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# Isospectrality for graph Laplacians under the change of coupling at graph vertices

Yulia Ershova, Irina I. Karpenko, and Alexander V. Kiselev[1]

**Abstract.** Laplacian operators on finite compact metric graphs are considered under the assumption that matching conditions at graph vertices are of $\delta$ and $\delta'$ types. An infinite set of trace formulae is obtained which link together two different quantum graphs under the assumption that their spectra coincide. The general case of graph Schrödinger operators is also considered, yielding the first trace formula. Tightness of results obtained under no additional restrictions on edge lengths is demonstrated by an example. Further examples are scrutinized when edge lengths are assumed to be rationally independent. In all but one of these impossibility of isospectral configurations is ascertained.

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## 1. Introduction

In the present paper we focus our attention on the so-called quantum graph, i.e., a metric graph $\Gamma$ with an associated second-order differential operator acting in Hilbert space $L^2(\Gamma)$ of square summable functions with an additional assumption that functions belonging to the domain of the operator are coupled by certain matching conditions at graph vertices. Recently these operators have attracted a considerable interest of both physicists and mathematicians due to a number of important physical applications. Extensive literature on the subject is surveyed in, e.g., [21, 6].

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The present paper is devoted to the study of an inverse spectral problem for Laplace and Schrödinger operators on finite compact metric graphs. One might classify the possible inverse problems on quantum graphs in the following way:

(i) given spectral data, edge potentials and matching conditions, to reconstruct the metric graph;

(ii) given the metric graph, edge potentials and spectral data, to reconstruct matching conditions;

(iii) given the metric graph, spectral data and matching conditions, to reconstruct edge potentials.

There exists an extensive literature devoted to problem (i). To name just a few, we would like to mention pioneering works [29, 18, 15] and later contributions [22, 24, 2, 16]. Different approaches to the same problem were developed, e.g., in [28, 4, 5], see also [23] for the analysis of a related problem. Problem (iii) is a generalization of the classical inverse problem for Sturm–Liouville operators and thus unsurprisingly has attracted by far the most interest; it is nonetheless beyond the scope of the present paper.

Problem (ii) has attracted much less attention. We believe it was first treated in [7] for the square of self-adjoint operator of the first derivative on a graph. Then, after being mentioned in [24, 2], it was treated in [3], but only in the case of star graphs. In our papers [10, 11] we suggested an approach based on the theory of boundary triples, leading to the asymptotic analysis of Weyl–Titchmarsh $M$-function of the graph.

Unlike [7, 3], in the present paper we consider the case of a general connected compact finite metric graph (in particular, this graph is allowed to possess cycles and loops), but only for two possible classes of matching conditions at graph vertices. Namely, each vertex is allowed to have matching of either $\delta$ or $\delta'$ type (see Section 2 for definitions). The named two classes singled out by us prove to be physically viable [12, 13].

## 2. Preliminaries

**2.1. Definition of the Laplace operator on a quantum graph.** We call $\Gamma = \Gamma(\mathbf{E}_\Gamma, \sigma)$ a *finite compact metric graph,* if it is a collection of a finite non-empty set $\mathbf{E}_\Gamma$ of compact intervals $\Delta_j = [x_{2j-1}, x_{2j}]$, $j = 1, 2, \dots, n$, called *edges*, and of a partition $\sigma$ of the set of endpoints $\{x_k\}_{k=1}^{2n}$ into $N$ classes, $\mathbf{V}_\Gamma = \bigcup_{m=1}^{N} V_m$. The equivalence classes $V_m$, $m = 1, 2, \dots, N$ will be called *vertices* and the number of elements belonging to the set $V_m$ will be called the *valence* (or, alternatively, *degree*) of the vertex $V_m$ (denoted $\deg V_m \equiv \gamma_m$).

With a finite compact metric graph $\Gamma$ we associate Hilbert spaces

$$L_2(\Gamma) = \bigoplus_{j=1}^{n} L_2(\Delta_j) \quad \text{and} \quad W_2^2(\Gamma) = \bigoplus_{j=1}^{n} W_2^2(\Delta_j).$$

These spaces obviously do not feel the graph connectivity, being the same for each graph with the same number of edges of same lengths.

In what follows, we single out two natural [12] classes of so-called *matching conditions* which lead to a properly defined self-adjoint operator on the graph $\Gamma$, namely, matching conditions of $\delta$ and $\delta'$ types. In order to describe these, we introduce the following notation. For a function $f \in W_2^2(\Gamma)$, we will use throughout the following definition of the normal derivative $\partial_n f(x_j)$ at the endpoints of the interval $\Delta_k$:

$$\partial_n f(x_j) = \begin{cases} f'(x_j) & \text{if } x_j \text{ is the left endpoint of the edge,} \\ -f'(x_j), & \text{if } x_j \text{ is the right endpoint of the edge.} \end{cases}$$

It will be convenient to introduce the following notation for a function $f \in W_2^2(\Gamma)$ at any graph vertex:

$$f^{\Sigma}(V_k) = \sum_{x_j \in V_k} f(x_j), \quad \partial_n^{\Sigma} f(V_k) = \sum_{x_j \in V_k} \partial_n f(x_j).$$

Associate either of the two symbols, $\delta$ or $\delta'$, to each vertex of the graph $\Gamma$. The graph thus obtained will be referred to as *marked* and denoted $\Gamma_\delta$. Any marked graph $\Gamma_\delta$ determines the lineal

$$\mathcal{D}(\Gamma_\delta) := \left\{ f \in W_2^2(\Gamma) \;\middle|\; \begin{array}{l} f \text{ is continuous at all internal vertices of } \delta \text{ type,} \\ \partial_n f(x_i) = \partial_n f(x_j) \text{ for all } i, j \text{ such that } x_i, x_j \in V \\ \text{at all internal vertices } V \text{ of } \delta' \text{ type} \end{array} \right\}.$$

Note that the latter definition imposes no restrictions on the functions from $\mathcal{D}(\Gamma_\delta)$ at boundary vertices of the graph, i.e., at vertices of valence 1. For reasons of convenience, we refer to all graph vertices of higher valence as *internal vertices* throughout.

We remark that if the vertex $V_k$ of valence $\gamma_k$ is of $\delta$ type, then obviously $f^{\Sigma}(V_k) = \gamma_k f(x_j),\ x_j \in V_k$. In the same way, for a vertex $V_k$ of $\delta'$ type one has $\partial_n^{\Sigma} f(V_k) = \gamma_k \partial_n f(x_j),\ x_j \in V_k$. In fact, throughout the rest of the paper we will only use the notation $f^{\Sigma}(V_k)$ and $\partial_n^{\Sigma}(V_k)$ in these two respective cases.

In Hilbert space $L_2(\Gamma)$ consider the operator $A_{\min}$, defined on each edge of the graph by the differential expression $-\frac{d^2}{dx^2}$, the domain of which $\operatorname{dom}(A_{\min})$ consists of all functions $f \in \mathcal{D}(\Gamma_\delta)$ such that

$$f^{\Sigma}(V_k) = 0, \quad \partial_n^{\Sigma} f(V_k) = 0 \quad \text{for all } k. \tag{1}$$

Obviously, $A_{\min}$ is a closed symmetric operator, which will be henceforth referred to as *the symmetric operator of the graph* $\Gamma_\delta$. The adjoint to it

$$A_{\max} := A_{\min}^*$$

is defined by the same differential expression on the domain $\mathcal{D}(\Gamma_\delta)$. The deficiency indices of $A_{\min}$ are equal to $(N, N)$.

