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Universal measurability and the Hochschild class of the Chern character

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Abstract. We study notions of measurability for singular traces, and characterise universal measurability for operators in Dixmier ideals. This measurability result is then applied to improve on the various proofs of Connes' identification of the Hochschild class of the Chern character of Dixmier summable spectral triples.

The measurability results show that the identification of the Hochschild class is independent of the choice of singular trace. As a corollary we obtain strong information on the asymptotics of the eigenvalues of operators naturally associated to spectral triples $(\mathcal{A}, H, D)$ and Hochschild cycles for $\mathcal{A}$.

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Contents

1. Introduction

In this article we exploit recent progress in the theory of singular traces to characterise operators in Dixmier ideals which are measurable with respect to wide classes of singular traces. The independence of the value obtained by applying a singular trace in one of these classes to a measurable operator places strong constraints on the asymptotics of the eigenvalues of such an operator.

We apply these measurability results to obtain improvements on Connes' Hochschild character theorem, [7, Theorem 8, IV.2.γ] and [1, 3, 10], identifying the Hochschild class of a (p,∞)-summable spectral triple, $p \in \mathbb{N}$. In particular we prove:

(i) Connes' result for arbitrary traces on $\mathcal{L}_{1,\infty}$ (other proofs hold only for the original trace discovered by Dixmier). This has interesting implications for the eigenvalues of the Hochschild cycles;

(ii) the analogue of this theorem for the (Macaev-Dixmier) ideal $\mathcal{M}_{1,\infty}$ as well as the p-convexifications $\mathcal{M}_{1,\infty}^{(p)}$ (introduced in [5], and denoted there by $\mathcal{Z}_p$). The latter ideal strictly contains $\mathcal{L}_{p,\infty}$. Our proof holds for a wide class of traces on $\mathcal{M}_{1,\infty}$, which we describe in the text.

The definition of (p,∞)-summability involves one of two ideals, denoted here by $\mathcal{L}_{1,\infty}$ and $\mathcal{M}_{1,\infty}$, or the related ideals $\mathcal{L}_{p,\infty}$ and $\mathcal{M}_{1,\infty}^{(p)}$. This is where potential confusion can arise, as well as much difficulty since the ideal $\mathcal{M}_{1,\infty}$ is more subtle than $\mathcal{L}_{1,\infty}$. The key technical improvement in this paper is the identification of a criterion guaranteeing measurability with respect to families of traces on these ideals.

As an indication of the improvements we have obtained, we state a consequence of our results which is applicable to numerous examples in the literature, including the case of Dirac operators on compact manifolds and the noncommutative torus.

Theorem. *Let $(\mathcal{A}, H, D)$ be a spectral triple with $(1+D^2)^{-1/2} \in \mathcal{L}_{p,\infty}$, where p is an integer of the same parity as the spectral triple. If the spectral triple is even we let Γ be the grading, and otherwise let $\Gamma = 1$. For every Hochschild cycle $c \in \mathcal{A}^{\otimes p+1}$, $c = \sum_i c_0^i \otimes c_1^i \otimes \cdots \otimes c_p^i$ set $\Omega(c) = \sum_i \Gamma c_0^i [D, c_1^i] \cdots [D, c_p^i]$. Then denoting the (suitably ordered)[1] eigenvalues of $\Omega(c)(1+D^2)^{-p/2}$ by λ_k we have*

$$\sum_{k=0}^{n} \lambda_k = \mathrm{Ch}(c)\log(n) + O(1),$$

where Ch *is the Chern character of the K-homology class of $(\mathcal{A}, H, D)$. In particular, $\Omega(c)(1+D^2)^{-p/2}$ is universally measurable in the sense of Definition 3.*

[1] The eigenvalues are counted with algebraic multiplicities and arranged so that their absolute values are non-increasing.

We remark that we began this investigation because there is a gap in the proof of Lemma 14 in [3] for the case $p = 1$ and the ideal denoted (and defined) below by $\mathcal{M}_{1,\infty}$. Rather than simply produce an erratum, we decided to revisit the whole argument in the light of progress made in the last 10 years [5, 6, 9, 12] which provides, amongst other contributions, a more powerful algebraic approach.

Moreover we make an interesting technical innovation in this current approach by exploiting recently discovered connections[2] between Dixmier traces and heat kernel functionals exposed in [19]. These connections result in a streamlining of the proof and a major reduction in the number of estimates needed (compared to the proof in [3]).

Our results are presented in the context of operator ideals in $\mathcal{L}(H)$ for a separable infinite dimensional Hilbert space. All of our results carry over to the general case of operator ideals of a semifinite von Neumann algebra although we do not present the argument in that generality here. We have simplified our approach, compared to [3], by assuming that our spectral triples (introduced in Section 3) are smooth, however, by taking more care in Lemma 30 we can recover the minimal smoothness requirements of [3, Lemma 2].

The necessary background on operator ideals, traces and measurability is presented in Section 2, and a key abstract measurability criterion is established in Subsection 2.4. Section 3 summarises what we need about spectral triples, Chern characters and Hochschild cohomology. We state our main results, Theorem 16 and Corollary 17 together with an outline of the proof in Subsection 3.2. Section 4 presents the proofs. An appendix shows how certain Hochschild coboundaries are computed.

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2. Preliminaries on operator ideals, traces and measurability

2.1. General notation. Fix throughout a separable infinite dimensional Hilbert space H. We let $\mathcal{L}(H)$ denote the algebra of all bounded operators on H. For a compact operator T on H, let $\lambda(k, T)$ and $\mu(k, T)$ denote its k-th eigenvalue[1] and k-th largest singular value (these are the eigenvalues of $|T|$). The sequence

[2] For a detailed exposition of the connections, we refer the reader to [14].

$\mu(T) = \{\mu(k,T)\}_{k\geq 0}$ is referred to as the singular value sequence of the operator T. The standard trace on $\mathcal{L}(H)$ is denoted by Tr. For an arbitrary operator $0 \leq T \in \mathcal{L}(H)$, we set

$$n_T(t) := \mathrm{Tr}(E_T(t,\infty)), \quad t > 0,$$

where $E_T(a,b)$ stands for the spectral projection of a self-adjoint operator T corresponding to the interval (a,b). Fix an orthonormal basis in H (the particular choice of a basis is inessential). We identify the algebra l_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence $\alpha \in l_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

2.2. Principal ideals $\mathcal{L}_{p,\infty}$ and the Macaev-Dixmier ideal $\mathcal{M}_{1,\infty}$. For a given $0 < p \leq \infty$, we let $\mathcal{L}_{p,\infty}$ denote the principal ideal in $\mathcal{L}(H)$ generated by the operator $\mathrm{diag}(\{(k+1)^{-1/p}\}_{k\geq 0})$. Equivalently,

$$\mathcal{L}_{p,\infty} = \{T \in \mathcal{L}(H) \colon \mu(k,T) = O((k+1)^{-1/p})\}.$$

These ideals, for different p, all admit an equivalent description in terms of spectral projections, namely

$$T \in \mathcal{L}_{p,\infty} \iff \mathrm{Tr}(E_{|T|}(1/n,\infty)) = O(n^p). \tag{1}$$

We also have

$$|T|^p \in \mathcal{L}_{1,\infty} \iff \mu^p(k,T) = O((k+1)^{-1}) \iff T \in \mathcal{L}_{p,\infty}.$$

We equip the ideal $\mathcal{L}_{p,\infty}$, $0 < p \leq \infty$, with a quasi-norm[3]

$$\|T\|_{p,\infty} = \sup_{k\geq 0}(k+1)^{1/p}\mu(k,T), \quad T \in \mathcal{L}_{p,\infty}.$$

The following Hölder property is widely used throughout the paper:

$$A_m \in \mathcal{L}_{p_m,\infty}, \quad 1 \leq m \leq n, \implies \prod_{m=1}^n A_m \in \mathcal{L}_{p,\infty}, \quad \frac{1}{p} = \sum_{m=1}^n \frac{1}{p_m}. \tag{2}$$

We also need the Macaev-Dixmier ideal $\mathcal{M}_{1,\infty}$, also known as a Lorentz space, given by

$$\mathcal{M}_{1,\infty} = \Big\{A \in \mathcal{L}(H) \colon \sup_{n\geq 0} \frac{1}{\log(2+n)} \sum_{k=0}^n \mu(k,A) < \infty\Big\}.$$

[3] A quasinorm satisfies the norm axioms, except that the triangle inequality is replaced by $||x+y|| \leq K(||x|| + ||y||)$ for some uniform constant $K > 1$.

The ideal $\mathcal{M}_{1,\infty}^{(p)}$ initially considered in [5] is the p-convexification of $\mathcal{M}_{1,\infty}$ defined as follows.

$$\mathcal{M}_{1,\infty}^{(p)} = \{A \in \mathcal{L}(H) \colon |A|^p \in \mathcal{M}_{1,\infty}\}.$$

The ideal $\mathcal{M}_{1,\infty}^{(p)}$ strictly contains $\mathcal{L}_{p,\infty}$. We refer the reader to the book [14] for a detailed discussion of the ideals $\mathcal{L}_{1,\infty}$ and $\mathcal{M}_{1,\infty}$.

2.3. Traces on $\mathcal{L}_{1,\infty}$.

Definition 1. If $\mathcal{I}$ is an ideal in $\mathcal{L}(H)$, then a unitarily invariant linear functional $\varphi\colon \mathcal{I} \to \mathbb{C}$ is said to be a trace.

Since $U^{-1}TU - T = [U^{-1}, TU]$ for all $T \in \mathcal{I}$ and for all unitaries $U \in \mathcal{L}(H)$, and since the unitaries span $\mathcal{L}(H)$, it follows that traces are precisely the linear functionals on $\mathcal{I}$ satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{I}, S \in \mathcal{L}(H).$$

The latter may be reinterpreted as the vanishing of the linear functional φ on the commutator subspace[4] which is denoted $[\mathcal{I}, \mathcal{L}(H)]$ and defined to be the linear span of all commutators $[T, S]\colon T \in \mathcal{I}, S \in \mathcal{L}(H)$. It is shown in [14, Lemma 5.2.2] that $\varphi(T_1) = \varphi(T_2)$ whenever $0 \leq T_1, T_2 \in \mathcal{I}$ are such that the singular value sequences $\mu(T_1)$ and $\mu(T_2)$ coincide. For $p > 1$, the ideal $\mathcal{L}_{p,\infty}$ does not admit a non-zero trace while for $p = 1$, there exists a plethora of traces on $\mathcal{L}_{1,\infty}$ (see e.g. [9] or [14]). An example of a trace on $\mathcal{L}_{1,\infty}$ is the restriction (from $\mathcal{M}_{1,\infty}$) of the Dixmier trace introduced in [8] that we now explain.

Definition 2. The dilation semigroup on l_∞ is defined by setting

$$\sigma_k(x_0, x_1, \cdots) = (\underbrace{x_0, \cdots, x_0}_{k \text{ times}}, \underbrace{x_1, \cdots, x_1}_{k \text{ times}}, \cdots)$$

for every $k \geq 1$. In this paper *a dilation invariant extended limit* means a state on the algebra l_∞ invariant under σ_k, $k = 2, 3, \ldots$.

[4] The commutator subspace of the ideal is, in general, not an ideal in $\mathcal{L}(H)$. For example, it follows from Theorem 5 below that

$$\operatorname{diag}\Big(\Big\{\frac{(-1)^k}{k+1}\Big\}_{k\geq 0}\Big) \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)], \quad \operatorname{diag}\Big(\Big\{\frac{1}{k+1}\Big\}_{k\geq 0}\Big) \notin [\mathcal{L}_{1,\infty}, \mathcal{L}(H)].$$

However, the commutator subspace of the ideal $\mathcal{L}_{1,\infty}$ is an ideal in $\mathcal{L}_{1,\infty}$ (as opposed to $\mathcal{L}(H)$). We refer the reader to [15] for the study of such subideals.

Example. Let ω be a dilation invariant extended limit. Then the functional

$$\mathrm{Tr}_\omega : \mathcal{M}^+_{1,\infty} \longrightarrow \mathbb{C}$$

defined by setting

$$\mathrm{Tr}_\omega(A) = \omega\Big(\Big\{\frac{1}{\log(2+n)}\sum_{k=0}^{n}\mu(k,A)\Big\}_{n\geq 0}\Big), \quad 0 \leq A \in \mathcal{M}_{1,\infty},$$

is additive and, therefore, extends to a trace on $\mathcal{M}_{1,\infty}$. We call such traces *Dixmier traces*. These traces clearly depend on the choice of the functional ω on l_∞. Using a slightly different definition, this notion of trace was applied by Connes [7] in noncommutative geometry. We also remark that the assumption used by Dixmier of translation invariance for the functional ω is redundant [14, Theorem 6.3.6]. An extensive discussion of traces, and more recent developments in the theory, may be found in [14] including a discussion of the following facts.

