \in \mathcal{S}(\mathcal{X} \times \mathbb{T})$. For fixed $x, y, \tilde{y}$ in $\mathcal{X}$ and η in $\mathcal{X}^*$, we consider the function of the argument $t \in \mathcal{X}$

$$\Phi(t) := \omega_A(x, x-y+t) q\Big(x + \frac{t-y}{2}, \eta\Big) u(x-y+t, \tilde{y}). \tag{4.3}$$

We use in (4.2) its Taylor expansion in $t = \tilde{y}$ with integral rest of order $n > d+s$ and eliminate the monomials $(\tilde{y}-y)^\alpha$ by integration by parts using the identity

$$(\tilde{y}-y)^\alpha e^{i\langle \eta, \tilde{y}-y\rangle} = (-D_\eta^\alpha) e^{i\langle \eta, \tilde{y}-y\rangle}.$$

We get

$$(\tilde{Q}u)(x,y) = \sum_{|\alpha|<n} \sum_{\beta\le\alpha} f_{\alpha\beta}(x)(T_{\alpha\beta}u)(x,y) + \sum_{|\alpha|=n} \sum_{\beta\le\alpha} \int_0^1 (R_{\alpha\beta}(\tau)u)(x,y)\, d\tau, \tag{4.4}$$

where

$$(T_{\alpha\beta}u)(x,y) := (2\pi)^{-d} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle \eta, y-\tilde{y}\rangle} t_{\alpha\beta}(x,\eta)(\partial_x^\beta u)(x,\tilde{y})\, d\tilde{y}\, d\eta, \tag{4.5}$$

and

$$\begin{aligned}(R_{\alpha\beta}(\tau)u)(x,y) := (2\pi)^{-d} \int_{\mathcal{X}} \int_{\mathcal{X}^*} & e^{i\langle \eta, y-\tilde{y}\rangle} h_{\tau,\alpha,\beta}(x, y-\tilde{y}) \\ & r_{\alpha\beta}\Big(x + (1-\tau)\frac{\tilde{y}-y}{2}, \eta\Big)(\partial_x^\beta u) \\ & (x-(1-\tau)(y-\tilde{y}), \tilde{y})\, d\tilde{y}\, d\eta,\end{aligned} \tag{4.6}$$

and where $f_{\alpha\beta} \in C^\infty_{\text{pol}}(\mathcal{X})$, $t_{\alpha\beta} \in S_1^{s-|\alpha|}(\mathcal{X})$, $r_{\alpha\beta} \in S_1^{s-n}(\mathcal{X})$, and finally $h_{\tau,\alpha,\beta} \in C^\infty_{\text{pol}}(\mathcal{X} \times \mathcal{X})$ uniformly for $\tau \in [0,1]$.

Let us use Lemma 1.12. Starting from (4.5) and considering $x \in \mathcal{X}$ as a parameter we conclude that there exists a semi-norm $c_{\alpha\beta}(q)$ of $q \in S_1^s(\mathcal{X})$ such that

$$\|(T_{\alpha\beta}u)(x,\cdot)\|^2_{L^2(\mathbb{T})} \le c_{\alpha\beta}(q)^2 \|(\partial_x^\beta u)(x,\cdot)\|^2_{\mathcal{H}^s(\mathbb{T})}, \quad \text{for all } x \in \mathcal{X}, u \in \mathcal{S}(\mathcal{X} \times \mathbb{T}). \tag{4.7}$$

Due to our hypothesis, there exists a constant $C(B)$ (bounded when the components of the magnetic field B take values in bounded subsets of $BC^\infty(\mathcal{X})$) and there exists $a \in \mathbb{Z}$ such that

$$|h_{\tau,\alpha,\beta}(x, y-\tilde{y})| \le C(B)\langle x\rangle^a \langle y-\tilde{y}\rangle^a, \quad \text{for all } (x,y,\tilde{y}) \in \mathcal{X}^3, \tau \in [0,1]. \tag{4.8}$$

We integrate by parts in (4.6), using the identity

$$e^{i\langle \eta, y-\tilde{y}\rangle} = \langle y-\tilde{y}\rangle^{-2N}(1-\Delta_\eta)^N e^{i\langle \eta, y-\tilde{y}\rangle}.$$

This allows us to conclude that there exists a seminorm $c'_{\alpha,\beta,N}(p)$ of the symbol $p \in S^s_1(\Xi)$ for which

$$|(R_{\alpha\beta}(\tau)u)(x,y)| \leq C(B)c'_{\alpha,\beta,N}(p)\langle x\rangle^a \int_{\mathcal{X}^*} \langle\eta\rangle^{s-n}\,d\eta \int_{\mathcal{X}} \langle z\rangle^{a-2N} |(\partial_x^\beta u)(x-(1-\tau)z, y-z)|\,dz, \tag{4.9}$$

for any $(x,y) \in \mathcal{X}^2$ and any $\tau \in [0,1]$.

We recall our choice $s-n < -d$, we choose further $2N \geq a+2d$ and we estimate the last integral by using the Cauchy–Schwartz inequality. We take the square of the inequality (4.9) and integrate with respect to $y \in E$. We conclude that for any Γ-periodic function $v \in L^2_{\mathrm{loc}}(\mathcal{X})$ and for any $z \in \mathcal{X}$ we have that for any $k \in \mathbb{N}$ there exists $C_k > 0$ such that, for any $\tau \in [0,1]$,

$$\begin{aligned}&\int_{\mathcal{X}} \langle x\rangle^{2k} \|(R_{\alpha\beta}(\tau)u)(x,\cdot)\|^2_{L^2(\mathbb{T})} \\ &\quad\leq C_k C(B)^2 c'_{\alpha,\beta,N}(p)^2 \int_{\mathcal{X}} \langle x\rangle^{2a+2k} \|(\partial_x^\alpha u)(x,\cdot)\|^2_{L^2(\mathbb{T})}\,dx.\end{aligned} \tag{4.10}$$

For the derivatives $\partial_x^\mu(T_{\alpha\beta}u)(x,\cdot)$ and $\partial_x^\mu(R_{\alpha\beta}(\tau)u)(x,\cdot)$ (for any $\mu \in \mathbb{N}^d$) we obtain in a similar way estimations of the same form. □

Lemma 4.2. *We have the topological embeddings (uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$)*

$$\mathcal{S}(\mathcal{X};\mathcal{H}^m(\mathbb{T})) \hookrightarrow \mathcal{K}^m_\epsilon(\mathcal{X}\times\mathcal{X}) \hookrightarrow \mathcal{S}'(\mathcal{X};\mathcal{H}^m(\mathbb{T})). \tag{4.11}$$

Proof. In order to prove the first embedding we take into account the density of $\mathcal{S}(\mathcal{X}\times\mathbb{T})$ into $\mathcal{S}(\mathcal{X};\mathcal{H}^m(\mathbb{T}))$ and the Definition 2.18 (c) of the space $\mathcal{K}^m_\epsilon(\mathcal{X}\times\mathcal{X})$ and we use Lemma 4.1 with the symbol q_m defining the Sobolev spaces.

For the second embedding let us note that the canonical sesquilinear map on $\mathcal{S}'(\mathcal{X};\mathcal{H}^m(\mathbb{T})) \times \mathcal{S}(\mathcal{X};\mathcal{H}^m(\mathbb{T}))$ is just a continuous extension of the scalar product

$$(u,v)_m := \int_{\mathcal{X}} (u(x,\cdot), v(x,\cdot))_{\mathcal{H}^m(\mathbb{T})}\,dx, \tag{4.12}$$

for all $(u,v) \in \mathcal{S}(\mathcal{X};\mathcal{H}^m(\mathbb{T})) \times \mathcal{S}(\mathcal{X};\mathcal{H}^m(\mathbb{T}))$.