We are now ready to define the Laplacian $A_{\vec{\alpha}}$ on the graph $\Gamma_\delta$ which is an operator of the negative second derivative on functions from $f \in \mathcal{D}(\Gamma_\delta)$ subject to the following additional *matching conditions*.

($\delta$) If $V_k$ is of $\delta$ type, then

$$\sum_{x_j \in V_k} \partial_n f(x_j) = \frac{\alpha_k}{\gamma_k} f^{\Sigma}(V_k).$$

($\delta'$) If $V_k$ is of $\delta'$ type, then

$$\sum_{x_j \in V_k} f(x_j) = -\frac{\alpha_k}{\gamma_k} \partial_n^{\Sigma} f(V_k).$$

Here $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_N)$ is a set of arbitrary real constants which we will refer to as *coupling constants*.

Note that matching conditions at internal vertices reflect the graph connectivity: if two graphs having the same set of edges have different topology, the resulting operators are different.

Provided that all coupling constants $\alpha_m$, $m = 1 \ldots N$, are real, it is easy to ascertain that the operator $A_{\vec{\alpha}}$ is a proper self-adjoint extension of the operator $A_{\min}$ in Hilbert space $L_2(\Gamma)$ [12, 17].

Clearly, the self-adjoint operator thus defined on a finite compact metric graph has purely discrete spectrum that accumulates to $+\infty$.

Note that without loss of generality each edge $\Delta_j$ of the graph $\Gamma$ can be considered to be an interval $[0, l_j]$, where $l_j = x_{2j} - x_{2j-1}$, $j = 1, \ldots, n$ is the length of the corresponding edge. Throughout the present paper we will therefore only consider this situation.

In order to treat the inverse spectral problem (ii) for graph Laplacians, we will first need to get an explicit expression for the generalized Weyl–Titchmarsh $M$-function of the operator considered. The most elegant and straightforward way of doing so is in our view by utilizing the apparatus of boundary triples developed in [14, 19, 20, 8]. We briefly recall what is essential for our work.

**2.2. Boundary triples and the Weyl–Titchmarsh matrix $M$-function.** Suppose that $A_{\min}$ is a symmetric densely defined closed linear operator acting in Hilbert space $H$. Assume that $A_{\min}$ is completely non-self-adjoint (simple),[1] i.e., there exists no reducing subspace $H_0$ in $H$ such that the restriction $A_{\min}|H_0$ is a selfadjoint operator in $H_0$. Further assume that the deficiency indices of $A_{\min}$ (possibly being infinite) are equal: $n_+(A_{\min}) = n_-(A_{\min}) \leq \infty$.

**Definition 2.1** ([14, 19, 8]). Let $\Gamma_0,\ \Gamma_1$ be linear mappings of $\operatorname{dom}(A_{\max})$ to $\mathcal{H}$, a separable Hilbert space. The triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is called *a boundary triple* for the operator $A_{\max}$ if

(1) for all $f, g \in \operatorname{dom}(A_{\max})$

$$(A_{\max} f, g)_H - (f, A_{\max} g)_H = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}};$$

(2) the mapping $\gamma$ defined as $f \longmapsto (\Gamma_0 f; \Gamma_1 f)$, $f \in \operatorname{dom}(A_{\max})$ is surjective, i.e., for all $Y_0, Y_1 \in \mathcal{H}$ there exists an element $y \in \operatorname{dom}(A_{\max})$ such that

$$\Gamma_0 y = Y_0, \quad \Gamma_1 y = Y_1.$$

A boundary triple can be constructed for any operator $A_{\max}$ of the class considered. Moreover, the space $\mathcal{H}$ can be chosen in a way such that $\dim \mathcal{H} = n_+ = n_-$. In particular one has $A_{\min} = A_{\max}|_{(\ker \Gamma_0 \cap \ker \Gamma_1)}$.

We further single out two proper self-adjoint extensions of the operator $A_{\min}$:

$$A_\infty := A_{\max}|\ker \Gamma_0, \quad A_0 := A_{\max}|\ker \Gamma_1.$$

[1] The condition of simplicity of $A_{\min}$ was studied in [1], where necessary and sufficient conditions of this property were obtained. In the simplest case, provided that all edge lengths are rationally independent and the graph contains no loops, simplicity is guaranteed. For the problems discussed in the present paper, simplicity is in fact irrelevant. Indeed, even if $A_{\min}$ is not simple, under the condition of isospectrality of two graph Laplacians the formula (11) of Section 4 still holds, see below for details, which is sufficient for our analysis, although in this case not all of the spectrum of a graph Laplacian is "visible" to both factors of (11).

A nontrivial extension $A_B$ of the operator $A_{\min}$ such that $A_{\min} \subset A_B \subset A_{\max}$ is called *almost solvable* if for every $f \in \operatorname{dom}(A_{\max})$

$$f \in \operatorname{dom}(A_B) \iff \Gamma_1 f = B\Gamma_0 f$$

for a bounded in $\mathcal{H}$ operator $B$.

The generalized Weyl–Titchmarsh $M$-function is then defined as follows.

**Definition 2.2** ([8, 14, 20]). Let $A_{\min}$ be a closed densely defined symmetric operator, $n_+(A_{\min}) = n_-(A_{\min})$, $(\mathcal{H}, \Gamma_0, \Gamma_1)$ being its boundary triple. The operator-function $M(\lambda)$, defined by

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \quad f_\lambda \in \ker(A_{\max} - \lambda),\ \lambda \in \mathbb{C}_\pm,$$

is called the *Weyl–Titchmarsh $M$-function* of a symmetric operator $A_{\min}$.

The following Theorem describing properties of the $M$-function clarifies its meaning.

**Theorem 2.1** ([14, 8], in the form adopted by Ryzhov [30]). *Let $M(\lambda)$ be the $M$-function of a symmetric operator $A_{\min}$ with equal deficiency indices $(n_+(A_{\min}) = n_-(A_{\min}) < \infty)$. Let $A_B$ be an almost solvable extension of $A_{\min}$ corresponding to a bounded operator $B$. Then for every $\lambda \in \mathbb{C}$:*

(1) *$M(\lambda)$ is an analytic operator-function when $\operatorname{Im}\lambda \neq 0$, its values being bounded linear operators in $\mathcal{H}$;*
(2) $(\operatorname{Im} M(\lambda))\operatorname{Im}\lambda > 0$ *when* $\operatorname{Im}\lambda \neq 0$;
(3) $M(\lambda)^* = M(\bar{\lambda})$ *when* $\operatorname{Im}\lambda \neq 0$;
(4) *$\lambda_0 \in \rho(A_B)$ if and only if $(B - M(\lambda))^{-1}$ admits analytic continuation into the point $\lambda_0$.*

## 3. Weyl–Titchmarsh function for the graph Laplacian

In this section we derive an explicit formula for Weyl–Titchmarsh $M$-function pertaining to graph Laplacians of the class considered.

**Theorem 3.1.** *Let $A_{\min}$ be the symmetric operator of the graph $\Gamma_\delta$ defined in (1). Then a boundary triple for the operator $A_{\max}$ such that any proper self-adjoint extension $A_{\vec{\alpha}}$ of section 2.1 is almost solvable can be defined as follows: $\mathcal{H} = \mathbb{C}^N$,*

$$(\Gamma_0 f)_k := \frac{1}{\gamma_k} \begin{cases} f^\Sigma(V_k) & \text{if } V_k \text{ is a vertex of } \delta \text{ type,} \\ \partial_n^\Sigma f(V_k) & \text{if } V_k \text{ is a vertex of } \delta' \text{ type,} \end{cases} \tag{2}$$

$$(\Gamma_1 f)_k := \begin{cases} \partial_n^\Sigma f(V_k) & \text{if } V_k \text{ is a vertex of } \delta \text{ type}, \\ -f^\Sigma(V_k) & \text{if } V_k \text{ is a vertex of } \delta' \text{ type}, \end{cases} \tag{3}$$

*where* $\gamma_k = \deg V_k$.