(1) All Dixmier traces on $\mathcal{L}_{1,\infty}$ are positive.

(2) All positive traces on $\mathcal{L}_{1,\infty}$ are continuous in the quasi-norm topology.

(3) There exist positive traces on $\mathcal{L}_{1,\infty}$ which are not (restrictions to $\mathcal{L}_{1,\infty}$ from $\mathcal{M}_{1,\infty}$ of) Dixmier traces (see [18]).

(4) There exist traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous (see [9]).

We are mostly interested in *normalised traces* $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$, that is, satisfying $\varphi(T) = 1$ whenever $0 \leq T$ is such that $\mu(k,T) = 1/(k+1)$ for all $k \geq 0$. We do not require continuity of a normalised trace.

The following definition, extending that originally introduced in [7], plays an important role here.

Definition 3. An operator $T \in \mathcal{L}_{1,\infty}$ will be said to be universally measurable if all normalised traces take the same value on T.

The following lemma characterises the universally measurable operators.

Lemma 4. *All normalised traces on $\mathcal{L}_{1,\infty}$ take the value $z \in \mathbb{C}$ on the operator T if and only if*

$$T - z\cdot \mathrm{diag}\Big(\Big\{\frac{1}{k+1}\Big\}_{k\geq 0}\Big) \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)].$$

Proof. Suppose that all normalised traces on $\mathcal{L}_{1,\infty}$ take the value z on the operator T. For brevity we write $T_0 = \mathrm{diag}(\{\frac{1}{k+1}\}_{k\geq 0})$. If $T - zT_0$ is not in the commutator subspace, then it follows from Zorn's lemma that there exists a linear functional φ on $\mathcal{L}_{1,\infty}$ such that $\varphi|_{[\mathcal{L}_{1,\infty},\mathcal{L}(H)]} = 0$ and such that $\varphi(T - zT_0) = 1$. By Definition 1, φ is a trace. Fix a normalised trace φ_0. The normalised trace $\varphi + (1 - \varphi(T_0))\varphi_0$ takes the value $z + 1$ at T, which contradicts the assumption. This proves that $T - zT_0 \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$. The converse assertion follows from the definitions. □

The description of the commutator subspace initially appeared in [11] in a very general situation. The statement below appeared first in [12] and for a detailed proof we refer the reader to Theorem 5.7.6 and Theorem 10.1.3 in [14].

Proposition 5. *An operator $T \in \mathcal{L}_{1,\infty}$ is universally measurable if and only if*

$$\sum_{k=0}^{n} \lambda(k, T) = z\log(n + 1) + O(1), \quad n \geq 0,$$

for some constant $z \in \mathbb{C}$. In this case, $\varphi(T) = z$ for every normalised trace φ. In particular

$$T \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)] \iff \sum_{k=0}^{n} \lambda(k, T) = O(1), \quad n \geq 0.$$

2.4. A universal measurability result. In this subsection, we prove a measurability criterion for operators of the form AV, $A \in \mathcal{L}(H)$, $V \in \mathcal{L}_{1,\infty}$, or $V \in \mathcal{M}_{1,\infty}$. This result links measurability with the heat semigroup, thus significantly improving the main result of [6]. More information on these links can be found in [14] (see also [19]). The precise statement of our measurability criterion is as follows.

Proposition 6. *Let $0 \leq V \in \mathcal{L}(H)$, $A \in \mathcal{L}(H)$ and $\alpha > 1$ be such that*

$$\mathrm{Tr}(AVe^{-(nV)^{-\alpha}}) = z\log(n) + O(1), \quad n \to \infty. \tag{3}$$

(a) *If $V \in \mathcal{L}_{1,\infty}$, then $\varphi(AV) = z$ for every normalised trace φ on $\mathcal{L}_{1,\infty}$.*

(b) *If $V \in \mathcal{M}_{1,\infty}$, then $\mathrm{Tr}_\omega(AV) = z$ for every Dixmier trace Tr_ω on $\mathcal{M}_{1,\infty}$.*

We require several Lemmas before presenting the proof of Proposition 6.

Lemma 7. *If* $0 \leq V \in \mathcal{L}_{1,\infty}$, *then, for every* $\alpha > 1$, *we have*

$$\mathrm{Tr}(V^\alpha(1 - e^{-(nV)^{-\alpha}})) = O(n^{1-\alpha}), \quad \mathrm{Tr}(e^{-(nV)^{-\alpha}})) = O(n), \quad n \to \infty.$$

Proof. By the assumption, we have $\mu(k, V) \leq \|V\|_{1,\infty}/(k+1)$ for all $k \geq 0$. Select $W \geq V$ (with the same eigenbasis) such that $\mu(k, W) = \|V\|_{1,\infty}/(k+1)$ for all $k \geq 0$. An elementary computation shows that the mapping

$$g \colon x \longrightarrow x(1 - e^{-x^{-1}}), \quad x \geq 0,$$

is increasing. Since V and W commute, $(nV)^\alpha \leq (nW)^\alpha$ for all $n \geq 1$ and it follows that $g((nV)^\alpha) \leq g((nW)^\alpha)$. Therefore,

$$\begin{aligned}
\mathrm{Tr}(V^\alpha(1 - e^{-(nV)^{-\alpha}})) &= n^{-\alpha}\mathrm{Tr}(g((nV)^\alpha)) \leq n^{-\alpha}\mathrm{Tr}(g((nW)^\alpha)) \\
&= \|V\|_{1,\infty}^\alpha \sum_{k=1}^{\infty} k^{-\alpha}(1 - \exp(-(n\|V\|_{1,\infty})^{-\alpha}k^\alpha)) \\
&\leq \|V\|_{1,\infty}^\alpha \int_0^\infty s^{-\alpha}(1 - \exp(-(n\|V\|_{1,\infty})^{-\alpha}s^\alpha))ds \\
&= n^{1-\alpha}\|V\|_{1,\infty} \int_0^\infty u^{-\alpha}(1 - \exp(-u^\alpha))du.
\end{aligned}$$

Here, in the last step we used the substitution $s = n\|V\|_{1,\infty}u$. This proves the first equality.

The second equality is proved as follows:

$$\begin{aligned}
\mathrm{Tr}(e^{-(nV)^{-\alpha}})) &\leq \mathrm{Tr}(e^{-(nW)^{-\alpha}})) \\
&= \sum_{k=1}^{\infty} \exp(-(n\|V\|_{1,\infty})^{-\alpha}k^\alpha) \\
&\leq \int_0^\infty \exp(-(n\|V\|_{1,\infty})^{-\alpha}s^\alpha)ds \\
&= n\|V\|_{1,\infty} \int_0^\infty \exp(-u^\alpha)du.
\end{aligned}$$

In the last step we again used the substitution $s = n\|V\|_{1,\infty}u$. □

Lemma 8. *If* $0 \le V \in \mathcal{L}_{1,\infty}$ *and if* $A \in \mathcal{L}(H)$, *then*

$$\sum_{k=0}^{n} \lambda(k, AV) = \mathrm{Tr}\Big(AVE_V\Big[\frac{1}{n}, \infty\Big)\Big) + O(1), \quad n \to \infty.$$

Proof. Recall that a Hilbert-Schmidt operator W is said to be V-modulated (in the sense of [14, Definition 11.2.1]) if

$$\sup_{t>0} t^{1/2} \| W(1 + tV)^{-1} \|_2 < \infty.$$

We show that the operator AV is V-modulated. Indeed, we have

$$\begin{aligned} \sup_{t>0} t^{1/2} \| AV(1 + tV)^{-1} \|_2 &\le \|A\|_\infty \sup_{t>0} t^{1/2} \| V(1 + tV)^{-1} \|_2 \\ &\le \|A\|_\infty \sup_{t>0} t^{1/2} \Big\| \Big\{ \frac{1}{\mu(k, V)^{-1} + t} \Big\}_{k \ge 0} \Big\|_2 \\ &< \infty. \end{aligned}$$

Let e_k, $k \ge 0$, be an eigenbasis of V. Since AV is V-modulated and since $V \ge 0$, it follows from Theorem 11.2.3 in [14] that

$$\sum_{k=0}^{n} \lambda(k, AV) = \sum_{k=0}^{n} \mu(k, V) \langle Ae_k, e_k \rangle + O(1).$$

By definition, $E_V[\frac{1}{n}, \infty)$ is the projection onto e_k, $0 \le k \le m(n)$, where $m(n) = \mathrm{Tr}(E_V[1/n, \infty))$. Since $V \in \mathcal{L}_{1,\infty}$, we have $\mu(k, V) \le \frac{C}{k+1}$ for some constant $C > 0$ and all $k \ge 0$. This inequality guarantees that $m(n) = O(n)$ as $n \to \infty$, by equation (1), in particular, there is a constant $C < \infty$ such that $m(n) \le Cn$, for all $n \ge 1$. It may also happen that $m(n) < n$.

If $m(n) < n$, then (since $\mu(k, V) < \frac{1}{n}$ for $k > m(n)$), we have

$$\sum_{k=m(n)+1}^{n} \mu(k, V) \le \sum_{k=m(n)+1}^{n} \frac{1}{n} \le 1.$$

If $m(n) \ge n$, then

$$\sum_{k=n+1}^{m(n)} \mu(k, V) \le \sum_{k=n+1}^{Cn} \mu(k, V) \le \|V\|_{1,\infty} \sum_{k=n+1}^{Cn} \frac{1}{k} = O(1).$$

In either case, we have

$$\Big| \sum_{k=0}^{n} \mu(k, V) - \sum_{k=0}^{m(n)} \mu(k, V) \Big| = O(1).$$

With these observations, we have the equality

$$\operatorname{Tr}\Big(AVE_V\Big[\frac{1}{n},\infty\Big)\Big)=\sum_{k=0}^{m(n)}\mu(k,V)\langle Ae_k,e_k\rangle.$$

It follows that

$$\begin{aligned}
&\Big|\sum_{k=0}^{n}\lambda(k,AV)-\operatorname{Tr}\Big(AVE_V\Big[\frac{1}{n},\infty\Big)\Big)\Big|\\
&\quad=\Big|\sum_{k=0}^{n}\mu(k,V)\langle Ae_k,e_k\rangle+O(1)-\sum_{k=0}^{m(n)}\mu(k,V)\langle Ae_k,e_k\rangle\Big|\\
&\quad\le\|A\|_\infty\Big|\sum_{k=0}^{n}\mu(k,V)-\sum_{k=0}^{m(n)}\mu(k,V)\Big|+O(1)\\
&\quad=O(1).
\end{aligned}$$

□

The above Lemmas allow us to prove the first statement of Proposition 6.

Proof of Proposition 6 (a). We start by showing that

$$|\operatorname{Tr}(AVe^{-(nV)^{-\alpha}})-\operatorname{Tr}(AVE_V[1/n,\infty))=O(1),\quad n\to\infty. \tag{4}$$

Indeed,

$$\begin{aligned}
&\Big|\operatorname{Tr}\Big(AVe^{-(nV)^{-\alpha}}\Big)-\operatorname{Tr}(AVE_V[1/n,\infty))\Big|\\
&\quad\le\Big|\operatorname{Tr}(AV(e^{-(nV)^{-\alpha}}-1)E_V[1/n,\infty))\Big|\\
&\qquad+\Big|\operatorname{Tr}(AVe^{-(nV)^{-\alpha}}E_V[0,1/n))\Big|\\
&\quad\le\|A\|_\infty\Big(\Big|\operatorname{Tr}(V(e^{-(nV)^{-\alpha}}-1)E_V[1/n,\infty))\Big|\\
&\qquad+\Big|\operatorname{Tr}(Ve^{-(nV)^{-\alpha}}E_V[0,1/n))\Big|\Big).
\end{aligned}$$

In order to complete the proof, we observe that the spectral theorem yields

$$VE_V[0,1/n)\le 1/n.$$

Similarly, for any $\alpha>1$ we have the inequality

$$\lambda\chi_{(1/n,\infty)}(\lambda)\le n^{\alpha-1}\lambda^\alpha,$$

where $\chi_{(1/n,\infty)}$ is the indicator function of the interval $(1/n,\infty)$, and so the spectral theorem yields $VE_V[1/n,\infty)\leq n^{\alpha-1}V^\alpha$.