Due to the density of $\mathcal{S}(\mathcal{X} \times \mathbb{T})$ into $\mathcal{K}^m_\epsilon(\mathcal{X} \times \mathcal{X})$, this amounts to prove that it exists a continuous seminorm $|\cdot|_{m,l}$ on $\mathcal{S}(\mathcal{X}; \mathcal{H}^m(\mathbb{T}))$ such that we have that

$$|(u, v)_m| \leq \|u\|_{\mathcal{K}^m_\epsilon} \cdot |v|_{m,l}, \tag{4.13}$$

for all $(u, v) \in \mathcal{S}(\mathcal{X}; \mathcal{H}^m(\mathbb{T})) \times \mathcal{S}(\mathcal{X}; \mathcal{H}^m(\mathbb{T}))$, where

$$\|u\|_{\mathcal{K}^m_\epsilon} = \|\widetilde{Q}_{m,\epsilon} u\|_{L^2(\mathcal{X}\times\mathbb{T})}.$$

Let us note that

$$\begin{aligned}(u, v)_m &= (u, (1 \otimes \langle D_\Gamma\rangle^{2m})v)_{L^2(\mathcal{X}\times\mathbb{T})} \\ &= (\widetilde{Q}_{m,\epsilon} u, \widetilde{Q}_{-m,\epsilon}(1 \otimes \langle D_\Gamma\rangle^{2m})v)_{L^2(\mathcal{X}\times\mathbb{T})}.\end{aligned}$$

We set

$$v_\Gamma := (1 \otimes \langle D_\Gamma\rangle^{2m})v \in \mathcal{S}(\mathcal{X} \times \mathbb{T})$$

and we observe that we have the inequality

$$|(u, v)_m| \leq \|\widetilde{Q}_{m,\epsilon} u\|_{L^2(\mathcal{X}\times\mathbb{T})} \|\widetilde{Q}_{-m,\epsilon} v_\Gamma\|_{L^2(\mathcal{X}\times\mathbb{T})}. \tag{4.14}$$

We conclude thus that the inequality (4.13) follows if we can prove that there exists a seminorm $|\cdot|_{m,l}$ on $\mathcal{S}(\mathcal{X}; \mathcal{H}^m(\mathbb{T}))$ such that

$$\|\widetilde{Q}_{-m,\epsilon} v_\Gamma\|_{L^2(\mathcal{X}\times\mathbb{T})} \leq C|v|_{m,l}, \quad \text{for all } v \in \mathcal{S}(\mathcal{X} \times \mathbb{T}). \tag{4.15}$$

By Lemma 4.1, we conclude that there exists a seminorm $|\cdot|_{-m,l}$ on $\mathcal{S}(\mathcal{X}\times\mathbb{T})$ such that

$$\|\widetilde{Q}_{-m,\epsilon} v_\Gamma\|_{L^2(\mathcal{X}\times\mathbb{T})} \leq C|v_\Gamma|_{-m,l}, \quad \text{for all } v \in \mathcal{S}(\mathcal{X} \times \mathbb{T}). \tag{4.16}$$

Now (4.15) follows by (4.16) once we note that $|v_\Gamma|_{-m,l} = |v|_{m,l}$. □

4.2. Pseudodifferential operators with operator-valued symbols

Definition 4.3. A family of Hilbert spaces $\{\mathcal{A}_\xi\}_{\xi\in\mathcal{X}^*}$ (indexed by the points in the *momentum space*) is said *to have temperate variation* when it verifies the two conditions:

(1) $\mathcal{A}_\xi = \mathcal{A}_\eta$ as complex vector spaces for all $(\xi, \eta) \in [\mathcal{X}^*]^2$;

(2) there exist $C > 0$ and $M \geq 0$ such that, for all $u \in \mathcal{A}_0$,

$$\|u\|_{\mathcal{A}_\xi} \leq C\langle\xi - \eta\rangle^M \|u\|_{\mathcal{A}_\eta}, \quad \text{for all } (\xi, \eta) \in [\mathcal{X}^*]^2. \tag{4.17}$$

Example 4.4. We can take

$$\mathcal{A}_\xi = \mathcal{H}^s(\mathcal{X}),$$

with any $s \in \mathbb{R}$ endowed with the ξ-dependent norm

$$\|u\|_{\mathcal{A}_\xi} := \left(\int_{\mathcal{X}} \langle \xi + \eta \rangle^{2s} |\hat{u}(\eta)|^2 d\eta \right)^{1/2},$$

for all $u \in \mathcal{H}^s(\mathcal{X})$ and all $\xi \in \mathcal{X}^*$. Inequality (4.17) clearly follows by the well known inequality

$$\langle \xi + \eta \rangle^{2s} \leq C_s \langle \zeta + \eta \rangle^{2s} \langle \xi - \zeta \rangle^{2|s|}, \quad \text{for all } (\xi, \eta, \zeta) \in [\mathcal{X}^*]^3, \tag{4.18}$$

where the constant C_s only depends on $s \in \mathbb{R}$. For this specific family we shall use the shorter notation $\mathcal{A}_\xi \equiv \mathcal{H}^s_\xi(\mathcal{X})$.

Definition 4.5. Suppose given two families of Hilbert spaces with tempered variation $\{\mathcal{A}_\xi\}_{\xi \in \mathcal{X}^*}$ and $\{\mathcal{B}_\xi\}_{\xi \in \mathcal{X}^*}$; suppose also given $m \in \mathbb{R}$, $\rho \in [0, 1]$ and $\mathcal{Y}$ a finite dimensional real vector space. A function $p \in C^\infty(\mathcal{Y} \times \mathcal{X}^*; \mathbb{B}(\mathcal{A}_0; \mathcal{B}_0))$ is called *an operator-valued symbol of class* $S^m_\rho(\mathcal{Y}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{B}_\bullet))$ when it verifies

$$\begin{aligned} &\text{for all } \alpha \in \mathbb{N}^{\dim \mathcal{Y}}, \beta \in \mathbb{N}^d, \text{ there exists } C_{\alpha,\beta} > 0 \text{ such that} \\ &\|(\partial^\alpha_y \partial^\beta_\xi p)(y, \xi)\|_{\mathbb{B}(\mathcal{A}_\xi; \mathcal{B}_\xi)} \leq C_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\beta|}, \text{ for all } (y, \xi) \in \mathcal{Y} \times \mathcal{X}^*. \end{aligned} \tag{4.19}$$

The space $S^m_\rho(\mathcal{Y}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{B}_\bullet))$ endowed with the family of seminorms $\nu_{\alpha,\beta}$ defined as being the smallest constants $C_{\alpha,\beta}$ that satisfy the defining property (4.19) is a metrizable locally convex linear topological space. In case we have for any $\xi \in \mathcal{X}^*$ that $\mathcal{A}_\xi = \mathcal{A}_0$ and $\mathcal{B}_\xi = \mathcal{B}_0$ as algebraic and topological structures, then we use the notation $S^m_\rho(\mathcal{Y}; \mathbb{B}(\mathcal{A}_0; \mathcal{B}_0))$. If moreover we have that $\mathcal{A}_0 = \mathcal{B}_0 = \mathbb{C}$, then we use the simple notation $S^m_\rho(\mathcal{Y})$.

Proposition 4.6. *If* $p \in S^m_1(\mathcal{X})$ *and if for any* $\xi \in \mathcal{X}^*$, *we set*

$$p_\xi := (\mathrm{id} \otimes \tau_{-\xi}) p,$$

$$P_\xi := \mathfrak{Op}(p_\xi),$$

and we denote by $\mathfrak{p}$ *the application*

$$\Xi \ni (x, \xi) \longmapsto P_\xi \in \mathbb{B}(\mathcal{H}^{s+m}_\xi(\mathcal{X}); \mathcal{H}^s_\xi(\mathcal{X})),$$

for some $s \in \mathbb{R}$, *we can prove that* $\mathfrak{p}$ *is an operator valued symbol of class* $S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X})))$. *Moreover the map*

$$S^m_1(\mathcal{X}) \ni p \longmapsto \mathfrak{p} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X})))$$

is continuous.