*In this setting, the matrix* $B$ *parameterizing the almost solvable extension* $A_{\vec{\alpha}}$ *is diagonal,* $B = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$*, where* $\{\alpha_k\}_{k=1}^N$ *are coupling constants of conditions* ($\delta$) *and* ($\delta'$).

*Proof.* It suffices to observe that

$$(A_{\max} f, g) - (f, A_{\max} g) = \sum_{k=1}^{N} \frac{1}{\gamma_k} (\partial_n^\Sigma f(V_k) \overline{g^\Sigma(V_k)} - f^\Sigma(V_k) \overline{\partial_n^\Sigma g(V_k)}).$$

Then formulae (2) and (3) and conditions ($\delta$) and ($\delta'$), immediately imply the claimed form of the matrix $B$. □

**Lemma 3.1.** *Assume that* $V_k$, $V_j$ *are two adjacent vertices of the graph* $\Gamma_\delta$ *connected by an edge* $e_{kj}$ *of length* $l_{kj}$, $\gamma_k = \deg V_k$. *Let the function* $f$ *be in* $\ker(A_{\max} - \lambda)$ *and put* $f_{kj} := f|e_{kj}$.

(i) *If* $V_k$, $V_j$ *are both of* $\delta$ *type and* $f^\Sigma(V_k) = \gamma_k$, $f^\Sigma(V_j) = 0$, *then*

$$\partial_n f_{kj}(V_k) = -\mu \cot \mu l_{kj}, \; \partial_n f_{kj}(V_j) = \frac{\mu}{\sin \mu l_{kj}} \tag{4}$$

(ii) *If* $V_k$, $V_j$ *are two vertices of* $\delta$ *and* $\delta'$ *types, respectively, and* $f^\Sigma(V_k) = \gamma_k$, $\partial_n^\Sigma f(V_j) = 0$, *then*

$$\partial_n f_{kj}(V_k) = \mu \tan \mu l_{kj}, \; f_{kj}(V_j) = \frac{1}{\cos \mu l_{kj}}. \tag{5}$$

(iii) *If* $V_k$, $V_j$ *are two vertices of* $\delta'$ *and* $\delta$ *types, respectively, and* $\partial_n^\Sigma f(V_k) = \gamma_k$, $f^\Sigma(V_j) = 0$, *then*

$$f_{kj}(V_k) = -\frac{1}{\mu} \tan \mu l_{kj}, \; \partial_n f_{kj}(V_j) = -\frac{1}{\cos \mu l_{kj}}. \tag{6}$$

(iv) *If* $V_k$, $V_j$ *are both of* $\delta'$ *type and* $\partial_n^\Sigma f(V_k) = 1$, $\partial_n^\Sigma f(V_j) = 0$, *then*

$$f_{kj}(V_k) = \frac{1}{\mu} \cot \mu l_{kj}, \; f_{kj}(V_j) = \frac{1}{\mu \sin \mu l_{kj}}. \tag{7}$$

*Here* $\mu = \sqrt{\lambda}$ *with the branch so chosen that* $\operatorname{Im} \mu \geq 0$.

The proof is an explicit computation.

**Theorem 3.2.** *Let $\Gamma_\delta$ be a marked compact metric graph and let the operator $A_{\min}$ be the symmetric operator* (1) *of the graph $\Gamma_\delta$. Assume that the boundary triple for $A_{\max}$ is $(\mathbb{C}^N, \Gamma_0, \Gamma_1)$, where $N$ is the number of vertices in $\Gamma$, whereas the operators $\Gamma_0$ and $\Gamma_1$ are defined by* (2) *and* (3)*. Then the generalized Weyl–Titchmarsh $M$-function is a $N \times N$ matrix with matrix elements given by the following formula for a vertex $V_k$ of $\delta$ type: $m_{jk}(\lambda)$ is given by*

$$\begin{cases} -\mu\Big(\sum_{\Delta_t\in E_k}\cot\mu l_t - \sum_{\Delta_t\in E'_k}\tan\mu l_t - 2\sum_{\Delta_t\in L_k}\tan\frac{\mu l_t}{2}\Big), & j=k,\\ \mu\sum_{\Delta_t\in C_{kj}}\frac{1}{\sin\mu l_t}, & j\neq k,\ V_j \text{ is a vertex of } \delta \text{ type adjacent to } V_k,\\ -\sum_{\Delta_t\in C'_{kj}}\frac{1}{\cos\mu l_t}, & j\neq k,\ V_j \text{ is a vertex of } \delta' \text{ type adjacent to } V_k,\\ 0, & j\neq k,\ V_j \text{ is a vertex not adjacent to } V_k, \end{cases} \tag{8}$$

*and by the following formula for a vertex $V_k$ of $\delta'$ type: $m_{jk}(\lambda)$ is given by*

$$\begin{cases} -\frac{1}{\mu}\Big(\sum_{\Delta_t\in E_k}\cot\mu l_t - \sum_{\Delta_t\in E'_k}\tan\mu l_t + 2\sum_{\Delta_t\in L_k}\cot\frac{\mu l_t}{2}\Big), & j=k,\\ -\sum_{\Delta_t\in C'_{kj}}\frac{1}{\cos\mu l_t}, & j\neq k,\ V_j \text{ is a vertex of } \delta \text{ type adjacent to } V_k,\\ -\frac{1}{\mu}\sum_{\Delta_t\in C_{kj}}\frac{1}{\sin\mu l_t}, & j\neq k,\ V_j \text{ is a vertex of } \delta' \text{ type adjacent to } V_k,\\ 0, & j\neq k,\ V_j \text{ is a vertex not adjacent to } V_k. \end{cases} \tag{9}$$

*Here $\mu = \sqrt{\lambda}$ (the branch such that $\operatorname{Im}\mu \geq 0$), $l_t$ is the length of the edge $\Delta_t$, $L_k$ is the set of loops at the vertex $V_k$, $E_k$ is the set of graph edges incident to the vertex $V_k$ with both endpoints of the same type, $E'_k$ is the set of graph edges incident to the vertex $V_k$ with endpoints of different type, $C_{kj}$ is the set of graph edges connecting vertices $V_k$ and $V_j$ of the same type, and finally, $C'_{kj}$ is the set of graph edges connecting vertices $V_k$ and $V_j$ of different types.*

*Proof.* The proof is an explicit computation. Consider a function

$$f_\lambda \in \ker(A_{\max} - \lambda).$$

Let

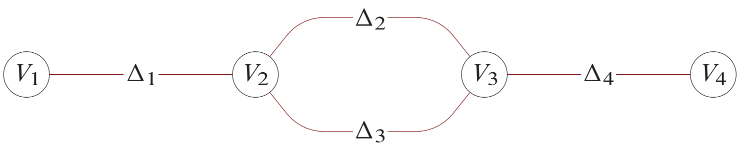
$$\Gamma_0 f_\lambda = e_k,$$

where $e_k = (0, 0, \dots, 1, 0, \dots, 0)^T$ with 1 in the $k$th position.

From Lemma 3.1 one gets the explicit description of $f_\lambda$; it is then possible to compute $\Gamma_1 f_\lambda$ which yields the $k$th column of the M-matrix sought provided that the graph has no loops.

