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Absence of l^1 eigenfunctions for lattice operators with fast local periodic approximation

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Abstract. We show that a lattice Schrödinger operator $\Delta + v$ whose potential $v\colon \mathbf{Z}^d \to \mathbf{C}$ admits fast local approximation by periodic functions does not have l^1 eigenfunctions. In particular, it does not exhibit Anderson localization. A special case of this result pertaining to quasi-periodic potentials states: Let $V\colon \mathbf{R}^d \to \mathbf{C}$ be a $(1,\dots,1)$-periodic function satisfying the Hölder condition. There is such $\theta > 0$ that if real numbers $\alpha_1,\dots,\alpha_d$ satisfy the inequality $\|n_1\alpha_1\| + \dots + \|n_d\alpha_d\| < \theta^{n_1 \cdots n_d}$ for infinitely many d-tuples $(n_1,\dots,n_d) \in \mathbf{N}^d$ ($\|\cdot\|$ is the distance from a real number to the nearest integer), then the operator $\Delta + v$ with $v(x) = V(\alpha_1 x_1,\dots,\alpha_d x_d)$ has no nontrivial eigenfunctions in $l^1(\mathbf{Z}^d)$. This statement contrasts the result of J. Bourgain: Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbf{Z}^d$, d arbitrary, *Geom. Funct. Anal.* **17** (2007), 682–706.

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1. Introduction

We consider a lattice Schrödinger operator $L = \Delta + v$ acting in $\mathbf{C}^{\mathbf{Z}^d}$ $(d \geq 2)$ as follows:

$$(Lu)(x) = \sum_{x' \in \mathbf{Z}^d \colon \|x'-x\|_1 = 1} u(x') + v(x)u(x), \quad x \in \mathbf{Z}^d.$$

We examine the case where the potential $v(\cdot)$ can be approximated with high accuracy by a sequence of periodic potentials with growing periods on a suitable increasing sequence of finite sets. We show that such operator does not have eigenfunctions in $l^1(\mathbf{Z}^d)$. In particular, there are no exponentially decaying eigenfunctions and hence no Anderson localization (the phenomenon where such eigenfunctions are complete in $l^2(\mathbf{Z}^d)$).

The class of potentials described above includes, among others, quasi-periodic potentials of the form

$$v(x) = V(\alpha_1 x_1, \dots, \alpha_d x_d), \quad x \in \mathbf{Z}^d, \tag{1}$$

where $V\colon \mathbf{R}^d \to \mathbf{C}$ is a $(1, \dots, 1)$-periodic function satisfying the Hölder condition, and α_i's are irrational numbers that admit very good approximation by rationals. The absence of fast decaying eigenfunctions for such potentials contrasts the result of Bourgain [1], according to which for a fixed real analytic function V on $\mathbf{T}^d$ satisfying a mild non-degeneracy condition, the operator $\Delta + \lambda v$ with $v(\cdot)$ given by (1) exhibits Anderson localization for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{T}^d \setminus \Omega_\lambda$, where $\operatorname{mes} \Omega_\lambda \to 0$ as $\lambda \to \infty$; see also the earlier work [2].

The main tool used in the paper is the inequality (Lemma 1) of the form $|u(0)| \le \sum_{x \in K} |u(x)|$, where $u(\cdot)$ is an arbitrary solution of a periodic linear homogeneous lattice equation and K is a certain finite subset of the group of periods not containing 0. This is a generalization of the one-dimensional inequality going back to [3]: for any solution of the equation

$$y(n-1) + y(n+1) + v(n)y(n) = \lambda y(n), \quad n \in \mathbf{Z},$$

with a T-periodic coefficient $v(\cdot)$ one has

$$|y(0)| \le 2 \max_{k=\pm 1, \pm 2} |y(kT)|$$

(in [3] this was proved for the equation

$$-y'' + v(t)y = \lambda y$$

with a real-valued T-periodic $v(t)$ and real λ; for the further history of this inequality, see [4]).

2. Periodic operators

From now on, given a set $X \subset \mathbf{Z}^d$, we will denote the set $X \setminus \{0\}$ by X^*. The cardinality of a finite set X will be denoted by $|X|$ or, alternatively, by $\#X$. The dimension d of the lattice $\mathbf{Z}^d$ will always be assumed to be ≥ 2 (except for Lemma 1 and Theorem 1, where d may also equal one).