It now follows that

$$\begin{aligned}&|\mathrm{Tr}(AVe^{-(nV)^{-\alpha}})-\mathrm{Tr}(AVE_V[1/n,\infty))|\\&\leq\|A\|_\infty\Big(n^{\alpha-1}\mathrm{Tr}(V^\alpha(1-e^{-(nV)^{-\alpha}}))+\frac{1}{n}\mathrm{Tr}(e^{-(nV)^{-\alpha}})\Big)\\&=O(1).\end{aligned}$$

Here, the last equality holds by Lemma 7. Appealing to the assumption (3) and Lemma 8, we rewrite the preceding inequality as

$$\sum_{k=0}^n\lambda(k,AV)=z\log(n)+O(1)$$

and conclude using Proposition 5. □

To prove the second part of Proposition 6, we need the following lemmas.

Lemma 9. *Let ω be a dilation invariant extended limit on l_∞. For every $0\leq V\in\mathcal{M}_{1,\infty}$ and $\alpha>1$, we have*

$$\omega\Big(\Big\{\frac{1}{n\log(n)}\mathrm{Tr}(e^{-(nV)^{-\alpha}})\Big\}_{n\geq0}\Big)=0.$$

Proof. Fix $\varepsilon\in[0,1]$ and observe that

$$e^{-t^{-\alpha}}\leq4\varepsilon t^2,\quad 0\leq t\leq\varepsilon.$$

Hence, for every $t>0$, we have

$$e^{-(nt)^{-\alpha}}\leq\chi_{(\frac{\varepsilon}{n},\infty)}(t)+4\varepsilon(nt)^2\chi_{[0,\frac{\varepsilon}{n}]}(t)\leq\chi_{(\frac{\varepsilon}{n},\infty)}(t)+4\varepsilon(\min\{nt,1\})^2.$$

Applying the functional calculus, we infer from the inequality above that

$$e^{-(nV)^{-\alpha}}\leq E_V(\varepsilon/n,\infty)+4\varepsilon\min\{(nV),1\}^2.$$

Hence, using the fact that ω is a positive functional, we obtain

$$\begin{aligned}&\omega\Big(\Big\{\frac{1}{n\log(n)}\mathrm{Tr}(e^{-(nV)^{-\alpha}})\Big\}_{n\geq0}\Big)\\&\leq\omega\Big(\Big\{\frac{n_V(\frac{\varepsilon}{n})}{n\log(n)}\Big\}_{n\geq0}\Big)+4\varepsilon\omega\Big(\Big\{\frac{\min\{(nV),1\}^2}{n\log(n)}\Big\}_{n\geq0}\Big).\end{aligned}$$

Here, the second term is well defined thanks to Lemma 8.4.2 (b) in [14]. By Lemma 8.2.8 in [14], the first term vanishes for every $\varepsilon>0$. Letting $\varepsilon\to0$, we conclude the proof. □

Lemma 10. *Let ω be a dilation invariant extended limit on l_∞, $\alpha > 1$, and introduce the notation T_+ for the positive part of a self adjoint operator T. For every $A \in \mathcal{L}(H)$ and for every $0 \leq V \in \mathcal{M}_{1,\infty}$, we have*

$$\omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}(AVe^{-(nV)^{-\alpha}})\Big\}_{n\geq 0}\Big) = \omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}(A(V-1/n)_+)\Big\}_{n\geq 0}\Big).$$

Proof. Without loss of generality, the operator A is positive. Fix $\varepsilon > 0$. Applying the functional calculus to the numerical inequality

$$e^{-\varepsilon^\alpha}(t-1/\varepsilon n)_+ \leq te^{-(nt)^{-\alpha}} \leq (t-1/n)_+ + \frac{1}{n}\chi_{[\frac{1}{n},\infty)}(t) + \frac{1}{n}e^{-(nt)^\alpha},$$

(the subscripted $+$ again denotes the positive part) we obtain an inequality involving trace class operators

$$e^{-\varepsilon^\alpha}(V-1/\varepsilon n)_+ \leq Ve^{-(nV)^{-\alpha}} \leq (V-1/n)_+ + \frac{1}{n}E_V[1/n,\infty) + \frac{1}{n}e^{-(nV)^\alpha}. \quad (5)$$

For any trace class operator T, cyclicity of the trace gives

$$\mathrm{Tr}(A^{1/2}TA^{1/2}) = \mathrm{Tr}(AT).$$

We apply this observation to the second inequality in (5) to infer that

$$\mathrm{Tr}(AVe^{-(nV)^{-\alpha}}) \leq \mathrm{Tr}\,(A(V-1/n)_+) + \frac{\|A\|_\infty}{n}n_V(1/n) + \frac{\|A\|_\infty}{n}\mathrm{Tr}(e^{-(nV)^\alpha}).$$

It follows from Lemma 8.2.8 in [14] and Lemma 9 that

$$\omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}(AVe^{-(nV)^{-\alpha}})\Big\}_{n\geq 0}\Big) \leq \omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}\,(A(V-1/n)_+)\Big\}_{n\geq 0}\Big). \quad (6)$$

Now we apply $\mathrm{Tr}(A^{1/2}TA^{1/2}) = \mathrm{Tr}(AT)$ to the first inequality in (5) to insert a positive operator A under the trace. So we infer that

$$\omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}(AVe^{-(nV)^{-\alpha}})\Big\}_{n\geq 0}\Big) \geq e^{-\varepsilon^\alpha}\omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}\,(A(V-1/n\varepsilon)_+)\Big\}_{n\geq 0}\Big).$$

Taking into account that ω is dilation invariant and passing to the limit $\varepsilon \to 0$, we infer that

$$\omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}(AVe^{-(nV)^{-\alpha}})\Big\}_{n\geq 0}\Big) \geq \omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}\,(A(V-1/n)_+)\Big\}_{n\geq 0}\Big). \quad (7)$$

The assertion follows by combining (6) and (7). □

We can now complete the proof of Proposition 6.

Proof of Proposition 6 (b). For every dilation invariant extended limit ω on l_∞, we define a heat semigroup functional

$$\xi_\omega \colon W \longrightarrow (\omega \circ M)\Big(\Big\{\frac{1}{n}\mathrm{Tr}(e^{-(nW)^{-1}})\Big\}_{n\geq 0}\Big), \quad 0 \leq W \in \mathcal{M}_{1,\infty}.$$

By Theorem 8.2.5 in [14], the functional ξ_ω extends to a Dixmier trace on $\mathcal{M}_{1,\infty}$. For every dilation invariant extended limit ω, we infer from Lemma 10 that

$$\omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}\Big(A(V-\frac{1}{n})_+\Big)\Big\}_{n\geq 0}\Big) = \omega\Big(\Big\{\frac{1}{\log(n)}\mathrm{Tr}(AVe^{-(nV)^{-\alpha}})\Big\}_{n\geq 0}\Big) = z.$$

Then, by Lemma 8.5.3 in [14], we have $\xi_\omega(AV) = z$ for every dilation invariant extended limit ω. Finally, by Theorem 8.3.6 in [14], the set of all Dixmier traces coincides with the set of all functionals ξ_ω, where ω runs through all dilation invariant extended limits on l_∞. The assertion follows immediately. □

3. Preliminaries on noncommutative geometry and the statements of the main results

3.1. Spectral triples and Hochschild (co)homology. Let $D\colon \mathrm{dom}(D) \to H$ be a self-adjoint operator with $\mathrm{dom}(D) \subset H$ a dense linear subspace. An operator D admits a polar decomposition $D = F|D|$, where the phase F is a self-adjoint unitary operator defined by

$$F := E_D([0,\infty)) - E_D(-\infty, 0)$$

and

$$|D|\colon \mathrm{dom}(D) \longrightarrow H$$

is a self-adjoint operator. The following definitions should be compared with Definition 1.20 in [2].

Definition 11. A spectral triple $(\mathcal{A}, H, D)$ consists of a subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ such that

(a) $a\colon \mathrm{dom}(D) \to \mathrm{dom}(D)$ for all $a \in \mathcal{A}$;

(b) $[D,a]\colon \mathrm{dom}(D) \to H$ extends to an operator $\partial(a) \in \mathcal{L}(H)$ for all $a \in \mathcal{A}$;

(c) $a(1+D^2)^{-1/2}$ is a compact operator for all $a \in \mathcal{A}$.

In what follows, if $a\colon \mathrm{dom}(D) \to \mathrm{dom}(D)$, then the (a priori unbounded) operator $[|D|, a]\colon \mathrm{dom}(D) \to H$ is denoted by $\delta(a)$.

Definition 12. A spectral triple is QC^∞ if

(a) $a\colon \mathrm{dom}(D^n) \to \mathrm{dom}(D^n)$ for all $a \in \mathcal{A}$ and

(b) for all $n \geq 0$ the operators

$$\delta^n(a)\colon \mathrm{dom}(D^n) \longrightarrow H, \quad \delta^n(\partial(a))\colon \mathrm{dom}(D^{n+1}) \longrightarrow H$$

extend to bounded operators for all $n \geq 0$ and for all $a \in \mathcal{A}$.

Definition 13. A spectral triple is said to be

(a) even if there exists $\Gamma \in \mathcal{L}(H)$ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$ and such that $[\Gamma, a] = 0$ for all $a \in \mathcal{A}$, $\{D, \Gamma\} = 0$. Here $\{\cdot,\cdot\}$ denotes anticommutator.

(b) odd if no such Γ exists. In this case, we set $\Gamma = 1$.

(c) (p,∞)-summable if $(1+D^2)^{-p/2} \in \mathcal{L}_{1,\infty}$.

(d) $\mathcal{M}^{(p)}_{1,\infty}$-summable if $(1+D^2)^{-p/2} \in \mathcal{M}_{1,\infty}$.

The following assertion is proved in many places, e.g. [4, Corollary 0.5], [3], and [16]. We prove a related statement in Lemma 30.

Proposition 14. *If $(\mathcal{A}, H, D)$ is a spectral triple that is QC^∞ and (p,∞)-summable, then $[F,a]$ and $[F,\delta^k(a)]$ lie in $\mathcal{L}_{p,\infty}$ for all $a \in \mathcal{A}$ and $k \geq 1$.*

Define multilinear mappings

$$\mathrm{ch}\colon \mathcal{A}^{\otimes(p+1)} \longrightarrow \mathcal{L}(H)$$

and

$$\Omega\colon \mathcal{A}^{\otimes(p+1)} \longrightarrow \mathcal{A}$$

by setting

$$\mathrm{ch}(a_0 \otimes \cdots \otimes a_p) = F\Gamma \prod_{k=0}^{p} [F, a_k],$$

and

$$\Omega(a_0 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=1}^{p} [D, a_k].$$

If a spectral triple $(\mathcal{A}, H, D)$ is (p,∞)-summable, then it follows from Proposition 14 and the Hölder inequality in equation (2) that $\mathrm{ch}(c) \in \mathcal{L}_{p/(p+1),\infty} \subset \mathcal{L}_1$ for all $c \in \mathcal{A}^{\otimes(p+1)}$. This justifies the following definition.

Definition 15. If $(\mathcal{A}, H, D)$ is a (p,∞)-summable spectral triple, then Connes' Chern character $\mathcal{A}^{\otimes(p+1)} \to \mathbb{C}$ is 'is defined (up to a constant factor) by setting

$$\mathrm{Ch}(c) = \frac{1}{2}\mathrm{Tr}(\mathrm{ch}(c)), \quad c \in \mathcal{A}^{\otimes(p+1)}.$$

In fact the Chern character is the class of Ch in periodic cyclic cohomology, but we shall ignore this distinction in the sequel.

We now turn to Hochschild (co)homology. The algebra $\mathcal{A}$ is equipped with the δ-topology, [17], determined by the seminorms

$$q_n : \mathcal{A} \longrightarrow [0,\infty)$$

given by

$$q_n(a) = \sum_{k=0}^{n} \|\delta^k(a)\| + \|\delta^k([D,a])\|.$$

The tensor powers of $\mathcal{A}$ are completed in the projective tensor product topology. If $\theta : \mathcal{A}^{\otimes n} \to \mathbb{C}$ is a continuous multilinear functional, then the multilinear functional

$$b\theta : \mathcal{A}^{\otimes(n+1)} \longrightarrow \mathbb{C}$$

is defined by

$$\begin{aligned}(b\theta)&(a_0 \otimes \cdots \otimes a_n)\\ &= \theta(a_0a_1 \otimes a_2 \otimes \cdots \otimes a_n)\\ &\quad + \sum_{k=1}^{n-1}(-1)^k\theta(a_0 \otimes a_1 \otimes \cdots a_{k-1} \otimes a_ka_{k+1} \otimes a_{k+1} \otimes \cdots \otimes a_n)\\ &\quad + (-1)^n\theta(a_na_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}).\end{aligned}$$

We call $b\theta$ the Hochschild coboundary of θ. If $b\theta = 0$, then we call θ a Hochschild cocycle. We also need the dual notion of Hochschild cycle. The Hochschild boundary

$$b : \mathcal{A}^{\otimes(n+1)} \longrightarrow \mathcal{A}^{\otimes n}$$

is defined by setting

$$\begin{aligned}b(a_0 \otimes \cdots \otimes a_n) &= a_0a_1 \otimes a_2 \otimes \cdots \otimes a_n\\ &\quad + \sum_{k=1}^{n-1}(-1)^k a_0 \otimes a_1 \otimes \cdots a_{k-1} \otimes a_ka_{k+1} \otimes a_{k+1} \otimes \cdots \otimes a_n\\ &\quad + (-1)^n a_na_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}.\end{aligned}$$

If $c \in \mathcal{A}^{\otimes(n+1)}$ is such that $bc = 0$, then c is called a *Hochschild cycle*. For example, if $n = 1$, then $b(a_0 \otimes a_1) = [a_0, a_1]$. Hence, an elementary tensor $a_0 \otimes a_1$ is a Hochschild cycle if and only if a_0 and a_1 commute. The definitions are dual in the sense that for any multilinear functional θ, $(b\theta)(a) = \theta(ba)$. In particular, a Hochschild coboundary vanishes on every Hochschild cycle.