If $\Gamma_\delta$ contains loops, it is easy to see that these lead to contributions in diagonal entries as claimed. $\square$

**Example 3.1.** Consider the following graph $\Gamma_\delta$:




Let the vertices $V_1$, $V_2$ be of $\delta$ type and the vertices $V_3$, $V_4$ – of $\delta'$ type. Then within the setup of Theorem 3.2 the Weyl–Titchmarsh function admits the form

$$M(\lambda) = \begin{pmatrix} -\mu\cot\mu l_1 & \mu\csc\mu l_1 & 0 & 0 \\ \mu\csc\mu l_1 & -\mu\left(\cot\mu l_1 - \sum_{t=2}^{3}\tan\mu l_t\right) & -\sum_{j=2}^{3}\sec\mu l_t & 0 \\ 0 & -\sum_{j=2}^{3}\sec\mu l_t & -\frac{1}{\mu}\left(\cot\mu l_4 - \sum_{t=2}^{3}\tan\mu l_t\right) & -\frac{1}{\mu}\csc\mu l_4 \\ 0 & 0 & -\frac{1}{\mu}\csc\mu l_4 & -\frac{1}{\mu}\cot\mu l_4 \end{pmatrix}.$$

## 4. Trace formulae for isospectral graph Laplacians

In the present section, we apply the mathematical apparatus developed in Section 3 in order to study isospectral (i.e., having the same spectrum, counting multiplicities) quantum Laplacians defined on a finite compact metric graph $\Gamma_\delta$.

Considering a pair of such Laplacians which differ only in coupling constants defining matching conditions we will derive an infinite series of trace formulae (cf. [10] for an analogous, although written in a different form, result in the case of all vertices being of $\delta$ type).

**Theorem 4.1.** *Let $\Gamma_\delta$ be a finite compact metric graph with $N$ vertices in which $V_1, V_2, \dots, V_{N_1}$ are of $\delta$ type, whereas $V_{N_1+1}, V_{N_1+2}, \dots, V_N$ are of $\delta'$ type. Let $A_{\vec\alpha}$, $A_{\vec{\tilde\alpha}}$ be two graph Laplacians on $\Gamma_\delta$ parameterized by coupling constants $\{\alpha_k\}$ and $\{\tilde\alpha_k\}$, $k = \overline{1,N}$, respectively. If (point) spectra of the operators $A_{\vec\alpha}$ and $A_{\vec{\tilde\alpha}}$ coincide counting multiplicities, the following infinite set of trace formulae holds*:

$$\sum_{i=1}^{N_1} \frac{(-\alpha_i)^m}{\gamma_i^m} + \sum_{i=N_1+1,\ \alpha_i\neq 0}^{N} \frac{\gamma_i^m}{\alpha_i^m} = \sum_{i=1}^{N_1} \frac{(-\tilde\alpha_i)^m}{\gamma_i^m} + \sum_{i=N_1+1,\ \tilde\alpha_i\neq 0}^{N} \frac{\gamma_i^m}{\tilde\alpha_i^m}, \tag{10}$$

*$m = 1, 2, \dots,$ where as above $\gamma_i = \deg V_i$. Moreover, the sets $\{\alpha_{N_1+1}, \dots, \alpha_N\}$ and $\{\tilde\alpha_{N_1+1}, \dots, \tilde\alpha_N\}$ have equal numbers of zero elements.*

*Proof.* Using results of [26, Chapter I], see [10] for details, it is easy to ascertain that the fraction $\det(B - M(\lambda))/\det(\widetilde B - M(\lambda))$ is an entire function of exponential type of order not greater than $1/2$ under the assumption that the spectra of $A_{\vec\alpha}$ and $A_{\vec{\tilde\alpha}}$ coincide (recall, that $B = \operatorname{diag}\{\alpha_1, \dots, \alpha_N\}$ and $\tilde B = \operatorname{diag}\{\tilde\alpha_1, \dots, \tilde\alpha_N\}$). Moreover, it has no finite zeroes. Therefore by Hadamard theorem [26] one has

$$\frac{\det(B - M(\lambda))}{\det(\widetilde B - M(\lambda))} = \exp(a) \tag{11}$$

for some constant $a$.

Consider asymptotic expansions of the functions $\det(B - M(\lambda))$ and $\det(\widetilde B - M(\lambda))$ as $\lambda \to -\infty$ along the real line. Using the asymptotic expansion for $M(\lambda)$ provided by Theorem 3.2 one has

$$\det(B - M(\lambda)) = \prod_{i=1}^{N_1} (\alpha_i + \gamma_i \tau) \prod_{i=N_1+1}^{N} \left(\alpha_i - \frac{\gamma_i}{\tau}\right) + o(\tau^{-M}),$$

$$\det(\widetilde B - M(\lambda)) = \prod_{i=1}^{N_1} (\tilde\alpha_i + \gamma_i \tau) \prod_{i=N_1+1}^{N} \left(\tilde\alpha_i - \frac{\gamma_i}{\tau}\right) + o(\tau^{-M})$$

for any natural $M > 0$, where $\tau = -i\sqrt{\lambda} \to +\infty$. Therefore, passing to the limit in (11) as $\tau \to +\infty$, one either gets

$$\exp(a) = \frac{\prod_{i=N_1+1,\ \alpha_i \neq 0}^{N} \alpha_i}{\prod_{i=N_1+1,\ \tilde{\alpha}_i \neq 0}^{N} \tilde{\alpha}_i} \frac{\prod_{i=N_1+1,\ \alpha_i = 0}^{N} \gamma_i}{\prod_{i=N_1+1,\ \tilde{\alpha}_i = 0}^{N} \gamma_i}$$

in the case when the sets $\{\alpha_{N_1+1}, \dots, \alpha_N\}$ and $\{\tilde{\alpha}_{N_1+1}, \dots, \tilde{\alpha}_N\}$ have equal numbers of zero elements, or faces a contradiction. Having divided both sides of (11) by $\exp(a)$ and then taking the logarithm of the result, one arrives at

$$\sum_{i=1}^{N_1} \ln\Big(1 + \frac{\alpha_i}{\gamma_i}\frac{1}{\tau}\Big) + \sum_{i=N_1+1,\ \alpha_i \neq 0}^{N} \ln\Big(1 - \frac{\gamma_i}{\alpha_i}\frac{1}{\tau}\Big)$$
$$- \sum_{i=1}^{N_1} \ln\Big(1 + \frac{\tilde{\alpha}_i}{\gamma_i}\frac{1}{\tau}\Big) - \sum_{i=N_1+1,\ \tilde{\alpha}_i \neq 0}^{N} \ln\Big(1 - \frac{\gamma_i}{\tilde{\alpha}_i}\frac{1}{\tau}\Big) + o(\tau^{-M}) = 0.$$

The Taylor expansion of logarithms yields that for any natural $M$

$$\begin{aligned} &-\sum_{j=1}^{M} \frac{(-1)^j}{j\tau^j} \sum_{i=1}^{N_1} \Big(\frac{\alpha_i}{\gamma_i}\Big)^j - \sum_{j=1}^{M} \frac{1}{j\tau^j} \sum_{i=N_1+1,\ \alpha_i \neq 0}^{N} \Big(\frac{\gamma_i}{\alpha_i}\Big)^j \\ &+ \sum_{j=1}^{M} \frac{(-1)^j}{j\tau^j} \sum_{i=1}^{N_1} \Big(\frac{\tilde{\alpha}_i}{\gamma_i}\Big)^j + \sum_{j=1}^{M} \frac{1}{j\tau^j} \sum_{i=N_1+1,\ \tilde{\alpha}_i \neq 0}^{N} \Big(\frac{\gamma_i}{\tilde{\alpha}_i}\Big)^j + o(\tau^{-M}) = 0. \end{aligned} \tag{12}$$

Comparing coefficients at equal powers of $\tau$ now completes the proof. □

We will revisit the trace formulae of Theorem 4.1 in the next section, see Corollary 5.1. The named Corollary reformulates necessary conditions of isospectrality for two graph Laplacians in a much more transparent and easier to check form. The remainder of the present section is devoted to a number of results which extend the applicability of the approach developed above in two different directions.