Lemma 1. *Let* Γ *be a subgroup of* $\mathbf{Z}^d$ *and* L *a* Γ*-periodic linear operator in* $\mathbf{C}^{\mathbf{Z}^d}$ (Γ*-periodicity means that, letting* $(T^\gamma u)(x) = u(x+\gamma)$*, we have* $T^\gamma L = LT^\gamma$ *for all* $\gamma \in \Gamma$.) *Suppose* $F \subset \Gamma$ *and* $Y \subset \mathbf{Z}^d$ *are such finite sets and* $\lambda \in \mathbf{C}$ *is such a number that*

(a) *if* $u(\cdot)$ *is a solution of the equation*

$$Lu = \lambda u \tag{2}$$

and $u|_Y = 0$*, then* $u|_F = 0$*;*

(b) $|F| > |Y|$.

Then for any solution $u(\cdot)$ *of* (2)

$$|u(0)| \le \sum_{x\in(F-F)^*} |u(x)|. \tag{3}$$

Proof. Let N denote the linear space of all solutions of (2), and let $M = N|_Y$. It follows from (a) that for each $x \in F$ the value of a solution $u \in N$ at x is uniquely determined by $u|_Y$ and is, therefore, given by a linear functional on M. Since $\dim M \le |Y|$, (b) implies that those functionals are linearly dependent: there are $b_x \in \mathbf{C}$ $(x \in F)$, not all of them 0, such that

$$\sum_{x\in F} b_x u(x) = 0 \quad \text{for all } u \in N.$$

There is such $a \in F$ that $|b_x| \le |b_a|$ for all $x \in F$. Then

$$u(a) = \sum_{x\in F\setminus\{a\}} c_x u(x),$$

where $c_x = -b_x/b_a$ and hence $|c_x| \le 1$; it follows that

$$|u(a)| \le \sum_{x\in F\setminus\{a\}} |u(x)|. \tag{4}$$

In view of the Γ-periodicity of L, the space N of solutions of (2) is invariant under translations by elements of Γ. Therefore, (4) implies the inequality

$$|u(0)| \le \sum_{x\in(F-a)\setminus\{0\}} |u(x)|,$$

from which (3) follows. □

Theorem 1. *Let* Γ *be a subgroup of* $\mathbf{Z}^d$ *and* L *a* Γ*-periodic linear operator in* $C^{\mathbf{Z}^d}$. *Let* $F \subset \Gamma$ *and* $Y \subset \mathbf{Z}^d$ *be two finite sets with the following properties*:

(a) *for any* $q \in \mathbf{Z}^d$ *and any solution* $u(\cdot)$ *of equation* (2) *such that* $u|_{Y+q} = 0$, *we also have* $u|_{F+q} = 0$;

(b) $|F| > |Y|$.

Then for any solution $u(\cdot)$ *of* (2) *and any* $q \in \mathbf{Z}^d$

$$|u(q)| \le \sum_{x \in (F-F)^*} |u(q+x)|. \tag{5}$$

Proof. For $q \in \mathbf{Z}^d$, let $L^q = T^q L T^{-q}$. The operator L^q and the sets F and Y satisfy the conditions of Lemma 1. Putting $u_q = T^q u$, where u is a solution of $Lu = \lambda u$, we have $L^q u_q = \lambda u_q$ and, by Lemma 1,

$$|u_q(0)| \le \sum_{x \in (F-F)^*} |u_q(x)|,$$

which is equivalent to (5). □

Let $L = \Delta + v$, where Δ is the lattice Laplacian,

$$(\Delta u)(x) = \sum_{z \in \mathbf{Z}^d :\, \|z\|_1 = 1} u(x+z), \quad x \in \mathbf{Z}^d,$$

and v is the operator of multiplication by a complex-valued function $v(\cdot)$ on $\mathbf{Z}^d$. Suppose $v(\cdot)$ is Γ-periodic, Γ being a subgroup of $\mathbf{Z}^d$ generated by d linearly independent vectors $f_1, \ldots, f_d$, where $f_j = \big(f_j^{(i)}\big)_{i=1}^d \in \mathbf{Z}^d$, $j = 1, \ldots, d$:

$$\Gamma = \left\{ \sum_{j=1}^d m_j f_j :\ m_j \in \mathbf{Z},\ j = 1, \ldots, d \right\}.$$