3.2. The main results and the plan of the proofs. For the statement of our main theorem, and the remainder of the paper, we assume that $p \in \mathbb{N}$.

Theorem 16. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple which is even or odd according to whether p is even or odd, and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle.*

(a) *If the spectral triple is (p,∞)-summable, then for every normalised trace φ on $\mathcal{L}_{1,\infty}$*
$$\varphi(\Omega(c)(1 + D^2)^{-p/2}) = \mathrm{Ch}(c). \tag{8}$$

(b) *If the spectral triple is $\mathcal{M}_{1,\infty}^{(p)}$-summable, then*
$$\mathrm{Tr}_\omega(\Omega(c)(1 + D^2)^{-p/2}) = \mathrm{Ch}(c)$$
for every Dixmier trace on $\mathcal{M}_{1,\infty}$.

Let us illustrate the assertion for $p = 1$. If elements $a_0, a_1 \in \mathcal{A}$ commute, then the elementary tensor $a_0 \otimes a_1$ is a Hochschild 1-cycle and
$$\varphi(a_0[D, a_1](1 + D^2)^{-1/2}) = \frac{1}{2}\mathrm{Tr}(F[F, a_0][F, a_1])$$
for every trace φ on $\mathcal{L}_{1,\infty}$. The corollary below follows from Theorem 16 and Proposition 5.

Corollary 17. *Suppose that the assumptions of Theorem 16 (a) hold. Suppose that $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle. Then*

(a) $\Omega(c)(1 + D^2)^{-p/2} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$ *if and only if* $\mathrm{Ch}(c) = 0$, *and more generally*
$$\Omega(c)(1 + D^2)^{-p/2} \in \mathrm{Ch}(c) \cdot \mathrm{diag}\Big(\Big\{\frac{1}{k+1}\Big\}_{k\geq 0}\Big) + [\mathcal{L}_{1,\infty}, \mathcal{L}(H)];$$

(b) *there is an equality*
$$\sum_{m=0}^{n} \lambda(m, \Omega(c)(1 + D^2)^{-p/2}) = \mathrm{Ch}(c)\log(n) + O(1), \quad n \geq 0.$$

Theorem 16 is initially proved under the assumption of invertibility of D in Subsection 4.4, after proving some intermediate steps. The first step is to replace $\Omega(c)|D|^{-p}$ by a new operator. More specifically, for $1 \leq m \leq p$, we define the multilinear mappings

$$\mathcal{W}_m : \mathcal{A}^{\otimes(p+1)} \longrightarrow \mathcal{L}(H)$$

by setting

$$\mathcal{W}_m(a_0 \otimes \cdots \otimes a_p) = \Gamma a_0 \Big(\prod_{k=1}^{m-1} [F, a_k] \Big) \delta(a_m) \Big(\prod_{k=m+1}^{p} [F, a_k] \Big). \tag{9}$$

By Proposition 14 and by the Hölder property in equation (2), $\mathcal{W}_m(c)D^{-1} \in \mathcal{L}_{1,p}$ (respectively, $\mathcal{W}_m(c)D^{-1} \in \mathcal{M}_{1,\infty}$). Then, by exploiting Hochschild cohomology (see Appendix A), we show in Subsection 4.1 that (for $D^{-1} \in \mathcal{L}_{1,\infty}$)

$$\Omega(c)|D|^{-p} - p\mathcal{W}_p(c)D^{-1} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)].$$

We prove the analogous result for $D^{-1} \in \mathcal{M}_{1,\infty}$ also. Then, in Subsection 4.2, we obtain a number of commutator estimates which allow us to prove, in Subsection 4.3, that for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$,

$$\operatorname{Tr}(\mathcal{W}_p(c)D^{-1}e^{-(s|D|)^{p+1}}) = \operatorname{Ch}(c) \log\big(1/s\big) + O(1), \quad s \to 0.$$

By invoking our abstract measurability criterion, Proposition 6, we can then assemble the pieces to prove the main result in Subsection 4.4. We also show at this point how to remove the invertibility assumption.

4. Proofs

Until Subsection 4.4, we will suppose that the operator D of a spectral triple $(\mathcal{A}, H, D)$ is invertible.

4.1. Exploiting Hochschild cohomology.

Our aim in this subsection is to prove the following result, by refining the approach of [3, Section 3.5].

Proposition 18. *Let $(\mathcal{A}, H, D)$ be an odd (respectively, even) QC^∞ spectral triple and let p be odd (respectively, even). For every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$,*

(a) *if $D^{-p} \in \mathcal{L}_{1,\infty}$, then $\Omega(c)|D|^{-p} - p\mathcal{W}_p(c)D^{-1} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$,*

(b) *if $D^{-p} \in \mathcal{M}_{1,\infty}$, then $\Omega(c)|D|^{-p} - p\mathcal{W}_p(c)D^{-1} \in [\mathcal{M}_{1,\infty}, \mathcal{L}(H)]$.*

We consider auxiliary multilinear mappings which generalise the mappings $\mathcal{W}_m$, $1 \leq m \leq p$, introduced above in equation (9). For $\mathcal{A} \subset \{1, \ldots, p\}$ define the multilinear mapping

$$\mathcal{W}_{\mathcal{A}} \colon \mathcal{A}^{\otimes(p+1)} \longrightarrow \mathcal{L}(H)$$

by setting

$$\mathcal{W}_{\mathcal{A}}(a_0 \otimes \cdots \otimes a_p) := \Gamma a_0 \prod_{k=1}^{p} [b_k, a_k], \quad a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)},$$

where $b_k = |D|$, for $k \in \mathcal{A}$, and $b_k = F$, for $k \notin \mathcal{A}$. Evidently, if $\mathcal{A} = \{m\}$, then $\mathcal{W}_{\mathcal{A}} = \mathcal{W}_m$. It follows from Proposition 14 and the Hölder property in equation (2) that

$$\mathcal{W}_{\mathcal{A}}(a) D^{-|\mathcal{A}|} \in \mathcal{L}_{1,\infty}, \quad \mathcal{A} \subset \{1, \ldots, p\}.$$

For every $\mathcal{A} \subset \{1, \ldots, p\}$, define the number

$$n_{\mathcal{A}} = |\{(i, j) \colon i < j,\ i \in \mathcal{A},\ j \notin \mathcal{A}\}|.$$

The following assertion explains the introduction of the mappings $\mathcal{W}_{\mathcal{A}}$, $\mathcal{A} \subset \{1, \ldots, p\}$ that are used for the proof of Proposition 18. We denote the cardinality of $\mathcal{A}$ by $|\mathcal{A}|$.

Lemma 19. *If $(\mathcal{A}, H, D)$ is QC^∞ spectral triple with $D^{-p} \in \mathcal{L}_{1,\infty}$, then for all $c \in \mathcal{A}^{\otimes(p+1)}$*

$$\Omega(c)|D|^{-p} - \sum_{\mathcal{A} \subset \{1,\ldots,p\}} (-1)^{n_{\mathcal{A}}} \mathcal{W}_{\mathcal{A}}(c) D^{-|\mathcal{A}|} \in \mathcal{L}_1.$$

Proof. We will proceed by proving that for $1 \leq q \leq p$ and $c = a_0 \otimes a_1 \otimes \cdots \otimes a_q$,

$$\Gamma a_0 [D, a_1] \cdots [D, a_q] |D|^{-q} = \sum_{\mathcal{A} \subset \{1,\ldots,q\}} (-1)^{n_{\mathcal{A}}} \mathcal{W}_{\mathcal{A}}(c) D^{-|\mathcal{A}|} \bmod \mathcal{L}_{p/(q+1),\infty}. \tag{10}$$

For $q = 1$, we consider $c = a_0 \otimes a_1 \in \mathcal{A}^{\otimes 2}$. We have

$$\begin{aligned}
[D, a_1] &= [F|D|, a_1] \\
&= F\delta(a_1) + [F, a_1]|D| \\
&= [F, \delta(a_1)] + \delta(a_1)F + [F, a_1]|D| \\
&= ([F, \delta(a_1)]|D|^{-1} + \delta(a_1)D^{-1} + [F, a_1])|D|.
\end{aligned}$$

By Proposition 14 and the assumption, the operator $[F,\delta(a_1)]|D|^{-1}$ is in $\mathcal{L}_{p,\infty}\cdot\mathcal{L}_{p,\infty}\subset\mathcal{L}_{p/2,\infty}$, while the other terms in parentheses are in $\mathcal{L}_{p,\infty}$, and give the right hand side of equation (10). Thus we have proved the case $q=1$.

Suppose then that we have proved the claim for some $q<p$. Since commutators with $|D|^{-1}$ improve summability, it follows that

$$\Big(\prod_{k=2}^{q+1}[D,a_k]\Big)|D|^{-1}=|D|^{-1}\Big(\prod_{k=2}^{q+1}[D,a_k]\Big)\bmod\mathcal{L}_{p/2,\infty}.$$

Therefore,

$$\begin{aligned}&\Gamma a_0\Big(\prod_{k=1}^{q+1}[D,a_k]\Big)|D|^{-q-1}\\&=\Gamma a_0[D,a_1]\Big(\Big(\prod_{k=2}^{q+1}[D,a_k]\Big)|D|^{-1}\Big)|D|^{-q}\\&=\Gamma a_0[D,a_1]\Big(|D|^{-1}\Big(\prod_{k=2}^{q+1}[D,a_k]\bmod\mathcal{L}_{p/2,\infty}\Big)\Big)|D|^{-q}\\&=\Gamma a_0[D,a_1]|D|^{-1}\Big(\Big(\prod_{k=2}^{q+1}[D,a_k]\Big)|D|^{-q}\Big)\bmod\mathcal{L}_{p/(q+2),\infty}.\end{aligned}$$

By induction, we have

$$\Big(\prod_{k=2}^{q+1}[D,a_k]\Big)|D|^{-q}=\mathfrak{S}$$

where

$$\mathfrak{S}=\sum_{\mathcal{A}\subset\{2,\dots,q+1\}}\Gamma\mathcal{W}_{\mathcal{A}}(1,a_2,\dots,a_{q+1})(-1)^{n_{\mathcal{A}}}D^{-|\mathcal{A}|}\bmod\mathcal{L}_{p/(q+1),\infty}.$$

Thus,

$$\begin{aligned}&\Gamma a_0\Big(\prod_{k=1}^{q+1}[D,a_k]\Big)|D|^{-q-1}\\&=\Gamma a_0[D,a_1]|D|^{-1}\mathfrak{S}\\&=\Gamma a_0(\delta(a_1)F+[F,a_1]|D|)|D|^{-1}\mathfrak{S}.\end{aligned}$$

Since commutators with $|D|^{-1}$ improve summability, it follows that

$$\begin{aligned}&|D|^{-1}\Gamma\mathcal{W}_{\mathcal{A}}(1,a_2,\dots,a_{q+1})\\&\quad=\Gamma\mathcal{W}_{\mathcal{A}}(1,a_2,\dots,a_{q+1})|D|^{-1}\bmod\mathcal{L}_{p/(q+2-|\mathcal{A}|),\infty}.\end{aligned}$$

Since $[F,\delta(a)]\in\mathcal{L}_{p,\infty}$ for all $a\in\mathcal{A}$, it follows that

$$\begin{aligned}&F\Gamma\mathcal{W}_{\mathscr{A}}(1,a_2,\dots,a_{q+1})\\&\quad=(-1)^{q-|\mathscr{A}|}\Gamma\mathcal{W}_{\mathscr{A}}(1,a_2,\dots,a_{q+1})F \bmod \mathcal{L}_{p/(q+1-|\mathscr{A}|),\infty}.\end{aligned}$$

Indeed, we have $F[F,a]=-[F,a]F$ for every $a\in\mathcal{A}$ and there are exactly $q-|\mathscr{A}|$ commutators $[F,a_j]$ in $\mathcal{W}_{\mathscr{A}}$.