Proof. In fact, let us recall that for any $\xi \in \mathcal{X}^*$ we have denoted by σ_ξ the multiplication operator with the function $e^{i\langle\xi,\cdot\rangle}$ on the space $\mathcal{S}'(\mathcal{X})$. Then, for any $u \in \mathcal{S}(\mathcal{X})$ and for any $\xi \in \mathcal{X}^*$ we have that $u \in \mathcal{H}^{s+m}_\xi(\mathcal{X})$ and we can write

$$\begin{aligned}(\sigma_{-\xi} P_0 \sigma_\xi u)(x) &= (2\pi)^{-d} \int_\Xi e^{i\langle\eta-\xi,x-y\rangle} p\Big(\frac{x+y}{2}, \eta\Big) u(y)\, dy\, d\eta \\ &= (2\pi)^{-d} \int_\Xi e^{i\langle\eta,x-y\rangle} p\Big(\frac{x+y}{2}, \eta+\xi\Big) u(y)\, dy\, d\eta \\ &= (P_\xi u)(x),\end{aligned}$$

and we conclude that

$$P_\xi = \sigma_{-\xi} P_0 \sigma_\xi, \quad \text{for all } \xi \in \mathcal{X}^*.$$

On the other side, for any $\xi \in \mathcal{X}^*$ we note that p_ξ is a symbol of class $S^m_1(\mathcal{X})$ and thus, the usual Weyl calculus implies that $P_\xi \in \mathbb{B}(\mathcal{H}^{s+m}(\mathcal{X}); \mathcal{H}^s(\mathcal{X}))$ for any $s \in \mathbb{R}$. We note easily that for any multi-index $\beta \in \mathbb{N}^d$ we can write

$$\partial^\beta_\xi P_\xi = \mathfrak{Op}(\partial^\beta_\xi p_\xi),$$

and we conclude that $P_\xi \in C^\infty(\Xi; \mathbb{B}(\mathcal{H}^{s+m}(\mathcal{X}); \mathcal{H}^s(\mathcal{X})))$ (constant with respect to the variable $x \in \mathcal{X}$) for any $s \in \mathbb{R}$. Let us further note that, for all $u \in \mathcal{S}(\mathcal{X})$ and all $\xi \in \mathcal{X}^*$,

$$\widehat{\sigma_\xi u} = \tau_\xi \hat{u},$$

and

$$\|\sigma_{-\xi} u\|^2_{\mathcal{H}^s_\xi(\mathcal{X})} = \int_{\mathcal{X}^*} \langle\xi+\eta\rangle^{2s} |\hat{u}(\xi+\eta)|^2\, d\eta = \|u\|^2_{\mathcal{H}^s(\mathcal{X})}. \tag{4.20}$$

Using these results we deduce that, for any $u \in \mathcal{S}(\mathcal{X})$ and any $\xi \in \mathcal{X}^*$,

$$\|P_\xi u\|^2_{\mathcal{H}^s_\xi(\mathcal{X})} = \|P_0 \sigma_\xi u\|^2_{\mathcal{H}^s(\mathcal{X})} \le C_s \|\sigma_\xi u\|^2_{\mathcal{H}^{s+m}(\mathcal{X})} = C_s \|u\|^2_{\mathcal{H}^{s+m}_\xi(\mathcal{X})},$$

and we obtain similar estimations for the derivatives of P_ξ. Finally we conclude that $\mathfrak{p} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X})))$ and we have the continuity of the map

$$S^m_1(\mathcal{X}) \ni p \longmapsto \mathfrak{p} \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X}))). \qquad \square$$

Definition 4.7. We denote by $S^m_{\rho,\epsilon}(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ the linear space of families $\{p_\epsilon\}_{|\epsilon|\leq\epsilon_0}$ satisfying the following conditions:

(1) for all $\epsilon\in[-\epsilon_0,\epsilon_0]$, $p_\epsilon\in S^m_\rho(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ uniformly with respect to $\epsilon\in[-\epsilon_0,\epsilon_0]$;

(2) $\lim_{\epsilon\to 0} p_\epsilon = p_0$ in $S^m_\rho(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$;

(3) denoting the variable in $\mathcal{X}^2$ by (x,y), for any multi-index $\alpha\in\mathbb{N}^d$ with $|\alpha|\geq 1$,

$$\lim_{\epsilon\to 0}\partial^\alpha_x p_\epsilon = 0 \quad \text{in } S^m_\rho(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet)).$$

$S^m_{\rho,\epsilon}(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ is endowed with the natural locally convex topology of symbols of Hörmander type.

As in the case of Definition 4.5, in case we have for any $\xi\in\mathcal{X}^*$ that $\mathcal{A}_\xi=\mathcal{A}_0$ and $\mathcal{B}_\xi=\mathcal{B}_0$ as algebraic and topological structures, then we use the notation $S^m_{\rho,\epsilon}(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_0;\mathcal{B}_0))$. If, moreover, $\mathcal{A}_0=\mathcal{B}_0=\mathbb{C}$, then we use the simple notation $S^m_{\rho,\epsilon}(\mathcal{X}^2)$. For the families of symbols of type $S^m_{\rho,\epsilon}(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ that do not depend on the first variable x in $\mathcal{X}^2$ we shall use the notation $S^m_{\rho,\epsilon}(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$. Let us also consider the following canonical injection

$$S^m_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))\ni p\longmapsto \mathrm{id}\otimes p\in S^m_{\rho,\epsilon}(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$$

as a constant family.

Remark 4.8. A symbol p belongs to $S^m_{\rho,\epsilon}(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ if and only if

$$p_\epsilon(x,y,\eta)=p_0(y,\eta)+r_\epsilon(x,y,\eta),$$

with $p_0\in S^m_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$, $r_\epsilon\in S^m_\rho(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$, $r_0=0$, and $\lim_{\epsilon\to 0} r_\epsilon=0$ in $S^m_\rho(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$. Evidently we have

$$p_0(y,\eta):=p_0(0,y,\eta).$$

4.2.1. Periodic operator valued symbols

Definition 4.9. We shall denote by $S^m_\rho(\mathcal{X}\times\mathbb{T};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ the space of symbols $p\in S^m_\rho(\mathcal{X}^2;\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ that are Γ-periodic with respect to the second variable, i.e.

$$p(x,y+\gamma,\xi)=p(x,y,\xi),\quad \text{for all } (x,y)\in\mathcal{X}^2,\ \xi\in\mathcal{X}^*,\ \gamma\in\Gamma.$$

In a similar way we define the spaces $S^m_\rho(\mathbb{T};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$, $S^m_\rho(\mathcal{X}\times\mathbb{T};\mathbb{B}(\mathcal{A}_0;\mathcal{B}_0))$, $S^m_\rho(\mathbb{T};\mathbb{B}(\mathcal{A}_0;\mathcal{B}_0))$, $S^m_\rho(\mathcal{X}\times\mathbb{T})$, $S^m_{\rho,\epsilon}(\mathcal{X}\times\mathbb{T};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$, $S^m_{\rho,\epsilon}(\mathcal{X}\times\mathbb{T};\mathbb{B}(\mathcal{A}_0;\mathcal{B}_0))$, $S^m_{\rho,\epsilon}(\mathcal{X}\times\mathbb{T})$.

Let us note that we have an evident identification of $S^m_{\rho,\epsilon}(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ with a subspace of $S^m_{\rho,\epsilon}(\mathcal{X}\times\mathbb{T};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$.

Proposition 4.10. *For any $s\in\mathbb{R}$ and any $p\in S^m_1(\mathbb{T})$, with the notations in Remark 1.13,*

$$P_{\Gamma,\xi}\in S^0_0(\mathbb{T};\mathbb{B}(\mathcal{K}_{s+m,\xi};\mathcal{K}_{s,\xi})),$$

and the application

$$S^m_1(\mathbb{T})\ni p\longmapsto P_{\Gamma,\xi}\in S^0_0(\mathbb{T};\mathbb{B}(\mathcal{K}_{s+m,\xi};\mathcal{K}_{s,\xi}))$$

is continuous.