Denote by A_Γ the fundamental region of the lattice Γ in $\mathbf{R}^d$,

$$A_\Gamma = \left\{ \sum_{j=1}^d \theta_j f_j :\ 0 \le \theta_j < 1,\ j = 1. \ldots, d \right\},$$

and by V_Γ its volume,

$$V_\Gamma = \mathrm{Vol}(A_\Gamma) = \big|\det\big[f_j^{(i)}\big]_{i,j=1}^d\big|.$$

Theorem 2. *Let $u(\cdot)$ be a solution of the equation $(\Delta + v)u = \lambda u$ with a Γ-periodic potential v. Then for any $q \in \mathbf{Z}^d$*

$$|u(q)| \le \sum_{x \in \Gamma^* : \, \|x\|_\infty \le 2d V_\Gamma} |u(q+x)|. \tag{6}$$

Proof. Fix an integer $n \ge 3$ and consider the following subset of $\mathbf{R}^d$:

$$D_n = \{x \in \mathbf{R}^d : 0 \le x_i < n, \ i = 1, \dots, d\}.$$

For $z \in \mathbf{Z}^d$, let

$$F_z = \Gamma \cap (D_n + z). \tag{7}$$

The lattice Γ has "density"

$$\lim_{r \to \infty} \frac{\#\{\gamma \in \Gamma : \|\gamma\|_\infty \le r\}}{(2r)^d} = 1/V_\Gamma,$$

and since

$$\mathbf{R}^d = \bigsqcup_{z \in n\mathbf{Z}^d} (D_n + z),$$

we have

$$\sup_{z \in n\mathbf{Z}^d} |F_z| \ge \mathrm{Vol}(D_n)/V_\Gamma = n^d/V_\Gamma.$$

Note that $|F_z|$, the cardinality of the set F_z, takes only finitely many values, and hence there are such points $z \in n\mathbf{Z}^d$ that

$$|F_z| \ge \frac{n^d}{V_\Gamma}.$$

Fix one such z and let $C_z = C_0 + z$, where $C_0 = \{0, 1, \dots, n-1\}^d$. Note that

$$F_z \subset \mathbf{Z}^d \cap (D_n + z) = (\mathbf{Z}^d \cap D_n) + z = C_0 + z = C_z.$$

Furthermore, let $Y_0 = C_0 \setminus S_0$, where

$$S_0 = \{1, \dots, n-2\}^{d-1} \times \{2, \dots, n-1\},$$

and

$$Y_z = Y_0 + z.$$

Given $q \in \mathbf{Z}^d$, for any solution $u(\cdot)$ of the equation $(\Delta + v)u = \lambda u$ such that $u|_{Y_z+q} = 0$ we have $u|_{C_z+q} = 0$ and consequently $u|_{F_z+q} = 0$.

We have

$$|Y_z| = |Y_0| = n^d - (n-2)^d.$$

Theorem 1 is applicable when $|Y_z| < |F_z|$, which is guaranteed if $n^d - (n-2)^d < n^d / V_\Gamma$, or

$$1 - \left(1 - \frac{2}{n}\right)^d < \frac{1}{V_\Gamma}.$$

Since the left-hand side is $< 2d/n$, we have $|Y_z| < |F_z|$ if

$$n \geq 2d V_\Gamma.$$

Theorem 1 then states that, given any solution $u(\cdot)$ of $(\Delta + v)u = \lambda u$ and any $q \in \mathbf{Z}^d$, we have

$$|u(q)| \leq \sum_{x \in (F_z - F_z)^*} |u(q+x)|. \tag{8}$$

Here, in view of (7),

$$F_z - F_z \subset \Gamma \cap (D_n - D_n) \subset \{x \in \Gamma \colon \|x\|_\infty \leq n\}.$$