First, we point out that the results obtained in this section so far allow generalization to the case of Shrödinger operators on finite compact metric graphs. These operators in the case when all edge potentials are assumed to be summable are correctly defined by the differential expression $-d^2/dx^2 + q(x)$, where $q \in L^1(\Gamma)$, on the same domain as the corresponding graph Laplacians (see conditions $(\delta)$ and $(\delta')$ of Section 2).

If no additional smoothness is required of edge potentials $q_j := q|\Delta_j$, it is only possible to obtain the first trace formula (i.e., for $m = 1$). If however edge potentials are assumed to be $C^\infty$, the full countable set of trace formulae is available. In the present paper we will confine ourselves to the former case.

**Theorem 4.2.** *Let $\Gamma_\delta$ be a marked finite compact metric graph having $N$ vertices in which $V_1, V_2, ..., V_{N_1}$ are vertices of $\delta$ type, whereas $V_{N_1+1}, V_{N_1+2}, ..., V_N$ are of $\delta'$ type. Let $\tilde{A}_{\tilde{\vec\alpha}}$ and $A_{\vec\alpha}$ be two Schrödinger operators on the graph $\Gamma_\delta$ parameterized by coupling constants $\{\alpha_k\}$ and $\{\tilde\alpha_k\}$, $k = \overline{1, N}$, respectively. Let all edge potentials[2] $\tilde q_i, q_i \in L_1(\Delta_i)$ for all $i = 1, \dots, n$. Let the (point) spectra of these two operators (counting multiplicities) be equal, $\sigma(\tilde{A}_{\tilde{\vec\alpha}}) = \sigma(A_{\vec\alpha})$. Then the numbers of zero coupling constants at vertices of $\delta'$ type are equal and*

$$-\sum_{i=1}^{N_1}\frac{\alpha_i}{\gamma_i} + \sum_{i=N_1+1:\ \alpha_i\neq 0}^{N}\frac{\gamma_i}{\alpha_i} = -\sum_{i=1}^{N_1}\frac{\tilde\alpha_i}{\gamma_i} + \sum_{i=N_1+1:\ \tilde\alpha_i\neq 0}^{N}\frac{\gamma_i}{\tilde\alpha_i},$$

*where $\gamma_i = \deg V_i$.*

The *proof* of this Theorem can be obtained using WKB asymptotics of regular Sturm–Liouville problem solutions [27], see [11] for more details.

**Corollary 4.1.** *Under the hypotheses of Theorem 4.2,*

(i) *if $\alpha_i = \alpha$ and $\tilde\alpha_i = \tilde\alpha$ for all graph vertices of $\delta$ type, $i = 1, \dots, N_1$, whereas $\alpha_i = -1/\alpha$ and $\tilde\alpha_i = -1/\tilde\alpha$ for all graph vertices of $\delta'$ type, $i = N_1 + 1, \dots, N$, then $\tilde\alpha = \alpha$;*

(ii) *if $\alpha_i = \alpha$ and $\tilde\alpha_i = \tilde\alpha$ for all graph vertices, $i = 1, \dots, N$, then either $\tilde\alpha = \alpha$ or $\tilde\alpha\alpha = -(\sum_{i=N_1+1}^{N}\gamma_i)/(\sum_{i=1}^{N_1}\gamma_i^{-1})$, where $\gamma_i = \deg V_i$.*

We finish this section with a complement to Theorem 4.1 which prohibits the "decoupled" Laplacian $A_\infty$ to have spectrum coinciding with that of an operator $A_{\vec\alpha}$ for any $\vec\alpha$ provided that all graph vertices are of $\delta$ type together with an analogous result for the case of $\delta'$ vertices. In order to prove this, one can use the following result.

**Proposition 4.1** ([8]). *Let $A_{\min}$ be a closed densely defined symmetric operator with equal deficiency indices, $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triple of the operator $A_{\max}$, and finally let $M(\lambda)$ be the corresponding Weyl–Titchmarsh function. Then for any bounded in $\mathcal{H}$ self-adjoint operator $K$*

[2] Note, that we do not need to assume here that the potentials are the same for operators $\tilde{A}_{\tilde{\vec\alpha}}$ and $A_{\vec\alpha}$.

(1) $(\mathcal{H},\ \Gamma_0,\ \Gamma_1 + K\Gamma_0)$ *is also a boundary triple for the operator* $A_{\max}$. *Moreover, the corresponding Weyl–Titchmarsh function admits the form* $\widehat{M}(\lambda) = M(\lambda) + K$;

(2) *If the operator* $A_B$ *is an almost solvable with respect to the boundary triple* $(\mathcal{H},\ \Gamma_0,\ \Gamma_1)$ *extension of the operator* $A_{\min}$, *this operator is almost solvable with respect to the boundary triple* $(\mathcal{H},\ \Gamma_0,\ \Gamma_1 + K\Gamma_0)$ *as well. Its parameterization is then* $\widehat{B} = B + K$.

This allows to prove the following

**Theorem 4.3.** *Let* $\Gamma_\delta$ *be a finite compact metric graph with* $N$ *vertices. Let* $A_{\vec{\alpha}}$ *be a graph Laplacian defined on* $\Gamma$ *with matching conditions of* $\delta$ *type at all vertices. Then the spectra of the operators* $A_{\vec{\alpha}}$ *and* $A_\infty$ *coincide for no non-zero parameterizing matrix* $B = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$.

*Proof.* First assume that the matrix $B$ is invertible. In this case, pass over to the boundary triple $(\mathcal{H},\ \widehat{\Gamma}_0 = \Gamma_1,\ \widehat{\Gamma}_1 = -\Gamma_0)$. The operator $A_B$ is clearly an almost solvable extension of $A_{\min}$ with respect to this triple, i.e., $A_B = A_{\widehat{B}}$, where $\widehat{B} = -B^{-1}$. The Weyl–Titchmarsh function admits the form $\widehat{M}(\lambda) = -M(\lambda)^{-1}$, and the operator $A_\infty$ in this "new" boundary triple corresponds to $A_0$. Hence in terms of the amended boundary triple one has isospectrality for the operators $A_{B^{-1}}$ and $A_0$.

In the case of all vertices having $\delta$ type one clearly has the following asymptotics for $M(\lambda)$,

$$M(\lambda) = -\tau\Gamma_N + o(\tau^{-M}) \qquad \text{for all } M > 0$$

and hence

$$\widehat{M}(\lambda) = (1/\tau)\Gamma_N^{-1} + o(\tau^{-M}) \quad \text{for all } M > 0,$$

where $\Gamma_N = \mathrm{diag}\{\gamma_1, \ldots, \gamma_N\}$.

As shown in the proof of Theorem 4.1, this asymptotic expansion together with the condition of isospectrality yields that both matrices parameterizing extensions must have equal ranks, which leads to a contradiction.