Therefore,

$$\begin{aligned}&\Gamma a_0\Big(\prod_{k=1}^{q+1}[D,a_k]\Big)|D|^{-q-1}\\&\quad=\sum_{\mathscr{A}\subset\{2,\dots,q+1\}}(-1)^{n_{\mathscr{A}}}(-1)^{q-|\mathscr{A}|}\Gamma a_0\delta(a_1)\Gamma\mathcal{W}_{\mathscr{A}}(1,a_2,\dots,a_{q+1})D^{-|\mathscr{A}|-1}\\&\qquad+\sum_{\mathscr{A}\subset\{2,\dots,q+1\}}(-1)^{n_{\mathscr{A}}}\Gamma a_0[F,a_1]\\&\qquad\qquad\Gamma\mathcal{W}_{\mathscr{A}}(1,a_2,\dots,a_{q+1})D^{-|\mathscr{A}|} \bmod \mathcal{L}_{p/(q+2),\infty}.\end{aligned}$$

For each $\mathscr{A}\subset\{2,\dots,q+1\}$ define

$$\tilde{\mathscr{A}}=\mathscr{A}\cup\{1\}\subset\{1,\dots,q+1\}$$

and

$$\hat{\mathscr{A}}=\mathscr{A}\subset\{1,\dots,q+1\}.$$

Then

$$n_{\tilde{\mathscr{A}}}=q-|\mathscr{A}|+n_{\mathscr{A}}$$

while

$$n_{\hat{\mathscr{A}}}=n_{\mathscr{A}}.$$

By definition, we have

$$\Gamma a_0\delta(a_1)\Gamma\mathcal{W}_{\mathscr{A}}(1,a_2,\dots,a_{q+1})=\mathcal{W}_{\tilde{\mathscr{A}}}(c)$$

and

$$\Gamma a_0[F,a_1]\Gamma\mathcal{W}_{\mathscr{A}}(1,a_2,\dots,a_{q+1})=\mathcal{W}_{\hat{\mathscr{A}}}(c).$$

Hence,

$$\begin{aligned}&\Gamma a_0\Big(\prod_{k=1}^{q+1}[D,a_k]\Big)|D|^{-q-1}\\&\quad=\sum_{\mathscr{A}\subset\{2,\dots,q+1\}}(-1)^{n_{\tilde{\mathscr{A}}}}\mathcal{W}_{\tilde{\mathscr{A}}}(c)+\sum_{\mathscr{A}\subset\{2,\dots,q+1\}}(-1)^{n_{\hat{\mathscr{A}}}}\mathcal{W}_{\hat{\mathscr{A}}}(c) \bmod \mathcal{L}_{p/(q+2),\infty}.\end{aligned}$$

Since every $\mathcal{B} \subset \{1, \dots, q+1\}$ coincides either with $\tilde{\mathcal{A}}$ or else with $\hat{\mathcal{A}}$ for a unique $\mathcal{A} \subset \{2, \dots, q+1\}$, the equation (10) follows for $q+1$. This proves the Lemma. $\square$

Lemma 20. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. Suppose that $|\mathcal{A}| \geq 2$ and $m-1, m \in \mathcal{A}$ for some m.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then $\mathcal{W}_{\mathcal{A}}(c)D^{-|\mathcal{A}|} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$.*

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then $\mathcal{W}_{\mathcal{A}}(c)D^{-|\mathcal{A}|} \in [\mathcal{M}_{1,\infty}, \mathcal{L}(H)]$.*

Proof. Let φ be a trace on $\mathcal{L}_{1,\infty}$ (respectively, on $\mathcal{M}_{1,\infty}$). The mapping on $\mathcal{A}^{\otimes(p+1)}$ given by

$$c \longmapsto \varphi(\mathcal{W}_{\mathcal{A}}(c)D^{-|\mathcal{A}|})$$

is the Hochschild coboundary (see Appendix A) of the multilinear mapping defined by

$$a_0 \otimes \cdots \otimes a_{p-1}$$
$$\longmapsto \frac{(-1)^{m-1}}{2}\varphi\Big(\Gamma a_0 \prod_{k=1}^{m-2}[b_k, a_k]\delta^2(a_{m-1}) \prod_{k=m}^{p-1}[b_{k+1}, a_k]D^{-|\mathcal{A}|}\Big).$$

Since a Hochschild coboundary vanishes on every Hochschild cycle, it follows that $\varphi(\mathcal{W}_{\mathcal{A}}(c)D^{-|\mathcal{A}|}) = 0$ for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$. Since φ is an arbitrary trace, the assertion follows. $\square$

Lemma 21. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. Suppose that $|\mathcal{A}_1| = |\mathcal{A}_2| \geq 2$ and that the symmetric difference $\mathcal{A}_1 \Delta \mathcal{A}_2 = \{m-1, m\}$ for some m.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then $\mathcal{W}_{\mathcal{A}_1}(c)D^{-|\mathcal{A}_1|} + \mathcal{W}_{\mathcal{A}_2}(c)D^{-|\mathcal{A}_2|} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$.*

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then $\mathcal{W}_{\mathcal{A}_1}(c)D^{-|\mathcal{A}_1|} + \mathcal{W}_{\mathcal{A}_2}(a)D^{-|\mathcal{A}_2|} \in [\mathcal{M}_{1,\infty}, \mathcal{L}(H)]$.*

Proof. Let φ be a trace on $\mathcal{L}_{1,\infty}$ (respectively, on $\mathcal{M}_{1,\infty}$). The mapping on $\mathcal{A}^{\otimes(p+1)}$ given by

$$c \longmapsto \varphi(\mathcal{W}_{\mathcal{A}_1}(c)D^{-|\mathcal{A}_1|}) + \varphi(\mathcal{W}_{\mathcal{A}_2}(c)D^{-|\mathcal{A}_2|})$$

is the Hochschild coboundary (see Appendix A) of the multilinear mapping defined by

$$a_0 \otimes \cdots \otimes a_{p-1}$$
$$\longmapsto (-1)^{m-1}\varphi\Big(\Gamma a_0 \prod_{k=1}^{m-2}[b_k, a_k][F, \delta(a_{m-1})] \prod_{k=m}^{p-1}[b_{k+1}, a_k] D^{-|\mathscr{A}_1|}\Big).$$

The proof is concluded by using the same argument as in the preceding lemma. □

Corollary 22. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. Suppose that $|\mathscr{A}| \geq 2$.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then $\mathcal{W}_{\mathscr{A}}(c)D^{-|\mathscr{A}|} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$.*

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then $\mathcal{W}_{\mathscr{A}}(c)D^{-|\mathscr{A}|} \in [\mathcal{M}_{1,\infty}, \mathcal{L}(H)]$.*

Proof. Let $n < m$ be such that $n, m \in \mathscr{A}$. Without loss of generality, $i + n \notin \mathscr{A}$ for all $0 < i < m - n$. Set

$$\mathscr{A}_i = (\mathscr{A} \backslash \{n\}) \cup \{i + n\}, \quad 0 \leq i < m - n.$$

We have

(1) $|\mathscr{A}_i| = |\mathscr{A}|$ and $|\mathscr{A}_i \Delta \mathscr{A}_{i-1}| = 2$ for all $1 \leq i < m - n$.

(2) $\mathscr{A}_0 = \mathscr{A}$ and $m - 1, m \in \mathscr{A}_{m-n-1}$.

It follows from Lemma 21 that $\mathcal{W}_{\mathscr{A}_{m-n-1}}(a)D^{-1} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$ (respectively, $\mathcal{W}_{\mathscr{A}_{m-n-1}}(a)D^{-1} \in [\mathcal{M}_{1,\infty}, \mathcal{L}(H)]$). The assertion follows by applying Lemma 20 $m - n - 1$ times. □

Lemma 23. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then $\mathcal{W}_{\varnothing}(c) \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$.*

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then $\mathcal{W}_{\varnothing}(c) \in [\mathcal{M}_{1,\infty}, \mathcal{L}(H)]$.*

Proof. We prove (a) only (the proof of (b) is identical). Let $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. We have

$$2\Gamma a_0 \prod_{k=1}^{p}[F, a_k] = [F, F\Gamma a_0 \prod_{k=1}^{p}[F, a_k]] + (-1)^{p-1} F\Gamma \prod_{k=0}^{p}[F, a_k] \tag{11}$$

so that

$$2\mathcal{W}_{\varnothing}(c) = [F, F\mathcal{W}_{\varnothing}(c)] + (-1)^{p-1}\operatorname{ch}(c). \tag{12}$$

Since $\mathcal{W}_{\varnothing}(c) \in \mathcal{L}_{1,\infty}$, it follows that $[F, F\mathcal{W}_{\varnothing}(c)] \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$. By Proposition 14 and the Hölder property in equation (2), $\operatorname{ch}(c) \in \mathcal{L}_1 \subset [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$. Thus, $\mathcal{W}_{\varnothing}(c) \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)]$. □

We are now ready to prove the main result of this subsection.

Proof of Proposition 18. As in preceding lemma, we prove (a) only (the proof of (b) is identical). For every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$, it follows from Lemma 19 that

$$\Omega(c)|D|^{-p} \in \sum_{\mathscr{A}\subset\{1,\dots,p\}} (-1)^{n_{\mathscr{A}}} \mathcal{W}_{\mathscr{A}}(c) D^{-|\mathscr{A}|} + \mathcal{L}_1.$$

Applying Corollary 22 to every summand in the sum $\sum_{|\mathscr{A}|\geq 2}$ and Lemma 23, we infer that

$$\Omega(c)|D|^{-p} \in \sum_{|\mathscr{A}|=1} (-1)^{n_{\mathscr{A}}} \mathcal{W}_{\mathscr{A}}(c) D^{-1} + [\mathcal{L}_{1,\infty}, \mathcal{L}(H)].$$

If $\mathscr{A} = \{m\}$, then $n_{\mathscr{A}} = p - m$. Therefore,

$$\Omega(c)|D|^{-p} \in \sum_{m=1}^{p} (-1)^{p-m} \mathcal{W}_m(c) D^{-1} + [\mathcal{L}_{1,\infty}, \mathcal{L}(H)].$$

Applying Lemma 21 $p - m$ times, we obtain

$$\mathcal{W}_m(c) D^{-1} - (-1)^{p-m} \mathcal{W}_p(c) D^{-1} \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)], \quad 1 \leq m < p.$$

This suffices to conclude the proof. □

4.2. Some commutator estimates. Our method of proof of Proposition 29 exploits some heat semigroup asymptotics. For this we need to introduce, in this subsection, a number of technical estimates for commutators involving the operator valued function $s \to f(s|D|)$, where $f(s) = e^{-|s|^{p+1}}$, and $s \in \mathbb{R}$. As before in the text, $p \in \mathbb{N}$. We make essential use of the fact that $\hat{f}'' \in L_1(-\infty, \infty)$ (this fact follows from Lemma 7 in [16]).

Lemma 24. *If* $(\mathcal{A}, H, D)$ *is a* QC^∞ *spectral triple, then*

$$\|[f(s|D|), a] - sf'(s|D|)\delta(a)\|_\infty \le s^2 \|\hat{f}''\|_1 \|\delta^2(a)\|_\infty$$

$$\|[f(s|D|), a] - s\delta(a) f'(s|D|)\|_\infty \le s^2 \|\hat{f}''\|_1 \|\delta^2(a)\|_\infty$$

for all $s > 0$ *and for all* $a \in \mathcal{A}$.