Let $n = 2d V_\Gamma$; then (8) implies (6). □

Corollary 1. *If $u(\cdot)$ is a solution of $(\Delta + v)u = \lambda u$, where the function $v(\cdot)$ on $\mathbf{Z}^d$ is $(\tau_1, \ldots, \tau_d)$-periodic ($\tau_1, \ldots, \tau_d \in \mathbf{N}$), then for any $q \in \mathbf{Z}^d$*

$$|u(q)| \leq \sum_{x \in \Gamma_\tau^* \colon \|x\|_\infty \leq 2d\tau_1 \ldots \tau_d} |u(q+x)|,$$

where

$$\Gamma_\tau = \{(j_1\tau_1, \ldots, j_d\tau_d) \colon j_1, \ldots, j_d \in \mathbf{Z}\}. \tag{9}$$

3. Operators approximable by periodic ones

We will denote a generic element of $\mathbf{N}^d$ as $\tau = (\tau_1, \ldots, \tau_d)$.

Theorem 3. *Let $L = \Delta + v$, where $v(\cdot)$ is a bounded complex-valued function on $\mathbf{Z}^d$. Suppose for some fixed $\varepsilon > 0$ and each $\tau = (\tau_1, \ldots, \tau_d)$ in some infinite set $\mathcal{T} \subset \mathbf{N}^d$ there is a $(\tau_1, \ldots, \tau_d)$-periodic function $v_\tau(\cdot)$ satisfying the inequality*

$$\max_{\|x\|_\infty \leq (2d+\varepsilon)\tau_1 \ldots \tau_d} |v_\tau(x) - v(x)| \leq \theta^{\tau_1 \cdots \tau_d}, \tag{10}$$

where θ is a constant such that

$$0 < \theta < (4d + 2\|v\|_\infty - 1)^{-2d}. \tag{11}$$

Then the equation $Lu = \lambda u$ with any $\lambda \in \mathbf{C}$ does not have nontrivial l^1 solutions.

Proof. Let

$$m_\tau = [(2d+\varepsilon)\tau_1 \dots \tau_d], \qquad \tau \in \mathcal{T},$$

where $[\,\cdot\,]$ denotes the integer part of a real number. Inequality (10) can be rewritten in the form

$$\rho_\tau \equiv \max_{x \in Q_\tau} |v_\tau(x) - v(x)| \le \theta^{\tau_1 \dots \tau_d}, \tag{12}$$

where

$$Q_\tau = \{x \in \mathbf{Z}^d : \|x\|_\infty \le m_\tau\}.$$

Suppose $u \colon \mathbf{Z}^d \to \mathbf{C}$ is a solution of the equation

$$(\Delta + v)u = \lambda u$$

such that

$$\|u\|_1 \le 1. \tag{13}$$

Pick any $\tau \in \mathcal{T}$ so there is a τ-periodic function $v_\tau(\cdot)$ satisfying (12). Define a subset Z_τ of Q_τ as follows:

$$Z_\tau = \{x \in Q_\tau : x_d \in \{-1, 0\} \text{ or } |x_i| = m_\tau \text{ for some } i \in \{1, \dots, d-1\}\}.$$

Also put

$$Q_\tau^{\circ} = \{x \in \mathbf{Z}^d : \|x\|_\infty \le m_\tau - 1\}.$$

Denote by $u_\tau(\cdot)$ the unique function on Q_τ such that

(i) $(\Delta u_\tau)(x) + v_\tau(x) u_\tau(x) = \lambda u_\tau(x)$ for all $x \in Q_\tau^{\circ}$;

(ii) $u_\tau|_{Z_\tau} = u|_{Z_\tau}$.

The function

$$w_\tau(x) = u_\tau(x) - u(x), \quad x \in Q_\tau,$$

satisfies the equations

$$w_\tau|_{Z_\tau} = 0$$

and

$$(\Delta w_\tau)(x) + (v(x) - \lambda) w_\tau(x) + r_\tau(x) u(x) + r_\tau(x) w_\tau(x) = 0, \quad x \in Q_\tau^{\circ},$$

where

$$r_\tau(x) = v_\tau(x) - v(x).$$

By representing any $x \in \mathbf{Z}^d$ in the form $x = (j,k)$, where $j \in \mathbf{Z}^{d-1}$ and $k \in \mathbf{Z}$, we transform the previous equation into

$$\begin{aligned} &w_\tau(j,k+1) + w_\tau(j,k-1) \\ &\quad + \sum_{j' \in \mathbf{Z}^{d-1}:\, \|j'-j\|_1 = 1} w_\tau(j',k) + (v(j,k)-\lambda) w_\tau(j,k) \\ &\quad + r_\tau(j,k) u(j,k) + r_\tau(j,k) w_\tau(j,k) = 0, \quad (j,k) \in Q_\tau^{\mathrm{o}}. \end{aligned}$$