Proof. These two last statements will follow once we have proved that for any $\alpha\in\mathbb{N}^d$ there exists $c_\alpha(p)$ defining seminorm of the topology of $S^m_1(\mathbb{T})$, such that

$$\|\partial^\alpha_\xi P_{\Gamma,\xi}\|_{\mathbb{B}(\mathcal{K}_{s+m,\xi};\mathcal{K}_{s,\xi})}\leq c_\alpha(p),\quad \text{for all } \xi\in\mathcal{X}^*.$$

It is clearly enough to prove the case $\alpha=0$. We deduce that for any $u\in\mathcal{K}_{s+m,\xi}$ we have that

$$\begin{aligned}\|P_{\Gamma,\xi}u\|_{\mathcal{K}_{s,\xi}}&=\|\langle D+\xi\rangle^s\sigma_{-\xi}P\sigma_\xi u\|_{L^2(E)}\\&=\|\langle D\rangle^s P\langle D\rangle^{-s-m}\sigma_\xi\langle D+\xi\rangle^{s+m}u\|_{L^2(E)}.\end{aligned}$$

As in the proof of Lemma 1.12 we deduce that

$$\|\langle D\rangle^s P\langle D\rangle^{-s-m}v\|_{L^2(E)}\leq C_0'(p)\|v\|_{L^2(F)},\quad \text{for any } v\in L^2_{\mathrm{loc}}(\mathcal{X})\cap\mathcal{S}'(\mathcal{X}).$$

We consider

$$w:=\langle D+\xi\rangle^{s+m}u\in L^2_{\mathrm{loc}}(\mathcal{X})\cap\mathcal{S}'_\Gamma(\mathcal{X})$$

and

$$v:=\sigma_\xi w,$$

and obtain

$$\begin{aligned}\|v\|^2_{L^2(F)}&=\|w\|^2_{L^2(F)}\\&\leq C_N^2\|w\|^2_{L^2(E)}\\&=C_N^2\|\langle D+\xi\rangle^{s+m}u\|^2_{L^2(E)}\\&=C_N^2\|u\|^2_{\mathcal{K}_{s+m,\xi}}.\end{aligned}$$

This gives us the desired estimation with $c_0(p)=C_NC_0'(p)$. □

Lemma 4.11. *Let $p \in S^m_1(\mathbb{T})$ be a real elliptic symbol (i.e. there exist $C > 0$ and $R > 0$ such that $p(y,\eta) \geq C|\eta|^m$ for any $(y,\eta) \in \Xi$ with $|\eta| \geq R$), with $m > 0$. Then the operator P_Γ defined in Lemma 1.12 is self-adjoint on the domain $\mathcal{K}_{m,0}$. Moreover, P_Γ is lower semi-bounded and its graph-norm on $\mathcal{K}_{m,0}$ gives a norm equivalent to the defining norm of $\mathcal{K}_{m,0}$.*

Proof. Let us first verify the symmetry of P_Γ on $\mathcal{K}_{m,0}$. Due to the density of $\mathcal{S}(\mathbb{T})$ in $\mathcal{K}_{m,0}$ and to the fact that $P_\Gamma \in \mathbb{B}(\mathcal{K}_{m,0}; L^2(\mathbb{T}))$, it is enough to verify the symmetry of P_Γ on $\mathcal{S}(\mathbb{T})$. Let u and v belong to $\mathcal{S}(\mathbb{T})$. Identifying $\mathcal{S}(\mathbb{T})$ with $\mathcal{E}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ and using the definition of the operator P on the space $\mathcal{S}'(\mathcal{X})$ one easily verifies that Pu also belongs to $\mathcal{E}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ and is explicitly given by the oscillating integral (for all $x \in \mathcal{X}$)

$$
\begin{aligned}
(Pu)(x) &= (2\pi)^{-d} \int_{\Xi} e^{i\langle \eta, x-y\rangle} p\Big(\frac{x+y}{2},\eta\Big) u(y)\,dy\,d\eta \\
&(2\pi)^{-d} \sum_{\gamma\in\Gamma} \int_{\tau_\gamma E} \int_{\mathcal{X}^*} e^{i\langle \eta, x-y\rangle} p\Big(\frac{x+y}{2},\eta\Big) u(y)\,dy\,d\eta \\
&= (2\pi)^{-d} \sum_{\gamma\in\Gamma} \int_{E} \int_{\mathcal{X}^*} e^{i\langle \eta, x-y+\gamma\rangle} p\Big(\frac{x+y-\gamma}{2},\eta\Big) u(y)\,dy\,d\eta,
\end{aligned}
\tag{4.21}
$$

the series converging in $\mathcal{E}(\mathcal{X})$. Using the Γ-periodicity of p we obtain that

$$
\begin{aligned}
&(Pu,v)_{L^2(E)} \\
&\quad= \int_E (Pu)(x)\overline{v(x)}\,dx \\
&\quad= (2\pi)^{-d} \sum_{\gamma\in\Gamma} \int_E \int_E \int_{\mathcal{X}^*} e^{i\langle \eta, x-y+\gamma\rangle} p\Big(\frac{x+y-\gamma}{2},\eta\Big) u(y)\overline{v(x)}\,dx\,dy\,d\eta \\
&\quad= (2\pi)^{-d} \int_E u(y) \overline{\Big[\sum_{\gamma\in\Gamma} \int_E \int_{\mathcal{X}^*} e^{i\langle \eta, y-x-\gamma\rangle} p\Big(\frac{x+y+\gamma}{2},\eta\Big) v(x)\,dx\,d\eta\Big]}\,dy \\
&\quad= \int_E u(y)\overline{(Pv)(y)}\,dy \\
&\quad= (u, Pv)_{L^2(E)}.
\end{aligned}
$$

In order to prove the self-adjointness of P_Γ let us choose some vector $u \in \mathcal{D}(P_\Gamma^*)$; thus it exists $f \in L^2(\mathbb{T})$ such that

$$(P_\Gamma\varphi, u)_{L^2(\mathbb{T})} = (\varphi, f)_{L^2(\mathbb{T})}, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{T}).$$

Using now the facts that $\mathcal{S}(\mathbb{T})$ is dense in $\mathcal{S}'(\mathbb{T})$ and P_Γ is symmetric on $\mathcal{S}(\mathbb{T})$, we deduce that

$$(\varphi, f)_\mathbb{T} = (P_\Gamma\varphi, u)_\mathbb{T} = (\varphi, P_\Gamma u)_\mathbb{T}, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{T}),$$

and thus we obtain the equality

$$P_\Gamma u = f \quad \text{in } \mathcal{S}'(\mathbb{T}).$$

By hypothesis P_Γ is an elliptic pseudodifferential operator of strictly positive order m, on the compact manifold $\mathbb{T}$, so that the usual regularity results imply that $u \in \mathcal{K}_{m,0} = \mathcal{D}(P_\Gamma)$. In conclusion P_Γ is self-adjoint on the domain $\mathcal{K}_{m,0}$. The lower semiboundedness property follows by the Gårding inequality and the equivalence of the norms stated as the last point of the lemma follows by the Closed Graph Theorem. □

Remark 4.12. Under the Hypothesis of Lemma 4.11, the same proof also shows that for any $\xi \in \mathcal{X}^*$, the operator $P_{\Gamma,\xi}$ from Remark 1.22 is self-adjoint and lower semibounded on $L^2(\mathbb{T})$ on the domain $\mathcal{K}_{m,\xi}$. As in Remark 1.10 we can identify $\mathcal{K}_{m,\xi}$ with $\mathcal{H}^m_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ (endowed with the norm $\|\langle D + \xi\rangle^m u\|_{L^2(E)}$) and thus we can deduce that the operator P_ξ is a self-adjoint operator in the space $L^2_{\mathrm{loc}}(\mathcal{X}) \cap \mathcal{S}'_\Gamma(\mathcal{X})$ on the domain $\mathcal{K}_{m,\xi}$. We know that

$$P = \sigma_\xi P_\xi \sigma_{-\xi}$$

and we also know that

$$\sigma_\xi : \mathcal{K}_{s,\xi} \longrightarrow \mathcal{F}_{s,\xi}$$

is a unitary operator for any $s \in \mathbb{R}$ and for any $\xi \in \mathcal{X}^*$ and we conclude that the operator induced by P in $\mathcal{F}_{0,\xi}$ is unitarily equivalent with the operator induced by P_ξ in $\mathcal{K}_{0,\xi} \cong L^2_{\mathrm{loc}}(X) \cap \mathcal{S}'_\Gamma(\mathcal{X})$. It follows that the operator P acting in $\mathcal{F}_{0,\xi}$ with domain $\mathcal{F}_{m,\xi}$ is self-adjoint and lower semibounded.