Proof. We use the method of [1, 3]. It is clear that

$$[f(s|D|), a] = \int_{-\infty}^{\infty} \hat{f}(u)[e^{ius|D|}, a]du. \tag{13}$$

An elementary computation shows that

$$[e^{ius|D|}, a] = ius \int_0^1 e^{iuvs|D|}\delta(a)e^{iu(1-v)s|D|}dv. \tag{14}$$

Combining (13) and (14), we obtain

$$[f(s|D|), a] = s \int_{-\infty}^{\infty} \int_0^1 \hat{f}'(u)e^{iuvs|D|}\delta(a)e^{iu(1-v)s|D|}dvdu.$$

Therefore,

$$\begin{aligned}
&[f(s|D|), a] - sf'(s|D|)\delta(a) \\
&\quad = s \int_{-\infty}^{\infty} \int_0^1 \hat{f}'(u)(e^{iuvs|D|}\delta(a)e^{iu(1-v)s|D|} - e^{ius|D|}\delta(a))dvdu \\
&\quad = s \int_{-\infty}^{\infty} \int_0^1 \hat{f}'(u)(e^{iuvs|D|}[\delta(a), e^{iu(1-v)s|D|}])dvdu.
\end{aligned}$$

As in equation (14), we have

$$\begin{aligned}
&[\delta(a), e^{iu(1-v)s|D|}] \\
&\quad = -iu(1-v)s \int_0^1 e^{iu(1-v)sw|D|}\delta^2(a)e^{iu(1-v)s(1-w)|D|}dw.
\end{aligned}$$

Hence,

$$[f(s|D|), a] - sf'(s|D|)\delta(a)$$
$$= -s^2 \int_{-\infty}^{\infty}\int_0^1\int_0^1 \hat{f}''(u)(1-v)e^{iu(1-v)sw|D|}\delta^2(a)e^{iu(1-v)s(1-w)|D|}dwdvdu.$$

The first inequality follows immediately. The proof of the second inequality is similar so we omit it. □

Lemma 25. *Let D be an invertible unbounded self-adjoint operator.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then*
$$\mathrm{Tr}(f(s|D|)) = O(s^{-p}), \quad \mathrm{Tr}(|D|^{-p-1}(1-f(s|D|))) = O(s),$$
as $s \to 0$,

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then, for every $\varepsilon > 0$,*
$$\mathrm{Tr}(f(s|D|)) = O(s^{-p-\varepsilon}), \quad \mathrm{Tr}(|D|^{-p-1}(1-f(s|D|))) = O(s^{1-\varepsilon}),$$
as $s \to 0$.

Proof. Using Lemma 7 with $V = |D|^{-p}$ and $\alpha = 1 + 1/p$, we obtain (a). We now prove (b). Since $D^{-p} \in \mathcal{M}_{1,\infty}$, it follows that

$$(k+1)\mu(k, D^{-p}) \leq \sum_{m=0}^{k} \mu(m, D^{-p}) \leq \mathrm{const}\cdot\log(k+2).$$

Hence,

$$\mu(k, D^{-p-\varepsilon}) \leq \Big(\mathrm{const}\cdot\frac{\log(k+2)}{k+1}\Big)^{(p+\varepsilon)/p} \leq \frac{\mathrm{const}}{k+1}, \quad k \geq 0.$$

Select an operator $D_0 \leq D$ (using the same eigenbasis) such that

$$\mu(k, D_0^{-p-\varepsilon}) = \frac{\mathrm{const}}{k+1}, \quad k \geq 0.$$

In what follows, we assume, to reduce the notation, that const $= 1$. For the first equality,

$$\begin{aligned}\mathrm{Tr}(f(s|D|)) &\leq \mathrm{Tr}(f(s|D_0|))\\ &= \sum_{k=1}^{\infty} e^{-(sk^{1/(p+\varepsilon)})^{p+1}}\\ &\leq \int_0^{\infty} e^{-(su^{1/(p+\varepsilon)})^{p+1}}du\\ &= s^{-p-\varepsilon}\int_0^{\infty} e^{-v^{(p+1)/(p+\varepsilon)}}dv.\end{aligned}$$

In order to prove the second equality, note that the mapping $s \to s^{-1}(1-e^{-s})$ is decreasing on $(0,\infty)$ and so is the mapping $s \to s^{-p-1}(1-f(s))$. It follows that

$$\begin{aligned}\mathrm{Tr}(|D|^{-p-1}(1-f(s|D|))) &\le \mathrm{Tr}(|D_0|^{-p-1}(1-f(s|D_0|)))\\ &= \sum_{k=1}^{\infty}(k^{1/(p+\varepsilon)})^{-p-1}(1-e^{-(sk^{1/(p+\varepsilon)})^{p+1}})\\ &\le \int_0^{\infty} u^{-(p+1)/(p+\varepsilon)}(1-e^{-(su^{1/(p+\varepsilon)})^{p+1}})du\\ &= s^{1-\varepsilon}\int_0^{\infty} v^{-(p+1)/(p+\varepsilon)}(1-e^{-v^{(p+1)/(p+\varepsilon)}})dv. \square\end{aligned}$$

Lemma 26. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $a \in \mathcal{A}$.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then*

$$\|[f'(s|D|), \delta(a)]\|_1 = O(s^{1-p})$$

as $s \to 0$.

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$ then (for every $\varepsilon > 0$)*

$$\|[f'(s|D|), \delta(a)]\|_1 = O(s^{1-p-\varepsilon}),$$

as $s \to 0$.

Proof. Suppose first that $p \ge 4$ or that $p = 2$. Define a positive function h by setting

$$f'(t) = -\mathrm{sgn}(t)h^2(t)$$

for all t. We have $h', h'' \in L_2(-\infty,\infty)$. It follows now from Lemma 7 in [16] that $\hat{h'} \in L_1(-\infty,\infty)$. Repeating the argument in the beginning of Lemma 24, we obtain

$$[h(s|D|), \delta(a)] = s\int_{-\infty}^{\infty}\int_0^1 \hat{h}'(u)e^{iuvs|D|}\delta^2(a)e^{iu(1-v)s|D|}dvdu$$

and, therefore,

$$\|[h(s|D|), \delta(a)]\|_\infty \le s\|\hat{h}'\|_1\|\delta^2(a)\|_\infty.$$

On the other hand, we have

$$[f'(s|D|), \delta(a)] = [h^2(s|D|), \delta(a)]$$
$$= h(s|D|)[h(s|D|), \delta(a)] + [h(s|D|), \delta(a)]h(s|D|).$$

Therefore,

$$\|[f'(s|D|), \delta(a)]\|_1 \leq 2\|h(s|D|)\|_1 \|[h(s|D|), \delta(a)]\|_\infty = \|h(s|D|)\|_1 \cdot O(s).$$

Recall that $h(s) \leq \text{const} \cdot f(s/2)$ for all $s \in \mathbb{R}$. If $D^{-p} \in \mathcal{L}_{1,\infty}$, then it follows from Lemma 25 (a) that $\|h(s|D|)\|_1 = O(s^{-p})$. Similarly, if $D^{-p} \in \mathcal{M}_{1,\infty}$, then it follows from Lemma 25 (b) that $\|h(s|D|)\|_1 = O(s^{-p-\varepsilon})$. This proves the assertion for $p \geq 4$ or $p = 2$.

If $p = 1$ or $p = 3$, then Lemma 7 in [16] is inapplicable and we have to proceed with a direct computation. Assume, for simplicity, that $p = 1$ and $D^{-1} \in \mathcal{L}_{1,\infty}$ (the proof is similar for $p = 3$ and for $\mathcal{M}_{1,\infty}$). Repeating the argument above, we obtain

$$\|[f^{1/2}(s|D|), \delta(a)]\|_1 = O(s), \quad \|[f^{1/2}(s|D|), \delta^2(a)]\|_1 = O(s).$$

Using the elementary equality

$$-\frac{1}{2}[f'(s|D|), \delta(a)] = \delta^2(a) \cdot sf(s|D|)$$
$$+ s|D|f^{1/2}(s|D|) \cdot [f^{1/2}(s|D|), \delta(a)]$$
$$+ [f^{1/2}(s|D|), \delta(a)] \cdot s|D|f^{1/2}(s|D|)$$
$$+ [f^{1/2}(s|D|), \delta^2(a)] \cdot sf^{1/2}(s|D|),$$

we infer that

$$\|[f'(s|D|), \delta(a)]\|_1$$
$$\leq \text{const} \cdot \text{Tr}(sf(s|D|) + 2s^2|D|f^{1/2}(s|D|) + s^2 f^{1/2}(s|D|)).$$

Recall that $sf(s), f^{1/2}(s) \leq \text{const} \cdot f(s/2)$ for all $s > 0$. By Lemma 25 (a), we have

$$s\text{Tr}(f(s|D|)) = O(1),$$
$$s\text{Tr}(s|D|f^{1/2}(s|D|)) = O(1),$$
$$s\text{Tr}(f^{1/2}(s|D|)) = O(1).$$

This proves the assertion for $p = 1$. ∎

Lemma 27. *Let* $(\mathcal{A}, H, D)$ *be a* QC^∞ *spectral triple and let* $a \in \mathcal{A}$.

(a) *If* $D^{-p} \in \mathcal{L}_{1,\infty}$, *then*

$$\|[f(s|D|), a] - s\delta(a) f'(s|D|)\|_1 = O(s^{2-p}),$$

as $s \to 0$.

(b) *If* $D^{-p} \in \mathcal{M}_{1,\infty}$, *then (for every* $\varepsilon > 0$*)*

$$\|[f(s|D|), a] - s\delta(a) f'(s|D|)\|_1 = O(s^{2-p-\varepsilon}),$$

as $s \to 0$.

Proof. Let $f = h^2$. Since h can be obtained from f by rescaling, the assertion of Lemma 24 also holds for h. We have

$$[f(s|D|), a] - \frac{s}{2}\{f'(s|D|), \delta(a)\} = h(s|D|)([h(s|D|), a] - sh'(s|D|)\delta(a)) \\ + ([h(s|D|), a] - s\delta(a)h'(s|D|))h(s|D|).$$

It follows that

$$\|[f(s|D|), a] - \frac{s}{2}\{f'(s|D|), \delta(a)\}\|_1 \\ \leq \|h(s|D|)\|_1(\|[h(s|D|), a] - sh'(s|D|)\delta(a)\|_\infty + \|[h(s|D|), a] \\ - s\delta(a)h'(s|D|)\|_\infty).$$

We infer from Lemma 24 that the expression in brackets is $O(s^2)$. If $D^{-p} \in \mathcal{L}_{1,\infty}$, then it follows from Lemma 25 (a) that $\|h(s|D|)\|_1 = O(s^{-p})$. Therefore,

$$\|[f(s|D|), a] - \frac{s}{2}\{f'(s|D|), \delta(a)\}\|_1 = O(s^{2-p}).$$

The assertion (a) follows now from Lemma 26. Similarly, if $D^{-p} \in \mathcal{M}_{1,\infty}$, then it follows from Lemma 25 (b) that $\|h(s|D|)\|_1 = O(s^{-p-\varepsilon})$. This proves the assertion (b). ∎

Proposition 28. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $a \in \mathcal{A}$.*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then*

$$\|[f(s|D|), a] - s\delta(a) f'(s|D|)\|_{p,1} = O(s),$$

as $s \to 0$.

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then (for every $\varepsilon > 0$)*

$$\|[f(s|D|), a] - s\delta(a) f'(s|D|)\|_{p-\varepsilon} = O(s^{1-2\varepsilon}),$$

as $s \to 0$.

Proof. We prove only the first assertion, as the proof of the second one is identical. If $p = 1$, then the assertion is proved in Lemma 27. Suppose $p > 1$ and set

$$T = [f(s|D|), a] - s\delta(a) f'(s|D|).$$

We infer from Lemma 24 that $\|T\|_\infty = O(s^2)$ and from Lemma 27 that $\|T\|_1 = O(s^{2-p})$ as $s \to 0$. The assertion follows from the interpolation inequality (see e.g. Theorem 2.g.18 and Corollary 2.g.14 in [13])

$$\|T\|_{p,1} \le \|T\|_1^{1/p} \|T\|_\infty^{1-1/p} = O(s^{(2-p)/p} \cdot s^{2(1-1/p)}) = O(s). \qquad \square$$

4.3. Asymptotics for the heat semigroup and the proof of Proposition 29. In order to study the operator $\mathcal{W}_p(c) D^{-1}$, which was introduced in Proposition 18, we now establish the following heat semigroup estimate.

Proposition 29. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple with $D^{-p} \in \mathcal{M}_{1,\infty}$. If the spectral triple and the integer p are both odd (respectively, even), then*

$$\mathrm{Tr}(\mathcal{W}_p(c) D^{-1} e^{-(s|D|)^{p+1}}) = \mathrm{Ch}(c) \log(1/s) + O(1), \quad s \to 0,$$

for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$.

In Lemma 30 and Lemma 31, we prepare the ground for the proof of Proposition 29.