This equation implies that

$$\begin{aligned} &|w_\tau(j,k\pm 1)| \\ &\quad \le |w_\tau(j,k\mp 1)| \\ &\qquad + \sum_{j' \in \mathbf{Z}^{d-1}:\, \|j'-j\|_1 = 1} |w_\tau(j',k)| + (|v(j,k)| + |\lambda|)|w_\tau(j,k)| \\ &\qquad + |r_\tau(j,k)||u(j,k)| + |r_\tau(j,k)||w_\tau(j,k)|, \quad (j,k) \in Q_\tau^{\mathrm{o}}. \end{aligned} \tag{14}$$

Putting

$$\sigma_\tau(k) = \sum_{x \in Q_\tau :\, x_d = k} |w_\tau(x)| \equiv \sum_{j \in \mathbf{Z}^{d-1}:\, \|j\|_\infty \le m_\tau - 1} |w_\tau(j,k)|, \quad -m_\tau \le k \le m_\tau,$$

we obtain from (14) by summation on j:

$$\sigma_\tau(k\pm 1) \le \sigma_\tau(k \mp 1) + B_\tau \sigma_\tau(k) + \rho_\tau, \quad -m_\tau + 1 \le k \le m_\tau - 1, \tag{15}$$

where

$$B_\tau = 2(d-1) + \|v\|_\infty + |\lambda| + \rho_\tau$$

(we use the facts that $|r_\tau(j,k)| \le \rho_\tau$ for all $(j,k) \in Q_\tau$ and $\sum_{j \in \mathbf{Z}^{d-1}} |u(j,k)| \le 1$ for all $k \in \mathbf{Z}$, due to (12) and (13), respectively).

Furthermore, $|\lambda|$ does not exceed the norm of the operator $L = \Delta + v$ acting in $l^1(\mathbf{Z}^d)$, which is $\le 2d + \|v\|_\infty$. Therefore,

$$B_\tau \le 4d + 2\|v\|_\infty - 2 + \rho_\tau, \quad \tau \in \mathcal{T}.$$

Pick such $D \in \mathbf{R}$ that

$$D > 4d + 2\|v\|_\infty - 1$$

and

$$D^{2d} < \frac{1}{\theta} \tag{16}$$

(which is possible due to (11)) and note that, by (12), we have

$$B_\tau < D - 1 \tag{17}$$

for all but finitely many $\tau \in \mathcal{T}$.

Inequalities (15) and (17) imply that

$$\sigma_\tau(k \pm 1) \le \sigma_\tau(k \mp 1) + (D-1)\sigma_\tau(k) + \rho_\tau, \quad -m_\tau + 1 \le k \le m_\tau - 1.$$

It follows by induction (using the equalities $\sigma_\tau(-1) = \sigma_\tau(0) = 0$) that

$$\sigma_\tau(k) \le D^{|k|-1}\rho_\tau, \quad -m_\tau \le k \le m_\tau. \tag{18}$$

Consequently,

$$\sum_{|k| \le m_\tau} \sigma_\tau(k) \le D^{m_\tau}\rho_\tau,$$

or, equivalently,

$$\sum_{x \in Q_\tau} |u_\tau(x) - u(x)| \le D^{m_\tau}\rho_\tau, \tag{19}$$

which holds for all $\tau \in \mathcal{T}$ with large enough $\|\tau\|_\infty$.

The function $u_\tau(\cdot)$ is defined on the cube Q_τ and satisfies the equation

$$\Delta u_\tau(x) + v_\tau(x)u_\tau(x) = \lambda u_\tau(x)$$

on Q_τ^{o}. According to the following lemma, this function has an extension to $\mathbf{Z}^d$ that satisfies the same equation for all $x \in \mathbf{Z}^d$.