Now let the matrix $B = \mathrm{diag}\{\alpha_1, \alpha_2, ..., \alpha_N\}$ degenerate. Consider the new boundary triple defined as follows: $(\mathcal{H},\ \widehat{\Gamma}_0 = \Gamma_0,\ \widehat{\Gamma}_1 = \Gamma_1 + \alpha\Gamma_0)$, where $\alpha > \max|\alpha_i|,\ i = \overline{1, N}$. Proposition 4.1 gives $A_B = A_{\widehat{B}}$, where $\widehat{B} = \mathrm{diag}\{\alpha_1 + \alpha, \alpha_2 + \alpha, ..., \alpha_N + \alpha\}$ is a non-degenerate matrix, whereas the operator $A_\infty$ remains the same. The Weyl–Titchmarsh function admits the form $\widehat{M}(\lambda) = M(\lambda) + \alpha I$. We have thus reduced the situation to the one already considered. $\square$

An analogous result holds with a similar proof in the case when all graph vertices are of $\delta'$ type.

**Theorem 4.4.** *Let $\Gamma_\delta$ be a finite compact metric graph with $N$ vertices. Let $A_{\vec\alpha}$ be a graph Laplacian defined on $\Gamma$ with matching conditions of $\delta'$ type at all vertices. Then the spectra of the operators $A_{\vec\alpha}$ and $A_\infty$ coincide for no non-zero parameterizing matrix $B = \operatorname{diag}\{\alpha_1, \alpha_2, \dots, \alpha_N\}$.*

## 5. Isospectral graphs by examples

**5.1. Further dissemination of trace formulae.** We start with the following

**Lemma 5.1.** *Every solution of the infinite system of equations*

$$\sum_{i=1}^{N} \beta_i^m = \sum_{i=1}^{N} \tilde\beta_i^m \quad m = 1, 2, \dots \tag{13}$$

*is one of the following*:

$$\beta_i = \tilde\beta_{k_i} \qquad i = 1, 2, \dots, N, \tag{14}$$

*where $\{k_1, k_2, \dots, k_N\}$ is a permutation of the finite sequence $\{1, 2, \dots, N\}$. Conversely, each of* (14) *solves the system* (13).

*Proof.* First assume that all $|\beta_i|$ are different. Let $\beta_{i_0}$ be the largest (by modulus) element of the sequence $\{\beta_1, \beta_2, \dots, \beta_N\}$. Divide both parts of (13) by $\beta_{i_0}^m$ and pass to the limit as $m \to \infty$. Then $\beta_{i_0} = \tilde\beta_{j_0}$ for some $j_0$, where $\tilde\beta_{j_0}$ is the largest (by modulus) element of the sequence $\{\tilde\beta_1, \tilde\beta_2, \dots, \tilde\beta_N\}$. (13) then takes the form

$$\sum_{i=1, i\neq i_0}^{N} \beta_i^m = \sum_{j=1, j\neq j_0}^{N} \tilde\beta_j^m \quad m = 1, 2, \dots.$$

Repeating the argument $N - 2$ times, one arrives at the claimed result. The generalization to the case of repeating $|\beta_i|$ is trivial. □

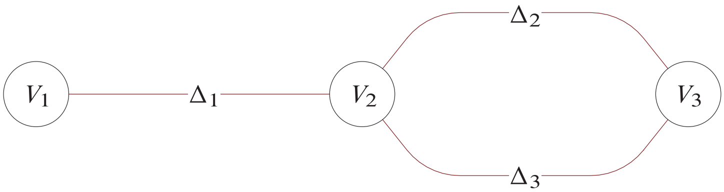
In particular together with Theorem 4.1 this yields that not only the numbers of zero coupling constants at $\delta'$ type vertices have to be equal in $\vec\alpha$ and $\vec{\tilde\alpha}$, but also the *total* numbers of zero coupling constants have to be the same under the condition of isospectrality.

Hence, Theorem 4.1, although leaving possibilities for isospectrality of graph Laplacians, significantly narrows down the set of "opportunities" for this isospectrality. Precisely, one has the following

**Corollary 5.1.** *Let $\Gamma_\delta$ be a finite compact metric graph with $N$ vertices. Let $A_{\vec{\alpha}}$, $A_{\vec{\tilde{\alpha}}}$ be two graph Laplacians on $\Gamma_\delta$ parameterized by coupling constants $\{\alpha_k\}$ and $\{\tilde{\alpha}_k\}$, $k = \overline{1, N}$, respectively. Then the graph Laplacians $A_{\vec{\alpha}}$ and $A_{\vec{\tilde{\alpha}}}$ are isospectral only if the sets $S$ and $\tilde{S}$ are permutations of each other, where $S = \{\sigma_i\}_{i=1}^N$ with $\sigma_j = -\alpha_j/\gamma_j$ for $V_j$ of $\delta$ type; $\sigma_j = \gamma_j/\alpha_j$ for $V_j$ of $\delta'$ type with non-zero coupling constant; $\sigma_j = 0$ in the remaining case. The set $\tilde{S}$ is defined in absolutely the same way based on coupling constants $\{\tilde{\alpha}_k\}$.*

Using the latter Corollary, for any given graph $\Gamma$ one could, at least theoretically, try every "allowed" configuration of coupling constants one by one and thus assess directly, which of these (if any) lead to isospectral configurations.

**Example 5.1.** Let $\Gamma_\delta$ be a decorated "lasso" graph of three vertices all of them being of $\delta$ type.




Let all three edges of the graph be of length 1. Let $\vec{\alpha} = (a, 3b, 2c)$ for arbitrary real $a, b, c \in \mathbb{R}$. Then $A_{\vec{\tilde{\alpha}}}$ is isospectral to $A_{\vec{\alpha}}$ for $\vec{\tilde{\alpha}} = (b, 3c, 2a)$ provided, that $2b = a + c$ and to $\vec{\tilde{\alpha}} = (c, 3a, 2b)$ provided, that $2a = b + c$.

Note, that both configurations $(b, 3c, 2a)$ and $(c, 3a, 2b)$ are among allowed by Corollary 5.1.

At the same time, if the lengths of graph edges are rationally independent, there are no isospectral configurations.

The *proof* is an explicit computation.

This example demonstrates that if one wishes to go one step further than Theorem 4.1 in the analysis of isospectrality for graph Laplacians, one needs to impose some additional restrictions on edge lengths. The analysis presented in Section 4, based on the asymptotic behavior of Weyl–Titchmarsh $M$-function, shows that the information derived from the leading term of its asymptotics carries no information on edge lengths. This asks for some additional considerations, to which the remainder of the present section is devoted.

*Henceforth we will assume that all edge lengths are rationally independent* and consider a number of examples of quantum graphs.