4.2.2. Magnetic pseudodifferential operators with operator-valued symbols.

We shall consider now *magnetic pseudodifferential operators associated to operator-valued symbols* and refer to the results in [17, 12, 13].

Proposition 4.13. *Let us consider $p \in S^m_\rho(\mathcal{X}; (\mathcal{A}_\bullet; \mathcal{B}_\bullet))$, a magnetic field B with components of class $BC^\infty(\mathcal{X})$ and a vector potential A with components of class $C^\infty_{\mathsf{pol}}(\mathcal{X})$.*

(1) *The integral in (1.6) exists for any $u \in \mathcal{S}(\mathcal{X}; \mathcal{A}_0)$ and any $x \in \mathcal{X}$ as oscillating Bochner integral and defines a function $\mathfrak{Op}^A(p)u \in \mathcal{S}(\mathcal{X}; \mathcal{B}_0)$.*

(2) *The map*

$$\mathfrak{Op}^A(p) \colon \mathcal{S}(\mathcal{X}; \mathcal{A}_0) \longrightarrow \mathcal{S}(\mathcal{X}; \mathcal{B}_0)$$

defined by (1.6) *and point* (1) *above is linear and continuous.*

(3) *The formal adjoint*

$$[\mathfrak{Op}^A(p)]^* \colon \mathcal{S}(\mathcal{X}; \mathcal{B}_0) \longrightarrow \mathcal{S}(\mathcal{X}; \mathcal{A}_0)$$

of the linear continuous operator defined in (2) *above is equal to $\mathfrak{Op}^A(p^*)$ where $p^* \in S^{m'}_\rho(\mathcal{X}; \mathbb{B}(\mathcal{B}_\bullet; \mathcal{A}_\bullet))$ where*

$$m' = m + 2(M_{\mathcal{A}} + M_{\mathcal{B}}) \quad \textit{and} \quad p^*(x, \xi) := [p(x, \xi)]^*$$

(the adjoint in $\mathbb{B}(\mathcal{A}_0; \mathcal{B}_0)$).

(4) *The operator $\mathfrak{Op}^A(p)$ extends in a natural way to a linear continuous operator*

$$\mathcal{S}'(\mathcal{X}; \mathcal{A}_0) \longrightarrow \mathcal{S}'(\mathcal{X}; \mathcal{B}_0),$$

that we denote in the same way.

Proof. Fix some $u \in \mathcal{S}(\mathcal{X}; \mathcal{A}_0)$ and for the beginning let us suppose that $p(y, \eta) = 0$ for $|\eta| \geq R$, with some $R > 0$. Then, for any $x \in \mathcal{X}$, the integral in (1.6) exists as a $\mathcal{B}_0$-valued Bochner integral. Let us note that in this case we can integrate by parts in (1.6) and use the identities

$$e^{i\langle \eta, x-y\rangle} = \langle x - y\rangle^{-2N_1}[(\mathrm{id} - \Delta_\eta)^{N_1} e^{i\langle \eta, x-y\rangle}],$$

and

$$e^{-i\langle \eta, y\rangle} = \langle \eta\rangle^{-2N_2}[(\mathrm{id} - \Delta_y)^{N_2} e^{-i\langle \eta, y\rangle}].$$

We deduce that there exist $C(N_1, N_2) > 0$ and $k(N_2) \in \mathbb{N}$ such that for any $l \in \mathbb{N}$ we have

$$\begin{aligned} &\|[\mathfrak{Op}^A(p)u](x)\|_{\mathcal{B}_0} \\ &\quad \leq C \Big[\int_{\Xi} \langle x - y\rangle^{-2N_1} \langle \eta\rangle^{-2N_2} (\langle x\rangle + \langle y\rangle)^{k(N_2)} \langle \eta\rangle^{m + M_{\mathcal{B}} + M_{\mathcal{A}}} \langle y\rangle^{-l}\, dy\, d\eta \Big] \\ &\qquad\qquad \sup_{|\alpha| \leq 2N_2} \sup_{y \in \mathcal{X}} \langle y\rangle^l \|(\partial^\alpha u)(y)\|_{\mathcal{A}_0}, \end{aligned}$$

where $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ are the constants from (4.17) with respect to each of the two families $\{A_\xi\}_{\xi\in\mathcal{X}^*}$ and $\{\mathcal{B}_\xi\}_{\xi\in\mathcal{X}^*}$. We choose

$$2N_2 \geq m + M_{\mathcal{A}} + M_{\mathcal{B}} + d,$$

$$l = 2N_1 + k(N_2) + d + 1,$$

$$2N_1 \geq k(N_2),$$

and we obtain that

$$\|[\mathfrak{Op}^A(p)u](x)\|_{\mathcal{B}_0} \leq C(N_1)\langle x\rangle^{-2N_1+k(N_2)} \sup_{|\alpha|\leq 2N_2} \sup_{y\in\mathcal{X}} \langle y\rangle^l \|(\partial^\alpha u)(y)\|_{\mathcal{A}_0}, \tag{4.22}$$

for all $x \in \mathcal{X}$.

Similar estimations may be obtained for the derivatives $\partial_x^\beta \mathfrak{Op}^A(p)u$ and this finishes the proof of the first two points of the Proposition for the "compact support" case. The general case follows by a usual cut-off and Dominated Convergence procedure. The proof of the last two points of the statement of the proposition is standard. □

Example 4.14. Let us consider a family $\{p_\epsilon\}_{|\epsilon|\leq\epsilon_0}$ of class $S^m_{1,\epsilon}(\mathcal{X}^2)$ and let us define, as in Subsection 1.2,

$$\tilde{p}_\epsilon(x, y, \xi, \eta) := p_\epsilon(x, y, \xi + \eta),$$

and

$$\mathfrak{q}_\epsilon(x, \xi) := \mathfrak{Op}(\tilde{p}_\epsilon(x, \cdot, \xi, \cdot)).$$

Then

(1) $\{\mathfrak{q}_\epsilon\}_{|\epsilon|\leq\epsilon_0} \in S^0_{0,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X}))$ for any $s \in \mathbb{R}$ and

(2) if the family of magnetic fields $\{B_\epsilon\}_{|\epsilon|\leq\epsilon_0}$ satisfies Hypothesis H.1 and if the associated vector potentials are choosen as in (1.4), then we have that

$$\mathfrak{Op}^{A_\epsilon}(\mathfrak{q}_\epsilon) \in \mathbb{B}(\mathcal{S}(\mathcal{X}; \mathcal{H}^{s+m}(\mathcal{X})); \mathcal{S}(\mathcal{X}; \mathcal{H}^s(\mathcal{X}))) \cap \mathbb{B}(\mathcal{S}'(\mathcal{X}; \mathcal{H}^{s+m}(\mathcal{X})); \mathcal{S}'(\mathcal{X}; \mathcal{H}^s(\mathcal{X}))), \quad \text{for all } s \in \mathbb{R}, \tag{4.23}$$

and

$$\mathfrak{Op}^{A_\epsilon}(\mathfrak{q}_\epsilon) \in \mathbb{B}(\mathcal{S}(\mathcal{X}^2); \mathcal{S}(\mathcal{X}^2)) \cap \mathbb{B}(\mathcal{S}'(\mathcal{X}^2); \mathcal{S}'(\mathcal{X}^2)), \tag{4.24}$$

and all the continuities are uniform with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. (1) By similar arguments as in Proposition 4.6 we prove that for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and $s \in \mathbb{R}$ we have that $\mathfrak{q}_\epsilon \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X})))$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ and the application

$$S^m_1(\mathcal{X}^2) \ni p_\epsilon \mapsto \mathfrak{q}_\epsilon \in S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{H}^{s+m}_\bullet(\mathcal{X}); \mathcal{H}^s_\bullet(\mathcal{X}))$$

is continuous for all $s \in \mathbb{R}$, uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. Point (1) follows then clearly.