Lemma 2. *Let*

$$Q = \{x \in \mathbf{Z}^d : a_i \le x_i \le b_i,\ i = 1, \dots, d\}$$

and

$$Q^{\mathrm{o}} = \{x \in \mathbf{Z}^d : a_i + 1 \le x_i \le b_i - 1,\ i = 1, \dots, d\}.$$

Let $v \colon \mathbf{Z}^d \to \mathbf{C}$ and $u \colon Q \to \mathbf{C}$ be functions such that

$$\Delta u(x) + v(x)u(x) = \lambda u(x) \quad \text{for all } x \in Q^{\mathrm{o}}.$$

Then there is a function $\tilde{u} \colon \mathbf{Z}^d \to \mathbf{C}$ such that

$$\tilde{u}|_Q = u \quad \text{and} \quad (\Delta + v)\tilde{u} = \lambda\tilde{u} \ \text{ on } \mathbf{Z}^d.$$

The proof is deferred to the appendix.

Due to the lemma, we can consider $u_\tau(\cdot)$ as a function defined on $\mathbf{Z}^d$ and satisfying the equation $(\Delta + v_\tau)u_\tau = \lambda u_\tau$ on the entire lattice $\mathbf{Z}^d$. The function $v_\tau(\cdot)$ is Γ_τ-periodic, where the lattice Γ_τ is defined by (9).

Pick any $q \in \mathbf{Z}^d$. According to Corollary 1,

$$|u_\tau(q)| \le \sum_{x \in q + P_\tau^*} |u_\tau(x)|, \tag{20}$$

where

$$P_\tau = \{x \in \Gamma_\tau \colon \ \|x\|_\infty \le 2d\,\tau_1 \dots \tau_d\}.$$

Assuming that

$$m_\tau - 2d\,\tau_1 \dots \tau_d \equiv [\varepsilon \tau_1 \dots \tau_d] \ge \|q\|_\infty$$

(which is true for all $\tau \in \mathcal{T}$ with large enough $\|\tau\|_\infty$), we have $q + P_\tau \subset Q_\tau$, so (20) and (19) imply that for all but finitely many $\tau \in \mathcal{T}$

$$|u(q)| \le \sum_{x \in q + P_\tau^*} |u(x)| + 2D^{m_\tau} \rho_\tau. \tag{21}$$

As $\mathcal{T} \ni \tau \to \infty$, the first summand on the right converges to 0 due to (13). The second summand does not exceed

$$2D^{(2d+\varepsilon)\tau_1 \dots \tau_d} \theta^{\tau_1 \dots \tau_d} = 2(D^{2d+\varepsilon}\theta)^{\tau_1 \dots \tau_d}.$$

Note that $\varepsilon > 0$ in (10) can be made arbitrarily small. Choose it so small that $D^{2d+\varepsilon} < \theta^{-1}$ (which is possible due to (16)). Then the right-hand side of (21) goes to 0 as $\mathcal{T} \ni \tau \to \infty$. Therefore, $u(q) = 0$. Since $q \in \mathbf{Z}^d$ was chosen arbitrarily, this completes the proof. □

We will apply now Theorem 3 to operators with quasi-periodic potentials. The distance from a real number a to the nearest integer will be denoted by $\|a\|$.

Theorem 4. *Let the potential $v(\cdot)$ of the Schrödinger operator $L = \Delta + v$ be of the form*

$$v(x) = V(\alpha_1 x_1, \dots, \alpha_d x_d), \quad x \in \mathbf{Z}^d,$$

where $\alpha_1, \dots, \alpha_d$ are real numbers and $V \colon \mathbf{R}^d \to \mathbf{C}$ is a $(1, \dots, 1)$-periodic function satisfying the Hölder condition

$$|V(t_1, \dots, t_d) - V(t_1', \dots, t_d')| \le C \sum_{i=1}^d |t_i - t_i'|^\beta, \tag{22}$$

where $0 < \beta \leq 1$. *Suppose there are* d *sequences of positive integers* $\nu_1^{(p)}, \dots, \nu_d^{(p)}$ $(p = 1, 2, \dots)$ *such that*

$$\nu_1^{(p)} + \dots + \nu_d^{(p)} \longrightarrow \infty \quad \text{as } p \to \infty$$

and

$$\|\nu_1^{(p)}\alpha_1\| + \dots + \|\nu_d^{(p)}\alpha_d\| \leq \eta^{\nu_1^{(p)} \dots \nu_d^{(p)}}, \quad p = 1, 2, \dots,$$

where

$$0 < \eta < (4d + 2\|V\|_\infty - 1)^{-2d/\beta}. \tag{23}$$

Then the operator $\Delta + v$ *does not have eigenfunctions in* $l^1(\mathbf{Z}^d)$.