Lemma 30. *If $(\mathcal{A}, H, D)$ is a QC^∞ spectral triple, then*

$$\Big(\prod_{k=0}^{m} [F, a_k]\Big) |D|^{m+1} \in \mathcal{L}(H), \quad a_k \in \mathcal{A},\ 0 \le k \le m.$$

Proof. Define the algebra

$$\mathcal{B} = \{A \in \mathcal{L}(H) \colon A \colon \mathrm{dom}(D^n) \longrightarrow \mathrm{dom}(D^n),\ \delta^n(A) \in \mathcal{L}(H) \text{ for all } n \geq 0\}.$$

An inductive argument shows that, for every $A \in \mathcal{B}$ and for every $n \geq 0$, there exists $B \in \mathcal{B}$ such that $A|D|^n = |D|^n B$. For all $k \leq m$ and for all $a_k \in \mathcal{A}$, we have $[D, a_k] \in \mathcal{B}$ and $F[|D|, a_k] \in \mathcal{B}$ (here, we used the fact that our spectral triple is QC^∞). Therefore,

$$[F, a_k] = [D, a_k]|D|^{-1} - F[|D|, a_k]|D|^{-1} = A_k|D|^{-1},$$

where $A_k \in \mathcal{B}$. Therefore,

$$\prod_{k=0}^{m}[F, a_k]|D|^{m+1} = \Big(\prod_{k=0}^{m-1}[F, a_k]\Big)A_m|D|^m.$$

Note that $A_m|D|^{-1} \cdot |D|^{m+1} = |D|^m B_m$ for some $B_m \in \mathcal{B}$. It follows that

$$\prod_{k=0}^{m}[F, a_k]|D|^{m+1} = \Big(\prod_{k=0}^{m-1}[F, a_k]\Big)|D|^m B_m.$$

The right hand side is bounded by induction. □

Note that the condition $D^{-p} \in \mathcal{M}_{1,\infty}$ guarantees that $D^{-p-2} \in \mathcal{L}_1$. Hence,

$$0 \leq -f'(s|D|) \leq \frac{4(p+1)}{e}(s|D|)^{-p-2} \in \mathcal{L}_1.$$

In particular, we have $f'(s|D|) \in \mathcal{L}_1$.

Lemma 31. *Let $(\mathcal{A}, H, D)$ be a QC^∞ spectral triple and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. Suppose that the spectral triple and p are both odd (respectively, even).*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then*

$$s\mathrm{Tr}(\mathcal{W}_p(c)Ff'(s|D|)) = -\mathrm{Ch}(c) + O(s),$$

as $s \to 0$.

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then, for every $\varepsilon > 0$,*

$$s\mathrm{Tr}(\mathcal{W}_p(c)Ff'(s|D|)) = -\mathrm{Ch}(c) + O(s^{1-\varepsilon}),$$

as $s \to 0$.

Here, $f(s) = e^{-|s|^{p+1}}$, $s \in \mathbb{R}$.

Proof. We only prove the first assertion. The proof of the second one is identical. Define the multilinear mappings

$$\mathcal{K}_s,\ \mathcal{H}_s\colon \mathcal{A}^{\otimes(p+1)} \longrightarrow \mathcal{L}(H)$$

by setting

$$\mathcal{K}_s(a_0 \otimes \cdots \otimes a_p) = \Gamma a_0\Big(\prod_{k=1}^{p-1}[F,a_k]\Big)[Ff(s|D|),a_p],$$

and

$$\mathcal{H}_s(a_0 \otimes \cdots \otimes a_p) = \Gamma a_0\Big(\prod_{k=1}^{p-1}[F,a_k]\Big)F[f(s|D|),a_p].$$

For all $c \in \mathcal{A}^{\otimes(p+1)}$, we have (see p. 293 in [7] for the second equality)

$$\mathcal{W}_{\varnothing}(c)f(s|D|) = \mathcal{K}_s(c) - \mathcal{H}_s(c), \quad \mathrm{ch}(c) = \mathcal{W}_{\varnothing}(c) + F\mathcal{W}_{\varnothing}(c)F.$$

Therefore,

$$\mathrm{Tr}(\mathrm{ch}(c)f(s|D|)) = 2\mathrm{Tr}(\mathcal{W}_{\varnothing}(c)f(s|D|)) = 2\mathrm{Tr}(\mathcal{K}_s(c)) - 2\mathrm{Tr}(\mathcal{H}_s(c)). \quad (15)$$

The mapping $c' \to \mathrm{Tr}(\mathcal{K}_s(c'))$ on $\mathcal{A}^{\otimes(p+1)}$ is the Hochschild coboundary[5] of the multilinear mapping defined by

$$a_0 \otimes \cdots \otimes a_{p-1} \longmapsto (-1)^p\mathrm{Tr}(\Gamma a_0\Big(\prod_{k=1}^{p-1}[F,a_k]\Big)Ff(s|D|)).$$

---

[5] For the sake of illustration, let $p = 2$ and let the multilinear mapping

$$\theta\colon \mathcal{A}^{\otimes 2} \longrightarrow \mathcal{L}(H)$$

be defined by setting

$$\theta(a_0 \otimes a_1) = \mathrm{Tr}(\Gamma a_0[F,a_1]T)$$

with $T \in \mathcal{L}_1$. We then have

$$\begin{aligned}(b\theta)(a_0 \otimes a_1 \otimes a_2) &= \theta(a_0a_1 \otimes a_2) - \theta(a_0 \otimes a_1a_2) + \theta(a_2a_0 \otimes a_1)\\ &= \mathrm{Tr}(\Gamma a_0a_1[F,a_2]T - \Gamma a_0[F,a_1a_2]T) + \mathrm{Tr}(\Gamma a_2a_0[F,a_1]T)\\ &= -\mathrm{Tr}(\Gamma a_0[F,a_1]a_2T) + \mathrm{Tr}(\Gamma a_0[F,a_1]Ta_2)\\ &= \mathrm{Tr}(\Gamma a_0[F,a_1][T,a_2]).\end{aligned}$$

Hence, it vanishes on every Hochschild cycle. On the other hand, we have

$$\mathrm{Tr}(\mathcal{H}_s(c')) = s\mathrm{Tr}(\mathcal{W}_p(c')Ff'(s|D|)) + O(s) \tag{16}$$

as may be seen by evaluating on $a_0 \otimes \cdots \otimes a_p$, using Proposition 28 to obtain

$$\begin{aligned}
&\Big|\mathrm{Tr}\Big(\Gamma a_0 \prod_{k=1}^{p-1}[F,a_k]F[f(s|D|),a_p]\Big) - s\mathrm{Tr}\Big(\Gamma a_0 \prod_{k=1}^{p-1}[F,a_k]F\delta(a_p)f'(s|D|)\Big)\Big| \\
&\quad \leq \|\Gamma a_0 \prod_{k=1}^{p-1}[F,a_k]F\|_{q,\infty}\,\|[f(s|D|),a_p] - s\delta(a_p)f'(s|D|)\|_{p,1} \\
&\quad = O(s)
\end{aligned}$$

and, since,

$$\begin{aligned}
&\Big|\mathrm{Tr}\Big(\Gamma a_0 \prod_{k=1}^{p-1}[F,a_k]F\delta(a_p)f'(s|D|)\Big) - \mathrm{Tr}(\mathcal{W}_p(a)Ff'(s|D|))\Big| \\
&\quad \leq \|\Gamma a_0\Big(\prod_{k=1}^{p-1}[F,a_k]\Big)[F,\delta(a_p)]|D|^p\|_\infty \cdot \||D|^{-p}f'(s|D|)\|_1 \\
&\quad = O(1),
\end{aligned}$$

the equality (16) follows. Combining the equalities (15), (16) and the fact that $\mathrm{Tr}(\mathcal{K}_s(c)) = 0$ for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$, we infer that

$$\mathrm{Tr}(\mathrm{ch}(c)f(s|D|)) = -2s\mathrm{Tr}(\mathcal{W}_p(c)Ff'(s|D|)) + O(s) \tag{17}$$

for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$. The operator $B = \mathrm{ch}(c)|D|^{p+1}$ is bounded by Lemma 30. Using Lemma 25 (a), we obtain

$$\begin{aligned}
|\mathrm{Tr}(Bf(s|D|)|D|^{-p-1}) - \mathrm{Tr}(B|D|^{-p-1})| &\leq \|B\|_\infty \mathrm{Tr}((1-f(s|D|))|D|^{-p-1}) \\
&= O(s).
\end{aligned}$$

Therefore,

$$\mathrm{Tr}(\mathrm{ch}(c)f(s|D|)) = \mathrm{Ch}(c) + O(s). \tag{18}$$

By combining (17) and (18), we conclude the proof. □

Proof of Proposition 29. By Lemma 31, we have

$$\mathrm{Tr}(\mathcal{W}_p(c)F|D|^p e^{-(s|D|)^{p+1}}) = \frac{1}{(p+1)}\mathrm{Ch}(c)s^{-p-1} + O(s^{\varepsilon-p}).$$

Setting $u = s^{p+1}$, we obtain

$$\mathrm{Tr}(\mathcal{W}_p(c)F|D|^p e^{-u|D|^{p+1}}) = \frac{1}{(p+1)u}\mathrm{Ch}(c) + O(u^{-(p-\varepsilon)/(p+1)}).$$

Integrating over $u \in [s,1]$, we obtain

$$\mathrm{Tr}(\mathcal{W}_p(c)F|D|^{-1}(e^{-s|D|^{p+1}} - e^{-|D|^{p+1}})) = \frac{1}{(p+1)}\mathrm{Ch}(c)\log\left(\frac{1}{s}\right) + O(1).$$

Taking into account that $D^{-p} \in \mathcal{M}_{1,\infty}$ implies that $\mathcal{W}_p(c)F|D|^{-1}e^{-|D|^{p+1}} \in \mathcal{L}_1$. Replacing s with s^{p+1}, we conclude the proof. ☐

4.4. Proof of the main result. In this subsection, we prove Theorem 16. Recall that the multilinear mapping $\mathcal{W}_p$ is defined in Section 3.2.

Lemma 32. *Let $(\mathcal{A}, H, D)$ be an odd (respectively, even) QC^∞ spectral triple and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. Suppose that p is odd (respectively, even).*

(a) *If $D^{-p} \in \mathcal{L}_{1,\infty}$, then*

$$\varphi(\mathcal{W}_p(c)D^{-1}) = \frac{\mathrm{Ch}(c)}{p}$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

(b) *If $D^{-p} \in \mathcal{M}_{1,\infty}$, then*

$$\mathrm{Tr}_\omega(\mathcal{W}_p(c)D^{-1}) = \frac{\mathrm{Ch}(c)}{p}$$

for every Dixmier trace on $\mathcal{M}_{1,\infty}$.

Proof. Recall the algebra

$$\mathcal{B} = \{A \in \mathcal{L}(H) \colon A \colon \mathrm{dom}(D) \longrightarrow \mathrm{dom}(D),\ \delta^n(A) \in \mathcal{L}(H) \text{ for all } n \geq 0\}.$$

It follows from Lemma 30 that $\mathcal{W}_p(a)|D|^{p-1} \in \mathcal{B}$ and is, therefore, bounded. Set $V = |D|^{-p}$ and $\alpha = 1 + 1/p$. It follows from Proposition 29 that

$$\mathrm{Tr}(\mathcal{W}_p(c)D^{-1}e^{-(nV)^{-\alpha}}) = \frac{\mathrm{Ch}(c)}{p}\log(n) + O(1)$$

as $n \to \infty$. By the previous paragraph, we have

$$A = \mathcal{W}_p(c)F|D|^{p-1} \in \mathcal{L}(H)$$

and, by assumption, $V \in \mathcal{L}_{1,\infty}$ (respectively, $V \in \mathcal{M}_{1,\infty}$). Therefore, Proposition 6 is applicable and yields

$$\varphi(\mathcal{W}_p(c)D^{-1}) = \frac{\mathrm{Ch}(c)}{p}$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$ or

$$\mathrm{Tr}_\omega(\mathcal{W}_p(c)D^{-1}) = \frac{\mathrm{Ch}(c)}{p}$$

for every Dixmier trace on $\mathcal{M}_{1,\infty}$, respectively. ☐

Lemma 33. *If* $(\mathcal{A}, H, D)$ *is a* QC^∞ *spectral triple, then so is* $(\mathcal{A}, H, D_0)$, *where* $D_0 = F(1 + D^2)^{1/2}$.