(2) Let us note that (4.23) and the uniformity with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ follow easily by Proposition 4.13 and its proof. In order to prove (4.24) let us note that

$$\tilde{p}'_\epsilon(x, \cdot, \xi, \cdot) := \langle\xi\rangle^{-|m|} \tilde{p}_\epsilon(x, \cdot, \xi, \cdot)$$

defines a symbol of class $S^m_0(\mathcal{X})$ uniformly with respect to $((x, \xi), \epsilon) \in \Xi \times [-\epsilon_0, \epsilon_0]$ and we can view the element $\tilde{p}'_\epsilon$ as a function in $BC^\infty(\Xi; S^m_0(\mathcal{X}))$. Then, the operator-valued symbol

$$\mathfrak{q}'_\epsilon(x, \xi) := \langle\xi\rangle^{-|m|} \mathfrak{q}_\epsilon(x, \xi)$$

has the property

$$(\partial^\alpha_x \partial^\beta_\xi \mathfrak{q}'_\epsilon)(x, \xi) \in \mathbb{B}(\mathcal{S}(\mathcal{X})),$$

for all $(\alpha, \beta) \in [\mathbb{N}^d]^2$, uniformly with respect to $((x, \xi), \epsilon) \in \Xi \times [-\epsilon_0, \epsilon_0]$. Denoting

$$\mathfrak{s}_s(x, \xi) := \langle\xi\rangle^s, \quad \text{for any } s \in \mathbb{R},$$

and writing

$$\mathfrak{Op}^{A_\epsilon}(\mathfrak{q}_\epsilon) = \mathfrak{Op}^{A_\epsilon}(\mathfrak{s}_{|m|}\mathfrak{q}'),$$

the proof of Proposition 4.13 implies (4.24) uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. □

Theorem 4.15. *Take three families of Hilbert spaces with temperate variation* $\{\mathcal{A}_\xi\}_{\xi\in\mathcal{X}^*}$, $\{\mathcal{B}_\xi\}_{\xi\in\mathcal{X}^*}$, *and* $\{\mathcal{C}_\xi\}_{\xi\in\mathcal{X}^*}$, *and two families of symbols* $\{p_\epsilon\}_{|\epsilon|\le\epsilon_0} \in S^m_{\rho,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{B}_\bullet; \mathcal{C}_\bullet))$ *and* $\{q_\epsilon\}_{|\epsilon|\le\epsilon_0} \in S^{m'}_{\rho,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{B}_\bullet))$, *and a family of magnetic fields* $\{B_\epsilon\}_{|\epsilon|\le\epsilon_0}$ *satisfying Hypothesis H.1 with an associated family of vector potentials* $\{A_\epsilon\}_{|\epsilon|\le\epsilon_0}$ *given by (1.4). Then*

(1) *There exist a family of symbols*

$$\{p_\epsilon \sharp^{B_\epsilon} q_\epsilon\}_{|\epsilon|\le\epsilon_0} \in S^{m+m'}_{\rho,\epsilon}(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{C}_\bullet)),$$

such that

$$\mathfrak{Op}^{A_\epsilon}(p_\epsilon)\mathfrak{Op}^{A_\epsilon}(q_\epsilon) = \mathfrak{Op}^{A_\epsilon}(p_\epsilon \sharp^{B_\epsilon} q_\epsilon);$$

(2) *the application*

$$S^m_\rho(\mathcal{X};\mathbb{B}(\mathcal{B}_\bullet;\mathcal{C}_\bullet))\times S^{m'}_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))\ni(p_\epsilon,q_\epsilon)$$
$$\longmapsto p_\epsilon\sharp^{B_\epsilon}q_\epsilon\in S^{m+m'}_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{C}_\bullet))$$

is continuous uniformly with respect to $\epsilon\in[-\epsilon_0,\epsilon_0]$;

(3) *there exists a family of symbols* $\{r_\epsilon\}_{|\epsilon|\le\epsilon_0}\in S^{m+m'-\rho}_{\rho,\epsilon}(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{C}_\bullet))$ *having the properties*

$$\lim_{\epsilon\to0} r_\epsilon=0\quad \textit{in } S^{m+m'-\rho}_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{C}_\bullet)) \tag{4.25}$$

and

$$p_\epsilon\sharp^{B_\epsilon}q_\epsilon=p_\epsilon\cdot q_\epsilon\;+\;r_\epsilon,\quad \textit{for all } \epsilon\in[-\epsilon_0,\epsilon_0]. \tag{4.26}$$

Proof. As in the proof of Proposition 4.13 we reduce the problem to the case of symbols with compact support in both arguments $(x,\xi)\in\Xi$. A direct computation using Stokes formula and the fact that $dB_\epsilon=0$ for any $\epsilon\in[-\epsilon_0,\epsilon_0]$ shows that for point (1) of the theorem we may take the definition of the composition operation to be the following well defined integral formula

$$(p_\epsilon\sharp^{B_\epsilon}q_\epsilon)(X)=\pi^{-2d}\int_\Xi\int_\Xi e^{-2i[\![Y,Z]\!]}\omega^{B_\epsilon}(x,y,z)\,p_\epsilon(X-Y)q_\epsilon(X-Z)\,dY\,dZ, \tag{4.27}$$

where we used the notation

$$X:=(x,\xi),\quad Y:=(y,\eta),\quad Z:=(z,\zeta),$$

$$[\![Y,Z]\!]:=\langle\eta,z\rangle-\langle\zeta,y\rangle,$$

and

$$\omega^{B_\epsilon}(x,y,z):=e^{-iF_\epsilon(x,y,z)},$$

where

$$F_\epsilon(x,y,z):=\int_{\langle x-y+z,x-y-z,x+y-z\rangle}B_\epsilon,$$

with $\langle a,b,c\rangle$ the triangle with vertices $a\in\mathcal{X}$, $b\in\mathcal{X}$ and $c\in\mathcal{X}$. A direct computation (see for example Lemma 1.1 in [12]) shows that all the vectors ∇_xF_ϵ, ∇_yF_ϵ and ∇_zF_ϵ have the form

$$C_\epsilon(x,y,z)y+D_\epsilon(x,y,z)z,$$

with C_ϵ and D_ϵ functions of class $BC^\infty(\mathcal{X}^3;\mathbb{B}(\mathcal{X}))$ satisfying the conditions

$$\lim_{\epsilon\to 0} C_\epsilon = \lim_{\epsilon\to 0} D_\epsilon = 0 \quad \text{in } BC^\infty(\mathcal{X}^3;\mathbb{B}(\mathcal{X})).$$

It follows easily then that the derivatives of $\omega^{B_\epsilon}(x,y,z)$ of order at least 1 are finite linear combinations of terms of the form

$$C_{(\alpha,\beta);\epsilon}\, y^\alpha z^\beta \omega^{B_\epsilon}(x,y,z),$$

with $C_{(\alpha,\beta);\epsilon} \in BC^\infty(\mathcal{X}^3)$ satisfying the property

$$\lim_{\epsilon\to 0} C_{(\alpha,\beta);\epsilon} = 0 \quad \text{in } BC^\infty(\mathcal{X}^3).$$