The theorem can be reformulated in the following equivalent way.

Theorem 4*. *Let* $V : \mathbf{R}^d \to \mathbf{C}$ *be a* $(1, \dots, 1)$*-periodic function satisfying the Hölder condition* (22). *Suppose* $\eta \in \mathbf{R}$ *satisfies inequality* (23). *If real numbers* $\alpha_1, \dots, \alpha_d$ *are such that*

$$\|\nu_1\alpha_1\| + \ldots + \|\nu_d\alpha_d\| \leq \eta^{\nu_1 \dots \nu_d} \tag{24}$$

for infinitely many d*-tuples* $(\nu_1, \dots, \nu_d) \in \mathbf{N}^d$, *then the operator* $\Delta + v$ *with* $v(x) = V(\alpha_1 x_1, \cdots, \alpha_d x_d)$ *has no eigenfunctions in* $l^1(\mathbf{Z}^d)$.

Proof. Denote by $\mathcal{T}$ the infinite set of those $\nu = (\nu_1, \dots, \nu_d) \in \mathbf{N}^d$ for which (24) holds. Given $\nu \in \mathcal{T}$, there is $\mu = (\mu_1, \dots, \mu_d) \in \mathbf{Z}^d$ such that $|\mu_i - \nu_i\alpha| = \|\nu_i\alpha\|$ for each $i = 1, \dots, d$. Let

$$\alpha_i^\nu = \frac{\mu_i}{\nu_i}, \quad i = 1, \dots, d,$$

and

$$v_\nu(x_1, \dots, x_d) = V(\alpha_1^\nu x_1, \dots, \alpha_d^\nu x_d).$$

The function $v_\nu(\cdot)$ is $(\nu_1, \dots, \nu_d)$-periodic. In order to apply Theorem 3, we need to estimate (for some fixed $\varepsilon > 0$) the number

$$M_\nu = \max_{\|x\|_\infty \leq (2d+\varepsilon)\, \nu_1 \dots \nu_d} |v_\nu(x) - v(x)|.$$

If $\|x\|_\infty \leq (2d+\varepsilon)\,\nu_1 \ldots \nu_d$, then

$$\begin{aligned} |v_\nu(x) - v(x)| &= |V(\alpha_1^\nu x_1, \ldots, \alpha_d^\nu x_d) - V(\alpha_1 x_1, \ldots, \alpha_d x_d)| \\ &\leq C \sum_{i=1}^d \left| \frac{\mu_i}{\nu_i} x_i - \alpha_i x_i \right|^\beta \\ &\leq C \sum_{i=1}^d |x_i|^\beta |\mu_i - \nu_i \alpha_i|^\beta \qquad (25) \\ &\leq C \|x\|_\infty^\beta \sum_{i=1}^d \|\nu_i \alpha_i\|^\beta \\ &\leq Cd((2d+\varepsilon)\nu_1 \ldots \nu_d)^\beta \, (\eta^{\nu_1 \ldots \nu_d})^\beta . \end{aligned}$$

Pick any $\theta \in \mathbf{R}$ such that

$$\eta^\beta < \theta < (4d + 2\|v\|_\infty - 1)^{-2d}, \qquad (26)$$

which is possible due to (23). It follows from (25) and the first inequality in (26) that $M_\nu < \theta^{\,\nu_1 \cdots \nu_d}$ for all but finitely many $\nu \in \mathcal{T}$. This and the second inequality in (26), in view of Theorem 3, guarantee that the equation $(\Delta + v)u = \lambda u$ does not have nontrivial l^1 solutions. □

Remark 1. By using, instead of (18), a better estimate of $\sigma_\tau(k)$, we can relax the requirement for the accuracy of periodic approximation in Theorem 3 by replacing the interval of possible values of θ, given by (11), with a larger interval $(0, \theta_0)$, where θ_0 can be found as follows. Let R be the radius of the smallest disk containing the set $v(\mathbf{Z}^d)$; then we put

$$A = 4d + 2R - 2, \quad B = (A + \sqrt{A^2 + 4})/2,$$

and

$$\theta_0 = B^{-2d}.$$

Similarly, the interval of possible values of η in Theorem 4, given by (23), can be replaced with a larger interval $(0, \eta_0)$, where $\eta_0 = \theta_0^{1/\beta}$. Here θ_0 is calculated in the way just described, R being the radius of the smallest disk containing the set $V(\mathbf{R}^d)$.