### 5.2. The case of a star-graph

**Theorem 5.1.** *Consider a finite compact metric graph $\Gamma_\delta$ which is a star, i.e., $\Gamma_\delta$ has exactly $N+1$, $N \geq 2$ vertices*[3] *with the only internal vertex being $V_{N+1}$. Assume that all edge lengths are rationally independent and that matching conditions at all vertices are of $\delta$ type. Let $A_{\vec{\alpha}}$ and $A_{\vec{\tilde{\alpha}}}$ be two graph Laplacians on $\Gamma_\delta$. Assume that their spectra coincide counting multiplicities. Then $\vec{\alpha} = \vec{\tilde{\alpha}}$.*

*Proof.* As in the proof of Theorem 4.1, one immediately ascertains the equality

$$\det(M(\lambda) - B) = \det(M(\lambda) - \widetilde{B}), \tag{15}$$

since in this case it is easy to see that $\exp(a) = 1$ in (11). By Theorem 3.2,

$$\det(M(\lambda) - B) = \begin{vmatrix} -\mu \cot \mu l_1 - \alpha_1 & 0 & \dots & \mu \csc \mu l_1 \\ 0 & -\mu \cot \mu l_2 - \alpha_2 & \dots & \mu \csc \mu l_2 \\ \vdots & \vdots & \dots & \vdots \\ \mu \csc \mu l_1 & \mu \csc \mu l_2 & \dots & -\mu \sum_{t=1}^{N} \cot \mu l_t - \alpha_{N+1} \end{vmatrix},$$

where $\mu = \sqrt{\lambda}$. Calculating this determinant explicitly, one has

$$\det(M(\lambda) - B) = K_{N+1}(\mu)\mu^{N+1} + K_N(\mu, \vec{\alpha})\mu^N + \dots + K_0(\mu, \vec{\alpha}),$$

where $K_j(\mu)$ are trigonometric functions. Here

$$\mu^{N+1} K_{N+1}(\mu) = \det M(\lambda)$$

and (up to the sign)

$$K_N(\mu) = \sum_{t=1}^{N+1} \alpha_t \prod_{j=1}^{N} \cot \mu l_j - \sum_{t,i=1;\ t<i}^{N} (\alpha_t + \alpha_i) \prod_{j \neq t,i} \cot \mu l_j, \tag{16}$$

---

[3] If $N = 2$, the coupling constant at the internal vertex is without loss of generality assumed to be non-zero, since otherwise the star-graph considered reduces to a single interval by graph cleaning, see, e.g., [25].

Utilizing the standard linear independence argument in conjunction with the condition of rational independence of edge lengths, one obtains

$$\sum_{t=1}^{N+1} \alpha_t = \sum_{t=1}^{N+1} \tilde{\alpha}_t; \tag{17}$$

$$\alpha_t + \alpha_i = \tilde{\alpha}_t + \tilde{\alpha}_i, \quad t, i = \overline{1, N}, \ t < i. \tag{18}$$

Using the first trace formula of Theorem 3.2, one additionally has

$$\sum_{t=1}^{N} \alpha_t + \frac{\alpha_{N+1}}{N} = \sum_{t=1}^{N} \tilde{\alpha}_t + \frac{\tilde{\alpha}_{N+1}}{N},$$

which together with (17) implies $\alpha_{N+1} = \tilde{\alpha}_{N+1}$. Then (17) admits the form

$$\sum_{t=1}^{N} \alpha_t = \sum_{t=1}^{N} \tilde{\alpha}_t. \tag{19}$$

On the other hand, (18) immediately yields

$$(N-1)\alpha_1 + \sum_{t=2}^{N} \alpha_t = (N-1)\tilde{\alpha}_1 + \sum_{t=2}^{N} \tilde{\alpha}_t. \tag{20}$$

Comparing (20) with (17), one obtains

$$\alpha_1 = \tilde{\alpha}_1$$

and, proceeding analogously, the same result for all $i = \overline{2, N}$. □

**5.3. The case of a chain graph.** We will demonstrate that unless the chain is also a star, it no longer suffices to consider linear relations between coupling constants in order to ascertain the fact that there are no isospectral graphs in this situation. In the present paper, we will only consider the chain of exactly 4 vertices, which is already enough to illustrate the point made. The general case of an arbitrary compact chain can in fact be reduced to this one by a corresponding recurrence relation.

**Proposition 5.1.** *Let $\Gamma_\delta$ be the metric $A_4$ graph with $\delta$ type matching conditions at all vertices and having rationally independent edge lengths. Let $A_{\vec{\alpha}}$ and $A_{\vec{\tilde{\alpha}}}$ be two graph Laplacians on $\Gamma_\delta$. Let additionally coupling constants at both internal vertices be non-zero for both Laplacians.*[4] *Assume that their spectra coincide counting multiplicities. Then $\vec{\alpha} = \vec{\tilde{\alpha}}$.*

[4] Again, this assumption just guarantees that the graph $\Gamma$ does not reduce by the procedure of cleaning [25] to either a star or an interval.

*Proof.* As in the proof of Theorem 5.1, (15) holds. By Theorem 3.2, $\det(M(\lambda)-B)$ is given by

$$\begin{vmatrix} -\mu\cot\mu l_1-\alpha_1 & \mu\csc\mu l_1 & 0 & 0 \\ \mu\csc\mu l_1 & -\mu\sum_{n=1}^{2}\cot\mu l_t-\alpha_2 & \mu\csc\mu l_2 & 0 \\ 0 & \mu\csc\mu l_2 & -\mu\sum_{n=2}^{3}\cot\mu l_t-\alpha_3 & \mu\csc\mu l_3 \\ 0 & 0 & \mu\csc\mu l_3 & -\mu\cot\mu l_3-\alpha_4 \end{vmatrix},$$

where the enumeration of vertices is chosen so that that the internal vertices are $V_2$ and $V_3$, the edge between them being of length $l_2$, whereas the edge of length $l_1$ connects $V_2$ to $V_1$ and consequently $l_3$ connects $V_3$ and $V_4$.

Proceeding exactly as in the proof of Theorem 5.1, one gets the following linear relations between $\vec{\alpha}$ and $\vec{\tilde{\alpha}}$ from consideration of the coefficient $K_3(\mu,\vec{\alpha})$:

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{\alpha}_4;$$

$$\alpha_1+\alpha_2+\alpha_4=\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_4;$$

$$\alpha_1+\alpha_4=\tilde{\alpha}_1+\tilde{\alpha}_4;$$

$$\alpha_1+\alpha_3+\alpha_4=\tilde{\alpha}_1+\tilde{\alpha}_3+\tilde{\alpha}_4.$$

The matrix of this system of linear equations has rank equal to 3 with kernel $(1,0,0,-1)^T$. This of course immediately implies $\alpha_2=\tilde{\alpha}_2$, $\alpha_3=\tilde{\alpha}_3$, but still does not prove the claim. Note that the first trace formula of Theorem 4.1 in fact follows from the relations above and thus provides no additional information.

Consider the coefficient $K_2(\mu,\vec{\alpha})$. This equips us with 4 quadratic relations on the coupling constants which together with the linear relations yield the claim. □

### 5.4. The case of mixed types with double edges

**Proposition 5.2.** *Let $\Gamma_8$ be the graph of Example 3.1. Let the edge lengths $l_j$, $j=1,\dots,4$ be rationally independent. Let $A_{\vec{\alpha}}$ and $A_{\vec{\tilde{\alpha}}}$ be two graph Laplacians on $\Gamma_8$. Assume that their spectra coincide counting multiplicities. Then $\vec{\alpha}=\vec{\tilde{\alpha}}$.*

*Proof.* Assume first that $\alpha_3,\alpha_4,\tilde{\alpha}_3,\tilde{\alpha}_4\neq 0$. As in the proof of Theorem 5.1, one has

$$\frac{1}{\alpha_3\alpha_4}\det(M(\lambda)-\operatorname{diag}\{\alpha_1,\dots,\alpha_4\})=\frac{1}{\tilde{\alpha}_3\tilde{\alpha}_4}\det(M(\lambda)-\operatorname{diag}\{\tilde{\alpha}_1,\dots,\tilde{\alpha}_4\}).$$

By Theorem 3.2, one gets the decomposition

$$\det(M(\lambda) - \operatorname{diag}\{\alpha_1, \dots, \alpha_4\}) \\ = K_2(\mu)\mu^2 + K_1(\mu, \vec{\alpha})\mu + K_0(\mu, \vec{\alpha}) + K_{-1}(\mu, \vec{\alpha})\mu^{-1} + K_{-2}(\mu)\mu^{-2}$$

with trigonometric coefficients $K_j$. The only difference with Theorem 5.1 is that here negative powers of $\mu$ appear. What's more, explicit calculation shows that both $K_1$ and $K_{-1}$ already are non-linear with respect to the coupling constants. Nevertheless, one ascertains almost immediately that these suffice to complete the proof. Consideration of the remaining cases differs only in minute details. □

We remark that the same result holds for the graph $\Gamma_8$ if all vertices are assumed to be of the same type (either $\delta$ or $\delta'$). Thus, a graph possessing double edges may prevent isospectrality.