Applying the usual integrations by parts with respect to the variables $\{y,z,\eta,\zeta\}$, we obtain (for some $C>0$ and any $N\in\mathbb{N}$)

$$\begin{aligned} &\|(p_\epsilon \sharp^{B_\epsilon} q_\epsilon)(X)\|_{\mathbb{B}(\mathcal{A}_\xi;\mathcal{C}_\xi)} \\ &\le C \max_{|\alpha|,|\beta|,|\gamma|,|\delta|\le N} \int_\Xi \int_\Xi \langle\eta\rangle^{-2N_1}\langle\zeta\rangle^{-2N_2}\langle y\rangle^{-2N_3}\langle z\rangle^{-2N_4} \\ &\qquad \|\partial_x^\alpha\partial_\xi^\beta p_\epsilon(X-Y)\|_{\mathbb{B}(\mathcal{B}_\xi;\mathcal{C}_\xi)} \\ &\qquad\qquad \|\partial_x^\gamma\partial_\xi^\delta q_\epsilon(X-Z)\|_{\mathbb{B}(\mathcal{A}_\xi;\mathcal{B}_\xi)}\, đY\, đZ, \end{aligned} \tag{4.28}$$

for any $\epsilon\in[-\epsilon_0,\epsilon_0]$. We use now (4.17) and (4.19) and obtain the following estimations valid for any $\epsilon\in[-\epsilon_0,\epsilon_0]$:

$$\begin{aligned} &\|\partial_x^\alpha\partial_\xi^\beta p_\epsilon(X-Y)\|_{\mathbb{B}(\mathcal{B}_\xi;\mathcal{C}_\xi)} \\ &\quad\le C\langle\eta\rangle^{2M}\|\partial_x^\alpha\partial_\xi^\beta p_\epsilon(X-Y)\|_{\mathbb{B}(\mathcal{B}_{\xi-\eta};\mathcal{C}_{\xi-\eta})} \\ &\quad\le C\langle\eta\rangle^{2M}\langle\xi-\eta\rangle^{m-\rho|\beta|}\big(\sup_{Z\in\Xi}\langle\zeta\rangle^{-m+\rho|\beta|}\|(\partial_z^\alpha\partial_\zeta^\beta p_\epsilon)(Z)\|_{\mathbb{B}(\mathcal{B}_\zeta;\mathcal{C}_\zeta)}\big). \end{aligned} \tag{4.29}$$

Repeating the same computations for the derivatives of q_ϵ and choosing suitable large exponents N_j ($1\le j\le 4$) in (4.28) we deduce the existence of two defining seminorms $|\cdot|_{n_1}$ and respectively $|\cdot|_{n_2}$ on the Fréchet space $S^m_\rho(\mathcal{X};\mathbb{B}(\mathcal{B}_\bullet;\mathcal{C}_\bullet))$ and respectively on $S^{m'}_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{B}_\bullet))$ such that

$$\sup_{X\in\Xi}\langle\xi\rangle^{-(m+m')}\|(p_\epsilon\sharp^{B_\epsilon}q_\epsilon)(X)\|_{\mathbb{B}(\mathcal{A}_\xi;\mathcal{C}_\xi)} \le |p_\epsilon|_{n_1}|q_\epsilon|_{n_2}, \quad \text{for all } \epsilon\in[-\epsilon_0,\epsilon_0]. \tag{4.30}$$

The derivatives of $p_\epsilon\sharp^{B_\epsilon}q_\epsilon$ can be estimated in a similar way in order to conclude that $p_\epsilon\sharp^{B_\epsilon}q_\epsilon \in S^{m+m'}_\rho(\mathcal{X};\mathbb{B}(\mathcal{A}_\bullet;\mathcal{C}_\bullet))$ uniformly with respect to $\epsilon\in[-\epsilon_0,\epsilon_0]$ and that property (2) is valid. Hypotheses (2) and (3) from the Definition 4.7 follow easily by (4.25) and (4.26). In conclusion there is only point (3) that remains to be proved. By the same arguments as above we can once again assume that the symbols p_ϵ and q_ϵ have compact support. We begin by using in (4.27) the equality

$$\begin{aligned} & p_\epsilon(X-Y)q_\epsilon(X-Z) \\ & \quad = p_\epsilon(X)q_\epsilon(X) - \int_0^1 [\langle Y, \nabla_X p_\epsilon(X-tY)\rangle q_\epsilon(X-tZ) \\ & \qquad\qquad + p_\epsilon(X-tY)\langle Z, \nabla_X q_\epsilon(X-tZ)\rangle]\, dt. \end{aligned} \tag{4.31}$$

The first term on the right side of equality (4.31) will produce the term $p_\epsilon q_\epsilon$ in equality (4.26) (see also Lemma 2.1 in [12]). Let us study now the term obtained by replacing the last term from (4.31) into (4.27). We eliminate Y and Z by integration by parts as in the beginning of this proof. These operations will produce derivatives of p_ϵ and q_ϵ with respect to $x \in \mathcal{X}$, that go to 0 for $\epsilon \to 0$ in their symbol spaces topology and derivatives of F_ϵ with respect to y and z; but these derivatives may be once again transformed by integrations by parts into factors of the form $C_\epsilon \in BC^\infty(\mathcal{X}^3)$ having limit 0 for $\epsilon \to 0$ as elements from $BC^\infty(\mathcal{X}^3)$. Thus, the estimations proved in the first part of the proof imply that equality (4.26) holds with

$$r_\epsilon = \int_0^1 s_\epsilon(t)dt,$$

where

$$s_\epsilon(t) \in S_\rho^{m+m'-\rho}(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{C}_\bullet))$$

uniformly with respect to $(\epsilon, t) \in [-\epsilon_0, \epsilon_0] \times [0, 1]$ and

$$\lim_{\epsilon\to 0} s_\epsilon(t) = 0 \quad \text{in } S_\rho^{m+m'-\rho}(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{C}_\bullet))$$

uniformly with respect to $t \in [0, 1]$. We conclude that r_ϵ has the properties stated in the theorem. □

Remark 4.16. The proof of Theorem 4.15 also implies the following fact: *the operation $\sharp^{B_\epsilon}$ is well defined also as operation*

$$S_\rho^m(\mathcal{X}; \mathbb{B}(\mathcal{B}_\bullet; \mathcal{C}_\bullet)) \times S_\rho^{m'}(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{B}_\bullet)) \longrightarrow S_\rho^{m+m'}(\mathcal{X}; \mathbb{B}(\mathcal{A}_\bullet; \mathcal{C}_\bullet))$$

being bilinear and continuous uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proposition 4.17. *Given two Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$ and, for any $\epsilon \in [-\epsilon_0, \epsilon_0]$, a symbol $p_\epsilon \in S^m_0(\mathcal{X}; \mathbb{B}(\mathcal{A}; \mathcal{B}))$, uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$, then, for any $s \in \mathbb{R}$, the operator $\mathfrak{Op}^{A_\epsilon}(p_\epsilon)$ belongs to the space $\mathbb{B}(\mathcal{H}^{s+m}_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{A}; \mathcal{H}^s_{A_\epsilon}(\mathcal{X}) \otimes \mathcal{B})$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. Moreover, the norm of $\mathfrak{Op}^{A_\epsilon}(p_\epsilon)$ in the above Banach space is bounded from above by a seminorm of p_ϵ in $S^m_0(\mathcal{X}; \mathbb{B}(\mathcal{A}; \mathcal{B}))$, uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$.*

Proof. For $m = s = 0$ the proposition may be proved by the same arguments as in the scalar case: $\mathcal{A} = \mathcal{B} = \mathbb{C}$ (see for example [12]). Also using the results from [12] we can see that for any $t \in \mathbb{R}$ the operator $Q_{s,\epsilon}$ belongs to the space $\mathbb{B}(\mathcal{H}^{t+s}_{A_\epsilon}(\mathcal{X}); \mathcal{H}^t_{A_\epsilon}(\mathcal{X}))$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$. The proof of the general case follows now by the identity

$$\mathfrak{Op}^{A_\epsilon}(p_\epsilon) = Q_{-s,\epsilon} Q_{s,\epsilon} \mathfrak{Op}^{A_\epsilon}(p_\epsilon) Q_{-(s+m),\epsilon} Q_{s+m,\epsilon}$$

and the fact that $q_{s,\epsilon} \sharp^{B_\epsilon} p_\epsilon \sharp^{B_\epsilon} q_{-(s+m),\epsilon}$ is a symbol of class $S^0_0(\mathcal{X}; \mathbb{B}(\mathcal{A}; \mathcal{B}))$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ (as implied by the Remark 4.16). □

Standard arguments allow us to prove the following statement.