Remark 2. The main results of the paper – Theorems 2, 3 and 4 – pertain to the lattice Schrödinger operator; however, the method we use can be adapted to other finite-range lattice operators, such as $\widetilde{\Delta} + v$, where $\widetilde{\Delta}$ is the diagonal Laplacian introduced in [5]:

$$(\widetilde{\Delta} u)(x) = \sum_{z \in \{-1,+1\}^d} u(x+z), \quad x \in \mathbf{Z}^d.$$

Appendix

Proof of Lemma 2. In this proof we will use the following notation: given two integers a and b ($a \leq b$), we will denote by $[\![a, b]\!]$ the finite set

$$\{x \in \mathbf{Z}\colon a \leq x \leq b\}.$$

We may assume that the set Q is nonempty so $a_i \leq b_i$ for all $i \in [\![1, d]\!]$. It suffices to show how $u(\cdot)$, which is initially defined on Q and satisfies the equation

$$\Delta u(x) + v(x) u(x) = \lambda u(x) \tag{27}$$

on Q°, can be extended to

$$Q_1 = [\![a_1 - 1, b_1]\!] \times [\![a_2, b_2]\!] \times \cdots \times [\![a_d, b_d]\!]$$

so it will satisfy (27) on

$$Q_1^{\circ} = [\![a_1, b_1 - 1]\!] \times [\![a_2 + 1, b_2 - 1]\!] \times \cdots \times [\![a_d + 1, b_d - 1]\!].$$

Let

$$R = \{a_1\} \times [\![a_2, b_2]\!] \times \cdots \times [\![a_d, b_d]\!]$$

and

$$R_- = \{a_1 - 1\} \times [\![a_2, b_2]\!] \times \cdots \times [\![a_d, b_d]\!] \equiv Q_1 \setminus Q.$$

We need to define u on R_- so that (27) will hold on $Q_1^{\circ} \cap R$.

Two cases are possible.

(i) $a_1 = b_1$. In this case the set Q_1° is empty, so we can define $u|_{R_-}$ arbitrarily.

(ii) $a_1 < b_1$. Equation (27), being applied at all points of the set

$$Q_1^{\circ} \cap R = \{a_1\} \times [\![a_2 + 1, b_2 - 1]\!] \times \cdots \times [\![a_d + 1, b_d - 1]\!],$$

determines the values of u on the set

$$\{a_1 - 1\} \times [\![a_2 + 1, b_2 - 1]\!] \times \cdots \times [\![a_d + 1, b_d - 1]\!] \subset R_-;$$

the values of u at the remaining points of R_- can be chosen arbitrarily. □

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References

[1] J. Bourgain, Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbf{Z}^d$, d arbitrary. *Geom. Funct. Anal.* **17** (2007), no. 3, 682–706. MR 2346272 Zbl 1152.82311

[2] J. Bourgain, M. Goldstein, and W. Schlag, Anderson localization for Schrödinger operators on $\mathbf{Z}^2$ with quasi-periodic potential. *Acta Math.* **188** (2002), no. 1, 41–86. MR 1947458 Zbl 1022.47023

[3] A. Y. Gordon, The point spectrum of the one-dimensional Schrödinger operator. *Uspehi Mat. Nauk* **31** (1976), no. 4 (190), 257–258. In Russian. MR 0458247 Zbl 0342.34012

[4] A. Y. Gordon, Imperfectly grown periodic medium: absence of localized states. *J. Spectr. Theory.* **5** (2015), no. 2, 279–294.

[5] S. Molchanov and B. Vainberg, Scattering on the system of the sparse bumps: multidimensional case. *Appl. Anal.* **71** (1999), no. 1-4, 167–185. MR 1690097 Zbl 1022.47510

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