**5.5. Isospectrality can happen if even cycles are allowed.** We will show that a cycle $C_4$ of four vertices even in the situation of rationally independent edge lengths allows for existence of isospectral Laplacians.

**Proposition 5.3.** *Let $\Gamma_8 = C_4$ with all four vertices of $\delta$ type and rationally independent edge lengths $l_j$, $j = 1, \dots, 4$. Assume that $A_{\vec{\alpha}}$ and $A_{\vec{\tilde{\alpha}}}$ are two graph Laplacians on $\Gamma_8$. Let additionally coupling constants at all vertices be non-zero for both Laplacians.*[5] *Assume that their spectra coincide counting multiplicities. Then $\vec{\alpha} = \vec{\tilde{\alpha}}$, unless $\vec{\alpha} = (\alpha, -\alpha, \alpha, -\alpha)$ and $\vec{\tilde{\alpha}} = (-\alpha, \alpha, -\alpha, \alpha)$ for some real $\alpha$. In the latter case $A_{\vec{\alpha}}$ and $A_{\vec{\tilde{\alpha}}}$ are isospectral.*

*Proof.* As in the proof of Theorem 5.1, (15) holds. Calculating this determinant explicitly on the basis of Theorem 3.2, one has

$$\det(M(\lambda) - B) = K_4(\mu)\mu^4 + K_3(\mu, \vec{\alpha})\mu^3 + \cdots + K_0(\mu, \vec{\alpha}),$$

where $K_j(\mu)$ are trigonometric functions. Calculating these functions explicitly and utilizing the standard linear independence argument in conjunction with the condition of rational independence of edge lengths, one obtains

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4; \tag{21}$$

[5] This assumption guarantees that the graph $\Gamma$ does not reduce by the procedure of cleaning [25] to a cycle of lower dimension.

$$\alpha_j \sum_{i\neq j} \alpha_i = \tilde{\alpha}_j \sum_{i\neq j} \tilde{\alpha}_i, \quad i, j = \overline{1,4}, \tag{22a}$$

$$\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_3 + \tilde{\alpha}_1\tilde{\alpha}_4 + \tilde{\alpha}_2\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{\alpha}_4, \tag{22b}$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_2 + \tilde{\alpha}_1\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{\alpha}_4 + \tilde{\alpha}_3\tilde{\alpha}_4, \tag{22c}$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_3\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_2 + \tilde{\alpha}_1\tilde{\alpha}_4 + \tilde{\alpha}_2\tilde{\alpha}_3 + \tilde{\alpha}_3\tilde{\alpha}_4; \tag{22d}$$

$$\alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_3\tilde{\alpha}_4 + \tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4, \tag{23a}$$

$$\alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_4 + \tilde{\alpha}_1\tilde{\alpha}_3\tilde{\alpha}_4, \tag{23b}$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3 + \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_4, \tag{23c}$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4; \tag{23d}$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4. \tag{24}$$

In variables

$$x_{i,j} := \alpha_i\alpha_j - \tilde{\alpha}_i\tilde{\alpha}_j, \quad i, j = 1, 4, \ i < j,$$

the system (22) turns out to be a linear system with matrix of rank 6. Hence,

$$\alpha_i\alpha_j = \tilde{\alpha}_i\tilde{\alpha}_j, \ i, j = \overline{1,4}. \tag{25}$$

As for (23), one reduces this system to a linear one by the following change of variables:

$$x_1 = \alpha_1\alpha_2\alpha_3 - \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_3, \quad x_2 = \alpha_1\alpha_2\alpha_4 - \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_4, \tag{26a}$$

$$x_3 = \alpha_1\alpha_3\alpha_4 - \tilde{\alpha}_1\tilde{\alpha}_3\tilde{\alpha}_4, \quad x_4 = \alpha_2\alpha_3\alpha_4 - \tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4. \tag{26b}$$

Since the rank of the corresponding matrix is equal to 3, the general solution of the latter system is non-trivial. Further, it has the form $(\beta, -\beta, \beta, -\beta)$. Consider the two possibilities separately.

(i) If $\beta = 0$, then (24) immediately implies $\alpha_i - \tilde{\alpha}_i = 0, \ j = \overline{1,4}$.

(ii) Let now $\beta \neq 0$. Denoting $u := \alpha_2\alpha_3,\ v := \alpha_2\alpha_4,\ w := \alpha_3\alpha_4$ and using (26) and (25) one has

$$(\alpha_1 - \tilde{\alpha}_1)u = \beta,\ (\alpha_1 - \tilde{\alpha}_1)v = -\beta,\ (\alpha_1 - \tilde{\alpha}_1)w = \beta,$$

whence $u = -v = w$, or equivalently $\alpha_2\alpha_3 = -\alpha_2\alpha_4 = \alpha_3\alpha_4$. Thus,

$$\alpha_3 = -\alpha_4,\ -\alpha_2 = \alpha_3.$$

In the same way one might also ascertain that $\alpha_1 = \alpha_3$. Finally one gets $\alpha_1 = \alpha_3 = \alpha,\ \alpha_2 = \alpha_4 = -\alpha, \tilde{\alpha}_1 = \tilde{\alpha}_3 = \tilde{\alpha},\ \tilde{\alpha}_2 = \tilde{\alpha}_4 = -\tilde{\alpha}$.

From (24) it follows that $|\alpha| = |\tilde{\alpha}|$. If $\alpha \neq \tilde{\alpha}$, the only remaining possibility of isospectral graph Laplacian turns out to be $\tilde{\alpha} = -\alpha$.

An explicit calculation shows that in this case one indeed gets the required property (15). □

Some remarks are in order.

1. Proposition 5.3 admits generalization to an arbitrary cycle $C_{2N}$, $M > 1$. As for the cycle $C_2$, technically one could prove isospectrality if and only if $\vec{\alpha} = (\alpha, -\alpha)$ and $\vec{\tilde{\alpha}} = -\vec{\alpha}$, however, this does not lead to non-trivial isospectral configurations since the corresponding two graphs are actually identical.

2. As for the case of cycles $C_{2N+1}$, $N \geq 1$, one can ascertain much along the same lines that such odd cycles do not permit isospectrality.

3. In [9] we show that essentially there are no other examples of isospectral graph Laplacians provided that all graph vertices are of $\delta$ type.

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Yulia Ershova, Department of Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY, U.K.

e-mail: julija.ershova@gmail.com

Irina I. Karpenko, Department of Algebra and Functional Analysis,
V. I. Vernadsky Taurida National university. 95007 Ukraine, Simferopol,
4 Vernadsky pr.

e-mail: i_karpenko@ukr.net

Alexander V. Kiselev, National Pedagogical Dragomanov University, 01601, Kiev,
Pirogova str 9, Ukraine

e-mail: alexander.v.kiselev@gmail.com