Proposition 4.18. *Suppose given a Hilbert space $\mathcal{A}$ and a bounded subset*

$$\{p_\epsilon\}_{|\epsilon| \le \epsilon_0} \subset S^0_\rho(\mathcal{X}; \mathbb{B}(\mathcal{A}))$$

such that

$$\lim_{\epsilon \to 0} p_\epsilon = 0$$

in this space of symbols. Then, for sufficiently small $\epsilon_0 > 0$,

(1) $\mathrm{id} + \mathfrak{Op}^{A_\epsilon}(p_\epsilon)$ *is invertible in $\mathbb{B}(L^2(\mathcal{X}) \otimes \mathcal{A})$ for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and*

(2) *it exists a bounded subset of symbols $\{q_\epsilon\}_{|\epsilon| \le \epsilon_0}$ from $S^0_\rho(\mathcal{X}; \mathbb{B}(\mathcal{A}))$ such that*

$$\lim_{\epsilon \to 0} q_\epsilon = 0 \quad \text{in } S^0_\rho(\mathcal{X}; \mathbb{B}(\mathcal{A}))$$

and

$$[\mathrm{id} + \mathfrak{Op}^{A_\epsilon}(p_\epsilon)]^{-1} = \mathrm{id} + \mathfrak{Op}^{A_\epsilon}(q_\epsilon).$$

4.3. Relativistic Hamiltonians. We shall close this subsection with the study of a property that connects the two relativistic Schrödinger Hamiltonians $\mathfrak{Op}^{A_\epsilon}(h_R)$ and $[\mathfrak{Op}^{A_\epsilon}(h_{NR})]^{1/2}$ with

$$h_R(x, \xi) := \langle \xi \rangle \equiv \sqrt{1 + |\xi|^2}$$

and

$$h_{NR}(x,\xi) := 1 + |\xi|^2 \equiv \langle\xi\rangle^2.$$

We shall use some arguments presented in §6.3 of [13]. The following proposition allows to prove that the operator in (1.11) verifies the hypothesis in Subsection 1.2.

Proposition 4.19. *There exists a bounded subset* $\{q_\epsilon\}_{|\epsilon|\leq\epsilon_0}$ *of symbols from* $S^0_1(\mathcal{X})$ *such that* $\lim_{\epsilon\to 0} q_\epsilon = 0$ *in* $S^0_1(\mathcal{X})$ *and*

$$[\mathfrak{Op}^{A_\epsilon}(h_{NR})]^{1/2} = \mathfrak{Op}^{A_\epsilon}(h_R) + \mathfrak{Op}^{A_\epsilon}(q_\epsilon). \tag{4.32}$$

Proof. Following [13], if we denote by p^- the inverse of the symbol p with respect to the composition $\sharp^{B_\epsilon}$,

$$[\mathfrak{Op}^{A_\epsilon}(h_{NR})]^{1/2} = \mathfrak{Op}^{A_\epsilon}(h_{NR})\mathfrak{Op}^{A_\epsilon}\left(-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathrm{z}^{-1/2}(\langle\xi\rangle^2 - \mathrm{z})^- d\mathrm{z}\right). \tag{4.33}$$

Recalling the proof of point (3) in Theorem 4.15 we can easily prove that

$$(\langle\xi\rangle^2 - \mathrm{z})\sharp^{B_\epsilon}(\langle\xi\rangle^2 - \mathrm{z})^{-1} = 1 + r_{\epsilon,\mathrm{z}} \tag{4.34}$$

where $\langle \mathrm{z}\rangle r_{\epsilon,\mathrm{z}} \in S^0_1(\mathcal{X})$ uniformly for $(\epsilon,\mathrm{z}) \in [-\epsilon_0,\epsilon_0]\times i\mathbb{R}$ and

$$\lim_{\epsilon\to 0}\langle \mathrm{z}\rangle r_{\epsilon,\mathrm{z}} = 0 \quad \text{in } S^0_1(\mathcal{X})$$

uniformly with respect to $\mathrm{z} \in i\mathbb{R}$. Following the proof of Proposition 4.18, for $\epsilon_0 > 0$ sufficiently small there exists a symbol $f_{\epsilon,\mathrm{z}}$ such that $\langle \mathrm{z}\rangle f_{\epsilon,\mathrm{z}} \in S^0_1(\mathcal{X})$ uniformly with respect to $(\epsilon,\mathrm{z}) \in [-\epsilon_0,\epsilon_0]\times i\mathbb{R}$,

$$\lim_{\epsilon\to 0}\langle \mathrm{z}\rangle f_{\epsilon,\mathrm{z}} = 0 \quad \text{in } S^0_1(\mathcal{X})$$

uniformly with respect to $\mathrm{z} \in i\mathbb{R}$ and we also have

$$(1 + r_{\epsilon,\mathrm{z}})^- = 1 + f_{\epsilon,\mathrm{z}}.$$

By (4.33) and the properties of the symbol $r_{\epsilon,\mathrm{z}}$, it follows that we can define

$$(\langle\xi\rangle^2 - \mathrm{z})^- := (\langle\xi\rangle^2 - \mathrm{z})^{-1}\sharp^{B_\epsilon}(1 + f_{\epsilon,\mathrm{z}}) = (\langle\xi\rangle^2 - \mathrm{z})^{-1} + (\langle\xi\rangle^2 - \mathrm{z})^{-1}\sharp^{B_\epsilon} f_{\epsilon,\mathrm{z}}. \tag{4.35}$$

Using (4.35) in (4.33) we note that the term $(\langle\xi\rangle^2 - \mathfrak{z})^{-1}$ produces by magnetic quantization a term of the form

$$\mathfrak{Op}^{A_\epsilon}(h_{NR})\mathfrak{Op}^{A_\epsilon}(h_R^{-1}),$$

and using Theorem 4.15 this operator may be put in the form

$$\mathfrak{Op}^{A_\epsilon}(h_R) + \mathfrak{Op}^{A_\epsilon}(q'_\epsilon),$$

where $q'_\epsilon \in S^0_1(\mathcal{X})$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ with

$$\lim_{\epsilon\to 0} q'_\epsilon = 0 \quad \text{in } S^0_1(\mathcal{X}).$$

If we note that $h_{NR}\sharp^{B_\epsilon}(h_{NR} - \mathfrak{z})^{-1} \in S^0_1(\mathcal{X})$ uniformly with respect to $(\epsilon, \mathfrak{z}) \in [-\epsilon_0, \epsilon_0] \times i\mathbb{R}$, then we can see that the last term of (4.35) gives in (4.33) by magnetic quantization an expression of the form $\mathfrak{Op}^{A_\epsilon}(q''_\epsilon)$ with $q''_\epsilon \in S^0_1(\mathcal{X})$ uniformly with respect to $\epsilon \in [-\epsilon_0, \epsilon_0]$ and such that

$$\lim_{\epsilon\to 0} q''_\epsilon = 0 \quad \text{in } S^0_1(\mathcal{X}). \qquad \square$$

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