Proof. Set $D_1 = D_0 - D \in \mathcal{L}(H)$. Define the operations

$$\delta_0 \colon a \longrightarrow [|D_0|, a], \quad \delta_1 \colon a \longrightarrow [|D_1|, a].$$

Noting that $|D_0| = |D| + |D_1|$, we infer that

$$\delta_0 = \delta + \delta_1.$$

Since the operations δ_0 and δ_1 commute, it follows that

$$\delta_0^n(a) = \sum_{k=0}^{n} \binom{n}{k} \delta_1^{n-k}(\delta^k(a)).$$

Since $\delta^k(a)$ is well defined and since $\delta_1 \colon \mathcal{L}(H) \to \mathcal{L}(H)$ is a bounded mapping, it follows that $\delta_0^n(a)$ is well defined. Similarly, $\delta_0^n(\partial(a))$ is well defined. Define the operations $\partial_0 \colon a \to [D_0, a]$ and $\partial_1 \colon a \to [D_1, a]$. We have

$$\delta_0^n(\partial_0(a)) = \delta_0^n(\partial(a)) + \delta_0^n(\partial_1(a)) = \delta_0^n(\partial(a)) + \partial_1(\delta_0^n(a)).$$

By Definition 12, $(\mathcal{A}, H, D_0)$ is a QC^∞ spectral triple. ☐

We are now ready to prove the main result of the paper. We present the detailed argument for the first part of the theorem.

CASE 1. Suppose that $(\mathcal{A}, H, D)$ is a QC^∞ odd (p,∞)-summable spectral triple and that p is even.

Let φ be a trace on $\mathcal{L}_{1,\infty}$. The mapping defined on $\mathcal{A}^{\otimes(p+1)}$ by

$$c \longmapsto \varphi(\Omega(c)(1+D^2)^{-p/2})$$

is the Hochschild coboundary (see Appendix A) of the multilinear mapping defined by

$$a_0 \otimes \cdots \otimes a_{p-1} \longmapsto \frac{1}{2}\varphi\Big(\prod_{k=0}^{p-1}[D,a_k](1+D^2)^{-p/2}\Big).$$

Every Hochschild coboundary vanishes on every Hochschild cycle, so that

$$\varphi(\Omega(c)(1+D^2)^{-p/2}) = 0$$

for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$. Thus, the left hand side of (8) vanishes. For $c' = a_0 \otimes \cdots \otimes a_p$, with p even,

$$F\prod_{k=0}^{p}[F,a_k] = -\prod_{k=0}^{p}[F,a_k]F$$

and, therefore,

$$\mathrm{Ch}(c') = \mathrm{Tr}\Big(F\prod_{k=0}^{p}[F,a_k]\Big) = -\mathrm{Tr}\Big(\prod_{k=0}^{p}[F,a_k]F\Big) = -\mathrm{Ch}(c').$$

Hence, $\mathrm{Ch}(c') = 0$ for all $c' \in \mathcal{A}^{\otimes(p+1)}$. Thus, the right hand side of (8) vanishes.

CASE 2. Suppose that $(\mathcal{A}, H, D)$ is a QC^∞ even (p,∞)-summable spectral triple and that p is odd.

Let φ be a trace on $\mathcal{L}_{1,\infty}$. By Definition 13, we have $\Gamma[D,a] = -[D,a]\Gamma$ and $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$. Since p is odd, it follows that

$$\begin{aligned}\Gamma a_0\prod_{k=1}^{p}[D,a_k](1+D^2)^{-p/2} &= a_0\Gamma\prod_{k=1}^{p}[D,a_k](1+D^2)^{-p/2}\\ &= -a_0\prod_{k=1}^{p}[D,a_k]\Gamma(1+D^2)^{-p/2}\\ &= -a_0\prod_{k=1}^{p}[D,a_k](1+D^2)^{-p/2}\Gamma.\end{aligned}$$

Applying the trace φ, we obtain

$$\varphi\Big(\Gamma a_0 \prod_{k=1}^{p}[D, a_k](1 + D^2)^{-p/2}\Big) = -\varphi\Big(\Gamma a_0 \prod_{k=1}^{p}[D, a_k](1 + D^2)^{-p/2}\Big).$$

Hence, the left hand side of (8) vanishes. Repeating the argument in Step 1, we infer that the right hand side of (8) vanishes as well.

CASE 3. Suppose that p and the (p,∞)-summable spectral triple $(\mathcal{A}, H, D)$ are simultaneously odd (or even).

If D is invertible, then we infer from Proposition 18 and Lemma 32 that

$$\varphi(\Omega(c)|D|^{-p}) = p\varphi(\mathcal{W}_p(c)D^{-1}) = \mathrm{Ch}(c)$$

and the assertion is proved. Suppose now that D is not invertible and consider the invertible operator

$$D_0 = F(1 + D^2)^{1/2}.$$

It follows from Lemma 33 that $(\mathcal{A}, H, D_0)$ is a spectral triple with $D_0^{-p} \in \mathcal{L}_{1,\infty}$. Clearly,

$$\begin{aligned} D_1 &:= D_0 - D \\ &= F((1 + |D|^2)^{1/2} - |D|) \\ &= F(|D| + (1 + |D|^2)^{1/2})^{-1} \in \mathcal{L}_{p,\infty}. \end{aligned}$$

We claim that

$$a_0 \prod_{k=1}^{p}[D, a_k]|D_0|^{-p} - a_0 \prod_{k=1}^{p}[D_0, a_k]|D_0|^{-p} \in \mathcal{L}_1 \tag{19}$$

for $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. To see the claim, let us write

$$\prod_{k=1}^{p}[D_0, a_k] = \sum_{\mathscr{A} \subset \{1,2,\ldots,p\}} \prod_{k=1}^{p} \begin{cases} [D, a_k] & \text{for } k \in \mathscr{A} \\ [D_1, a_k] & \text{for } k \notin \mathscr{A} \end{cases}$$

The summand corresponding to the case $\mathscr{A} = \{1, 2, \ldots, p\}$ coincides with $a_0 \prod_{k=1}^{p}[D, a_k]|D_0|^{-p}$, while all other summands belong to $\mathcal{L}_1$. Indeed, since there exists $k \notin \mathscr{A}$, it follows that the product contains the term $[D_1, a_k] \in \mathcal{L}_{p,\infty}$. Thus, such a summand belongs to $\mathcal{L}_{p,\infty} \cdot \mathcal{L}_{1,\infty} \subset \mathcal{L}_1$ (by equation (2)). Since the assertion holds for the spectral triple $(\mathcal{A}, H, D_0)$, we infer that it also holds for the spectral triple $(\mathcal{A}, H, D)$.

Case 4. If the spectral triple is $\mathcal{M}^{(p)}_{1,\infty}$-summable, then the proof of Theorem 16 (b) follows that of Theorem 16 (a) (see Cases 1, 2, 3 above) *mutatis mutandi*.

A. Computation of coboundaries

Computation 1. Let $\mathscr{A} \subset \{1, \dots, p\}$ be such that $m-1, m \in \mathscr{A}$. Let φ be a trace on $\mathcal{L}_{1,\infty}$ (respectively, on $\mathcal{M}_{1,\infty}$). The mapping on $\mathcal{A}^{\otimes(p+1)}$ defined by

$$c \longmapsto \varphi(\mathcal{W}_{\mathscr{A}}(c) D^{-|\mathscr{A}|})$$

is a Hochschild coboundary of the multilinear mapping

$$\begin{aligned}\theta\colon & a_0 \otimes \cdots \otimes a_{p-1} \\ &\longmapsto \frac{(-1)^{m-1}}{2}\varphi\Big(\Gamma a_0 \prod_{k=1}^{m-2}[b_k, a_k]\delta^2(a_{m-1}) \prod_{k=m}^{p-1}[b_{k+1}, a_k] D^{-|\mathscr{A}|}\Big).\end{aligned}$$

Proof. For brevity, we prove the assertion for $p = 2$ as the proof in the general case is very similar. We have

$$\begin{aligned}(b\theta)(a_0, a_1, a_2) &= \theta(a_0 a_1, a_2) - \theta(a_0, a_1 a_2) + \theta(a_2 a_0, a_1) \\ &= -\frac{1}{2}\varphi(\Gamma a_0 a_1 \delta^2(a_2)|D|^{-2}) \\ &\qquad + \frac{1}{2}\varphi(\Gamma a_0 \delta^2(a_1 a_2)|D|^{-2}) \\ &\qquad - \frac{1}{2}\varphi(\Gamma a_2 a_0 \delta^2(a_1)|D|^{-2}).\end{aligned}$$

Since Γ commutes with a_2 and since φ is a trace, it follows that

$$\begin{aligned}\varphi(\Gamma a_2 a_0 \delta^2(a_1)|D|^{-2}) &= \varphi(\Gamma a_0 \delta^2(a_1)|D|^{-2} a_2) \\ &= \varphi(\Gamma a_0 \delta^2(a_1) a_2 |D|^{-2}) + \varphi(\Gamma a_0 \delta^2(a_1)[|D|^{-2}, a_2]).\end{aligned}$$

We have

$$[|D|^{-2}, a_2] = -|D|^{-1}\delta(a_2)|D|^{-2} - |D|^{-2}\delta(a_2)|D|^{-1} \in \mathcal{L}_{1/3,\infty} \subset \mathcal{L}_1.$$

Therefore,

$$\varphi(\Gamma a_2 a_0 \delta^2(a_1)|D|^{-2}) = \varphi(\Gamma a_0 \delta^2(a_1) a_2 |D|^{-2} a_2).$$

Finally, we have

$$(b\theta)(a_0, a_1, a_2) = \frac{1}{2}\varphi(\Gamma a_0(\delta^2(a_1 a_2) - a_1\delta^2(a_2) - \delta^2(a_1)a_2)|D|^{-2})$$

and since

$$\delta^2(a_1 a_2) - a_1\delta^2(a_2) - \delta^2(a_1)a_2 = 2\delta(a_1)\delta(a_2),$$

the assertion follows. ☐

Computation 2. Let $\mathcal{A}_1, \mathcal{A}_2 \subset \{1, \dots, p\}$ be such that $|\mathcal{A}_1| = |\mathcal{A}_2|$ and

$$\mathcal{A}_1 \Delta \mathcal{A}_2 = \{m-1, m\}.$$

Let φ be a trace on $\mathcal{L}_{1,\infty}$ (respectively, on $\mathcal{M}_{1,\infty}$). The mapping on $\mathcal{A}^{\otimes(p+1)}$ defined by

$$c \longmapsto \varphi(\mathcal{W}_{\mathcal{A}_1}(c) D^{-|\mathcal{A}_1|}) + \varphi(\mathcal{W}_{\mathcal{A}_2}(c) D^{-|\mathcal{A}_2|})$$

is a Hochschild coboundary of the multilinear mapping

$$\begin{aligned}\theta\colon a_0 \otimes \cdots \otimes a_{p-1} \\ \longmapsto (-1)^{m-1}\varphi\Big(\Gamma a_0 \prod_{k=1}^{m-2}[b_k, a_k][F, \delta(a_{m-1})] \prod_{k=m}^{p-1}[b_{k+1}, a_k] D^{-|\mathcal{A}_1|}\Big).\end{aligned}$$

Proof. For brevity, we prove the assertion for $p = 2$ as the proof in the general case is a slight extension of this argument. We have

$$\begin{aligned}(b\theta)(a_0, a_1, a_2) &= \theta(a_0 a_1, a_2) - \theta(a_0, a_1 a_2) + \theta(a_2 a_0, a_1) \\ &= -\varphi(\Gamma a_0 a_1 [F, \delta(a_2)]|D|^{-1}) \\ &\quad + \varphi(\Gamma a_0 [F, \delta(a_1 a_2)]|D|^{-1}) \\ &\quad - \varphi(\Gamma a_2 a_0 [F, \delta(a_1)]|D|^{-1}).\end{aligned}$$

Since Γ commutes with a_2 and since φ is a trace, it follows that

$$\begin{aligned}\varphi(\Gamma a_2 a_0[F,\delta(a_1)]|D|^{-1}) &= \varphi(\Gamma a_0[F,\delta(a_1)]|D|^{-1}a_2) \\ &= \varphi(\Gamma a_0[F,\delta(a_1)]a_2|D|^{-1}) \\ &\quad + \varphi(\Gamma a_0[F,\delta(a_1)][|D|^{-1},a_2]).\end{aligned}$$

We have

$$[|D|^{-1},a_2] = -|D|^{-1}\delta(a_2)|D|^{-1} \in \mathcal{L}_{1/2,\infty} \subset \mathcal{L}_1.$$

Therefore,

$$\varphi(\Gamma a_2 a_0[F,\delta(a_1)]|D|^{-1}) = \varphi(\Gamma a_0[F,\delta(a_1)]a_2|D|^{-1}).$$

Finally, we have

$$(b\theta)(a_0,a_1,a_2) = \varphi(\Gamma a_0([F,\delta(a_1a_2)] - a_1[F,\delta(a_2)] - [F,\delta(a_1)]a_2)|D|^{-1}).$$

Since

$$[F,\delta(a_1a_2)] - a_1[F,\delta(a_2)] - [F,\delta(a_1)]a_2 = [F,a_1]\delta(a_2) + \delta(a_1)[F,a_2],$$

the assertion follows. □

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